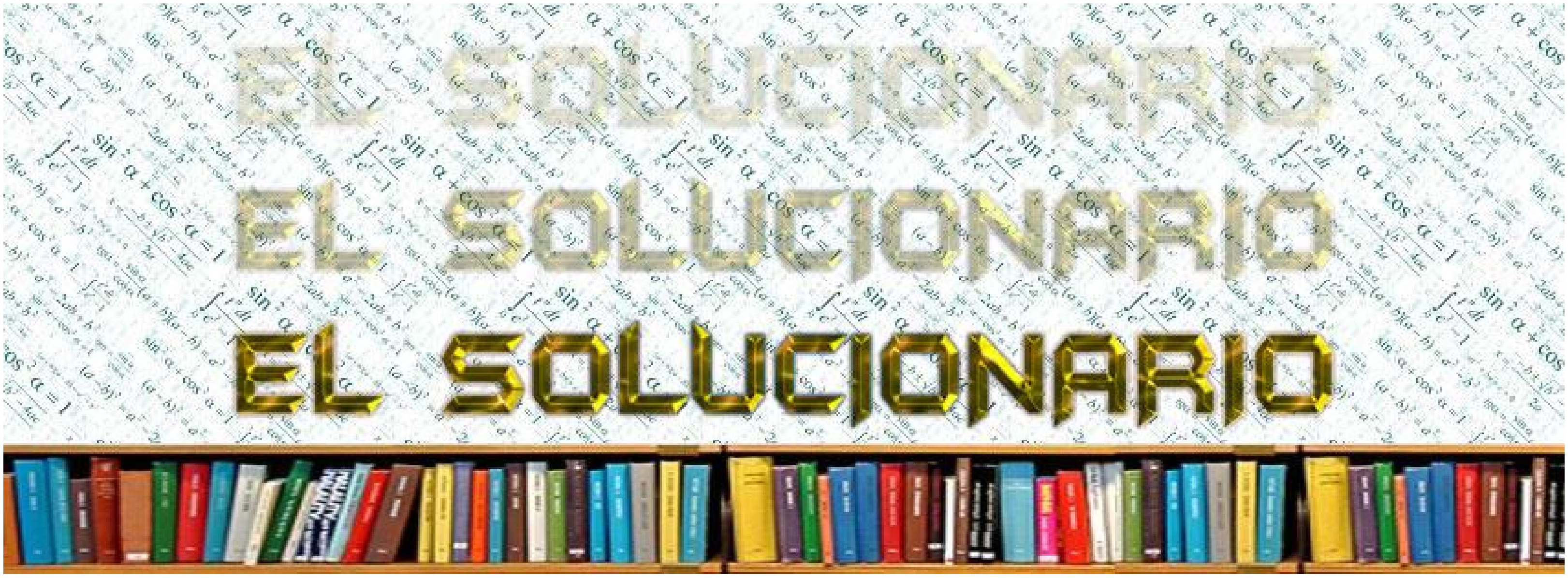
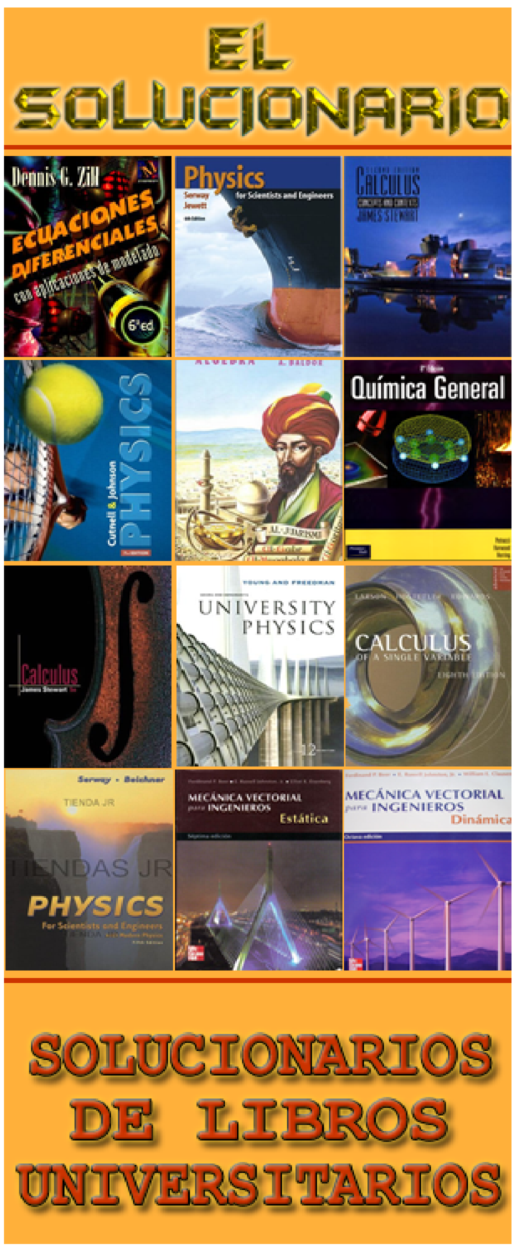


http://www.elsolucionario.blogspot.com



LIBROS UNIVERISTARIOS
Y SOLUCIONARIOS DE
MUCHOS DE ESTOS LIBROS

LOS SOLUCIONARIOS
CONTIENEN TODOS LOS
EJERCICIOS DEL LIBRO
RESUELTOS Y EXPLICADOS
DE FORMA CLARA

VISITANOS PARA
DESARGALOS GRATIS.

Section 1.1

C01S01.001: If $f(x) = \dfrac{1}{x}$, then:

(a) $f(-a) = \dfrac{1}{-a} = -\dfrac{1}{a}$;

(b) $f(a^{-1}) = \dfrac{1}{a^{-1}} = a$;

(c) $f(\sqrt{a}\,) = \dfrac{1}{\sqrt{a}} = \dfrac{1}{a^{1/2}} = a^{-1/2}$;

(d) $f(a^2) = \dfrac{1}{a^2} = a^{-2}$.

C01S01.002: If $f(x) = x^2 + 5$, then:

(a) $f(-a) = (-a)^2 + 5 = a^2 + 5$;

(b) $f(a^{-1}) = (a^{-1})^2 + 5 = a^{-2} + 5 = \dfrac{1}{a^2} + 5 = \dfrac{1 + 5a^2}{a^2}$;

(c) $f(\sqrt{a}\,) = (\sqrt{a}\,)^2 + 5 = a + 5$;

(d) $f(a^2) = (a^2)^2 + 5 = a^4 + 5$.

C01S01.003: If $f(x) = \dfrac{1}{x^2 + 5}$, then:

(a) $f(-a) = \dfrac{1}{(-a)^2 + 5} = \dfrac{1}{a^2 + 5}$;

(b) $f(a^{-1}) = \dfrac{1}{(a^{-1})^2 + 5} = \dfrac{1}{a^{-2} + 5} = \dfrac{1 \cdot a^2}{a^{-2} \cdot a^2 + 5 \cdot a^2} = \dfrac{a^2}{1 + 5a^2}$;

(c) $f(\sqrt{a}\,) = \dfrac{1}{(\sqrt{a}\,)^2 + 5} = \dfrac{1}{a + 5}$;

(d) $f(a^2) = \dfrac{1}{(a^2)^2 + 5} = \dfrac{1}{a^4 + 5}$.

C01S01.004: If $f(x) = \sqrt{1 + x^2 + x^4}$, then:

(a) $f(-a) = \sqrt{1 + (-a)^2 + (-a)^4} = \sqrt{1 + a^2 + a^4}$;

(b) $f(a^{-1}) = \sqrt{1 + (a^{-1})^2 + (a^{-1})^4} = \sqrt{1 + a^{-2} + a^{-4}} = \sqrt{\dfrac{(a^4) \cdot (1 + a^{-2} + a^{-4})}{a^4}}$

$$= \sqrt{\frac{a^4 + a^2 + 1}{a^4}} = \frac{\sqrt{a^4 + a^2 + 1}}{\sqrt{a^4}} = \frac{\sqrt{a^4 + a^2 + 1}}{a^2};$$

(c) $f(\sqrt{a}\,) = \sqrt{1 + (\sqrt{a}\,)^2 + (\sqrt{a}\,)^4} = \sqrt{1 + a + a^2}$;

(d) $f(a^2) = \sqrt{1 + (a^2)^2 + (a^4)^2} = \sqrt{1 + a^4 + a^8}$.

C01S01.005: If $g(x) = 3x + 4$ and $g(a) = 5$, then $3a + 4 = 5$, so $3a = 1$; therefore $a = \frac{1}{3}$.

C01S01.006: If $g(x) = \dfrac{1}{2x - 1}$ and $g(a) = 5$, then:

$$\begin{aligned}
\frac{1}{2a-1} &= 5;\\
1 &= 5\cdot(2a-1);\\
1 &= 10a-5;\\
10a &= 6;\\
a &= \frac{3}{5}.
\end{aligned}$$

C01S01.007: If $g(x) = \sqrt{x^2+16}$ and $g(a) = 5$, then:

$$\begin{aligned}
\sqrt{a^2+16} &= 5;\\
a^2+16 &= 25;\\
a^2 &= 9;\\
a &= 3 \text{ or } a = -3.
\end{aligned}$$

C01S01.008: If $g(x) = x^3 - 3$ and $g(a) = 5$, then $a^3 - 3 = 5$, so $a^3 = 8$. Hence $a = 2$.

C01S01.009: If $g(x) = \sqrt[3]{x+25} = (x+25)^{1/3}$ and $g(a) = 5$, then

$$\begin{aligned}
(a+25)^{1/3} &= 5;\\
a+25 &= 5^3 = 125;\\
a &= 100.
\end{aligned}$$

C01S01.010: If $g(x) = 2x^2 - x + 4$ and $g(a) = 5$, then:

$$\begin{aligned}
2a^2 - a + 4 &= 5;\\
2a^2 - a - 1 &= 0;\\
(2a+1)(a-1) &= 0;\\
2a+1 &= 0 \text{ or } a-1 = 0;\\
a &= -\frac{1}{2} \text{ or } a = 1.
\end{aligned}$$

C01S01.011: If $f(x) = 3x - 2$, then

$$\begin{aligned}
f(a+h) - f(a) &= [3(a+h)-2] - [3a-2]\\
&= 3a + 3h - 2 - 3a + 2 = 3h.
\end{aligned}$$

C01S01.012: If $f(x) = 1 - 2x$, then

$$f(a+h) - f(a) = [1 - 2(a+h)] - [1 - 2a] = 1 - 2a - 2h - 1 + 2a = -2h.$$

C01S01.013: If $f(x) = x^2$, then

$$\begin{aligned} f(a+h) - f(a) &= (a+h)^2 - a^2 \\ &= a^2 + 2ah + h^2 - a^2 = 2ah + h^2 = h \cdot (2a+h). \end{aligned}$$

C01S01.014: If $f(x) = x^2 + 2x$, then

$$\begin{aligned} f(a+h) - f(a) &= [(a+h)^2 + 2(a+h)] - [a^2 + 2a] \\ &= a^2 + 2ah + h^2 + 2a + 2h - a^2 - 2a = 2ah + h^2 + 2h = h \cdot (2a + h + 2). \end{aligned}$$

C01S01.015: If $f(x) = \dfrac{1}{x}$, then

$$\begin{aligned} f(a+h) - f(a) &= \frac{1}{a+h} - \frac{1}{a} = \frac{a}{a(a+h)} - \frac{a+h}{a(a+h)} \\ &= \frac{a - (a+h)}{a(a+h)} = \frac{-h}{a(a+h)}. \end{aligned}$$

C01S01.016: If $f(x) = \dfrac{2}{x+1}$, then

$$\begin{aligned} f(a+h) - f(a) &= \frac{2}{a+h+1} - \frac{2}{a+1} = \frac{2(a+1)}{(a+h+1)(a+1)} - \frac{2(a+h+1)}{(a+h+1)(a+1)} \\ &= \frac{2a+2}{(a+h+1)(a+1)} - \frac{2a+2h+2}{(a+h+1)(a+1)} = \frac{(2a+2) - (2a+2h+2)}{(a+h+1)(a+1)} \\ &= \frac{2a+2-2a-2h-2}{(a+h+1)(a+1)} = \frac{-2h}{(a+h+1)(a+1)}. \end{aligned}$$

C01S01.017: If $x > 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = 1.$$

If $x < 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1.$$

We are given $f(0) = 0$, so the range of f is $\{-1, 0, 1\}$. That is, the range of f is the set consisting of the three real numbers -1, 0, and 1.

C01S01.018: Given $f(x) = [\![3x]\!]$, we see that

$$f(x) = 0 \quad \text{if} \quad 0 \leqq x < \tfrac{1}{3},$$
$$f(x) = 1 \quad \text{if} \quad \tfrac{1}{3} \leqq x < \tfrac{2}{3},$$
$$f(2) = 2 \quad \text{if} \quad \tfrac{2}{3} \leqq x < 1;$$

moreover,

$$f(x) = -3 \quad \text{if} \quad -1 \leqq x < -\tfrac{2}{3},$$
$$f(x) = -2 \quad \text{if} \quad -\tfrac{2}{3} \leqq x < -\tfrac{1}{3},$$
$$f(x) = -1 \quad \text{if} \quad -\tfrac{1}{3} \leqq x < 0.$$

In general, if m is any integer, then

$$f(x) = 3m \quad \text{if} \quad m \leqq x < m + \tfrac{1}{3},$$
$$f(x) = 3m + 1 \quad \text{if} \quad m + \tfrac{1}{3} \leqq x < m + \tfrac{2}{3},$$
$$f(x) = 3m + 2 \quad \text{if} \quad m + \tfrac{2}{3} \leqq x < m + 1.$$

Because every integer is equal to $3m$ or to $3m + 1$ or to $3m + 2$ for some integer m, we see that the range of f includes the set $\boldsymbol{Z}$ of all integers. Because f can assume no values other than integers, we can conclude that the range of f is exactly $\boldsymbol{Z}$.

C01S01.019: Given $f(x) = (-1)^{[\![x]\!]}$, we first note that the values of the exponent $[\![x]\!]$ consist of all the integers and no other numbers. So all that matters about the exponent is whether it is an even integer or an odd integer, for if even then $f(x) = 1$ and if odd then $f(x) = -1$. No other values of $f(x)$ are possible, so the range of f is the set consisting of the two numbers -1 and 1.

C01S01.020: If $0 < x \leqq 1$, then $f(x) = 34$. If $1 < x \leqq 2$ then $f(x) = 34 + 21 = 55$. If $2 < x \leqq 3$ then $f(x) = 34 + 2 \cdot 21 = 76$. We continue in this way and conclude with the observation that if $11 < x < 12$ then $f(x) = 34 + 11 \cdot 21 = 265$. So the range of f is the set

$$\{34,\ 55,\ 76,\ 97,\ 118,\ 139,\ 160,\ 181,\ 202,\ 223,\ 244,\ 265\}.$$

C01S01.021: Given $f(x) = 10 - x^2$, note that for every real number x, x^2 is defined, and for every such real number x^2, $10 - x^2$ is also defined. Therefore the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S01.022: Given $f(x) = x^3 + 5$, we note that for each real number x, x^3 is defined; moreover, for each such real number x^3, $x^3 + 5$ is also defined. Thus the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S01.023: Given $f(t) = \sqrt{t^2}$, we observe that for every real number t, t^2 is defined and nonnegative, and hence that $\sqrt{t^2}$ is defined as well. Therefore the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S01.024: Given $g(t) = \left(\sqrt{t}\,\right)^2$, we observe that $\sqrt{t}$ is defined exactly when $t \geqq 0$. In this case, $\left(\sqrt{t}\,\right)^2$ is also defined, and hence the domain of g is the set $[0, +\infty)$ of all nonnegative real numbers.

C01S01.025: Given $f(x) = \sqrt{3x - 5}$, we note that $3x - 5$ is defined for all real numbers x, but that its square root will be defined when and only when $3x - 5$ is nonnegative; that is, when $3x - 5 \geqq 0$, so that $x \geqq \frac{5}{3}$. So the domain of f consists of all those real numbers x in the interval $\left[\frac{5}{3}, +\infty\right)$.

C01S01.026: Given $g(t) = \sqrt[3]{t+4} = (t+4)^{1/3}$, we note that $t+4$ is defined for every real number t and the cube root of $t+4$ is defined for every possible resulting value of $t+4$. Therefore the domain of g is the set $\boldsymbol{R}$ of all real numbers.

C01S01.027: Given $f(t) = \sqrt{1-2t}$, we observe that $1-2t$ is defined for every real number t, but that its square root is defined only when $1-2t$ is nonnegative. We solve the inequality $1-2t \geqq 0$ to find that $f(t)$ is defined exactly when $t \leqq \frac{1}{2}$. Hence the domain of f is the interval $\left(-\infty, \frac{1}{2}\right]$.

C01S01.028: Given

$$g(x) = \frac{1}{(x+2)^2},$$

we see that $(x+2)^2$ is defined for every real number x, but that $g(x)$, its reciprocal, will be defined only when $(x+2)^2 \neq 0$; that is, when $x+2 \neq 0$. So the domain of g consists of those real numbers $x \neq -2$.

C01S01.029: Given

$$f(x) = \frac{2}{3-x},$$

we see that $3-x$ is defined for all real values of x, but that $f(x)$, double its reciprocal, is defined only when $3-x \neq 0$. So the domain of f consists of those real numbers $x \neq 3$.

C01S01.030: Given

$$g(t) = \sqrt{\frac{2}{3-t}},$$

it is necessary that $3-t$ be both nonzero (so that its reciprocal is defined) and nonnegative (so that the square root is defined). Thus $3-t > 0$, and therefore the domain of g consists of those real numbers $t < 3$.

C01S01.031: Given $f(x) = \sqrt{x^2+9}$, observe that for each real number x, x^2+9 is defined and, moreover, is positive. So its square root is defined for every real number x. Hence the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S01.032: Given

$$h(z) = \frac{1}{\sqrt{4-z^2}},$$

we note that $4-z^2$ is defined for every real number z, but that its square root will be defined only if $4-z^2 \geqq 0$. Moreover, the square root cannot be zero, else its reciprocal will be undefined, so we need to solve the inequality $4-z^2 > 0$; that is, $z^2 < 4$. The solution is $-2 < z < 2$, so the domain of h is the open interval $(-2, 2)$.

C01S01.033: Given $f(x) = \sqrt{4-\sqrt{x}}$, note first that we require $x \geqq 0$ in order that $\sqrt{x}$ be defined. In addition, we require $4-\sqrt{x} \geqq 0$ so that *its* square root will be defined as well. So we solve [simultaneously] $x \geqq 0$ and $\sqrt{x} \leqq 4$ to find that $0 \leqq x \leqq 16$. So the domain of f is the closed interval $[0, 16]$.

C01S01.034: Given

$$f(x) = \sqrt{\frac{x+1}{x-1}},$$

we require that $x \neq 1$ so that the fraction is defined. In addition we require that the fraction be nonnegative so that its square root will be defined. These conditions imply that both numerator and denominator be positive or that both be negative; moreover, the numerator may also be zero. But if the denominator is positive then the [larger] numerator will be positive as well; if the numerator is nonpositive then the [smaller] denominator will be negative. So the domain of f consists of those real numbers for which *either* $x - 1 > 0$ *or* $x + 1 \leqq 0$; that is, either $x > 1$ or $x \leqq -1$. So the domain of f is the union of the two intervals $(-\infty, -1]$ and $(1, +\infty)$. Alternatively, it consists of those real numbers x *not* in the interval $(-1, 1]$.

C01S01.035: Given:

$$g(t) = \frac{t}{|t|}.$$

This fraction will be defined whenever its denominator is nonzero, thus for all real numbers $t \neq 0$. So the domain of g consists of the nonzero real numbers; that is, the union of the two intervals $(-\infty, 0)$ and $(0, +\infty)$.

C01S01.036: If a square has edge length x, then its area A is given by $A = x^2$ and its perimeter P is given by $P = 4x$. To express A in terms of P:

$$x = \tfrac{1}{4}P;$$

$$A = x^2 = \left(\tfrac{1}{4}P\right)^2 = \tfrac{1}{16}P^2.$$

Thus to express A as a function of P, we write

$$A(P) = \tfrac{1}{16}P^2, \quad 0 \leqq P < +\infty.$$

(It will be convenient later in the course to allow the possibility that P, x, and A are zero. If this produces an answer that fails to meet real-world criteria for a solution, then that possibility can simply be eliminated when the answer to the problem is stated.)

C01S01.037: If a circle has radius r, then its circumference C is given by $C = 2\pi r$ and its area A by $A = \pi r^2$. To express C in terms of A, we first express r in terms of A, then substitute in the formula for C:

$$A = \pi r^2; \qquad r = \sqrt{\frac{A}{\pi}};$$

$$C = 2\pi r = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{\pi^2 A}{\pi}} = 2\sqrt{\pi A}.$$

Therefore to express C as a function of A, we write

$$C(A) = 2\sqrt{\pi A}, \qquad 0 \leqq A < +\infty.$$

It is also permissible simply to write $C(A) = 2\sqrt{\pi A}$ without mentioning the domain, because the "default" domain is correct. In the first displayed equation we do not write $r = \pm\sqrt{A/\pi}$ because we know that r is never negative.

C01S01.38: If r denotes the radius of the sphere, then its volume is given by $V = \frac{4}{3}\pi r^3$ and its surface area by $S = 4\pi r^2$. Hence

$$r = \frac{1}{2}\sqrt{\frac{S}{\pi}}\ ;$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot \frac{1}{8}\left(\frac{S}{\pi}\right)^{3/2} = \frac{1}{6}\pi\left(\frac{S}{\pi}\right)^{3/2}.$$

Answer: $V(S) = \dfrac{1}{6}\pi\left(\dfrac{S}{\pi}\right)^{3/2}, \quad 0 \leqq S < +\infty.$

C01S01.039: To avoid decimals, we note that a change of 5°C is the same as a change of 9°F, so when the temperature is 10°C it is $32 + 18 = 50$°F; when the temperature is 20°C then it is $32 + 2 \cdot 18 = 68$°F. In general we get the Fahrenheit temperature F by adding 32 to the product of $\frac{1}{10}C$ and 18, where C is the Celsius temperature. That is,

$$F = 32 + \frac{9}{5}C,$$

and therefore $C = \frac{5}{9}(F - 32)$. Answer:

$$C(F) = \frac{5}{9}(F - 32), \quad F > -459.67.$$

C01S01.040: Suppose that a rectangle has base length x and perimeter 100. Let h denote the height of such a rectangle. Then $2x + 2h = 100$, so that $h = 50 - x$. Because $x \geqq 0$ and $h \geqq 0$, we see that $0 \leqq x \leqq 50$. The area A of the rectangle is xh, so that

$$A(x) = x(50 - x), \quad 0 \leqq x \leqq 50.$$

C01S01.041: Let y denote the height of such a rectangle. The rectangle is inscribed in a circle of diameter 4, so the bottom side x and the left side y are the two legs of a right triangle with hypotenuse 4. Consequently $x^2 + y^2 = 16$, so $y = \sqrt{16 - x^2}$ (not $-\sqrt{16 - x^2}$ because $y \geqq 0$). Because $x \geqq 0$ and $y \geqq 0$, we see that $0 \leqq x \leqq 4$. The rectangle has area $A = xy$, so

$$A(x) = x\sqrt{16 - x^2}, \quad 0 \leqq x \leqq 4.$$

C01S042.042: We take the problem to mean that current production is 200 barrels per day per well, that if one new well is drilled then the 21 wells will produce 195 barrels per day per well; in general, that if x new wells are drilled then the $20 + x$ wells will produce $200 - 5x$ barrels per day per well. So total production would be $p = (20 + x)(200 - 5x)$ barrels per day. But because $200 - 5x \geqq 0$, we see that $x \leqq 40$. Because $x \geqq 0$ as well (you don't "undrill" wells), here's the answer:

$$p(x) = 4000 + 100x - 5x^2, \quad 0 \leqq x \leqq 40, \quad x \text{ an integer.}$$

C01S01.043: The square base of the box measures x by x centimeters; let y denote its height (in centimeters). Because the volume of the box is 324 cm^3, we see that $x^2y = 324$. The base of the box costs $2x^2$ cents, each of its four sides costs xy cents, and its top costs x^2 cents. So the total cost of the box is

$$C = 2x^2 + 4xy + x^2 = 3x^2 + 4xy. \tag{1}$$

Because $x > 0$ and $y > 0$ (the box has positive volume), but because y can be arbitrarily close to zero (as well as x), we see also that $0 < x < +\infty$. We use the equation $x^2y = 324$ to eliminate y from Eq. (1) and thereby find that

$$C(x) = 3x^2 + \frac{1296}{x}, \quad 0 < x < +\infty.$$

C01S01.044: If the rectangle is rotated around its side S of length x to produce a cylinder, then x will also be the height of the cylinder. Let y denote the length of the two sides perpendicular to S; then y will be the radius of the cylinder; moreover, the perimeter of the original rectangle is $2x + 2y = 36$. Hence $y = 18 - x$. Note also that $x \geqq 0$ and that $x \leqq 18$ (because $y \geqq 0$). The volume of the cylinder is $V = \pi y^2 x$, and so

$$V(x) = \pi x(18 - x)^2, \quad 0 \leqq x \leqq 18.$$

C01S01.045: Let h denote the height of the cylinder. Its radius is r, so its volume is $\pi r^2 h = 1000$. The total surface area of the cylinder is

$$A = 2\pi r^2 + 2\pi rh \qquad \text{(look inside the front cover of the book);}$$

$$h = \frac{1000}{\pi r^2}, \qquad \text{so}$$

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

Now r cannot be negative; r cannot be zero, else $\pi r^2 h \neq 1000$. But r can be arbitrarily small positive as well as arbitrarily large positive (by making h sufficiently close to zero). Answer:

$$A(r) = 2\pi r^2 + \frac{2000}{r}, \quad 0 < r < +\infty.$$

C01S01.046: Let y denote the height of the box (in centimeters). Then

$$2x^2 + 4xy = 600, \qquad \text{so that} \qquad y = \frac{600 - 2x^2}{4x}. \tag{1}$$

The volume of the box is

$$V = x^2 y = \frac{(600 - 2x^2) \cdot x^2}{4x} = \frac{1}{4}(600x - 2x^3) = \frac{1}{2}(300x - x^3)$$

by Eq. (1). Also $x > 0$ by Eq. (1), but the maximum value of x is attained when Eq. (1) forces y to be zero, at which point $x = \sqrt{300} = 10\sqrt{3}$. Answer:

$$V(x) = \frac{300x - x^3}{2}, \quad 0 < x \leqq 10\sqrt{3}.$$

C01S01.047: The base of the box will be a square measuring $50 - 2x$ in. on each side, so the open-topped box will have that square as its base and four rectangular sides each measuring $50 - 2x$ by x (the height of the box). Clearly $0 \leqq x$ and $2x \leqq 50$. So the volume of the box will be

$$V(x) = x(50 - 2x)^2, \quad 0 \leqq x \leqq 25.$$

C01S01.048: Recall that $A(x) = x(50 - x)$, $0 \leqq x \leqq 50$. Here is a table of a few values of the function A at some special numbers in its domain:

x	0	5	10	15	20	25	30	35	40	45	50
A	0	225	400	525	600	625	600	525	400	225	0

It appears that when $x = 25$ (so the rectangle is a square), the rectangle has maximum area 625.

C01S01.049: Recall that the total daily production of the oil field is $p(x) = (20 + x)(200 - 5x)$ if x new wells are drilled (where x is an integer satisfying $0 \leqq x \leqq 40$). Here is a table of *all* of the values of the production function p:

x	0	1	2	3	4	5	6	7
p	4000	4095	4180	4255	4320	4375	4420	4455
x	8	9	10	11	12	13	14	15
p	4480	4495	4500	4495	4480	4455	4420	4375
x	16	17	18	19	20	21	22	23
p	4320	4255	4180	4095	4000	3895	3780	3655
x	24	25	26	27	28	29	30	31
p	3520	3375	3220	3055	2880	2695	2500	2295
x	32	33	34	35	36	37	38	39
p	2080	1855	1620	1375	1120	855	580	295

and, finally, $p(40) = 0$. Answer: Drill ten new wells.

C01S01.050: The surface area A of the box of Example 8 was

$$A(x) = 2x^2 + \frac{500}{x}, \qquad 0 < x < \infty.$$

The restrictions $x \geqq 1$ and $y \geqq 1$ imply that $1 \leqq x \leqq \sqrt{125}$. A small number of values of A, rounded to three places, are given in the following table.

x	1	2	3	4	5	6	7	8	9	10	11
A	502	258	185	157	150	155	169	191	218	250	287

It appears that A is minimized when $x = y = 5$.

C01S01.051: If x is an integer, then $\text{CEILING}(x) = x$ and $-\text{FLOOR}(-x) = -(-x) = x$. If x is not an integer, then choose the integer n so that $n < x < n + 1$. Then $\text{CEILING}(x) = n + 1$, $-(n + 1) < -x < -n$, and

$$-\text{FLOOR}(-x) = -[-(n + 1)] = n + 1.$$

In both cases we see that $\text{CEILING}(x) = -\text{FLOOR}(-x)$.

C01S01.052: The range of $\text{ROUND}(x)$ is the set $\boldsymbol{Z}$ of all integers. If k is a nonzero constant, then as x varies through all real number values, so does kx. Hence the range of $\text{ROUND}(kx)$ is $\boldsymbol{Z}$ if $k \neq 0$. If $k = 0$ then the range of $\text{ROUND}(kx)$ consists of the single number zero.

C01S01.053: By the result of Problem 52, the range of $\text{Round}(10x)$ is the set of all integers, so the range of $g(x) = \frac{1}{10}\text{Round}(10x)$ is the set of all integral multiple of $\frac{1}{10}$.

C01S01.054: What works for π will work for every real number; let $\text{Round2}(x) = \frac{1}{100}\text{Round}(100x)$. To be certain that this is correct (we will verify it only for positive numbers), write the [positive] real number x in the form

$$x = k + \frac{t}{10} + \frac{h}{100} + \frac{m}{1000} + r,$$

where k is a nonnnegative integer, t (the "tenths" digit) is a nonnegative integer between 0 and 9, h (the "hundredths" digit) is a nonnegative integer between 0 and 9, as is m, and $0 \leqq r < 0.001$. Then

$$\begin{aligned}\text{Round2}(x) &= \tfrac{1}{100}\text{Floor}(100x + 0.5)\\ &= \tfrac{1}{100}\text{Floor}(100k + 10t + h + \tfrac{1}{10}(m+5) + 100r).\end{aligned}$$

If $0 \leqq m \leqq 4$, the last expression becomes

$$\frac{1}{100}(100k + 10t + h) = k + \frac{t}{10} + \frac{h}{100},$$

which is the correct two-digit rounding of x. If $5 \leqq m \leqq 9$, it becomes

$$\frac{1}{100}(100k + 10t + h + 1) = k + \frac{t}{10} + \frac{h+1}{100},$$

also the correct two-digit rounding of x in this case.

C01S01.055: Let $\text{Round4}(x) = \frac{1}{10000}\text{Round}(10000x)$. To verify that Round4 has the desired property for [say] positive values of x, write such a number x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + \frac{d_5}{100{,}000} + r,$$

where k is a nonnegative integer, each d_i is an integer between 0 and 9, and $0 \leqq r < 0.00001$. Application of Round4 to x then produces

$$\frac{1}{10000}\text{Floor}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + \tfrac{1}{10}(d_5 + 5) + 10000r).$$

Then consideration of the two cases $0 \leqq d_5 \leqq 4$ and $5 \leqq d_5 \leqq 9$ will show that Round4 produces the correct four-place rounding of x in both cases.

C01S01.056: Let $\text{Chop4}(x) = \frac{1}{10000}\text{Floor}(10000x)$. Suppose that $x > 0$. Write x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + r,$$

where k is a nonnegative integer, each of the d_i is an integer between 0 and 9, and $0 \leqq r < 0.0001$. Then $\text{Chop4}(x)$ produces

$$\tfrac{1}{10000}\text{Floor}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + 10000r)$$

$$= \tfrac{1}{10000}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4)$$

because $10000r < 1$. It follows that CHOP4 has the desired effect.

C01S01.057:

x	0.0	0.2	0.4	0.6	0.8	1.0
y	1.0	0.44	-0.04	-0.44	-0.76	-1.0

The sign change occurs between $x = 0.2$ and $x = 0.4$.

x	0.20	0.25	0.30	0.35	0.40
y	0.44	0.3125	0.19	0.0725	-0.04

The sign change occurs between $x = 0.35$ and $x = 0.40$.

x	0.35	0.36	0.37	0.38	0.39	0.40
y	0.0725	0.0496	0.0269	0.0044	-0.0179	-0.04

From this point on, the data for y will be rounded.

x	0.380	0.382	0.384	0.386	0.388	0.390
y	0.0044	-0.0001	-0.0045	-0.0090	-0.0135	-0.0179

Answer (rounded to two places): 0.38. The quadratic formula yields the two roots $\frac{1}{2}\left(3 \pm \sqrt{5}\,\right)$; the smaller of these is approximately 0.3819660112501051517954.

Problems 58 through 66 are worked in the same way as Problem 57.

C01S01.058: The sign change intervals are [2, 3], [2.6, 2.8], [2.60, 2.64], and [2.616, 2.624]. Answer: $\frac{1}{2}\left(3 + \sqrt{5}\right) \approx 2.62$.

C01S01.059: The sign change intervals are [1, 2], [1.2, 1.4], [1.20, 1.24], [1.232, 1.240], and [1.2352, 1.2368]. Answer: $-1 + \sqrt{5} \approx 1.24$.

C01S01.060: The sign change intervals are $[-4, -3]$, $[-3.4, -3.2]$, $[-3.24, -3.20]$, $[-3.240, -3.232]$, and $[-3.2368, -3.2352]$. Answer: $-1 - \sqrt{5} \approx -3.24$.

C01S01.061: The sign change intervals are [0, 1], [0.6, 0.8], [0.68, 0.72], [0.712, 0.720], and [0.7184, 0.7200]. Answer: $\frac{1}{4}\left(7 - \sqrt{17}\right) \approx 0.72$.

C01S01.062: The sign change intervals are [2, 3], [2.6, 2.8], [2.76, 2.80], [2.776, 2.784], and [2.7792, 2.7808]. Answer: $\frac{1}{4}\left(7 + \sqrt{17}\right) \approx 2.78$.

C01S01.063: The sign change intervals are [3, 4], [3.2, 3.4], [3.20, 3.24], [3.208, 3.216], and [3.2080, 3.2096]. Answer: $\frac{1}{2}\left(11 - \sqrt{21}\right) \approx 3.21$.

C01S01.064: The sign change intervals are [7, 8], [7.6, 7.8], [7.76, 7.80], [7.784, 7.792], and [7.7904, 7.7920]. Answer: $\frac{1}{2}\left(11 + \sqrt{21}\right) \approx 7.79$.

C01S01.065: The sign change intervals are [1, 2], [1.6, 1.8], [1.60, 1.64], [1.608, 1.616], [1.6144, 1.6160], and [1.61568, 1.61600]. Answer: $\frac{1}{6}\left(-23 + \sqrt{1069}\right) \approx 1.62$.

C01S01.066: The sign change intervals are $[-10, -9]$, $[-9.4, -9.2]$, $[-9.32, -9.28]$, $[-9.288, -9.280]$, and $[-9.2832, -9.2816]$. Answer: $\frac{1}{6}\left(-23 - \sqrt{1069}\right) \approx -9.28$.

Section 1.2

C01S02.001: The slope of L is $m = (3-0)/(2-0) = \frac{3}{2}$, so L has equation

$$y - 0 = \frac{3}{2}(x-0); \qquad \text{that is,} \qquad 2y = 3x.$$

C01S02.002: Because L is vertical and $(7,\ 0)$ lies on L, every point on L has Cartesian coordinates $(7,\ y)$ for some number y (and every such point lies on L). Hence an equation of L is $x = 7$.

C01S02.003: Because L is horizontal, it has slope zero. Hence an equation of L is

$$y - (-5) = 0 \cdot (x-3); \qquad \text{that is,} \qquad y = -5.$$

C01S02.004: Because $(2,\ 0)$ and $(0,\ -3)$ lie on L, it has slope $(0+3)/(2-0) = \frac{3}{2}$. Hence an equation of L is

$$y - 0 = \tfrac{3}{2}(x-2); \qquad \text{that is,} \qquad y = \tfrac{3}{2}x - 3.$$

C01S02.005: The slope of L is $(3-(-3))/(5-2) = 2$, so an equation of L is

$$y - 3 = 2(x-5); \qquad \text{that is,} \qquad y = 2x - 7.$$

C01S02.006: An equation of L is $y - (-4) = \frac{1}{2}(x-(-1))$; that is, $2y + 7 = x$.

C01S02.007: The slope of L is $\tan(135^\circ) = -1$, so L has equation

$$y - 2 = -1 \cdot (x-4); \qquad \text{that is,} \qquad x + y = 6.$$

C01S02.008: Equation: $y - 7 = 6(x-0)$; that is, $y = 6x + 7$.

C01S02.009: The second line's equation can be written in the form $y = -2x + 10$ to show that it has slope -2. Because L is parallel to the second line, L also has slope -2 and thus equation $y - 5 = -2(x-1)$.

C01S02.010: The equation of the second line can be rewritten as $y = -\frac{1}{2}x + \frac{17}{2}$ to show that it has slope $-\frac{1}{2}$. Because L is perpendicular to the second line, L has slope 2 and thus equation $y - 4 = 2(x+2)$.

C01S02.011: $x^2 - 4x + 4 + y^2 = 4$: $(x-2)^2 + (y-0)^2 = 2^2$. Center $(2, 0)$, radius 2.

C01S02.012: $x^2 + y^2 + 6y + 9 = 9$: $(x-0)^2 + (y+3)^2 = 3^2$. Center $(0, -3)$, radius 3.

C01S02.013: $x^2 + 2x + 1 + y^2 + 2y + 1 = 4$: $(x+1)^2 + (y+1)^2 = 2^2$. Center $(-1,\ -1)$, radius 2.

C01S02.014: $x^2 + 10x + 25 + y^2 - 20y + 100 = 25$: $(x+5)^2 + (y-10)^2 = 5^2$. Center $(-5, 10)$, radius 5.

C01S02.015: $x^2 + y^2 + x - y = \frac{1}{2}$: $x^2 + x + \frac{1}{4} + y^2 - y + \frac{1}{4} = 1$; $(x+\frac{1}{2})^2 + (y-\frac{1}{2})^2 = 1$. Center: $(-\frac{1}{2},\ \frac{1}{2})$, radius 1.

C01S02.016: $x^2 + y^2 - \frac{2}{3}x - \frac{4}{3}y = \frac{11}{9}$: $x^2 - \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{16}{9}$; $(x-\frac{1}{3})^2 + (y-\frac{2}{3})^2 = (\frac{4}{3})^2$. Center $(\frac{1}{3}, \frac{2}{3})$, radius $\frac{4}{3}$.

C01S02.017: $y = (x-3)^2$: Opens upward, vertex at $(3, 0)$.

C01S02.018: $y - 16 = -x^2$: Opens downward, vertex at $(0, 16)$.

C01S02.019: $y - 3 = (x+1)^2$: Opens upward, vertex at $(-1, 3)$.

C01S02.020: $2y = x^2 - 4x + 4 + 4$: $y - 2 = \frac{1}{2}(x-2)^2$. Opens upward, vertex at $(2, 2)$.

C01S02.021: $y = 5(x^2 + 4x + 4) + 3 = 5(x+2)^2 + 3$: Opens upward, vertex at $(-2, 3)$.

C01S02.022: $y = -(x^2 - x) = -(x^2 - x + \frac{1}{4}) + \frac{1}{4}$: $y - \frac{1}{4} = -(x - \frac{1}{2})^2$. Opens downward, vertex at $(\frac{1}{2}, \frac{1}{4})$.

C01S02.023: $x^2 - 6x + 9 + y^2 + 8y + 16 = 25$: $(x-3)^2 + (y+4)^2 = 5^5$. Circle, center $(3, -4)$, radius 5.

C01S02.024: $(x-1)^2 + (y+1)^2 = 0$: The graph consists of the single point $(1, -1)$.

C01S02.025: $(x+1)^2 + (y+3)^2 = -10$: There are no points on the graph.

C01S02.026: $x^2 + y^2 - x + 3y + 2.5 = 0$: $x^2 - x + 0.25 + y^2 + 3y + 2.25 = 0$: $(x-0.5)^2 + (y+1.5)^2 = 0$. The graph consists of the single point $(0.5, -1.5)$.

C01S02.027: The graph is the straight line segment connecting the two points $(-1, 7)$ and $(1, -3)$ (including those two points).

C01S02.028: The graph is the straight line segment connecting the two points $(0, 2)$ and $(2, -8)$, including the first of these two points but not the second.

C01S02.029: The graph is the parabola that opens downward, symmetric around the y-axis, with vertex at $(0, 10)$ and x-intercepts $\pm\sqrt{10}$.

C01S02.030: The graph of $y = 1 + 2x^2$ is a parabola that opens upwards, is symmetric around the y-axis, and has vertex at $(0, 1)$.

C01S02.031: The graph of $y = x^3$ can be visualized by modifying the familiar graph of the parabola with equation $y = x^2$: The former may be obtained by multiplying the y-coordinate of the latter's point (x, x^2) by x. Thus both have flat spots at the origin. For $0 < x < 1$, the graph of $y = x^3$ is below that of $y = x^2$. They cross at $(1, 1)$, and for $x > 1$ the graph of $y = x^2$ is below that of $y = x^3$, with the difference becoming arbitrarily large as x increases without bound. If the graph of $y = x^3$ for $x \geqq 0$ is rotated $180°$ around the point $(0, 0)$, the graph of $y = x^3$ for $x < 0$ is the result.

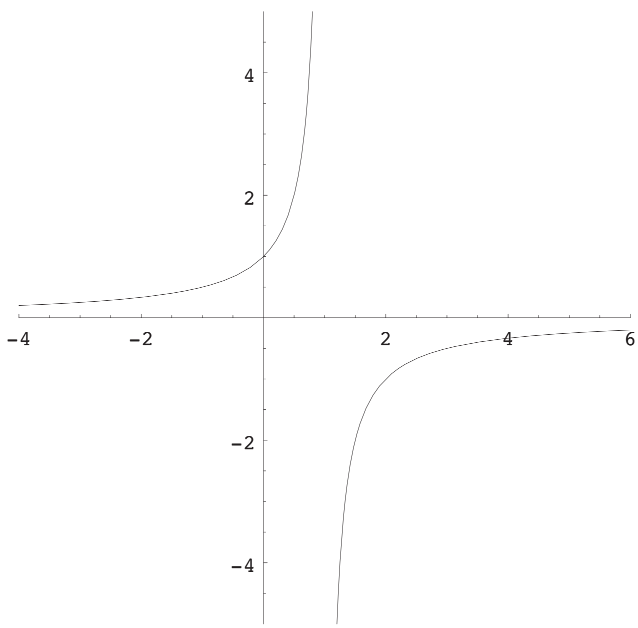
C01S02.032: The graph of $f(x) = x^4$ can be visualized by first visualizing the graph of $y = x^2$. If the y-coordinate of each point on this graph is replaced with its square (x^4), the result is the graph of f. The effect on the graph of $y = x^2$ is to multiply the y-coordinate by x^2, which is between 0 and 1 for $0 < |x| < 1$ and which is larger than 1 for $|x| > 1$. Thus the graph of f superficially resembles that of $y = x^2$, but is much closer to the x-axis for $|x| < 1$ and much farther away for $|x| > 1$. The two graphs cross at $(0, 0)$ (where each has a flat spot) and at $(\pm 1, 1)$, but the graph of f is much steeper at the latter two points.

C01S02.033: To graph $y = f(x) = \sqrt{4 - x^2}$, note that $y \geqq 0$ and that $y^2 = 4 - x^2$; that is, $x^2 + y^2 = 4$. Hence the graph of f is the *upper half* of the circle with center $(0, 0)$ and radius 2.

C01S02.034: To graph $y = f(x) = -\sqrt{9 - x^2}$, note that $y \leqq 0$ and that $y^2 = 9 - x^2$; that is, that $x^2 + y^2 = 9$. Hence the graph of f is the *lower half* of the circle with center $(0, 0)$ and radius 3.

C01S02.035: To graph $f(x) = \sqrt{x^2 - 9}$, note that there is no graph for $-3 < x < 3$, that $f(\pm 3) = 0$, and that $f(x) > 0$ for $x < -3$ and for $x > 3$. If x is large positive, then $\sqrt{x^2 - 9} \approx \sqrt{x^2} = x$, so the graph of f has x-intercept $(3, 0)$ and rises as x increases, nearly coinciding with the graph of $y = x$ for x large positive. The case $x < -3$ is trickier. In this case, if x is a large negative number, then $f(x) = \sqrt{x^2 - 9} \approx \sqrt{x^2} = -x$ (Note the minus sign!). So for $x \leqq -3$, the graph of f has x-intercept $(-3, 0)$ and, for x large negative, almost coincides with the graph of $y = -x$. Later we will see that the graph of f becomes arbitrarily steep as x gets closer and closer to ± 3.

C01S02.036: As x increases without bound—either positively or negatively—$f(x)$ gets arbitrarily close to zero. Moreover, if x is large positive then $f(x)$ is negative and close to zero, so the graph of f lies just below the x-axis for such x. Similarly, the graph of f lies just above the x-axis for x large negative. If x is slightly less than 1 but very close to 1, then $f(x)$ is the reciprocal of a tiny positive number, hence is a large positive number. So the graph of f just to the left of the vertical line $x = 1$ almost coincides with the top half of that line. Similarly, just to the right of the line $x = 1$, then graph of f almost coincides with the bottom half of that line. There is no graph where $x = 1$, so the graph resembles the one in the next figure. The only intercept is the y-intercept $(0, 1)$. The graph correctly shows that the graph of f is increasing for $x < 1$ and for $x > 1$.



C01S02.037: Note that $f(x)$ is positive and close to zero for x large positive, so that the graph of f is just above the x-axis—and nearly coincides with it—for such x. Similarly, the graph of f is just below the x-axis and nearly coincides with it for x large negative. There is no graph where $x = -2$, but if x is slightly greater than -2 then $f(x)$ is the reciprocal of a very small positive number, so $f(x)$ is large and nearly coincides with the upper half of the vertical line $x = -2$. Similarly, if x is slightly less than -2, then the graph of $f(x)$ is large negative and nearly coincides with the the lower half of the line $x = -3$. The graph of f is decreasing for $x < -2$ and for $x > -2$ and its only intercept is the y-intercept $\left(0, \frac{1}{2}\right)$.

C01S02.038: Note that $f(x)$ is very small but positive if x is either large positive or large negative. There is no graph for $x = 0$, but if x is very close to zero, then $f(x)$ is the reciprocal of a very small positive number, and hence is large positive. So the graph of f is just above the x-axis and almost coincides with it if $|x|$ is large, whereas the graph of f almost coincides with the positive y-axis for x near zero. There are no intercepts; the graph of f is increasing for $x < 0$ and is decreasing for $x > 0$.

C01S02.039: Note that $f(x) > 0$ for all x other than $x = 1$, where f is not defined. If $|x|$ is large, then $f(x)$ is near zero, so the graph of f almost coincides with the x-axis for such x. If x is very close to 1, then $f(x)$ is the reciprocal of a very small positive number, hence $f(x)$ is large positive. So for such x, the graph of $f(x)$ almost coincides with the upper half of the vertical line $x = 1$. The only intercept is $(0, 1)$.

C01S02.040: Note first that $f(x)$ is undefined at $x = 0$. To handle the absolute value symbol, we look at two cases: If $x > 0$, then $f(x) = 1$; if $x < 0$, then $f(x) = -1$. So the graph of f consists of the part of the horizontal line $y = 1$ for which $x > 0$, together with the part of the horizontal line $y = -1$ for which $x < 0$.

C01S02.041: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis. If x is large negative, then $f(x)$ is negative and close to zero, so the graph of f is slightly below the x-axis and almost coincides with the x-axis. If x is slightly greater than $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. If x is slightly less than $-\frac{3}{2}$ then $f(x)$ is very large negative, so the graph of f almost coincides with the lower half of that vertical line. The graph of f is decreasing for $x < -\frac{3}{2}$ and also decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{3}\right)$.

C01S02.042: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive or large negative, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis for $|x|$ large. If x is close to $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. The graph of f is increasing for $x < -\frac{3}{2}$ and decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{9}\right)$.

C01S02.043: Given $y = f(x) = \sqrt{1-x}$, note that $y \geqq 0$ and that $y^2 = 1 - x$; that is, $x = 1 - y^2$. So the graph is the part of the parabola $x = 1 - y^2$ for which $y \geqq 0$. This parabola has horizontal axis of symmetry the y-axis, opens to the left (because the coefficient of y^2 is negative), and has vertex $(1, 0)$. Therefore the graph of f is the upper half of this parabola.

C01S02.044: Note that the interval $x < 1$ is the domain of f, so there is no graph for $x \geqq 1$. If x is a large negative number, then the denominator is large positive, so that its reciprocal $f(x)$ is very small positive. As x gets closer and closer to 1 (while $x < 1$), the denominator approaches zero, so its reciprocal $f(x)$ takes on arbitrarily large positive values. So the graph of f is slightly above the x-axis and almost coincides with that axis for x large negative; the graph of f almost coincides with the upper half of the vertical line $x = 1$ for x near (and less than) 1. The graph of f is increasing for all $x < 1$ and $(0, 1)$ is the only intercept.

C01S02.045: Note that $f(x)$ is defined only if $2x + 3 > 0$; that is, if $x > -\frac{3}{2}$. Note also that $f(x) > 0$ for all such x. If x is large positive, then $f(x)$ is positive but near zero, so the graph of f is just above the x-axis and almost coincides with it. If x is very close to $-\frac{3}{2}$ (but larger), then the denominator in $f(x)$ is very tiny positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$ for such x. The graph of f is decreasing for all $x > -\frac{3}{2}$.

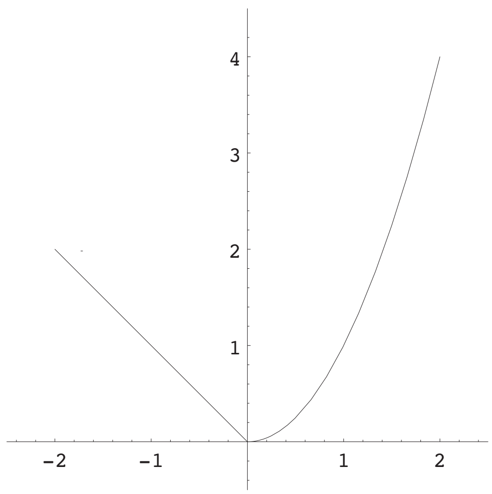
C01S02.046: Given: $f(x) = |2x - 2|$. Case 1: $x \geqq 1$. Then $2x - 2 \geqq 0$, so that $f(x) = 2x - 2$. Because $f(1) = 0$, the graph of f for $x \geqq 1$ consists of the part of the straight line through $(1, 0)$ with slope 2. Case 2: $x < 1$. Then $2x - 2 < 0$, so that $f(x) = -2x + 2$. The line $y = -2x + 2$ passes through $(1, 0)$, so the graph of f for $x < 1$ consists of the part of the straight line through $(1, 0)$ with slope -2.

C01S02.047: Given: $f(x) = |x| + x$. If $x \geqq 0$ then $f(x) = x + x = 2x$, so if $x \geqq 0$ then the graph of f is the part of the straight line through $(0, 0)$ with slope 2 for which $x \geqq 0$. If $x < 0$ then $f(x) = -x + x = 0$, so the rest of the graph of f coincides with the negative x-axis.

C01S02.048: Given: $f(x) = |x - 3|$. If $x \geqq 0$ then $f(x) = x - 3$, so the graph of f consists of the straight line through $(3, 0)$ with slope 1 for $x \geqq 3$. If $x < 0$ then $f(x) = -x + 3$, so the graph of f consists of that part of the straight line with slope -3 and y-intercept $(0, 3)$. These two line segments fit together perfectly at the point $(3, 0)$; there is no break or gap or discontinuity in the graph of f.

C01S02.049: Given: $f(x) = |2x + 5|$. The two cases are determined by the point where $2x + 5$ changes sign, which is where $x = -\frac{5}{2}$. If $x \geqq -\frac{5}{2}$, then $f(x) = 2x + 5$, so the graph of f consists of the part of the line with slope 2 and y-intercept 5 for which $x \geqq -\frac{5}{2}$. If $x < -\frac{5}{2}$, then the graph of f is the part of the straight line $y = -2x - 5$ for which $x < -\frac{5}{2}$.

C01S02.050: The graph consists of the part of the line $y = -x$ for which $x < 0$ together with the part of the parabola $y = x^2$ for which $x \geqq 0$. The two graphs fit together perfectly at the point $(0, 0)$; there is no break, gap, jump, or discontinuity there. The graph is shown next.



C01S02.051: The graph consists of the horizontal line $y = 0$ for $x < 0$ together with the horizontal line $y = 1$ for $x \geqq 0$. As x moves from left to right through the value zero, there is an abrupt and unavoidable "jump" in the value of f from 0 to 1. That is, f is discontinuous at $x = 0$. To see part of the graph of f, enter the *Mathematica* commands

```
f[x_] := If[x < 0, 0, 1]
Plot[ f[x], {x, −3.5, 3.5 }, AspectRatio −> Automatic, PlotRange −> { { −3.5, 3.5 }, {−1.5, 2.5 }} ];
```

C01S02.052: The graph of f consists of the open intervals $\dots$, $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$, $\dots$ on the x-axis together with the isolated points $\dots$, $(-1, 1)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, $\dots$. There is a discontinuity at every integral value of x. A *Mathematica* plot of

```
f[ x_] := If[IntegerQ[x], 1, 0 ]
```

will produce a graph that's completely different because *Mathematica,* like most plotting programs, "connects the dots," in effect assuming that every function is continuous at every point in its domain.

C01S02.053: Because the graph of the greatest integer function changes at each integral value of x, the graph of $f(x) = [\![2x]\!]$ changes twice as often—at each integral multiple of $\frac{1}{2}$. So as x moves from left to right through such points, the graph jumps upward one unit. Thus there is a discontinuity at each integral multiple of $\frac{1}{2}$. Because f is constant otherwise, these are the only discontinuities. To see something like the graph of f, enter the *Mathematica* commands

```
f[x_] := Floor[ 2*x ];
Plot[ f[x], {x, −3.5, 3.5 }, AspectRatio −> Automatic, PlotRange −> { { −3.5, 3.5 }, {−4.5, 4.5 }} ];
```

Mathematica will draw vertical lines connecting points that it shouldn't, making the graph look like treads and risers of a staircase, whereas only the treads are on the graph.

C01S02.054: The function f is undefined at $x = 1$. The graph consists of the horizontal line $y = 1$ for $x > 1$ together with the horizontal line $y = -1$ for $x < 1$. There is a discontinuity at $x = 1$.

C01S02.055: Given: $f(x) = [\![x]\!]$. If n is an integer and $n \leqq x < n+1$, then express x as $x = n + (\!(x)\!)$ where $(\!(x)\!) = x - [\![x]\!]$ is the *fractional part* of x. Then $f(x) = n - x = n - [n + (\!(x)\!)] = -(\!(x)\!)$. So $f(x)$ is the negative of the fractional part of x. So as x ranges from n up to (but not including) $n+1$, $f(x)$ begins at 0 and drops linearly down not quite to -1. That is, on the interval $(n, n+1)$, the graph of f is the straight line segment connecting the two points $(n, 0)$ and $(n+1, -1)$ with the first of these points included and the second excluded. There is a discontinuity at each integral value of x.

C01S02.056: Given: $f(x) = [\![x]\!] + [\![-x]\!] + 1$. If x is an integer, then $f(x) = x + (-x) + 1 = 1$. If x is not an integer, then choose the integer n such that $n < x < n+1$. Then $-(n+1) < -x < -n$, so

$$f(x) = [\![x]\!] + [\![-x]\!] + 1 = n - (n+1) + 1 = 0.$$

So f is the same function as the one defined in Problem 52 and has the same discontinuities: one at each integral value of x.

C01S02.057: Because $y = 2x^2 - 6x + 7 = 2(x^2 - 3x + 3.5) = 2(x^2 - 3x + 2.25 + 1.25) = 2(x - 1.5)^2 + 2.5$, the vertex of the parabola is at $(1.5, 2.5)$.

C01S02.058: Because $y = 2x^2 - 10x + 11 = 2(x^2 - 5x + 5.5) = 2(x^2 - 5x + 6.25 - 0.75) = 2(x - 2.5)^2 - 1.5$, the vertex of the parabola is at $(2.5, -1.5)$.

C01S02.059: Because $y = 4x^2 - 18x + 22 = 4(x^2 - (4.5)x + 5.5) = 4(x^2 - (4.5)x + 5.0625 + 0.4375)$ $= 4(x - 2.25)^2 + 1.75$, the vertex of the parabola is at $(2.25, 1.75)$.

C01S02.060: Because $y = 5x^2 - 32x + 49 = 5(x^2 - (6.4)x + 9.8) = 5(x^2 - (6.4)x + 10.24 - 0.44)$ $= 5(x - 3.2)^2 - 2.2$, the vertex of the parabola is at $(3.2, -2.2)$.

C01S02.061: Because $y = -8x^2 + 36x - 32 = -8(x^2 - (4.5)x + 4) = -8(x^2 - (4.5)x + 5.0625 - 1.0625)$ $= -8(x - 2.25)^2 + 8.5$, the vertex of the parabola is at $(2.25, 8.5)$.

C01S02.062: Because $y = -5x^2 - 34x - 53 = -5(x^2 + (6.8)x + 10.6) = -5(x^2 + (6.8)x + 11.56 - 0.96)$ $= -5(x + 3.4)^2 + 4.8$, the vertex of the parabola is at $(-3.4, 4.8)$.

C01S02.063: Because $y = -3x^2 - 8x + 3 = -3\left(x^2 + \frac{8}{3}x - 1\right) = -3\left(x^2 + \frac{8}{3}x + \frac{16}{9} - \frac{25}{9}\right) = -3\left(x + \frac{4}{3}\right)^2 + \frac{25}{3}$, the vertex of the parabola is at $\left(-\frac{4}{3}, \frac{25}{3}\right)$.

C01S02.064: Because $y = -9x^2 + 34x - 28 = -9\left(x^2 - \frac{34}{9}x + \frac{28}{9}\right) = -9\left(x^2 - \frac{34}{9}x + \frac{289}{81} - \frac{37}{81}\right)$ $= -9\left(x - \frac{17}{9}\right)^2 + \frac{37}{9}$, the vertex of the parabola is at $\left(\frac{17}{9}, \frac{37}{9}\right)$.

C01S02.065: To find the maximum height $y = -16t^2 + 96t$ of the ball, we find the vertex of the parabola: $y = -16(t^2 - 6t) = -16(t^2 - 6t + 9 - 9) = -16(t - 3)^2 + 144$. The vertex of the parabola is at $(3, 144)$ and therefore the maximum height of the ball is 144 ft.

C01S02.066: Recall that the area of the rectangle is given by $y = A(x) = x(50 - x)$. To maximize $A(x)$ we find the vertex of the parabola: $y = 50x - x^2 = -(x^2 - 50x) = -(x^2 - 50x + 625 - 625) = -(x - 25)^2 + 625$. Because the vertex of the parabola is at $(25, 625)$ and $x = 25$ is in the domain of the function A, the maximum value of $A(x)$ occurs at $x = 25$ and is $A(25) = 625$ (ft^2).

C01S02.067: If two positive numbers x and y have sum 50, then $y = 50 - x$ and $x < 50$ (because $y > 0$). To maximize their product $p(x)$ we find the vertex of the parabola

$$\begin{aligned} y = p(x) = x(50 - x) &= -(x^2 - 50x) \\ &= -(x^2 - 50x + 625 - 625) = -(x - 25)^2 + 625, \end{aligned}$$

which is at $(25, 625)$. Because $0 < 25 < 50$, $x = 25$ is in the domain of the product function $p(x) = x(50 - x)$, and hence the maximum value of the product of x and y is $p(25) = 625$.

C01S02.068: Recall that if x new wells are drilled, then the resulting total production p is given by $p(x) = 4000 + 100x - 5x^2$. To maximize $p(x)$ we find the vertex of the parabola

$$\begin{aligned} y = p(x) = -5x^2 + 100x + 4000 &= -5(x^2 - 20x - 800) \\ &= -5(x^2 - 20x + 100 - 900) = -5(x - 10)^2 + 4500. \end{aligned}$$

The vertex of the parabola $y = p(x)$ is therefore at $(10, 4500)$. Because $x = 10$ is in the domain of p (it is an integer between 0 and 40) and because the parabola opens downward (the coefficient of x^2 is negative), $x = 10$ indeed maximizes $p(x)$.

C01S02.069: The graph looks like the graph of $y = |x|$ because the slope of the left-hand part is -1 and that of the right-hand part is 1; but the vertex is shifted to $(-1, 0)$, so—using the translation principle—the graph in Fig. 1.2.29 must be the graph of $f(x) = |x + 1|$, $-2 \leqq x \leqq 2$.

C01S02.070: Because the graph in Fig. 1.2.30 is composed of three straight-line segments, it can be described most easily using a "three-part" function:

$$f(x) = \begin{cases} 2x + 6 & \text{if } -3 \leqq x < -2; \\ 2 & \text{if } -2 \leqq x < 2; \\ \frac{1}{3}(10 - 2x) & \text{if } \quad 2 \leqq x \leqq 5. \end{cases}$$

C01S02.071: The graph in Fig. 1.2.31 is much like the graph of the greatest integer function—it takes on only integral values—but the "jumps" occur twice as often, so this must be very like—indeed, it is exactly—the graph of $f(x) = [\![2x]\!]$, $-1 \leqq x < 2$.

C01S02.072: The graph in Fig. 1.2.32 resembles the graph of the greatest integer function in that it takes on all integral values and only those, but it is decreasing rather than increasing and the "jumps" occur only at the even integers. Thus it must be the graph of something similar to $f(x) = -[\![\frac{1}{2}x]\!]$, $-4 \leqq x < 4$. Comparing values of f at $x = -4$, -3, -2.1, -2, -1, -0.1, 0, 1, 1.9, 2, 3, and 3.9 with points on the graph is sufficient evidence that the graph of f is indeed that shown in the figure.

C01S02.073: Clearly $x(t) = 45t$ for the first hour; that is, for $0 \leqq t \leqq 1$. In the second hour the graph of $x(t)$ must be a straight line (because of constant speed) of slope 75, thus with equation $x(t) = 75t + C$ for some constant C. The constant C is determined by the fact that $45t$ and $75t + C$ must be equal at time

$t = 1$, as the automobile cannot suddenly jump from one position to a completely different position in an instant. Hence $45 = 75 + C$, so that $C = -30$. Therefore

$$x(t) = \begin{cases} 45t & \text{if } 0 \leqq t \leqq 1; \\ 75t - 30 & \text{if } 1 < t \leqq 2. \end{cases}$$

To see the graph of $x(t)$, plot in *Mathematica*

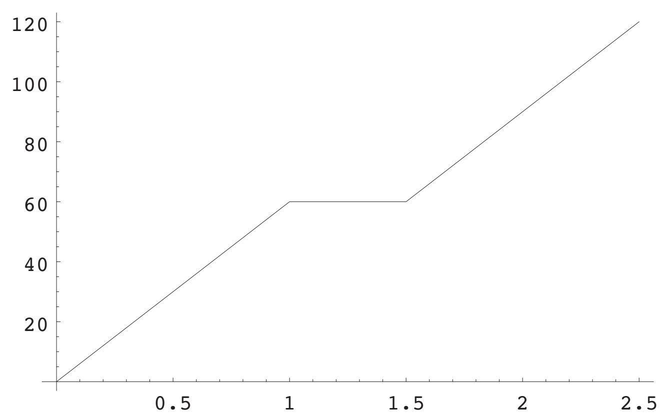
```
x[t_] := If[t < 1, 45*t, 75*t - 30 ]
```

on the interval $0 \leqq t \leqq 2$.

C01S02.074: The graph of $x(t)$ will consist of three straight-line segments (because of the constant speeds), the first of slope 60 for $0 \leqq t \leqq 1$, the second of slope zero for $1 \leqq t \leqq 1.5$, and the third of slope 60 for $1.5 \leqq t \leqq 2.5$. The first pair must coincide when $t = 1$ and the second pair must coincide when $t = 1.5$ because the graph of $x(t)$ can have no discontinuities. So if we write $x(t) = 60$ for $0 \leqq t \leqq 1$, we must have $x(t) = 60$ for $1 \leqq t \leqq 1.5$. Finally, $x(t) = 60t + C$ for some constant C if $1.5 \leqq t \leqq 2.5$, but the latter must equal 60 when $t = 1.5$, so that $C = -30$. Hence

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 60 & \text{if } 1 < t \leqq 1.5, \\ 60t - 30 & \text{if } 1.5 < t \leqq 2.5. \end{cases}$$

The graph of $x(t)$ is shown next.



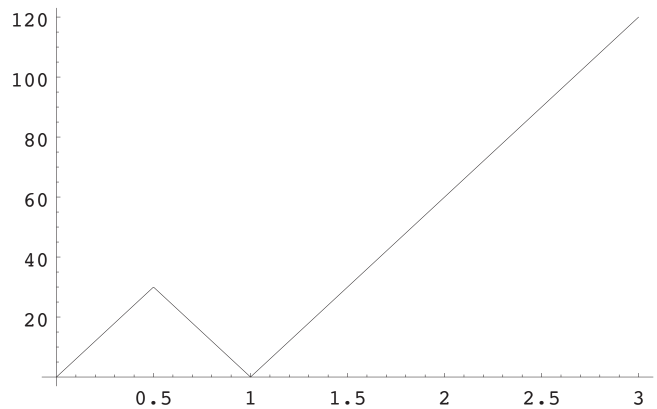
C01S02.075: The graph must consist of two straight-line segments (because of the constant speeds). The first must have slope 60, so we have $x(t) = 60t$ for $0 \leqq t \leqq 1$. The second must have slope -30, negative because you're driving in the reverse direction, so $x(t) = -30t + C$ for some constant C if $1 \leqq t \leqq 3$. The two segments must coincide when $t = 1$, so that $60 = -30 + C$. Thus $C = 90$ and thus a formula for $x(t)$ is

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 90 - 30t & \text{if } 1 < t \leqq 3. \end{cases}$$

C01S02.076: We need three straight line segments, the first of slope 60 for $0 \leqq t \leqq 0.5$, the second of slope -60 for $0.5 \leqq t \leqq 1$, and the third of slope 60 for $1 \leqq t \leqq 3$. Clearly the first must be $x(t) = 60t$ for $0 \leqq t \leqq 0.5$. The second must have the form $x(t) = -60t + C$ for some constant C, and the first and second must coincide when $t = 0.5$, so that $30 = -30 + C$, and thus $C = 60$. The third segment must have the form $x(t) = 60t + K$ for some constant K, and the second and third must coincide when $t = 1$, so that $0 = 60 + K$, and so $K = -60$. Therefore a formula for $x(t)$ is

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 0.5, \\ 60 - 60t & \text{if } 0.5 < t \leqq 1, \\ 60t - 60 & \text{if } 1 < t \leqq 3. \end{cases}$$

The graph of $x(t)$ is shown next.



C01S02.077: Initially we work in units of pages and cents (to avoid decimals and fractions). The graph of C, as a function of p, must be a straight line segment, and its slope is (by information given)

$$\frac{C(79) - C(34)}{79 - 34} = \frac{305 - 170}{79 - 34} = \frac{135}{45} = 3.$$

Thus $C(p) = 3p + K$ for some constant K. So $3 \cdot 34 + K = 170$, and it follows that $K = 68$. So $C(p) = 3p + 68$, $1 \leqq p \leqq 100$, if C is to be expressed in cents. If C is to be expressed in dollars, we have

$$C(p) = (0.03)p + 0.68, \quad 1 \leqq p \leqq 100.$$

The "fixed cost" is incurred regardless of the number of pamphlets printed; it is \$0.68. The "marginal cost" of printing each additional page of the pamphlet is the coefficient \$0.03 of p.

C01S02.078: We are given $C(x) = a + bx$ where a and b are constants; we are also given

$$99.45 = C(207) = a + 207b \qquad \text{and}$$

$$79.15 = C(149) = a + 149b.$$

Subtraction of the second equation from the first yields $20.3 = 58b$, so that $b = 0.35$. Substitution of this datum in the first of the preceding equations then yields

$$99.45 = a + 207 \cdot 0.35 = a + 72.45, \qquad \text{so that} \qquad a = 27.$$

Therefore $C(x) = 27 + (0.35)x$, $0 \leqq x < +\infty$. Thus if you drive 175 miles on the third day, the cost for that day will be $C(175) = 88.25$ (in dollars). The slope $b = 0.35$ represents a cost of \$0.35 per mile. The C-intercept $a = 27$ represents the daily base cost of renting the car. In civil engineering and in some branches of applied mathematics, the intercept $a = 27$ is sometimes called the *offset*, representing the vertical amount by which $C(0)$ is "offset" from zero.

C01S02.079: Suppose that the letter weighs x ounces, $0 < x \leqq 16$. If $x \leqq 8$, then the cost is simply 8 (dollars). If $8 < x \leqq 9$, add \$0.80; if $9 < x \leqq 10$, add \$1.60, and so on. Very roughly, one adds \$0.80 if

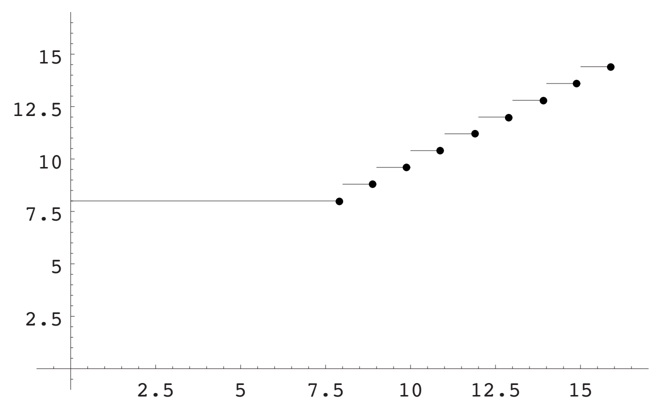
$[\![x-8]\!] = 1$, \$1.60 if $[\![x-8]\!] = 2$, and so on. But this isn't quite right—we are using the FLOOR function of Section 1.1, whereas we should really be using the CEILING function. By the result of Problem 51 of that section, we see that instead of cost

$$C(x) = 8 + (0.8)[\![x-8]\!]$$

for $8 < x \leqq 16$, we should instead write

$$C(x) = \begin{cases} 8 & \text{if } 0 < x \leqq 8, \\ 8 - (0.8)[\![-(x-8)]\!] & \text{if } 8 < x \leqq 16. \end{cases}$$

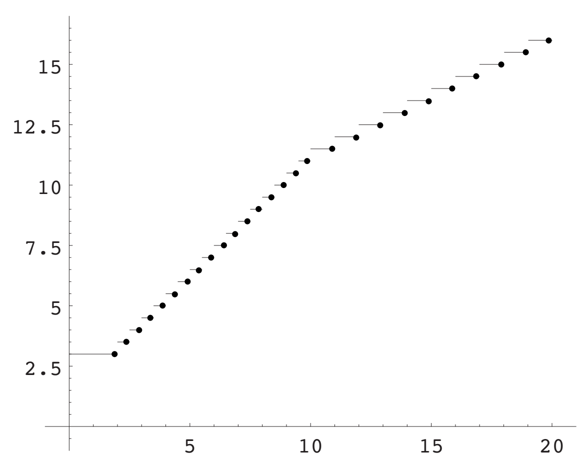
The graph of the cost function is shown next.



C01S02.080: Solve this problem like Problem 79 (but it is more complicated). Result:

$$C(x) = \begin{cases} 3 & \text{if } 0 < x \leqq 2; \\ 3 - 0.5[\![-2(x-2)]\!] & \text{if } 2 < x \leqq 10; \\ 11 - 0.5[\![-(x-10)]\!] & \text{if } 10 < x \leqq 20. \end{cases}$$

The graph of C is shown below.



C01S02.081: Boyle's law states that under conditions of constant temperature, the product of the pressure p and the volume V of a fixed mass of gas remains constant. If we assume that $pV = c$, a constant, for the given data, we find that the given five data points yield the values $c = 1.68$, 1.68, 1.675, 1.68, and 1.62. The

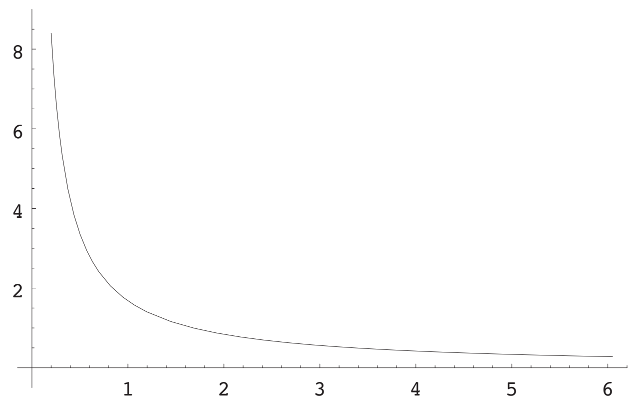
average of these is 1.65 (to two places) and should be a good estimate of the true value of c. Alternatively, you can use a computer algebra program to find c; in *Mathematica*, for example, the command Fit will fit given data points to a sum of constant multiples of functions you specify. We used the commands

```
data = {{0.25, 6.72}, {1.0, 1.68}, {2.5, 0.67}, {4.0, 0.42}, {6.0, 0.27}};
Fit [data, { 1/p }, p]
```

to find that

$$V(p) = \frac{1.67986}{p}$$

yields the best *least-squares* fit of the given data to a function of the form $V(p) = c/p$. We rounded the numerator to 1.68 to find the estimates $V(0.5) \approx 3.36$ and $V(5) \approx 0.336$ (L). The graph of $V(p)$ is shown next.



C01S02.082: It seems reasonable to assume that the maximum average temperature occurs on July 15 and the minimum on January 15, so that a multiple of a cosine function should fit the given data if we take $t = 0$ on July 15. So we assume a solution of the form

$$T(t) = c_1 + c_2 \cos\left(\frac{2\pi t}{365}\right).$$

Also assuming that the average year-round daily temperature is the average of the minimum and the maximum, we find that $c_1 = 61.25$, so we could find c_2 by the averaging method of Problem 81. Alternatively, we could use the Fit command in *Mathematica* to find both c_1 and c_2 simultaneously as follows:

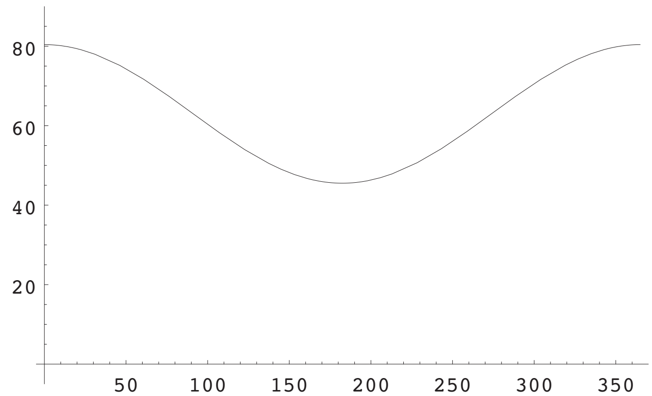
```
data = {{0, 79.1}, {62, 70.2}, {123, 52.3}, {184, 43.4}, {224, 52.2}, {285, 70.1}};
Fit [data, { 1, Cos[2*Pi*t/365] }, t]
```

The result is the formula

$$T(t) = 62.9602 + (17.437)\cos\left(\frac{2\pi t}{365}\right).$$

The values predicted by this function at the six dates in question are [approximately] 80.4, 71.4, 53.9, 45.5, 49.8, and 66.3. Not bad, considering we are dealing with weather, a most unpredictable phenomenon. The graph of $T(t)$ is shown next. Units on the horizontal axis are days, measured from July 15. Units on

the vertical axis are degrees Fahrenheit. Remember that these are *average* daily temperatures; it is not uncommon for a winter low in Athens to be below 28°F and for a summer high to be as much as 92°F.



Section 1.3

C01S03.001: The domain of f is $\boldsymbol{R}$, the set of all real numbers; so is the domain of g, but $g(x) = 0$ when $x = 1$ and when $x = -3$. So the domain of $f + g$ and $f \cdot g$ is the set $\boldsymbol{R}$ and the domain of f/g is the set of all real numbers other than 1 and -3. Their formulas are

$$\begin{aligned}
(f+g)(x) &= x^2 + 3x - 2, \\
(f \cdot g)(x) &= (x+1)(x^2+2x-3) = x^3 + 3x^2 - x - 3, \quad \text{and} \\
\left(\frac{f}{g}\right)(x) &= \frac{x+1}{x^2+2x-3}.
\end{aligned}$$

C01S03.002: The domain of f consists of all real numbers other than 1 and the domain of g consists of all real numbers other than $-\frac{1}{2}$. Hence the domain of $f+g$, $f \cdot g$, and f/g consists of all real numbers other than $-\frac{1}{2}$ and 1. For such x,

$$\begin{aligned}
(f+g)(x) &= \frac{1}{x-1} + \frac{1}{2x+1} = \frac{3x}{(x-1)(2x+1)}, \\
(f \cdot g)(x) &= \frac{1}{(x-1)(2x+1)}, \quad \text{and} \\
\left(\frac{f}{g}\right)(x) &= \frac{2x+1}{x-1}.
\end{aligned}$$

Note that, in spite of the last equation, the domain of f/g does *not* include the number $-\frac{1}{2}$.

C01S03.003: The domain of f is the interval $[0, +\infty)$ and the domain of g is the interval $[2, +\infty)$. Hence the domain of $f+g$ and $f \cdot g$ is the interval $[2, +\infty)$, but because $g(2) = 0$, the domain of f/g is the open interval $(2, +\infty)$. The formulas for these combinations are

$$\begin{aligned}
(f+g)(x) &= \sqrt{x} + \sqrt{x-2}, \\
(f \cdot g)(x) &= \sqrt{x}\,\sqrt{x-2} = \sqrt{x^2-2x}, \quad \text{and} \\
\left(\frac{f}{g}\right)(x) &= \frac{\sqrt{x}}{\sqrt{x-2}} = \sqrt{\frac{x}{x-2}}.
\end{aligned}$$

C01S03.004: The domain of f is the interval $[-1, +\infty)$ and the domain of g is the interval $(-\infty, 5]$. Hence the domain of $f+g$ and $f \cdot g$ is the closed interval $[-1, 5]$, but because $g(5) = 0$, the domain of f/g is the half-open interval $[-1, 5)$. Their formulas are

$$(f+g)(x) = \sqrt{x+1} + \sqrt{5-x}, \qquad (f \cdot g)(x) = \sqrt{x+1}\,\sqrt{5-x} = \sqrt{5+4x-x^2}, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+1}}{\sqrt{5-x}} = \sqrt{\frac{x+1}{5-x}}.$$

C01S03.005: The domain of f is the set $\boldsymbol{R}$ of all real numbers; the domain of g is the open interval $(-2, 2)$. Hence the domain of $f+g$ and $f \cdot g$ is the open interval $(-2, 2)$; because $g(x)$ is never zero, the domain of f/g is the same. Their formulas are

$$(f+g)(x) = \sqrt{x^2+1} + \frac{1}{\sqrt{4-x^2}}, \qquad (f \cdot g)(x) = \frac{\sqrt{x^2+1}}{\sqrt{4-x^2}}, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \sqrt{x^2+1}\,\sqrt{4-x^2} = \sqrt{4+3x^2-x^4}.$$

C01S03.006: The domain of f is the set of all real numbers other than 2 and the domain of g is the set of all real numbers other than -2. Hence the domain of $f+g$ and $f \cdot g$ is the set of all real numbers other than ± 2. But because $g(-1) = 0$, -1 does not belong to the domain of f/g, which therefore consists of all real numbers other than -2, -1, and 2. The formulas of these combinations are

$$(f+g)(x) = \frac{x-1}{x-2} + \frac{x+1}{x+2} = \frac{2x^2-4}{x^2-4}, \qquad (f \cdot g)(x) = \frac{x-1}{x-2} \cdot \frac{x+1}{x+2} = \frac{x^2-1}{x^2-4}, \quad \text{and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{x-2} \cdot \frac{x+2}{x+1} = \frac{x^2+x-2}{x^2-x-2}.$$

C01S03.007: $f(x) = x^3 - 3x + 1$ has 1, 2, or 3 zeros, approaches $+\infty$ as x does, and approaches $-\infty$ as x does. Because $f(0) \neq 0$, the graph does not match Fig. 1.3.26, so it must match Fig. 1.3.30.

C01S03.008: $f(x) = 1 + 4x - x^3$ has one, two, or three zeros, approaches $-\infty$ as $x \to +\infty$ and approaches $+\infty$ as $x \to -\infty$. Hence its graph must be the one shown in Fig. 1.3.28.

C01S03.009: $f(x) = x^4 - 5x^3 + 13x + 1$ has four or fewer zeros and approaches $+\infty$ as x approaches either $+\infty$ or $-\infty$. Hence its graph must be the one shown in Fig. 1.3.31.

C01S03.010: $f(x) = 2x^5 - 10x^3 + 6x - 1$ has between one and five zeros, approaches $+\infty$ as x does, and approaches $-\infty$ as x does. So its graph might be the one shown in Fig. 1.3.26, the one in Fig. 1.3.29, or the one in Fig. 1.3.30. But $f(0) \neq 0$, so Fig. 1.3.26 is ruled out, and we have already found that the graph in Fig. 1.3.30 matches the function in Problem 7. Therefore the graph of f must be the one shown in Fig. 1.3.29. Alternatively, the observation that $f(x)$ changes sign on the *five* intervals $[-3, -2]$, $[-1, 0]$, $[0, 0.5]$, $[0.5, 1]$, and $[2, 3]$ shows that $f(x)$ has five zeros; therefore the graph must be the one shown in Fig. 1.3.29.

C01S03.011: $f(x) = 16 + 2x^2 - x^4$ approaches $-\infty$ as x approaches either $+\infty$ or $-\infty$, so its graph must be the one shown in Fig. 1.3.27.

C01S03.012: $f(x) = x^5 + x$ approaches $+\infty$ as x does and approaches $-\infty$ as x does. Moreover, $f(x) > 0$ if $x > 0$ and $f(x) < 0$ if $x < 0$, which rules out every graph except for the one shown in Fig. 1.3.26.

C01S03.013: The graph of f has vertical asymptotes at $x = -1$ and at $x = 2$, so its graph must be the one shown in Fig. 1.3.34.

C01S03.014: The graph of $f(x)$ has vertical asymptotes at $x = \pm 3$, so its graph must be the one shown in Fig. 1.3.32.

C01S03.015: The graph of f has no vertical asymptotes and has maximum value 3 when $x = 0$. Hence its graph must be the one shown in Fig. 1.3.33.

C01S03.016: The denominator $x^3 - 1 = (x-1)(x^2+x+1)$ of $f(x)$ is zero only when $x = 1$ (because $x^2 + x + 1 > x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 \geqq 0$ for all x), so its graph must be the one shown in Fig. 1.3.35.

C01S03.017: The domain of $f(x) = x\sqrt{x+2}$ is the interval $[-2, +\infty)$, so its graph must be the one shown in Fig. 1.3.38.

C01S03.018: The domain of $f(x) = \sqrt{2x - x^2}$ consists of those numbers for which $2x - x^2 \geqq 0$; that is, $x(2 - x) \geqq 0$. This occurs when x and $2 - x$ have the same sign and also when either is zero. If $x > 0$ and $2 - x > 0$, then $0 < x < 2$. If $x < 0$ and $2 - x < 0$, then $x < 0$ and $x > 2$, which is impossible. Hence the domain of f is the closed interval $[0, 2]$. So the graph of f must be the one shown in Fig. 1.3.36.

C01S03.019: The domain of $f(x) = \sqrt{x^2 - 2x}$ consists of those numbers x for which $x^2 - 2x \geqq 0$; that is, $x(x - 2) \geqq 0$. This occurs when x and $x - 2$ have the same sign and also when either is zero. If $x > 0$ and $x - 2 > 0$, then $x > 2$; if $x < 0$ and $x - 2 < 0$, then $x < 0$. So the domain of f is the union of the two intervals $(-\infty, 0]$ and $[2, +\infty)$. So the graph of f must be the one shown in Fig. 1.3.39.

C01S03.020: The domain of $f(x) = 2(x^2 - 2x)^{1/3}$ is the set $\boldsymbol{R}$ of all real numbers because every real number has a [unique] cube root. By the analysis in the solution of Problem 19, $x^2 - 2x < 0$ if $0 < x < 2$ and $x^2 - 2x \geqq 0$ otherwise. Hence $f(x) < 0$ if $0 < x < 2$ and $f(x) \geqq 0$ otherwise. This makes it certain that the graph of f is the one shown in Fig. 1.3.37.

C01S03.021: Good viewing window: $-2.5 \leqq x \leqq 2.5$. Three zeros, approximately -1.88, 0.35, and 1.53.

C01S03.022: Good viewing window: $-3 \leqq x \leqq 3$. Two zeros: -2 and 1.

C01S03.023: Good viewing window: $-3.5 \leqq x \leqq 2.5$. One zero, approximately -2.10.

C01S03.024: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Four zeros, approximately -1.28, 0.61, 1.46, and 2.20.

C01S03.025: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Three zeros: approximately -1.30, exactly 1, and approximately 2.30.

C01S03.026: Good viewing window: $-1.6 \leqq x \leqq 2.8$. Two zeros, approximately -1.33 and 2.37.

C01S03.027: Good viewing window: $-7.5 \leqq x \leqq 8.5$. Three zeros: Approximately -5.70, -2.22, and 7.91.

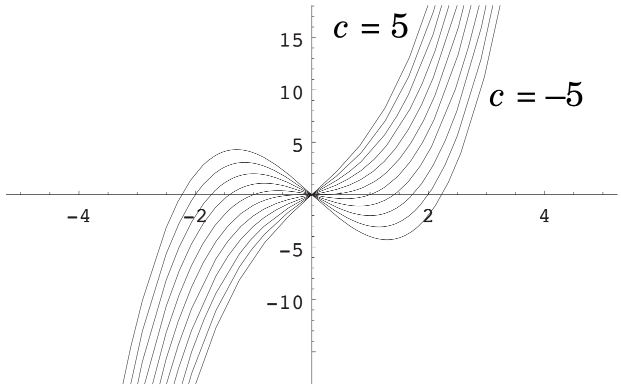
C01S03.028: Good viewing window: None; it takes three: $-22 \leqq x \leqq 8$ shows that there is a zero near -20 and that the graph crosses the x-axis somewhere in the vicinity of $x = 0$. The window $-3 \leqq x \leqq 3$ shows that something interesting happens near $x = -1$ and that there is a zero near 1.8. The window $-1.4 \leqq x \leqq 0.4$ shows that there are zeros near -1.1 and -0.8. Closer approximations to these four zeros are -19.88, -1.09, -0.79, and 1.76.

C01S03.029: The viewing window $-11 \leqq x \leqq 8$ shows that there are five zeros, although the two near 2.5 may be only one. The window $1.5 \leqq x \leqq 3.5$ shows that there are in fact two zeros near 2.5. Approximate values of the five zeros are -10.20, -7.31, 1.98, 3.25, and 7.28.

C01S03.030: The viewing window $-16 \leqq x \leqq 16$ shows that there are zeros near ± 15 and perhaps a few more near $x = 0$. The window $-4 \leqq x \leqq 4$ shows that there are in fact four zeros near $x = 0$. Approximate values of the six are ± 15.48, ± 3.04, and ± 1.06.

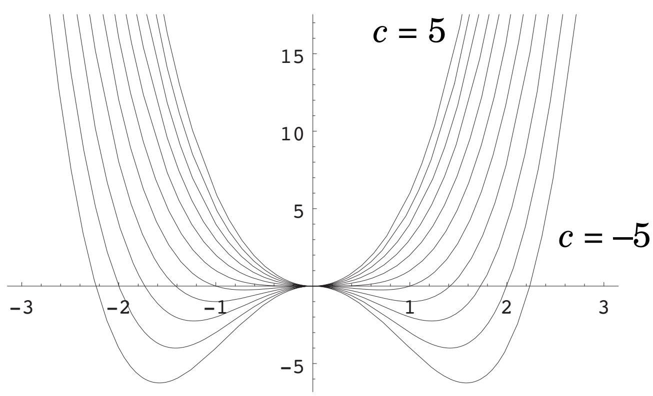
C01S03.031: Every time c increases by 1, the graph is raised 1 unit (in the positive y-direction), but there is no other change.

C01S03.032: The graph starts with two "bends" when $c = -5$. As c increases the bends become narrower and narrower and disappear when $c = 0$. Then the graph gets steeper and steeper. See the following figure.



C01S03.033: The graph always passes through (0, 0) and is tangent to the x-axis there. When $c = -5$ there is another zero at $x = 5$. As c increases this zero shifts to the left until it coincides with the one at $x = 0$ when $c = 0$. At this point the "bend" in the graph disappears. As c increases from 1 to 5, the bend reappears to the left of the x-axis and the second zero reappears at $-c$.

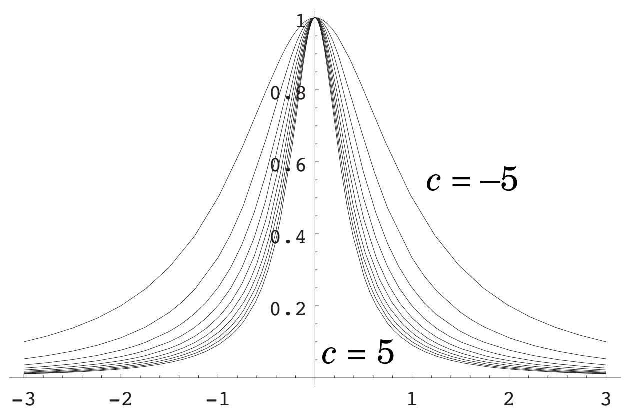
C01S03.034: The graph is always tangent to the x-axis at $x = 0$ and is always symmetric around the y-axis. When $c = -5$ there is another pair of zeros near ± 2.2. As c increases these zeros move closer to $x = 0$ and the bends in the graph get smaller and smaller. They disappear when $c = 0$ and, at the same time, the zeros merge with the one at $x = 0$. Thereafter the graph simply becomes steeper and steeper. See the following figure.



C01S03.035: The graph is always symmetric around the origin (and, consequently, always passes through the origin). When $c = -5$ there is another pair of zeros near ± 2.2. As c increases the graph develops positive slope at $x = 0$, two more bends, and two more zeros on either side of the origin. They move outward and, when $c = -2$, they coincide with the outer pair of zeros, which have also been moving toward the origin. They reach the origin when $c = 0$ and thereafter the graph simply becomes steeper and steeper.

C01S03.036: As c increases the "mountain" around the y-axis gets narrower and steeper. See the following

figure.



C01S03.037: As c increases the graph becomes wider and taller; its shape does not seem to change very much.

C01S03.038: The length of the airfoil is approximately 1.0089 and its width is approximately 0.200057.

Section 1.4

C01S04.001: Because $g(x) = 2^x$ increases—first slowly, then rapidly—on the set of all real numbers, with values in the range $(0, +\infty)$, the given function $f(x) = 2^x - 1$ must increase in the same way, but with values in the range $(-1, +\infty)$. Therefore its graph is the one shown in Fig. 1.4.29.

C01S04.002: Given: $f(x) = 2 - 3^{-x}$. The graph of $g(x) = 3^x$ increases, first slowly, then rapidly, on its domain the set $\boldsymbol{R}$ of all real numbers. Hence $h(x) = 3^{-x}$ decreases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(0, +\infty)$. Hence $j(x) = -3^{-x}$ increases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(-\infty, 0)$. Therefore $f(x) = 2 - 3^{-x}$ increases, first rapidly, then slowly, on $\boldsymbol{R}$, with values in the interval $(-\infty, 2)$. Therefore its graph must be the one shown in Fig. 1.4.33.

C01S04.003: The graph of $f(x) = 1 + \cos x$ is simply the graph of the ordinary cosine function raised 1 unit—moved upward 1 unit in the positive y-direction. Hence its graph is the one shown in Fig. 1.4.27.

C01S04.004: The graph of $g(x) = 2 \sin x$ resembles the graph of the ordinary sine function, but with values ranging from -2 to 2. The graph of $h(x) = -2 \sin x$ is the same, but turned "upside down." Add 2 to get $f(x) = 2 - 2 \sin x$ and the graph of h is raised 2 units, thus taking values in the range $[0, 4]$. So the graph of f is the one shown in Fig. 1.4.32.

C01S04.005: The graph of $g(x) = 2 \cos x$ resembles the graph of the cosine function, but with all values doubled, so that its range is the interval $[-2, 2]$. Add 1 to get $f(x) = 1 + 2 \cos x$ and the range is now the interval $[-1, 3]$. So the graph of f is the one shown in Fig. 1.4.35.

C01S04.006: Turn the graph of the sine function upside down, then add 2 to get $f(x) = 2 - \sin x$, with range the interval $[1, 3]$. Hence the graph of f is the one shown in Fig. 1.4.28.

C01S04.007: The graph of $g(x) = 2^x$ increases, first slowly, then rapidly, on the set of all real numbers, with range the interval $(0, +\infty)$. So its reciprocal $h(x) = 2^{-x}$ decreases, first rapidly, then slowly, with the same domain and range. Multiply by x to obtain $f(x) = x \cdot 2^{-x}$. The effect of multiplication by x is to change large positive values into large negative values for $x < 0$, to cause $f(0)$ to be zero, and to multiply very small positive values (of 2^{-x}) by somewhat large positive values (of x) for $x > 0$, resulting in values that are still small and positive, even when x is quite large. So the graph of f must increase rapidly through negative values, pass through $(0, 0)$, rise to a maximum, then decrease rapidly through positive values toward zero. Hence the graph of f must be the one shown in Fig. 1.4.31.

C01S04.008: The graph of $g(x) = \log x$ has domain the set $(0, +\infty)$ of all positive real numbers; it rises, first rapidly, then more slowly, with range the set of all real numbers, and its graph passes through the point $(1, 0)$. Division by $x > 0$ will have little effect if x is near zero, as this will merely multiply large negative values of $\log x$ by large positive numbers. But when x is large positive, it will be much larger than $\log x$, and thus the graph of $f(x)$ will rise to a maximum somewhere to the right of $x = 1$, then decreases fairly rapidly toward zero. So the graph of f is the one shown in Fig. 1.4.36.

C01S04.009: The graph of $g(x) = 1 + \cos 6x$ will resemble the graph of the cosine function, but raised 1 unit (so that its range is the interval $[0, 2]$) and with much more "activity" on the x-axis (because of the factor 6). Division by $1 + x^2$ will have little effect until x is no longer close to zero, and then the effect will be to divide values of $g(x)$ by larger and larger positive numbers, so that the cosine oscillations have a much smaller range that $0 \leqq x \leqq 2$; they will range from 0 to smaller and smaller positive values as $|x|$ increases. So the graph of f is the one shown in Fig. 1.4.34.

C01S04.010: The graph of $g(x) = \sin 10x$ resembles that of the sine function, but with much more "activity" because of the factor 10. Multiply by the rapidly decreasing positive numbers 2^{-x} and you will see the sine oscillations decreasing from the range $[-1, 1]$ when x is near zero to very small oscillations—near zero—as x increases. So the graph of f is the one shown in Fig. 1.4.30.

C01S04.011: Given $f(x) = 1 - x^2$ and $g(x) = 2x + 3$,

$$f(g(x)) = 1 - (g(x))^2 = 1 - (2x+3)^2 = -4x^2 - 12x - 8 \quad \text{and}$$
$$g(f(x)) = 2f(x) + 3 = 2(1 - x^2) + 3 = -2x^2 + 5.$$

C01S04.012: Given $f(x) = -17$ and $g(x) = |x|$,

$$f(g(x)) = -17 \quad \text{and}$$
$$g(f(x)) = |f(x)| = |-17| = 17.$$

The first result is a little puzzling until one realizes that to obtain $f(g(x))$, one substitutes $g(x)$ for x for every occurrence of x in the formula for f. No x there means there's no place to put $g(x)$. Indeed, $f(h(x)) = -17$ no matter what the formula of h.

C01S04.013: If $f(x) = \sqrt{x^2 - 3}$ and $g(x) = x^2 + 3$, then

$$f(g(x)) = \sqrt{(g(x))^2 - 3} = \sqrt{(x^2+3)^2 - 3} = \sqrt{x^4 + 6x^2 + 6} \quad \text{and}$$
$$g(f(x)) = (f(x))^2 + 3 = \left(\sqrt{x^2 - 3}\right)^2 + 3 = x^2 - 3 + 3 = x^2.$$

The domain of $f(g)$ is the set $\boldsymbol{R}$ of all real numbers, but the domain of $g(f)$ is the same as the domain of f, the set of all real numbers x such that $x^2 \geqq 3$.

C01S04.014: If $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x^2+1}$, then

$$f(g(x)) = (g(x))^2 + 1 = \frac{1}{(x^2+1)^2} + 1 = \frac{x^4 + 2x^2 + 2}{x^4 + 2x^2 + 1} \quad \text{and}$$
$$g(f(x)) = \frac{1}{(f(x))^2 + 1} = \frac{1}{(x^2+1)^2 + 1} = \frac{1}{x^4 + 2x^2 + 2}.$$

C01S04.015: If $f(x) = x^3 - 4$ and $g(x) = (x+4)^{1/3}$, then

$$f(g(x)) = (g(x))^3 - 4 = \left((x+4)^{1/3}\right)^3 - 4 = x + 4 - 4 = x \quad \text{and}$$
$$g(f(x)) = (f(x) + 4)^{1/3} = \left(x^3 - 4 + 4\right)^{1/3} = \left(x^3\right)^{1/3} = x.$$

The domain of both $f(g)$ and $g(f)$ is the set $\boldsymbol{R}$ of all real numbers, so here is an example of the highly unusual case in which $f(g)$ and $g(f)$ are the same function.

C01S04.016: If $f(x) = \sqrt{x}$ and $g(x) = \cos x$, then

$$f(g(x)) = f(\cos x) = \sqrt{\cos x} \quad \text{and}$$
$$g(f(x)) = g\left(\sqrt{x}\right) = \cos\left(\sqrt{x}\right).$$

C01S04.017: If $f(x) = \sin x$ and $g(x) = x^3$, then

$$f(g(x)) = f\left(x^3\right) = \sin\left(x^3\right) = \sin x^3 \quad \text{and}$$
$$g(f(x)) = g(\sin x) = (\sin x)^3 = \sin^3 x.$$

We note in passing that $\sin x^3$ and $\sin^3 x$ don't mean the same thing!

C01S04.018: If $f(x) = \sin x$ and $g(x) = \cos x$, then $f(g(x)) = f(\cos x) = \sin(\cos x)$ and $g(f(x)) = g(\sin x) = \cos(\sin x)$.

C01S04.019: If $f(x) = 1 + x^2$ and $g(x) = \tan x$, then $f(g(x)) = f(\tan x) = 1 + (\tan x)^2 = 1 + \tan^2 x$ and $g(f(x)) = g(1 + x^2) = \tan(1 + x^2)$.

C01S04.020: If $f(x) = 1 - x^2$ and $g(x) = \sin x$, then

$$f(g(x)) = f(\sin x) = 1 - (\sin x)^2 = 1 - \sin^2 x = \cos^2 x \quad \text{and}$$
$$g(f(x)) = g(1 - x^2) = \sin(1 - x^2).$$

Note: The answers to Problems 21 through 30 are not unique. We have generally chosen the simplest and most natural answer.

C01S04.021: $h(x) = (2 + 3x)^2 = (g(x))^k = f(g(x))$ where $f(x) = x^k$, $k = 2$, and $g(x) = 2 + 3x$.

C01S04.022: $h(x) = (4 - x)^3 = (g(x))^3 = f(g(x))$ where $f(x) = x^k$, $k = 3$, and $g(x) = 4 - x$.

C01S04.023: $h(x) = (2x - x^2)^{1/2} = (g(x))^{1/2} = f(g(x))$ where $f(x) = x^k$, $k = \frac{1}{2}$, and $g(x) = 2x - x^2$.

C01S04.024: $h(x) = (1 + x^4)^{17} = (g(x))^{17} = f(g(x))$ where $f(x) = x^k$, $k = 17$, and $g(x) = 1 + x^4$.

C01S04.025: $h(x) = (5 - x^2)^{3/2} = (g(x))^{3/2} = f(g(x))$ where $f(x) = x^k$, $k = \frac{3}{2}$, and $g(x) = 5 - x^2$.

C01S04.026: $h(x) = \left[(4x - 6)^{1/3}\right]^4 = (4x - 6)^{4/3} = (g(x))^{4/3} = f(g(x))$ where $f(x) = x^k$, $k = \frac{4}{3}$, and $g(x) = 4x - 6$. Alternatively, $h(x) = (g(x))^4 = f(g(x))$ where $f(x) = x^k$, $k = 4$, and $g(x) = (4x - 6)^{1/3}$.

C01S04.027: $h(x) = (x + 1)^{-1} = (g(x))^{-1} = f(g(x))$ where $f(x) = x^k$, $k = -1$, and $g(x) = x + 1$.

C01S04.028: $h(x) = (1 + x^2)^{-1} = (g(x))^{-1} = f(g(x))$ where $f(x) = x^k$, $k = -1$, and $g(x) = 1 + x^2$.

C01S04.029: $h(x) = (x + 10)^{-1/2} = (g(x))^{-1/2} = f(g(x))$ where $f(x) = x^k$, $k = -\frac{1}{2}$, and $g(x) = x + 10$.

C01S04.030: $h(x) = (1+x+x^2)^{-3} = (g(x))^{-3} = f(g(x))$ where $f(x) = -x^k$, $k = -3$, and $g(x) = 1+x+x^2$.

C01S04.031: Recommended window: $-2 \leqq x \leqq 2$. The graph makes it evident that the equation has exactly one solution (approximately 0.641186).

C01S04.032: Recommended window: $-5 \leqq x \leqq 5$. The graph makes it evident that the equation has exactly three solutions (approximately -3.63796, -1.86236, and 0.88947).

C01S04.033: Recommended window: $-5 \leqq x \leqq 5$. The graph makes it evident that the equation has exactly one solution (approximately 1.42773).

C01S04.034: Recommended window: $-6 \leqq x \leqq 6$. The graph makes it evident that the equation has exactly three solutions (approximately -3.83747, -1.97738, and 1.30644).

C01S04.035: Recommended window: $-8 \leqq x \leqq 8$. The graph makes it evident that the equation has exactly five solutions (approximately -4.08863, -1.83622, 1.37333, 5.65222, and 6.61597).

C01S04.036: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly one solution (approximately 1.32432).

C01S04.037: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly three solutions (approximately 1.41841, 5.55211, and 6.86308).

C01S04.038: Recommended window: $-4 \leqq x \leqq 4$. The graph makes it evident that the equation has exactly two solutions (approximately ± 1.37936).

C01S04.039: Recommended window: $-11 \leqq x \leqq 11$. The graph makes it evident that the equation has exactly six solutions (approximately -5.92454, -3.24723, 3.04852, 6.75738, 8.59387, and [exactly] 0).

C01S04.040: Recommended window: $0.1 \leqq x \leqq 20$. The graph makes it evident that the equation has exactly six solutions (approximately 0.372968, 1.68831, 4.29331, 8.05637, 11.1288, and 13.6582).

C01S04.041: Graphical methods show that the solution of $10 \cdot 2^t = 100$ is slightly less than 3.322. We began with the viewing window $0 \leqq t \leqq 6$ and gradually narrowed it to $3.321 \leqq t \leqq 3.323$.

C01S04.042: Under the assumption that the interest is compounded continuously at a rate of 7.696% (for an annual yield of 8%), we solved the equation $5000 \cdot (1.07696)^t = 15000$ for $t \approx 14.8176$. We began with the viewing window $10 \leqq t \leqq 20$ and gradually narrowed it to $14.81762 \leqq y \leqq 14.81763$. Under the assumption that the interest is compounded yearly at an annual rate of 8%, we solved the equation $A(t) = 5000 \cdot (1.08)^t = 15000$ by evaluating $A(14) \approx 14686$ and $A(15) \approx 15861$. Thus in this case you'd have to wait a full 15 years for your money to triple.

C01S04.043: Graphical methods show that the solution of $(67.4) \cdot (1.026)^t = 134.8$ is approximately 27.0046. We began with the viewing window $20 \leqq t \leqq 30$ and gradually narrowed it to $27.0045 \leqq t \leqq 27.0047$.

C01S04.044: Graphical methods show that the solution of $A(t) = (0.9975)^t = 0.5$ is approximately 276.912. We began with the viewing window $200 \leqq t \leqq 300$ and gradually narrowed it to $276.910 \leqq t \leqq 276.914$.

C01S04.045: Graphical methods show that the solution of $A(t) = 12 \cdot (0.975)^t = 1$ is approximately 98.149. We began with the viewing window $50 \leqq t \leqq 250$ and gradually narrowed it to $98.148 \leqq t \leqq 98.150$.

C01S04.046: Graphical methods show that the negative solution of $x^2 = 2^x$ is approximately -0.76666. We began with the viewing window $-1 \leqq x \leqq 0$ and gradually narrowed it to $-0.7667 \leqq t \leqq -0.7666$.

C01S04.047: We plotted $y = \log_{10} x$ and $y = \frac{1}{2}x^{1/5}$ simultaneously. We began with the viewing window $1 \leqq x \leqq 10$ and gradually narrowed it to $4.84890 \leqq x \leqq 4.84892$. Answer: $x \approx 4.84891$.

C01S04.048: We began with the viewing window $-2 \leqq x \leqq 2$, which showed the two smaller solutions but not the larger solution. We first narrowed this window to $-0.9054 \leqq x \leqq -0.9052$ to get the first solution, $x \approx -0.9053$. We returned to the original window and narrowed it to $1.1324 \leqq x \leqq 1.1326$ to get the second solution, $x \approx 1.1325$. We looked for a solution in the window $20 \leqq x \leqq 30$ but there was none. But the exponential graph was still below the polynomial graph, so we checked the window $30 \leqq x \leqq 32$. A solution was evident, and we gradually narrowed this window to $31.3636 \leqq x \leqq 31.3638$ to discover the third solution, $x \approx 31.3637$.

Chapter 1 Miscellaneous Problems

C01S0M.001: The domain of $f(x) = \sqrt{x-4}$ is the set of real numbers x for which $x - 4 \geqq 0$; that is, the interval $[4, +\infty)$.

C01S0M.002: The domain of f consists of those real numbers x for which $2 - x \neq 0$; that is, the set of all real numbers other than 2.

C01S0M.003: The domain of f consists of those real numbers for which the denominator is nonzero; that is, the set of all real numbers other than ± 3.

C01S0M.004: Because $x^2 + 1$ is never zero, the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S0M.005: If $x \geqq 0$, then $\sqrt{x}$ exists; there is no obstruction to adding 1 to $\sqrt{x}$ nor to cubing the sum. Hence the domain of f is the set $[0, +\infty)$ of all nonnegative real numbers.

C01S0M.006: Given:

$$f(x) = \frac{x+1}{x^2 - 2x}.$$

The only obstruction to computing the number $f(x)$ is the possibility that the denominator is zero. Thus we must eliminate from the set of all real numbers those for which $x^2 - 2x = 0$; that is, $x(x-2) = 0$. Therefore the domain of f is the set of all real numbers other than 0 and 2.

C01S0M.07: The function $f(x) = \sqrt{2 - 3x}$ is defined whenever the radicand is nonnegative; that is, whenever

$$\begin{aligned} 2 - 3x &\geqq 0; \\ 3x &\leqq 2; \\ x &\leqq \tfrac{2}{3}. \end{aligned}$$

Hence the domain of f is the interval $\left(-\infty, \frac{2}{3}\right]$.

C01S0M.008: In order that the square root is defined, we require $9 - x^2 \geqq 0$; we also need the denominator in $f(x)$ to be nonzero, so we further require that $9 - x^2 \neq 0$. Hence $9 - x^2 > 0$; that is, $x^2 < 9$, so that $-3 < x < 3$. Hence the domain of f is the open interval $(-3, 3)$.

C01S0M.009: Regardless of the value of x, it's always possible to subtract 2 from x, to subtract x from 4, and to multiply the results. Hence the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S0M.010: The domain of f consists of those real numbers x for which $(x-2)(4-x)$ is nonnegative. That is, $x - 2$ and $4 - x$ are both positive, or $x - 2$ and $4 - x$ are both negative, or either is zero. First case: $x - 2 > 0$ and $4 - x > 0$. Then $2 < x < 4$, so the interval $(2, 4)$ is part of the domain of f. Second case: $x - 2 < 0$ and $4 - x < 0$. These inequalities imply that $x < 2$ and $4 < x$. No real numbers satisfy both these inequalities. So the second case contributes no numbers to the domain of f. Third case: $x - 2 = 0$ or $4 - x = 0$. That is, $x = 2$ or $x = 4$. Therefore the domain of f is the closed interval $[2, 4]$.

C01S0M.011: Because $100 \leqq V \leqq 200$ and $p > 0$, it follows that $100p \leqq pV \leqq 200p$. Because $pV = 800$, we see that $100p \leqq 800 \leqq 200p$, so that $p \leqq 8 \leqq 2p$. That is, $p \leqq 8$ and $4 \leqq p$, so that $4 \leqq p \leqq 8$. This is the range of possible values of p.

C01S0M.012: If $70 \leqq F \leqq 90$, then $70 \leqq 32 + \frac{9}{5}C \leqq 90$. Hence

$$\begin{aligned} 70 - 32 \leqq \tfrac{9}{5}C &\leqq 90 - 32; \\ 38 \leqq \tfrac{9}{5}C &\leqq 58; \\ 190 \leqq 9C &\leqq 290; \\ \tfrac{190}{9} \leqq C &\leqq \tfrac{290}{9}. \end{aligned}$$

Answer: The Celsius temperature ranged from a low of about 21.1°C to a high of about 32.2°C.

C01S0M.013: Because $25 < R < 50$, $25I < IR < 50I$, so that

$$\begin{aligned} &25I < E < 50I; \\ &25I < 100 < 50I; \\ &\quad I < 4 < 2I; \\ &\quad I < 4 \quad \text{and} \quad 2 < I. \end{aligned}$$

Therefore the current I lies in the range $2 < I < 4$.

C01S0M.014: Because $3 < L < 4$, we see that

$$\begin{aligned} \frac{3}{32} < \frac{L}{32} &< \frac{4}{32}; \\ \sqrt{\frac{3}{32}} < \sqrt{\frac{L}{32}} &< \sqrt{\frac{1}{8}}\,; \\ 2\pi\sqrt{\frac{3}{32}} < 2\pi\sqrt{\frac{L}{32}} &< 2\pi\sqrt{\frac{1}{8}}\,; \\ \frac{\pi}{2}\sqrt{\frac{3}{2}} < T &< \pi\sqrt{\frac{1}{2}}\,. \end{aligned}$$

In approximate terms, $1.923825 < T < 2.221441$.

C01S0M.015: If a cube has edge length x, then its volume is $V = x^3$ and its total surface area is $S = 6x^2$ (because each of its six faces has area x^2). Hence $x = \sqrt{S/6}$, and therefore

$$V(S) = \left(\sqrt{\frac{S}{6}}\right)^3 = \left(\frac{S}{6}\right)^{3/2}, \qquad 0 < S < +\infty.$$

Under certain circumstances it would be both permissible and desirable to let the domain of V be the interval $[0, +\infty)$.

C01S0M.016: Let r denote the radius, and h the height, of the cylinder. Then its volume V and total surface area A are given by

$$V = \pi r^2 h \qquad \text{and} \qquad A = 2\pi rh + 2\pi r^2$$

(look inside the front cover of the textbook). In this problem we are given $h = r$, so that $V = \pi r^3$ and $A = 4\pi r^2$. Therefore

$$r = \left(\frac{V}{\pi}\right)^{1/3} \qquad \text{and so} \qquad A = 4\pi\left(\frac{V}{\pi}\right)^{2/3}.$$

$$\text{Answer:} \qquad A(V) = 4\pi\left(\frac{V}{\pi}\right)^{2/3}, \quad 0 < V < +\infty.$$

It is permissible, and sometimes desirable, to use instead the domain $0 \leqq V < +\infty$.

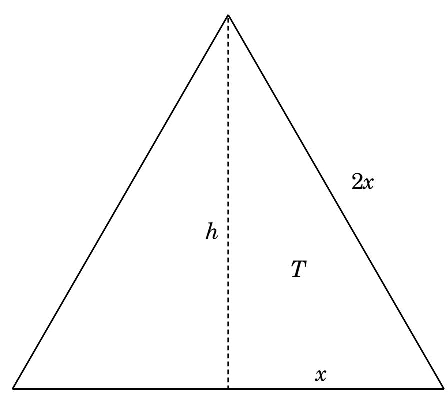
C01S0M.017: The following figure shows an equilateral triangle with sides of length $2x$ and an altitude of length h. Because T is a right triangle, we see that

$$x^2 + h^2 = (2x)^2, \qquad \text{so that} \qquad h = x\sqrt{3}.$$

The area of this triangle is $A = hx$ and its perimeter is $P = 6x$. So

$$A = x^2\sqrt{3} \qquad \text{and} \qquad x = \frac{P}{6}.$$

Therefore $A(P) = \dfrac{P^2\sqrt{3}}{36}, \quad 0 < P < \infty.$



C01S0M.018: The square has perimeter x and thus edge length $y = \frac{1}{4}x$. The circle has circumference $100 - x$. Thus if z is the radius of the circle, then $2\pi z = 100 - x$, so that $z = (100 - x)/(2\pi)$. The area of the square is y^2 and the area of the circle is πz^2, so that the sum of the areas of the square and the circle is given by

$$A(x) = \frac{x^2}{16} + \pi\left(\frac{100 - x}{2\pi}\right)^2, \quad 0 < x < 100.$$

Looking ahead to Chapter 3, it will be advantageous to use the *closed* interval [0, 100] for the domain of the function A.

C01S0M.019: The slope of L is $\dfrac{13 - 5}{1 - (-3)} = 2$, so an equation of L is

$$y - 5 = 2(x + 3); \qquad \text{that is,} \qquad y = 2x + 11.$$

C01S0M.020: An equation of L is $y - (-1) = -3(x - 4)$; that is, $3x + y = 11$.

C01S0M.021: The point $(0, -5)$ lies on L, so an equation of L is

$$y - (-5) = \frac{1}{2}(x - 0); \qquad \text{alternatively,} \qquad 2y + 10 = x.$$

C01S0M.022: The equation $3x - 2y = 4$ of the other line may be written in the form $y = \frac{3}{2}x - 2$, revealing that it and L have slope $\frac{3}{2}$. Hence an equation of L is

$$y - (-3) = \frac{3}{2}(x - 2); \qquad \text{that is,} \qquad y = \frac{3}{2}x - 6.$$

C01S0M.023: The equation $y - 2x = 10$ may be written in the form $y = 2x + 10$, showing that it has slope 2. Hence the perpendicular line L has slope $-\frac{1}{2}$. Therefore an equation of L is

$$y - 7 = -\frac{1}{2}(x - (-3)); \qquad \text{that is,} \qquad x + 2y = 11.$$

C01S0M.024: The segment S joining $(1, -5)$ and $(3, -1)$ has slope $(-1 - (-5))/(3 - 1) = 2$ and midpoint $(2, -3)$, and hence L has slope $-\frac{1}{2}$ and passes through $(2, -3)$. So an equation of L is

$$y - (-3) = -\frac{1}{2}(x - 2); \qquad \text{that is,} \qquad x + 2y = -4.$$

C01S0M.025: The graph of $y = f(x) = 2 - 2x - x^2$ is a parabola opening downward. The only such graph is shown in Fig. 1.MP.6.

C01S0M.026: Given: $f(x) = x^3 - 4x^2 + 5$. Because $f(-1) = 0$, $f(1) = 2 > 0 > -3 = f(2)$, and $f(3) = -4 < 0 < 5 = f(4)$, the graph of f crosses the x-axis at $x = -1$, between $x = 1$ and $x = 2$, and between $x = 3$ and $x = 4$. Hence the graph of f is the one shown in Fig. 1.MP.9.

C01S0M.027: Given: $f(x) = x^4 - 4x^3 + 5$. Because the graph of f has no vertical asymptotes and because $f(x)$ approaches $+\infty$ as x approaches either $+\infty$ or $-\infty$, the graph of f must be the one shown in Fig. 1.MP.4.

C01S0M.028: Given:

$$f(x) = \frac{5}{x^2 - x - 6} = \frac{5}{(x - 3)(x + 2)}.$$

The denominator in $f(x)$ is zero when $x = 3$ and when $x = -2$ (and the numerator is not zero), so the graph of $y = f(x)$ has vertical asymptotes at $x = -2$ and at $x = 3$. Also $f(x)$ approaches zero as x approaches either $+\infty$ or $-\infty$. Therefore the graph of $y = f(x)$ must be the one shown in Fig. 1.MP.11.

C01S0M.029: Given:

$$f(x) = \frac{5}{x^2 - x + 6} = \frac{20}{4x^2 - 4x + 1 + 23} = \frac{20}{(2x - 1)^2 + 23}.$$

The algebra displayed here shows that the denominator in $f(x)$ is never zero, so there are no vertical asymptotes. It also shows that the maximum value of $f(x)$ occurs when the denominator is minimal; that is, when $x = \frac{1}{2}$. Finally, $f(x)$ approaches zero as x approaches either $+\infty$ or $-\infty$. So the graph of $y = f(x)$ must be the one shown in Fig. 1.MP.3.

C01S0M.030: If $y = f(x) = \sqrt{8 + 2x - x^2}$, then

$$y^2 = 8 + 2x - x^2;$$
$$x^2 - 2x + 1 + y^2 = 9;$$
$$(x-1)^2 + (y-0)^2 = 3^2.$$

The last is the equation of a circle with center $(1, 0)$ and radius 3. But $y \geqq 0$, so the graph of f is the upper half of that circle, and it is shown in Fig. 1.MP.10.

C01S0M.031: Given: $f(x) = 2^{-x} - 1$. The graph of $y = 2^x$ is an increasing exponential function, so the graph of $y = 2^{-x}$ is a decreasing exponential function, approaching 0 as x approaches $+\infty$. So the graph of f approaches -1 as x approaches $+\infty$. Moreover, $f(0) = 0$. Therefore the graph of f is the one shown in Fig. 1.MP.7.

C01S0M.032: The graph of $f(x) = \log_{10}(x+1)$ is obtained from the graph of $g(x) = \log_{10} x$ by translation one unit to the left; note also that $f(0) = 0$. Therefore the graph of f is the one shown in Fig. 1.MP.2.

C01S0M.033: The graph of $y = 3\sin x$ oscillates between its minimum value -3 and its maximum value 3, so the graph of $f(x) = 1 + 3\sin x$ oscillates between -2 and 4. This graph is shown in Fig. 1.MP.8.

C01S0M.034: The graph of $f(x) = x + 3\sin x$ viewed at a great distance resembles the graph of $y = x$. A closer view shows oscillations, due to the sine function, superposed on the graph of $y = x$. Thus the graph of f is the one shown in Fig. 1.MP.5.

C01S0M.035: The graph of $2x - 5y = 7$ is the straight line with x-intercept $\frac{7}{2}$ and y-intercept $-\frac{7}{5}$.

C01S0M.036: If $|x - y| = 1$, then $x - y = 1$ or $x - y = -1$. The graph of the first of these is the straight line $y = x - 1$ with slope 1 and y-intercept -1; the graph of the second is the straight line $y = x + 1$ with slope 1 and y-intercept 1. So the graph of $|x - y| = 1$ consists of these two parallel lines.

C01S0M.037: We complete the square: $x^2 - 2x + 1 + y^2 = 1$, so that $(x-1)^2 + (y-0)^2 = 1^2$. Thus the graph of the given equation is the circle with center (1, 0) and radius 1.

C01S0M.038: We complete the square in x and in y to obtain

$$x^2 + 6x + 9 + y^2 - 4y + 4 = 16;$$
$$(x+3)^2 + (y-2)^2 = 4^2.$$

Therefore the graph of the given equation is the circle with center $(-3,\ 2)$ and radius 4.

C01S0M.039: The graph is a parabola opening upward. To find its vertex, we complete the square:

$$\begin{aligned} y &= 2\left(x^2 - 2x - \tfrac{1}{2}\right) \\ &= 2\left(x^2 - 2x + 1 - \tfrac{3}{2}\right) = 2(x-1)^2 - 3. \end{aligned}$$

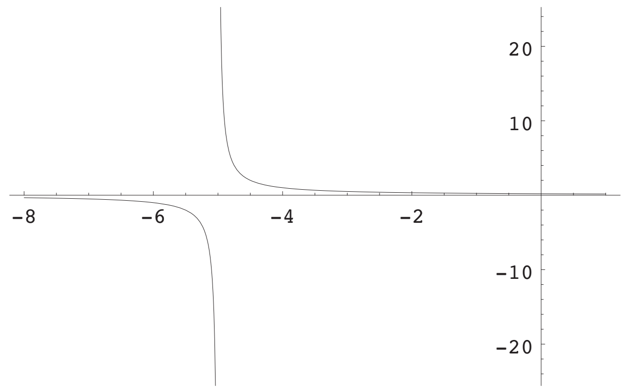
So the vertex of this parabola is at the point $(1,\ -3)$.

C01S0M.040: The graph is a parabola opening downward. To find its vertex, we complete the square:

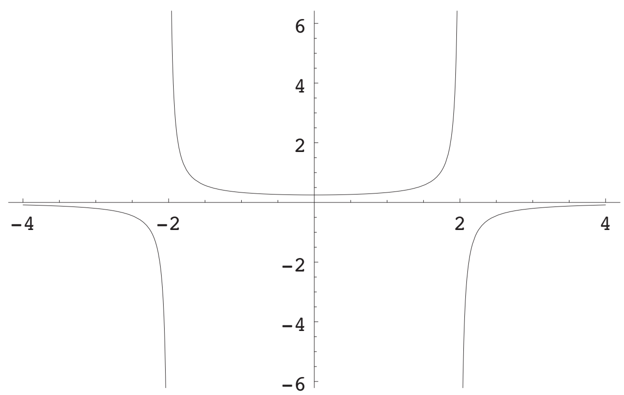
$$y = 4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = 4 - (x-2)^2.$$

Thus the vertex of this parabola is at the point (2, 4).

C01S0M.041: The graph has a vertical asymptote at $x = -5$ and is shown next.

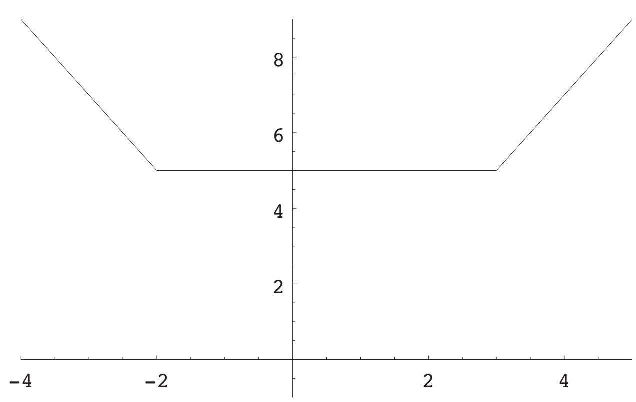


C01S0M.042: The graph has vertical asymptotes at $x = \pm 2$ and is shown next.



C01S0M.043: The graph of f is obtained by shifing the graph of $g(x) = |x|$ three units to the right, so that the graph of f has its "vertex" at the point (3, 0).

C01S0M.044: Given: $f(x) = |x - 3| + |x + 2|$. If $x \geqq 3$ then $f(x) = x - 3 + x + 2 = 2x - 1$, so the graph is the unbounded line segment with slope 2 and endpoint (3, 5) for $x \geqq 3$. If $-2 \leqq x \leqq 3$ then $f(x) = 3 - x + x + 2 = 5$, so another part of the graph is the horizontal line segment joining (−2, 5) with (3, 5). If $x \leqq -2$ then $f(x) = 3 - x - x - 2 = -2x + 1$, so the rest of the graph is the unbounded line segment with slope −2 and endpoint (−2, 5) for $x \leqq -2$. The graph is shown next.



C01S0M.045: Suppose that a, b, and c are arbitrary real numbers. Then

$$|a + b + c| = |(a + b) + c| \leqq |a + b| + |c| \leqq |a| + |b| + |c| \,.$$

.

C01S0M.046: Suppose that a and b are arbitrary real numbers. Then $|a| = |(a - b) + b| \leqq |a - b| + |b|$. Therefore $|a| - |b| \leqq |a - b|$.

C01S0M.047: If $x - 3 > 0$ and $x + 2 > 0$, then $x > 3$ and $x > -2$, so $x > 3$. If $x - 3 < 0$ and $x + 2 < 0$, then $x < 3$ and $x < -2$, so $x < -2$. Answer: $(-\infty, -2) \cup (3, \infty)$.

C01S0M.048: $(x - 1)(x - 2) < 0$: $x - 1$ and $x - 2$ have opposite signs, so either $x < 1$ and $x > 2$ (which leads to *no* values of x) or $x > 1$ and $x < 2$. Answer: $(1,\ 2)$.

C01S0M.049: $(x - 4)(x + 2) > 0$: Either $x > 4$ and $x > -2$ (so that $x > 4$) or $x < 4$ and $x < -2$ (so that $x < -2$). Answer: $(-\infty,\ -2) \cup (4,\ +\infty)$.

C01S0M.050: $2x \geqq 15 - x^2$: $x^2 + 2x - 15 \geqq 0$, so $(x - 3)(x + 5) \geqq 0$. Now $x + 5 > x - 3$, so $x - 3 \geqq 0$ or $x + 5 \leqq 0$. Thus $x \geqq 3$ or $x \leqq -5$. Answer: $(-\infty,\ -5\,] \cup [\,3,\ +\infty)$.

C01S0M.051: The viewing window $-3 \leqq x \leqq 8$ shows a solution near -1 and another near 5. Gradual magnification of the region near -1 shows a solution between -1.1405 and -1.1395. Similarly, the other solution is between 6.1395 and 6.1405. So the solutions are approximately -1.140 and 6.140.

C01S0M.052: The viewing window $-2 \leqq x \leqq 5$ shows a solution near -1 and another near 4. To approximate the first more closely, we used the method of repeated tabulation on $[-1.0,\ -0.8]$, then on $[-0.88,\ -0,86]$, then on $[-0.872,\ -0.870]$. To approximate the second, we used the interval $[4.1,\ 4.3]$, then $[4.20,\ 4.22]$, then $[4.204,\ 4.206]$. To three places, the solutions are -0.872 and 4.205.

C01S0M.053: The viewing window $0.5 \leqq x \leqq 3$ shows one solution near 1.2 and another near 2.3. The method of repeated tabulation with successive intervals [1.1, 1.3], [1.18, 1.20], and [1.190, 1.192] yields the approximation 1.191 to the first solution. The successive intervals [2.2, 2.4], [2.30, 2.32], and [2.308, 2.310] yield the approximation 2.309 to the second solution.

C01S0M.054: The viewing window $-7 \leqq x \leqq 2$ shows one solution near -6 and another near 1. The method of repeated tabulation with successive intervals $[-6.1,\ -5.9]$, $[-5.98,\ -5.96]$, and $[-5.974,\ -5.970]$ yield the approximation -5.972 to the first solution. Simlarly, we find the second solution to be approximately 1.172.

C01S0M.055: The viewing window $-6 \leqq x \leqq 2$ shows one solution near -5 and another near 1. The method of repeated tabulation with successive intervals $[-5.1,\ -4.9]$, $[-5.04,\ -5.02]$, and $[-5.022,\ -5.020]$, then with the intervals [0.8, 1.0], [0.88, 0.90], and [0.896, 0.898], yields the two approximations -5.021 and 0.896 to the two solutions.

C01S0M.056: The viewing window $-11 \leqq x \leqq 3$ shows one solution near -10 and another near 1.7. The method of repeated tabulation, first with the intervals $[-10.0, -9.9]$, $[-9.97,\ -9.96]$, and $[-9.963,\ -9.962]$, then with [1.7, 1.8], [1.73, 1.74], and [1.739, 1.741], yields the two approximations -9.962 and 1.740 to the two solutions.

C01S0M.057: The viewing window $2 \leqq x \leqq 3$ shows the low point with x-coordinate near 2.5. The method of repeated tabulation, using the successive intervals [2.4, 2.6], [2.48, 2.52], and [2.496, 2.504], indicates that the low point is very close to (2.5, 0.75).

C01S0M.058: The viewing window $-1 \leqq x \leqq 4$ shows the low point with x-coordinate near 1.7. The method of repeated tabulation, with the successive intervals [1.6, 1.8], [1.64, 1.68], and [1.664, 1.672], shows that the low point is quite close to (1.66, 2.67).

C01S0M.059: The viewing window $-0.5 \leqq x \leqq 4$ shows that the low point has x-coordinate near 1.8. The method of repeated tabulation, with the successive intervals [1.7, 1.9], [1.72, 1.78], and [1.744, 1.756], shows that the low point is very close to $(1.75, -1.25)$.

C01S0M.060: The viewing window $-5 \leqq x \leqq 1$ shows that the low point has x-coordinate near -2.5. The method of repeated tabulation indicates that the low point is very close to $(-2.4, 6.2)$.

C01S0M.061: The viewing window $-5 \leqq x \leqq 1$ show that the x-coordinate of the low point is close to -2. The method of repeated tabulation shows that the low point is very close to $(-2.0625, 0.96875)$.

C01S0M.062: The viewing window $-7 \leqq x \leqq 1$ shows that the x-coordinate of the low point is close to -4. The method of repeated tabulation indicates that the low point is very close to $(-4.111, 3.889)$.

C01S0M.063: The small rectangle has dimensions $10 - 4x$ by $7 - 2x$; $(7)(10) - (10 - 4x)(7 - 2x) = 20$, which leads to the quadratic equation $8x^2 - 48x + 20 = 0$. One solution of this equation is approximately 5.5495, which must be rejected; it is too large. The value of x is the other solution: $x \approx 0.4505$.

C01S0M.064: After shrinking, the tablecloth has dimensions $60 - x$ by $35 - x$. The area of this rectangle is 93% of the area of the original tablecloth, so $(60 - x)(35 - x) = (0.93)(35)(60)$. The larger solution of this quadratic equation is approximately 93.43, which we reject as too large. Answer: $x \approx 1.573$.

C01S0M.065: The viewing window $-4 \leqq x \leqq 4$ shows three solutions (and there can be no more).

C01S0M.066: The viewing window $-3 \leqq x \leqq 3$ shows two solutions, and there can be no more because $x^4 > |-3x^2 + 4x - 5|$ if $|x| > 3$.

C01S0M.067: We plotted $y = \sin x$ and $y = x^3 - 3x + 1$ simultaneously to see where they crossed. The viewing window $-2.2 \leqq x \leqq 2.2$ shows three solutions, and there can be no more because $|x^3 - 3x + 1| > 1$ if $|x| > 2.2$.

C01S0M.068: We plotted $y = \cos x$ and $y = x^4 - x$ simultaneously to see where they crossed. The viewing window $-2 \leqq x \leqq 2$ shows two solutions, and there can be no more because $x^4 - x > 1$ if $|x| > 2$.

C01S0M.069: We plotted $y = \cos x$ and $y = \log_{10} x$ simultaneously to see where they crossed. The viewing window $0.1 \leqq x \leqq 14$ shows three solutions, and there can be no more because $\log_{10} x < -1$ if $0 < x < 0.1$ and $\log_{10} x > 1$ if $x > 14$.

C01S0M.070: We plotted $y = 10^{-x}$ and $y = \log_{10} x$ simultaneously to see where they crossed. The viewing window $0.1 \leqq x \leqq 3$ shows one solution, and there can be no more because the exponential function is decreasing for all x and the logarithm function is increasing for all $x > 0$.

Section 2.1

C02S01.001: $f(x) = 0 \cdot x^2 + 0 \cdot x + 5$, so $m(a) = 0 \cdot 2 \cdot a + 0 \equiv 0$. In particular, $m(2) = 0$, so the tangent line has equation $y - 5 = 0 \cdot (x - 0)$; that is, $y \equiv 5$.

C02S01.002: $f(x) = 0 \cdot x^2 + 1 \cdot x + 0$, so $m(a) = 0 \cdot 2 \cdot a + 1 \equiv 1$. In particular, $m(2) = 1$, so the tangent line has equation $y - 2 = 1 \cdot (x - 2)$; that is, $y = x$.

C02S01.003: Because $f(x) = 1 \cdot x^2 + 0 \cdot x + 0$, the slope-predictor is $m(a) = 2 \cdot 1 \cdot a + 0 = 2a$. Hence the line L tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = 4$. So an equation of L is $y - f(2) = 4(x - 2)$; that is, $y = 4x - 4$.

C02S01.004: Because $f(x) = -2x^2 + 0 \cdot x + 1$, the slope-predictor for f is $m(a) = 2 \cdot (-2)a + 0 = -4a$. Thus the line L tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = -8$ and equation $y - f(2) = -8(x - 2)$; that is, $y = -8x + 9$.

C02S01.005: Because $f(x) = 0 \cdot x^2 + 4x - 5$, the slope-predictor for f is $m(a) = 2 \cdot 0 \cdot a + 4 = 4$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope 4 and therefore equation $y - 3 = 4(x - 2)$; that is, $y = 4x - 5$.

C02S01.006: Because $f(x) = 0 \cdot x^2 - 3x + 7$, the slope-predictor for f is $m(a) = 2 \cdot 0 \cdot a - 3 = -3$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope -3 and therefore equation $y - 1 = -3(x - 2)$; that is, $y = -3x + 7$.

C02S01.007: Because $f(x) = 2x^2 - 3x + 4$, the slope-predictor for f is $m(a) = 2 \cdot 2 \cdot a - 3 = 4a - 3$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope 5 and therefore equation $y - 6 = 5(x - 2)$; that is, $y = 5x - 4$.

C02S01.008: Because $f(x) = (-1) \cdot x^2 - 3x + 5$, the slope-predictor for f is $m(a) = 2 \cdot (-1) \cdot a - 3 = -2a - 3$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope -7 and therefore equation $y + 5 = -7(x - 2)$; that is, $y = -7x + 9$.

C02S01.009: Because $f(x) = 2x^2 + 6x$, the slope-predictor for f is $m(a) = 4a + 6$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = 14$ and therefore equation $y - 20 = 14(x - 2)$; that is, $y = 14x - 8$.

C02S01.010: Because $f(x) = -3x^2 + 15x$, the slope-predictor for f is $m(a) = -6a + 15$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = 3$ and therefore equation $y - 18 = 3(x - 2)$; that is, $y = 3x + 12$.

C02S01.011: Because $f(x) = -\frac{1}{100}x^2 + 2x$, the slope-predictor for f is $m(a) = -\frac{2}{100}a + 2$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = -\frac{1}{25} + 2 = \frac{49}{25}$ and therefore equation $y - \frac{99}{25} = \frac{49}{25}(x - 2)$; that is, $25y = 49x + 1$.

C02S01.012: Because $f(x) = -9x^2 - 12x$, the slope-predictor for f is $m(a) = -18a - 12$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = -48$ and therefore equation $y + 60 = -48(x - 2)$; that is, $y = -48x + 36$.

C02S01.013: Because $f(x) = 4x^2 + 1$, the slope-predictor for f is $m(a) = 8a$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = 16$ and therefore equation $y - 17 = 16(x - 2)$; that is, $y = 16x - 15$.

C02S01.014: Because $f(x) = 24x$, the slope-predictor for f is $m(a) = 24$. So the line tangent to the graph of f at $(2,\ f(2))$ has slope $m(2) = 24$ and therefore equation $y - 48 = 24(x - 2)$; that is, $y = 24x$.

C02S01.015: If $f(x) = -x^2 + 10$, then the slope-predictor for f is $m(a) = -2a$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 0$. So the tangent line is horizontal at the point (0, 10) and at no other point of the graph of f.

C02S01.016: If $f(x) = -x^2 + 10x$, then the slope-predictor for f is $m(a) = -2a + 10$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 5$. So the tangent line is horizontal at the point (5, 25) and at no other point of the graph of f.

C02S01.017: If $f(x) = x^2 - 2x + 1$, then the slope-predictor for f is $m(a) = 2a - 2$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 1$. So the tangent line is horizontal at the point (1, 0) and at no other point of the graph of f.

C02S01.018: If $f(x) = x^2 + x - 2$, then the slope-predictor for f is $m(a) = 2a + 1$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = -\frac{1}{2}$. So the tangent line is horizontal at the point $\left(-\frac{1}{2},\, -\frac{9}{4}\right)$ and at no other point of the graph of f.

C02S01.019: If $f(x) = -\frac{1}{100}x^2 + x$, then the slope-predictor for f is $m(a) = -\frac{1}{50}a + 1$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 50$. So the tangent line is horizontal at the point (50, 25) and at no other point of the graph of f.

C02S01.020: If $f(x) = -x^2 + 100x$, then the slope-predictor for f is $m(a) = -2a + 100$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 50$. So the tangent line is horizontal at the point (50, 2500) and at no other point of the graph of f.

C02S01.021: If $f(x) = x^2 - 2x - 15$, then the slope-predictor for f is $m(a) = 2a - 2$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 1$. So the tangent line is horizontal at the point (1, -16) and at no other point of the graph of f.

C02S01.022: If $f(x) = x^2 - 10x + 25$, then the slope-predictor for f is $m(a) = 2a - 10$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 5$. So the tangent line is horizontal at the point (5, 0) and at no other point of the graph of f.

C02S01.023: If $f(x) = -x^2 + 70x$, then the slope-predictor for f is $m(a) = -2a + 70$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 35$. So the tangent line is horizontal at the point (35, 1225) and at no other point of the graph of f.

C02S01.024: If $f(x) = x^2 - 20x + 100$, then the slope-predictor for f is $m(a) = 2a - 20$. A line tangent to the graph of f will be horizontal when $m(a) = 0$, thus when $a = 10$. So the tangent line is horizontal at the point (10, 0) and at no other point of the graph of f.

C02S01.025: If $f(x) = x^2$, then the slope-predictor for f is $m(a) = 2a$. So the line tangent to the graph of f at the point $P(-2, 4)$ has slope $m(-2) = -4$ and the normal line at P has slope $\frac{1}{4}$. Hence an equation for the line tangent to the graph of f at P is $y - 4 = -4(x + 2)$; that is, $y = -4x - 4$. An equation for the line normal to the graph of f at P is $y - 4 = \frac{1}{4}(x + 2)$; that is, $4y = x + 18$.

C02S01.026: If $f(x) = -2x^2 - x + 5$, then the slope-predictor for f is $m(a) = -4a - 1$. So the line tangent to the graph of f at the point $P(-1, 4)$ has slope $m(-1) = 3$ and the normal line at P has slope $-\frac{1}{3}$. Hence an equation for the line tangent to the graph of f at P is $y - 4 = 3(x + 1)$; that is, $y = 3x + 7$. An equation for the line normal to the graph of f at P is $y - 4 = -\frac{1}{3}(x + 1)$; that is, $x + 3y = 11$.

C02S01.027: If $f(x) = 2x^2 + 3x - 5$, then the slope-predictor for f is $m(a) = 4a + 3$. So the line tangent to the graph of f at the point $P(2, 9)$ has slope $m(2) = 11$ and the normal line at P has slope $-\frac{1}{11}$. Hence

an equation for the line tangent to the graph of f at P is $y - 9 = 11(x - 2)$; that is, $y = 11x - 13$. An equation for the line normal to the graph of f at P is $y - 9 = -\frac{1}{11}(x - 2)$; that is, $x + 11y = 101$.

C02S01.028: If $f(x) = x^2$, then the slope-predictor for f is $m(a) = 2a$. Hence the line L tangent to the graph of f at the point (x_0, y_0) has slope $m(x_0) = 2x_0$. Because $y_0 = x_0^2$, an equation of L is $y - x_0^2 = 2x_0(x - x_0)$. To find where L meets the x-axis, we substitute $y = 0$ in the equation of L and solve for x:

$$0 - x_0^2 = 2x_0(x - x_0);$$

$$x - x_0 = -\tfrac{1}{2}x_0 \qquad (\text{if } x_0 \neq 0);$$

$$x = x_0 - \tfrac{1}{2}x_0 = \tfrac{1}{2}x_0.$$

Therefore if $x_0 \neq 0$, L meets the x-axis at the point $\left(\frac{1}{2}x_0,\, 0\right)$. If $x_0 = 0$, then L *is* the x-axis and therefore meets the x-axis at $\left(\frac{1}{2}x_0,\, 0\right) = (0,\, 0)$ as well as at every other point.

C02S01.029: If the ball has height $y(t) = -16t^2 + 96t$ (feet) at time t (s), then the slope-predictor for y is $m(a) = -32a + 96$. Assuming that the maximum height of the ball occurs at the point on the graph of y where the tangent line is horizontal, we find that point by solving $m(a) = 0$ and find that $a = 3$. So the highest point on the graph of y is the point $(3,\, y(3)) = (3,\, 144)$. Therefore the ball reaches a maximum height of 144 (ft).

C02S01.030: The slope-predictor for $A(x) = -x^2 + 50x$ is $m(a) = -2a + 50$. The highest point on the graph of A occurs where the tangent line is horizontal; that is, when $2a = 50$, so that $a = 25$. (We know it's the high point rather than the low point because the graph of $y = A(x)$ is a parabola that opens downward.) So the highest point on the graph of A is the point $(25,\, A(25)) = (25,\, 625)$. Because $a = 25$ is in the domain $[0,\, 50]$ of the function A, the maximum possible area of the rectangle is 625 (ft^2).

C02S01.031: If the two positive numbers x and y have sum 50, then $y = 50 - x$, $x > 0$, and $x < 50$ (because $y > 0$). So the product of two such numbers is given by

$$p(x) = x(50 - x), \qquad 0 < x < 50.$$

The graph of $p(x) = -x^2 + 50x$ has a highest point because the graph of $y = p(x)$ is a parabola that opens downward. The slope-predictor for the function p is $m(a) = -2a + 50$. The highest point on the graph of p will occur when the tangent line is horizontal, so that $m(a) = 0$. This leads to $a = 25$, which does lie in the domain of p. Therefore the highest point on the graph of p is $(25,\, p(25)) = (25,\, 625)$. Hence the maximum possible value of $p(x)$ is 625. So the maximum possible product of two positive numbers with sum 50 is 625.

C02S01.032: If $y = f(x) = -\frac{1}{625}x^2 + x$, then the slope predictor for f is $m(a) = -\frac{2}{625}a + 1$. (a) The projectile hits the ground at that point x for which $f(x) = 0$; that is, $x^2 = 625x$, so that $x = 0$ (which we reject; this is where the projectile *leaves* the ground) or $x = 625$. Because the projectile travels from $x = 0$ to $x = 625$, the horizontal distance it travels is 625 (ft). (b) To find the maximum height of the projectile, we find where the line tangent to the graph of f is horizontal. This occurs when $m(a) = 0$, so that $a = 312.5$. So the maximum height of the projectile is $f(312.5) = 156.25$ (ft). (It's a maximum rather than a minimum because the graph of $y = f(x)$ is a parabola that opens downwards *and* $x = 312.5$ does lie in the domain $[0,\, 625]$ of the function f.)

C02S01.033: Suppose that the "other" line L is tangent to the parabola at the point $(a,\, a^2)$. The slope-predictor for $y = f(x) = x^2$ is $m(a) = 2a$, so the line L has slope $m(a) = 2a$. (Note that a changes from a variable to a constant in the last sentence. This is dangerous but the notation has forced this situation

upon us.) Using the two-point formula for slope, we can compute the slope of L in another way and equate our two results:

$$\begin{aligned}
\frac{a^2-0}{a-3} &= 2a; \\
a^2 &= 2a(a-3); \\
a &= 2a-6; \qquad \text{(because } a \neq 0\text{);} \\
a &= 6.
\end{aligned}$$

Therefore L has slope $m(6) = 12$. Because L passes through $(3,\,0)$, an equation of L is $y - 0 = 12(x-3)$; that is, $y = 12x - 36$.

C02S01.034: If $y = f(x) = -x^2 + 4x$, then the slope-predictor for f is $m(a) = -2a + 4$. Suppose that the line L passes through the point $P(2,\,5)$ and is tangent to the graph of f. Let $Q(c,\,f(c)) = (c,\,4c - c^2)$ be the point of tangency. We can use the two points P and Q to compute the slope of L. We can also use the slope-predictor. We do so and equate the results:

$$\begin{aligned}
\frac{4c-c^2-5}{c-2} &= -2c+4; \\
4c-c^2-5 &= (c-2)(-2c+4) = -2c^2+8c-8; \\
c^2-4c+3 &= 0; \\
(c-1)(c-3) &= 0.
\end{aligned}$$

Therefore $c = 1$ or $c = 3$. We have discovered that there are two points at which L may be tangent to the graph of f: $(1,\,f(1)) = (1,\,3)$ and $(3,\,f(3)) = (3,\,3)$. Thus one tangent line has slope 2 and the other has slope -2; their equations may be written as

$$y - 5 = 2(x-2) \qquad \text{and} \qquad y - 5 = -2(x-2).$$

C02S01.035: Suppose that $(a,\,a^2)$ is the point on the graph of $y = x^2$ closest to $(3,\,0)$. Let L be the line segment from $(3,\,0)$ to $(a,\,a^2)$. Under the plausible assumption that L is normal to the tangent line at $(a,\,a^2)$, we infer that the slope m of L is $-1/(2a)$ because the slope of the tangent line is $2a$. Because we can also compute m by using the two points known to lie on it, we find that

$$m = -\frac{1}{2a} = \frac{a^2-0}{a-3}.$$

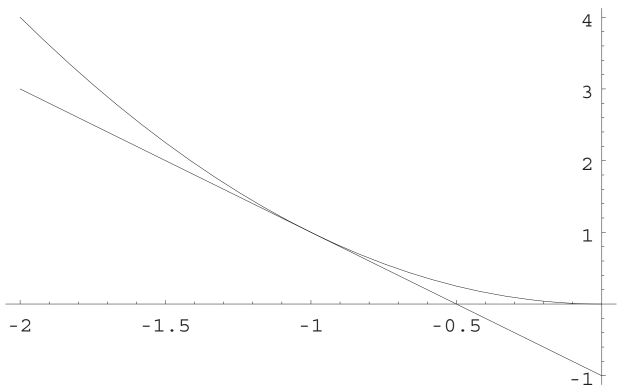
This leads to the equation $0 = 2a^3 + a - 3 = (a-1)(2a^2 + 2a + 3)$, which has $a = 1$ as its only real solution (note that the discriminant of $2a^2 + 2a + 3$ is negative). Intuitively, it's clear that there *is* a point on the graph nearest $(3\,,0)$, so we have found it: That point is $(1,\,1)$.

Alternatively, if $(x,\,x^2)$ is an arbitrary point on the given parabola, then the distance from $(x,\,x^2)$ to $(3,\,0)$ is the square root of $f(x) = (x^2-0)^2 + (x-3)^2 = x^4 + x^2 - 6x + 9$. A positive quantity is minimized when its square is minimized, so we minimize the distance from $(x,\,x^2)$ to $(3,\,0)$ by minimizing $f(x)$. The slope-predictor for f is $m(a) = 4a^3 + 2a - 6 = 2(a-1)(2a^2 + 2x + 3)$, and (as before) the equation $m(a) = 0$ has only one real solution, $a = 1$. Again appealing to intuition for the existence of a point on the parabola nearest to $(3,\,0)$, we see that it can only be the point $(1,\,1)$. In Chapter 3 we will see how the existence of the closest point can be established without an appeal to the intuition.

C02S01.036: Given: $f(x) = x^2$ and $a = -1$. We computed

$$\frac{f(a+h)-f(a-h)}{2h} \tag{1}$$

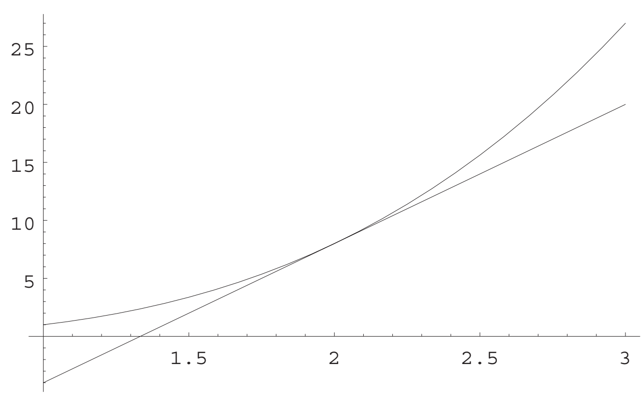
for $h = 10^{-1},\ 10^{-2},\ \ldots,$ and 10^{-10}. The values of the expression in (1) were all -2.00000000000000000000 (to twenty places). The numerical evidence overwhelmingly suggests that the slope of the tangent line is exactly -2 and thus that it has equation $y = -2x - 1$. The graph of this line and $y = f(x)$ are shown next.



C02S01.037: Given: $f(x) = x^3$ and $a = 2$. We computed

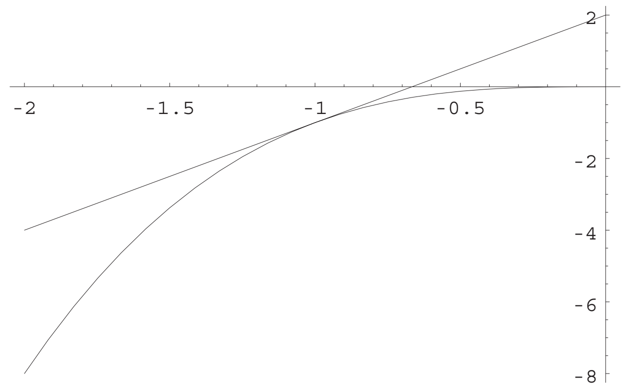
$$\frac{f(a+h)-f(a-h)}{2h} \tag{1}$$

for $h = 10^{-1},\ 10^{-2},\ \ldots,\ 10^{-10}$. The values of the expression in (1) were $12.01,\ 12.0001,\ 12.000001,\ \ldots,$ 12.00000000000000000001. The numerical evidence overwhelmingly suggests that the slope of the tangent line is 12 and thus that it has equation $y = 12x - 16$. The graph of this line and $y = f(x)$ are shown next.

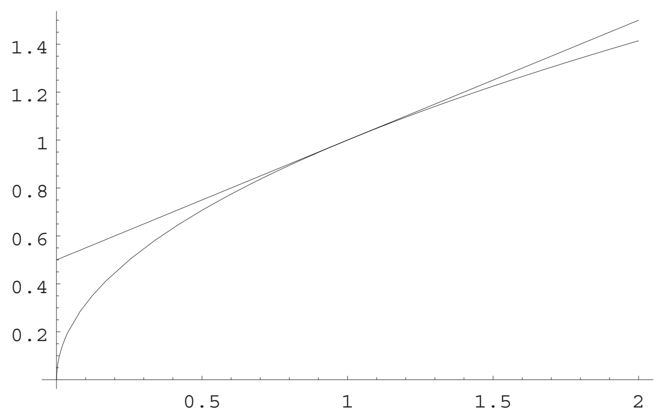


C02S01.038: Using the techniques in the previous solution produced strong evidence that the slope of the tangent line is 3, so that its equation is $y = 3x + 2$. The graphs of $f(x) = x^3$ and the tangent line are shown

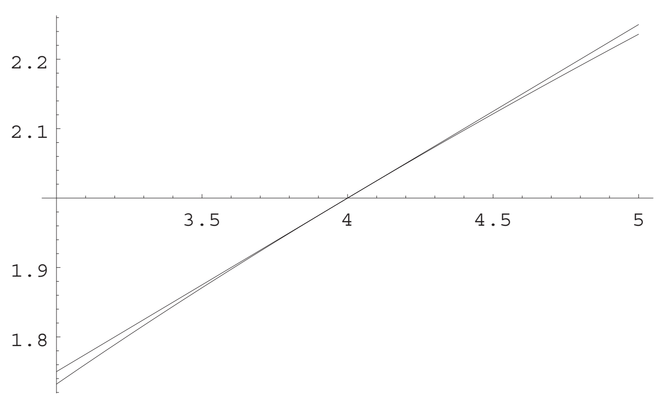
next.



C02S01.039: The numerical evidence suggests that the slope of the tangent line is $\frac{1}{2}$, so that its equation is $y = \frac{1}{2}(x + 1)$. The graph of the tangent line and the graph of $f(x) = \sqrt{x}$ are shown next.

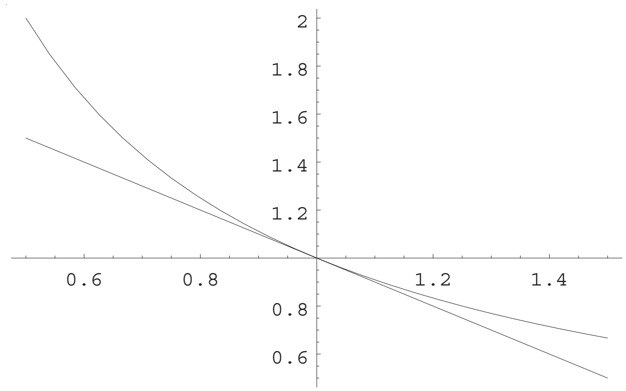


C02S01.040: The numerical evidence suggests that the slope of the tangent line is $\frac{1}{4}$, so that its equation is $y = \frac{1}{4}(x + 4)$. The graph of the tangent line and the graph of $f(x) = \sqrt{x}$ are shown next.

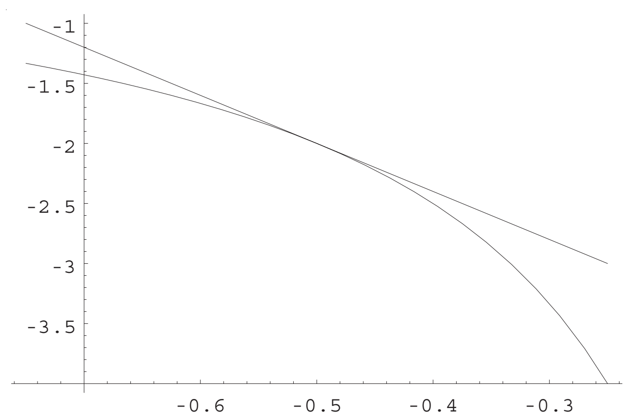


C02S01.041: The numerical evidence suggests that the slope of the tangent line is -1, so that its equation

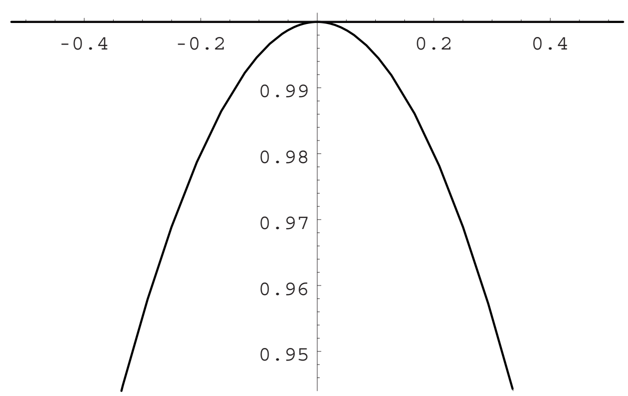
is $y = -x + 2$. The graph of the tangent line and the graph of $f(x) = 1/x$ are shown next.



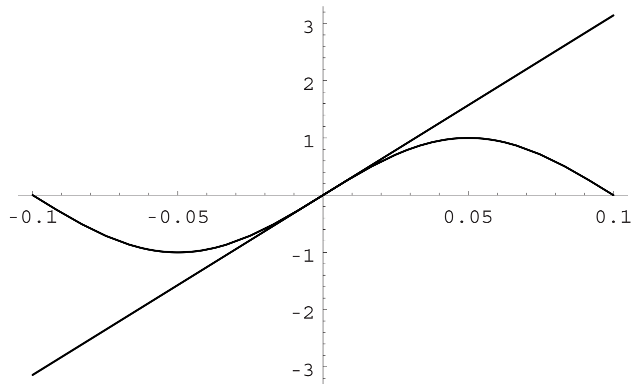
C02S01.042: The numerical evidence suggests that the slope of the tangent line is -4, so that its equation is $y = -4x - 4$. The graph of the tangent line and the graph of $f(x) = 1/x$ are shown next.



C02S01.043: The numerical evidence suggests that the slope of the tangent line is 0, so that its equation is $y = 1$. The graph of the tangent line and the graph of $f(x) = \cos x$ are shown next.



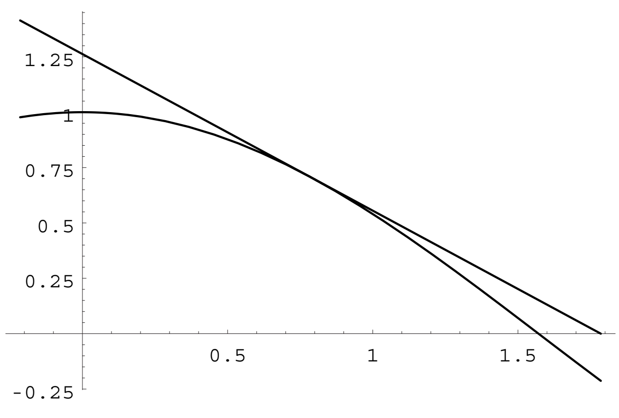
C02S01.044: The numerical evidence suggests that the slope of the tangent line is 10π, so that its equation is $y = 10\pi x$. The graph of the tangent line and the graph of $f(x) = 10\pi x$ are shown next.



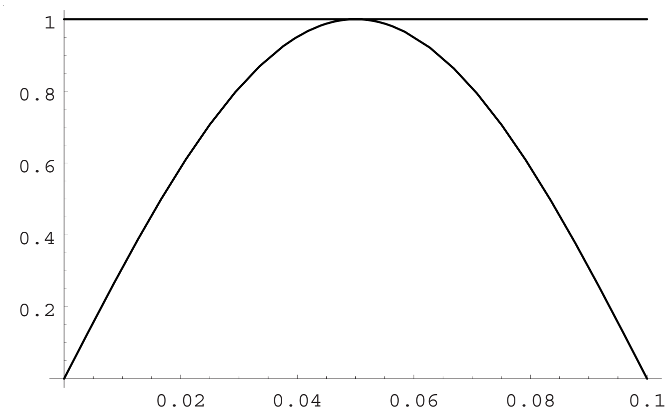
C02S01.045: The numerical evidence suggests that the slope of the tangent line is $-1/\sqrt{2}$, so that its equation is

$$y - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right).$$

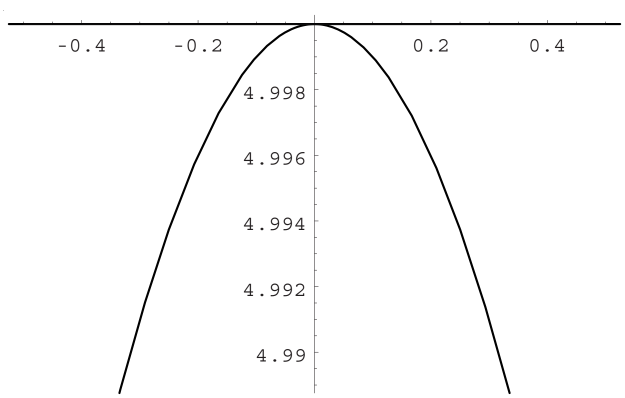
The graph of the tangent line and the graph of $f(x) = \cos x$ are shown next.



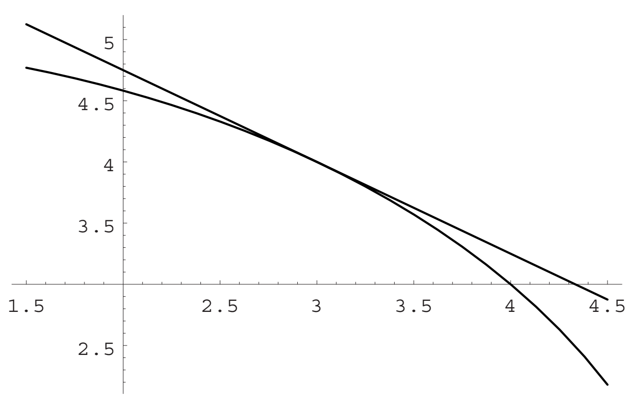
C02S01.046: The numerical evidence suggests that the tangent line is horizontal, so that its equation is $y \equiv 1$. The graph of the tangent line and the graph of $f(x) = \sin 10\pi x$ are shown next.



C02S01.047: The numerical evidence suggests that the tangent line is horizontal, so that its equation is $y \equiv 5$. The graph of the tangent line and the graph of $f(x) = \sqrt{25 - x^2}$ are shown next.



C02S01.048: The numerical evidence suggests that the tangent line has slope $-\frac{3}{4}$, so that its equation is $3x + 4y = 25$. The graph of the tangent line and the graph of $f(x) = \sqrt{25 - x^2}$ are shown next.



Section 2.2

C02S02.001: $\displaystyle\lim_{x\to 3}(3x^2+7x-12) = 3\left(\lim_{x\to 3} x\right)^2 + 7\left(\lim_{x\to 3} x\right) - \lim_{x\to 3} 12 = 3\cdot 3^2 + 7\cdot 3 - 12 = 36.$

C02S02.002: $\displaystyle\lim_{x\to -2}\left(x^3-3x^2+5\right) = \lim_{x\to -2} x^3 - 3\lim_{x\to -2} x^2 + \lim_{x\to -2} 5 = -15.$

C02S02.003: $\displaystyle\lim_{x\to 1}\left(x^2-1\right)\left(x^7+7x-4\right) = \lim_{x\to 1}\left(x^2-1\right)\cdot\lim_{x\to 1}\left(x^7+7x-4\right) = 0\cdot 4 = 0.$

C02S02.004: $\displaystyle\lim_{x\to -2}\left(x^3-3x+3\right)\left(x^2+2x+5\right) = \lim_{x\to -2}\left(x^3-3x+3\right)\cdot\lim_{x\to -2}\left(x^2+2x+5\right) = 1\cdot 5 = 5.$

C02S02.005: $\displaystyle\lim_{x\to 1}\frac{x+1}{x^2+x+1} = \frac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(x^2+x+1)} = \frac{2}{3}.$

C02S02.006: $\displaystyle\lim_{t\to -2}\frac{t+2}{t^2+4} = \frac{\lim_{t\to -2}(t+2)}{\lim_{t\to -2}(t^2+4)} = \frac{0}{8} = 0.$

C02S02.007: $\displaystyle\lim_{x\to 3}\frac{\left(x^2+1\right)^3}{\left(x^3-25\right)^3} = \frac{\lim_{x\to 3}\left(x^2+1\right)^3}{\lim_{x\to 3}\left(x^3-25\right)^3} = \frac{\left(\lim_{x\to 3}\left(x^2+1\right)\right)^3}{\left(\lim_{x\to 3}\left(x^3-25\right)\right)^3} = \frac{10^3}{2^3} = \frac{1000}{8} = 125.$

C02S02.008: $\displaystyle\lim_{z\to -1}\frac{\left(3z^2+2z+1\right)^{10}}{\left(z^3+5\right)^5} = \frac{\lim_{z\to -1}\left(3z^2+2z+1\right)^{10}}{\lim_{z\to -1}\left(z^3+5\right)^5} = \frac{\left(\lim_{z\to -1}\left(3z^2+2z+1\right)\right)^{10}}{\left(\lim_{z\to -1}\left(z^3+5\right)\right)^5} = \frac{2^{10}}{4^5} = 1.$

C02S02.009: $\displaystyle\lim_{x\to 1}\sqrt{4x+5} = \sqrt{\lim_{x\to 1}(4x+5)} = \sqrt{9} = 3.$

C02S02.010: $\displaystyle\lim_{y\to 4}\sqrt{27-\sqrt{y}} = \sqrt{\lim_{y\to 4}\left(27-\sqrt{y}\,\right)} = \sqrt{25} = 5.$

C02S02.011: $\displaystyle\lim_{x\to 3}\left(x^2-1\right)^{3/2} = \left(\lim_{x\to 3}\left(x^2-1\right)\right)^{3/2} = 8^{3/2} = 16\sqrt{2}.$

C02S02.012: $\displaystyle\lim_{t\to -4}\sqrt{\frac{t+8}{25-t^2}} = \frac{\sqrt{\lim_{t\to -4}(t+8)}}{\sqrt{\lim_{t\to -4}\left(25-t^2\right)}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}.$

C02S02.013: $\displaystyle\lim_{z\to 8}\frac{z^{2/3}}{z-\sqrt{2z}} = \frac{\lim_{z\to 8} z^{2/3}}{\lim_{z\to 8}\left(z-\sqrt{2z}\,\right)} = \frac{4}{4} = 1.$

C02S02.014: $\displaystyle\lim_{t\to 2}\sqrt[3]{3t^3+4t-5} = \sqrt[3]{\lim_{t\to 2}\left(3t^3+4t-5\right)} = 3.$

C02S02.015: $\displaystyle\lim_{w\to 0}\sqrt{(w-2)^4} = \sqrt{\lim_{w\to 0}(w-2)^4} = \sqrt{(-2)^4} = 4.$

C02S02.016: $\displaystyle\lim_{t\to -4}\sqrt[3]{(t+1)^6} = \sqrt[3]{\lim_{t\to -4}(t+1)^6} = 9.$

C02S02.017: $\displaystyle\lim_{x\to-2}\sqrt[3]{\frac{x+2}{(x-2)^2}}=\sqrt[3]{\lim_{x\to-2}\frac{(x+2)}{(x-2)^2}}=0.$

C02S02.018: $\displaystyle\lim_{y\to5}\left(\frac{2y^2+2y+4}{6y-3}\right)^{1/3}=\left(\frac{64}{27}\right)^{1/3}=\frac{4}{3}.$

C02S02.019: $\displaystyle\lim_{x\to-1}\frac{x+1}{x^2-x-2}=\lim_{x\to-1}\frac{x+1}{(x+1)(x-2)}=\lim_{x\to-1}\frac{1}{x-2}=-\frac{1}{3}.$

C02S02.020: $\displaystyle\lim_{t\to3}\frac{t^2-9}{t-3}=\lim_{t\to3}\frac{(t-3)(t+3)}{t-3}=\lim_{t\to3}(t+3)=6.$

C02S02.021: $\displaystyle\lim_{x\to1}\frac{x^2+x-2}{x^2-4x+3}=\lim_{x\to1}\frac{(x+2)(x-1)}{(x-3)(x-1)}=\lim_{x\to1}\frac{x+2}{x-3}=-\frac{3}{2}.$

C02S02.022: $\displaystyle\lim_{y\to-1/2}\frac{4y^2-1}{4y^2+8y+3}=\lim_{y\to-1/2}\frac{(2y-1)(2y+1)}{(2y+3)(2y+1)}=\lim_{y\to-1/2}\frac{2y-1}{2y+3}=-\frac{2}{2}=-1.$

C02S02.023: $\displaystyle\lim_{t\to-3}\frac{t^2+6t+9}{t^2-9}=\lim_{t\to-3}\frac{(t+3)(t+3)}{(t+3)(t-3)}=\lim_{t\to-3}\frac{t+3}{t-3}=0.$

C02S02.024: $\displaystyle\lim_{x\to2}\frac{x^2-4}{3x^2-2x-8}=\lim_{x\to2}\frac{(x-2)(x+2)}{(x-2)(3x+4)}=\lim_{x\to2}\frac{x+2}{3x+4}=\frac{2}{5}.$

C02S02.025: $\displaystyle\lim_{z\to-2}\frac{(z+2)^2}{z^4-16}=\lim_{z\to-2}\frac{(z+2)(z+2)}{(z+2)(z-2)(z^2+4)}=\lim_{z\to-2}\frac{z+2}{(z-2)(z^2+4)}=0.$

C02S02.026: $\displaystyle\lim_{t\to3}\frac{t^3-9t}{t^2-9}=\lim_{t\to3}\frac{t(t^2-9)}{t^2-9}=3.$

C02S02.027: $\displaystyle\lim_{x\to1}\frac{x^3-1}{x^4-1}=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)(x^2+1)}=\lim_{x\to1}\frac{x^2+x+1}{(x+1)(x^2+1)}=\frac{3}{4}.$

C02S02.028: $\displaystyle\lim_{y\to-3}\frac{y^3+27}{y^2-9}=\lim_{y\to-3}\frac{(y+3)(y^2-3y+9)}{(y+3)(y-3)}=\lim_{y\to-3}\frac{y^2-3y+9}{y-3}=-\frac{27}{6}=-\frac{9}{2}.$

C02S02.029: $\displaystyle\lim_{x\to3}\frac{\dfrac{1}{x}-\dfrac{1}{3}}{x-3}=\lim_{x\to3}\left(\frac{3-x}{3x}\right)\left(\frac{1}{x-3}\right)=\lim_{x\to3}\frac{-1}{3x}=-\frac{1}{9}.$

C02S02.030: $\displaystyle\lim_{t\to0}\frac{\dfrac{1}{2+t}-\dfrac{1}{2}}{t}=\lim_{t\to0}\left(\frac{2-(2+t)}{2(2+t)}\right)\left(\frac{1}{t}\right)=\lim_{t\to0}\left(\frac{2-2-t)}{2(2+t)}\right)\left(\frac{1}{t}\right)=\lim_{t\to0}\frac{-1}{2(2+t)}=-\frac{1}{4}.$

C02S02.031: $\displaystyle\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2}=\lim_{x\to4}\sqrt{x}+2=4.$

C02S02.032: $\displaystyle\lim_{x\to9}\frac{3-\sqrt{x}}{9-x}=\lim_{x\to9}\frac{3-\sqrt{x}}{(3-\sqrt{x}\,)(3+\sqrt{x}\,)}=\lim_{x\to9}\frac{1}{3+\sqrt{x}}=\frac{1}{6}.$

C02S02.033: $\displaystyle \lim_{t\to 0}\frac{\sqrt{t+4}-2}{t} = \lim_{t\to 0}\left(\frac{\sqrt{t+4}-2}{t}\right)\cdot\left(\frac{\sqrt{t+4}+2}{\sqrt{t+4}+2}\right)$

$$= \lim_{t\to 0}\frac{t+4-4}{t\left(\sqrt{t+4}+2\right)}$$

$$= \lim_{t\to 0}\frac{t}{t\left(\sqrt{t+4}+2\right)} = \lim_{t\to 0}\frac{1}{\sqrt{t+4}+2} = \frac{1}{4}.$$

C02S02.034: $\displaystyle \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sqrt{9+h}}-\frac{1}{3}\right) = \lim_{h\to 0}\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}$

$$= \lim_{h\to 0}\left(\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}\right)\cdot\left(\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}}\right)$$

$$= \frac{9-(9+h)}{3h\sqrt{9+h}\left(3+\sqrt{9+h}\,\right)} = \lim_{h\to 0}\frac{-1}{3\sqrt{9+h}\left(3+\sqrt{9+h}\,\right)} = -\frac{1}{54}.$$

C02S02.035: $\displaystyle \lim_{x\to 4}\frac{x^2-16}{2-\sqrt{x}} = \lim_{x\to 4}\frac{(x+4)(\sqrt{x}-2)(\sqrt{x}+2)}{2-\sqrt{x}} = \lim_{x\to 4}\left[-(x+4)(\sqrt{x}+2)\right] = -32.$

C02S02.036: $\displaystyle \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\right)\cdot\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= \lim_{x\to 0}\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\,\right)}$$

$$= \lim_{x\to 0}\frac{2}{\left(\sqrt{1+x}+\sqrt{1-x}\,\right)} = 1.$$

C02S02.037: $\displaystyle \frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h} = \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = 3x^2+3xh+h^2 \to 3x^2$ as $h\to 0$. When $x=2$, $y=f(2)=x^3=8$ and the slope of the tangent line to this curve at $x=2$ is $3x^2=12$, so an equation of this tangent line is $y=12x-16$.

C02S02.038: $\displaystyle \frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{1}{x+h}\right)-\left(\dfrac{1}{x}\right)}{h} = \frac{x-(x+h)}{hx(x+h)} = \frac{-1}{x(x+h)} \to -\frac{1}{x^2}$ as $h\to 0$. When $x=2$, $y=f(2)=\frac{1}{2}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{4}$, so an equation of this tangent line is $y-\frac{1}{2}=-\frac{1}{4}(x-2)$; that is, $y=-\frac{1}{4}(x-4)$.

C02S02.039: $\displaystyle \frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \frac{x^2-(x+h)^2}{hx^2(x+h)^2} = \frac{-2x-h}{x^2(x+h)^2} \to -\frac{2}{x^3}$ as $h\to 0$. When $x=2$, $y=f(2)=\frac{1}{4}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{4}$, so an equation of this tangent line is $y-\frac{1}{4}=-\frac{1}{4}(x-2)$; that is, $y=-\frac{1}{4}(x-3)$.

C02S02.040: $\displaystyle \frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{1}{x+h+1}\right)-\left(\dfrac{1}{x+1}\right)}{h} = \frac{x+1-x-h-1}{h(x+1)(x+h+1)} = \frac{-1}{(x+1)(x+h+1)}.$

This approaches $-\dfrac{1}{(x+1)^2}$ as h approaches 0. When $x=2$, $y=f(2)=\frac{1}{3}$ and the slope of the line tangent to this curve at $x=2$ is $-\frac{1}{9}$, so an equation of this tangent line is $y-\frac{1}{3}=-\frac{1}{9}(x-2)$; that is, $y=-\frac{1}{9}(x-5)$.

C02S02.041: $\displaystyle \frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{2}{x+h-1}\right)-\left(\dfrac{2}{x-1}\right)}{h} = \frac{2(x-1-x-h+1)}{h(x-1)(x+h-1)} = \frac{-2}{(x-1)(x+h-1)}.$

This approaches $\dfrac{-2}{(x-1)^2}$ as h approaches 0. When $x = 2$, $y = f(2) = 2$ and the slope of the line tangent to this curve at $x = 2$ is -2, so an equation of this tangent line is $y-2 = -2(x-2)$; alternatively, $y = -2(x-3)$.

C02S02.042:
$$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{x+h}{x+h-1}\right) - \left(\dfrac{x}{x-1}\right)}{h} = \frac{(x-1)(x+h) - x^2 - xh + x}{h(x-1)(x+h-1)}$$
$$= \frac{-1}{(x-1)(x+h-1)} \to \frac{-1}{(x-1)^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = 2$ and the slope of the line tangent to this curve at $x = 2$ is -1, so an equation of this tangent line is $y - 2 = -1(x-2)$; that is, $y = -x + 4$.

C02S02.043:
$$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{1}{\sqrt{x+h+2}}\right) - \left(\dfrac{1}{\sqrt{x+2}}\right)}{h}$$
$$= \left(\frac{\sqrt{x+2} - \sqrt{x+h+2}}{h\sqrt{x+2}\,\sqrt{x+h+2}}\right) \cdot \left(\frac{\sqrt{x+2} + \sqrt{x+h+2}}{\sqrt{x+2} + \sqrt{x+h+2}}\right)$$
$$= \frac{-h}{h\sqrt{x+2}\,\sqrt{x+h+2}\,\left(\sqrt{x+2} + \sqrt{x+h+2}\,\right)} \to \frac{-1}{(x+2)\left(2\sqrt{x+2}\,\right)}$$

as $h \to 0$. When $x = 2$, $y = f(2) = \frac{1}{2}$ and the slope of the line tangent to this curve at $x = 2$ is $-\frac{1}{16}$, so an equation of this tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-2)$; that is, $y = -\frac{1}{16}(x-10)$.

C02S02.044:
$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + \dfrac{3}{x+h} - x^2 - \dfrac{3}{x}}{h}$$
$$= (2x+h) + \frac{-3}{x(x+h)} \to 2x - \frac{3}{x^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = \frac{11}{2}$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{13}{4}$, so an equation of this tangent line is $y - \frac{11}{2} = \frac{13}{4}(x-2)$.

C02S02.045:
$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}$$
$$= \left(\frac{\sqrt{2(x+h)+5} - \sqrt{2x+5}}{h}\right) \cdot \left(\frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}}\right)$$
$$= \frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} \to \frac{1}{\sqrt{2x+5}} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = 3$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{1}{3}$, so an equation of this tangent line is $y - 3 = \frac{1}{3}(x-2)$; if you prefer, $y = \frac{1}{3}(x+7)$.

C02S02.046:
$$\frac{f(x+h)-f(x)}{h} = \frac{\left(\dfrac{(x+h)^2}{x+h+1}\right) - \left(\dfrac{x^2}{x+1}\right)}{h} = \frac{(x+1)\,(x+h)^2 - (x+h+1)\left(x^2\right)}{h(x+1)(x+h+1)}$$
$$= \frac{x^2 + xh + 2x + h}{(x+1)(x+h+1)} \to \frac{x^2+2x}{(x+1)^2} \quad \text{as } h \to 0.$$

When $x = 2$, $y = f(2) = \frac{4}{3}$ and the slope of the line tangent to this curve at $x = 2$ is $\frac{8}{9}$, so an equation of this tangent line is $y - \frac{4}{3} = \frac{8}{9}(x-2)$; that is, $9y = 8x - 4$.

C02S02.047:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	2.01	2.001	2.	2.	2.

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	1.99	1.9999	2.	2.	2.

The limit appears to be 2.

C02S02.048:

x	$1+10^{-2}$	$1+10^{-4}$	$1+10^{-6}$	$1+10^{-8}$	$1+10^{-10}$
$f(x)$	4.0604	4.0006	4.00001	4.	4.

x	$1-10^{-2}$	$1-10^{-4}$	$1-10^{-6}$	$1-10^{-8}$	$1-10^{-10}$
$f(x)$	3.9404	3.9994	3.99999	4.	4.

The limit appears to be 4.

C02S02.049:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.16662	0.166666	0.166667	0.166667	0.166667

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.166713	0.166667	0.166667	0.166667	0.166667

The limit appears to be $\frac{1}{6}$.

C02S02.050:

x	$4+10^{-2}$	$4+10^{-4}$	$4+10^{-6}$	$4+10^{-8}$	$4+10^{-10}$
$f(x)$	3.00187	3.00002	3.	3.	3.

x	$4-10^{-2}$	$4-10^{-4}$	$4-10^{-6}$	$4-10^{-8}$	$4-10^{-10}$
$f(x)$	2.99812	2.99998	3.	3.	3.

The limit appears to be 3.

C02S02.051:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	-0.37128	-0.374963	-0.375	-0.375	-0.375

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	-0.378781	-0.375038	-0.375	-0.375	-0.375

The limit appears to be $-\frac{3}{8}$.

C02S02.052:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	-0.222225	-0.222222	-0.222222	-0.222225	-0.222222

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	-0.222225	-0.222222	-0.222222	-0.222222	-0.222222

Limit: $-\frac{2}{9}$.

C02S02.053:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.999983	1.	1.	1.	1.

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.999983	1.	1.	1.	1.

The limit appears to be 1.

C02S02.054:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	0.49996	0.5	0.5	0.499817	0

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	0.499996	0.5	0.5	0.499817	0

Beware of round-off errors.
The limit is 0.5.

C02S02.055:

x	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$f(x)$	0.166666	0.166667	0.166667	0.166667	0.166667

x	-10^{-2}	-10^{-3}	-10^{-4}	-10^{-5}	-10^{-6}
$f(x)$	0.166666	0.166667	0.166667	0.166667	0.166667

The limit appears to be $\frac{1}{6}$.

C02S02.056:

x	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$f(x)$	1.04723	1.00092	1.00001	1.	1.

x	-10^{-2}	-10^{-4}	-10^{-6}	-10^{-8}	-10^{-10}
$f(x)$	.954898	.999079	.999986	1.	1.

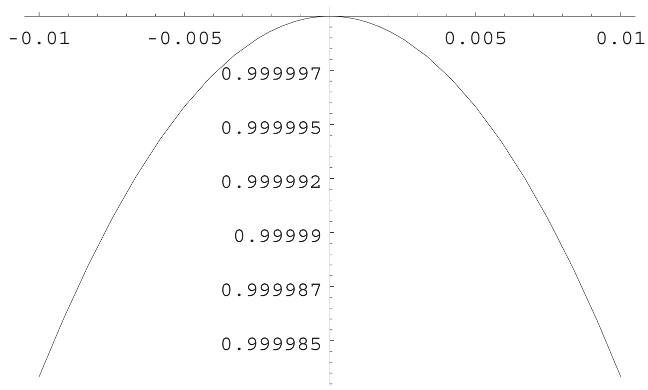
The limit appears to be 1.

C02S02.057:

x	2^{-1}	2^{-5}	2^{-10}	2^{-15}	2^{-20}
$(1+x)^{1/x}$	2.25	2.67699	2.71696	2.71824	2.71828

x	-2^{-1}	-2^{-5}	-2^{-10}	-2^{-15}	-2^{-20}
$(1+x)^{1/x}$	4.	2.76210	2.71961	2.71832	2.71828

C02S02.058: The graph of $y = (\sin x)/x$ on the interval $[-0.01,\ 0.01]$ is next.

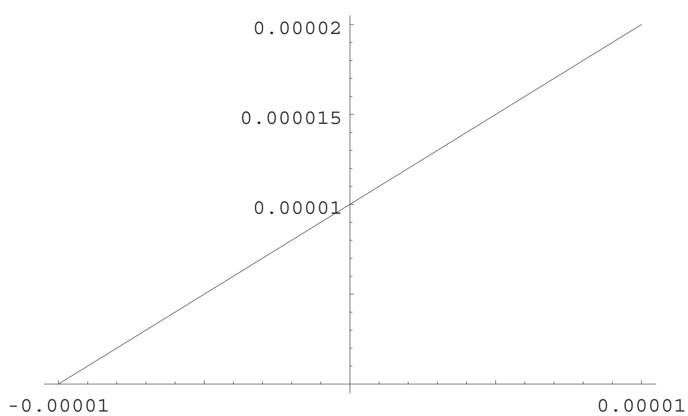


C02S02.059: $\displaystyle\lim_{x\to 0} \frac{x - \tan x}{x^3} = -\frac{1}{3}.$ Answer: -0.3333.

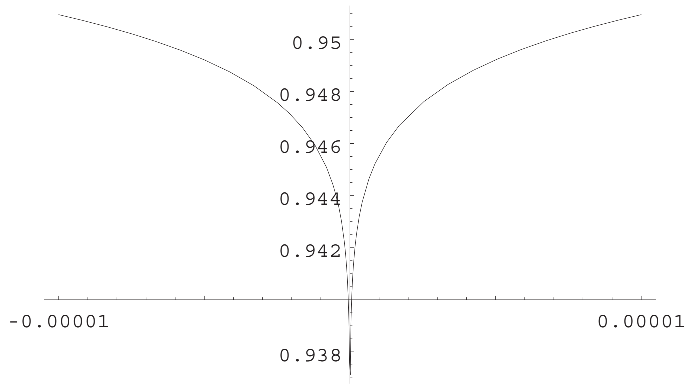
C02S02.060: $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{\tan 5x} = \frac{2}{5}.$

C02S02.061: $\sin\left(\dfrac{\pi}{2^{-n}}\right) = \sin\left(2\pi \cdot 2^{(n-1)}\right) = 0$ for every positive integer n. Therefore $\displaystyle\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$, if it were to exist, would be 0. Notice however that $\sin\left(3^n \cdot \dfrac{\pi}{2}\right)$ alternates between $+1$ and -1 for $n = 1, 2, 3, \ldots$. Therefore $\displaystyle\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ does not exist.

C02S02.062: The graph of $f(x) = \sin x + 10^{-5}\cos x$ on the interval $[-0.00001,\ 0.00001]$ is shown next. The graph makes it clear that the limit is certainly not zero and almost certainly is 10^{-5}.



C02S02.063: The graph of $f(x) = \left(\log_{10}(1/|x|)\right)^{-1/32}$ is shown next, as well as a table of values of $f(x)$ for x very close to zero. The table was generated by *Mathematica*, version 3.0, but virtually any computer algebra system will produce similar results.



x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
10^{-1}	1.0000	10^{-6}	0.9455	10^{-11}	0.9278	10^{-16}	0.9170
10^{-2}	0.9786	10^{-7}	0.9410	10^{-12}	0.9253	10^{-17}	0.9153
10^{-3}	0.9663	10^{-8}	0.9371	10^{-13}	0.9230	10^{-18}	0.9136
10^{-4}	0.9576	10^{-9}	0.9336	10^{-14}	0.9208	10^{-19}	0.9121
10^{-5}	0.9509	10^{-10}	0.9306	10^{-15}	0.9189	10^{-20}	0.9106

C02S02.064: The slope of the line tangent to the graph of $y = 10^x$ at the point $(0, 1)$ is

$$L = \lim_{h \to 0} \frac{10^{0+h} - 10^0}{h} = \lim_{h \to 0} \frac{10^h - 1}{h}.$$

With $h = 0.1,\ 0.01,\ 0.001,\ \ldots,\ 0.000001,$ a calculator reports that the corresponding values of $(10^h - 1)/h$ are (approximately) $2.58925,\ 2.32930,\ 2.30524,\ \ldots,$ and 2.30259. This is fair evidence that $L = \ln 10 \approx 2.302585$. The slope-predictor for $y = 10^x$ is

$$m(x) = \lim_{h \to 0} \frac{10^{x+h} - 10^x}{h} = 10^x \cdot \left(\lim_{h \to 0} \frac{10^h - 1}{h} \right) = L \cdot 10^x.$$

The line tangent to the graph of $y = 10^x$ at the point $P(a,\ 10^a)$ has predicted equation

$$y - 10^a = L \cdot 10^a \cdot (x - a).$$

To see the graph of $y = 10^x$ near P and the line predicted to be tangent to that graph at P, enter the *Mathematica* commands

```
a = 2;      (* or any other value you please *)
Plot[ { 10^x, 10^a + (10^a)*Log[10]*(x - a) }, { x, a - 1, a + 1 }];
```

Section 2.3

C02S03.001: $\theta \cdot \dfrac{\theta}{\sin\theta} \to 0 \cdot 1 = 0$ as $\theta \to 0$.

C02S03.002: $\dfrac{\sin\theta}{\theta} \cdot \dfrac{\sin\theta}{\theta} \to 1 \cdot 1 = 1$ as $\theta \to 0$.

C02S03.003: Multiply numerator and denominator by $1+\cos\theta$ (the *conjugate* of the numerator) to obtain

$$\lim_{\theta\to 0} \frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{1+\cos\theta} = 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

C02S03.004: $\dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta} = \dfrac{\sin\theta}{\theta} \cdot \dfrac{1}{\cos\theta} \to 1 \cdot \dfrac{1}{1} = 1$ as $\theta \to 0$.

C02S03.005: Divide each term in numerator and denominator by t. Then it's clear that the denominator is approaching zero whereas the numerator is not, so the limit does not exist. Because the numerator is positive and the denominator is approaching zero through negative values, the answer $-\infty$ is also correct.

C02S03.006: As $\theta \to 0$, so does $\omega = \theta^2$, and

$$\frac{\sin 2\omega}{\omega} = \frac{2\sin\omega\cos\omega}{\omega} = \frac{\sin\omega}{\omega} \cdot 2\cos\omega \to 1 \cdot 2 \cdot 1 = 2.$$

C02S03.007: Let $z = 5x$. Then $z \to 0$ as $x \to 0$, and $\dfrac{\sin 5x}{x} = \dfrac{5\sin z}{z} \to 5 \cdot 1 = 5$.

C02S03.008: $\dfrac{\sin 2z}{z\cos 3z} = \dfrac{2\sin z\cos z}{z\cos 3z} = \dfrac{2\cos z}{\cos 3z} \cdot \dfrac{\sin z}{z} \to 2 \cdot \dfrac{1}{1} \cdot 1 = 2$ as $z \to 0$.

C02S03.009: This limit does not exist because $\sqrt{x}$ is not defined for x near 0 if $x < 0$. But

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \left(\sqrt{x}\right) \cdot \left(\frac{\sin x}{x}\right) = 0 \cdot 1 = 0.$$

C02S03.010: Using the identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ (inside the front cover), we obtain

$$\lim_{x\to 0} \frac{1-\cos 2x}{x} = \lim_{x\to 0} \frac{2(1-\cos 2x)}{2x} = \lim_{x\to 0} \frac{2\sin^2 x}{x} = \lim_{x\to 0} (2\sin x)\,\frac{\sin x}{x} = 2 \cdot 0 \cdot 1 = 0.$$

Alternatively, you could multiply numerator and denominator by $1+\cos 2x$ (the conjugate of the numerator).

C02S03.011: Let $x = 3z$. Then $x \to 0$ is equivalent to $z \to 0$, and therefore

$$\lim_{x\to 0} \frac{1}{x} \sin\frac{x}{3} = \lim_{z\to 0} \frac{1}{3z} \sin z = \lim_{z\to 0} \frac{1}{3} \cdot \frac{\sin z}{z} = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

C02S03.012: Let $x = 3\theta$. Then $\theta = \frac{1}{3}x$ and $\theta \to 0$ is equivalent to $x \to 0$. Hence

$$\lim_{\theta\to 0} \frac{(\sin 3\theta)^2}{\theta^2\cos\theta} = \lim_{x\to 0} \frac{(\sin x)^2}{\frac{1}{9}x^2\cos\left(\frac{1}{3}x\right)} = \lim_{x\to 0} 9 \cdot \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos\left(\frac{1}{3}x\right)} = 9 \cdot 1 \cdot 1 \cdot \frac{1}{1} = 9.$$

C02S03.013: Multiply numerator and denominator by $1+\cos x$ (the conjugate of the numerator) to obtain

$$\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{(\sin x)(1+\cos x)}=\lim_{x\to 0}\frac{1-\cos^2 x}{(\sin x)(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{(\sin x)(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{1+\cos x}=\frac{0}{1+1}=0.$$

C02S03.014: By Problem 4, $(\tan x)/x \to 1$ as $x \to 0$. This observation implies that

$$\lim_{x\to 0}\frac{\tan kx}{kx}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{kx}{\tan kx}=1$$

for any nonzero constant k. Hence

$$\lim_{x\to 0}\frac{\tan 3x}{\tan 5x}=\lim_{x\to 0}\frac{3}{5}\cdot\frac{\tan 3x}{3x}\cdot\frac{5x}{\tan 5x}=\frac{3}{5}\cdot 1\cdot 1=\frac{3}{5}.$$

C02S03.015: Recall that $\sec x=\dfrac{1}{\cos x}$ and $\csc x=\dfrac{1}{\sin x}$. Hence

$$\lim_{x\to 0} x\sec x\csc x=\lim_{x\to 0}\frac{1}{\cos x}\cdot\frac{x}{\sin x}=\frac{1}{1}\cdot 1=1.$$

We also used the fact that

$$\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{1}{\dfrac{\sin x}{x}}=\frac{1}{1}=1.$$

C02S03.016: $\displaystyle\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=\lim_{\theta\to 0}\frac{2\sin\theta\cos\theta}{\theta}=\lim_{\theta\to 0}(2\cos\theta)\cdot\frac{\sin\theta}{\theta}=2\cdot 1\cdot 1=2.$

C02S03.017: Multiply numerator and denominator by $1+\cos\theta$ (the conjugate of the numerator) to obtain

$$\begin{aligned}\lim_{\theta\to 0}\frac{(1-\cos\theta)(1+\cos\theta)}{(\theta\sin\theta)(1+\cos\theta)}&=\lim_{\theta\to 0}\frac{\sin^2\theta}{(\theta\sin\theta)(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta(1+\cos\theta)}\\&=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{1}{1+\cos\theta}=1\cdot\frac{1}{1+1}=\frac{1}{2}.\end{aligned}$$

C02S03.018: $\displaystyle\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\sin\theta=1\cdot 0=0.$

C02S03.019: $\displaystyle\lim_{z\to 0}\frac{\tan z}{\sin 2z}=\lim_{z\to 0}\frac{\sin z}{(\cos z)(2\sin z\cos z)}=\lim_{z\to 0}\frac{1}{2\cos^2 z}=\frac{1}{2\cdot 1^2}=\frac{1}{2}.$

C02S03.020: $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{3x}=\lim_{x\to 0}\frac{2}{3}\cdot\frac{\tan 2x}{2x}=\frac{2}{3}\cdot 1=\frac{2}{3}$ (with the aid of Problem 4).

C02S03.021: $\displaystyle\lim_{x\to 0} x\cot 3x=\lim_{x\to 0}\frac{x\cos 3x}{\sin 3x}=\lim_{x\to 0}\frac{3x}{\sin 3x}\cdot\frac{\cos 3x}{3}=1\cdot\frac{1}{3}=\frac{1}{3}$ (see Problem 15, last line).

C02S03.022: $\displaystyle\lim_{x\to 0}\frac{x-\tan x}{\sin x}=\lim_{x\to 0}\frac{\left(\dfrac{x}{x}-\dfrac{\tan x}{x}\right)}{\left(\dfrac{\sin x}{x}\right)}=\frac{1-1}{1}=0$ (with the aid of Problem 4).

C02S03.023: Let $x=\frac{1}{2}t$. Then $x\to 0$ is equivalent to $t\to 0$, so

$$\lim_{t\to 0} \frac{\sin\left(\dfrac{t}{2}\right)}{\dfrac{t}{2}} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Therefore

$$\lim_{t\to 0} \frac{1}{t^2}\sin^2\left(\frac{t}{2}\right) = \lim_{t\to 0} \frac{1}{4}\cdot\frac{4}{t^2}\sin^2\left(\frac{t}{2}\right) = \lim_{t\to 0} \frac{1}{4}\cdot\left[\frac{\sin\left(\dfrac{t}{2}\right)}{\dfrac{t}{2}}\right]^2 = \frac{1}{4}\cdot 1^2 = \frac{1}{4}.$$

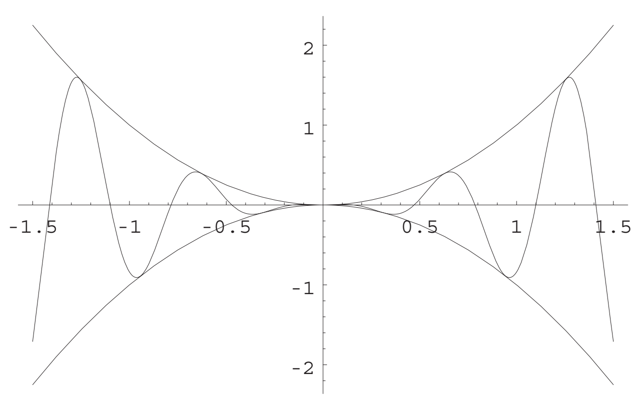
C02S03.024: Because $\dfrac{\sin x}{x} \to 1$ as $x \to 0$, it follows that

$$\lim_{x\to 0} \frac{\sin kx}{kx} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{kx}{\sin kx} = 1$$

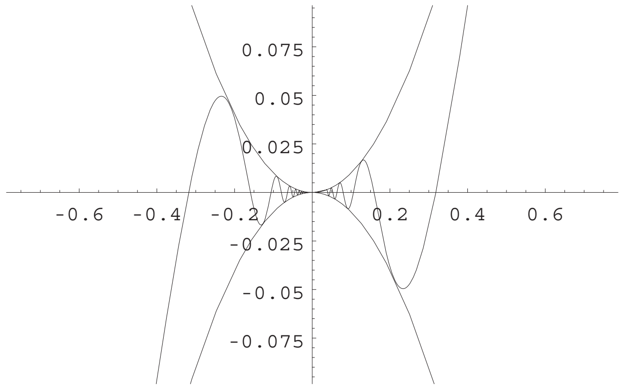
for any nonzero constant k. Hence

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 5x} = \lim_{x\to 0} \frac{2}{5}\cdot\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x} = \frac{2}{5}\cdot 1\cdot 1 = \frac{2}{5}.$$

C02S03.025: Because $-1 \leqq \cos 10x \leqq 1$ for all x, $-x^2 \leqq x^2\cos 10x \leqq x^2$ for all x. But both $-x^2$ and x^2 approach zero as $x \to 0$. Therefore $\displaystyle\lim_{x\to 0} x^2\cos 10x = 0$. The second inequality is illustrated next.



C02S03.026: Because $-1 \leqq \sin x \leqq 1$ for all x, also $-x^2 \leqq x^2 \sin\dfrac{1}{x} \leqq x^2$ for all $x \neq 0$. Because both $-x^2$ and x^2 approach zero as $x \to 0$, it follows from the squeeze law that $\displaystyle\lim_{x\to 0} x^2\sin\frac{1}{x} = 0$. The second inequality is illustrated next.



C02S03.027: First, $-1 \leqq \cos x \leqq 1$ for all x. Therefore

$$-x^2 \leqq x^2 \cos \frac{1}{\sqrt[3]{x}} \leqq x^2$$

for all $x \neq 0$. Finally, both x^2 and $-x^2$ approach zero as $x \to 0$. Therefore, by the squeeze law,

$$\lim_{x \to 0} x^2 \cos \frac{1}{\sqrt[3]{x}} = 0.$$

C02S03.028: Because $-1 \leqq \sin x \leqq 1$ for all x,

$$-\left| \sqrt[3]{x} \right| \leqq \sqrt[3]{x} \sin \frac{1}{x} \leqq \left| \sqrt[3]{x} \right|$$

for all $x \neq 0$. Because both $-\left| \sqrt[3]{x} \right|$ and $\left| \sqrt[3]{x} \right|$ approach zero as $x \to 0$, it follows from the squeeze law that

$$\lim_{x \to 0} \sqrt[3]{x} \sin \frac{1}{x} = 0.$$

C02S03.029: $\displaystyle \lim_{x \to 0^+} \left(3 - \sqrt{x}\right) = 3 - \sqrt{\lim_{x \to 0^+} x} = 3 - 0 = 3.$

C02S03.030: $\displaystyle \lim_{x \to 0^+} \left(4 + 3x^{3/2}\right) = 4 + 3 \cdot \left(\lim_{x \to 0^+} x\right)^{3/2} = 4 - 3 \cdot 0 = 4.$

C02S03.031: $\displaystyle \lim_{x \to 1^-} \sqrt{x - 1}$ does not exist because if $x < 1$, then $x - 1 < 0$.

C02S03.032: Because $x \to 4^-$, $x < 4$, so that $\sqrt{4 - x}$ is defined for all such x. Therefore the limit exists and $\displaystyle \lim_{x \to 4^-} \sqrt{4 - x} = \sqrt{4 - 4} = 0$.

C02S03.033: Because $x \to 2^+$, $x > 2$, so that $x^2 > 4$. Hence $\sqrt{x^2 - 4}$ is defined for all such x and $\displaystyle \lim_{x \to 2^+} \sqrt{x^2 - 4} = \sqrt{4 - 4} = 0$.

C02S03.034: Because $x \to 3^+$, $x > 3$, so that $9 - x^2 < 0$ for all such x. Thus the given limit does not exist.

C02S03.035: Because $x \to 5^-$, $x < 5$ and $x > 0$ for x sufficiently close to 5. Therefore $x(5-x) > 0$ for such x, so that $\sqrt{x(5-x)}$ exists for such x. Therefore

$$\lim_{x \to 5^-} \sqrt{x(5-x)} = \sqrt{\left(\lim_{x \to 5^-} x\right)\left(5 - \lim_{x \to 5^-} x\right)} = \sqrt{5 \cdot 0} = 0.$$

C02S03.036: As $x \to 2^-$, $x < 2$, and $x > -2$ for x sufficiently close to 2. For such x, $4 - x^2 > 0$, so that $\sqrt{4-x^2}$ exists. Therefore $\lim\limits_{x \to 2^-} \sqrt{4-x^2} = \sqrt{4-2^2} = \sqrt{0} = 0$.

C02S03.037: As $x \to 4^+$, $x > 4$, so that both $4x$ and $x - 4$ are positive. Hence the radicand is positive and the square root exists. But the denominator in the radicand is approaching zero through positive values while the numerator is approaching 16. So the fraction is approaching $+\infty$. Therefore

$$\lim_{x \to 4^+} \sqrt{\frac{4x}{x-4}} = +\infty.$$

It is also correct to say that this limit does not exist.

C02S03.038: First, $6 - x - x^2 = (3+x)(2-x)$, so that as $x \to -3^+$, $x > -3$, and thus $3 + x > 3 - 3 = 0$. Also $x < 2$ if x is sufficiently close to -3, so that $2 - x > 0$. Therefore $(3+x)(2-x) > 0$, and so the square root is defined. Finally,

$$\lim_{x \to -3^+} \sqrt{6-x-x^2} = \lim_{x \to -3^+} \sqrt{(3+x)(2-x)} = \sqrt{0 \cdot 5} = \sqrt{0} = 0.$$

C02S03.039: If $x < 5$, then $x - 5 < 0$, so $\dfrac{x-5}{|x-5|} = \dfrac{x-5}{-(x-5)} = -1$. Therefore the limit is -1.

C02S03.040: If $-4 < x < 4$, then $16 - x^2 > 0$, so $\dfrac{16-x^2}{\sqrt{16-x^2}} = \sqrt{16-x^2} \to 0$ as $x \to -4^+$.

C02S03.041: If $x > 3$, then $x^2-6x+9 = (x-3)^2 > 0$ and $x-3 > 0$, so $\dfrac{\sqrt{x^2-6x+9}}{x-3} = \dfrac{|x-3|}{x-3} = \dfrac{x-3}{x-3} \to 1$ as $x \to 3^+$.

C02S03.042: $\dfrac{x-2}{x^2-5x+6} = \dfrac{x-2}{(x-2)(x-3)} = \dfrac{1}{x-3} \to -1$ as $x \to 2^+$. Indeed, the two-sided limit exists and is equal to -1.

C02S03.043: If $x > 2$ then $x - 2 > 0$, so $\dfrac{2-x}{|x-2|} = \dfrac{2-x}{x-2} = -1$. Therefore the limit is also -1.

C02S03.044: If $x < 7$ then $x - 7 < 0$, so $\dfrac{7-x}{|x-7|} = \dfrac{7-x}{-(x-7)} = 1$. So the limit is 1.

C02S03.045: $\dfrac{1-x^2}{1-x} = \dfrac{(1+x)(1-x)}{1-x} = 1 + x$, so the limit is 2.

C02S03.046: As $x \to 0^-$, $x < 0$, so that $x - |x| = x - (-x) = 2x$. Therefore

$$\lim_{x \to 0^-} \frac{x}{x-|x|} = \lim_{x \to 0^-} \frac{x}{2x} = \lim_{x \to 0^-} \frac{1}{2} = \frac{1}{2}.$$

C02S03.047: Recall first that $\sqrt{z^2} = |z|$ for every real number z. Because $x \to 5^+$, $x > 5$, so $5 - x < 0$. Therefore $\sqrt{(5-x)^2} = |5 - x| = -(5 - x) = x - 5$. Therefore

$$\lim_{x \to 5^+} \frac{\sqrt{(5-x)^2}}{5-x} = \lim_{x \to 5^+} \frac{x-5}{5-x} = \lim_{x \to 5^+} (-1) = -1.$$

C02S03.048: Recall that $\sqrt{z^2} = |z|$ for every real number z. Because $x \to -4^-$, we know that $x < -4$. Hence $4 + x < 4 + (-4) = 0$. Therefore

$$\lim_{x \to -4^-} \frac{4+x}{\sqrt{(4+x)^2}} = \lim_{x \to -4^-} \frac{4+x}{|4+x|} = \lim_{x \to -4^-} \frac{4+x}{-(4+x)} = \lim_{x \to -4^-} (-1) = -1.$$

C02S03.049: The right-hand and left-hand limits both fail to exist at $a = 1$. The behavior of f near a is best described by observing that

$$\lim_{x \to 1^+} \frac{1}{x-1} = +\infty \qquad \text{and} \qquad \lim_{x \to 1^-} \frac{1}{x-1} = -\infty.$$

C02S03.050: The right-hand and left-hand limits both fail to exist at $a = 3$. The behavior of f near a is best described by observing that

$$\lim_{x \to 3^+} \frac{2}{3-x} = -\infty \qquad \text{and} \qquad \lim_{x \to 3^-} \frac{2}{3-x} = +\infty.$$

C02S03.051: The right-hand and left-hand limits both fail to exist at $a = -1$. The behavior of f near a is best described by observing that

$$\lim_{x \to -1^+} \frac{x-1}{x+1} = -\infty \qquad \text{and} \qquad \lim_{x \to -1^-} \frac{x-1}{x+1} = +\infty.$$

C02S03.052: The right-hand and left-hand limits both fail to exist at $a = 5$. The behavior of f near a is best described by observing that

$$\lim_{x \to 5^+} \frac{2x-5}{5-x} = -\infty \qquad \text{and} \qquad \lim_{x \to 5^-} \frac{2x-5}{5-x} = +\infty.$$

C02S03.053: The right-hand and left-hand limits both fail to exist at $a = -2$. If x is slightly greater than -2, then $1 - x^2$ is close to $1 - 4 = -3$, while $x + 2$ is a positive number close to zero. In this case $f(x)$ is a large negative number. Similarly, if x is slightly less than -2, then $1 - x^2$ is close to -3, while $x + 2$ is a negative number close to zero. In this case $f(x)$ is a large positive number. The behavior of f near -2 is best described by observing that

$$\lim_{x \to -2^+} \frac{1-x^2}{x+2} = -\infty \qquad \text{and} \qquad \lim_{x \to -2^-} \frac{1-x^2}{x+2} = +\infty.$$

C02S03.054: The right-hand and left-hand limits fail to exist at $a = 5$. If x is close to 5 but $x \neq 5$, then $x - 5$ is close to zero, so that $(x - 5)^2$ is a *positive* number still very close to zero. Its reciprocal is therefore a very large positive number. That is,

$$\lim_{x \to 5^+} \frac{1}{(x-5)^2} = \lim_{x \to 5^-} \frac{1}{(x-5)^2} = +\infty. \tag{1}$$

Unlike the previous problems of this sort, we may in this case also write

$$\lim_{x \to 5} \frac{1}{(x-5)^2} = +\infty.$$

Nevertheless, Eq. (1) implies that neither the left-hand nor the right-hand limit of $f(x)$ *exists* (is a real number) at $x = 5$.

C02S03.055: The left-hand and right-hand limits both fail to exist at $x = 1$. To simplify $f(x)$, observe that

$$f(x) = \frac{|1-x|}{(1-x)^2} = \frac{|1-x|}{|1-x|^2} = \frac{1}{|1-x|}.$$

Therefore we can describe the behavior of $f(x)$ near $a = 1$ in this way:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{|1-x|} = +\infty.$$

C02S03.056: Because $x^2 + 6x + 9 = (x+3)^2$, the denominator in $f(x)$ is zero when $x = -3$, and so the left-hand and right-hand limits fail to exist at $a = -3$. When x is close to -3 but $x \neq -3$, $(x+3)^2$ is a *positive* number very close to zero, while the numerator $x + 1$ is close to -2. Therefore $f(x)$ is a very large negative number. That is,

$$\lim_{x \to -3} \frac{x+1}{x^2 + 6x + 9} = -\infty.$$

C02S03.057: First simplify $f(x)$: If $x^2 \neq 4$ (that is, if $x \neq \pm 2$), then

$$f(x) = \frac{x-2}{4-x^2} = \frac{x-2}{(2+x)(2-x)} = \frac{-1}{2+x}.$$

So even though $f(2)$ does not exist, there is no real problem with the limit of $f(x)$ as $x \to 2$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{-1}{2+x} = -\frac{1}{4}.$$

But the left-hand and right-hand limits of $f(x)$ fail to exist at $x = -2$, because

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{-1}{2+x} = -\infty \quad \text{and} \quad \lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{-1}{2+x} = +\infty.$$

C02S03.058: First simplify:

$$f(x) = \frac{x-1}{x^2 - 3x + 2} = \frac{x-1}{(x-1)(x-2)} = \frac{1}{x-2}$$

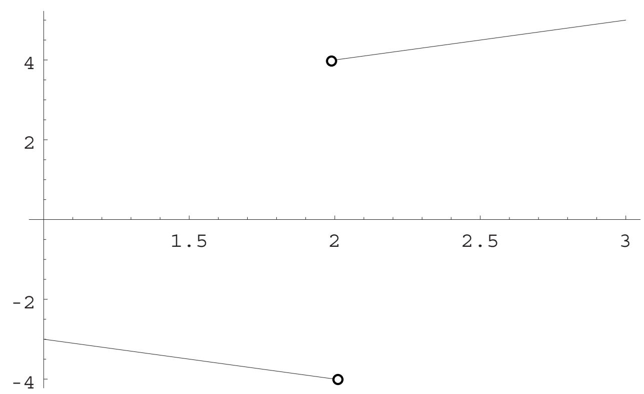
if $x \neq 1$ and $x \neq 2$. But even though $f(1)$ is undefined,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x-2} = \frac{1}{1-2} = -1.$$

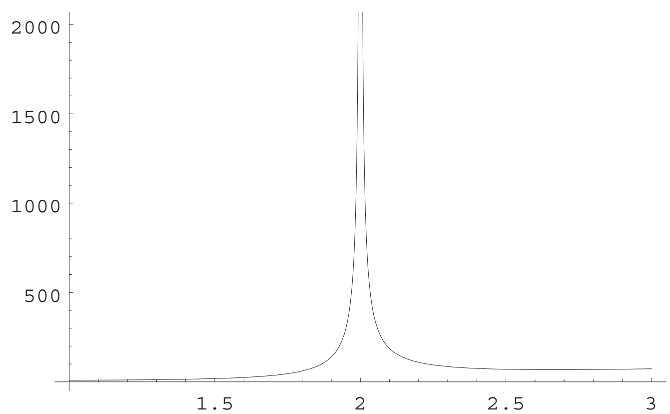
But the one-sided limits fail to exist at $x = 2$:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{1}{x-2} = +\infty \quad \text{and} \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{1}{x-2} = -\infty.$$

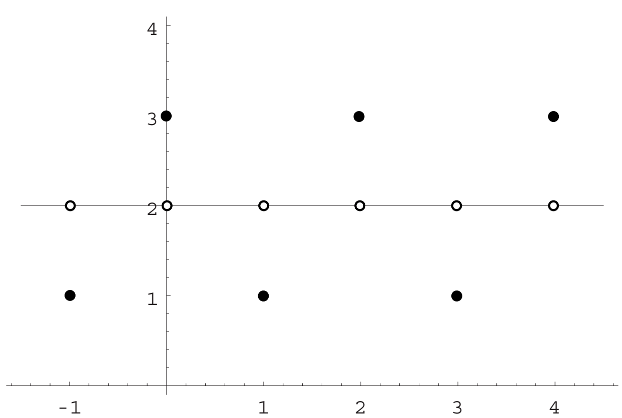
C02S03.059: $\lim_{x\to 2^+} \dfrac{x^2-4}{|x-2|} = 4$ and $\lim_{x\to 2^-} \dfrac{x^2-4}{|x-2|} = -4$. The two-sided limit does not exist. The graph is shown next.



C02S03.060: Because $\lim_{x\to 2^+} \dfrac{x^4-8x+16}{|x-2|} = +\infty$ and $\lim_{x\to 2^-} \dfrac{x^4-8x+16}{|x-2|} = +\infty$, the two-sided limit also fails to exist. The graph is shown next.

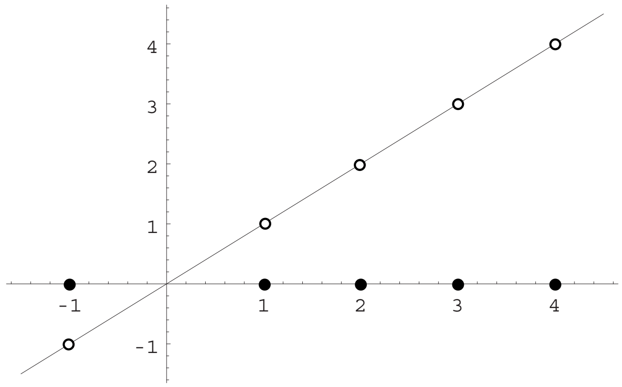


C02S03.061: If x is an even integer then $f(x) = 3$, if x is an odd integer then $f(x) = 1$, and $\lim_{x\to a} f(x) = 2$ for *all* real number values of a. The graph of f is shown next.

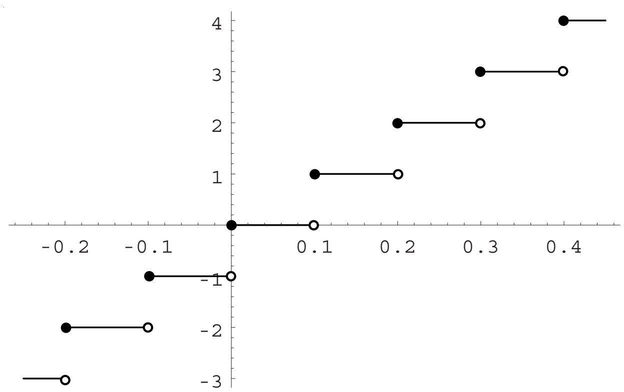


C02S03.062: If n is any integer then $f(x) \to n$ as $x \to n$. Note: $\lim_{x\to a} f(x) = a$ for *all* real number values

of a. The graph of f is shown next.



C02S03.063: For any integer n, $\lim_{x \to n^-} f(x) = 10n - 1$ and $\lim_{x \to n^+} f(x) = 10n$. Note: $\lim_{x \to a} f(x)$ exists if and only if $10a$ is not an integer. The graph is shown next.



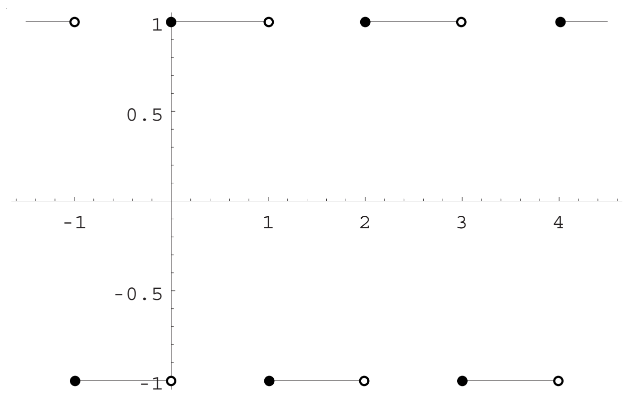
C02S03.064: If n is an odd integer, then $f(x) = n - 1$, an even integer, for $n - 1 \leqq x < n$ and $f(x) = n$, an odd integer, for $n \leqq x < n + 1$. Therefore

$$\lim_{x \to n^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = -1.$$

Similarly, if n is an even integer, then

$$\lim_{x \to n^-} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = 1.$$

Finally, $\lim_{x \to a} f(x)$ exists if and only if a is *not* an integer. The graph of f is shown next.



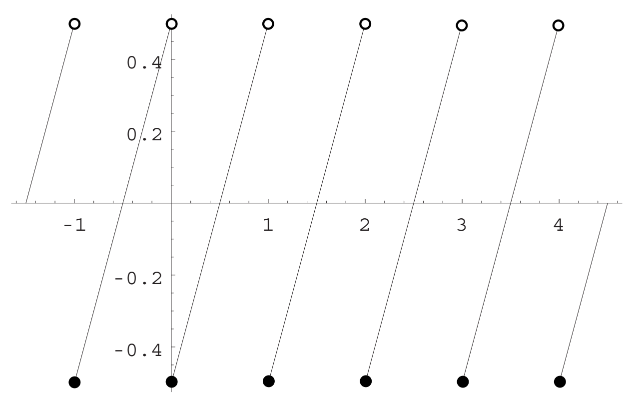
C02S03.065: If n is an integer and $n < x < n+1$, then write $x = n + t$ where $0 < t < 1$. Then $f(x) = n + t - n - \frac{1}{2} = t - \frac{1}{2}$. Moreover, $x \to n^+$ is equivalent to $t \to 0^+$. Therefore

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} \left(t - \tfrac{1}{2}\right) = \lim_{t \to 0^+} \left(t - \tfrac{1}{2}\right) = -\frac{1}{2}.$$

Similar reasoning, with $n - 1 < x < n$, shows that if n is an integer, then

$$\lim_{x \to n^-} f(x) = \frac{1}{2}.$$

Finally, if a is a real number other than an integer, then $\lim\limits_{x \to a} f(x)$ exists. The graph of f is next.

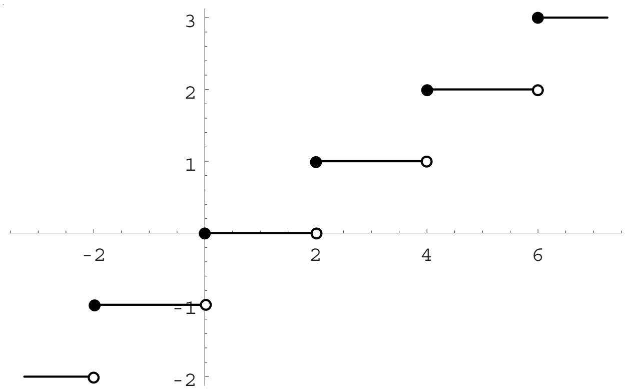


C02S03.066: Given the real number x, there is a [unique] integer n such that $2n \leqq x < 2n + 2$. Thus $n \leqq \frac{1}{2}x < n + 1$, and in this case $f(x) = n$. So if $m = 2n$ is an even integer, then $f(x) \to m$ as $x \to m^+$ and $x \to a$ for every real number a strictly between $2n$ and $2n + 2$. But if $2n - 2 < x < 2n$, then $n - 1 < \frac{1}{2}x < n$, so that $f(x) = n - 1$; in this case $f(x) \to n - 1$ as $x \to m^-$. Therefore:

If k is an odd integer, then $\lim\limits_{x \to k} f(x) = \frac{1}{2}(k - 1)$.

If k is an even integer, then $\lim\limits_{x \to k^+} f(x) = \frac{1}{2}k$ and $\lim\limits_{x \to k^-} f(x) = \frac{1}{2}(k - 2)$.

Finally, $\lim\limits_{x \to a} f(x)$ exists if and only if a is not an even integer. The graph of f is next.



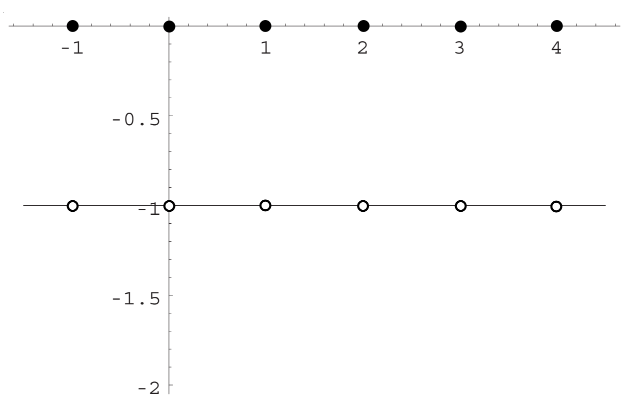
C02S03.067: If x is an integer, then $f(x) = x - x = 0$. If x is not an integer, choose the [unique] integer n such that $n < x < n + 1$. Then $-(n+1) < -x < -n$, so $f(x) = n - (n+1) = -1$. Therefore

$$\lim_{x \to a} f(x) = \lim_{x \to a} (-1) = -1$$

for every real number a. In particular, for every integer n,

$$\lim_{x \to n^-} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = -1.$$

The graph of f is shown next.



C02S03.068: If n is a positive integer, then

$$\lim_{x \to n^-} f(x) = \frac{n-1}{n} \qquad \text{and} \qquad \lim_{x \to n^+} f(x) = 1.$$

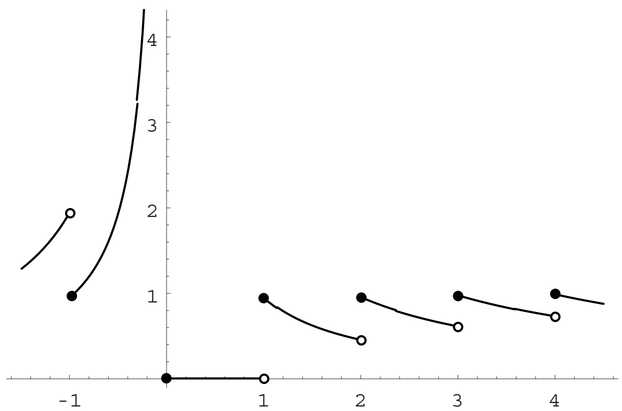
For any integer $n < 0$,

$$\lim_{x \to n^-} f(x) = \frac{n+1}{n} \qquad \text{and} \lim_{x \to n^+} f(x) = 1.$$

Also,

$$\lim_{x \to 0^-} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to 0^+} f(x) = 0.$$

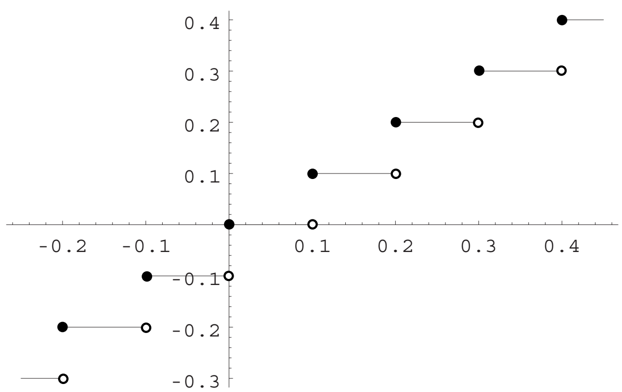
Note: $\lim_{x \to a} f(x)$ exists if and only if a is not an integer. The graph of f is next.



C02S03.069: The values of a for which $\lim_{x \to a} g(x)$ exists are those real numbers not integral multiples of $\frac{1}{10}$. If b is an integral multiple of $\frac{1}{10}$, then

$$\lim_{x \to b^-} g(x) = b - \frac{1}{10} \qquad \text{and} \qquad \lim_{x \to b^+} g(x) = b.$$

The graph of g is shown next.



C02S03.070: Let $f(x) = \operatorname{sgn}(x)$ and $g(x) = -\operatorname{sgn}(x)$. Clearly neither $f(x)$ nor $g(x)$ has a limit as $x \to 0$ (for example, $f(x) \to 1$ as $x \to 1^+$ but $f(x) \to -1$ as $x \to 1^-$). But

$$f(x) + g(x) = \begin{cases} 1 - 1 & \text{if } x > 0, \\ -1 + 1 & \text{if } x < 0, \\ 0 + 0 & \text{if } x = 0, \end{cases}$$

so that $f(x) + g(x) \equiv 0$, and therefore $f(x) + g(x) \to 0$ as $x \to 0$. Also,

$$f(x) \cdot g(x) = \begin{cases} -1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Therefore $\lim_{x \to 0} f(x) \cdot g(x) = -1$.

C02S03.071: Because $-x^2 \leqq f(x) \leqq x^2$ for all x and because $-x^2 \to 0$ and $x^2 \to 0$ as $x \to 0$, it follows from the squeeze law for limits that $\lim_{x \to 0} f(x) = 0 = f(0)$.

C02S03.072: As $x \to 0^+$, $1/x \to +\infty$, so $1 + 2^{1/x} \to +\infty$ as well. Therefore

$$\lim_{x\to 0^+} \frac{1}{1+2^{1/x}} = 0.$$

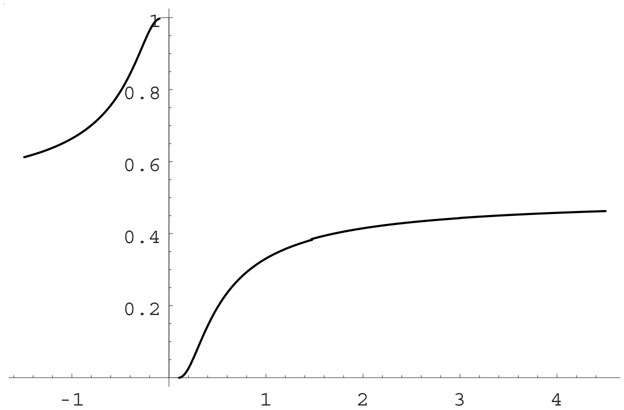
As $x \to 0^-$, $1/x \to -\infty$, so $2^{1/x} \to 0$. Consequently,

$$\lim_{x\to 0^-} \frac{1}{1+2^{1/x}} = 1.$$

Therefore $\lim\limits_{x\to 0} f(x)$ does not exist. This function approaches its one-sided limits at $x = 0$ very rapidly. For example,

$$f(0.01) \approx 7.888609052 \times 10^{-31} \quad \text{and} \quad f(-0.01) \approx 1 - 7.888601052 \times 10^{-31}.$$

The graph of f for x near zero is next.



C02S03.073: Given: $f(x) = x \cdot [\![1/x]\!]$. Let's first study the right-hand limit of $f(x)$ at $x = 0$. We need consider only values of x in the interval $(0, 1)$, and if $0 < x < 1$ then

$$1 < \frac{1}{x}, \qquad \text{so that} \qquad n \leqq \frac{1}{x} < n + 1$$

for some [unique] positive integer n. Moreover, if so then

$$\frac{1}{n+1} < x \leqq \frac{1}{n}.$$

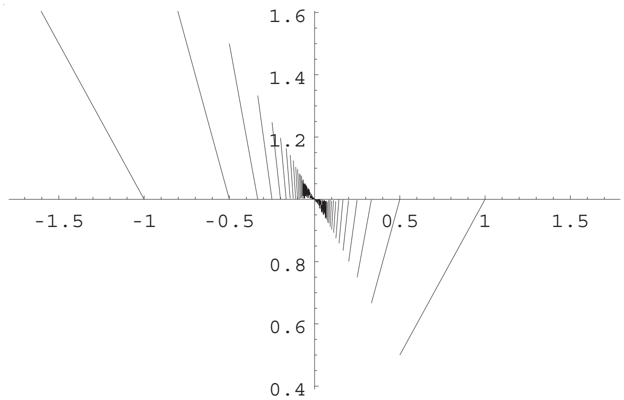
Therefore $f(x) = x \cdot n$, so that

$$\frac{n}{n+1} < f(x) \leqq \frac{n}{n} = 1. \tag{1}$$

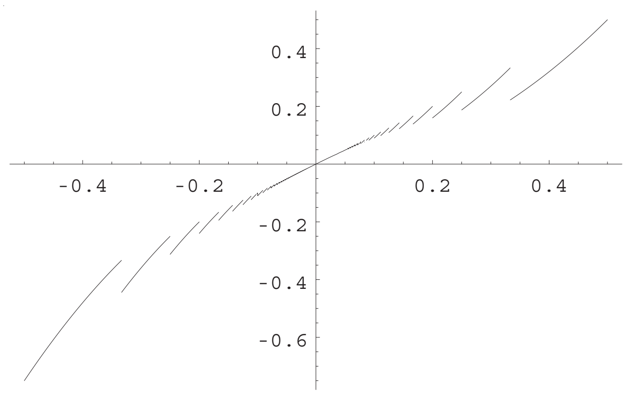
As $x \to 0^+$, $n \to \infty$, so the bounds on $f(x)$ in (1) both approach 1. Therefore

$$\lim_{x\to 0^+} f(x) = 1.$$

A similar (but slightly more delicate) argument shows that $f(x) \to 1$ as $x \to 0^-$ as well. Therefore $\lim\limits_{x\to 0} f(x)$ exists and is equal to 1. The graph of f is next.



C02S03.074: Here, $f(x)$ is obtained from the function in Problem 73 by multiplication by x. Therefore, because the function in Problem 73 had limit 1 as $x \to 0$, the product rule for limits implies that $f(x) \to 0 \cdot 1 = 0$ as $x \to 0$. The graph of f near zero is next.



C02S03.075: Given $\epsilon > 0$, let $\delta = \epsilon/7$. Suppose that

$$0 < |x - (-3)| < \delta.$$

Then

$$\begin{aligned} |x + 3| &< \frac{\epsilon}{7}; \\ |7x + 21| &< \epsilon; \\ |7x - 9 + 30| &< \epsilon; \\ |(7x - 9) - (-30)| &< \epsilon. \end{aligned}$$

Therefore, by definition, $\lim_{x \to -3} (7x - 9) = -30$.

C02S03.076: Given $\epsilon > 0$, let $\delta = \epsilon/17$. Suppose that $0 < |x - 5| < \delta$. Then

$$|17x - 85| < 17\delta;$$

$$|(17x - 35) - 50| < \epsilon.$$

Therefore, by definition, $\displaystyle \lim_{x \to 5} (17x - 35) = 50$.

C02S03.077: Definition: We say that the number L is the **right-hand limit** of the function f at $x = a$ provided that, for every $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < |x - a| < \delta$ and $x > a$, then $|f(x) - L| < \epsilon$.

To prove that $\displaystyle \lim_{x \to 0^+} \sqrt{x} = 0$, suppose that $\epsilon > 0$ is given. Let $\delta = \epsilon^2$. Suppose that $|x - 0| < \delta$ and that $x > 0$. Then $0 < x < \delta = \epsilon^2$. Hence $\sqrt{x} < \epsilon$, and therefore

$$|\sqrt{x} - 0| < \epsilon.$$

So, by definition, $\displaystyle \lim_{x \to 0^+} \sqrt{x} = 0$.

C02S03.078: Let $\epsilon > 0$ be given. Let $\delta = \sqrt{\epsilon}$. Suppose that $0 < |x - 0| < \delta$. Then $0 < x^2 < \delta^2 = \epsilon$. Hence $|x^2 - 0| < \epsilon$. Therefore, by definition,

$$\lim_{x \to 0} x^2 = 0.$$

C02S03.079: Suppose that $\epsilon > 0$ is given. Let δ be the minimum of the two numbers 1 and $\epsilon/5$ and suppose that $0 < |x - 2| < \delta$. Then

$$|x - 2| < 1;$$

$$-1 < x - 2 < 1;$$

$$3 < x + 2 < 5;$$

$$|x + 2| < 5.$$

Therefore

$$|x^2 - 4| = |x + 2| \cdot |x - 2| < 5 \cdot \delta \leqq 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Hence, by definition, $\displaystyle \lim_{x \to 2} x^2 = 4$.

C02S03.080: Given $\epsilon > 0$, choose δ to be the minimum of 1 and $\epsilon/10$. Suppose that $0 < |x - 7| < \delta$. Then

$$|x - 7| < 1;$$

$$-1 < x - 7 < 1;$$

$$8 < x + 2 < 10;$$

$$|x + 2| < 10.$$

Therefore

$$|(x^2 - 5x - 4) - 10| = |x + 2| \cdot |x - 7| < 10 \cdot \delta \leqq 10 \cdot \frac{\epsilon}{10} = \epsilon.$$

Thus, by definition, $\lim_{x \to 7} (x^2 - 5x - 4) = 10$.

C02S03.081: Given $\epsilon > 0$, let δ be the minimum of 1 and $\epsilon/29$. Suppose that $0 < |x - 10| < \delta$. Then

$$\begin{aligned} &0 < |x - 10| < 1; \\ &-1 < x - 10 < 1; \\ &-2 < 2x - 20 < 2; \\ &25 < 2x + 7 < 29; \\ &|2x + 7| < 29. \end{aligned}$$

Thus

$$|(2x^2 - 13x - 25) - 45| = |2x + 7| \cdot |x - 10| < 29 \cdot \delta \leqq 29 \cdot \frac{\epsilon}{29} = \epsilon.$$

Therefore, by definition, $\lim_{x \to 10} (2x^2 - 13x - 25) = 45$.

C02S03.082: Given $\epsilon > 0$, choose δ to be the minimum of 1 and $\epsilon/19$. Suppose that $0 < |x - 2| < \delta$. Then

$$\begin{aligned} &0 < |x - 2| < 1; \\ &-1 < x - 2 < 1; \\ &1 < x < 3; \\ &1 < x^2 < 9 \qquad \text{and} \qquad 2 < 2x < 6; \\ &3 < x^2 + 2x < 15; \\ &7 < x^2 + 2x + 4 < 19; \\ &|x^2 + 2x + 4| < 19. \end{aligned}$$

Consequently,

$$|x^3 - 8| = |x^2 + 2x + 4| \cdot |x - 2| < 19 \cdot \delta \leqq 19 \cdot \frac{\epsilon}{19} = \epsilon.$$

Therefore, by definition, $\lim_{x \to 2} x^3 = 8$.

C02S03.083: In Problem 78 we showed that if $a = 0$, then

$$\lim_{x \to a} x^2 = \lim_{x \to 0} x^2 = 0 = 0^2 = a^2,$$

so the result we are to prove here holds when $a = 0$. Next case: Suppose that $a > 0$. Let $\epsilon > 0$ be given. Choose δ to be the minimum of the numbers 1 and $\epsilon/(2a+1)$. Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. Then

$$\begin{aligned} &|x - a| < 1; \\ &-1 < x - a < 1; \end{aligned}$$

$$2a - 1 < x + a < 2a + 1;$$

$$|x + a| < 2a + 1.$$

Thus

$$|x^2 - a^2| = |x + a| \cdot |x - a| < (2a + 1) \cdot \frac{\epsilon}{2a + 1} = \epsilon.$$

Therefore, by definition, $\lim_{x \to a} x^2 = a^2$ if $a > 0$.

Final case: $a < 0$. Given $\epsilon > 0$, let

$$\delta = \min \left\{ 1, \frac{\epsilon}{|2a - 1|} \right\}.$$

Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. Then

$$|x - a| < 1;$$

$$-1 < x - a < 1;$$

$$2a - 1 < x + a < 2a + 1;$$

$$|x + a| < |2a - 1|$$

(because $|2a - 1| > |2a + 1|$ if $a < 0$). It follows that

$$|x^2 - a^2| = |x + a| \cdot |x - a| < |2a - 1| \cdot \frac{\epsilon}{|2a - 1|} = \epsilon.$$

Therefore, by definition, $\lim_{x \to a} x^2 = a^2$ if $a < 0$.

C02S03.084: Suppose that $\epsilon > 0$ is given. Case (1): $a = 0$. Let $\delta = \sqrt[3]{\epsilon}$ and proceed much as in the solution of Problem 78. Case (2): $a > 0$. Let

$$\delta = \min \left\{ \frac{a}{2}, \frac{4\epsilon}{19a^2} \right\}.$$

Note that $\delta > 0$. Suppose that $0 < |x - a| < \delta$. Then:

$$|x - a| < \frac{a}{2};$$

$$-\frac{a}{2} < x - a < \frac{a}{2};$$

$$\frac{a}{2} < x < \frac{3a}{2};$$

$$\frac{a^2}{4} < x^2 < \frac{9a^2}{4} \qquad \text{(because } x > 0\text{)};$$

$$\frac{a^2}{2} < ax < \frac{3a^2}{2};$$

$$\frac{3a^2}{4} < x^2 + ax < \frac{15a^2}{4};$$

$$\frac{7a^2}{4} < x^2 + ax + a^2 < \frac{19a^2}{4};$$

$$|x^2 + ax + a^2| < \frac{19a^2}{4}.$$

Therefore

$$|x^3 - a^2| = |x^2 + ax + a^2| \cdot |x - a| < \frac{19a^2}{4} \cdot \frac{4\epsilon}{19a^2} = \epsilon.$$

Thus, by definition, $\lim\limits_{x \to a} x^2 = a^2$ if $a > 0$. Case (3), in which $a < 0$, is similar.

Section 2.4

C02S04.001: Suppose that a is a real number. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} (2x^5 - 7x^2 + 13) = \left(\lim_{x\to a} 2x^5\right) - \left(\lim_{x\to a} 7x^2\right) + \left(\lim_{x\to a} 13\right)$$

$$= \left(\lim_{x\to a} 2\right)\left(\lim_{x\to a} x\right)^5 - \left(\lim_{x\to a} 7\right)\left(\lim_{x\to a} x\right)^2 + 13 = 2a^5 - 7a^2 + 13 = f(a).$$

Therefore f is continuous at x for every real number x.

C02S04.002: Suppose that a is a real number. Then

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a} 7x^3\right) - \left(\lim_{x\to a} (2x+1)^5\right) = \left(\lim_{x\to a} 7\right)\left(\lim_{x\to a} x\right)^3 - \left(\lim_{x\to a} (2x+1)\right)^5$$

$$= 7a^3 - \left[\left(\lim_{x\to a} 2\right)\left(\lim_{x\to a} x\right) + \left(\lim_{x\to a} 1\right)\right]^5 = 7a^3 - (2a+1)^5 = f(a).$$

Therefore f is continuous at x for every real number x.

C02S04.003: Suppose that a is a real number. Then

$$\lim_{x\to a} g(x) = \lim_{x\to a} \frac{2x-1}{4x^2+1} = \frac{\lim_{x\to a} (2x-1)}{\lim_{x\to a} (4x^2+1)}$$

$$= \frac{\left(\lim_{x\to a} 2\right)\left(\lim_{x\to a} x\right) - \left(\lim_{x\to a} 1\right)}{\left(\lim_{x\to a} 4\right)\left(\lim_{x\to a} x\right)^2 + \left(\lim_{x\to a} 1\right)} = \frac{2a-1}{4a^2+1} = g(a).$$

Therefore g is continuous at x for every real number x.

C02S04.004: Suppose that a is a fixed real number. Then

$$\lim_{x\to a} g(x) = \frac{\lim_{x\to a} x^3}{\lim_{x\to a} x^2 + 2\lim_{x\to a} x + 5}$$

$$= \frac{(\lim_{x\to a} x)^3}{(\lim_{x\to a} x)^2 + 2\lim_{x\to a} x + 5} = \frac{a^3}{a^2+2a+5} = g(a).$$

Therefore g is continuous at x for all real x.

C02S04.005: Suppose that a is a fixed real number. Then $a^2+4a+5 = (a+2)^2+1 > 0$, so $h(a)$ is defined. Moreover,

$$\lim_{x\to a} h(x) = \lim_{x\to a} \sqrt{x^2+4x+5} = \left(\lim_{x\to a} (x^2+4x+5)\right)^{1/2}$$

$$= \left[\left(\lim_{x\to a} x\right)^2 + \left(\lim_{x\to a} 4\right)\left(\lim_{x\to a} x\right) + \left(\lim_{x\to a} 5\right)\right]^{1/2} = \sqrt{a^2+4a+5} = h(a).$$

Therefore, by definition, h is continuous at $x = a$. Because a is arbitrary, h is continuous at x for every real number x.

C02S04.006: Suppose that a is a real number. Then $h(a)$ exists because $x^{1/3}$ is defined for every real number x. Moreover,

$$\lim_{x\to a} h(x) = \lim_{x\to a} (1-5x)^{1/3} = \left(\lim_{x\to a} (1-5x)\right)^{1/3}$$

$$= \left[\left(\lim_{x\to a} 1\right) - \left(\lim_{x\to a} 5\right)\left(\lim_{x\to a} x\right)\right]^{1/3} = (1-5a)^{1/3} = h(a).$$

Therefore h is continuous at $x = a$, and thus continuous at every real number x.

C02S04.007: Suppose that a is a real number. Then $1 + \cos^2 a \neq 0$, so that $f(a)$ is defined. Note also that

$$\lim_{x\to a} \sin x = \sin a \qquad \text{and} \qquad \lim_{x\to a} \cos x = \cos a$$

because the sine and cosine functions are continuous at every real number (Theorem 1). Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1-\sin x}{1+\cos^2 x} = \frac{\lim_{x\to a} (1-\sin x)}{\lim_{x\to a} (1+\cos^2 x)}$$

$$= \frac{\left(\lim_{x\to a} 1\right) - \left(\lim_{x\to a} \sin x\right)}{\left(\lim_{x\to a} 1\right) + \left(\lim_{x\to a} \cos x\right)^2} = \frac{1-\sin a}{1+\cos^2 a} = f(a).$$

Therefore f is continuous at a. Because a is arbitrary, f is continuous at x for every real number x.

C02S04.008: Suppose that a is a real number. Then $0 \leqq \sin^2 a \leqq 1$, so that $1 - \sin^2 a \geqq 0$, and thus $g(a) = (1-\sin^2 a)^{1/4}$ exists. Because the sine function is continuous on the set of all real numbers (Theorem 1), we know also that $\sin x \to \sin a$ as $x \to a$. Therefore

$$\lim_{x\to a} g(x) = \lim_{x\to a} (1-\sin^2 x)^{1/4} = \left(\lim_{x\to a} (1-\sin^2 x)\right)^{1/4}$$

$$= \left[\left(\lim_{x\to a} 1\right) - \left(\lim_{x\to a} \sin x\right)^2\right]^{1/4} = \left(1-\sin^2 a\right)^{1/4} = g(a).$$

Therefore g is continuous at a for every real number a.

C02S04.009: If $a > -1$, then $f(a)$ exists because $a \neq -1$. Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{1}{x+1} = \frac{\lim_{x\to a} 1}{\left(\lim_{x\to a} x\right) + \left(\lim_{x\to a} 1\right)} = \frac{1}{a+1} = f(a).$$

Therefore f is continuous on the interval $x > -1$.

C02S04.010: If $-2 < a < 2$, then $f(a)$ exists because $a^2 - 4 \neq 0$. Moreover,

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{x-1}{x^2-4} = \frac{\left(\lim_{x\to a} x\right) - \left(\lim_{x\to a} 1\right)}{\left(\lim_{x\to a} x\right)^2 - \left(\lim_{x\to a} 4\right)} = \frac{a-1}{a^2-4} = f(a).$$

Therefore f is continuous at x if $-2 < x < 2$.

C02S04.011: Because $-\frac{3}{2} \leqq t \leqq \frac{3}{2}$, $0 \leqq 4t^2 \leqq 9$, so that the radicand in $g(t)$ is never negative. Therefore $g(a)$ is defined for every real number a in the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$, and

$$\begin{aligned}\lim_{t\to a} g(t) &= \lim_{t\to a} (9-4t^2)^{1/2} = \left(\lim_{t\to a} (9-4t^2)\right)^{1/2} \\ &= \left[\left(\lim_{t\to a} 9\right) - 4\left(\lim_{t\to a} t\right)^2\right]^{1/2} = (9-4a^2)^{1/2} = g(a).\end{aligned}$$

Therefore g is continuous at a for every real number a in $\left[-\frac{3}{2}, \frac{3}{2}\right]$.

C02S04.012: If $1 \leqq z \leqq 3$, then $0 \leqq z-1 \leqq 2$, $-3 \leqq -z \leqq -1$, and $0 \leqq 3-z \leqq 2$. So the radicand in $h(z)$ is nonnegative for such values of z, and therefore if $1 \leqq a \leqq 3$ then

$$\begin{aligned}\lim_{z\to a} h(z) &= \lim_{z\to a} [(z-1)(3-z)]^{1/2} = \left[\lim_{z\to a} (z-1)(3-z)\right]^{1/2} \\ &= \left[\left(\left[\lim_{z\to a} z\right] - \left[\lim_{z\to a} 1\right]\right)\left(\left[\lim_{z\to a} 3\right] - \left[\lim_{z\to a} z\right]\right)\right]^{1/2} = [(a-1)(3-a)]^{1/2} = h(a).\end{aligned}$$

Therefore h is continuous at a for all real numbers a in the interval $[1, 3]$.

C02S04.013: If $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, then $\cos x \neq 0$, so $f(x)$ is defined for all such x. In addition, $\cos x \to \cos a$ as $x \to a$ because the cosine function is continuous everywhere (Theorem 1). Therefore

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{x}{\cos x} = \frac{\lim_{x\to a} x}{\lim_{x\to a} \cos x} = \frac{a}{\cos a} = f(a).$$

Therefore f is continuous at x if $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

C02S04.014: If $-\frac{1}{6}\pi < t < \frac{1}{6}\pi$, then $-\frac{1}{2} < \sin t < \frac{1}{2}$, so that $1 - 2\sin t > 0$ for such values of t. Therefore $g(a)$ is defined if $-\frac{1}{6}\pi < a < \frac{1}{6}\pi$. Moreover, for such values of a, we have

$$\begin{aligned}\lim_{t\to a} g(t) &= \lim_{t\to a} (1-2\sin t)^{1/2} = \left(\lim_{t\to a} (1-2\sin t)\right)^{1/2} \\ &= \left[\left(\lim_{t\to a} 1\right) - \left(\lim_{t\to a} 2\right)\left(\lim_{t\to a} \sin t\right)\right]^{1/2} = (1-2\sin a)^{1/2} = g(a).\end{aligned}$$

Therefore g is continuous at a for each real number a in the interval $\left(-\frac{1}{6}\pi, \frac{1}{6}\pi\right)$.

C02S04.015: The root law of Section 2.2 implies that $g(x) = \sqrt[3]{x}$ is continuous on the set $\boldsymbol{R}$ of all real numbers. We know that the polynomial $h(x) = 2x$ is continuous on $\boldsymbol{R}$ (Section 2.4, page 88). Hence the sum $f(x) = h(x) + g(x)$ is continuous on $\boldsymbol{R}$.

C02S04.016: The polynomial $f(x) = x^2$ is continuous on $\boldsymbol{R}$ (the set of all real numbers) and the quotient

$$h(x) = \frac{1}{x}$$

of continuous functions is continuous where its denominator is not zero. Hence the sum $g(x) = f(x) + h(x)$ is continuous on its domain, the set of all nonzero real numbers.

C02S04.017: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than -3.

C02S04.018: Because $f(t)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 5.

C02S04.019: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers.

C02S04.020: Because $g(z)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), g is continuous wherever its denominator is nonzero. Therefore g is continuous on its domain, the set of all real numbers other than ± 1.

C02S04.021: Note that $f(x)$ is not defined at $x = 5$, so it is not continuous there. Because $f(x) = 1$ for $x > 5$ and $f(x) = -1$ for $x < 5$, f is a polynomial on the interval $(5, +\infty)$ and a [another] polynomial on the interval $(-\infty, 5)$. Therefore f is continuous on its domain, the set of all real numbers other than 5.

C02S04.022: Because $h(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), h is continuous wherever its denominator is nonzero. Therefore h is continuous on its domain, the set of all real numbers.

C02S04.023: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 2.

C02S04.024: Because $g(t) = 4 + t^4$ is a polynomial, it is continuous everywhere (and never negative). Because $h(t) = \sqrt[4]{t}$ is a root function, it is continuous wherever $t \geqq 0$. Therefore (by Theorem 2) the composition $f(t) = h(g(t))$ is continuous everywhere.

C02S04.025: Let

$$h(x) = \frac{x+1}{x-1}.$$

Because $h(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), h is continuous wherever its denominator is nonzero. Therefore h is continuous on its domain, the set of all real numbers other than 1. Now let $g(x) = \sqrt[3]{x}$. By the root rule of Section 2.2, g is continuous everywhere. Therefore the composition $f(x) = g(h(x))$ is continuous on the set of all real numbers other than 1.

C02S04.026: Here, $F(u) = g(h(u))$ where $g(u) = \sqrt[3]{u}$ and $h(u) = 3 - u^3$. Now g is continuous everywhere by the root rule of Section 2.2; h is continuous everywhere because $h(u)$ is a polynomial. Therefore the composition $F(u) = g(h(u))$ of continuous functions is continuous where defined; namely, on the set $\boldsymbol{R}$ of all real numbers.

C02S04.027: Because $f(x)$ is the quotient of continuous functions (the numerator and denominator are polynomials, continuous everywhere), f is continuous wherever its denominator is nonzero. Therefore f is continuous on its domain, the set of all real numbers other than 0 and 1.

C02S04.028: The domain of f is the interval $-3 \leqq z \leqq 3$, and on that domain $f(z)$ is the composition of continuous functions, thus f is continuous there. Because $f(z)$ is not defined if $|z| > 3$, it is not continuous for $z < -3$ nor for $z > 3$. But it is still correct to say simply that "f is continuous" (see the definition of **continuous** on page 88).

C02S04.029: Let $h(x) = 4 - x^2$. Then h is continuous everywhere because $h(x)$ is a polynomial. The root function $g(x) = \sqrt{x}$ is continuous for $x \geqq 0$ by the root rule of Section 2.2. Hence $g(h(x)) = \sqrt{4 - x^2}$ is continuous wherever $x^2 \leqq 4$; that is, on the interval $[-2,\ 2]$. The quotient

$$f(x) = \frac{x}{\sqrt{4 - x^2}} = \frac{x}{g(h(x))}$$

is continuous wherever the numerator is continuous (that's everywhere) and the denominator is both continuous and nonzero (that's the open interval $(-2,\ 2)$). Therefore f is continuous on the open interval $(-2,\ 2)$. That is, f is continuous on its domain.

C02S04.030: Because $f(x)$ is formed by the composition and quotient of continuous functions (polynomials and root functions), it will be continuous wherever the denominator in the fraction is nonzero and the fraction is nonnegative. So continuity of f will occur when both

$$4 - x^2 \neq 0 \qquad \text{and} \qquad \frac{1 - x^2}{4 - x^2} \geqq 0.$$

The first inequality is equivalent to $x \neq \pm 2$ and the second will hold when $1 - x^2$ and $4 - x^2$ have the same sign (both positive or both negative) or the numerator is zero. If both are positive, then $x^2 < 1$ and $x^2 < 4$, so that $-1 < x < 1$. If both are negative, then $x^2 > 1$ and $x^2 > 4$, so that $x^2 > 4$; that is, $x < -2$ or $x > 2$. Finally, $1 - x^2 = 0$ when $x = \pm 1$. Therefore f is continuous on its domain, $(-\infty,\ -2) \cup [-1,\ 1] \cup (2,\ +\infty)$.

C02S04.031: Because $f(x)$ is the quotient of continuous functions, it is continuous where its denominator is nonzero; that is, if $x \neq 0$. Thus f is continuous on its domain and not continuous at $x = 0$ (because it is undefined there).

C02S04.032: Given:

$$g(\theta) = \frac{\theta}{\cos\theta}.$$

Because $g(\theta)$ is the quotient of continuous functions, it is continuous wherever its denominator is nonzero; that is, at every real number x not an odd integral multiple of $\pi/2$. That is, g is discontinuous (because it is undefined) at

$$\dots,\ -\frac{5\pi}{2},\ -\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2},\ \dots.$$

Therefore g is continuous on its domain, the set

$$\cdots \cup \left(-\tfrac{5}{2}\pi,\ -\tfrac{3}{2}\pi\right) \cup \left(-\tfrac{3}{2}\pi,\ -\tfrac{1}{2}\pi\right) \cup \left(-\tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi\right) \cup \left(\tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi\right) \cup \left(\tfrac{3}{2}\pi,\ \tfrac{5}{2}\pi\right) \cup \left(\tfrac{5}{2}\pi,\ \tfrac{7}{2}\pi\right) \cup \cdots.$$

C02S04.033: Given:

$$f(x) = \frac{1}{\sin 2x}.$$

The numerator in $f(x)$ is a polynomial, thus continuous everywhere. The denominator is the composition of a function continuous on the set of all real numbers (the sine function) with another continuous function (a polynomial), hence is also continuous everywhere. Thus because $f(x)$ is the quotient of continuous functions, it is continuous wherever its denominator is nonzero; that is, its only discontinuities occur when $\sin 2x = 0$. Thus f is continuous at every real number other than an integral multiple of $\pi/2$.

C02S04.034: Because $f(x) = \sqrt{\sin x}$ is the composition of continuous functions, it is continuous wherever it is defined; that is, wherever $\sin x \geqq 0$. Hence f is continuous on its domain, the set

$$\cdots \cup [-4\pi,\ -3\pi] \cup [-2\pi,\ -\pi] \cup [0,\ \pi] \cup [2\pi,\ 3\pi] \cup [4\pi,\ 5\pi] \cup \cdots.$$

C02S04.035: Given: $f(x) = \sin|x|$. The sine function is continuous on the set of all real numbers, as is the absolute value function. Therefore their composition f is continuous on the set $\boldsymbol{R}$ of all real numbers.

C02S04.036: Given:

$$G(u) = \frac{1}{\sqrt{1 + \cos u}}.$$

Because $G(u)$ is the sum, composition, and quotient of continuous functions, it is continuous where it is defined. There is no obstruction to computing $\sqrt{1 + \cos u}$ because $1 + \cos u \geqq 0$ for every real number u. Hence G will be undefined, and thus not continuous, exactly when its denominator is zero, which is exactly when $1 + \cos u = 0$. Therefore G is continuous except at the odd integral multiples of π. Put another way, G is continuous on the union of open intervals of the form $([2n-1]\pi,\ [2n+1]\pi)$ where n runs through all integral values.

C02S04.037: The function

$$f(x) = \frac{x}{(x+3)^3}$$

is not continuous when $x = -3$. This discontinuity is not removable because $f(x) \to -\infty$ as $x \to -3^+$, so that the limit of $f(x)$ at $x = -3$ does not exist.

C02S04.038: The function

$$f(t) = \frac{t}{t^2 - 1}$$

is not continuous when $t = \pm 1$ because $t^2 - 1 = 0$ then. These discontinuities are not removable because $f(t) \to +\infty$ as $t \to 1^+$ and as $t \to -1^+$, so that f has no limit at either $t = 1$ or $t = -1$.

C02S04.039: First simplify $f(x)$:

$$f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2} \quad \text{if} \quad x \neq 2.$$

Now $f(x)$ is not defined at $x = \pm 2$ because $x^2 - 4 = 0$ for such x. The discontinuity at -2 is not removable because $f(x) \to +\infty$ as $x \to -2^+$. But $f(x) \to \frac{1}{4}$ as $x \to 2$, so the discontinuity at $x = 2$ is removable; f can be made continuous at $x = 2$ by defining its value there to be its limit there, $\frac{1}{4}$.

C02S04.040: First try to simplify the formula of G:

$$G(u) = \frac{u+1}{u^2 - u - 6} = \frac{u+1}{(u-3)(u+2)}.$$

This computation shows that G is not continuous at $u = 3$ and at $u = -2$. It also shows that these discontinuities are not removable because $G(u) \to +\infty$ as $u \to 3^+$ and as $u \to -2^+$, so that G has no limit at either of its discontinuities.

C02S04.041: Given:

$$f(x) = \frac{1}{1 - |x|}.$$

The function f is not continuous at ± 1 because its denominator is zero if $x = -1$ and if $x = 1$. Because $f(x) \to +\infty$ as $x \to 1^-$ and as $x \to -1^+$ (consider separately the cases $x > 0$ and $x < 0$), these discontinuities are not removable; $f(x)$ has no limit at -1 or at 1.

C02S04.042: If $x > 1$, then

$$h(x) = \frac{|x - 1|}{(x - 1)^3} = \frac{x - 1}{(x - 1)^3} = \frac{1}{(x - 1)^2}.$$

Therefore h is discontinuous at $x = 1$ and, because $h(x) \to +\infty$ as $x \to 1^+$, this discontinuity is not removable.

C02S04.043: If $x > 17$, then $x - 17 > 0$, so that

$$f(x) = \frac{x - 17}{|x - 17|} = \frac{x - 17}{x - 17} = 1.$$

But if $x < 17$, then $x - 17 < 0$, and thus

$$f(x) = \frac{x - 17}{|x - 17|} = \frac{x - 17}{-(x - 17)} = -1.$$

Therefore $h(x)$ has no limit as $x \to 17$ because its left-hand and right-hand limits there are unequal. Thus the discontinuity at $x = 17$ is not removable.

C02S04.044: First simplify:

$$g(x) = \frac{x^2 + 5x + 6}{x + 2} = \frac{(x + 2)(x + 3)}{x + 2} = x + 3 \quad \text{if} \quad x \neq -2.$$

Therefore, although g is discontinuous at $x = -2$ (because it is not defined there), this discontinuity is removable; simply define $g(-2)$ to be 1, the limit of $g(x)$ as $x \to -2$.

C02S04.045: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 0$ as $x \to 0^+$ and as $x \to 0^-$, so defining $f(0)$ to be 0, the limit of $f(x)$ at $x = 0$, will make f continuous there.

C02S04.046: Although $f(x)$ is not continuous at $x = 1$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 2$ as $x \to 1^+$ and as $x \to 1^-$, so defining $f(1)$ to be 2, the limit of $f(x)$ at $x = 1$, will make f continuous there.

C02S04.047: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For it is clear that $f(x) \to 1$ as $x \to 0^+$ and as $x \to 0^-$, so defining $f(0)$ to be 1, the limit of $f(x)$ at $x = 0$, will make f continuous there.

C02S04.048: Although $f(x)$ is not continuous at $x = 0$ (because it is not defined there), this discontinuity is removable. For

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1 - \cos x}{x} = \lim_{x \to 0^-} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \to 0^-} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = 1 \cdot \frac{0}{1 + 1} = 0,$$

and it is clear that $f(x) \to 0$ as $x \to 0^+$. So defining $f(0)$ to be 0, the limit of $f(x)$ at $x = 0$, will make f continuous there.

C02S04.049: The given function is clearly continuous for all x *except possibly* for $x = 0$. For continuity at $x = 0$, the left-hand and right-hand limits of $f(x)$ must be the same there. But

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + c) = c$$

and $f(x) \to 4$ as $x \to 0^+$. So continuity of f at $x = 0$ can occur only if $c = 4$. Moreover, if $c = 4$, then (as we have seen) $f(x) \to 4$ as $x \to 0$ and $f(0) = 4$, so f will be continuous at $x = 0$ if and only if $c = 4$. Answer: $c = 4$.

C02S04.050: Clearly f is continuous if $x \neq 3$, for if $x < 3$ or if $x > 3$, then $f(x)$ is a polynomial, regardless of the value of c. For continuity at $x = 3$, we require that the one-sided limits of $f(x)$ at $x = 3$ be equal. But $f(x) \to 6 + c$ as $x \to 3^-$ and $f(x) \to 2c - 3$ as $x \to 3^+$. Equality of the one-sided limits is equivalent to

$$6 + c = 2c - 3; \qquad \text{that is,} \qquad c = 9.$$

Finally, if $c = 9$, then the two-sided limit of $f(x)$ at $x = 3$ is 15 and $f(3) = 2 \cdot 3 + 9 = 15$, so f will be continuous at $x = 3$ if $c = 9$. Answer: $c = 9$.

C02S04.051: Note that f is continuous at x if $x \neq 0$, because $f(x)$ is a polynomial for $x < 0$ and for $x > 0$ regardless of the value of c. To be continuous at $x = 0$, it's necessary that the left-hand and right-hand limits exist and are equal there. Now

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (c^2 - x^2) = c^2 \qquad \text{and} \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2(x - c)^2 = 2c^2,$$

and therefore continuity at $x = 0$ will hold if and only if $c^2 = 2c^2$; that is, if $c = 0$. And if so, then $f(0) = \lim_{x \to 0} f(x)$ as well, so f will be continuous at $x = 0$. Answer: $c = 0$.

C02S04.052: Note that f is continuous if $x < \pi$ because $f(x)$ is a polynomial for such x; also, f is continuous for $x > \pi$ because (regardless of the value of c) $f(x)$ is a constant multiple of a continuous function. For continuity of f at $x = \pi$, the left-hand and right-hand limits must be equal there. But

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} (c^3 - x^3) = c^3 - \pi^3 \qquad \text{and} \qquad \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} c \sin x = \lim_{x \to \pi^+} c \sin \pi = 0.$$

So continuity of f at $x = \pi$ requires $c^3 - \pi^3 = 0$; that is, $c = \pi$. And if so, then $f(\pi) = \pi^3 - \pi^3 = 0 = \lim_{x \to \pi} f(x)$, so f will be continuous at $x = \pi$. Answer: $c = \pi$.

C02S04.053: Let $f(x) = x^2 - 5$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [2, 3]. Also $f(2) = -1 < 0 < 4 = f(3)$, so $f(c) = 0$ for some number c in [2, 3]. That is, $c^2 - 5 = 0$. Hence the equation $x^2 - 5 = 0$ has a solution in [2, 3].

C02S04.054: Let $f(x) = x^3 + x + 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval $[-1, 0]$. Also $f(-1) = -1 < 0 < 1 = f(0)$, so $f(c) = 0$ for some number c in $[-1, 0]$. That is, $c^3 + c + 1 = 0$. Hence the equation $x^3 + x + 1 = 0$ has a solution in $[-1, 0]$.

C02S04.055: Let $f(x) = x^3 - 3x^2 + 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [0, 1]. Also $f(0) = 1 > 0 > -1 = f(1)$, so $f(c) = 0$ for

some number c in [0, 1]. That is, $c^3 - 3c^2 + 1 = 0$. Hence the equation $x^3 - 3x^2 + 1 = 0$ has a solution in [0, 1].

C02S04.056: Let $f(x) = x^3 - 5$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [1, 2]. Also $f(1) = -4 < 0 < 3 = f(2)$, so $f(c) = 0$ for some number c in [1, 2]. That is, $c^3 - 5 = 0$. Hence the equation $x^3 = 5$ has a solution in [1, 2].

C02S04.057: Let $f(x) = x^4 + 2x - 1$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval [0, 1]. Also $f(0) = -1 < 0 < 2 = f(1)$, so $f(c) = 0$ for some number c in [0, 1]. That is, $c^4 + 2c - 1 = 0$. Hence the equation $x^4 + 2x - 1 = 0$ has a solution in [0, 1].

C02S04.058: Let $f(x) = x^5 - 5x^3 + 3$. Then f is continuous everywhere because $f(x)$ is a polynomial. So f has the intermediate value property on the interval $[-3, -2]$. Also $f(-3) = -105 < 0 < 11 = f(-2)$, so $f(c) = 0$ for some number c in $[-3, -2]$. That is, $c^5 - 5c^3 + 3 = 0$. Hence the equation $x^5 - 5x^3 + 3 = 0$ has a solution in $[-3, -2]$.

C02S04.059: Given: $f(x) = x^3 - 4x + 1$. Values of $f(x)$:

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

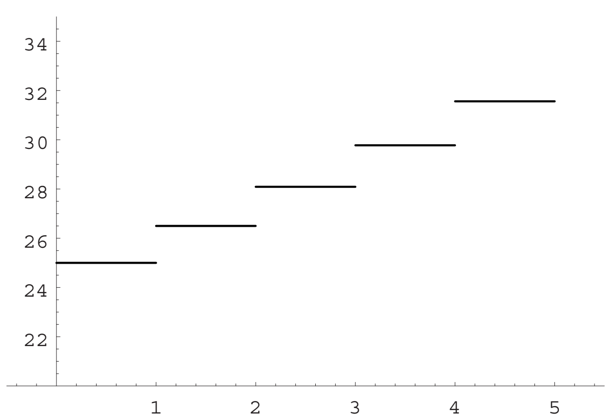
So $f(x_i) = 0$ for x_1 in $(-3, -2)$, x_2 in $(0, 1)$, and x_3 in $(1, 2)$. Because these intervals do not overlap, the equation $f(x) = 0$ has at least three real solutions. Because $f(x)$ is a polynomial of degree 3, that equation also has at most three real solutions. Therefore the equation $x^3 - 4x + 1 = 0$ has exactly three real solutions.

C02S04.060: Given: $f(x) = x^3 - 3x^2 + 1$. Values of $f(x)$:

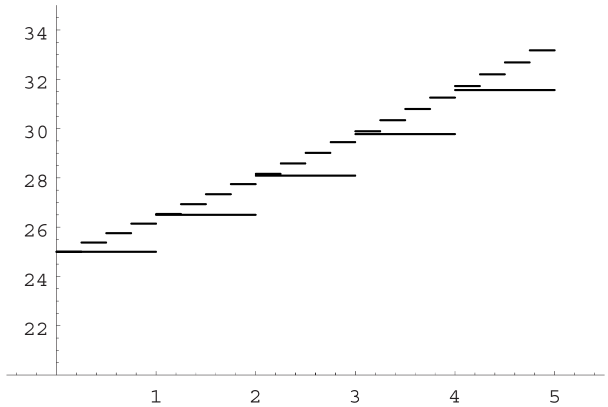
x	-3	-2	-1	0	1	2	3
$f(x)$	-53	-19	-3	1	-1	-3	1

So $f(x_i) = 0$ for x_1 in $(-1, 0)$, x_2 in $(0, 1)$, and x_3 in $(2, 3)$. Because these intervals do not overlap, the equation $f(x) = 0$ has at least three real solutions. Because $f(x)$ is a polynomial of degree 3, that equation also has at most three real solutions. Therefore it has exactly three real solutions.

C02S04.061: At time t, $[\![t]\!]$ years have elapsed, and at that point your starting salary has been multiplied by 1.06 exactly t times. Thus it is $S(t) = 25 \cdot (1.06)^{[\![t]\!]}$. Of course S is discontinuous exactly when t is an integer between 1 and 5. The graph is next.

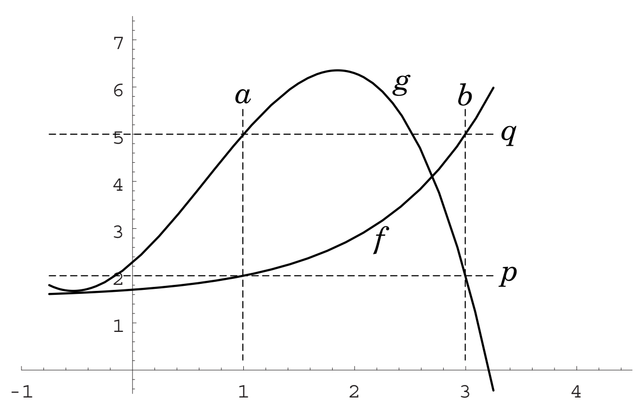


C02S04.062: The salary function is $P(t) = 25 \cdot (1.015)^{[\![4t]\!]}$. It is discontinuous at the end of every three-month period; that is, at integral multiples of $\frac{1}{4}$. You will accumulate more money with three-month raises of 1.5% than with yearly raises of 6%; the total salary received by the end of the first five years with yearly raises would be \$140,930 but with quarterly raises it would be \$144,520. The graphs of the function $S(t)$ of Problem 61 and $P(t)$ are shown next. Although the graphs do not make it perfectly clear, it turns out that $P(t) > S(t)$ if $t \geqq \frac{1}{4}$, so that the quarterly raise is better for you financially after the first three months and continues to outpace the yearly raise as long as you keep the job.



C02S04.063: The next figure shows the graphs of two such functions f and g, with $[a, b] = [1, 3]$, $p = 2$, and $q = 5$. Because f and g are continuous on $[a, b]$, so is $h = f - g$. Because $p \neq q$, $h(a) = p - q$ and $h(b) = q - p$ have opposite signs, so that 0 is an intermediate value of the continuous function h. Therefore $h(c) = 0$ for some number c in (a, b). That is, $f(c) = g(c)$. This concludes the proof. To construct the figure, we used (the given coefficients are approximate)

$$f(x) = 1.53045 + (0.172739)e^x \qquad \text{and} \qquad g(x) = 2.27857 + (2.05)x + (1.36429)x^2 - (0.692857)x^3.$$



C02S04.064: Let $f(t)$ denote your distance from Estes Park during your trip today, with f measured in kilometers and t in hours, $t = 0$ corresponding to time 1 P.M. Let $g(t)$ denote your distance from Estes Park during your trip tomorrow, with g in kilometers, t in hours, and $t = 0$ corresponding to 1 P.M. tomorrow. Assuming that both f and g are continuous, we use the facts that $f(0) = 0$, $f(1) = M$ (where M is the distance from Estes Park to Grand Lake), $g(0) = M$, and $g(1) = 0$ and apply the result of Problem 63 to conclude that $f(c) = g(c)$ for some number c in $(0, 1)$. That is, at time $t = c$ tomorrow you will be at exactly the same spot (at distance $g(c)$ from Estes Park) as you will be at the same time $t = c$ today at distance $f(c) = g(c)$ from Estes Park.

The 1999 *National Geographic Road Atlas* indicates that $M \approx 101$ (km). Making this trip in a single hour is unforgivable given the magnificent scenery (and probably impossible as well given the dozens of tight turns on the highway).

C02S04.065: Given $a > 0$, let $f(x) = x^2 - a$. Then f is continuous on $[0, a+1]$ because $f(x)$ is a polynomial. Also $f(a+1) > 0$ because

$$f(a+1) = (a+1)^2 - a = a^2 + a + 1 > 1 > 0.$$

So $f(0) = -a < 0 < f(a+1)$. Therefore, because f has the intermediate value property on the interval $[0, a+1]$, there exists a number r in $(0, a+1)$ such that $f(r) = 0$. That is, $r^2 - a = 0$, so that $r^2 = a$. Therefore a has a square root.

Our proof shows that a has a positive square root. Can you modify it to show that a also has a negative square root? Do you see why we used the interval $[0, a+1]$ rather than the simpler $[0, a]$?

C02S04.066: Clearly $a = 0$ has a cube root. Suppose first that $a > 0$. Let $f(x) = x^3 - a$. Then f has the intermediate value property on $[0, a+1]$ because $f(x)$ is a polynomial. Moreover,

$$f(a+1) = (a+1)^3 - a = a^3 + 3a^2 + 2a + 1 > 1 > 0 > -a = f(0).$$

Therefore there exist a number c in $[0, a+1]$ such that $f(c) = 0$. That is, $c^3 - a = 0$, so that $c^3 = a$. Thus the positive real number a has a cube root. Moreover, $(-c)^3 = -(c^3) = -a$, so that every negative real number has a cube root as well.

C02S04.067: Given the real number a, we need to show that

$$\lim_{x \to a} \cos x = \cos a.$$

Let $h = x - a$, so that $x = a + h$. Then $x \to a$ is equivalent to $h \to 0$; also, $\cos x = \cos(a + h)$. Thus

$$\lim_{x \to a} \cos x = \lim_{h \to 0} \cos(a+h) = \lim_{h \to 0} (\cos a \cos h - \sin a \sin h) = (\cos a) \cdot 1 - (\sin a) \cdot 0 = \cos a.$$

Therefore the cosine function is continuous at $x = a$ for every real number a.

C02S04.068: If x is not an integer, choose that [unique] integer $n = [\![x]\!]$ such that $n < x < n + 1$. Then $f(x) = x + n$ on the interval $(n, n+1)$, thus $f(x)$ is effectively a polynomial on that interval. So f is continuous at x. But if m is an integer, then

$$\lim_{x \to m^-} f(x) = \lim_{x \to m^-} (x + [\![x]\!]) = \lim_{x \to m^-} (x + m - 1) = m + m - 1 = 2m - 1,$$

whereas

$$\lim_{x \to m^+} f(x) = \lim_{x \to m^+} (x + [\![x]\!]) = \lim_{x \to m^+} (x + m) = m + m = 2m.$$

Because the left-hand and right-hand limits of $f(x)$ differ at m, f is not continuous there. Thus f is discontinuous at each integer and continuous at every other real number.

C02S04.069: Suppose that a is a real number. We appeal to the formal definition of the limit in Section 2.2 (page 74) to show that $f(x)$ has no limit as $x \to a$. Suppose by way of contradiction that $f(x) \to L$ as $x \to a$. Then, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - L| < \epsilon$ for every number x such that $0 < |x - a| < \delta$. So this statement must hold if $\epsilon = \frac{1}{4}$.

Case 1: $L = 0$. Then there must exist a number $\delta > 0$ such that

$$|f(x) - 0| < \tfrac{1}{4}$$

if $0 < |x - a| < \delta$. But, regardless of the value of δ, there exist irrational values of x satisfying this last inequality. (We'll explain why in a moment.) Choose such a number x. Then

$$|f(x) - 0| = |1 - 0| = 1 < \tfrac{1}{4}.$$

This is impossible. So $L \neq 0$.

Case 2: $L = 1$. Proceed exactly as in Case 1, except choose a *rational* value of x such that $0 < |x-a| < \delta$. Then

$$|f(x) - 1| = |0 - 1| = 1 < \tfrac{1}{4}.$$

This, too, is impossible. So $L \neq 1$.

Case 3: L is neither 0 nor 1. Let $\epsilon = \frac{1}{3}|L|$. Note that $\epsilon > 0$. Then suppose that there exists $\delta > 0$ such that,

$$\text{if} \qquad 0 < |x - a| < \delta, \qquad \text{then} \qquad |f(x) - L| < \epsilon. \tag{1}$$

Choose a rational number x satisfying the left-hand inequality. Then

$$|f(x) - L| = |0 - L| = |L| = 3\epsilon.$$

It follows from (1) that $3\epsilon < \epsilon$, which is impossible because $\epsilon > 0$.

In summary, L cannot be 0, nor can it be 1, nor can it be any other real number. Therefore $f(x)$ has no limit as $x \to a$. Consequently f is not continuous at $x = a$.

In this proof we relied heavily on the fact that if a is any real number, then we can find both rational and irrational numbers arbitrarily close to a. Rather than providing a formal proof, we illustrate how to do this in the case that

$$a = 1.23456789101112131415\cdots\,.$$

(It doesn't matter whether a is rational or irrational.) To produce rational numbers arbitrarily close to a, use

$$1.2,\ 1.23,\ 1.234,\ 1.2345,\ 1.23456,\ 1.234567,\ \dots\,. \tag{2}$$

The numbers in (2) are all rational because they all have terminating decimal expansions, and the nth number in (2) differs from a by less than 10^{-n}, so there are rational numbers arbitrarily close to a. To get irrational numbers with the same properties, use

$$1.201001000100001000001000\cdots\,,\ 1.230100100010000100000100\cdots\,,$$
$$1.2340100100010000100000100\cdots\,,\ 1.23450100100010000100000100\cdots\,,\ \dots\,.$$

These numbers are irrational because every one of them has a nonrepeating decimal expansion.

C02S04.070: You can modify the argument in the solution of Problem 69 to show that $f(x)$ has no limit at $x = a$ if $a \neq 0$. Simply use a^2 in place of 1 in that argument. Because $0 \leqq f(x) \leqq x^2$ for all x, and because $0 \to 0$ and $x^2 \to 0$ as $x \to 0$, it follows from the squeeze theorem that

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

Therefore f is continuous at $x = 0$.

C02S04.071: Let $g(x) = x - \cos x$. Then $g(x)$ is the sum of continuous functions, thus continuous everywhere, and in particular on the interval $\left[0, \frac{1}{2}\pi\right]$. So g has the intermediate value property there. Also $g(0) = -1 < 0$ and $g(1) = 1 > 0$. Therefore there exists a number c in $\left(0, \frac{1}{2}\pi\right)$ such that $g(c) = 0$. Thus $c - \cos x = 0$, so that $c = \cos c$. Thus the equation $x = \cos x$ has a solution in $\left(0, \frac{1}{2}\pi\right)$. (This solution is approximately 0.7390851.)

C02S04.072: Let $h(x) = x + 5\cos x$. Then $h(x)$ is the sum and product of continuous functions, thus is continuous everywhere, including the intervals $[-\pi, 0]$, $[0, \pi]$, and $[\pi, 2\pi]$. Moreover,

$$h(-\pi) = -\pi - 5 < 0 < 5 = h(0), \qquad h(0) = 5 > 0 > \pi - 5 = h(\pi),$$
$$\text{and} \qquad h(\pi) = \pi - 5 < 0 < 2\pi + 5 = h(2\pi).$$

By the intermediate value property of continuous functions, $h(a) = 0$ for some number a in $I = (-\pi, 0)$, $h(b) = 0$ for some number b in $J = (0, \pi)$, and $h(c) = 0$ for some number c in $K = (\pi, 2\pi)$. Because no two of I, J, and K have any points in common, the numbers a, b, and c are distinct. Therefore the equation $h(x) = 0$ three distinct solutions; in other words, the equation $x = -5\cos x$ has three distinct solutions. (We have *not* shown that there are no additional solutions, but this was not required.) Finally, $a \approx -1.30644$, $b \approx 1.97738$, and $c \approx 3.83747$.

C02S04.073: Because

$$\lim_{x\to 0^+} 2^{1/x} = +\infty,$$

f is not right continuous at $x = 0$. Because

$$\lim_{x\to 0^-} 2^{1/x} = \lim_{u\to -\infty} 2^u = \lim_{z\to +\infty} \frac{1}{2^z} = 0 = f(0),$$

f is left continuous at $x = 0$.

C02S04.074: Because

$$\lim_{x\to 0} 2^{-1/x^2} = 0 = f(0),$$

the function f is both left and right continuous—thus continuous—at $x = 0$.

C02S04.075: Because

$$\lim_{x\to 0^+} \frac{1}{1 + 2^{1/x}} = 0 \neq f(0),$$

f is not right continuous at $x = 0$. But

$$\lim_{x\to 0^-} \frac{1}{1 + 2^{1/x}} = \frac{1}{1 + 0} = 1 = f(0),$$

f is left continuous at $x = 0$.

C02S04.076: Because

$$\lim_{x \to 0} \frac{1}{1 + 2^{-1/x^2}} = \frac{1}{1+0} = 1 = f(0),$$

the function f is both left and right continuous at $x = 0$—thus it is continuous there.

C02S04.077: We consider only the discontinuity at $x = a = \pi/2$; the behavior of f is the same near all of its discontinuities (the odd integral multiples of a). Because

$$\lim_{x \to a^+} \frac{1}{1 + 2^{\tan x}} = \frac{1}{1+0} = 1 = f(1),$$

the function f is right continuous at $x = a$. But

$$\lim_{x \to a^-} \frac{1}{1 + 2^{\tan x}} = 0 \neq f(0),$$

so f is not left continuous at $x = a$.

C02S04.078: We consider only the discontinuities at $x = 0$ and $x = \pi$, because the behavior of f at every even integral multiple of π is the same as its behavior at $x = 0$, and its behavior at every odd integral multiple of π is the same as its behavior at $x = \pi$. We first note that

$$\lim_{x \to 0^+} \frac{1}{1 + 2^{1/\sin x}} = 0 = f(0),$$

so that f is right continuous at $x = 0$. But

$$\lim_{x \to 0^-} \frac{1}{1 + 2^{1/\sin x}} = 1 \neq f(0),$$

and thus f is not left continuous at $x = 0$. The situation is reversed at π, as one might gather from examining the graph of the sine function:

$$\lim_{x \to \pi^-} \frac{1}{1 + 2^{1/\sin x}} = 0 = f(\pi),$$

so that f is left continuous at $x = \pi$, but

$$\lim_{x \to \pi^+} \frac{1}{1 + 2^{1/\sin x}} = 1 \neq f(\pi).$$

Therefore f is not right continuous at $x = \pi$.

Chapter 2 Miscellaneous Problems

C02S0M.001: $\lim_{x\to 0}(x^2-3x+4)=\left(\lim_{x\to 0} x\right)^2-3\cdot\left(\lim_{x\to 0} x\right)+4=0^2-3\cdot 0+4=4.$

C02S0M.002: $\lim_{x\to -1}(3-x+x^3)=3-\left(\lim_{x\to -1} x\right)+\left(\lim_{x\to -1} x\right)^3=3-(-1)+(-1)^3=3.$

C02S0M.003: $\lim_{x\to 2}(4-x^2)^{10}=\left[4-\left(\lim_{x\to 2} x\right)^2\right]^{10}=\left(4-2^2\right)^{10}=0^{10}=0.$

C02S0M.004: $\lim_{x\to 1}(x^2+x-1)^{17}=\left[\left(\lim_{x\to 1} x\right)^2+\left(\lim_{x\to 1} x\right)-1\right]^{17}=(1^2+1-1)^{17}=1.$

C02S0M.005: $\displaystyle\lim_{x\to 2}\frac{1+x^2}{1-x^2}=\frac{1+\left(\lim_{x\to 2} x\right)^2}{1-\left(\lim_{x\to 2} x\right)^2}=\frac{1+2^2}{1-2^2}=\frac{1+4}{1-4}=-\frac{5}{3}.$

C02S0M.006: $\displaystyle\lim_{x\to 3}\frac{2x}{x^2-x-3}=\frac{2\cdot\left(\lim_{x\to 3} x\right)}{\left(\lim_{x\to 3} x\right)^2-\left(\lim_{x\to 3} x\right)-3}=\frac{2\cdot 3}{3^2-3-3}=\frac{6}{3}=2.$

C02S0M.007: $\displaystyle\frac{x^2-1}{1-x}=-\frac{(x+1)(x-1)}{x-1}=-(x+1)\to -2$ as $x\to 1$.

C02S0M.008: $\displaystyle\frac{x+2}{x^2+x-2}=\frac{x+2}{(x+2)(x-1)}=\frac{1}{x-1}\to -\frac{1}{3}$ as $x\to -2$.

C02S0M.009: $\displaystyle\frac{t^2+6t+9}{9-t^2}=-\frac{(t+3)^2}{(t+3)(t-3)}=-\frac{t+3}{t-3}\to -\frac{-3+3}{-3-3}=0$ as $t\to -3$.

C02S0M.010: $\displaystyle\frac{4x-x^3}{3x+x^2}=\frac{x(4-x^2)}{x(3+x)}=\frac{4-x^2}{3+x}\to\frac{4-0}{3+0}=\frac{4}{3}$ as $x\to 0$.

C02S0M.011: $\lim_{x\to 3}(x^2-1)^{2/3}=\left[\left(\lim_{x\to 3} x\right)^2-1\right]^{2/3}=(3^2-1)^{2/3}=8^{2/3}=\left(8^{1/3}\right)^2=2^2=4.$

C02S0M.012: $\displaystyle\lim_{x\to 2}\left(\frac{2x^2+1}{2x}\right)^{1/2}=\left[\frac{2\cdot\left(\lim_{x\to 2} x\right)^2+1}{2\cdot\left(\lim_{x\to 2} x\right)}\right]^{1/2}=\left(\frac{2\cdot 4+1}{2\cdot 2}\right)^{1/2}=\left(\frac{9}{4}\right)^{1/2}=\frac{3}{2}.$

C02S0M.013: $\displaystyle\lim_{x\to 3}\left(\frac{5x+1}{x^2-8}\right)^{3/4}=\left[\frac{5\cdot\left(\lim_{x\to 3} x\right)+1}{\left(\lim_{x\to 3} x\right)^2-8}\right]^{3/4}=\left(16^{1/4}\right)^3=8.$

C02S0M.014: $\displaystyle\frac{x^4-1}{x^2+2x-3}=\frac{(x^2+1)(x+1)(x-1)}{(x+3)(x-1)}=\frac{(x^2+1)(x+1)}{x+3}\to\frac{2\cdot 2}{4}=1$ as $x\to 1$.

C02S0M.015: First multiply numerator and denominator by $\sqrt{x+2}+3$ (the *conjugate* of the numerator) to obtain

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} = \lim_{x\to 7} \frac{x+2-9}{(x-7)\left(\sqrt{x+2}+3\right)} = \lim_{x\to 7} \frac{x-7}{(x-7)\left(\sqrt{x+2}+3\right)}$$

$$= \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3} = \frac{\displaystyle\lim_{x\to 7} 1}{\displaystyle\lim_{x\to 7} \left(\sqrt{x+2}+3\right)} = \frac{1}{\left(2+\displaystyle\lim_{x\to 7} x\right)^{1/2} + \displaystyle\lim_{x\to 7} 3} = \frac{1}{3+3} = \frac{1}{6}.$$

C02S0M.016: Note that $x > 1$ as $x \to 1^+$, so that $\sqrt{x^2-1}$ is defined for such x. Therefore

$$\lim_{x\to 1^+} \left(x - \sqrt{x^2-1}\right) = \left(\lim_{x\to 1^+} x\right) - \left[\left(\lim_{x\to 1^+} x\right)^2 - \left(\lim_{x\to 1^+} 1\right)\right]^{1/2} = 1 - \sqrt{1^2-1} = 1 - 0 = 1.$$

C02S0M.017: First simplify:

$$\frac{\dfrac{1}{\sqrt{13+x}} - \dfrac{1}{3}}{x+4} = \frac{1}{x+4} \cdot \frac{3-\sqrt{13+x}}{3\sqrt{13+x}} = \frac{3-\sqrt{13+x}}{3(x+4)\sqrt{13+x}}.$$

Then multiply numerator and denominator by $3+\sqrt{13+x}$, the conjugate of the numerator, to obtain

$$\frac{9-(13+x)}{3(x+4)\left(\sqrt{13+x}\,\right)\left(3+\sqrt{13+x}\,\right)} = \frac{-(x+4)}{3(x+4)\left(\sqrt{13+x}\,\right)\left(3+\sqrt{13+x}\,\right)} = -\frac{1}{3\left(\sqrt{13+x}\,\right)\left(3+\sqrt{13+x}\,\right)}.$$

Now let $x \to -4$ to obtain the limit $-\dfrac{1}{3\cdot 3\cdot (3+3)} = -\dfrac{1}{54}$.

C02S0M.018: Because $x \to 1^+$, $x > 1$, so that $1 - x < 0$. Therefore

$$\lim_{x\to 1^+} \frac{1-x}{|1-x|} = \lim_{x\to 1^+} \frac{1-x}{-(1-x)} = \lim_{x\to 1^+} (-1) = -1.$$

C02S0M.019: First, $4-4x+x^2 = (2-x)^2 = (x-2)^2$. Because $x \to 2^+$, $x > 2$, so that $x - 2 > 0$. Hence $\sqrt{4-4x+x^2} = \sqrt{(x-2)^2} = |x-2| = x-2$. Therefore

$$\lim_{x\to 2^+} \frac{2-x}{\sqrt{4-4x+x^2}} = \lim_{x\to 2^+} \frac{2-x}{x-2} = \lim_{x\to 2^+} (-1) = -1.$$

C02S0M.020: As $x \to -2^-$, $x < -2$, so that $x + 2 < 0$. Hence $|x+2| = -(x+2)$. Thus

$$\lim_{x\to -2^-} \frac{x+2}{|x+2|} = \lim_{x\to -2^-} \frac{x+2}{-(x+2)} = \lim_{x\to -2^-} (-1) = -1.$$

C02S0M.021: As $x \to 4^+$, $x > 4$, so that $x - 4 > 0$. Therefore $|x-4| = x-4$, and thus

$$\lim_{x\to 4^+} \frac{x-4}{|x-4|} = \lim_{x\to 4^+} \frac{x-4}{x-4} = \lim_{x\to 4^+} 1 = 1.$$

C02S0M.022: As $x \to 3^-$, $x < 3$, so that $x^2 - 9 < 0$ for $-3 < x < 3$. Therefore $\sqrt{x^2-9}$ is undefined for such x, and consequently $\displaystyle\lim_{x\to 3^-} \sqrt{x^2-9}$ does not exist.

C02S0M.023: As $x \to 2^+$, $x > 2$, so that $4 - x^2 < 0$. Therefore $\sqrt{4 - x^2}$ is undefined for all such x, and consequently $\displaystyle\lim_{x \to 2^+} \sqrt{4 - x^2}$ does not exist.

C02S0M.024: As $x \to -3$, $(x+3)^2 \to 0$ whereas the numerator x approaches -3. Therefore this limit does not exist. Because $(x+3)^2$ is approaching zero through positive values, it is also correct to write

$$\lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty.$$

C02S0M.025: As $x \to 2$, the denominator $(x-2)^2$ is approaching zero, while the numerator $x + 2$ is approaching 4. So this limit does not exist. Because the denominator is approaching zero through positive values, it is also correct (and more informative) to write

$$\lim_{x \to 2} \frac{x+2}{(x-2)^2} = +\infty.$$

C02S0M.026: As $x \to 1^-$, the denominator $x-1$ is approaching zero, but the numerator x is not. Therefore this limit does not exist. Because the numerator is approaching 1 and the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x}{x-1} = -\infty.$$

C02S0M.027: Because $x \to 3^+$, the denominator $x - 3$ is approaching zero, but the numerator x is not. Therefore this limit does not exist. Because the denominator is approaching zero through positive values while the numerator is approaching 3, it is also correct to write

$$\lim_{x \to 3^+} \frac{x}{x-3} = +\infty.$$

C02S0M.028: Because

$$\frac{x-2}{x^2 - 3x + 2} = \frac{x-2}{(x-1)(x-2)} = \frac{1}{x-1}$$

if $x \neq 2$, the limit of this fraction as $x \to 1^-$ does not exist: The numerator is approaching 1 while the denominator is approaching zero. Because the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x-2}{x^2 - 3x + 2} = -\infty.$$

C02S0M.029: As $x \to 1^-$, the numerator of the fraction is approaching 2, but the denominator is approaching zero. Therefore this limit does not exist. Because the denominator is approaching zero through negative values, it is also correct to write

$$\lim_{x \to 1^-} \frac{x+1}{(x-1)^3} = -\infty.$$

C02S0M.030: Note first that

$$\frac{25-x^2}{x^2-10x+25} = \frac{(5+x)(5-x)}{(x-5)^2} = \frac{(5+x)(5-x)}{(5-x)^2} = \frac{5+x}{5-x}.$$

Thus as $x \to 5^+$, the numerator approaches 10 while the denominator is approaching zero through negative values. Therefore this limit does not exist. It is also correct to write

$$\lim_{x\to 5^+} \frac{25-x^2}{x^2-10x+25} = -\infty.$$

C02S0M.031: Let $u = 3x$. Then $x = \frac{1}{3}u$; also, $x \to 0$ is equivalent to $u \to 0$. Thus

$$\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{u\to 0} \frac{\sin u}{\frac{1}{3}u} = \lim_{u\to 0} \frac{3\sin u}{u} = \left(\lim_{u\to 0} 3\right)\cdot\left(\lim_{u\to 0} \frac{\sin u}{u}\right) = 3\cdot 1 = 3.$$

C02S0M.032: Let $u = 5x$; then $x = \frac{1}{5}u$; moreover, $x \to 0$ is equivalent to $u \to 0$. Therefore

$$\lim_{x\to 0} \frac{\tan 5x}{x} = \lim_{u\to 0} \frac{\tan u}{\frac{1}{5}u} = \lim_{u\to 0} \frac{5\sin u}{u\cos u} = 5\cdot\left(\lim_{u\to 0} \frac{\sin u}{u}\right)\cdot\left(\lim_{u\to 0} \frac{1}{\cos u}\right) = 5\cdot 1\cdot \frac{1}{1} = 5.$$

C02S0M.033: The substitution $u = kx$ shows that if $k \neq 0$, then

$$\lim_{x\to 0} \frac{\sin kx}{kx} = 1.$$

It also follows that $\displaystyle\lim_{x\to 0} \frac{kx}{\sin kx} = 1$. Therefore

$$\lim_{x\to 0} \frac{\sin 3x}{\sin 2x} = \lim_{x\to 0} \frac{\sin 3x}{3x}\cdot\frac{2x}{\sin 2x}\cdot\frac{3x}{2x} = \lim_{x\to 0} \frac{\sin 3x}{3x}\cdot\frac{2x}{\sin 2x}\cdot\frac{3}{2} = 1\cdot 1\cdot\frac{3}{2} = \frac{3}{2}.$$

C02S0M.034: We saw in the solution of Problem C02S03.014 that if k is a nonzero constant, then

$$\lim_{x\to 0} \frac{\tan kx}{kx} = 1 = \lim_{x\to 0} \frac{kx}{\tan kx}.$$

Therefore

$$\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x\to 0} \frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}\cdot\frac{2x}{3x} = \lim_{x\to 0} \frac{\tan 2x}{2x}\cdot\frac{3x}{\tan 3x}\cdot\frac{2}{3} = 1\cdot 1\cdot\frac{2}{3} = \frac{2}{3}.$$

C02S0M.035: Let $x = u^2$ where $u > 0$. Then $x \to 0^+$ is equivalent to $u \to 0^+$. Hence

$$\lim_{x\to 0^+} \frac{x}{\sin\sqrt{x}} = \lim_{u\to 0^+} \frac{u^2}{\sin u} = \lim_{u\to 0^+} u\cdot\frac{u}{\sin u} = 0\cdot 1 = 0.$$

C02S0M.036: First multiply numerator and denominator by the conjugate $1+\cos 3x$ of the numerator:

$$\begin{aligned}\frac{1-\cos 3x}{2x} &= \frac{(1-\cos 3x)(1+\cos 3x)}{2x(1+\cos 3x)} = \frac{1-\cos^2 3x}{2x(1+\cos 3x)} = \frac{\sin^2 3x}{2x(1+\cos 3x)}\\ &= \frac{\sin 3x}{2x}\cdot\frac{\sin 3x}{1+\cos 3x} = \frac{\sin 3x}{3x}\cdot\frac{3x}{2x}\cdot\frac{\sin 3x}{1+\cos 3x} = \frac{\sin 3x}{3x}\cdot\frac{3}{2}\cdot\frac{\sin 3x}{1+\cos 3x}.\end{aligned}$$

Now let $x \to 0$ to obtain the limit $1 \cdot \dfrac{3}{2} \cdot \dfrac{0}{1+1} = 0$.

C02S0M.037: First multiply numerator and denominator by the conjugate $1 + \cos 3x$ of the numerator:

$$\begin{aligned}
\frac{1-\cos 3x}{2x^2} &= \frac{(1-\cos 3x)(1+\cos 3x)}{2x^2(1+\cos 3x)} = \frac{1-\cos^2 3x}{2x^2(1+\cos 3x)} = \frac{\sin^2 3x}{2x^2(1+\cos 3x)} \\
&= \frac{\sin 3x}{2x} \cdot \frac{\sin 3x}{x} \cdot \frac{1}{1+\cos 3x} = \frac{\sin 3x}{3x} \cdot \frac{3x}{2x} \cdot \frac{\sin 3x}{3x} \cdot \frac{3x}{x} \cdot \frac{1}{1+\cos 3x} \\
&= \frac{\sin 3x}{3x} \cdot \frac{3}{2} \cdot \frac{\sin 3x}{3x} \cdot \frac{3}{1} \cdot \frac{1}{1+\cos 3x}.
\end{aligned}$$

Now let $x \to 0$ to obtain the limit $1 \cdot \dfrac{3}{2} \cdot 1 \cdot \dfrac{3}{1} \cdot \dfrac{1}{1+1} = \dfrac{9}{4}$.

C02S0M.038: Express the cotangent and cosecant functions in terms of the sine and cosine functions to obtain

$$\lim_{x\to 0} x^3 \cot x \csc x = \lim_{x\to 0} (x^3) \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = \lim_{x\to 0} x \cdot \frac{x}{\sin x} \cdot (\cos x) \cdot \frac{x}{\sin x} = 0 \cdot 1 \cdot 1 \cdot 1 = 0.$$

C02S0M.039: Let $u = 2x$; then $x = \frac{1}{2}u$, and $x \to 0$ is then equivalent to $u \to 0$. Also express the secant and tangent functions in terms of the sine and cosine functions. Result:

$$\lim_{x\to 0} \frac{\sec 2x \tan 2x}{x} = \lim_{u\to 0} \frac{\sec u \tan u}{\frac{1}{2}u} = \lim_{u\to 0} \frac{2\sin u}{u\cos^2 u} = \lim_{u\to 0} \frac{2}{\cos^2 u} \cdot \frac{\sin u}{u} = \frac{2}{1} \cdot 1 = 2.$$

C02S0M.040: Let $u = 3x$; then $x = \frac{1}{3}u$, and $x \to 0$ is then equivalent to $u \to 0$. Also express the cotangent function in terms of sines and cosines. Result:

$$\lim_{x\to 0} x^2 \cot^2 3x = \lim_{x\to 0} \frac{x^2 \cos^2 3x}{\sin^2 3x} = \lim_{u\to 0} \frac{\frac{1}{9}u^2 \cos^2 u}{\sin^2 u} = \lim_{u\to 0} \frac{\cos^2 u}{9} \cdot \frac{u}{\sin u} \cdot \frac{u}{\sin u} = \frac{1}{9} \cdot 1 \cdot 1 = \frac{1}{9}.$$

C02S0M.041: Given $f(x) = 2x^2 + 3$, a slope-predictor for f is $m(x) = 4x$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, 5)$ is therefore $m(1) = 4$. So an equation of that line is $y - 5 = 4(x-1)$; that is, $y = 4x + 1$.

C02S0M.042: Given $f(x) = -5x^2 + x$, a slope-predictor for f is $m(x) = -10x + 1$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, -4)$ is therefore $m(1) = -9$. So an equation of that line is $y + 4 = -9(x-1)$; that is, $y = -9x + 5$.

C02S0M.043: Given $f(x) = 3x^2 + 4x - 5$, a slope-predictor for f is $m(x) = 6x + 4$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, 2)$ is therefore $m(1) = 10$. So an equation of that line is $y - 2 = 10(x-1)$; that is, $y = 10x - 8$.

C02S0M.044: Given $f(x) = -3x^2 - 2x + 1$, a slope-predictor for f is $m(x) = -6x - 2$. The slope of the line tangent to the graph of f at $(1, f(1)) = (1, -4)$ is therefore $m(1) = -8$. So an equation of that line is $y + 4 = -8(x-1)$; that is, $y = -8x + 4$.

C02S0M.045: Given $f(x) = (x-1)(2x-1) = 2x^2 - 3x + 1$, a slope-predictor for f is $m(x) = 4x - 3$. The slope of the line tangent to the graph of f at $(1,\ f(1)) = (1,\ 0)$ is therefore $m(1) = 1$. So an equation of that line is $y = x - 1$.

C02S0M.046: Given $f(x) = \frac{1}{3}x - \left(\frac{1}{4}x\right)^2 = -\frac{1}{16}x^2 + \frac{1}{3}x$, a slope-predictor for f is $m(x) = -\frac{1}{8}x + \frac{1}{3}$. The slope of the line tangent to the graph of f at $(1,\ f(1)) = \left(1,\ \frac{13}{48}\right)$ is therefore $m(1) = -\frac{1}{8} + \frac{1}{3} = \frac{5}{24}$. So an equation of that line is $y - \frac{13}{48} = \frac{5}{24}(x-1)$; that is, $48y = 10x + 3$.

C02S0M.047: If $f(x) = 2x^2 + 3x$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 + 3(x+h) - (2x^2 + 3x)}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 3x + 3h - 2x^2 - 3x}{h} = \lim_{h\to 0} \frac{4xh + 2h^2 + 3h}{h} = \lim_{h\to 0} \frac{h(4x + 2h + 3)}{h}$$

$$= \lim_{h\to 0} (4x + 2h + 3) = 4x + 3.$$

C02S0M.048: If $f(x) = x - x^3$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h) - (x+h)^3 - (x - x^3)}{h}$$

$$= \lim_{h\to 0} \frac{x + h - (x^3 + 3x^2h + 3xh^2 + h^3) - x + x^3}{h} = \lim_{h\to 0} \frac{x + h - x^3 - 3x^2h - 3xh^2 - h^3 - x + x^3}{h}$$

$$= \lim_{h\to 0} \frac{h - 3x^2h - 3xh^2 - h^3}{h} = \lim_{h\to 0} \frac{h(1 - 3x^2 - 3xh - h^2)}{h}$$

$$= \lim_{h\to 0} (1 - 3x^2 - 3xh - h^2) = 1 - 3x^2.$$

C02S0M.049: If $f(x) = \dfrac{1}{3-x}$, then the slope-predicting function for f is

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{3-(x+h)} - \dfrac{1}{3-x}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{(3-x) - (3-x-h)}{(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{3 - x - 3 + x + h}{(3-x-h)(3-x)} = \lim_{h\to 0} \frac{h}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{(3-x-h)(3-x)} = \frac{1}{(3-x)^2}.$$

C02S0M.050: If $f(x) = \dfrac{1}{2x+1}$, then the slope-predicting function for f is

$$m(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{2(x+h)+1}-\dfrac{1}{2x+1}}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(2x+1)-(2x+2h+1)}{(2x+2h+1)(2x+1)}$$

$$=\lim_{h\to 0}\frac{2x+1-2x-2h-1}{h(2x+2h+1)(2x+1)}=\lim_{h\to 0}\frac{-2h}{h(2x+2h+1)(2x+1)}$$

$$=\lim_{h\to 0}\frac{-2}{(2x+2h+1)(2x+1)}=-\frac{2}{(2x+1)^2}.$$

C02S0M.051: If $f(x)=x-\dfrac{1}{x}$, then the slope-predicting function for f is

$$m(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)-\dfrac{1}{x+h}-\left(x-\dfrac{1}{x}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\left(x+h-\frac{1}{x+h}-x+\frac{1}{x}\right)=\lim_{h\to 0}\frac{1}{h}\cdot\left(h+\frac{1}{x}-\frac{1}{x+h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\left(h+\frac{x+h-x}{x(x+h)}\right)=\lim_{h\to 0}\left(1+\frac{h}{hx(x+h)}\right)$$

$$=\lim_{h\to 0}\left(1+\frac{1}{x(x+h)}\right)=1+\frac{1}{x^2}=\frac{x^2+1}{x^2}.$$

C02S0M.052: If $f(x)=\dfrac{x}{x+1}$, then the slope-predicting function for f is

$$m(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+h}{x+h+1}-\frac{x}{x+1}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x+1)-(x+h+1)(x)}{(x+h+1)(x+1)}=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(x^2+x+hx+h)-(x^2+xh+x)}{(x+h+1)(x+1)}$$

$$=\lim_{h\to 0}\frac{x^2+x+hx+h-x^2-xh-x}{h(x+h+1)(x+1)}=\lim_{h\to 0}\frac{h}{h(x+h+1)(x+1)}$$

$$=\lim_{h\to 0}\frac{1}{(x+h+1)(x+1)}=\frac{1}{(x+1)^2}.$$

C02S0M.053: If $f(x)=\dfrac{x+1}{x-1}$, then the slope-predicting function for f is

$$m(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h+1)(x-1) - (x+h-1)(x+1)}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{x^2 + hx - h - 1 - x^2 - hx - h + 1}{h(x+h-1)(x-1)} = \lim_{h \to 0} \frac{-2h}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(x+h-1)(x-1)} = -\frac{2}{(x-1)^2}.$$

C02S0M.054: We must deal with $|2x+3|$, and to do so we need to know when $2x+3$ changes sign: When $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If $x > -\frac{3}{2}$, then $2x + 3 > 0$, so that

$$f(x) = 3x - x^2 + (2x+3) = -x^2 + 5x + 3 \qquad \text{if} \qquad x > -\tfrac{3}{2}.$$

By the theorem on page 58 (Section 2.1), the slope-predicting function for f will be $m_1(x) = -2x + 5$ if $x > -\frac{3}{2}$. But if $x < -\frac{3}{2}$, then $2x + 3 < 0$, so that

$$f(x) = 3x - x^2 - (2x+3) = -x^2 + x - 3 \qquad \text{if} \qquad x < -\tfrac{3}{2}.$$

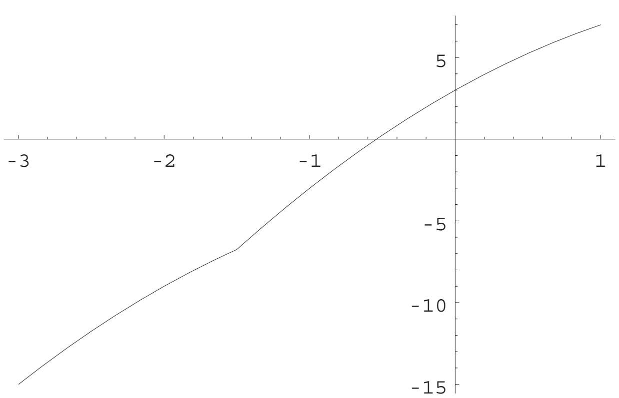
By the theorem just cited, the slope-predicting function for f will be $m_2(x) = -2x+1$ if $x < -\frac{3}{2}$. Therefore the general slope-predicting function for f will be

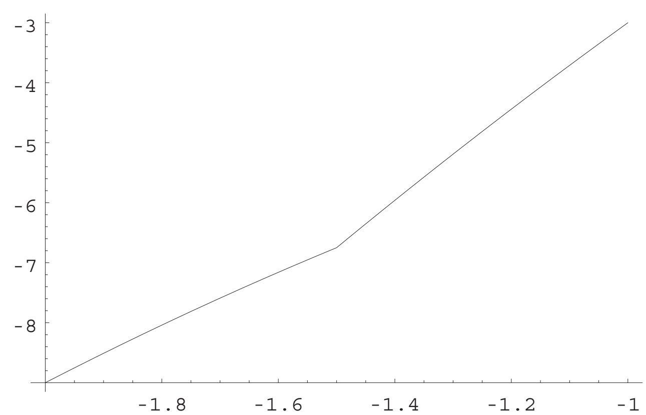
$$m(x) = \begin{cases} -2x + 5 & \text{if } x > -\frac{3}{2}, \\ -2x + 1 & \text{if } x < -\frac{3}{2}. \end{cases}$$

There will be no tangent line at $x = -\frac{3}{2}$. The reason is that

$$\lim_{x \to -1.5^-} \frac{f(x+h) - f(x)}{h} = 4 \quad \text{whereas} \quad \lim_{x \to -1.5^+} \frac{f(x+h) - f(x)}{h} = 8.$$

Therefore there is no tangent line at the point $\left(-\frac{3}{2}, -\frac{27}{4}\right)$. But f is continuous at that point; indeed, f is continuous on the set $\boldsymbol{R}$ of all real numbers. The graph of f is shown next on the left; the part of the graph near the corner point at $\left(-\frac{3}{2}, -\frac{27}{4}\right)$ is shown magnified on the right.





C02S0M.055: Following the suggestion, the line tangent to the graph of $y = x^2$ at (a, a^2) has slope $2a$ (because the slope-predicting function for $f(x) = x^2$ is $m(x) = 2x$). But using the two-point formula for slope, this line also has slope

$$\frac{a^2 - 4}{a - 3} = 2a,$$

so that $a^2 - 4 = 2a^2 - 6a$; that is, $a^2 - 6a + 4 = 0$. The quadratic formula yields the two solutions $a = 3 \pm \sqrt{5}$, so one of the two lines in question has slope $2(3 + \sqrt{5})$ and the other has slope $2(3 - \sqrt{5})$. Both lines pass through $(3, 4)$, so their equations are

$$y - 4 = 2\left(3 + \sqrt{5}\right)(x - 3) \quad \text{and} \quad y - 4 = 2\left(3 - \sqrt{5}\right)(x - 3).$$

C02S0M.056: The given line has equation $y = -x - 3$, so its slope is -1. The radius of the circle from its center $(2, 3)$ to the point (a, b) of tangency is perpendicular to that line, so has slope 1. So the radius lies on the line $y - 3 = x - 2$; that is, $y = x + 1$. We solve $y = x + 1$ and $y = -x - 3$ simultaneously to find the point of tangency (a, b) to be $(-2, -1)$. The distance from the center of the circle to this point is $4\sqrt{2}$. Therefore an equation of the circle is $(x - 2)^2 + (y - 3)^2 = 32$.

C02S0M.057: First simplify $f(x)$:

$$f(x) = \frac{1 - x}{1 - x^2} = \frac{1 - x}{(1 + x)(1 - x)} = \frac{1}{1 + x} \tag{1}$$

if $x \neq 1$. Every rational function is continuous wherever it is defined, so f is continuous except at ± 1. The computations in (1) show that $f(x)$ has no limit as $x \to -1$, so f cannot be made continuous at $x = -1$. But the discontinuity at $x = 1$ is removable; if we redefine f at $x = 1$ to be its limit $\frac{1}{2}$ there, then f will be continuous there as well.

C02S0M.058: Every rational function is continuous where it is defined; that is, where its denominator is nonzero. So

$$f(x) = \frac{1 - x}{(2 - x)^2}$$

is continuous except at $x = 2$. This discontinuity is not removable because $f(x)$ has no limit at $x = 2$.

C02S0M.059: First simplify $f(x)$:

$$f(x) = \frac{x^2 + x - 2}{x^2 + 2x - 3} = \frac{(x - 1)(x + 2)}{(x - 1)(x + 3)} = \frac{x + 2}{x + 3} \tag{1}$$

provided that $x \neq 1$. Note that f is a rational function, so f is continuous wherever it is defined: at every number other than 1 and -3. The computations in (1) show that $f(x)$ has no limit at $x = -3$, so it cannot be redefined in such a way to be continuous there. But the discontinuity at $x = 1$ is removable; if we redefine f at $x = 1$ to be its limit $\frac{3}{4}$ there, then f will be continuous everywhere except at $x = -3$.

C02S0M.060: Note that $f(x) = 1$ if $x^2 > 1$; that is, if $x > 1$ or $x < -1$. But if $x^2 < 1$, so that $-1 < x < 1$, then $f(x) = -1$. Hence f is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$. But f cannot be made continuous at either $x = 1$ or $x = -1$, because its left-hand and right-hand limits are unequal at each of these points. In any case, f is continuous wherever it is defined.

C02S0M.061: Let $f(x) = x^5 + x - 1$. Then $f(0) = -1 < 0 < 1 = f(1)$. Because $f(x)$ is a polynomial, it is continuous on $[0, 1]$, so f has the intermediate value property there. Hence there exists a number c in $(0, 1)$ such that $f(c) = 0$. Thus $c^5 + c - 1 = 0$, and so the equation $x^5 + x - 1 = 0$ has a solution. (The value of c is approximately 0.754877666.)

C02S0M.062: Let $f(x) = x^5 - 4x^2 + 1$. Here are some values of $f(x)$:

x	-1	0	1	2
$f(x)$	-4	1	-2	17

Because $f(x)$ is a polynomial, it is continuous everywhere. Therefore $f(x_1) = 0$ for some number x_1 in $(-1, 0)$, $f(x_2) = 0$ for some number x_2 in $(0, 1)$, and $f(x_3) = 0$ for some number x_3 in $(1, 2)$. The numbers x_1, x_2, and x_3 are distinct because they lie in nonoverlapping intervals. Therefore the equation $x^5 - 4x^2 + 1 = 0$ has at least three real solutions. (The actual values are $x_1 \approx 0.50842209$, $x_2 \approx 1.52864292$, and $x_3 \approx -0.49268877$.)

C02S0M.063: Let $g(x) = x - \cos x$. Then $g(0) = -1 < 0 < \pi/2 = g(\pi/2)$. Because g is continuous, $g(c) = 0$ for some number c in $(0, \pi/2)$. That is, $c - \cos c = 0$, so that $c = \cos c$.

C02S0M.064: Let $h(x) = x + \tan x$. Then $h(\pi) = \pi > 0$ and $h(x) \to -\infty$ as x approaches $\pi/2$ from above (from the right). This implies that $h(r) < 0$ for some number r slightly larger than $\pi/2$. Because h is continuous on the interval $[r, \pi]$, h has the intermediate value property there, so $h(c) = 0$ for some number c between r and π, and thus between $\pi/2$ and π. That is, $c + \tan c = 0$, so that $\tan c = -c$, and c does lie in the required interval $\left(\frac{1}{2}\pi, \pi\right)$.

C02S0M.065: Suppose that L is a straight line through $\left(12, \frac{15}{2}\right)$ that is normal to the graph of $y = x^2$ at the point (a, a^2). The line tangent to the graph of $y = x^2$ at that point has slope $2a$, and the slope of L is then $-1/(2a)$. We can equate this to the slope of L found by using the two-point formula:

$$\frac{a^2 - \frac{15}{2}}{a - 12} = -\frac{1}{2a};$$

$$2a\left(a^2 - \tfrac{15}{2}\right) = -(a - 12);$$

$$2a^3 - 15a = -a + 12;$$

$$2a^3 - 14a - 12 = 0;$$

$$a^3 - 7a - 6 = 0.$$

By inspection, one solution of the last equation is $a = -1$. By the factor theorem of algebra, we know that $a - (-1) = a + 1$ is a factor of the polynomial $a^3 - 7a - 6$, and division of the former into the latter yields

$$a^3 - 7a - 6 = (a + 1)(a^2 - a - 6) = (a + 1)(a - 3)(a + 2).$$

So the equation $a^3 - 7a - 6 = 0$ has the three solutions $a = -1$, $a = 3$, and $a = -2$. Therefore there are *three* lines through $\left(12, \frac{15}{2}\right)$ that are normal to the graph of $y = x^2$, and their slopes are $\frac{1}{4}$, $\frac{1}{2}$, and $-\frac{1}{6}$.

C02S0M.066: Let $(0, c)$ be the center of a too-big circle, r its radius, and (a, a^2) the point in the first quadrant where the too-big circle and the parabola are tangent. The idea is to solve for a in terms of c and (possibly) r, then to impose the condition that there is *exactly one* solution for a! This means that the circle just reaches to the bottom of the parabola and not beyond.

Consider the radius of the circle connecting $(0,\, c)$ with $(a,\, a^2)$. The circle and the parabola are mutually tangent at $(a,\, a^2)$, so this radius must be normal not only to the circle, but also to the parabola at the point $(a,\, a^2)$. We compute the slope of this radius in two ways to find that

$$\frac{a^2 - c}{a - 0} = -\frac{1}{2a};$$

$$a^2 - c = -\frac{1}{2};$$

$$a^2 = c - \frac{1}{2}.$$

Now we impose the condition that there is only one point at which the circle and the parabola meet. The last equation will have exactly one solution when $c = \frac{1}{2}$, and in this case the radius of the circle—because it touches the parabola only at $(0, 0)$—will also be $r = \frac{1}{2}$. Answer: $\frac{1}{2}$.

Section 3.1

C03S01.001: Given $f(x) = 4x - 5$, we have $a = 0$, $b = 4$, and $c = -5$, so $f'(x) = 2ax + b = 4$.

C03S01.002: Given $g(t) = -16t^2 + 100$, we have $a = -16$, $b = 0$, and $c = 100$, so $g'(t) = 2at + b = -32t$.

C03S01.003: If $h(z) = z(25-z) = -z^2+25z$, then $a = -1$, $b = 25$, and $c = 0$, so $h'(z) = 2az+b = -2z+25$.

C03S01.004: If $f(x) = -49x + 16$, then $a = 0$, $b = -49$, and $c = 16$, so $f'(x) = -49$.

C03S01.005: If $y = 2x^2 + 3x - 17$, then $a = 2$, $b = 3$, and $c = -17$, so $\dfrac{dy}{dx} = 2ax + b = 4x + 3$.

C03S01.006: If $x = -100t^2 + 16t$, then $a = -100$, $b = 16$, and $c = 0$, so $\dfrac{dx}{dt} = 2at + b = -200t + 16$.

C03S01.007: If $z = 5u^2 - 3u$, then $a = 5$, $b = -3$, and $c = 0$, so $\dfrac{dz}{du} = 2au + b = 10u - 3$.

C03S01.008: If $v = -5y^2 + 500y$, then $a = -5$, $b = 500$, and $c = 0$, so $\dfrac{dv}{dy} = 2ay + b = -10y + 500$.

C03S01.009: If $x = -5y^2 + 17y + 300$, then $a = -5$, $b = 17$, and $c = 300$, so $\dfrac{dx}{dy} = 2ay + b = -10y + 17$.

C03S01.010: If $u = 7t^2 + 13t$, then $a = 7$, $b = 13$, and $c = 0$, so $\dfrac{du}{dt} = 2at + b = 14t + 13$.

C03S01.011: $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h) - 1 - (2x-1)}{h} = \lim_{h\to 0} \frac{2x + 2h - 1 - 2x + 1}{h}$

$= \displaystyle\lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$

C03S01.012: $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2 - 3(x+h) - (2-3x)}{h} = \lim_{h\to 0} \frac{2 - 3x - 3h - 2 + 3x}{h}$

$= \displaystyle\lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} (-3) = -3.$

C03S01.013: $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 5 - (x^2+5)}{h}$

$= \displaystyle\lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x.$

C03S01.014: $f(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3 - 2(x+h)^2 - (3 - 2x^2)}{h}$

$= \displaystyle\lim_{h\to 0} \frac{3 - 2x^2 - 4xh - 2h^2 - 3 + 2x^2}{h} = \lim_{h\to 0} \frac{-4xh - 2h^2}{h} = \lim_{h\to 0} (-4x - 2h) = -4x.$

C03S01.015: $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)+1} - \dfrac{1}{2x+1}}{h}$

$= \displaystyle\lim_{h\to 0} \frac{2x + 1 - (2x + 2h + 1)}{h(2x+2h+1)(2x+1)} = \lim_{h\to 0} \frac{2x + 1 - 2x - 2h - 1}{h(2x+2h+1)(2x+1)} = \lim_{h\to 0} \frac{-2h}{h(2x+2h+1)(2x+1)}$

$$= \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)} = \frac{-2}{(2x+1)^2}.$$

C03S01.016: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{3-(x+h)} - \dfrac{1}{3-x}}{h}$

$$= \lim_{h\to 0} \frac{(3-x)-(3-x-h)}{h(3-x-h)(3-x)} = \lim_{h\to 0} \frac{3-x-3+x+h}{h(3-x-h)(3-x)} = \lim_{h\to 0} \frac{h}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{(3-x-h)(3-x)} = \frac{1}{(3-x)^2}.$$

C03S01.017: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2x+2h+1} - \sqrt{2x+1}\right)\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)} = \lim_{h\to 0} \frac{(2h+2h+1)-(2x+1)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

C03S01.018: $f'(x) = \lim_{h\to 0} \dfrac{1}{h}\left(f(x+h)-f(x)\right) = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h+1}} - \dfrac{1}{\sqrt{x+1}}\right)$

$$= \lim_{h\to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} = \lim_{h\to 0} \frac{\left(\sqrt{x+1} - \sqrt{x+h+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)} = \frac{-1}{\left(\sqrt{x+1}\right)^2\left(2\sqrt{x+1}\right)} = -\frac{1}{2(x+1)^{3/2}}.$$

C03S01.019: $f'(x) = \lim_{h\to 0} \dfrac{1}{h}\left(f(x+h)-f(x)\right) = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{x+h}{1-2(x+h)} - \dfrac{x}{1-2x}\right)$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x+h)(1-2x)-(1-2x-2h)(x)}{(1-2x-2h)(1-2x)} = \lim_{h\to 0} \frac{(x-2x^2+h-2xh)-(x-2x^2-2xh)}{h(1-2x-2h)(1-2x)}$$

$$= \lim_{h\to 0} \frac{x-2x^2+h-2xh-x+2x^2+2xh}{h(1-2x-2h)(1-2x)} = \lim_{h\to 0} \frac{h}{h(1-2x-2h)(1-2x)}$$

$$= \lim_{h\to 0} \frac{1}{(1-2x-2h)(1-2x)} = \frac{1}{(1-2x)^2}.$$

C03S01.020: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{x+h+1}{x+h-1} - \dfrac{x+1}{x-1}\right)$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(x+h+1)(x-1)-(x+h-1)(x+1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{(x^2-x+hx-h+x-1)-(x^2+x+hx+h-x-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{x^2 - x + hx - h + x - 1 - x^2 - x - hx - h + x + 1}{h(x+h-1)(x-1)} = \lim_{h \to 0} \frac{-2h}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-2}{(x+h-1)(x-1)} = -\frac{2}{(x-1)^2}.$$

C03S01.021: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t$, so $v(t) = 0$ when $t = 0$. The position of the particle then is $x(0) = 100$.

C03S01.022: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t + 160$, so $v(t) = 0$ when $t = 5$. The position of the particle then is $x(5) = 425$.

C03S01.023: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t + 80$, so $v(t) = 0$ when $t = 2.5$. The position of the particle then is $x(2.5) = 99$.

C03S01.024: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = 200t$, so $v(t) = 0$ when $t = 0$. The position of the particle then is $x(0) = 50$.

C03S01.025: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -20 - 10t$, so $v(t) = 0$ when $t = -2$. The position of the particle then is $x(-2) = 120$.

C03S01.026: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 160$ is zero, and $v(t) = 0$ when $t = 5$. The height of the ball then is $y(5) = 400$ (ft).

C03S01.027: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 64$ is zero, and $v(t) = 0$ when $t = 2$. The height of the ball then is $y(2) = 64$ (ft).

C03S01.028: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 128$ is zero, and $v(t) = 0$ when $t = 4$. The height of the ball then is $y(4) = 281$ (ft).

C03S01.029: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 96$ is zero, and $v(t) = 0$ when $t = 3$. The height of the ball then is $y(3) = 194$ (ft).

C03S01.030: Figure 3.1.22 shows a graph first increasing, then with a horizontal tangent at $x = 0$, then decreasing. Hence its derivative must be first positive, then zero when $x = 0$, then negative. This matches Fig. 3.1.28(c).

C03S01.031: Figure 3.1.23 shows a graph first decreasing, then with a horizontal tangent where $x = 1$, then increasing thereafter. So its derivative must be negative for $x < 1$, zero when $x = 1$, and positive for $x > 1$. This matches Fig. 3.1.28(e).

C03S01.032: Figure 3.1.24 shows a graph increasing for $x < -1.5$, decreasing for $-1.5 < x < 1.5$, and increasing for $1.5 < x$. So its derivative must be positive for $x < -1.5$, negative for $-1.5 < x < 1.5$, and positive for $1.5 < x$. This matches Fig. 3.1.28(b).

C03S01.033: Figure 3.1.25 shows a graph decreasing for $x < -1.5$, increasing for $-1.5 < x < 0$, decreasing for $0 < 1 < 1.5$, and increasing for $1.5 < x$. Hence its derivative is negative for $x < -1.5$, positive for

$-1.5 < x < 0$, negative for $0 < x < 1.5$, and positive for $1.5 < x$. Only the graph in Fig. 3.1.28(f) shows these characteristics.

C03S01.034: Figure 3.1.26 shows a graph with horizontal tangents near where $x = -3$, $x = 0$, and $x = 3$. So the graph of the derivative must be zero near these three points, and this behavior is matched by Fig. 3.1.28(a).

C03S01.035: Figure 3.1.27 shows a graph that increases, first slowly, then rapidly. So its derivative must exhibit the same behavior, and thus its graph is the one shown in Fig. 3.1.28(d).

C03S01.036: Note that

$$C(F) = \frac{5}{9}F - \frac{160}{9} \qquad \text{and so} \qquad F(C) = \frac{9}{5}C + 32.$$

So the rate of change of C with respect to F is

$$C'(F) = \frac{dC}{dF} = \frac{5}{9}$$

and the rate of change of F with respect to C is

$$F'(C) = \frac{dF}{dC} = \frac{9}{5}.$$

C03S01.037: Let r note the radius of the circle. Then $A = \pi r^2$ and $C = 2\pi r$. Thus

$$r = \frac{C}{2\pi}, \qquad \text{and so} \qquad A(C) = \frac{1}{4\pi}C^2, \quad C > 0.$$

Therefore the rate of change of A with respect to C is

$$A'(C) = \frac{dA}{dC} = \frac{1}{2\pi}C.$$

C03S01.038: Let r denote the radius of the circular ripple in feet at time t (seconds). Then $r = 5t$, and the area within the ripple at time t is $A = \pi r^2 = 25\pi t^2$. The rate at which this area is increasing at time t is $A'(t) = 50\pi t$, so at time $t = 10$ the area is increasing at the rate of $A'(10) = 50\pi \cdot 10 = 500\pi$ (ft^2/s).

C03S01.039: The velocity of the car (in feet per second) at time t (seconds) is $v(t) = x'(t) = 100 - 10t$. The car comes to a stop when $v(t) = 0$; that is, when $t = 10$. At that time the car has traveled a distance $x(10) = 500$ (ft). So the car skids for 10 seconds and skids a distance of 500 ft.

C03S01.040: Because $V(t) = 10 - \frac{1}{5}t + \frac{1}{1000}t^2$, $V'(t) = -\frac{1}{5} + \frac{1}{500}t$ and so the rate at which the water is leaking out one minute later ($t = 60$) is $V'(60) = -\frac{2}{25}$ (gal/s); that is, -4.8 gal/min. The average rate of change of V from $t = 0$ until $t = 100$ is

$$\frac{V(100) - V(0)}{100 - 0} = \frac{0 - 10}{100} = -\frac{1}{10}.$$

The instantaneous rate of change of V will have this value when $V'(t) = -\frac{1}{10}$, which we easily solve for $t = 50$.

C03S01.041: First, $P(t) = 100 + 30t + 4t^2$. The initial population is 100, so doubling occurs when $P(t) = 200$; that is, when $4t^2 + 30t - 100 = 0$. The quadratic formula yields $t = 2.5$ as the only positive solution

of this equation, so the population will take two and one-half months to double. Because $P'(t) = 30 + 8t$, the rate of growth of the population when $P = 200$ will be $P'(2.5) = 50$ (chipmunks per month).

C03S01.042: In our construction, the tangent line at 1989 passes through the points $(1984, 259)$ and $(1994, 423)$, and so has slope 16.4; this yields a rate of growth of approximately 16.4 thousand per year in 1989. Alternatively, using the *Mathematica* function `Fit` to fit the given data to a sixth-degree polynomial, we find that P (in thousands) is given in terms of t (as a four-digit year) by

$$\begin{aligned}P(t) \approx &(1.0453588 \times 10^{-14})x^6 - (4.057899 \times 10^{-11})x^5 + (3.9377735 \times 10^{-8})x^4 \\ &+ (5.93932556 \times 10^{-11})x^3 + (5.972176 \times 10^{-14})x^2 + (5.939427 \times 10^{-17})x + (3.734218 \times 10^{-12}),\end{aligned}$$

and that $P'(1989) \approx 16.4214$. Of course neither method is exact.

C03S01.043: On our graph, the tangent line at the point $(20, 810)$ has slope $m_1 \approx 0.6$ and the tangent line at $(40, 2686)$ has slope $m_2 \approx 0.9$. A line of slope 1 on our graph corresponds to a velocity of 125 ft/s (because the line through $(0, 0)$ and $(10, 1250)$ has slope 1), and thus we estimate the velocity of the car at time $t = 20$ to be about $(0.6)(125) = 75$ ft/s, and at time $t = 40$ it is traveling at about $(0.9)(125) = 112.5$ ft/s. The method is crude; the answer in the back of the textbook is quite different simply because it was obtained by someone else. When we used the *Mathematica* function `Fit` to fit the data to a sixth-degree polynomial, we obtained

$$\begin{aligned}x(t) \approx 0.0000175721 &+ (6.500002)x + (1.112083)x^2 + (0.074188)x^3 \\ &- (0.00309375)x^4 + (0.0000481250)x^5 - (0.000000270834)x^6,\end{aligned}$$

which yields $x'(20) \approx 74.3083$ and $x'(40) \approx 109.167$. Of course neither method is exact.

C03S01.044: With volume V and edge x, the volume of the cube is given by $V(x) = x^3$. Now $\dfrac{dV}{dx} = 3x^2$, which is indeed half the total surface area $6x^2$ of the cube.

C03S01.045: With volume V and radius r, the volume of the sphere is $V(r) = \frac{4}{3}\pi r^3$. Then $\dfrac{dV}{dr} = 4\pi r^2$, and this is indeed the surface area of the sphere.

C03S01.046: A right circular cylinder of radius r and height h has volume $V = \pi r^2 h$ and total surface area S obtained by adding the areas of its top, bottom, and curved side: $S = 2\pi r^2 + 2\pi rh$. We are given $h = 2r$, so $V(r) = 2\pi r^3$ and $S(r) = 6\pi r^2$. Also $dV/dr = 6\pi r^2 = S(r)$, so the rate of change of volume with respect to radius is indeed equal to total surface area.

C03S01.047: We must compute dV/dt when $t = 30$; $V(r) = \frac{4}{3}\pi r^3$ is the volume of the balloon when its radius is r. We are given $r = \dfrac{60 - t}{12}$, and thus

$$V(t) = \frac{4}{3}\pi\left(\frac{60 - t}{12}\right)^3 = \frac{\pi}{1296}(216000 - 10800t + 180t^2 - t^3).$$

Therefore

$$\frac{dV}{dt} = \frac{\pi}{1296}(-10800 + 360t - 3t^2),$$

and so $V'(t) = -\dfrac{25\pi}{12}$ in.^{3}/s; that is, air is leaking out at approximately 6.545 in.^{3}/s.

C03S01.048: From $V(p) = \dfrac{1.68}{p}$ we derive $V'(p) = -\dfrac{1.68}{p^2}$. The rate of change of V with respect to p when $p = 2$ (atm) is then $V'(2) = -0.42$ (liters/atm).

C03S01.049: Let $V(t)$ denote the volume (in cm^3) of the snowball at time t (in hours) and let $r(t)$ denote its radius then. From the data given in the problem, $r = 12 - t$. The volume of the snowball is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(12-t)^3 = \frac{4}{3}\pi\left(1728 - 432t + 36t^2 - t^3\right),$$

so its instantaneous rate of change is

$$V'(t) = \frac{4}{3}\pi\left(-432 + 72t - 3t^2\right).$$

Hence its rate of change of volume when $t = 6$ is $V'(6) = -144\pi\ \text{cm}^3/\text{h}$. Its average rate of change of volume from $t = 3$ to $t = 9$ in cm^3/h is

$$\frac{V(9) - V(3)}{9 - 3} = \frac{36\pi - 972\pi}{6} = -156\pi\ (\text{cm}^3/\text{h}).$$

C03S01.050: The velocity of the ball at time t is $\dfrac{dy}{dt} = -32t + 96$, which is zero when $t = 3$. So the maximum height of the ball is $y(3) = 256$ (ft). It hits the ground when $y(t) = 0$; that is, when $-16t^2 + 96t + 112 = 0$. The only positive solution of this equation is $t = 7$, so the impact speed of the ball is $|y'(7)| = 128$ (ft/s).

C03S01.051: The spaceship hits the ground when $25t^2 - 100t + 100 = 0$, which has solution $t = 2$. The velocity of the spaceship at time t is $y'(t) = 50t - 100$, so the speed of the spaceship at impact is (fortunately) zero.

C03S01.052: Because $P(t) = 100 + 4t + \frac{3}{10}t^2$, we have $P'(t) = 4 + \frac{3}{5}t$. The year 1986 corresponds to $t = 6$, so the rate of change of P then was $P'(6) = 7.6$ (thousands per year). The average rate of change of P from 1983 ($t = 3$) to 1988 ($t = 8$) was

$$\frac{P(8) - P(3)}{8 - 3} = \frac{151.2 - 114.7}{5} = 7.3 \text{ (thousands per year).}$$

C03S01.053: The average rate of change of the population from January 1, 1990 to January 1, 2000 was

$$\frac{P(10) - P(0)}{10 - 0} = \frac{6}{10} = 0.6 \quad \text{(thousands per year).}$$

The instantaneous rate of change of the population (in thousands per year, again) at time t was

$$P'(t) = 1 - (0.2)t + (0.018)t^2.$$

Using the quadratic formula to solve the equation $P'(t) = 0.6$, we find two solutions:

$$t = \frac{50 - 10\sqrt{7}}{9} \approx 2.6158318766 \quad \text{and} \quad t = \frac{50 + 10\sqrt{7}}{9} \approx 8.4952792345.$$

These values of t correspond to August 12, 1992 and June 30, 1998, respectively.

C03S01.054: (a) If $f(x) = |x|$, then

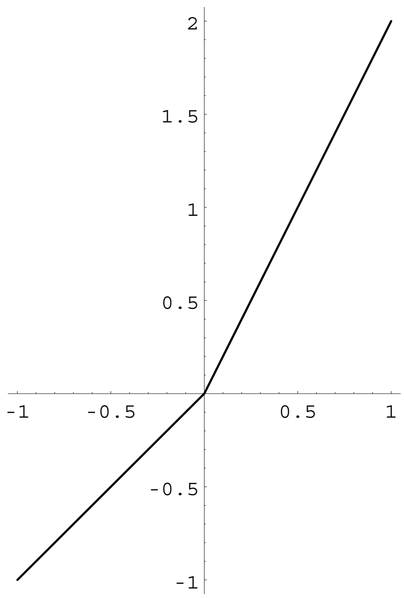
$$f'_-(0) = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1;$$

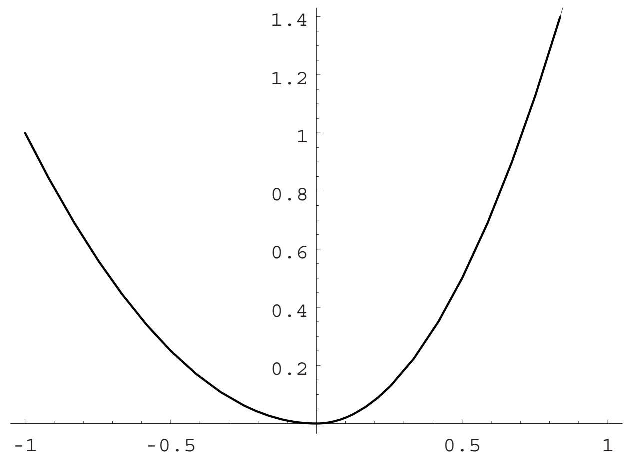
similarly, $f'_+(0) = 1$. (b) The function $f(x) = |2x - 10|$ is not differentiable at $x = 5$. Its right-hand derivative there is

$$f'_+(5) = \lim_{h\to 0^+} \frac{|2\cdot(5+h) - 10| - 0}{h} = \lim_{h\to 0^+} \frac{10 + 2h - 10}{h} = 2.$$

Similarly, $f'_-(5) = -2$.

C03S01.055: The graphs of the function of part (a) is shown next, on the left; the graph of the function of part (b) is on the right.





(a) $f'_-(0) = 1$ while $f'_+(0) = 2$. Hence f is not differentiable at $x = 0$. (b) In contrast,

$$f'_-(0) = \lim_{h\to 0^-} \frac{(0+h)^2 - 2\cdot 0^2}{h} = 0$$

and

$$f'_+(0) = \lim_{h\to 0^+} \frac{2\cdot(0+h)^2 - 2\cdot 0^2}{h} = 0;$$

therefore f is differentiable at $x = 0$ and $f'(0) = 0$.

C03S01.056: The function f is clearly differentiable except possibly at $x = 1$. But

$$f'_-(1) = \lim_{h\to 0^-} \frac{2\cdot(1+h) + 1 - 3}{h} = 2$$

and

$$f'_+(1) = \lim_{h\to 0^+} \frac{4\cdot(1+h) - (1+h)^2 - 3}{h} = \lim_{h\to 0^+} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h\to 0^+} (2-h) = 2.$$

Therefore f is differentiable at $x = 1$ as well.

C03S01.057: Clearly f is differentiable except possibly at $x = 3$. Moreover,

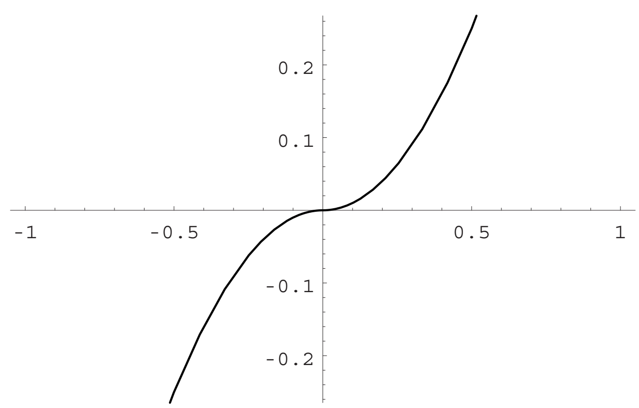
$$f'_-(3) = \lim_{h\to 0^-} \frac{11 + 6\cdot(3+h) - (3+h)^2 - 20}{h} = \lim_{h\to 0^-} \frac{11 + 18 + 6h - 9 - 6h - h^2 - 20}{h} = \lim_{h\to 0^-} \frac{-h^2}{h} = 0$$

and

$$f'_+(3) = \lim_{h\to 0^+} \frac{(3+h)^2 - 6\cdot(3+h) + 29 - 20}{h} = \lim_{h\to 0^+} \frac{9 + 6h + h^2 - 18 - 6h + 29 - 20}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0.$$

Therefore the function f is also differentiable at $x = 3$; moreover, $f'(3) = 0$.

C03S01.058: The graph of $f(x) = x\cdot|x|$ is shown next.



Because $f(x) = x^2$ if $x > 0$ and $f(x) = -x^2$ if $x < 0$, f is differentiable except possibly at $x = 0$. But

$$f'_-(0) = \lim_{h\to 0^-} \frac{(0+h)\cdot|0+h|}{h} = \lim_{h\to 0^-} \frac{-h^2}{h} = 0$$

and

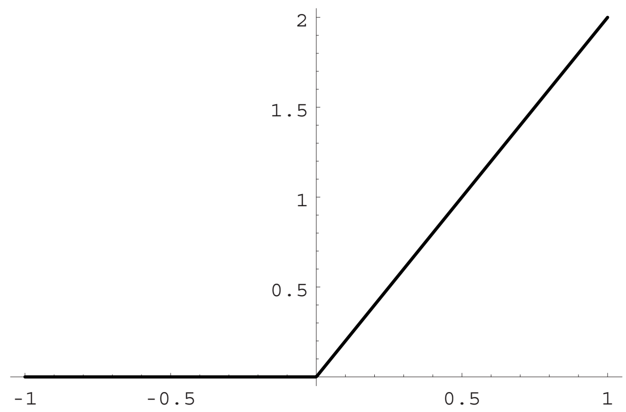
$$f'_+(0) = \lim_{h\to 0^+} \frac{(0+h)\cdot|0+h|}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0.$$

Therefore f is differentiable at $x = 0$ and $f'(0) = 0$. Because

$$f'(x) = \begin{cases} 2x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -2x & \text{if } x < 0, \end{cases}$$

we see that $f'(x) = 2|x|$ for all x.

C03S01.059: The graph of $f(x) = x + |x|$ is shown next.



Because $f(x) = 0$ if $x < 0$ and $f(x) = 2x$ if $x > 0$, clearly f is differentiable except possibly at $x = 0$. Next,

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{0 + h + |0 + h|}{h} = \lim_{h \to 0^-} \frac{h - h}{h} = 0,$$

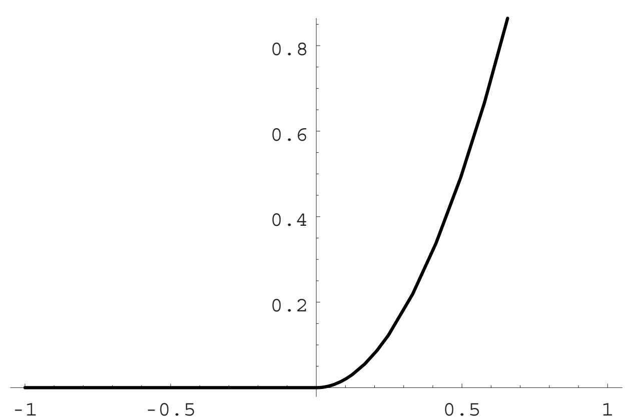
whereas

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{0 + h + |0 + h|}{h} = \lim_{h \to 0^+} \frac{2h}{h} = 2 \neq 0.$$

Therefore f is not differentiable at $x = 0$. In summary, $f'(x) = 2$ if $x > 0$ and $f'(x) = 0$ if $x < 0$. For a "single-formula" version of the derivative, consider

$$f'(x) = 1 + \frac{|x|}{x}.$$

C03S01.060: The graph of $f(x) = x \cdot (x + |x|)$, is shown next.



Because $f(x) = 2x^2$ if $x > 0$ and $f(x) = 0$ if $x < 0$, $f'(x)$ exists except possibly at $x = 0$. But

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{(0 + h) \cdot (0 + h + |0 + h|)}{h} = \lim_{h \to 0^-} \frac{h \cdot (h - h)}{h} = 0$$

and

$$f'_+(0) = \lim_{h\to 0^+} \frac{(0+h)\cdot(0+h+|0+h|)}{h} = \lim_{h\to 0^+} \frac{h\cdot(h+h)}{h} = 0.$$

Therefore f is differentiable at $x=0$ and $f'(0)=0$. Finally, $f'(x)=4x$ if $x>0$ and $f'(x)=0$ if $x<0$. For a "single-formula" version of the derivative, consider $f'(x) = 2\cdot(x+|x|)$.

Section 3.2

C03S02.001: Given: $f(x) = 3x^2 - x + 5$. We apply the rule for differentiating a linear combination and the power rule to obtain

$$f'(x) = 3D_x(x^2) - D_x(x) + D_x(5) = 3 \cdot 2x - 1 + 0 = 6x - 1.$$

C03S02.002: Given: $g(t) = 1 - 3t^2 - 2t^4$. We apply the rule for differentiating a linear combination and the power rule to obtain

$$g'(t) = D_t(1) - 3D_t(t^2) - 2D_t(t^4) = 0 - 3 \cdot 2t - 2 \cdot 4t^3 = -6t - 8t^3.$$

C03S02.003: Given: $f(x) = (2x+3)(3x-2)$. We apply the product rule to obtain

$$f'(x) = (2x+3)D_x(3x-2) + (3x-2)D_x(2x+3) = (2x+3) \cdot 3 + (3x-2) \cdot 2 = 12x + 5.$$

C03S02.004: Given: $g(x) = (2x^2 - 1)(x^3 + 2)$. We apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$g'(x) = (2x^2-1)D_x(x^3+2) + (x^3+2)D_x(2x^2-1) = (2x^2-1)(3x^2) + (x^3+2)(4x) = 10x^4 - 3x^2 + 8x.$$

C03S02.005: Given: $h(x) = (x+1)^3$. We rewrite $h(x)$ in the form

$$h(x) = (x+1)(x+1)(x+1)$$

and then apply the extended product rule in Eq. (16) to obtain

$$\begin{aligned} h'(x) &= (x+1)(x+1)D_x(x+1) + (x+1)(x+1)D_x(x+1) + (x+1)(x+1)D_x(x+1) \\ &= (x+1)(x+1)(1) + (x+1)(x+1)(1) + (x+1)(x+1)(1) = 3(x+1)^2. \end{aligned}$$

Alternatively, we could rewrite $h(x)$ in the form

$$h(x) = x^3 + 3x^2 + 3x + 1$$

and then apply the power rule and the rule for differentiating a linear combination to obtain

$$h'(x) = D_x(x^3) + 3D_x(x^2) + 3D_x(x) + D_x(1) = 3x^2 + 6x + 3.$$

The first method gives the answer in a more useful form because it is easier to determine where $h'(x)$ is positive, where negative, and where zero. Zeugma!

C03S02.006: Given: $g(t) = (4t-7)^2 = (4t-7) \cdot (4t-7)$. We apply the product rule and the rule for differentiating a linear combination to obtain

$$g'(t) = (4t-7)D_t(4t-7) + (4t-7)D_t(4t-7) = 4 \cdot (4t-7) + 4 \cdot (4t-7) = 8 \cdot (4t-7) = 32t - 56.$$

C03S02.007: Given: $f(y) = y(2y-1)(2y+1)$. We apply the extended product rule in Eq. (16) to obtain

$$
\begin{aligned}
f'(y) &= (2y-1)(2y+1)\,D_y(y) + y(2y+1)\,D_y(2y-1) + y(2y-1)\,D_y(2y+1)\\
&= (2y-1)(2y+1)\cdot 1 + y(2y+1)\cdot 2 + y(2y-1)\cdot 2 = 4y^2 - 1 + 4y^2 + 2y + 4y^2 - 2y = 12y^2 - 1.
\end{aligned}
$$

Alternatively, we could first expand: $f(y) = 4y^3 - y$. Then we could apply the rule for differentiating a linear combination and the power rule to obtain $f'(y) = 4\,D_y(y^3) - D_y(y) = 12y^2 - 1$.

C03S02.008: Given: $f(x) = 4x^4 - \dfrac{1}{x^2}$. We apply various rules, including the reciprocal rule, to obtain

$$
f'(x) = 4\,D_x(x^4) - \left(-\frac{D_x(x^2)}{(x^2)^2}\right) = 4\cdot 4x^3 + \frac{2x}{x^4} = 16x^3 + \frac{2}{x^3}.
$$

Alternatively, we could rewrite: $f(x) = 4x^4 - x^{-2}$. Then we could apply the rule for differentiating a linear combination and the power rule (both for positive and for negative integral exponents) to obtain

$$
f'(x) = 4\,D_x(x^4) - D_x(x^{-2}) = 4\cdot 4x^3 - (-2)x^{-3} = 16x^3 + 2x^{-3} = 16x^3 + \frac{2}{x^3}.
$$

C03S02.009: We apply the rule for differentiating a linear combination and the reciprocal rule (twice) to obtain

$$
\begin{aligned}
g'(x) &= D_x\!\left(\frac{1}{x+1}\right) - D_x\!\left(\frac{1}{x-1}\right)\\
&= -\frac{D_x(x+1)}{(x+1)^2} + \frac{D_x(x-1)}{(x-1)^2} = -\frac{1}{(x+1)^2} + \frac{1}{(x-1)^2}.
\end{aligned}
$$

Looking ahead to later sections and chapters—in which we will want to find where $g'(x)$ is positive, negative, or zero—it would be good practice to simplify $g'(x)$ to

$$
g'(x) = \frac{(x+1)^2 - (x-1)^2}{(x+1)^2(x-1)^2} = \frac{4x}{(x+1)^2(x-1)^2}.
$$

C03S02.010: We apply the reciprocal rule to $f(t) = \dfrac{1}{4-t^2}$ to obtain

$$
f'(t) = -\frac{D_t(4-t^2)}{(4-t^2)^2} = -\frac{-2t}{(4-t^2)^2} = \frac{2t}{(4-t^2)^2}.
$$

C03S02.011: First write (or think of) $h(x)$ as

$$
h(x) = 3\cdot\frac{1}{x^2+x+1},
$$

then apply the rule for differentiating a linear combination and the reciprocal rule to obtain

$$
h'(x) = 3\cdot\left(-\frac{D_x(x^2+x+1)}{(x^2+x+1)^2}\right) = \frac{-3\cdot(2x+1)}{(x^2+x+1)^2}.
$$

Alternatively apply the quotient rule directly to obtain

$$
h'(x) = \frac{(x^2+x+1)D_x(3) - 3\,D_x(x^2+x+1)}{(x^2+x+1)^2} = \frac{-3\cdot(2x+1)}{(x^2+x+1)^2}.
$$

C03S02.012: Multiply numerator and denominator in $f(x)$ by x to obtain

$$f(x) = \frac{1}{1 - \dfrac{2}{x}} = \frac{x}{x-2}.$$

Then apply the quotient rule to obtain

$$f'(x) = \frac{(x-2)\,D_x(x) - x\,D_x(x-2)}{(x-2)^2} = \frac{(x-2)\cdot 1 - x \cdot 1}{(x-2)^2} = \frac{-2}{(x-2)^2}.$$

C03S02.013: Given $g(t) = (t^2+1)(t^3+t^2+1)$, apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$\begin{aligned} g'(t) &= (t^2+1)\,D_t(t^3+t^2+1) + (t^3+t^2+1)\,D_t(t^2+1) = (t^2+1)(3t^2+2t+0) + (t^3+t^2+1)(2t+0) \\ &= (3t^4+2t^3+3t^2+2t) + (2t^4+2t^3+2t) = 5t^4+4t^3+3t^2+4t. \end{aligned}$$

Alternatively, first expand: $g(t) = t^5+t^4+t^3+2t^2+1$, then apply the rule for differentiating a linear combination and the power rule.

C03S02.014: Given $f(x) = (2x^3-3)(17x^4-6x+2)$, apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$\begin{aligned} f'(x) &= (2x^3-3)(68x^3-6) + (6x^2)(17x^4-6x+2) \\ &= (136x^6-216x^3+18) + (102x^6-36x^3+12x^2) = 238x^6-252x^3+12x^2+18. \end{aligned}$$

Alternatively, first expand $f(x)$, then apply the linear combination rule and the power rule.

C03S02.015: The easiest way to find $g'(z)$ is first to rewrite $g(z)$:

$$g(z) = \frac{1}{2z} - \frac{1}{3z^2} = \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}.$$

Then apply the linear combination rule and the power rule (for negative integral exponents) to obtain

$$g'(z) = \frac{1}{2}(-1)z^{-2} - \frac{1}{3}(-2)z^{-3} = -\frac{1}{2z^2} + \frac{2}{3z^3} = \frac{4-3z}{6z^3}.$$

The last step is advisable should it be necessary to find where $g'(z)$ is positive, where it is negative, and where it is zero. Hypozeuxis!

C03S02.016: The quotient rule yields

$$\begin{aligned} f'(x) &= \frac{x^2\,D_x(2x^3-3x^2+4x-5) - (2x^3-3x^2+4x-5)\,D_x(x^2)}{(x^2)^2} \\ &= \frac{(x^2)(6x^2-6x+4) - (2x^3-3x^2+4x-5)(2x)}{x^4} = \frac{(6x^4-6x^3+4x^2) - (4x^4-6x^3+8x^2-10x)}{x^4} \\ &= \frac{6x^4-6x^3+4x^2-4x^4+6x^3-8x^2+10x}{x^4} = \frac{2x^4-4x^2+10x}{x^4} = \frac{2x^3-4x+10}{x^3}. \end{aligned}$$

But if the objective is to obtain the correct answer as quickly as possible, regardless of its appearance, you could proceed as follows (using the linear combination rule and the power rule for negative integral exponents):

$$f(x) = 2x - 3 + 4x^{-1} - 5x^{-2}, \quad \text{so} \quad f'(x) = 2 - 4x^{-2} + 10x^{-3}.$$

C03S02.017: Apply the extended product rule in Eq. (16) to obtain

$$\begin{aligned}
g'(y) &= (3y^2 - 1)(y^2 + 2y + 3)\,D_y(2y) + (2y)(y^2 + 2y + 3)\,D_y(3y^2 - 1) + (2y)(3y^2 - 1)\,D_y(y^2 + 2y + 3) \\
&= (3y^2 - 1)(y^2 + 2y + 3)(2) + (2y)(y^2 + 2y + 3)(6y) + (2y)(3y^2 - 1)(2y + 2) \\
&= (6y^4 + 12y^3 + 18y^2 - 2y^2 - 4y - 6) + (12y^4 + 24y^3 + 36y^2) + (12y^4 - 4y^2 + 12y^3 - 4y) \\
&= 30y^4 + 48y^3 + 48y^2 - 8y - 6.
\end{aligned}$$

Or if you prefer, first expand $g(y)$, then apply the linear combination rule and the power rule to obtain

$$\begin{aligned}
&g(y) = (6y^3 - 2y)(y^2 + 2y + 3) = 6y^5 + 12y^4 + 16y^3 - 4y^2 - 6y, \quad \text{so} \\
&g'(y) = 30y^4 + 48y^3 + 48y^2 - 8y - 6.
\end{aligned}$$

C03S02.018: By the quotient rule,

$$f'(x) = \frac{(x^2 + 4)\,D_x(x^2 - 4) - (x^2 - 4)\,D_x(x^2 + 4)}{(x^2 + 4)^2} = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}.$$

C03S02.019: Apply the quotient rule to obtain

$$\begin{aligned}
g'(t) &= \frac{(t^2 + 2t + 1)\,D_t(t - 1) - (t - 1)\,D_t(t^2 + 2t + 1)}{(t^2 + 2t + 1)^2} = \frac{(t^2 + 2t + 1)(1) - (t - 1)(2t + 2)}{[\,(t + 1)^2]^2} \\
&= \frac{(t^2 + 2t + 1) - (2t^2 - 2)}{(t + 1)^4} = \frac{3 + 2t - t^2}{(t + 1)^4} = -\frac{(t + 1)(t - 3)}{(t + 1)^4} = \frac{3 - t}{(t + 1)^3}.
\end{aligned}$$

C03S02.020: Apply the reciprocal rule to obtain

$$u'(x) = -\frac{D_x(x^2 + 4x + 4)}{(x + 2)^4} = -\frac{2x + 4}{(x + 2)^4} = -\frac{2}{(x + 2)^3}.$$

C03S02.021: Apply the reciprocal rule to obtain

$$v'(t) = -\frac{D_t(t^3 - 3t^2 + 3t - 1)}{(t - 1)^6} = -\frac{3t^2 - 6t + 3}{(t - 1)^6} = -\frac{3(t - 1)^2}{(t - 1)^6} = -\frac{3}{(t - 1)^4}.$$

C03S02.022: The quotient rule yields

$$\begin{aligned}
h(x) &= \frac{(2x-5)\,D_x(2x^3+x^2-3x+17)-(2x^3+x^2-3x+17)\,D_x(2x-5)}{(2x-5)^2}\\
&= \frac{(2x-5)(6x^2+2x-3)-(2x^3+x^2-3x+17)(2)}{(2x-5)^2}\\
&= \frac{(12x^3-26x^2-16x+15)-(4x^3+2x^2-6x+34)}{(2x-5)^2}\\
&= \frac{12x^3-26x^2-16x+15-4x^3-2x^2+6x-34}{(2x-5)^2} = \frac{8x^3-28x^2-10x-19}{(2x-5)^2}.
\end{aligned}$$

C03S02.023: The quotient rule yields

$$g'(x) = \frac{(x^3+7x-5)(3)-(3x)(3x^2+7)}{(x^3+7x-5)^2} = \frac{3x^3+21x-15-9x^3-21x}{(x^3+7x-5)^2} = -\frac{6x^3+15}{(x^3+7x-5)^2}.$$

C03S02.024: First expand the denominator, then multiply numerator and denominator by t^2, to obtain

$$f(t) = \frac{1}{\left(t+\dfrac{1}{t}\right)^2} = \frac{1}{t^2+2+\dfrac{1}{t^2}} = \frac{t^2}{t^4+2t^2+1}.$$

Then apply the quotient rule to obtain

$$f'(t) = \frac{(t^4+2t^2+1)(2t)-(t^2)(4t^3+4t)}{\left[\,(t^2+1)^2\right]^2} = \frac{2t^5+4t^3+2t-4t^5-4t^3}{(t^2+1)^4} = \frac{2t-2t^5}{(t^2+1)^4}.$$

A modest simplification is possible:

$$f'(t) = -\frac{2t(t^4-1)}{(t^2+1)^4} = -\frac{2t(t^2+1)(t^2-1)}{(t^2+1)^4} = -\frac{2t(t^2-1)}{(t^2+1)^3}.$$

C03S02.025: First multiply each term in numerator and denominator by x^4 to obtain

$$g(x) = \frac{x^3-2x^2}{2x-3}.$$

Then apply the quotient rule to obtain

$$g'(x) = \frac{(2x-3)(3x^2-4x)-(x^3-2x^2)(2)}{(2x-3)^2} = \frac{(6x^3-17x^2+12x)-(2x^3-4x^2)}{(2x-3)^2} = \frac{4x^3-13x^2+12x}{(2x-3)^2}.$$

It is usually wise to simplify an expression before differentiating it.

C03S02.026: First multiply each term in numerator and denominator by x^2+1 to obtain

$$f(x) = \frac{x^3(x^2+1)-1}{x^4(x^2+1)+1} = \frac{x^5+x^3-1}{x^6+x^4+1}.$$

Then apply the quotient rule to obtain

$$
\begin{aligned}
f'(x) &= \frac{(x^6+x^4+1)(5x^4+3x^2)-(x^5+x^3-1)(6x^5+4x^3)}{(x^6+x^4+1)^2}\\
&= \frac{(5x^{10}+8x^8+3x^6+5x^4+3x^2)-(6x^{10}+10x^8+4x^6-6x^5-4x^3)}{(x^6+x^4+1)^2}\\
&= \frac{5x^{10}+8x^8+3x^6+5x^4+3x^2-6x^{10}-10x^8-4x^6+6x^5+4x^3}{(x^6+x^4+1)^2}\\
&= \frac{-x^{10}-2x^8-x^6+6x^5+5x^4+4x^3+3x^2}{(x^6+x^4+1)^2}.
\end{aligned}
$$

C03S02.027: If $y(x) = x^3 - 6x^5 + \frac{3}{2}x^{-4} + 12$, then the linear combination rule and the power rules yield $h'(x) = 3x^2 - 30x^4 - 6x^{-5}$.

C03S02.028: Given:

$$x(t) = \frac{3}{t} - \frac{4}{t^2} - 5 = 3t^{-1} - 4t^{-2} - 5,$$

it follows from the linear combination rule and the power rule for negative integral exponents that

$$x'(t) = -3t^{-2} + 8t^{-3} = \frac{8}{t^3} - \frac{3}{t^2} = \frac{8-3t}{t^3}.$$

C03S02.029: Given:

$$y(x) = \frac{5-4x^2+x^5}{x^3} = \frac{5}{x^3} - \frac{4x^2}{x^3} + \frac{x^5}{x^3} = 5x^{-3} - 4x^{-1} + x^2,$$

it follows from the linear combination rule and the power rules that

$$y'(x) = -15x^{-4} + 4x^{-2} + 2x = 2x + \frac{4}{x^2} - \frac{15}{x^4} = \frac{2x^5+4x^2-15}{x^4}.$$

C03S02.030: Given

$$u(x) = \frac{2x-3x^2+2x^4}{5x^2} = \frac{2}{5}x^{-1} - \frac{3}{5} + \frac{2}{5}x^2,$$

it follows from the linear combination rule and the power rules that

$$u'(x) = -\frac{2}{5}x^{-2} + \frac{4}{5}x = \frac{4x^3-2}{5x^2}.$$

C03S02.031: Because $y(x)$ can be written in the form $y(x) = 3x - \frac{1}{4}x^{-2}$, the linear combination rule and the power rules yield $y'(x) = 3 + \frac{1}{2}x^{-3}$.

C03S02.032: We use the reciprocal rule, the linear combination rule, and the power rule for positive integral exponents:

$$f'(z) = -\frac{D_z(z^3+2z^2+2z)}{z^2(z^2+2z+2)^2} = -\frac{3z^2+4z+2}{z^2(z^2+2z+2)^2}.$$

C03S02.033: If we first combine the two fractions, we will need to use the quotient rule only once:

$$y(x) = \frac{x}{x-1} + \frac{x+1}{3x} = \frac{3x^2 + x^2 - 1}{3x(x-1)} = \frac{4x^2 - 1}{3x^2 - 3x},$$

and therefore

$$y'(x) = \frac{(3x^2 - 3x)(8x) - (4x^2 - 1)(6x - 3)}{(3x^2 - 3x)^2} = \frac{24x^3 - 24x^2 - 24x^3 + 12x^2 + 6x - 3}{(3x^2 - 3x)^2} = \frac{-12x^2 + 6x - 3}{(3x^2 - 3x)^2}.$$

C03S02.034: First multiply each term in numerator and denominator by t^2 to obtain

$$u(t) = \frac{1}{1 - 4t^{-2}} = \frac{t^2}{t^2 - 4},$$

then apply the quotient rule:

$$u'(t) = \frac{(t^2 - 4)(2t) - (t^2)(2t)}{(t^2 - 4)^2} = -\frac{8t}{(t^2 - 4)^2}.$$

C03S02.035: The quotient rule (and other rules, such as the linear combination rule and the power rule) yield

$$\begin{aligned} y'(x) &= \frac{(x^2 + 9)(3x^2 - 4) - (x^3 - 4x + 5)(2x)}{(x^2 + 9)^2} \\ &= \frac{3x^4 + 23x^2 - 36 - 2x^4 + 8x^2 - 10x}{(x^2 + 9)^2} = \frac{x^4 + 31x^2 - 10x - 36}{(x^2 + 9)^2}. \end{aligned}$$

C03S02.036: Expand $w(z)$ and take advantage of negative exponents:

$$w(z) = z^2\left(2z^3 - \frac{3}{4z^4}\right) = 2z^5 - \frac{3}{4}z^{-2},$$

and so

$$w'(z) = 10z^4 + \frac{3}{2}z^{-3} = 10z^4 + \frac{3}{2z^3} = \frac{20z^7 + 3}{2z^3}.$$

C03S02.037: First multiply each term in numerator and denominator by $5x^4$ to obtain

$$y(x) = \frac{10x^6}{15x^5 - 4}.$$

Then apply the quotient rule (among others):

$$y'(x) = \frac{(15x^5 - 4)(60x^5) - (10x^6)(75x^4)}{(15x^5 - 4)^2} = \frac{900x^{10} - 240x^5 - 750x^{10}}{(15x^5 - 4)^2} = \frac{150x^{10} - 240x^5}{(15x^5 - 4)^2} = \frac{30x^5(5x^5 - 8)}{(15x^5 - 4)^2}.$$

C03S02.038: First rewrite

$$z(t) = 4 \cdot \frac{1}{t^4 - 6t^2 + 9},$$

then apply the linear combination rule and the reciprocal rule to obtain

$$z'(t) = -4 \cdot \frac{4t^3 - 12t}{(t^4 - 6t^2 + 9)^2} = \frac{48t - 16t^3}{(t^2 - 3)^4} = -\frac{16t(t^2 - 3)}{(t^2 - 3)^4} = -\frac{16t}{(t^2 - 3)^3}.$$

C03S02.039: The quotient rule yields

$$y'(x) = \frac{(x+1)(2x) - (x^2)(1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}.$$

C03S02.040: Use the quotient rule, or if you prefer write $h(w) = w^{-1} + 10w^{-2}$, so that

$$h'(w) = -w^{-2} - 20w^{-3} = -\left(\frac{1}{w^2} + \frac{20}{w^3}\right) = -\frac{w + 20}{w^3}.$$

C03S02.041: Given $f(x) = x^3$ and $P(2, 8)$ on its graph, $f'(x) = 3x^2$, so that $f'(2) = 12$ is the slope of the line L tangent to the graph of f at P. So L has equation $y - 8 = 12(x - 2)$; that is, $12x - y = 16$.

C03S02.042: Given $f(x) = 3x^2 - 4$ and $P(1, -1)$ on its graph, $f'(x) = 6x$, so that $f'(1) = 6$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 1 = 6(x - 1)$; that is, $6x - y = 7$.

C03S02.043: Given $f(x) = 1/(x - 1)$ and $P(2, 1)$ on its graph,

$$f'(x) = -\frac{D_x(x-1)}{(x-1)^2} = -\frac{1}{(x-1)^2},$$

so that $f'(2) = -1$ is the slope of the line L tangent to the graph of f at P. So L has equation $y-1 = -(x-2)$; that is, $x + y = 3$.

C03S02.044: Given $f(x) = 2x - x^{-1}$ and $P(0.5, -1)$ on its graph, $f'(x) = 2 + x^{-2}$, so that $f'(0.5) = 6$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 1 = 6\left(x - \frac{1}{2}\right)$; that is, $6x - y = 4$.

C03S02.045: Given $f(x) = x^3 + 3x^2 - 4x - 5$ and $P(1, -5)$ on its graph, $f'(x) = 3x^2 + 6x - 4$, so that $f'(1) = 5$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 5 = 5(x - 1)$; that is, $5x - y = 10$.

C03S02.046: Given

$$f(x) = \left(\frac{1}{x} - \frac{1}{x^2}\right)^{-1} = \left(\frac{x-1}{x^2}\right)^{-1} = \frac{x^2}{x-1},$$

and $P(2, 4)$ on its graph,

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2},$$

so that $f'(2) = 0$ is the slope of the line L tangent to the graph of f at P. So L has equation $y-4 = 0\cdot(x-2)$; that is, $y = 4$.

C03S02.047: Given $f(x) = 3x^{-2} - 4x^{-3}$ and $P(-1, 7)$ on its graph, $f'(x) = 12x^{-4} - 6x^{-3}$, so that $f'(-1) = 18$ is the slope of the line L tangent to the graph of f at P. So L has equation $y - 7 = 18(x + 1)$; that is, $18x - y = -25$.

C03S02.048: Given

$$f(x) = \frac{3x - 2}{3x + 2}$$

and $P(2, 0.5)$ on its graph,

$$f'(x) = \frac{(3x + 2)(3) - (3x - 2)(3)}{(3x + 2)^2} = \frac{12}{(3x + 2)^2},$$

so that $f'(2) = \frac{3}{16}$ is the slope of the line L tangent to the graph of f at P. So L has equation $y - \frac{1}{2} = \frac{3}{16}(x - 2)$; that is, $3x - 16y = -2$.

C03S02.049: Given

$$f(x) = \frac{3x^2}{x^2 + x + 1}$$

and $P(-1, 3)$ on its graph,

$$f'(x) = \frac{(x^2 + x + 1)(6x) - (3x^2)(2x + 1)}{(x^2 + x + 1)^2} = \frac{3x^2 + 6x}{(x^2 + x + 1)^2},$$

so that $f'(-1) = -3$ is the slope of the line L tangent to the graph of f at P. So an equation of the line L is $y - 3 = -3(x + 1)$; that is, $3x + y = 0$.

C03S02.050: Given

$$f(x) = \frac{6}{1 - x^2}$$

and $P(2, -2)$ on its graph,

$$f'(x) = -6 \cdot \frac{-2x}{(1 - x^2)^2} = \frac{12x}{(1 - x^2)^2},$$

so that $f'(2) = \frac{8}{3}$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 2 = \frac{8}{3}(x - 2)$; that is, $8x - 3y = 22$.

C03S02.051: $V = V_0(1 + \alpha T + \beta T^2 + \gamma T^3)$ where $\alpha \approx -0.06427 \times 10^{-3}$, $\beta \approx 8.5053 \times 10^{-6}$, and $\gamma \approx -6.79 \times 10^{-8}$. Now $dV/dt = V_0(\alpha + 2\beta T + 3\gamma T^2)$; $V = V_0 = 1000$ when $T = 0$. Because $V'(0) = \alpha V_0 < 0$, the water contracts when it is first heated. The rate of change of volume at that point is $V'(0) \approx -0.06427$ cm^3 per °C.

C03S02.052: $W = \dfrac{2 \times 10^9}{R^2} = (2 \times 10^9)R^{-2}$, so $\dfrac{dW}{dR} = -\dfrac{4 \times 10^9}{R^3}$; when $R = 3960$, $\dfrac{dW}{dR} = -\dfrac{62500}{970299}$ (lb/mi). Thus W decreases initially at about 1.03 ounces per mile.

C03S02.053: Draw a cross section of the tank through its axis of symmetry. Let r denote the radius of the (circular) water surface when the height of water in the tank is h. Draw a typical radius, label it r, and label the height h. From similar triangles in your figure, deduce that $h/r = 800/160 = 5$, so $r = h/5$. The volume of water in a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$, so in this case we have $V = V(h) = \frac{1}{75}\pi h^3$. The rate of change of V with respect to h is $dV/dh = \frac{1}{25}\pi h^2$, and therefore when $h = 600$, we have $V'(600) = 14400\pi$; that is, approximately 45239 cm^3 per cm.

C03S02.054: Because $y'(x) = 3x^2+2x+1$, the slope of the tangent line at $(1, 3)$ is $y'(1) = 6$. The equation of the tangent line at $(1, 3)$ is $y - 3 = 6(x - 1)$; that is, $y = 6x - 3$. The intercepts of the tangent line are $(0, -3)$ and $\left(\frac{1}{2}, 0\right)$.

C03S02.055: The slope of the tangent line can be computed using dy/dx at $x = a$ and also by using the two points known to lie on the line. We thereby find that

$$3a^2 = \frac{a^3 - 5}{a - 1}.$$

This leads to the equation $(a+1)(2a^2 - 5a + 5) = 0$. The quadratic factor has negative discriminant, so the only real solution of the cubic equation is $a = -1$. The point of tangency is $(-1, -1)$, the slope there is 3, and the equation of the line in question is $y = 3x + 2$.

C03S02.056: Let (a, a^3) be a point of tangency. The tangent line therefore has slope $3a^2$ and, because it passes through $(2, 8)$, we have

$$3a^2 = \frac{a^3 - 8}{a - 2}; \qquad \text{that is,} \qquad 3a^2(a - 2) = a^3 - 8.$$

This leads to the equation $2a^2 - 2a - 4 = 0$, so that $a = -1$ or $a = 2$. The solution $a = 2$ yields the line tangent at $(2, 8)$ with slope 12. The solution $a = -1$ gives the line tangent at $(-1, -1)$ with slope 3. The two lines have equations $y - 8 = 12(x - 2)$ and $y + 1 = 3(x + 1)$; that is, $y = 12x - 16$ and $y = 3x + 2$.

C03S02.057: Suppose that some straight line L is tangent to the graph of $f(x) = x^2$ at the points (a, a^2) and (b, b^2). Our plan is to show that $a = b$, and we may conclude that L cannot be tangent to the graph of f at two *different* points. Because $f'(x) = 2x$ and because (a, a^2) and (b, b^2) both lie on L, the slope of L is equal to both $f'(a)$ and $f'(b)$; that is, $2a = 2b$. Hence $a = b$, so that (a, a^2) and (b, b^2) are the same point. Conclusion: No straight line can be tangent to the graph of $y = x^2$ at two different points.

C03S02.058: Let $(a, 1/a)$ be a point of tangency. The slope of the tangent there is $-1/a^2$, so $-1/a^2 = -2$. Thus there are two possible values for a: $\pm\frac{1}{2}\sqrt{2}$. These lead to the equations of the two lines: $y = -2x+2\sqrt{2}$ and $y = -2x - 2\sqrt{2}$.

C03S02.059: Given $f(x) = x^n$, we have $f'(x) = nx^{n-1}$. The line tangent to the graph of f at the point $P(x_0, y_0)$ has slope that we compute in two ways and then equate:

$$\frac{y - (x_0)^n}{x - x_0} = n(x_0)^{n-1}.$$

To find the x-intercept of this line, substitute $y = 0$ into this equation and solve for x. It follows that the x-intercept is $x = \dfrac{n-1}{n}x_0$.

C03S02.060: Because $dy/dx = 5x^4 + 2 \geqq 2 > 0$ for all x, the curve has no horizontal tangent line. The minimal slope occurs when dy/dx is minimal, and this occurs when $x = 0$. So the smallest slope that a line tangent to this graph can have is 2.

C03S02.061: $D_x[f(x)]^3 = f'(x)f(x)f(x) + f(x)f'(x)f(x) + f(x)f(x)f'(x) = 3[f(x)]^2f'(x)$.

C03S02.062: Suppose that u_1, u_2, u_3, u_4, and u_5 are differentiable functions of x. Let primes denote derivatives with respect to x. Then

$$\begin{aligned}D_x[u_1u_2u_3u_4] &= D_x[(u_1u_2u_3)u_4] = (u_1u_2u_3)'u_4 + (u_1u_2u_3)u_4'\\ &= (u_1'u_2u_3 + u_1u_2'u_3 + u_1u_2u_3')u_4 + (u_1u_2u_3)u_4'\\ &= u_1'u_2u_3u_4 + u_1u_2'u_3u_4 + u_1u_2u_3'u_4 + u_1u_2u_3u_4'.\end{aligned}$$

Next, using this result,

$$\begin{aligned}D_x[u_1u_2u_3u_4u_5] &= D_x[(u_1u_2u_3u_4)u_5] = (u_1u_2u_3u_4)'u_5 + (u_1u_2u_3u_4)u_5'\\ &= (u_1'u_2u_3u_4 + u_1u_2'u_3u_4 + u_1u_2u_3'u_4 + u_1u_2u_3u_4')u_5 + (u_1u_2u_3u_4)u_5'\\ &= u_1'u_2u_3u_4u_5 + u_1u_2'u_3u_4u_5 + u_1u_2u_3'u_4u_5 + u_1u_2u_3u_4'u_5 + u_1u_2u_3u_4u_5'.\end{aligned}$$

C03S02.063: Let $u_1(x) = u_2(x) = u_3(x) = \cdots = u_{n-1}(x) = u_n(x) = f(x)$. Then the left-hand side of Eq. (16) is $D_x[\,(f(x))^n]$ and the right-hand side is

$$f'(x)[f(x)]^{n-1} + f(x)f'(x)[f(x)]^{n-2} + [f(x)]^2f'(x)[f(x)]^{n-3} + \cdots + [f(x)]^{n-1}f'(x) = n[f(x)]^{n-1}\cdot f'(x).$$

Therefore if n is a positive integer and $f'(x)$ exists, then

$$D_x[\,(f(x))^n] = n(f(x))^{n-1}\cdot f'(x).$$

C03S02.064: Substitution of $f(x) = x^2 + x + 1$ and $n = 100$ in the result of Problem 63 yields

$$D_x[\,(x^2+x+1)^{100}] = D_x[\,(f(x))^n] = n(f(x))^{n-1}\cdot f'(x) = 100(x^2+x+1)^{99}\cdot(2x+1).$$

C03S02.065: Let $f(x) = x^3 - 17x + 35$ and let $n = 17$. Then $g(x) = (f(x))^n$. Hence, by the result in Problem 63,

$$g'(x) = D_x[\,(f(x))^n] = n(f(x))^{n-1}\cdot f'(x) = 17(x^3-17x+35)^{16}\cdot(3x^2-17).$$

C03S02.066: We begin with $f(x) = ax^3 + bx^2 + cx + d$. Then $f'(x) = 3ax^2 + 2bx + c$. The conditions in the problem require that (simultaneously)

$$\begin{array}{lcl} 1 = f(0) = d, & & 0 = f(1) = a+b+c+d,\\ 0 = f'(0) = c, & \text{and} & 0 = f'(1) = 3a+2b+c.\end{array}$$

These equations have the unique solution $a = 2$, $b = -3$, $c = 0$, and $d = 1$. Therefore $f(x) = 2x^3 - 3x^2 + 1$ is the only possible solution. It is easy to verify that $f(x)$ satisfies the conditions required in the problem.

C03S02.067: If n is a positive integer and

$$f(x) = \frac{x^n}{1+x^2},$$

then

$$f'(x) = \frac{(1+x^2)(nx^{n-1}) - (2x)(x^n)}{(1+x^2)^2} = \frac{nx^{n-1} + nx^{n+1} - 2x^{n+1}}{(1+x^2)^2} = \frac{x^{n-1}[n+(n-2)x^2]}{(1+x^2)^2}. \tag{1}$$

If $n = 0$, then (by the reciprocal rule)

$$f'(x) = -\frac{2x}{(1+x^2)^2}.$$

If $n = 2$, then by Eq. (1)

$$f'(x) = \frac{2x}{(1+x^2)^2}.$$

In each case there can be but one solution of $f'(x) = 0$, so there is only one horizontal tangent line. If $n = 0$ it is tangent to the graph of f at the point $(0,\ 1)$; if $n = 2$ it is tangent to the graph of f at the point $(0,\ 0)$.

C03S02.068: If $n = 1$, then Eq. (1) of the solution of Problem 67 yields

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

The equation $f'(x) = 0$ has the two solutions $x = \pm 1$, so there are two points on the graph of f where the tangent line is horizontal: $\left(-1,\ -\frac{1}{2}\right)$ and $\left(1,\ \frac{1}{2}\right)$.

C03S02.069: If n is a positive integer and $n \geqq 3$, $f'(x) = 0$ only when the numerator is zero in Eq. (1) of the solution of Problem 67; that is, when $x^{n-1}(n + [n-2]x^2) = 0$. But this implies that $x = 0$ (because $n \geqq 3$) or that $n + [n-2]x^2 = 0$. The latter is impossible because $n > 0$ and $[n-2]x^2 \geqq 0$. Therefore the only horizontal tangent to the graph of f is at the point $(0,\ 0)$.

C03S02.070: By Eq. (1) in the solution of Problem 67, if

$$f(x) = \frac{x^3}{1+x^2}, \qquad \text{then} \qquad f'(x) = \frac{x^2(3+x^2)}{(1+x^2)^2}.$$

So $f'(x) = 1$ when

$$\begin{aligned} \frac{x^2(3+x^2)}{(1+x^2)^2} &= 1; \\ x^2(3+x^2) &= (1+x^2)^2; \\ x^4 + 3x^2 &= x^4 + 2x^2 + 1; \\ x^2 &= 1. \end{aligned}$$

Therefore there are two points where the line tangent to the graph of f has slope 1; they are $\left(-1,\ -\frac{1}{2}\right)$ and $\left(1,\ \frac{1}{2}\right)$.

C03S02.071: If

$$f(x) = \frac{x^3}{1+x^2}, \qquad \text{then} \qquad f'(x) = \frac{x^2(3+x^2)}{(1+x^2)^2} = \frac{x^4+3x^2}{(1+x^2)^2},$$

by Eq. (1) in the solution of Problem 67. A line tangent to the graph of $y = f'(x)$ will be horizontal when the derivative $f''(x)$ of $f'(x)$ is zero. But

$$D_x[f'(x)] = f''(x) = \frac{(1+x^2)^2(4x^3+6x) - (x^4+3x^2)(4x^3+4x)}{(1+x^2)^4}$$

$$= \frac{(1+x^2)^2(4x^3+6x) - (x^4+3x^2)(4x)(x^2+1)}{(1+x^2)^4} = \frac{(1+x^2)(4x^3+6x) - (x^4+3x^2)(4x)}{(1+x^2)^3}$$

$$= \frac{4x^3+6x+4x^5+6x^3-4x^5-12x^3}{(1+x^2)^3} = \frac{6x-2x^3}{(1+x^2)^3} = \frac{2x(3-x^2)}{(1+x^2)^3}.$$

So $f''(x) = 0$ when $x = 0$ and when $x = \pm\sqrt{3}$. Therefore there are three points on the graph of $y = f'(x)$ at which the tangent line is horizontal: $(0, 0)$, $\left(-\sqrt{3}, \frac{9}{8}\right)$, and $\left(\sqrt{3}, \frac{9}{8}\right)$.

C03S02.072: (a) Using the quadratic formula, $V'(T) = 0$ when

$$T = T_m = \frac{170100 - 20\sqrt{59243226}}{4074} \approx 3.9668034952936377057 2 \quad (\text{in } ^\circ\text{C})$$

and substitution in the formula for $V(T)$ (Example 5) yields

$$V_m = V(T_m) \approx 999.87464592037071155281 \quad (\text{cm}^3).$$

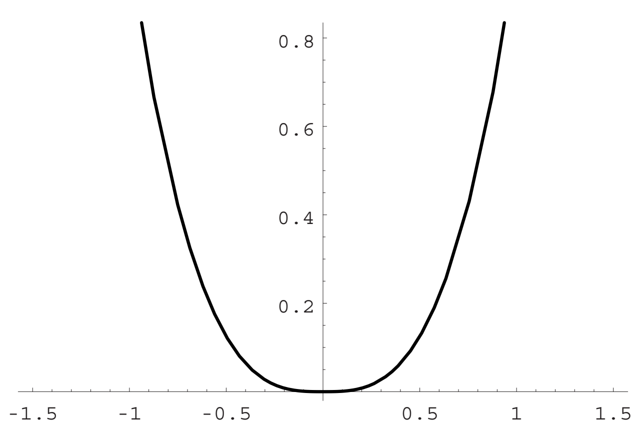
(b) The *Mathematica* command

```
Solve[ V(T) == 1000, T ]
```

yielded three solutions, the only one of which is close to $T = 8$ was

$$T = \frac{85050 - 10\sqrt{54879293}}{1358} \approx 8.07764394099814733845 \quad (\text{in } ^\circ\text{C}).$$

C03S02.073: The graph of $f(x) = |x^3|$ is shown next.

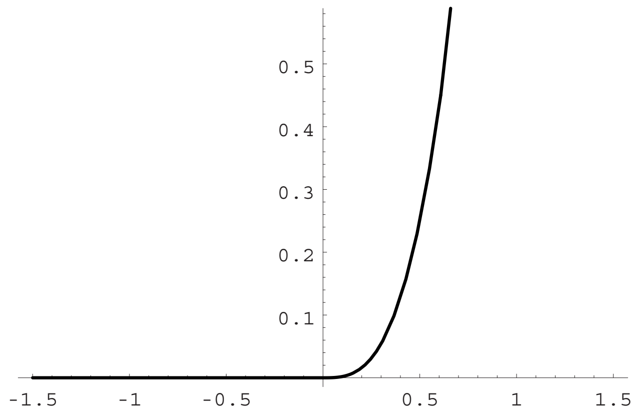


Clearly f is differentiable at x if $x \neq 0$. Moreover,

$$f'_-(0) = \lim_{h\to 0^-} \frac{|(0+h)^3| - 0}{h} = \lim_{h\to 0^-} \frac{-h^3}{h} = 0$$

and $f_+(0) = 0$ by a similar computation. Therefore f is differentiable everywhere.

C03S02.074: The graph of $f(x) = x^3 + |x^3|$ is shown next.



Clearly f is differentiable except possibly at $x = 0$. Moreover,

$$f'_-(0) = \lim_{h \to 0^-} \frac{(0+h)^3 + |(0+h)^3|}{h} = \lim_{h \to 0^-} \frac{h^3 - h^3}{h} = 0$$

and

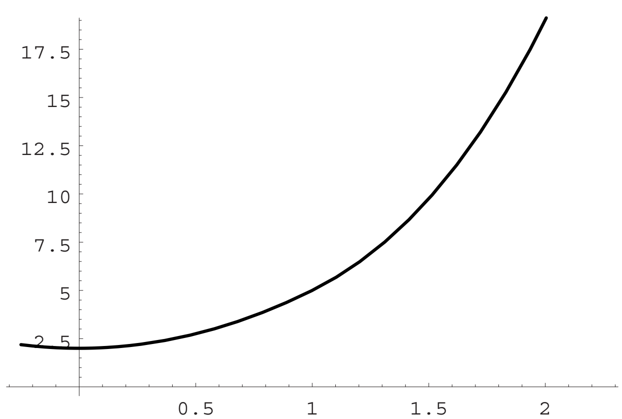
$$f_+(0) = \lim_{h \to 0^+} \frac{(0+h)^3 + |(0+h)^3|}{h} = \lim_{h \to 0^+} \frac{2h^3}{h} = 0.$$

Therefore f is differentiable at x for all x in $\boldsymbol{R}$.

C03S02.075: The graph of

$$f(x) = \begin{cases} 2 + 3x^2 & \text{if } x < 1, \\ 3 + 2x^3 & \text{if } x \geqq 1, \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But

$$f'_-(1) = \lim_{h \to 0^-} \frac{2 + 3(1+h)^2 - 5}{h} = \lim_{h \to 0^-} \frac{6h + 3h^2}{h} = 6$$

and

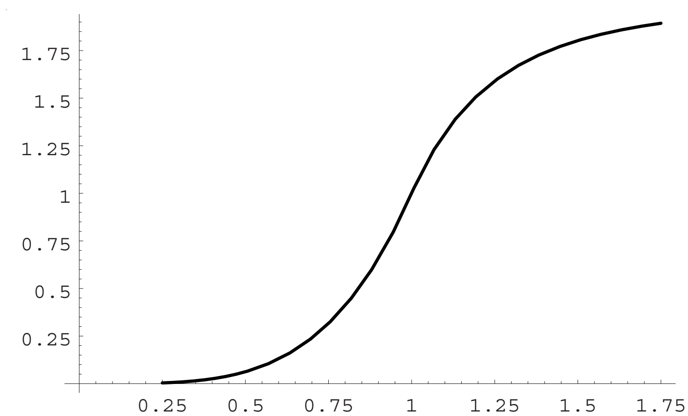
$$f'_+(1) = \lim_{h \to 0^+} \frac{3 + 2(1+h)^3 - 5}{h} = \lim_{h \to 0^+} \frac{6h + 6h^2 + 2h^3}{h} = 6.$$

Therefore $f'(1)$ exists (and $f'(1) = 6$), and hence $f'(x)$ exists for every real number x.

C03S02.076: The graph of

$$f(x) = \begin{cases} x^4 & \text{if } x < 1, \\ 2 - \dfrac{1}{x^4} & \text{if } x \geqq 1 \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But here we have

$$f'_-(1) = \lim_{h \to 0^-} \frac{(1+h)^4 - 1}{h} = \lim_{h \to 0^-} \frac{4h + 6h^2 + 4h^3 + h^4}{h} = 4$$

and

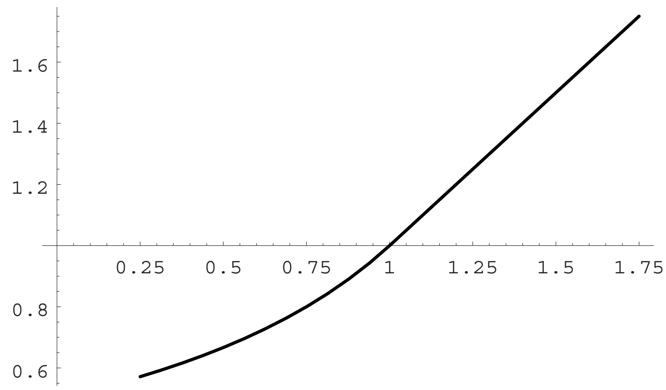
$$\begin{aligned} f'_+(1) &= \lim_{h \to 0^+} \frac{1}{h} \cdot \left(2 - \frac{1}{(1+h)^4} - 1\right) = \lim_{h \to 0^+} \frac{1}{h} \cdot \left(1 - \frac{1}{(1+h)^4}\right) \\ &= \lim_{h \to 0^+} \frac{4h + 6h^2 + 4h^3 + h^4}{h(1+h)^4} = \lim_{h \to 0^+} \frac{4 + 6h + 4h^2 + h^3}{(1+h)^4} = \frac{4}{1} = 4. \end{aligned}$$

Therefore f is differentiable at $x = 1$ as well.

C03S02.077: The graph of

$$f(x) = \begin{cases} \dfrac{1}{2 - x} & \text{if } x < 1, \\ x & \text{if } x \geqq 1 \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But

$$f'_-(1) = \lim_{h\to 0^-} \frac{1}{h} \cdot \left(\frac{1}{2-(1+h)} - 1 \right) = \lim_{h\to 0^+} \frac{1}{h} \cdot \left(\frac{1}{1-h} - 1 \right) = \lim_{h\to 0^-} \frac{h}{h(1-h)} = \lim_{h\to 0^-} \frac{1}{1-h} = 1$$

and

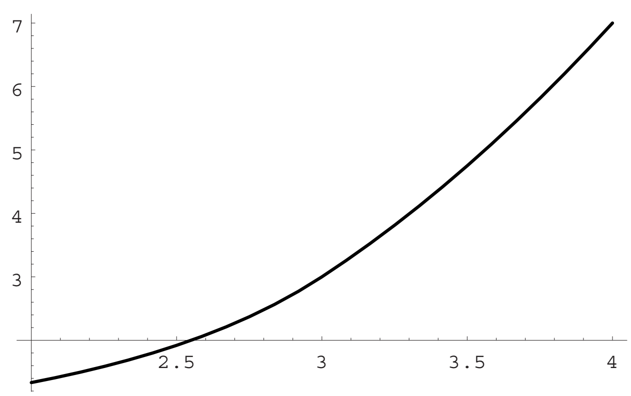
$$f'_+(1) = \lim_{h\to 0^+} \frac{1+h-1}{h} = 1.$$

Thus f is differentiable at $x = 1$ as well (and $f'(1) = 1$).

C03S02.078: The graph of the function

$$f(x) = \begin{cases} \dfrac{12}{(5-x)^2} & \text{if } x < 3, \\ x^2 - 3x + 3 & \text{if } x \geqq 3 \end{cases}$$

is next.



Clearly f is differentiable except possibly at $x = 3$. But

$$f'_-(3) = \lim_{h\to 0^-} \frac{1}{h} \cdot \left(\frac{12}{(5-3-h)^2} - 3 \right) = \lim_{h\to 0^-} \frac{12-3(2-h)^2}{h(2-h)^2} = \lim_{h\to 0^-} \frac{12h-3h^2}{h(2-h)^2} = \lim_{h\to 0^-} \frac{12-3h}{(2-h)^2} = 3$$

and

$$f'_+(3) = \lim_{h\to 0^+} \frac{(3+h)^2 - 3(3+h) + 3 - 3}{h} = \lim_{h\to 0^+} \frac{6h + h^2 - 3h}{h} = 3.$$

Thus f is differentiable at $x = 3$ as well.

Section 3.3

C03S03.001: Given $y = (3x+4)^5$, the chain rule yields

$$\frac{dy}{dx} = 5 \cdot (3x+4)^4 \cdot D_x(3x+4) = 5 \cdot (3x+4)^4 \cdot 3 = 15(3x+4)^4.$$

C03S03.002: Given $y = (2-5x)^3$, the chain rule yields

$$\frac{dy}{dx} = 3 \cdot (2-5x)^2 \cdot D_x(2-5x) = 3 \cdot (2-5x)^2 \cdot (-5) = -15(2-5x)^2.$$

C03S03.003: Rewrite the given function in the form $y = (3x-2)^{-1}$ in order to apply the chain rule. The result is

$$\frac{dy}{dx} = (-1)(3x-2)^{-2} \cdot D_x(3x-2) = (-1)(3x-2)^{-2} \cdot 3 = -3(3x-2)^{-2} = -\frac{3}{(3x-2)^2}.$$

C03S03.004: Rewrite the given function in the form $y = (2x+1)^{-3}$ in order to apply the chain rule. The result is

$$\frac{dy}{dx} = (-3)(2x+1)^{-4} \cdot D_x(2x+1) = (-3)(2x+1)^{-4} \cdot 2 = -6(2x+1)^{-4} = -\frac{6}{(2x+1)^4}.$$

C03S03.005: Given $y = (x^2+3x+4)^3$, the chain rule yields

$$\frac{dy}{dx} = 3(x^2+3x+4)^2 \cdot D_x(x^2+3x+4) = 3(x^2+3x+4)^2(2x+3).$$

C03S03.006: $\dfrac{dy}{dx} = -4 \cdot (7-2x^3)^{-5} \cdot D_x(7-2x^3) = -4 \cdot (7-2x^3)^{-5} \cdot (-6x^2) = 24x^2(7-2x^3)^{-5} = \dfrac{24x^2}{(7-2x^3)^5}.$

C03S03.007: We use the product rule, and in the process of doing so must use the chain rule twice: Given $y = (2-x)^4(3+x)^7$,

$$\begin{aligned}
\frac{dy}{dx} &= (2-x)^4 \cdot D_x(3+x)^7 + (3+x)^7 \cdot D_x(2-x)^4 \\
&= (2-x)^4 \cdot 7 \cdot (3+x)^6 \cdot D_x(3+x) + (3+x)^7 \cdot 4(2-x)^3 \cdot D_x(2-x) \\
&= (2-x)^4 \cdot 7 \cdot (3+x)^6 \cdot 1 + (3+x)^7 \cdot 4(2-x)^3 \cdot (-1) = 7(2-x)^4(3+x)^6 - 4(2-x)^3(3+x)^7 \\
&= (2-x)^3(3+x)^6(14-7x-12-4x) = (2-x)^3(3+x)^6(2-11x).
\end{aligned}$$

The last simplifications would be necessary only if you needed to find where $y'(x)$ is positive, where negative, and where zero.

C03S03.008: Given $y = (x+x^2)^5(1+x^3)^2$, the product rule—followed by two applications of the chain rule—yields

$$
\begin{aligned}
\frac{dy}{dx} &= (x+x^2)^5 \cdot D_x(1+x^3)^2 + (1+x^3)^2 \cdot D_x(x+x^2)^5 \\
&= (x+x^2)^5 \cdot 2 \cdot (1+x^3) \cdot D_x(1+x^3) + (1+x^3)^2 \cdot 5 \cdot (x+x^2)^4 \cdot D_x(x+x^2) \\
&= (x+x^2)^5 \cdot 2 \cdot (1+x^3) \cdot 3x^2 + (1+x^3)^2 \cdot 5 \cdot (x+x^2)^4 \cdot (1+2x) \\
&= \cdots = x^4(x+1)^6(x^2-x+1)(16x^3-5x^2+5x+5).
\end{aligned}
$$

Sometimes you have to factor an expression as much as you can to determine where it is positive, negative, or zero.

C03S03.009: We will use the quotient rule, which will require use of the chain rule to find the derivative of the denominator:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{(3x-4)^3 D_x(x+2) - (x+2)D_x(3x-4)^3}{[\,(3x-4)^3]^2} \\
&= \frac{(3x-4)^3 \cdot 1 - (x+2) \cdot 3 \cdot (3x-4)^2 \cdot D_x(3x-4)}{(3x-4)^6} \\
&= \frac{(3x-4)^3 - 3(x+2)(3x-4)^2 \cdot 3}{(3x-4)^6} = \frac{(3x-4) - 9(x+2)}{(3x-4)^4} = -\frac{6x+22}{(3x-4)^4}.
\end{aligned}
$$

C03S03.010: We use the quotient rule, and need to use the chain rule twice along the way:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{(4+5x+6x^2)^2 \cdot D_x(1-x^2)^3 - (1-x^2)^3 \cdot D_x(4+5x+6x^2)^2}{(4+5x+6x^2)^4} \\
&= \frac{(4+5x+6x^2)^2 \cdot 3 \cdot (1-x^2)^2 \cdot D_x(1-x^2) - (1-x^2)^3 \cdot 2 \cdot (4+5x+6x^2) \cdot D_x(4+5x+6x^2)}{(4+5x+6x^2)^4} \\
&= \frac{(4+5x+6x^2)^2 \cdot 3 \cdot (1-x^2)^2 \cdot (-2x) - (1-x^2)^3 \cdot 2 \cdot (4+5x+6x^2) \cdot (5+12x)}{(4+5x+6x^2)^4} \\
&= \frac{(4+5x+6x^2) \cdot 3 \cdot (1-x^2)^2 \cdot (-2x) - (1-x^2)^3 \cdot 2 \cdot (5+12x)}{(4+5x+6x^2)^3} = -\frac{2(x^2-1)^2(6x^3+10x^2+24x+5)}{(4+5x+6x^2)^3}.
\end{aligned}
$$

C03S03.011: Here is a problem in which use of the chain rule contains another use of the chain rule. Given $y = [1+(1+x)^3]^4$,

$$
\begin{aligned}
\frac{dy}{dx} &= 4[1+(1+x)^3]^3 \cdot D_x[1+(1+x)^3] = 4[1+(1+x)^3]^3 \cdot [0 + D_x(1+x)^3] \\
&= 4[1+(1+x)^3]^3 \cdot 3 \cdot (1+x)^2 \cdot D_x(1+x) = 12[1+(1+x)^3]^3(1+x)^2.
\end{aligned}
$$

C03S03.012: Again a "nested chain rule" problem:

$$\begin{aligned}\frac{dy}{dx} &= -5\cdot\left[x+(x+x^2)^{-3}\right]^{-6}\cdot D_x\left[x+(x+x^2)^{-3}\right]\\ &= -5\cdot\left[x+(x+x^2)^{-3}\right]^{-6}\left[1+(-3)\cdot(x+x^2)^{-4}\cdot D_x(x+x^2)\right]\\ &= -5\cdot\left[x+(x+x^2)^{-3}\right]^{-6}\left[1+(-3)\cdot(x+x^2)^{-4}\cdot(1+2x)\right].\end{aligned}$$

C03S03.013: Given: $y=(u+1)^3$ and $u=\dfrac{1}{x^2}$. The chain rule yields

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=3(u+1)^2\cdot\frac{-2}{x^3}=-\frac{6}{x^3}\left(\frac{1}{x^2}+1\right)^2=-\frac{6(x^2+1)^2}{x^7}.$$

C03S03.014: Write $y=\frac{1}{2}u^{-1}-\frac{1}{3}u^{-2}$. Then, with $u=2x+1$, the chain rule yields

$$\begin{aligned}\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} &= \left(-\frac{1}{2}u^{-2}+\frac{2}{3}u^{-3}\right)\cdot 2=-2\left(\frac{1}{2u^2}-\frac{2}{3u^3}\right)\\ &= -2\left(\frac{1}{2(2x+1)^2}-\frac{2}{3(2x+1)^3}\right)=\cdots=\frac{1-6x}{3(1+2x)^3}.\end{aligned}$$

C03S03.015: Given $y=(1+u^2)^3$ and $u=(4x-1)^2$, the chain rule yields

$$\begin{aligned}\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} &= 6u(1+u^2)^2\cdot 8\cdot(4x-1)\\ &= 48\cdot(4x-1)^2(1+(4x-1)^4)^2(4x-1)=48(4x-1)^3(1+(4x-1)^4)^2.\end{aligned}$$

Without the chain rule, our only way to differentiate $y(x)$ would be first to expand it:

$$\begin{aligned}y(x) &= 8-192x+2688x^2-25600x^3+181248x^4-983040x^5+4128768x^6\\ &\quad -13369344x^7+32636928x^8-57671680x^9+69206016x^{10}-50331648x^{11}+16777216x^{12}.\end{aligned}$$

Then we could differentiate $y(x)$ using the linear combination and power rules:

$$\begin{aligned}y'(x) &= -192+5376x-76800x^2+724992x^3-4915200x^4+24772608x^5-93585408x^6\\ &\quad +261095424x^7-519045120x^8+692060160x^9-553648128x^{10}+201326592x^{11}.\end{aligned}$$

Fortunately, the chain rule is available—and even if not, we still have *Maple*, *Derive*, *Mathematica*, and MATLAB.

C03S03.016: If $y=u^5$ and $u=\dfrac{1}{3x-2}$, then the chain rule yields

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=5u^4\cdot\left(-\frac{D_x(3x-2)}{(3x-2)^2}\right)=-\frac{5}{(3x-2)^4}\cdot\frac{3}{(3x-2)^2}=-\frac{15}{(3x-2)^6}.$$

C03S03.017: If $y=u(1-u)^3$ and $u=\dfrac{1}{x^4}$, then the chain rule yields

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left[(1-u)^3-3u(1-u)^2\right]\cdot\left(-4x^{-5}\right)=\left[(1-x^{-4})^3-3x^{-4}(1-x^{-4})^2\right]\cdot\left(-4x^{-5}\right),$$

which a very patient person can simplify to

$$\frac{dy}{dx} = \frac{16 - 36x^4 + 24x^8 - 4x^{12}}{x^{17}}.$$

C03S03.018: If $y = \dfrac{u}{u+1}$ and $u = \dfrac{x}{x+1}$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{(u+1)\cdot 1 - u \cdot 1}{(u+1)^2} \cdot \frac{(x+1)\cdot 1 - x \cdot 1}{(x+1)^2} = \frac{1}{(u+1)^2} \cdot \frac{1}{(x+1)^2}$$

$$= \frac{1}{\left(\dfrac{x}{x+1} + 1\right)^2} \cdot \frac{1}{(x+1)^2} = \frac{1}{\left(\dfrac{2x+1}{x+1}\right)^2} \cdot \frac{1}{(x+1)^2} = \frac{1}{(2x+1)^2}.$$

C03S03.019: If $y = u^2(u - u^4)^3$ and $u = x^{-2}$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \big[2u(u-u^4)^3 + 3u^2(u-u^4)^2(1-4u^3)\,\big] \cdot (-2x^{-3})$$

$$= \big[2x^{-2}(x^{-2} - x^{-8})^3 + 3x^{-4}(x^{-2} - x^{-8})^2(1 - 4x^{-6})\,\big] \cdot (-2x^{-3}) = \cdots = \frac{28 - 66x^6 + 48x^{12} - 10x^{18}}{x^{29}}.$$

C03S03.020: If $y = \dfrac{u}{(2u+1)^4}$ and $u = x - 2x^{-1}$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{(2u+1)^4 - 8u(2u+1)^3}{(2u+1)^8} \cdot (1 + 2x^{-2}) = \frac{2u + 1 - 8u}{(2u+1)^5} \cdot (1 + 2x^{-2})$$

$$= \frac{1 - 6u}{(2u+1)^5} \cdot \frac{x^2+2}{x^2} = \frac{1 - 6x + \dfrac{12}{x}}{\left(2x - \dfrac{4}{x} + 1\right)^5} \cdot \frac{x^2+2}{x^2} = \frac{x - 6x^2 + 12}{x\left(\dfrac{2x^2 - 4 + x}{x}\right)^5} \cdot \frac{x^2+2}{x^2}$$

$$= \frac{x^4(12 + x - 6x^2)}{(2x^2 + x - 4)^5} \cdot \frac{x^2+2}{x^2} = \frac{x^2(12 + x - 6x^2)(x^2+2)}{(2x^2+x-4)^5}$$

$$= \frac{x^2(12x^2 + x^3 - 6x^4 + 24 + 2x - 12x^2)}{(2x^2+x-4)^5} = \frac{x^2(24 + 2x + x^3 - 6x^4)}{(2x^2+x-4)^5}.$$

C03S03.021: Let $u(x) = 2x - x^2$ and $n = 3$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 3u^2 \cdot (2 - 2x) = 3(2x - x^2)^2(2 - 2x).$$

C03S03.022: Let $u(x) = 2 + 5x^3$ and $n = -1$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = (-1)u^{-2} \cdot 15x^2 = -\frac{15x^2}{(2+5x^3)^2}.$$

C03S03.023: Let $u(x) = 1 - x^2$ and $n = -4$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = (-4)u^{-5} \cdot (-2x) = \frac{8x}{(1-x^2)^5}.$$

C03S03.024: Let $u(x) = x^2 - 4x + 1$ and $n = 3$. Then $f(x) = u^n$, so

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 3u^2 \cdot (2x - 4) = 3(x^2 - 4x + 1)^2(2x - 4).$$

C03S03.025: Let $u(x) = \dfrac{x+1}{x-1}$ and $n = 7$. Then $f(x) = u^n$, and therefore

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 7u^6 \cdot \frac{(x-1)-(x+1)}{(x-1)^2} = 7\left(\frac{x+1}{x-1}\right)^6 \cdot \frac{-2}{(x-1)^2} = -14 \cdot \frac{(x+1)^6}{(x-1)^8}.$$

C03S03.026: Let $n = 4$ and $u(x) = \dfrac{x^2+x+1}{x+1}$. Thus

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 4\left(\frac{x^2+x+1}{x+1}\right)^3 \cdot \frac{(x+1)(2x+1)-(x^2+x+1)}{(x+1)^2}$$

$$= \frac{4(x^2+x+1)^3}{(x+1)^3} \cdot \frac{2x^2+3x+1-x^2-x-1}{(x+1)^2} = \frac{4(x^2+x+1)^3}{(x+1)^3} \cdot \frac{x^2+2x}{(x+1)^2} = \frac{4(x^2+x+1)^3(x^2+2x)}{(x+1)^5}.$$

C03S03.027: $g'(y) = 1 + 5(2y-3)^4 \cdot 2 = 1 + 10(2y-3)^4$.

C03S03.028: $h'(z) = 2z(z^2+4)^3 + 3z^2(z^2+4)^2 \cdot 2z = (2z^3 + 8z + 6z^3)(z^2+4)^2 = 8z(z^2+1)(z^2+4)^2$.

C03S03.029: If $F(s) = (s - s^{-2})^3$, then

$$F'(s) = 3(s-s^{-2})^2(1+2s^{-3}) = 3\left(s - \frac{1}{s^2}\right)^2 \cdot \left(1 + \frac{2}{s^3}\right) = 3\left(\frac{s^3-1}{s^2}\right)^2 \cdot \frac{s^3+2}{s^3}$$

$$= 3 \cdot \frac{(s^3-1)^2(s^3+2)}{s^7} = 3 \cdot \frac{(s^6-2s^3+1)(s^3+2)}{s^7} = \frac{3(s^9-3s^3+2)}{s^7}.$$

C03S03.030: If $G(t) = \left(t^2 + 1 + \dfrac{1}{t}\right)^2$, then

$$G'(t) = 2\left(2t - \frac{1}{t^2}\right) \cdot \left(t^2 + 1 + \frac{1}{t}\right) = 4t^3 + 4t + 2 - \frac{2}{t^2} - \frac{2}{t^3} = \frac{4t^6 + 4t^4 + 2t^3 - 2t - 2}{t^3}.$$

C03S03.031: If $f(u) = (1+u)^3(1+u^2)^4$, then

$$f'(u) = 3(1+u)^2(1+u^2)^4 + 8u(1+u)^3(1+u^2)^3 = (1+u)^2(1+u^2)^3(11u^2 + 8u + 3).$$

C03S03.032: If $g(w) = (w^2 - 3w + 4)(w+4)^5$, then

$$g'(w) = 5(w+4)^4(w^2 - 3w + 4) + (w+4)^5(2w-3) = (w+4)^4(7w^2 - 10w + 8).$$

C03S03.033: If $h(v) = \left[v - \left(1 - \dfrac{1}{v}\right)^{-1}\right]^{-2}$, then

$$h'(v) = (-2)\left[v - \left(1 - \frac{1}{v}\right)^{-1}\right]^{-3}\left[1 + \left(1 - \frac{1}{v}\right)^{-2}\left(\frac{1}{v^2}\right)\right] = \frac{2(v-1)(v^2-2v+2)}{v^3(2-v)^3}.$$

C03S03.034: If $p(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-4}$, then

$$p'(t) = 4\left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}\right)^{-5}\left(\frac{1}{t^2} + \frac{2}{t^3} + \frac{3}{t^4}\right) = 4\left(\frac{t^2+t+1}{t^3}\right)^{-5}\left(\frac{t^2+2t+3}{t^4}\right)$$

$$= 4\left(\frac{t^3}{t^2+t+1}\right)^5\left(\frac{t^2+2t+3}{t^4}\right) = \frac{4t^{15}(t^2+2t+3)}{(t^2+t+1)^5t^4} = \frac{4t^{11}(t^2+2t+3)}{(t^2+t+1)^5}.$$

C03S03.035: If $F(z) = (5z^5 - 4z + 3)^{-10}$, then

$$F'(z) = -10(5z^5 - 4z + 3)^{-11}(25z^4 - 4) = \frac{40 - 250z^4}{(5z^5 - 4z + 3)^{11}}.$$

C03S03.036: Given $G(x) = (1 + [x + (x^2 + x^3)^4]^5)^6$,

$$G'(x) = 6(1 + [x + (x^2 + x^3)^4]^5)^5 \cdot 5[x + (x^2 + x^3)^4]^4 \cdot [1 + 4(x^2 + x^3)^3(2x + 3x^2)].$$

When $G'(x)$ is expanded completely (written in polynomial form), it has degree 359 and the term with largest coefficient is $7431394213599636006965105906903841744 0x^{287}$.

C03S03.037: Chain rule: $\dfrac{dy}{dx} = 4(x^3)^3 \cdot 3x^2$. Power rule: $\dfrac{dy}{dx} = 12x^{11}$.

C03S03.038: Chain rule: $\dfrac{dy}{dx} = (-1)\left(\dfrac{1}{x}\right)^{-2}\left(-\dfrac{1}{x^2}\right)$. Power rule: $\dfrac{dy}{dx} = 1$.

C03S03.039: Chain rule: $\dfrac{dy}{dx} = 2(x^2 - 1) \cdot 2x$. Without chain rule: $\dfrac{dy}{dx} = 4x^3 - 4x$.

C03S03.040: Chain rule: $\dfrac{dy}{dx} = -3(1 - x)^2$. Without chain rule: $\dfrac{dy}{dx} = -3 + 6x - 3x^2$.

C03S03.041: Chain rule: $\dfrac{dy}{dx} = 4(x + 1)^3$. Without chain rule: $\dfrac{dy}{dx} = 4x^3 + 12x^2 + 12x + 4$.

C03S03.042: Chain rule: $\dfrac{dy}{dx} = -2(x + 1)^{-3}$. Reciprocal rule: $\dfrac{dy}{dx} = -\dfrac{2x + 2}{(x^2 + 2x + 1)^2}$.

C03S03.043: Chain rule: $\dfrac{dy}{dx} = -2x(x^2 + 1)^{-2}$. Reciprocal rule: $\dfrac{dy}{dx} = -\dfrac{2x}{(x^2 + 1)^2}$.

C03S03.044: Chain rule: $\dfrac{dy}{dx} = 2(x^2 + 1) \cdot 2x$. Product rule: $\dfrac{dy}{dx} = 2x(x^2 + 1) + 2x(x^2 + 1)$.

C03S03.045: If $f(x) = \sin(x^3)$, then $f'(x) = \left[\cos(x^3)\right] \cdot D_x(x^3) = 3x^2\cos(x^3) = 3x^2\cos x^3$.

C03S03.046: If $g(t) = (\sin t)^3$, then $g'(t) = 3(\sin t)^2 \cdot D_t \sin t = (3\sin^2 t)(\cos t) = 3\sin^2 t\,\cos t$.

C03S03.047: If $g(z) = (\sin 2z)^3$, then

$$g'(z) = 3(\sin 2z)^2 \cdot D_z(\sin 2z) = 3(\sin 2z)^2(\cos 2z) \cdot D_z(2z) = 6\sin^2 2z\,\cos 2z.$$

C03S03.048: If $k(u) = \sin(1+\sin u)$, then $k'(u) = [\cos(1+\sin u)] \cdot D_u(1+\sin u) = [\cos(1+\sin u)] \cdot \cos u$.

C03S03.049: The radius of the circular ripple is $r(t) = 2t$ and its area is $a(t) = \pi(2t)^2$; thus $a'(t) = 8\pi t$. When $r = 10$, $t = 5$, and at that time the rate of change of area with respect to time is $a'(5) = 40\pi$ (in.^{2}/s).

C03S03.050: If the circle has area A and radius r, then $A = \pi r^2$, so that $r = \sqrt{A/\pi}$. If t denotes time in seconds, then the rate of change of the radius of the circle is

$$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt} = \frac{1}{2\sqrt{\pi A}} \cdot \frac{dA}{dt}. \tag{1}$$

We are given the values $A = 75\pi$ and $dA/dt = -2\pi$; when we substitute these values into the last expression in Eq. (1), we find that $\dfrac{dr}{dt} = -\frac{1}{15}\sqrt{3}$. Hence the radius of the circle is decreasing at the rate of $-\frac{1}{15}\sqrt{3}$ (cm/s) at the time in question.

C03S03.051: Let A denote the area of the square and x the length of each edge. Then $A = x^2$, so $dA/dx = 2x$. If t denotes time (in seconds), then

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 2x\frac{dx}{dt}.$$

All that remains is to substitute the given data $x = 10$ and $dx/dt = 2$ to find that the area of the square is increasing at the rate of 40 in.^{2}/s at the time in question.

C03S03.052: Let x denote the length of each side of the triangle. Then its altitude is $\frac{1}{2}x\sqrt{3}$, and so its area is $A = \frac{1}{4}x^2\sqrt{3}$. Therefore the rate of change of its area with respect to time t (in seconds) is

$$\frac{dA}{dt} = \left(\frac{1}{2}x\sqrt{3}\right) \cdot \frac{dx}{dt}.$$

We are given $x = 10$ and $dx/dt = 2$, so at that point the area is increasing at $10\sqrt{3}$ (in.^{2}/s).

C03S03.053: The volume of the block is $V = x^3$ where x is the length of each edge. So $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. We are given $dx/dt = -2$, so when $x = 10$ the volume of the block is decreasing at 600 in.^{3}/h.

C03S03.054: By the chain rule, $f'(y) = h'(g(y)) \cdot g'(y)$. Then substitution of the data given in the problem yields $f'(-1) = h'(g(-1)) \cdot g'(-1) = h'(2) \cdot g'(-1) = -1 \cdot 7 = -7$.

C03S03.055: $G'(t) = f'(h(t)) \cdot h'(t)$. Now $h(1) = 4$, $h'(1) = -6$, and $f'(4) = 3$, so $G'(1) = 3 \cdot (-6) = -18$.

C03S03.056: The derivative of $f(f(f(x)))$ is the product of the three expressions $f'(f(f(x)))$, $f'(f(x))$, and $f'(x)$. When $x = 0$, $f(x) = 0$ and $f'(x) = 1$. Thus when $x = 0$, each of those three expressions has value 1, so the answer is 1.

C03S03.057: The volume of the balloon is given by $V = \frac{4}{3}\pi r^3$, so

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Answer: When $r = 10$, $dV/dt = 4\pi \cdot 10^2 \cdot 1 = 400\pi \approx 1256.64$ (cm^3/s).

C03S03.058: Let V denote the volume of the balloon and r its radius at time t (in seconds). We are given $dV/dt = 200\pi$. Now

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

When $r = 5$, we have $200\pi = 4\pi \cdot 25 \cdot (dr/dt)$, so $dr/dt = 2$. Answer: When $r = 5$ (cm), the radius of the balloon is increasing at 2 cm/s.

C03S03.059: Given: $\dfrac{dr}{dt} = -3$. Now $\dfrac{dV}{dt} = -300\pi = 4\pi r^2 \cdot (\dfrac{dr}{dt})$. So $4\pi r^2 = 100\pi$, and thus $r = 5$ (cm) at the time in question.

C03S03.060: Let x denote the radius of the hailstone and let V denote its volume. Then

$$V = \frac{4}{3}\pi x^3, \quad \text{and so} \quad \frac{dV}{dt} = 4\pi x^2 \frac{dx}{dt}.$$

When $x = 2$, $\dfrac{dV}{dt} = -0.1$, and therefore $-\dfrac{1}{10} = 4\pi \cdot 2^2 \cdot \dfrac{dx}{dt}$. So $\dfrac{dx}{dt} = -\dfrac{1}{160\pi}$. Answer: At the time in question, the radius of the hailstone is decreasing at $\dfrac{1}{160\pi}$ cm/s —that is, at about 0.002 cm/s.

C03S03.061: Let V denote the volume of the snowball and A its surface area at time t (in hours). Then

$$\frac{dV}{dt} = kA \quad \text{and} \quad A = cV^{2/3}$$

(the latter because A is proportional to r^2, whereas V is proportional to r^3). Therefore

$$\frac{dV}{dt} = \alpha V^{2/3} \quad \text{and thus} \quad \frac{dt}{dV} = \beta V^{-2/3}$$

(α and β are constants). From the last equation we may conclude that $t = \gamma V^{1/3} + \delta$ for some constants γ and δ, so that $V = V(t) = (Pt + Q)^3$ for some constants P and Q. From the information $500 = V(0) = Q^3$ and $250 = V(1) = (P + Q)^3$, we find that $Q = 5\sqrt[3]{4}$ and that $P = -5 \cdot (\sqrt[3]{4} - \sqrt[3]{2})$. Now $V(t) = 0$ when $PT + Q = 0$; it turns out that

$$T = \frac{\sqrt[3]{2}}{\sqrt[3]{2} - 1} \approx 4.8473.$$

Therefore the snowball finishes melting at about 2:50:50 P.M. on the same day.

C03S03.062: Let V denote the volume of the block, x the length of each of its edges. Then $V = x^3$. In 8 hours x decreases from 20 to 8, and dx/dt is steady, so t hours after 8:00 A.M. have

$$x = 20 - \frac{3}{2}t.$$

Also

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = 3x^2 \cdot (-\frac{3}{2}) = -\frac{9}{2} \cdot \left(20 - \frac{3}{2}t\right)^2.$$

At 12 noon we have $t = 4$, so at noon $\dfrac{dV}{dt} = -\frac{9}{2}(20-6)^2 = -882$. Answer: The volume is decreasing at 882 in.^{3}/h then.

C03S03.063: By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{dw} \cdot \frac{dw}{dx}, \quad \text{and therefore} \quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}.$$

C03S03.064: Given: n is a fixed integer, f is differentiable, $f(1) = 1$, $F(x) = f(x^n)$, and $G(x) = [\,f(x)\,]^n$. Then

$$F(1) = f(1^n) = f(1) = 1 = 1^n = [\,f(1)\,]^n = G(1).$$

Next,

$$F'(x) = D_x f(x^n) = f'(x^n) \cdot nx^{n-1} \qquad \text{and} \qquad G'(x) = D_x\,[\,f(x)\,]^n = n\,[\,f(x)\,]^{n-1} \cdot f'(x).$$

Therefore

$$F'(1) = f'(1^n) \cdot n \cdot 1 = nf'(1) = n \cdot 1^{n-1} \cdot f'(1) = n \cdot [\,f(1)\,]^{n-1} \cdot f'(1) = G'(1).$$

C03S03.065: If $h(x) = \sqrt{x+4}$, then

$$h'(x) = \frac{1}{2\sqrt{x+4}} \cdot D_x(x+4) = \frac{1}{2\sqrt{x+4}} \cdot 1 = \frac{1}{2\sqrt{x+4}}.$$

C03S03.066: If $h(x) = x^{3/2} = x \cdot \sqrt{x}$, then

$$h'(x) = 1 \cdot \sqrt{x} + x \cdot D_x\left(\sqrt{x}\right) = \sqrt{x} + \frac{x}{2\sqrt{x}} = \sqrt{x} + \frac{1}{2}\sqrt{x} = \frac{3}{2}\sqrt{x}.$$

C03S03.067: If $h(x) = (x^2+4)^{3/2} = (x^2+4)\sqrt{x^2+4}$, then

$$h'(x) = 2x\sqrt{x^2+4} + (x^2+4) \cdot \frac{1}{2\sqrt{x^2+4}} \cdot 2x = 2x\sqrt{x^2+4} + x\sqrt{x^2+4} = 3x\sqrt{x^2+4}.$$

C03S03.068: If $h(x) = |x| = \sqrt{x^2}$, then

$$h'(x) = \frac{1}{2\sqrt{x^2}} \cdot D_x\left(x^2\right) = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|}.$$

Section 3.4

C03S04.001: Write $f(x) = 4x^{5/2} + 2x^{-1/2}$ to find

$$f'(x) = 10x^{3/2} - x^{-3/2} = 10x^{3/2} - \frac{1}{x^{3/2}} = \frac{10x^3 - 1}{x^{3/2}}.$$

C03S04.002: Write $g(t) = 9t^{4/3} - 3t^{-1/3}$ to find

$$g'(t) = 12t^{1/3} + t^{-4/3} = 12t^{1/3} + \frac{1}{t^{4/3}} = \frac{12t^{5/3} + 1}{t^{4/3}}.$$

C03S04.003: Write $f(x) = (2x+1)^{1/2}$ to find

$$f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x+1}}.$$

C03S04.004: Write $h(z) = (7-6z)^{-1/3}$ to find that

$$h'(z) = -\frac{1}{3}(7-6z)^{-4/3} \cdot (-6) = \frac{2}{(7-6z)^{4/3}}.$$

C03S04.005: Write $f(x) = 6x^{-1/2} - x^{3/2}$ to find that $f'(x) = -3x^{-3/2} - \dfrac{3}{2}x^{1/2} = -\dfrac{3(x^2+2)}{2x^{3/2}}$.

C03S04.006: Write $\phi(u) = 7u^{-2/3} + 2u^{1/3} - 3u^{10/3}$ to find that

$$\phi'(u) = -\frac{14}{3}u^{-5/3} + \frac{2}{3}u^{-2/3} - 10u^{7/3} = -\frac{2(15u^4 - u + 7)}{3u^{5/3}}.$$

C03S04.007: $D_x(2x+3)^{3/2} = \dfrac{3}{2}(2x+3)^{1/2} \cdot 2 = 3\sqrt{2x+3}$.

C03S04.008: $D_x(3x+4)^{4/3} = \dfrac{4}{3}(3x+4)^{1/3} \cdot 3 = 4(3x+4)^{1/3} = 4\sqrt[3]{3x+4}$.

C03S04.009: $D_x(3-2x^2)^{-3/2} = -\dfrac{3}{2}(3-2x^2)^{-5/2} \cdot (-4x) = \dfrac{6x}{(3-2x^2)^{5/2}}$.

C03S04.010: $D_y(4-3y^3)^{-2/3} = -\dfrac{2}{3}(4-3y^3)^{-5/3} \cdot (-9y^2) = \dfrac{6y^2}{(4-3y^3)^{5/3}}$.

C03S04.011: $D_x(x^3+1)^{1/2} = \dfrac{1}{2}(x^3+1)^{-1/2} \cdot (3x^2) = \dfrac{3x^2}{2\sqrt{x^3+1}}$.

C03S04.012: $D_z(z^4+3)^{-2} = -2(z^4+3)^{-3} \cdot 4z^3 = -\dfrac{8z^3}{(z^4+3)^3}$.

C03S04.013: $D_x(2x^2+1)^{1/2} = \dfrac{1}{2}(2x^2+1)^{-1/2} \cdot 4x = \dfrac{2x}{\sqrt{2x^2+1}}$.

C03S04.014: $D_t\left(t(1+t^4)^{-1/2}\right) = (1+t^4)^{-1/2} - \dfrac{1}{2}t(1+t^4)^{-3/2} \cdot 4t^3 = \dfrac{1}{(1+t^4)^{1/2}} - \dfrac{2t^4}{(1+t^4)^{3/2}}$

$$= \frac{1-t^4}{(1+t^4)^{3/2}}.$$

C03S04.015: $D_t\left(t^{3/2}\sqrt{2}\right) = \frac{3}{2}t^{1/2}\sqrt{2} = \frac{3\sqrt{t}}{\sqrt{2}}.$

C03S04.016: $D_t\left(\frac{1}{\sqrt{3}}\cdot t^{-5/2}\right) = -\frac{5}{2\sqrt{3}}\cdot t^{-7/2} = -\frac{5}{2t^{7/2}\sqrt{3}}.$

C03S04.017: $D_x(2x^2-x+7)^{3/2} = \frac{3}{2}(2x^2-x+7)^{1/2}\cdot(4x-1) = \frac{3}{2}(4x-1)\sqrt{2x^2-x+7}.$

C03S04.018: $D_z(3z^2-4)^{97} = 97(3z^2-4)^{96}\cdot 6z = 582z(3z^2-4)^{96}.$

C03S04.019: $D_x(x-2x^3)^{-4/3} = -\frac{4}{3}(x-2x^3)^{-7/3}\cdot(1-6x^2) = \frac{4(6x^2-1)}{3(x-2x^3)^{7/3}}.$

C03S04.020: $D_t\left[\,t^2+(1+t)^4\,\right]^5 = 5\left[\,t^2+(1+t)^4\,\right]^4\cdot D_t\left[\,t^2+(1+t)^4\,\right]$

$$= 5\left[\,t^2+(1+t)^4\,\right]^4\cdot\left[\,2t+4(1+t)^3\cdot 1\,\right] = 5\left[\,t^2+(1+t)^4\,\right]^4\cdot\left[\,2t+4(1+t)^3\,\right].$$

C03S04.021: If $f(x) = x(1-x^2)^{1/2}$, then (by the product rule and the chain rule, among others)

$$\begin{aligned} f'(x) &= 1\cdot(1-x^2)^{1/2} + x\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot D_x(1-x^2) \\ &= (1-x^2)^{1/2} + x\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}. \end{aligned}$$

C03S04.022: Write $g(x) = \dfrac{(2x+1)^{1/2}}{(x-1)^{1/2}}$ to find

$$\begin{aligned} g'(x) &= \frac{(x-1)^{1/2}\cdot\frac{1}{2}(2x+1)^{-1/2}\cdot 2 - \frac{1}{2}(x-1)^{-1/2}\cdot(2x+1)^{1/2}}{\left[\,(x-1)^{1/2}\,\right]^2} \\ &= \frac{2(x-1)^{1/2}(2x+1)^{-1/2} - (x-1)^{-1/2}(2x+1)^{1/2}}{2(x-1)} \\ &= \frac{2(x-1)-(2x+1)}{2(x-1)(x-1)^{1/2}(2x+1)^{1/2}} = -\frac{3}{2(x-1)^{3/2}\sqrt{2x+1}}. \end{aligned}$$

C03S04.023: If $f(t) = \sqrt{\dfrac{t^2+1}{t^2-1}} = \left(\dfrac{t^2+1}{t^2-1}\right)^{1/2}$, then

$$f'(t) = \frac{1}{2}\left(\frac{t^2+1}{t^2-1}\right)^{-1/2}\cdot\frac{(t^2-1)(2t)-(t^2+1)(2t)}{(t^2-1)^2} = \frac{1}{2}\left(\frac{t^2-1}{t^2+1}\right)^{1/2}\cdot\frac{-4t}{(t^2-1)^2} = -\frac{2t}{(t^2-1)^{3/2}\sqrt{t^2+1}}.$$

C03S04.024: If $h(y) = \left(\dfrac{y+1}{y-1}\right)^{17}$, then

$$h'(y) = 17\left(\frac{y+1}{y-1}\right)^{16}\cdot\frac{(y-1)\cdot 1-(y+1)\cdot 1}{(y-1)^2} = 17\left(\frac{y+1}{y-1}\right)^{16}\cdot\frac{-2}{(y-1)^2} = -\frac{34(y+1)^{16}}{(y-1)^{18}}.$$

C03S04.025: $D_x\left(x-\dfrac{1}{x}\right)^3 = 3\left(x-\dfrac{1}{x}\right)^2\left(1+\dfrac{1}{x^2}\right) = 3\left(\dfrac{x^2-1}{x}\right)^2 \cdot \dfrac{x^2+1}{x^2} = \dfrac{3(x^2-1)^2(x^2+1)}{x^4}.$

C03S04.026: Write $g(z) = z^2(1+z^2)^{-1/2}$, then apply the product rule and the chain rule to obtain

$$g'(z) = 2z(1+z^2)^{-1/2} + z^2 \cdot \left(-\frac{1}{2}\right)(1+z^2)^{-3/2} \cdot 2z = \frac{2z}{(1+z^2)^{1/2}} - \frac{z^3}{(1+z^2)^{3/2}} = \frac{z^3+2z}{(1+z^2)^{3/2}}.$$

C03S04.027: Write $f(v) = \dfrac{(v+1)^{1/2}}{v}$. Then

$$f'(v) = \frac{v \cdot \frac{1}{2}(v+1)^{-1/2} - 1 \cdot (v+1)^{1/2}}{v^2} = \frac{v \cdot (v+1)^{-1/2} - 2(v+1)^{1/2}}{2v^2} = \frac{v-2(v+1)}{2v^2(v+1)^{1/2}} = -\frac{v+2}{2v^2(v+1)^{1/2}}.$$

C03S04.028: $h'(x) = \dfrac{5}{3}\left(\dfrac{x}{1+x^2}\right)^{2/3} \cdot \dfrac{(1+x^2)\cdot 1 - x \cdot 2x}{(1+x^2)^2} = \dfrac{5}{3}\left(\dfrac{x}{1+x^2}\right)^{2/3} \cdot \dfrac{1-x^2}{(1+x^2)^2}.$

C03S04.029: $D_x(1-x^2)^{1/3} = \dfrac{1}{3}(1-x^2)^{-2/3} \cdot (-2x) = -\dfrac{2x}{3(1-x^2)^{2/3}}.$

C03S04.030: $D_x(x+x^{1/2})^{1/2} = \dfrac{1}{2}(x+x^{1/2})^{-1/2}\left(1+\dfrac{1}{2}x^{-1/2}\right) = \dfrac{1+2\sqrt{x}}{4\sqrt{x}\,\sqrt{x+\sqrt{x}}}.$

C03S04.031: If $f(x) = x(3-4x)^{1/2}$, then (with the aid of the product rule and the chain rule)

$$f'(x) = 1 \cdot (3-4x)^{1/2} + x \cdot \frac{1}{2}(3-4x)^{-1/2} \cdot (-4) = (3-4x)^{1/2} - \frac{2x}{(3-4x)^{1/2}} = \frac{3(1-2x)}{\sqrt{3-4x}}.$$

C03S04.032: Given $g(t) = \dfrac{t-(1+t^2)^{1/2}}{t^2}$,

$$g'(t) = \frac{t^2\left(1-\frac{1}{2}(1+t^2)^{-1/2}\cdot 2t\right) - 2t\left(t-(1+t^2)^{1/2}\right)}{(t^2)^2} = \frac{t\left(1-t(1+t^2)^{-1/2}\right) - 2\left(t-(1+t^2)^{1/2}\right)}{t^3}$$

$$= \frac{t-t^2(1+t^2)^{-1/2} - 2t + 2(1+t^2)^{1/2}}{t^3} = \frac{-t(1+t^2)^{1/2} - t^2 + 2(1+t^2)}{t^3(1+t^2)^{1/2}} = \frac{t^2+2-t(1+t^2)^{1/2}}{t^3(1+t^2)^{1/2}}.$$

C03S04.033: If $f(x) = (1-x^2)(2x+4)^{1/3}$, then the product rule (among others) yields

$$\begin{aligned} f'(x) &= -2x(2x+4)^{1/3} + \frac{2}{3}(1-x^2)\cdot(2x+4)^{-2/3} \\ &= \frac{-6x(2x+4)+2(1-x^2)}{3(2x+4)^{2/3}} = \frac{-12x^2-24x+2-2x^2}{3(2x+4)^{2/3}} = \frac{2-24x-14x^2}{3(2x+4)^{2/3}}. \end{aligned}$$

C03S04.034: If $f(x) = (1-x)^{1/2}(2-x)^{1/3}$, then

$$f'(x) = \tfrac{1}{2}(1-x)^{-1/2}(-1)\cdot(2-x)^{1/3} + \tfrac{1}{3}(2-x)^{-2/3}(-1)\cdot(1-x)^{1/2} = -\left(\frac{(2-x)^{1/3}}{2(1-x)^{1/2}} + \frac{(1-x)^{1/2}}{3(2-x)^{2/3}}\right)$$

$$= -\frac{3(2-x)+2(1-x)}{6(2-x)^{2/3}(1-x)^{1/2}} = \frac{5x-8}{6(2-x)^{2/3}(1-x)^{1/2}}.$$

C03S04.035: If $g(t) = \left(1 + \dfrac{1}{t}\right)^2 (3t^2 + 1)^{1/2}$, then

$$g'(t) = \left(1 + \frac{1}{t}\right)^2 \cdot \frac{1}{2}(3t^2 + 1)^{-1/2}(6t) + 2\cdot\left(1 + \frac{1}{t}\right)\left(-\frac{1}{t^2}\right)(3t^2 + 1)^{1/2}$$

$$= 3t \cdot \frac{(t+1)^2}{t^2(3t^2+1)^{1/2}} - \frac{2}{t^2} \cdot \frac{t+1}{t}(3t^2+1)^{1/2} = \frac{3t^2(t+1)^2}{t^3(3t^2+1)^{1/2}} - \frac{2(t+1)(3t^2+1)}{t^3(3t^2+1)^{1/2}} = \frac{3t^4 - 3t^2 - 2t - 2}{t^3\sqrt{3t^2+1}}.$$

C03S04.036: If $f(x) = x(1 + 2x + 3x^2)^{10}$, then

$$f'(x) = (1 + 2x + 3x^2)^{10} + 10x(1 + 2x + 3x^2)^9(2 + 6x) = (3x^2 + 2x + 1)^9(63x^2 + 22x + 1).$$

C03S04.037: If $f(x) = \dfrac{2x - 1}{(3x + 4)^5}$, then

$$f'(x) = \frac{2(3x+4)^5 - (2x-1)\cdot 5(3x+4)^4 \cdot 3}{(3x+4)^{10}} = \frac{2(3x+4) - 15(2x-1)}{(3x+4)^6} = \frac{23 - 24x}{(3x+4)^6}.$$

C03S04.038: If $h(z) = (z - 1)^4(z + 1)^6$, then

$$h'(z) = 4(z-1)^3(z+1)^6 + 6(z+1)^5(z-1)^4 = (z-1)^3(z+1)^5(4(z+1) + 6(z-1)) = (z-1)^3(z+1)^5(10z-2).$$

C03S04.039: If $f(x) = \dfrac{(2x + 1)^{1/2}}{(3x + 4)^{1/3}}$, then

$$f'(x) = \frac{(3x+4)^{1/3}(2x+1)^{-1/2} - (2x+1)^{1/2}(3x+4)^{-2/3}}{(3x+4)^{2/3}}$$

$$= \frac{(3x+4) - (2x+1)}{(3x+4)^{4/3}(2x+1)^{1/2}} = \frac{x+3}{(3x+4)^{4/3}(2x+1)^{1/2}}.$$

C03S04.040: If $f(x) = (1 - 3x^4)^5(4 - x)^{1/3}$, then

$$f'(x) = 5(1 - 3x^4)^4(-12x^3)(4 - x)^{1/3} + (1 - 3x^4)^5 \cdot \frac{1}{3}(4 - x)^{-2/3}(-1)$$

$$= -60x^3(1 - 3x^4)^4(4 - x)^{1/3} - \frac{(1 - 3x^4)^5}{3(4 - x)^{2/3}} = \frac{-180x^3(1 - 3x^4)^4(4 - x)}{3(4 - x)^{2/3}} - \frac{(1 - 3x^4)^5}{3(4 - x)^{2/3}}$$

$$= \frac{\big[\,(180x^4 - 720x^3) - (1 - 3x^4)\,\big]\,(1 - 3x^4)^4}{3(4 - x)^{2/3}} = \frac{(183x^4 - 720x^3 - 1)(1 - 3x^4)^4}{3(4 - x)^{2/3}}.$$

C03S04.041: If $h(y) = \dfrac{(1 + y)^{1/2} + (1 - y)^{1/2}}{y^{5/3}}$, then

$$\begin{aligned}
h'(y) &= \frac{y^{5/3}\left[\,\frac{1}{2}(1+y)^{-1/2} - \frac{1}{2}(1-y)^{-1/2}\,\right] - \frac{5}{3}y^{2/3}\left[\,(1+y)^{1/2} + (1-y)^{1/2}\,\right]}{y^{10/3}} \\
&= \frac{y\left[\,\frac{1}{2}(1+y)^{-1/2} - \frac{1}{2}(1-y)^{-1/2}\,\right] - \frac{5}{3}\left[\,(1+y)^{1/2} + (1-y)^{1/2}\,\right]}{y^{8/3}} \\
&= \frac{y\left[\,3(1+y)^{-1/2} - 3(1-y)^{-1/2}\,\right] - 10\left[\,(1+y)^{1/2} + (1-y)^{1/2}\,\right]}{6y^{8/3}} \\
&= \frac{y\left[\,3(1-y)^{1/2} - 3(1+y)^{1/2}\,\right] - 10\left[\,(1+y)(1-y)^{1/2} + (1-y)(1+y)^{1/2}\,\right]}{6y^{8/3}(1-y)^{1/2}(1+y)^{1/2}} \\
&= \frac{(7y-10)\sqrt{1+y} - (7y+10)\sqrt{1-y}}{6y^{8/3}\sqrt{1-y}\,\sqrt{1+y}}.
\end{aligned}$$

C03S04.042: If $f(x) = (1-x^{1/3})^{1/2}$, then

$$f'(x) = \frac{1}{2}(1-x^{1/3})^{-1/2}\left(-\frac{1}{3}x^{-2/3}\right) = -\frac{1}{6x^{2/3}\sqrt{1-x^{1/3}}}.$$

C03S04.043: If $g(t) = \left[t + (t+t^{1/2})^{1/2}\right]^{1/2}$, then

$$g'(t) = \frac{1}{2}\left[t + (t+t^{1/2})^{1/2}\right]^{-1/2} \cdot \left[1 + \frac{1}{2}(t+t^{1/2})^{-1/2}\left(1 + \frac{1}{2}t^{-1/2}\right)\right].$$

It is possible to write the derivative without negative exponents. The symbolic algebra program *Mathematica* yields

$$g'(t) = -\frac{(t+(t+t^{1/2})^{1/2})^{1/2}\left[1 - 4t^{3/2} - 4t^2 + 3t^{1/2}\left(1+(t+t^{1/2})^{1/2}\right) + 2t\left(1+(t+t^{1/2})^{1/2}\,\right)\right]}{8t(1+t^{1/2})(t^{3/2}-t^{1/2}-1)}.$$

But the first answer that *Mathematica* gives is

$$g'(t) = \frac{1 + \dfrac{1 + \dfrac{1}{2\sqrt{t}}}{2\sqrt{t+\sqrt{t}}}}{2\sqrt{t+\sqrt{t+\sqrt{t}}}}.$$

C03S04.044: If $f(x) = x^3\sqrt{1 - \dfrac{1}{x^2+1}}$, then

$$f'(x) = 3x^2\sqrt{1-\frac{1}{x^2+1}} + \frac{1}{2}x^3\left(1 - \frac{1}{x^2+1}\right)^{-1/2} \cdot \frac{2x}{(x^2+1)^2}.$$

The symbolic algebra program *Mathematica* simplifies this to

$$f'(x) = (3x^2+4)\left(\frac{x^2}{x^2+1}\right)^{3/2}.$$

C03S04.045: Because

$$y'(x) = \frac{dy}{dx} = \frac{2}{3x^{1/3}}$$

is never zero, there are no horizontal tangents. Because $y(x)$ is continuous at $x = 0$ and $|y'(x)| \to +\infty$ as $x \to 0$, there is a vertical tangent at $(0, 0)$.

C03S04.046: If $f(x) = x\sqrt{4 - x^2}$, then

$$f'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{2(2 - x^2)}{\sqrt{4 - x^2}}.$$

Hence there are horizontal tangents at $\left(-\sqrt{2}, -2\right)$ and at $\left(\sqrt{2}, 2\right)$. Because f is continuous at ± 2 and

$$\lim_{x \to -2^+} |f(x)| = +\infty = \lim_{x \to 2^-} |f(x)|,$$

there are vertical tangents at $(-2, 0)$ and $(2, 0)$.

C03S04.047: If $g(x) = x^{1/2} - x^{3/2}$, then

$$g'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1 - 3x}{2\sqrt{x}}.$$

Thus there is a horizontal tangent at $\left(\frac{1}{3}, \frac{2}{9}\sqrt{3}\right)$. Also, because g is continuous at $x = 0$ and

$$\lim_{x \to 0^+} |g'(x)| = \lim_{x \to 0^+} \frac{1 - 3x}{2\sqrt{x}} = +\infty,$$

the graph of g has a vertical tangent at $(0, 0)$.

C03S04.048: If $h(x) = (9 - x^2)^{-1/2}$, then

$$h'(x) = -\frac{1}{2}(9 - x^2)^{-3/2} \cdot (-2x) = \frac{x}{(9 - x^2)^{3/2}}.$$

So the graph of h has a horizontal tangent at $\left(0, \frac{1}{3}\right)$. There are no vertical tangents because, even though $|h'(x)| \to +\infty$ as $x \to 3^-$ and as $x \to -3^+$, h is not continuous at 3 or at -3, and there are no other values of x at which $|h'(x)| \to +\infty$.

C03S04.049: If $y(x) = x(1 - x^2)^{-1/2}$, then

$$y'(x) = \frac{dy}{dx} = (1 - x^2)^{-1/2} - \frac{1}{2}x(1 - x^2)^{-3/2} \cdot (-2x) = \frac{1}{(1 - x^2)^{1/2}} + \frac{x^2}{(1 - x^2)^{3/2}} = \frac{1}{(1 - x^2)^{3/2}}.$$

Thus the graph of $y(x)$ has no horizontal tangents because $y'(x)$ is never zero. The only candidates for vertical tangents are at $x = \pm 1$, but there are none because $y(x)$ is not continuous at either of those two values of x.

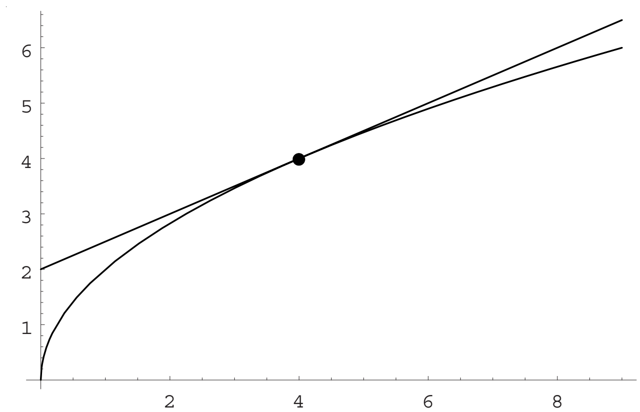
C03S04.050: If $f(x) = \sqrt{(1 - x^2)(4 - x^2)} = (x^4 - 5x^2 + 4)^{1/2}$, then

$$f'(x) = \frac{1}{2}(x^4 - 5x^2 + 4)^{-1/2} \cdot (4x^3 - 10x) = \frac{x(2x^2 - 5)}{\sqrt{(1 - x^2)(4 - x^2)}}.$$

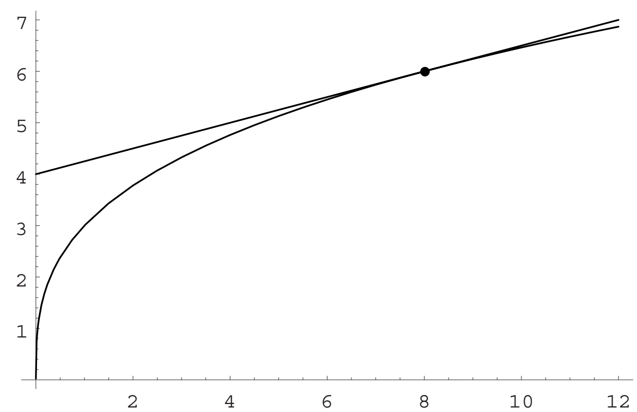
There are no horizontal tangents where $2x^2 = 5$ because the two corresponding values of x are not in the domain $(-\infty, -2] \cup [-1, 1] \cup [2, +\infty)$ of f. There is a horizontal tangent at $(0, 2)$. There are vertical

tangents at $(-2, 0)$, $(-1, 0)$, $(1, 0)$, and $(2, 0)$ because the appropriate one-sided limits of $|f'(x)|$ are all $+\infty$.

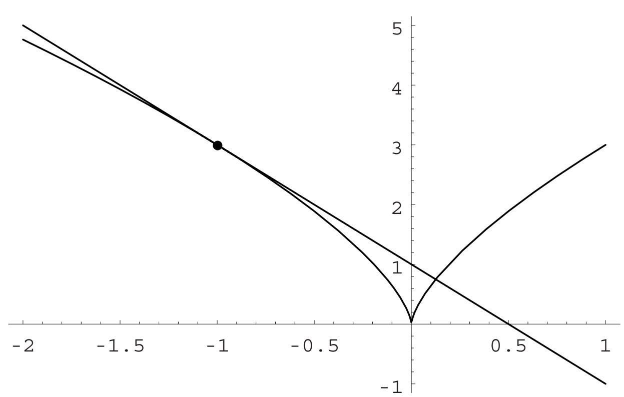
C03S04.051: Let $f(x) = 2\sqrt{x}$. Then $f'(x) = x^{-1/2}$, so an equation of the required tangent line is $y - f(4) = f'(4)(x-4)$; that is, $y = \frac{1}{2}(x+4)$. The graph of f and this tangent line are shown next.



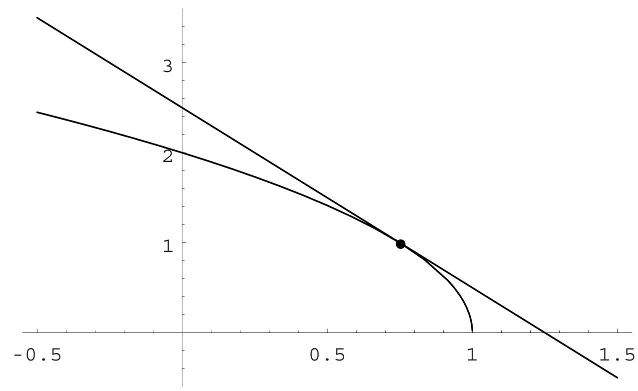
C03S04.052: If $f(x) = 3x^{1/3}$ then $f'(x) = x^{-2/3}$, so an equation of the required tangent line is $y - f(8) = f'(8)(x-8)$; that is, $y = \frac{1}{4}(x+16)$. A graph of f and this tangent line are shown next.



C03S04.053: If $f(x) = 3x^{2/3}$, then $f'(x) = 2x^{-1/3}$. Therefore an equation of the required tangent line is $y - f(-1) = f'(-1)(x+1)$; that is, $y = -2x + 1$. A graph of f and this tangent line are shown next.



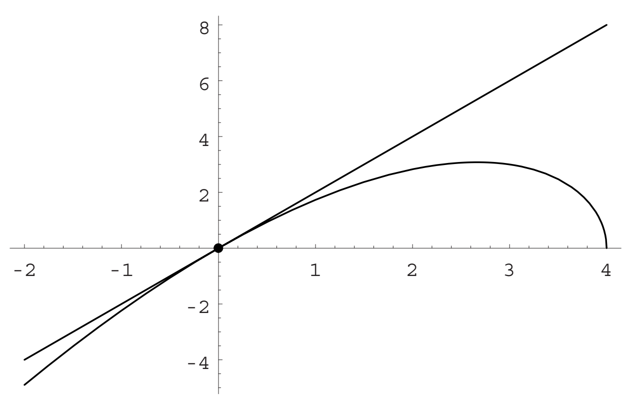
C03S04.054: If $f(x) = 2(1-x)^{1/2}$, then $f'(x) = -(1-x)^{-1/2}$, and therefore an equation of the required tangent line is $y - f\left(\frac{3}{4}\right) = f'\left(\frac{3}{4}\right)\left(x - \frac{3}{4}\right)$; that is, $y = -2x + \frac{5}{2}$. The graph of f and this tangent line are shown next.



C03S04.055: If $f(x) = x(4-x)^{1/2}$, then

$$f'(x) = (4-x)^{1/2} - \frac{1}{2}x(4-x)^{-1/2} = (4-x)^{1/2} - \frac{x}{2(4-x)^{1/2}} = \frac{8-3x}{2\sqrt{4-x}}.$$

So an equation of the required tangent line is $y - f(0) = f'(0)(x-0)$; that is, $y = 2x$. A graph of f and this tangent line are shown next.

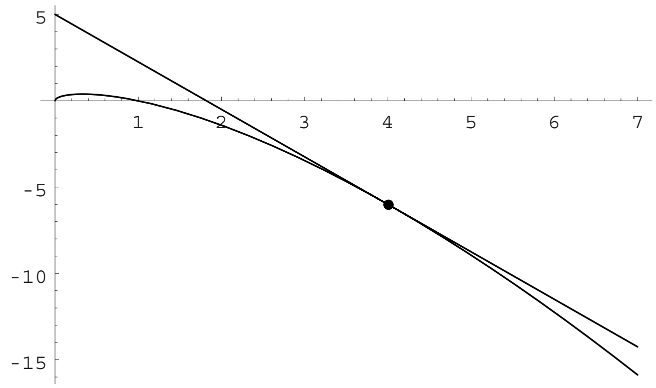


C03S04.056: If $f(x) = x^{1/2} - x^{3/2}$, then (as in the solution of Problem 47)

$$f'(x) = \frac{1-3x}{2\sqrt{x}}.$$

Therefore an equation of the required tangent line is $y - f(4) = f'(4)(x-4)$; that is, $y = -\frac{11}{4}x + 5$. A graph

of f and this tangent line are shown next.



C03S04.057: If $x < 0$ then $f'(x) < 0$; as $x \to 0^-$, $f'(x)$ appears to approach $-\infty$. If $x > 0$ then $f'(x) > 0$; as $x \to 0^+$, $f'(x)$ appears to approach $+\infty$. So the graph of f' must be the one shown in Fig. 3.4.13(d).

C03S04.058: If $x \neq 0$, then $f'(x) > 0$; moreover, $f'(x)$ appears to be approaching zero as $|x|$ increases without bound. In contrast, $f'(x)$ appears to approach $+\infty$ as $x \to 0$. Hence the graph of f' must be the one shown in Fig. 3.4.13(f).

C03S04.059: Note that $f'(x) > 0$ if $x < 0$ whereas $f'(x) < 0$ if $x > 0$. Moreover, as $x \to 0$, $|f'(x)|$ appears to approach $+\infty$. So the graph of f' must be the one shown in Fig. 3.4.13(b).

C03S04.060: We see that $f'(x) > 0$ for $x < 1.4$ (approximately), that $f(x) = 0$ when $x \approx 1.4$, and that $f'(x) < 0$ for $1.4 < x < 2$; moreover, $f'(x) \to -\infty$ as $x \to 2^-$. So the graph of f' must be the one shown in Fig. 3.4.13(a).

C03S04.061: We see that $f'(x) < 0$ for $-2 < x < -1.4$ (approximately), that $f'(x) > 0$ for $-1.4 < x < 1.4$ (approximately), and that $f'(x) < 0$ for $1.4 < x < 2$. Also $f'(x) = 0$ when $x \approx \pm 1.4$. Therefore the graph of f must be the one shown in Fig. 3.4.13(e).

C03S04.062: Figure 3.4.12 shows a graph whose derivative is negative for $x < -1$, positive for $-1 < x < -0.3$ (approximately), negative for $-0.3 < x < 0$, positive for $0 < x < 0.3$ (approximately), negative for $0.3 < x < 1$, and positive for $1 < x$. Moreover, $f'(x) = 0$ when $x = \pm 1$ and when $x \approx \pm 0.3$. Finally, $f'(x) \to -\infty$ as $x \to 0^-$ whereas $f'(x) \to \infty$ as $x \to 0^+$. Therefore the graph of f' must be the one shown in Fig. 3.4.13(c).

C03S04.063: $L = \dfrac{P^2 g}{4\pi^2}$, so $\dfrac{dL}{dP} = \dfrac{Pg}{2\pi^2}$, and hence $\dfrac{dP}{dL} = \dfrac{2\pi^2}{Pg}$. Given $g = 32$ and $P = 2$, we find the value of the latter to be $\frac{1}{32}\pi^2 \approx 0.308$ (seconds per foot).

C03S04.064: $dV/dA = \frac{1}{4}\sqrt{A/\pi}$, and $A = 400\pi$ when the radius of the sphere is 10, so the answer is 5 (in appropriate units, such as cubic meters per square meter).

C03S04.065: Whether $y = +\sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$, it follows easily that $dy/dx = -x/y$. The slope of the tangent is -2 when $x = 2y$, so from the equation $x^2 + y^2 = 1$ we see that $x^2 = 4/5$, so that $x = \pm\frac{2}{5}\sqrt{5}$. Because $y = \frac{1}{2}x$, the two points we are to find are $\left(-\frac{2}{5}\sqrt{5},\ -\frac{1}{5}\sqrt{5}\right)$ and $\left(\frac{2}{5}\sqrt{5},\ \frac{1}{5}\sqrt{5}\right)$.

C03S04.066: Using some of the results in the preceding solution, we find that the slope of the tangent is 3 when $x = -3y$, so that $y^2 = \frac{1}{10}$. So the two points of tangency are $\left(-\frac{3}{10}\sqrt{10},\ \frac{1}{10}\sqrt{10}\right)$ and $\left(\frac{3}{10}\sqrt{10},\ -\frac{1}{10}\sqrt{10}\right)$.

C03S04.067: The line tangent to the parabola $y = x^2$ at the point $Q(a,\ a^2)$ has slope $2a$, so the normal to the parabola at Q has slope $-1/(2a)$. The normal also passes through $P(18,\ 0)$, so we can find its slope another way—by using the two-point formula. Thus

$$-\frac{1}{2a} = \frac{a^2 - 0}{a - 18};$$

$$18 - a = 2a^3;$$

$$2a^3 + a - 18 = 0.$$

By inspection, $a = 2$ is a solution of the last equation. Thus $a - 2$ is a factor of the cubic, and division yields

$$2a^3 + a - 18 = (a - 2)(2a^2 + 4a + 9).$$

The quadratic factor has negative discriminant, so $a = 2$ is the only real solution of $2a^3 + a - 18 = 0$. Therefore the normal line has slope $-\frac{1}{4}$ and equation $x + 4y = 18$.

C03S04.068: Let $Q(a,\ a^2)$ be a point on the parabola $y = x^2$ at which some line through $P(3,\ 10)$ is normal to the parabola. Then, as in the solution of Problem 67, we find that

$$\frac{a^2 - 10}{a - 3} = -\frac{1}{2a}.$$

This yields the cubic equation $2a^3 - 19a - 3 = 0$, and after a little computation we find one of its small integral roots to be $r = -3$. So $a + 3$ is a factor of the cubic; by division, the other factor is $2a^2 - 6a - 1$, which is zero when $a = \frac{1}{2}\left(3 \pm \sqrt{11}\right)$. So the three lines have slopes

$$\frac{1}{6}, \qquad -\frac{1}{3 - \sqrt{11}}, \qquad \text{and} \qquad -\frac{1}{3 + \sqrt{11}}.$$

Their equations are

$$y - 10 = \frac{1}{6}(x - 3), \qquad y - 10 = -\frac{1}{3 - \sqrt{11}}(x - 3), \qquad \text{and} \qquad y - 10 = -\frac{1}{3 + \sqrt{11}}(x - 3).$$

C03S04.069: If a line through $P\left(0,\ \frac{5}{2}\right)$ is normal to $y = x^{2/3}$ at $Q(a,\ a^{2/3})$, then it has slope $-\frac{3}{2}a^{1/3}$. As in the two previous solutions, we find that

$$\frac{a^{2/3} - \frac{5}{2}}{a} = -\frac{3}{2}a^{1/3},$$

which yields $3a^{4/3} + 2a^{2/3} - 5 = 0$. Put $u = a^{2/3}$; we obtain $3u^2 + 2u - 5 = 0$, so that $(3u + 5)(u - 1) = 0$. Because $u = a^{2/3} > 0$, $u = 1$ is the only solution, so $a = 1$ and $a = -1$ yield the two possibilities for the point Q, and therefore the equations of the two lines are

$$y - \frac{5}{2} = -\frac{3}{2}x \qquad \text{and} \qquad y - \frac{5}{2} = \frac{3}{2}x.$$

C03S04.070: Suppose that $P = P(u, v)$, so that $u^2 + v^2 = a^2$. Then the slope of the radius OP is $m_r = v/u$ if $u \neq 0$; if $u = 0$ then OP lies on the y-axis. Also, whether $y = +\sqrt{a^2 - x^2}$ or $y = -\sqrt{a^2 - x^2}$, it follows that

$$\frac{dy}{dx} = \pm \frac{-x}{\sqrt{a^2 - x^2}} = \pm \frac{-x}{\pm y} = -\frac{x}{y}. \tag{1}$$

Thus if $u \neq 0$ and $v \neq 0$, then the slope of the line tangent L to the circle at $P(u, v)$ is $m_t = -u/v$. In this case

$$m_r \cdot m_t = \frac{v}{u} \cdot \left(-\frac{u}{v}\right) = -1,$$

so that OP is perpendicular to L if $u \neq 0$ and $v \neq 0$. If $u = 0$ then Eq. (1) shows that L has slope 0, so that L and OP are also perpendicular in this case. Finally, if $v = 0$ then OP lies on the x-axis and L is vertical, so the two are also perpendicular in this case. In every case we see that L and OP are perpendicular.

C03S04.071: Equation (3) is an *identity*, and if two functions have identical graphs on an interval, then their derivatives will also be identically equal to each other on that interval. (That is, if $f(x) \equiv g(x)$ on an interval I, then $f'(x) \equiv g'(x)$ there.) There is no point in differentiating both sides of an algebraic *equation*.

C03S04.072: If $f(x) = x^{1/2}$ and $a > 0$, then

$$f'(a) = \lim_{x \to a} \frac{x^{1/2} - a^{1/2}}{x - a} = \lim_{x \to a} \frac{x^{1/2} - a^{1/2}}{(x^{1/2} - a^{1/2})(x^{1/2} + a^{1/2})} = \lim_{x \to a} \frac{1}{x^{1/2} + a^{1/2}} = \frac{1}{2a^{1/2}}.$$

Therefore $D_x\, x^{1/2} = \dfrac{1}{2} x^{-1/2}$ if $x > 0$.

C03S04.073: If $f(x) = x^{1/3}$ and $a > 0$, then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \\ &= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{a^{2/3} + a^{2/3} + a^{2/3}} = \frac{1}{3a^{2/3}}. \end{aligned}$$

Therefore $D_x\, x^{1/3} = \dfrac{1}{3} x^{-2/3}$ if $x > 0$.

This formula is of course valid for $x < 0$ as well. To show this, observe that the previous argument is valid if $a < 0$, or—if you prefer—you can use the chain rule, laws of exponents, and the preceding result, as follows. Suppose that $x < 0$. Then $-x > 0$; also, $x^{1/3} = -(-x)^{1/3}$. So

$$D_x(x^{1/3}) = D_x\left[-(-x)^{1/3}\right] = -D_x(-x)^{1/3} = -\left[\frac{1}{3}(-x)^{-2/3} \cdot (-1)\right] = \frac{1}{3}(-x)^{2/3} = \frac{1}{3}x^{-2/3}.$$

Therefore $D_x(x^{1/3}) = \dfrac{1}{3} x^{-2/3}$ if $x \neq 0$.

C03S04.074: If $f(x) = x^{1/5}$ and $a > 0$, then

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{x^{1/5} - a^{1/5}}{x - a} = \lim_{x \to a} \frac{x^{1/5} - a^{1/5}}{(x^{1/5} - a^{1/5})(x^{4/5} + x^{3/5}a^{1/5} + x^{2/5}a^{2/5} + x^{1/5}a^{3/5} + a^{4/5})} \\ &= \lim_{x \to a} \frac{1}{x^{4/5} + x^{3/5}a^{1/5} + x^{2/5}a^{2/5} + x^{1/5}a^{3/5} + a^{4/5}} = \frac{1}{a^{4/5} + a^{4/5} + a^{4/5} + a^{4/5} + a^{4/5}} = \frac{1}{5a^{4/5}}. \end{aligned}$$

Therefore $D_x(x^{1/5}) = \dfrac{1}{5}x^{-4/5}$ if $x > 0$.

As in the concluding paragraph in the previous solution, it is easy to show that this formula holds for all $x \neq 0$.

C03S04.075: The preamble to Problems 72 through 75 implies that if q is a positive integer and x and a are positive real numbers, then

$$x - a = (x^{1/q} - a^{1/q})(x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + x^{1/q}a^{(q-2)/q} + a^{(q-1)/q}).$$

Thus if $f(x) = x^{1/q}$ and $a > 0$, then

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{x^{1/q} - a^{1/q}}{x - a} \\
&= \lim_{x \to a} \frac{x^{1/q} - a^{1/q}}{(x^{1/q} - a^{1/q})(x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + x^{1/q}a^{(q-2)/q} + a^{(q-1)/q})} \\
&= \lim_{x \to a} \frac{1}{x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + a^{(q-1)/q}} \qquad (q \text{ terms in the denominator}) \\
&= \frac{1}{a^{(q-1)/q} + a^{(q-1)/q} + a^{(q-1)/q} + \cdots + a^{(q-1)/q}} \qquad (\text{still } q \text{ terms in the denominator}) \\
&= \frac{1}{qa^{(q-1)/q}} = \frac{1}{q}a^{-(q-1)/q}.
\end{aligned}$$

Therefore $D_x(x^{1/q}) = \dfrac{1}{q}x^{-(q-1)/q}$ if $x > 0$ and q is a positive integer. This result is easy to extend to the case $x < 0$. Therefore if q is a positive integer and $x \neq 0$, then

$$D_x(x^{1/q}) = \frac{1}{q}x^{(1/q)-1}.$$

Section 3.5

C03S05.001: Because $f(x) = 1 - x$ is decreasing everywhere, it can attain a maximum only at a left-hand endpoint of its domain and a minimum only at a right-hand endpoint of its domain. Its domain $[-1, 1)$ has no right-hand endpoint, so f has no minimum value. Its maximum value occurs at -1, is $f(-1) = 2$, and is the global maximum value of f on its domain.

C03S05.002: Because $f(x) = 2x + 1$ is increasing everywhere, it can have a minimum only at a left-hand endpoint of its domain and a maximum only at a right-hand endpoint of its domain. But its domain $[-1, 1)$ has no right-hand endpoint, so f has no maximum. It has the global minimum value $f(-1) = -1$ at the left-hand endpoint of its domain.

C03S05.003: Because $f(x) = |x|$ is decreasing for $x < 0$ and increasing for $x > 0$, it can have a maximum only at a left-hand or a right-hand endpoint of its domain $(-1, 1)$. But its domain has no endpoints, so f has no maximum value. It has the global minimum value $f(0) = 0$.

C03S05.004: Because $g(x) = \sqrt{x}$ is increasing on $(0, 1]$, its reciprocal $f(x) = 1/\sqrt{x}$ is decreasing there. But

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty,$$

so f has no maximum value. It has the global minimum value $f(1) = 1$ at the right-hand endpoint of its domain.

C03S05.005: Given: $f(x) = |x - 2|$ on $(1, 4]$. If $x > 2$ then $f(x) = x - 2$, which is increasing for $x > 2$; if $x < 2$ then $f(x) = 2 - x$, which is decreasing for $x < 2$. So f can have a maximum only at an endpoint of its domain; the only endpoint is at $x = 4$, where $f(x)$ has the maximum value $f(4) = 2$. Because $f(x) \to 1$ as $x \to 1^+$, the extremum at $x = 4$ is in fact a global maximum. Finally, $f(2) = 0$ is the global minimum value of f.

C03S05.006: If $f(x) = 5 - x^2$, then $f'(x) = -2x$, so $x = 0$ is the only critical point of f. We note that f is increasing for $x < 0$ and decreasing for $x > 0$, so $f(0) = 5$ is the global maximum value of f. Because $f(-1) = 4$ and $f(x) \to 1$ as $x \to 2^-$, the minimum at $x = -1$ is local but not global.

C03S05.007: Given: $f(x) = x^3 + 1$ on $[-1, 1]$. The only critical point of f occurs where $f'(x) = 3x^2$ is zero; that is, at $x = 0$. But $f(x) < 1 = f(0)$ if $x < 0$ whereas $f(x) > 1 = f(0)$ if $x > 0$, so there is no extremum at $x = 0$. By Theorem 1 (page 142), f must have a global maximum and a global minimum. The only possible locations are at the endpoints of the domain of f, and therefore $f(-1) = 0$ is the global minimum value of f and $f(1) = 2$ is its global maximum value.

C03S05.008: If

$$f(x) = \frac{1}{x^2 + 1}, \qquad \text{then} \qquad f'(x) = -\frac{2x}{(x^2 + 1)^2},$$

so $x = 0$ is the only critical point of f. Because $g(x) = x^2 + 1$ is increasing for $x > 0$ and decreasing for $x < 0$, we may conclude that f is decreasing for $x > 0$ and increasing for $x < 0$. Therefore f has the global maximum value $f(0) = 1$ at $x = 0$ and no other extrema of any kind.

C03S05.009: If

$$f(x) = \frac{1}{x(1-x)}, \qquad \text{then} \qquad f'(x) = \frac{2x-1}{x^2(1-x)^2},$$

which does not exist at $x = 0$ or at $x = 1$ and is zero when $x = \frac{1}{2}$. But none of these points lies in the domain $[2, 3]$ of f, so there are no extrema at those three points. By Theorem 1 f must have a global maximum and a global minimum, which therefore must occur at the endpoints of its domain. Because $f(2) = -\frac{1}{2}$ and $f(3) = -\frac{1}{6}$, the former is the global minimum value of f and the latter is its global maximum value.

C03S05.010: If

$$f(x) = \frac{1}{x(1-x)}, \qquad \text{then} \qquad f'(x) = \frac{2x-1}{x^2(1-x)^2},$$

which does not exist at $x = 0$ or at $x = 1$ and is zero when $x = \frac{1}{2}$. But the domain $(0, 1)$ of f includes only the last of these three points. We note that

$$f\left(\tfrac{1}{2}\right) = 4 \qquad \text{and that} \qquad \lim_{x \to 0^+} f(x) = +\infty = \lim_{x \to 1^-} f(x),$$

and therefore f has no global maximum value. The reciprocal of $f(x)$ is

$$g(x) = x - x^2 = -(x^2 - x) = -\left(x^2 - x + \tfrac{1}{4}\right) + \tfrac{1}{4} = \tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2,$$

which has the global maximum value $\frac{1}{4}$ at $x = \frac{1}{2}$. Therefore $f(x)$ has the global minimum value 4 at $x = \frac{1}{2}$.

C03S05.011: $f'(x) = 3$ is never zero and always exists. Therefore $f(-2) = -8$ is the global minimum value of f and $f(3) = 7$ is its global maximum value.

C03S05.012: $f'(x) = -3$ always exists and is never zero. Therefore $f(5) = -11$ is the global minimum value of f and $f(-1) = 7$ is its global maximum value.

C03S05.013: $h'(x) = -2x$ always exists and is zero only at $x = 0$, which is not in the domain of h. Therefore $h(1) = 3$ is the global maximum value of h and $h(3) = -5$ is its global minimum value.

C03S05.014: $f'(x) = 2x$ always exists and is zero only at $x = 0$, an endpoint of the domain of f. Therefore $f(0) = 3$ is the global minimum value of f and $f(5) = 28$ is its global maximum value.

C03S05.015: $g'(x) = 2(x - 1)$ always exists and is zero only at $x = 1$. Because $g(-1) = 4$, $g(1) = 0$, and $g(4) = 9$, the global minimum value of g is 0 and the global maximum is 9. If $-1 < x < 0$ then $g(x) = (x - 1)^2 < 4$, so the extremum at $x = -1$ is a local maximum.

C03S05.016: $h'(x) = 2x + 4$ always exists and is zero only at $x = -2$. Because $h(-3) = 4$, $h(-2) = 3$, and $h(0) = 7$, the global minimum value of h is 3 and its global maximum is 7. Because the graph of h is a parabola opening upward, $h(-3) = 4$ is a local (but not global) maximum value of h.

C03S05.017: $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$ always exists and is zero when $x = -1$ and when $x = 1$. Because $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, and $f(4) = 52$, the latter is the global maximum value of f and -2 is its global minimum value—note that the minimum occurs at two different points on the graph. Because f is continuous on $[-2, 2]$, it must have a global maximum there, and our work shows that it occurs at $x = -1$. But because $f(4) = 52 > 2 = f(-1)$, $f(-1) = 2$ is only a local maximum for f on its domain $[-2, 4]$. Summary: Global minimum value -2, local maximum value 2, global maximum value 52.

C03S05.018: $g'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$ always exists and is zero when $x = 1$ and when $x = 2$. Because $g(0) = 0$, $g(1) = 5$, $g(2) = 4$, and $g(4) = 32$, the global minimum value of g is $g(0) = 0$ and its global maximum is $g(4) = 32$. Because g is continuous on $[0, 2]$, it must have a global maximum there, so $g(1) = 5$ is a *local* maximum for g on $[0, 4]$. Because g is continuous on $[1, 4]$, it must have a global minimum there, so $g(2) = 4$ is a *local* minimum for g on $[0, 4]$.

C03S05.019: If

$$h(x) = x + \frac{4}{x}, \qquad \text{then} \qquad h'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}.$$

Therefore h is continuous on $[1, 4]$ and $x = 2$ is the only critical point of h in its domain. Because $h(1) = 5$, $h(2) = 4$, and $h(4) = 5$, the global maximum value of h is 5 and its global minimum value is 4.

C03S05.020: If $f(x) = x^2 + \dfrac{16}{x}$, then $f'(x) = 2x - \dfrac{16}{x^2} = \dfrac{2x^3 - 16}{x^2} = \dfrac{2(x-2)(x^2+2x+4)}{x^2}$. So f is continuous on its domain $[1, 3]$ and its only critical point is $x = 2$. Because $f(1) = 17$, $f(2) = 12$, and $f(3) = \frac{43}{3} \approx 14.333$, the global maximum value of f is 17 and its global minimum value is 12. Because f is continuous on $[2, 3]$, it must have a global maximum there, and therefore $f(3) = \frac{43}{3}$ is a *local* maximum value of f on $[1, 3]$.

C03S05.021: $f'(x) = -2$ always exists and is never zero, so $f(1) = 1$ is the global minimum value of f and $f(-1) = 5$ is its global maximum value.

C03S05.022: $f'(x) = 2x - 4$ always exists and is zero when $x = 2$, which is an endpoint of the domain of f. Hence $f(2) = -1$ is the global minimum value of f and $f(0) = 3$ is its global maximum value.

C03S05.023: $f'(x) = -12 - 18x$ always exists and is zero when $x = -\frac{2}{3}$. Because $f(-1) = 8$, $f\left(-\frac{2}{3}\right) = 9$, and $f(1) = -16$, the global maximum value of f is 9 and its global minimum value is -16. Consideration of the interval $\left[-1, -\frac{2}{3}\right]$ shows that $f(-1) = 8$ is a local minimum of f.

C03S05.024: $f'(x) = 4x - 4$ always exists and is zero when $x = 1$. Because $f(0) = 7$, $f(1) = 5$, and $f(2) = 7$, the global maximum value of f is 7 and its global minimum value is 5.

C03S05.025: $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$ always exists and is zero when $x = -1$ and when $x = 3$. Because $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$, and $f(4) = -15$, the global minimum value of f is -22 and its global maximum is 10. Consideration of the interval $[-2, -1]$ shows that $f(-2) = 3$ is a *local* minimum of f; consideration of the interval $[3, 4]$ shows that $f(4) = -15$ is a *local* maximum of f.

C03S05.026: $f'(x) = 3x^2 + 1$ always exists and is never zero, so $f(-1) = -2$ is the global minimum value of f and $f(2) = 10$ is its global maximum value.

C03S05.027: $f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$ always exists and is zero at $x = -1$, at $x = 0$, and at $x = 1$. We note that $f(-2) = -56$, $f(-1) = 2$, $f(0) = 0$, $f(1) = -2$, and $f(2) = 56$. So the global minimum value of f is -56 and its global maximum value is 56. Consideration of the interval $[-2, 0]$ shows that $f(-1) = 2$ is a *local* maximum of f on its domain $[-2, 2]$. Similarly, $f(1) = -2$ is a *local* minimum of f there. Suppose that x is near, but not equal, to zero. Then $f(x) = x^3(3x^2 - 5)$ is negative if $x > 0$ and positive if $x < 0$. Therefore there is *no extremum* at $x = 0$.

C03S05.028: Given: $f(x) = |2x - 3|$ on $[1, 2]$. If $x > \frac{3}{2}$ then $f(x) = 2x - 3$, so that $f'(x) = 2$. If $x < \frac{3}{2}$ then $f(x) = 3 - 2x$, so that $f'(x) = -2$. Therefore $f'(x)$ is never zero. But it fails to exist at $x = \frac{3}{2}$. Because $f(1) = 1$, $f\left(\frac{3}{2}\right) = 0$, and $f(2) = 1$, the global maximum value of f is 1 and its global minimum value is 0.

C03S05.029: Given: $f(x) = 5 + |7 - 3x|$ on $[1, 5]$. If $x < \frac{7}{3}$, then $-3x > -7$, so that $7 - 3x > 0$; in this case, $f(x) = 12 - 3x$ and so $f'(x) = -3$. Similarly, if $x > \frac{7}{3}$, then $f(x) = 3x - 2$ and so $f'(x) = 3$. Hence $f'(x)$ is never zero, but it fails to exist at $x = \frac{7}{3}$. Now $f(1) = 9$, $f\left(\frac{7}{3}\right) = 5$, and $f(5) = 13$, so 13 is the global maximum value of f and 5 is its global minimum value. Consideration of the continuous function f on the interval $\left[1, \frac{7}{3}\right]$ shows that $f(1) = 9$ is a *local* maximum of f on its domain.

C03S05.030: Given: $f(x) = |x+1| + |x-1|$ on $[-2, 2]$. If $x < -1$ then $f(x) = -(x+1) - (x-1) = -2x$, so that $f'(x) = -2$. If $x > 1$ then $f(x) = x + 1 + x - 1 = 2x$, so that $f'(x) = 2$. If $-1 \leqq x \leqq 1$ then $f(x) = x + 1 - (x - 1) = 2$, so that $f'(x) = 0$. But $f'(x)$ does not exist at $x = -1$ or at $x = 1$. We note that $f(-2) = 4$, $f(x) = 2$ for all x such that $-1 \leqq x \leqq 1$, and that $f(2) = 4$. So 4 is the global maximum value of f and 2 is its global minimum value. Observe that f has infinitely many critical points: every number in the interval $[-1, 1]$.

C03S05.031: $f'(x) = 150x^2 - 210x + 72 = 6(5x - 3)(5x - 4)$ always exists and is zero at $x = \frac{3}{5}$ and at $x = \frac{4}{5}$. Now $f(0) = 0$, $f\left(\frac{3}{5}\right) = 16.2$, $f\left(\frac{4}{5}\right) = 16$, and $f(1) = 17$. Hence 17 is the global maximum value of f and 0 is its global minimum value. Consideration of the intervals $\left[0, \frac{4}{5}\right]$ and $\left[\frac{3}{5}, 1\right]$ shows that 16.2 is a *local* maximum value of f on $[0, 1]$ and that 16 is a *local* minimum value of f there.

C03S05.032: If $f(x) = 2x + \dfrac{1}{2x}$, then $f'(x) = 2 - \dfrac{1}{2x^2} = \dfrac{4x^2 - 1}{2x^2}$. Therefore $f'(x)$ exists for all x in the domain $[1, 4]$ of f and there are no points in the domain of f at which $f'(x) = 0$. Thus the global minimum value of f is $f(1) = 2.5$ and its global maximum value is $f(4) = 8.125$.

C03S05.033: If

$$f(x) = \frac{x}{x+1}, \qquad \text{then} \qquad f'(x) = \frac{1}{(x+1)^2},$$

so $f'(x)$ exists for all x in the domain $[0, 3]$ of f and is never zero there. Hence $f(0) = 0$ is the global minimum value of f and $f(3) = \frac{3}{4}$ is its global maximum value.

C03S05.034: If $f(x) = \dfrac{x}{x^2 + 1}$, then $f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$, so $f'(x)$ exists for all x; the only point in the domain of f at which $f'(x) = 0$ is $x = 1$. Now $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(3) = \frac{3}{10}$, so 0 is the global minimum value of f and $\frac{1}{2}$ is its global maximum value. By the usual argument, there is a local minimum at $x = 3$.

C03S05.035: If

$$f(x) = \frac{1 - x}{x^2 + 3}, \qquad \text{then} \qquad f'(x) = \frac{(x+1)(x-3)}{(x^2+3)^2},$$

so $f'(x)$ always exists and is zero when $x = -1$ and when $x = 3$. Now $f(-2) = \frac{3}{7}$, $f(-1) = \frac{1}{2}$, $f(3) = -\frac{1}{6}$, and $f(5) = -\frac{1}{7}$. So the global minimum value of f is $-\frac{1}{6}$ and its global maximum value is $\frac{1}{2}$. Consideration of the interval $[-2, -1]$ shows that $\frac{3}{7}$ is a *local* minimum value of f; consideration of the interval $[3, 5]$ shows that $-\frac{1}{7}$ is a *local* maximum value of f.

C03S05.036: If $f(x) = 2 - x^{1/3}$, then

$$f'(x) = -\frac{1}{3x^{2/3}},$$

so $f'(x)$ is never zero and $f'(x)$ does not exist when $x = 0$. Nevertheless, f is continuous on its domain $[-1, 8]$. And $f(-1) = 3$, $f(0) = 2$, and $f(8) = 0$, so the global maximum value of f is 3 and its global

minimum value is 0. Because $g(x) = x^{1/3}$ is an increasing function, $f(x) = 2 - x^{1/3}$ is decreasing on its domain, and therefore there is no extremum at $x = 0$.

C03S05.037: Given: $f(x) = x(1 - x^2)^{1/2}$ on $[-1, 1]$. First,

$$f'(x) = (1 - x^2)^{1/2} + x \cdot \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = (1 - x^2)^{1/2} - \frac{x^2}{(1 - x^2)^{1/2}} = \frac{1 - 2x^2}{\sqrt{1 - x^2}}.$$

Hence $f'(x)$ exists for $-1 < x < 1$ and not otherwise, but we will check the endpoints ± 1 of the domain of f separately. Also $f'(x) = 0$ when $x = \pm\frac{1}{2}\sqrt{2}$. Now $f(-1) = 0$, $f\left(-\frac{1}{2}\sqrt{2}\right) = -\frac{1}{2}$, $f\left(\frac{1}{2}\sqrt{2}\right) = \frac{1}{2}$, and $f(1) = 0$. Therefore the global minimum value of f is $-\frac{1}{2}$ and its global maximum value is $\frac{1}{2}$. Consideration of the interval $\left[-1, -\frac{1}{2}\sqrt{2}\right]$ shows that $f(-1) = 0$ is a *local* maximum value of f on $[-1, 1]$; similarly, $f(1) = 0$ is a *local* minimum value of f there.

C03S05.038: Given: $f(x) = x(4 - x^2)^{1/2}$ on $[0, 2]$. Then

$$f'(x) = (4 - x^2)^{1/2} + x \cdot \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x) = (4 - x^2)^{1/2} - \frac{x^2}{(4 - x^2)^{1/2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}},$$

so $f'(x)$ exists if $0 \leqq x < 2$ and is zero when $x = \sqrt{2}$. Now $f(0) = 0 = f(2)$ and $f\left(\sqrt{2}\right) = 2$, so the former is the global minimum value of f on $[0, 2]$ and the latter is its global maximum value there.

C03S05.039: Given: $f(x) = x(2 - x)^{1/3}$ on $[1, 3]$. Then

$$f'(x) = (2 - x)^{1/3} + x \cdot \frac{1}{3}(2 - x)^{-2/3} \cdot (-1) = (2 - x)^{1/3} - \frac{x}{3(2 - x)^{2/3}} = \frac{6 - 4x}{3(2 - x)^{2/3}}.$$

Then $f'(2)$ does not exist and $f'(x) = 0$ when $x = \frac{3}{2}$. Also f is continuous everywhere, and $f(1) = 1$, $f\left(\frac{3}{2}\right) \approx 1.19$, and $f(3) = -3$. Hence the global minimum value of f is -3 and its global maximum value is $f\left(\frac{3}{2}\right) = 3 \cdot 2^{-4/3} \approx 1.190551$. Consideration of the interval $\left[1, \frac{3}{2}\right]$ shows that $f(1) = 1$ is a *local* minimum value of f.

C03S05.040: Given: $f(x) = x^{1/2} - x^{3/2}$ on $[0, 4]$. Then

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2x^{1/2}} - \frac{3x^{1/2}}{2} = \frac{1 - 3x}{2\sqrt{x}}.$$

Then $f'(x)$ does not exist when $x = 0$, although f is continuous on its domain; also, $f'(x) = 0$ when $x = \frac{1}{3}$. Now $f(0) = 0$, $f\left(\frac{1}{3}\right) = \frac{2}{9}\sqrt{3}$, and $f(4) = -6$. So -6 is the global minimum value of f and its global maximum value is $\frac{2}{9}\sqrt{3}$. Consideration of the interval $\left[0, \frac{1}{3}\right]$ shows that $f(0) = 0$ is a *local* minimum value of f.

C03S05.041: If $A \neq 0$, then $f'(x) \equiv A$ is never zero, but because f is continuous it must have global extrema. Therefore they occur at the endpoints. If $A = 0$, then f is a constant function, and its maximum and minimum value B occurs at every point of the interval, including the two endpoints.

C03S05.042: The hypotheses imply that f has no critical points in (a, b), but f must have global extrema. Therefore they occur at the endpoints.

C03S05.043: $f'(x) = 0$ if x is not an integer; $f'(x)$ does not exist if x is an integer (we saw in Chapter 2 that $f(x) = [\![x]\!]$ is discontinuous at each integer).

C03S05.044: If $f(x) = ax^2 + bx + c$ and $a \neq 0$, then $f'(x) = 2ax + b$. Clearly $f'(x)$ exists for all x, and $f'(x) = 0$ has the unique solution $x = -b/(2a)$. Therefore f has exactly one critical point on the real number line.

C03S05.045: If $f(x) = ax^3 + bx^2 + cx + d$ and $a \neq 0$, then $f'(x) = 3ax^2 + 2bx + c$ exists for all x, but the quadratic equation $3ax^2 + 2bx + c = 0$ has two solutions if the discriminant $\Delta = 4b^2 - 12ac$ is positive, one solution if $\Delta = 0$, and no [real] solutions if $\Delta < 0$. Therefore f has either no critical points, exactly one critical point, or exactly two. Examples:

$$\begin{aligned} f(x) = x^3 + x \qquad & \text{has no critical points,} \\ f(x) = x^3 \qquad & \text{has exactly one critical point, and} \\ f(x) = x^3 - 3x \qquad & \text{has exactly two critical points.} \end{aligned}$$

C03S05.046: A formula for f is

$$f(x) = \min\{x - [\![x]\!],\ 1 + [\![x]\!] - x\}. \tag{1}$$

If you are not comfortable with the idea that "min" is a "function," an equivalent way of defining f is this:

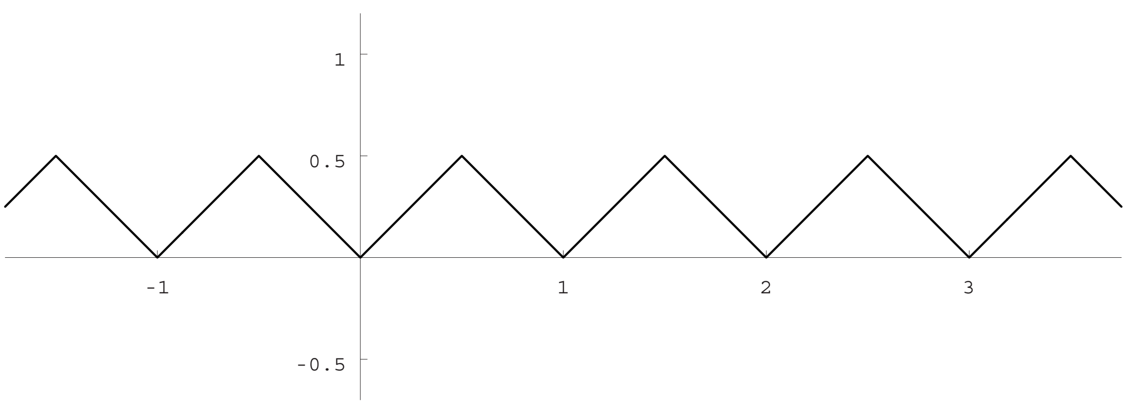
$$f(x) = \frac{1}{2}\Big(1 - \big|2x - 1 - 2[\![x]\!]\big|\Big).$$

To verify that f performs as advertised, suppose that x is a real number and that $n = [\![x]\!]$, so that $n \leqq x < n + 1$. Case (1): $n \leqq x \leqq n + \frac{1}{2}$. Then

$$x - [\![x]\!] = x - n \leqq \frac{1}{2} \quad \text{and} \quad 1 + [\![x]\!] - x = 1 + n - x = 1 - (x - n) \geqq \frac{1}{2},$$

so that Eq. (1) yields $f(x) = x - n$, which is indeed the distance from x to the nearest integer, because in Case (1) the nearest integer is n. Case (2), in which $n + \frac{1}{2} < x < n + 1$, is handled similarly.

The graph of f is shown next. It should be clear that $f'(x)$ fails to exist at every integral multiple of $\frac{1}{2}$ and that its derivative is either $+1$ or -1 otherwise. Hence its critical points are the integral multiples of $\frac{1}{2}$.



C03S05.047: The derivative is positive on $(-\infty, -1.3)$, negative on $(-1.3, 1.3)$, and positive on $(1.3, +\infty)$. So its graph must be the one in Fig. 3.5.15(c). (Numbers with decimal points are approximations.)

C03S05.048: The derivative is negative on $(-\infty, -1.0)$, positive on $(-1.0, 1.0)$, negative on $(1.0, 3.0)$, and positive on $(3.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(f). (Numbers with decimal points are approximations.)

C03S05.049: The derivative is positive on $(-\infty, 0.0)$, negative on $(0.0, 2.0)$, and positive on $(2.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(d). (Numbers with decimal points are approximations.)

C03S05.050: The derivative is positive on $(-\infty, -2.0)$, negative on $(-2.0, 0.0)$, positive on $(0.0, 2.0)$, and negative on $(2.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(b). (Numbers with decimal points are approximations.)

C03S05.051: The derivative is negative on $(-\infty, -2.0)$, positive on $(-2.0, 1.0)$, and negative on $(1.0, +\infty)$. Therefore its graph must be the one shown in Fig. 3.5.15(a). (Numbers with decimal points are approximations.)

C03S05.052: The derivative is negative on $(-\infty, -2.2)$, positive on $(-2.2, 2.2)$, and negative again on $(2.2, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(e). (Numbers with decimal points are approximations.)

Note: In Problems 53 through 60, we used *Mathematica* 3.0 and Newton's method (when necessary), carrying 40 decimal digits throughout all computations. Answers are correct or correctly rounded to the number of digits shown. Your answers may differ in the last (or last few) digits beause of differences in hardware or software. Using a graphing calculator or computer to zoom in on solutions has more limited accuracy when using certain machines.

C03S05.053: Global maximum value 28 at the left endpoint $x = -2$, global minimum value approximately 6.828387610996 at the critical point where $x = -1 + \frac{1}{3}\sqrt{30} \approx 0.825741858351$, local maximum value 16 at the right endpoint $x = 2$.

C03S05.054: Local minimum value 22 at the left endpoint $x = -4$, global maximum value approximately 31.171612389004 at the critical point $x = -1 - \frac{1}{3}\sqrt{30} \approx -2.825741858351$, global minimum value approximately 6.828387610996 at the critical point $x = -1 + \frac{1}{3}\sqrt{30} \approx 0.825741858351$, local maximum value 16 at the right endpoint $x = 2$.

C03S05.055: Global maximum value 136 at the left endpoint $x = -3$, global minimum value approximately -8.669500829438 at the critical point $x \approx -0.762212740507$, local maximum value 16 at the right endpont $x = 3$.

C03S05.056: Global maximum value 160 at the left endpoint $x = -3$, global minimum value approximately -16.048632589199 at the critical point $x \approx -0.950838582066$, local maximum value approximately 8.976226903748 at the critical point $x \approx 1.323417756580$, local minimum value -8 at the right endpoint $x = 3$.

C03S05.057: Global minimum value -5 at the left endpoint $x = 0$, global maximum value approximately 8.976226903748 at the critical point $x \approx 1.323417756580$, local minimum value 5 at the right endpoint $x = 2$.

C03S05.058: Local maximum value 3 at the left endpoint $x = -1$, global minimum value approximately -5.767229705222 at the critical point $x \approx -0.460141424682$, global maximum value approximately 21.047667292488 at the critical point $x \approx 0.967947424014$, local minimum value 21 at the right endpoint $x = 1$.

C03S05.059: Local minimum value -159 at the left endpoint $x = -3$, global maximum value approxiately 30.643243080334 at the critical point $x \approx -1.911336401963$, local minimum value approximately -5.767229705222 at the critical point $x \approx -0.460141424682$, local maximum value approximately 21.047667292488 at the critical point $x \approx 0.967947424014$, global minimum value -345 at the right endpoint $x = 3$.

C03S05.060: Local minimum value 0 at the left endpoint $x = 0$, local maximum value approximately 21.047667292488 at the critical point $x \approx 0.967947424014$, global minimum value approximately -1401.923680667600 at the critical point $x \approx 5.403530402632$, global maximum value 36930 at the right endpoint $x = 10$.

Section 3.6

C03S06.001: With $x > 0$, $y > 0$, and $x + y = 50$, we are to maximize the product $P = xy$.

$$P = P(x) = x(50 - x) = 50x - x^2, \qquad 0 < x < 50$$

($x < 50$ because $y > 0$.) The product is not maximal if we let $x = 0$ or $x = 50$, so we adjoin the endpoints to the domain of P; thus the continuous function $P(x) = 50x - x^2$ has a global maximum on the closed interval $[0, 50]$, and the maximum does *not* occur at either endpoint. Because f is differentiable, the maximum must occur at a point where $P'(x) = 0$: $50 - 2x = 0$, and so $x = 25$. Because this is the only critical point of P, it follows that $x = 25$ maximizes $P(x)$. When $x = 25$, $y = 50 - 25 = 25$, so the two positive real numbers with sum 50 and maximum possible product are 25 and 25.

C03S06.002: If two parallel sides of the rectangle both have length x and the other two sides both have length y, then we are to maximize the area $A = xy$ given that $2x + 2y = 200$. So

$$A = A(x) = x(100 - x), \qquad 0 \leqq x \leqq 100.$$

Clearly the maximum value of A occurs at a critical point of A in the interval (0, 100). But $A'(x) = 100 - 2x$, so $x = 50$ is the location of the maximum. When $x = 50$, also $y = 50$, so the rectangle of maximal area is a square of area $50^2 = 2500$ ft^2.

C03S06.003: If the coordinates of the "fourth vertex" are (x, y), then $y = 100 - 2x$ and the area of the rectangle is $A = xy$. So we are to maximize

$$A(x) = x(100 - 2x) \qquad 0 \leqq x \leqq 50.$$

By the usual argument the solution occurs where $A'(x) = 0$, thus where $x = 25$, $y = 50$, and the maximum area is 1250.

C03S06.004: If the side of the pen parallel to the wall has length x and the two perpendicular sides both have length y, then we are to maximize area $A = xy$ given $x + 2y = 600$. Thus

$$A = A(y) = y(600 - 2y), \qquad 0 \leqq y \leqq 300.$$

Adjoining the endpoints to the domain is allowed because the maximum we seek occurs at neither endpoint. Therefore the maximum occurs at an interior critical point. We have $A'(y) = 600 - 4y$, so the only critical point of A is $y = 150$. When $y = 150$, we have $x = 300$, so the maximum possible area that can be enclosed is 45000 m^2.

C03S06.005: If x is the length of each edge of the base of the box and y denotes the height of the box, then its volume is given by $V = x^2y$. Its total surface area is the sum of the area x^2 of its bottom and four times the area xy of each of its vertical sides, so $x^2 + 4xy = 300$. Thus

$$V = V(x) = x^2 \cdot \frac{300 - x^2}{4x} = \frac{300x - x^3}{4}, \qquad 1 \leqq x \leqq 10\sqrt{3}.$$

Hence

$$V'(x) = \frac{300 - 3x^2}{4},$$

so $V'(x)$ always exists and $V'(x) = 0$ when $x = 10$ (we discard the solution $x = -10$; it's not in the domain of V). Then

$$V(1) = \frac{299}{4} = 74.75, \qquad V(10) = 500, \qquad \text{and} \qquad V\left(10\sqrt{3}\right) = 0,$$

so the maximum possible volume of the box is 500 in.^{3}.

C03S06.006: The excess of the number x over its square is $f(x) = x - x^2$. In this problem we also know that $0 \leqq x \leqq 1$. Then $f(x) = 0$ at the endpoints of its domain, so the maximum value of $f(x)$ must occur at an interior critical point. But $f'(x) = 1 - 2x$, so the only critical point of f is $x = \frac{1}{2}$, which must yield a maximum because f is continuous on $[0, 1]$. So the maximum value of $x - x^2$ for $0 \leqq x \leqq 1$ is $\frac{1}{4}$.

C03S06.007: If the two numbers are x and y, then we are to minimize $S = x^2 + y^2$ given $x > 0$, $y > 0$, and $x + y = 48$. So $S(x) = x^2 + (48 - x)^2$, $0 \leqq x \leqq 48$. Here we adjoin the endpoints to the domain of S to ensure the existence of a maximum, but we must test the values of S at these endpoints because it is not immediately clear that neither $S(0)$ nor $S(48)$ yields the maximum value of S. Now $S'(x) = 2x - 2(48 - x)$; the only interior critical point of S is $x = 24$, and when $x = 24$, $y = 24$ as well. Finally, $S(0) = (48)^2 = 2304 = S(48) > 1152 = S(24)$, so the answer is 1152.

C03S06.008: Let x be the length of the side around which the rectangle is rotated and let y be the length of each perpendicular side. Then $2x + 2y = 36$. The radius of the cylinder is y and its height is x, so its volume is $V = \pi y^2 x$. So

$$V = V(y) = \pi y^2(18 - y) = \pi(18y^2 - y^3),$$

with natural domain $0 < y < 18$. We adjoin the endpoints to the domain because neither $y = 0$ nor $y = 18$ maximizes $V(y)$, and deduce the existence of a global maximum at an interior critical point. Now

$$V'(y) = \pi(36y - 3y^2) = 3\pi y(12 - y).$$

So $V'(y) = 0$ when $y = 0$ and when $y = 12$. The former value of y *minimizes* $V(y)$, so the maximum possible volume of the cylinder is $V(12) = 864\pi$.

C03S06.009: Let x and y be the two numbers. Then $x + y = 10$, $x \geqq 0$, and $y \geqq 0$. We are to minimize the sum of their cubes,

$$S = x^3 + y^3: \qquad S(x) = x^3 + (10 - x)^3, \qquad 0 \leqq x \leqq 10.$$

Now $S'(x) = 3x^2 - 3(10 - x)^2$, so the values of x to be tested are $x = 0$, $x = 5$, and $x = 10$. At the endpoints, $S = 1000$; when $x = 5$, $S = 250$ (the minimum).

C03S06.010: Draw a cross section of the cylindrical log—a circle of radius r. Inscribe in this circle a cross section of the beam—a rectangle of width w and height h. Draw a diagonal of the rectangle; the Pythagorean theorem yields $x^2 + h^2 = 4r^2$. The strength S of the beam is given by $S = kwh^2$ where k is a positive constant. Because $h^2 = 4r^2 - w^2$, we have

$$S = S(w) = kw(4r^2 - w^2) = k(4wr^2 - w^3)$$

with natural domain $0 < w < 2r$. We adjoin the endpoints to this domain; this is permissible because $S = 0$ at each, and so is not maximal. Next, $S'(w) = k(4r^2 - 3w^2)$; $S'(w) = 0$ when $3w^2 = 4r^2$, and the corresponding (positive) value of w yields the maximum of S (we know that $S(w)$ must have a maximum on $[0, 2r]$ because

of the continuity of S on this interval, and we also know that the maximum does not occur at either endpoint, so there is only one possible location for the maximum). At maximum, $h^2 = 4r^2 - w^2 = 3w^2 - w^2$, so $h = w\sqrt{2}$ describes the shape of the beam of greatest strength.

C03S06.011: As in Fig. 3.6.18, let y denote the length of each of the internal dividers and of the two sides parallel to them; let x denote the length of each of the other two sides. The total length of all the fencing is $2x + 4y = 600$ and the area of the corral is $A = xy$. Hence

$$A = A(y) = \frac{600 - 4y}{2} \cdot y = 300y - 2y^2, \qquad 0 \leqq y \leqq 150.$$

Now $A'(y) = 0$ only when $y = 75$, and $A(0) = 0 = A(150)$, and therefore the maximum area of the corral is $A(75) = 11250 \text{ yd}^2$.

C03S06.012: Let r denote the radius of the cylinder and h its height. We are to maximize its volume $V = \pi r^2 h$ given the constraint that the total surface area is 150π:

$$2\pi r^2 + 2\pi r h = 150\pi, \qquad \text{so that} \qquad h = \frac{75 - r^2}{r}.$$

Thus

$$V = V(r) = \pi r(75 - r^2) = \pi(75r - r^3), \qquad 0 < r < \sqrt{75}.$$

We may adjoin both endpoints to this domain without creating a spurious maximum, so we use $\left[0, 5\sqrt{3}\,\right]$ as the domain of V. Next, $V'(r) = \pi(75 - 3r^2)$. Hence $V'(r)$ always exists and its only zero in the domain of V occurs when $r = 5$ (and $h = 10$). But V is zero at the two endpoints of its domain, so $V(5) = 250\pi$ is the maximum volume of such a cylinder.

C03S06.013: If the rectangle has sides x and y, then $x^2 + y^2 = 16^2$ by the Pythagorean theorem. The area of the rectangle is then

$$A(x) = x\sqrt{256 - x^2}\,, \qquad 0 \leqq x \leqq 16.$$

A positive quantity is maximized exactly when its square is maximized, so in place of A we maximize

$$f(x) = (A(x))^2 = 256x^2 - x^4.$$

The only solutions of $f'(x) = 0$ in the domain of A are $x = 0$ and $x = 8\sqrt{2}$. But $A(0) = 0 = A(16)$, so $x = 8\sqrt{2}$ yields the maximum value 128 of A.

C03S06.014: If the far side of the rectangle has length $2x$ (this leads to simpler algebra than length x), and the sides perpendicular to the far side have length y, then by the Pythagorean theorem, $x^2 + y^2 = L^2$. The area of the rectangle is $A = 2xy$, so we maximize

$$A(x) = 2x\sqrt{L^2 - x^2}\,, \qquad 0 \leqq x \leqq L$$

by maximizing

$$f(x) = (A(x))^2 = 4(L^2x^2 - x^4).$$

Now $f'(x) = 4(2L^2x - 4x^3) = 8x(L^2 - 2x^2)$ is zero when $x = 0$ (rejected; $A(0) = 0$) and when $x = \frac{1}{2}L\sqrt{2}$. Note also that $A(L) = 0$. By the usual argument, $x = \frac{1}{2}L\sqrt{2}$ maximizes $f(x)$ and thus $A(x)$. The answer is $A\left(\frac{1}{2}L\sqrt{2}\right) = L^2$.

C03S06.015: $V'(T) = -0.06426 + (0.0170086)T - (0.0002037)T^2$. The equation $V'(T) = 0$ is quadratic with the two (approximate) solutions $T \approx 79.532$ and $T \approx 3.967$. The formula for $V(T)$ is valid only in the range $0 \leqq T \leqq 30$, so we reject the first solution. Finally, $V(0) = 999.87$, $V(30) \approx 1003.763$, and $V(3.967) \approx 999.71$. Thus the volume is minimized when $T \approx 3.967$, and therefore water has its greatest density at about 3.967°C.

C03S06.016: Let $P(x, 0)$ be the lower right-hand corner point of the rectangle. The rectangle then has base $2x$, height $4 - x^2$, and thus area

$$A(x) = 2x(4 - x^2) = 8x - 2x^3, \qquad 0 \leqq x \leqq 2.$$

Now $A'(x) = 8 - 6x^2$; $A'(x) = 0$ when $x = \frac{2}{3}\sqrt{3}$. Because $A(0) = 0$, $A(2) = 0$, and $A\left(\frac{2}{3}\sqrt{3}\right) > 0$, the maximum possible area is $A\left(\frac{2}{3}\sqrt{3}\right) = \frac{32}{9}\sqrt{3}$.

C03S06.017: Let x denote the length of each edge of the base and let y denote the height of the box. We are to maximize its volume $V = x^2y$ given the constraint $2x^2 + 4xy = 600$. Solve the latter for y to write

$$V(x) = 150x - \frac{1}{2}x^3, \qquad 1 \leqq x \leqq 10\sqrt{3}.$$

The solution of $V'(x) = 0$ in the domain of V is $x = 10$. Because $V(10) = 1000 > V(1) = 149.5 > V\left(10\sqrt{3}\right) = 0$, this shows that $x = 10$ maximizes V and that the maximum value of V is 1000 cm^3.

C03S06.018: Let x denote the radius of the cylinder and y its height. Then its total surface area is $\pi x^2 + 2\pi xy = 300\pi$, so $x^2 + 2xy = 300$. We are to maximize its volume $V = \pi x^2 y$. Because

$$y = \frac{300 - x^2}{2x}, \qquad \text{it follows that} \qquad V = V(x) = \frac{\pi}{2}(300x - x^3), \quad 0 \leqq x \leqq 10\sqrt{3}.$$

It is then easy to show that $x = 10$ maximizes $V(x)$, that $y = x = 10$ as well, and thus that the maximum possible volume of the can is 1000π in.^{3}

C03S06.019: Let x be the length of the edge of each of the twelve small squares. Then each of the three cross-shaped pieces will form boxes with base length $1 - 2x$ and height x, so each of the three will have volume $x(1 - 2x)^2$. Both of the two cubical boxes will have edge x and thus volume x^3. So the total volume of all five boxes will be

$$V(x) = 3x(1 - 2x)^2 + 2x^3 = 14x^3 - 12x^2 + 3x, \qquad 0 \leqq x \leqq \frac{1}{2}.$$

Now $V'(x) = 42x^2 - 24x + 3$; $V'(x) = 0$ when $14x^2 - 8x - 1 = 0$. The quadratic formula gives the two solutions $x = \frac{1}{14}\left(4 \pm \sqrt{2}\right)$. These are approximately 0.3867 and 0.1847, and both lie in the domain of V. Finally, $V(0) = 0$, $V(0.1847) \approx 0.2329$, $V(0.3867) \approx 0.1752$, and $V(0.5) = 0.25$. Therefore, to maximize V, one must cut each of the three large squares into four smaller squares of side length $\frac{1}{2}$ each and form the resulting twelve squares into two cubes. At maximum volume there will be only two boxes, not five.

C03S06.020: Let x be the length of each edge of the square base of the box and let h denote its height. Then its volume is $V = x^2h$. The total cost of the box is \$144, hence

$$4xh + x^2 + 2x^2 = 144 \qquad \text{and thus} \qquad h = \frac{144 - 3x^2}{4x}.$$

Therefore

$$V = V(x) = \frac{x}{4}\left(144 - 3x^2\right) = 36x - \frac{3}{4}x^3.$$

The natural domain of V is the open interval $\left(0,\ 4\sqrt{3}\right)$, but we may adjoin the endpoints as usual to obtain a closed interval. Also

$$V'(x) = 36 - \frac{9}{4}x^2,$$

so $V'(x)$ always exists and is zero only at $x = 4$ (reject the other root $x = -4$). Finally, $V(x) = 0$ at the endpoints of its domain, so $V(4) = 96$ (ft^3) is the maximum volume of such a box. The dimensions of the largest box are 4 ft square on the base by 6 ft high.

C03S06.021: Let x denote the edge length of one square and y that of the other. Then $4x + 4y = 80$, so $y = 20 - x$. The total area of the two squares is $A = x^2 + y^2$, so

$$A = A(x) = x^2 + (20 - x)^2 = 2x^2 - 40x + 400,$$

with domain (0, 20); adjoin the endpoints as usual. Then $A'(x) = 4x - 40$, which always exists and which vanishes when $x = 10$. Now $A(0) = 400 = A(20)$, whereas $A(10) = 200$. So to minimize the total area of the two squares, make two equal squares. To maximize it, make only one square.

C03S06.022: Let r be the radius of the circle and x the edge of the square. We are to maximize total area $A = \pi r^2 + x^2$ given the side condition $2\pi r + 4x = 100$. From the last equation we infer that

$$x = \frac{100 - 2\pi r}{4} = \frac{50 - \pi r}{2}.$$

So

$$A = A(r) = \pi r^2 + \frac{1}{4}(50 - \pi r)^2 = \left(\pi + \frac{1}{4}\pi^2\right)r^2 - 25\pi r + 625$$

for $0 \leqq r \leqq 50/\pi$ (because $x \geqq 0$). Now

$$A'(r) = 2\left(\pi + \frac{1}{4}\pi^2\right)r - 25\pi;$$

$$A'(r) = 0 \quad \text{when} \quad r = \frac{25}{2 + \frac{\pi}{2}} = \frac{50}{\pi + 4};$$

that is, when $r \approx 7$. Finally,

$$A(0) = 625, \quad A\left(\frac{50}{\pi}\right) \approx 795.77 \quad \text{and} \quad A\left(\frac{50}{\pi + 4}\right) \approx 350.06.$$

Results: For minimum area, construct a circle of radius $50/(\pi+4) \approx 7.00124$ (cm) and a square of edge length $100/(\pi + 4) \approx 14.00248$ (cm). For maximum area, bend all the wire into a circle of radius $50/\pi \approx 15.91549$ (cm).

C03S06.023: Let x be the length of each segment of fence perpendicular to the wall and let y be the length of each segment parallel to the wall.

Case 1: The internal fence is perpendicular to the wall. Then $y = 600 - 3x$ and the enclosure will have area $A(x) = 600x - 3x^2$, $0 \leqq x \leqq 200$. Then $A'(x) = 0$ when $x = 100$; $A(100) = 30000$ (m^2) is the maximum in Case 1.

Case 2: The internal fence is parallel to the wall. Then $y = 300 - x$, and the area of the enclosure is given by $A(x) = 300x - x^2$, $0 \leqq x \leqq 300$. Then $A'(x) = 0$ when $x = 150$; $A(150) = 22500$ (m^2) is the maximum in Case 2.

Answer: The maximum possible area of the enclosure is 30000 m^2. The divider must be perpendicular to the wall and of length 100 m. The side parallel to the wall is to have length 300 m.

C03S06.024: See Fig. 3.6.22 of the text. Suppose that the pen measures x (horizontal) by y (vertical). Then it has area $A = xy$.

Case 1: $x \geqq 10$, $y \geqq 5$. Then

$$x + (x - 10) + y + (y - 5) = 85, \quad \text{so} \quad x + y = 50.$$

Therefore

$$A = A(x) = x(50 - x) = 50x - x^2, \qquad 10 \leqq x \leqq 45.$$

Then $A'(x) = 0$ when $x = 25$; $A(25) = 625$. Note that $A(10) = 400$ and that $A(45) = 225$.

Case 2: $0 \leqq x \leqq 10$, $y \geqq 5$. Then

$$x + y + (y - 5) = 85, \quad \text{so} \quad x + 2y = 90.$$

Therefore

$$A = A(x) = x\frac{90 - x}{2} = \frac{1}{2}(90x - x^2), \qquad 0 \leqq x \leqq 10.$$

In this case, $A'(x) = 0$ when $x = 45$, but 45 doesn't lie in the domain of A. Note that $A(0) = 0$ and that $A(10) = 400$.

Case 3: $x \geqq 10$, $0 \leqq y \leqq 5$. Then

$$x + (x - 10) + y = 85, \quad \text{so} \quad 2x + y = 95.$$

Therefore

$$A = A(x) = x(95 - 2x) = 95x - 2x^2, \qquad 45 \leqq x \leqq 47.5.$$

In this case $A'(x) = 0$ when $x = 23.75$, not in the domain of A. Note that $A(45) = 225$ and that $A(47.5) = 0$.

Conclusion: The area of the pen is maximized when the pen is square, 25 m on each side (the maximum from Case 1).

C03S06.025: Let the dimensions of the box be x by x by y. We are to maximize $V = x^2y$ subject to some conditions on x and y. According to the poster on the wall of the Bogart, Georgia Post Office, the *length* of the box is the larger of x and y, and the *girth* is measured around the box in a plane perpendicular to its length.

Case 1: $x < y$. Then the length is y, the girth is $4x$, and the mailing constraint is $4x + y \leqq 100$. It is clear that we take $4x + y = 100$ to maximize V, so that

$$V = V(x) = x^2(100 - 4x) = 100x^2 - 4x^3, \qquad 0 \leqq x \leqq 25.$$

Then $V'(x) = 4x(50 - 3x)$; $V'(x) = 0$ for $x = 0$ and for $x = 50/3$. But $V(0) = 0$, $V(25) = 0$, and $V(50/3) = 250000/27 \approx 9259$ (in.^{3}). The latter is the maximum in Case 1.

Case 2: $x \geqq y$. Then the length is x and the girth is $2x + y$, although you may get some argument from a postal worker who may insist that it's $4x$. So $3x + 2y = 100$, and thus

$$V = V(x) = x^2 \left(\frac{100 - 3x}{2} \right) = 50x^2 - \frac{3}{2}x^3, \qquad 0 \leqq x \leqq 100/3.$$

Then $V'(x) = 100x - \frac{9}{2}x^2$; $V'(x) = 0$ when $x = 0$ and when $x = 200/9$. But $V(0) = 0$, $V(100/3) = 0$, and $V(200/9) = 2000000/243 \approx 8230$ (in.^{3}).

Case 3: You lose the argument in Case 2. Then the box has length x and girth $4x$, so $5x = 100$; thus $x = 20$. To maximize the total volume, no calculus is needed—let $y = x$. Then the box of maximum volume will have volume $20^3 = 8000$ (in.^{3}).

Answer: The maximum is $\dfrac{250000}{27}$ in.^{3}

C03S06.026: In this problem the girth of the package is its circumference; no one would interpret "girth" in any other way. So suppose that the package has length x and radius r. Then it has volume $V = \pi r^2 x$ where $x + 2\pi r = 100$. We seek to maximize

$$V = V(r) = \pi r^2 (100 - 2\pi r) = \pi(100r^2 - 2\pi r^3), \qquad 0 \leqq r \leqq \frac{50}{\pi}.$$

Now

$$V'(r) = \pi(200r - 6\pi r^2) = 2\pi r(100 - 3\pi r);$$

$V'(r) = 0$ when $r = 0$ and when $r = 100/(3\pi)$. But

$$V(0) = 0, \qquad V\left(\frac{50}{\pi}\right) = 0, \qquad \text{and} \qquad V\left(\frac{100}{3\pi}\right) = \frac{1000000}{27\pi} \approx 11789 \ \ (\text{in.}^3),$$

and the latter is clearly the maximum of V.

C03S06.027: Suppose that n presses are used, $1 \leqq n \leqq 8$. The total cost of the poster run would then be

$$C(n) = 5n + (10 + 6n)\left(\frac{50000}{3600n}\right) = 5n + \frac{125}{9}\left(\frac{10}{n} + 6\right)$$

dollars. Temporarily assume that n can take on every real number value between 1 and 8. Then

$$C'(n) = 5 - \frac{125}{9} \cdot \frac{10}{n^2};$$

$C'(n) = 0$ when $n = \frac{5}{3}\sqrt{10} \approx 5.27$ presses. But an integral number of presses must be used, so the actual number that will minimize the cost is either 5 or 6, unless the minimum occurs at one of the two endpoints. The values in question are $C(1) \approx 227.2$, $C(5) \approx 136.1$, $C(6) \approx 136.5$, and $C(8) \approx 140.7$. So to minimize cost and thereby maximize the profit, five presses should be used.

C03S06.028: Let x denote the number of workers hired. Each worker will pick $900/x$ bushels; each worker will spend $180/x$ hours picking beans. The supervisor cost will be $1800/x$ dollars, and the cost per worker will be $8 + (900/x)$ dollars. Thus the total cost will be

$$C(x) = 8x + 900 + \frac{1800}{x}, \qquad 1 \leqq x.$$

It is clear that large values of x make $C(x)$ large, so the global minimum of $C(x)$ occurs either at $x = 1$ or where $C'(x) = 0$. Assume for the moment that x can take on all real number values in $[1, +\infty)$, not merely integral values, so that C' is defined. Then

$$C'(x) = 8 - \frac{1800}{x^2}; \;\; C'(x) = 0 \quad \text{when} \quad x^2 = 225.$$

Thus $C'(15) = 0$. Now $C(1) = 2708$ and $C(15) = 1140$, so fifteen workers should be hired; the cost to pick each bushel will be approximately \$1.27.

C03S06.029: We are to minimize the total cost C over a ten-year period. This cost is the sum of the initial cost and ten times the annual cost:

$$C(x) = 150x + 10\left(\frac{1000}{2+x}\right), \qquad 0 \leqq x \leqq 10.$$

Next,

$$C'(x) = 150 - \frac{10000}{(2+x)^2}; \quad C'(x) = 0 \quad \text{when} \quad 150 = \frac{10000}{(2+x)^2},$$

so that $(2+x)^2 = \frac{200}{3}$. One of the resulting values of x is negative, so we reject it. The other is $x = -2 + \sqrt{200/3} \approx 6.165$ (in.). The problem itself suggests that x must be an integer, so we check $x = 6$ and $x = 7$ along with the endpoints of the domain of C. In dollars, $C(0) = 5000$, $C(6) \approx 2150$, $C(7) \approx 2161$, and $C(10) \approx 2333$. Result: Install six inches of insulation. The annual savings over the situation with no insulation at all then will be one-tenth of $5000 - 2150$, about \$285 per year.

C03S06.030: We assume that each one-cent increase in price reduces sales by 50 burritos per night. Let x be the amount, in cents, by which the price is increased. The resulting profit is

$$\begin{aligned}
P(x) &= (50+x)(5000-5x) - 25(5000-50x) - 100000 \\
&= (25+x)(5000-50x) - 100000 \\
&= 25000 + 3750x - 50x^2, \qquad -50 \leqq x.
\end{aligned}$$

Because $P(x) < 0$ for large values of x and for $x = -50$, P will be maximized where $P'(x) = 0$:

$$P'(x) = 3750 - 100x; \quad P'(x) = 0 \quad \text{when} \quad x = 37.5.$$

Now $P(37) = 953$, $P(37.5) \approx 953.13$, and $P(38) = 953$. Therefore profit is maximized when the selling price is either 87¢ or 88¢, and the maximum profit will be \$953.

C03S06.031: Let x be the number of five-cent fare increases. The resulting revenue will be

$$R(x) = (150+5x)(600-40x), \qquad -15 \leqq x \leqq 15$$

(the revenue is the product of the price and the number of passengers). Now

$$\begin{aligned}
&R(x) = 90000 - 3000x - 200x^2; \\
&R'(x) = -3000 - 400x; \qquad R'(x) = 0 \quad \text{when} \quad x = -7.5.
\end{aligned}$$

Because the fare must be an integral number of cents, we check $R(-7) = 1012 = R(-8)$ (dollars). Answer: The fare should be either \$1.10 or \$1.15; this is a reduction of 40 or 35 cents, respectively, and each results in the maximum possible revenue of \$1012 per day.

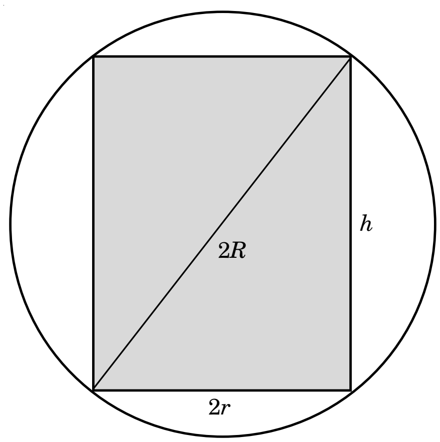
C03S06.032: The following figure shows a central cross section of the sphere and inscribed cylinder. The radius of the cylinder is r and its height is h; the radius of the sphere is R. From the Pythagorean theorem we see that $4r^2 + h^2 = 4R^2$. The volume of the cylinder is $V = \pi r^2 h$, and therefore we find that

$$\begin{aligned} V = V(h) &= \pi\left(R^2 - \frac{1}{4}h^2\right)h \\ &= \frac{\pi}{4}(4R^2h - h^3), \qquad 0 \leqq h \leqq 2R. \end{aligned}$$

Then

$$V'(h) = \frac{\pi}{4}(4R^2 - 3h^2),$$

so $V(h) = 0$ when $3h^2 = 4R^2$, so that $h = \frac{2}{3}R\sqrt{3}$. This value of h maximizes V because $V(0) = 0$ and $V(2R) = 0$. The corresponding value of r is $\frac{1}{3}R\sqrt{6}$, so the ratio of the height of the cylinder to its radius is $h/r = \sqrt{2}$. The volume of the maximal cylinder is $\frac{4}{9}\pi R^3\sqrt{3}$ and the volume of the sphere is $\frac{4}{3}\pi R^3$; the ratio of the volume of the sphere to that of the maximal inscribed cylinder is thus $\sqrt{3}$.



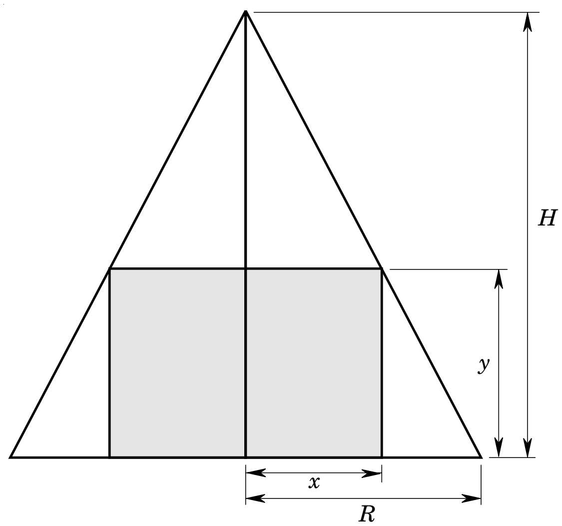
C03S06.033: The following figure shows a cross section of the cone and inscribed cylinder. Let x be the radius of the cylinder and y its height. By similar triangles in the figure,

$$\frac{H}{R} = \frac{y}{R-x}, \quad \text{so} \quad y = \frac{H}{R}(R-x).$$

We are to maximize the volume $V = \pi x^2 y$ of the cylinder, so we write

$$\begin{aligned} V = V(x) &= \pi x^2 \frac{H}{R}(R-x) \\ &= \pi\frac{H}{R}(Rx^2 - x^3), \qquad 0 \leqq x \leqq R. \end{aligned}$$

Because $V(0) = 0 = V(r)$, V is maximized when $V'(x) = 0$; this leads to the equation $2xR = 3x^2$ and thus to the results $x = \frac{2}{3}R$ and $y = \frac{1}{3}H$.



C03S06.034: Let the circle have equation $x^2 + y^2 = 1$ and let (x, y) denote the coordinates of the upper right-hand vertex of the trapezoid (Fig. 3.6.25). Then the area A of the trapezoid is the product of its altitude y and the average of the lengths of its two bases, so

$$A = \frac{1}{2}y(2x+2) \quad \text{where} \quad y^2 = 1 - x^2.$$

A positive quantity is maximized when its square is maximized, so we maximize instead

$$\begin{aligned} f(x) = A^2 &= (x+1)^2(1-x^2) \\ &= 1 + 2x - 2x^3 - x^4, \qquad 0 \leqq x \leqq 1. \end{aligned}$$

Because $f(0) = 0 = f(1)$, f is maximized when $f'(x) = 0$:

$$0 = 2 - 6x^2 - 4x^3 = 2(1+x)^2(1-2x).$$

But the only solution of $f'(x) = 0$ in the domain of f is $x = \frac{1}{2}$. Finally, $f\left(\frac{1}{2}\right) = \frac{27}{16}$, so the maximum possible area of the trapezoid is $\frac{3}{4}\sqrt{3}$. This is just over 41% of the area of the circle, so the answer meets the test of plausibility.

C03S06.035: Draw a circle in the plane with center at the origin and with radius R. Inscribe a rectangle with vertical and horizontal sides and let (x, y) be its vertex in the first quadrant. The base of the rectangle has length $2x$ and its height is $2y$, so the perimeter of the rectangle is $P = 4x + 4y$. Also $x^2 + y^2 = R^2$, so

$$P = P(x) = 4x + 4\sqrt{R^2 - x^2}\,, \qquad 0 \leqq x \leqq R.$$

$$P'(x) = 4 - \frac{4x}{\sqrt{R^2 - x^2}};$$

$$P'(x) = 0 \quad \text{when} \quad 4\sqrt{R^2 - x^2} = 4x;$$

$$R^2 - x^2 = x^2;$$

$$x^2 = \frac{1}{2}R^2.$$

Because $x > 0$, $x = \frac{1}{2}R\sqrt{2}$. The corresponding value of $P(x)$ is $4R\sqrt{2}$, and $P(0) = 4R = P(R)$. So the former value of x maximizes the perimeter P. Because $y^2 = R^2 - x^2$ and because $R^2 - x^2 = x^2$ at maximum, $y = x$ at maximum. Therefore the rectangle of largest perimeter that can be inscribed in a circle is a square.

C03S06.036: Let $(x,\, y)$ be the coordinates of the vertex of the rectangle in the first quadrant. Then, by symmetry, the area of the rectangle is $A = (2x)(2y) = 4xy$. But from the equation of the ellipse we find that

$$\begin{aligned} y &= \tfrac{3}{5}\sqrt{25 - x^2}, \quad \text{so} \\ A = A(x) &= \tfrac{12}{5}x\sqrt{25 - x^2}\,, \qquad 0 \leqq x \leqq 5. \end{aligned}$$

We can simplify the algebra by maximizing instead

$$\begin{aligned} f(x) &= \tfrac{25}{144}A^2 = 25x^2 - x^4; \\ f'(x) &= 50x - 4x^3; \\ f'(x) &= 0 \quad \text{when} \quad x = 0 \quad \text{and when} \quad x = \tfrac{5}{2}\sqrt{2}. \end{aligned}$$

Now $A(0) = 0 = A(5)$, whereas $A\left(\frac{5}{2}\sqrt{2}\right) = 30$. So the rectangle of maximum area has base $2x = 5\sqrt{2}$ and height $2y = 3\sqrt{2}$.

C03S06.037: We are to maximize volume $V = \frac{1}{3}\pi r^2 h$ given $r^2 + h^2 = 100$. The latter relation enables us to write

$$V = V(h) = \frac{1}{3}\pi(100 - h^2)h = \frac{1}{3}\pi(100h - h^3), \qquad 0 \leqq h \leqq 10.$$

Now $V'(h) = \frac{1}{3}\pi(100 - 3h^2)$, so $V'(h) = 0$ when $3h^2 = 100$, thus when $h = \frac{10}{3}\sqrt{3}$. But $V(h) = 0$ at the endpoints of its domain, so the latter value of h maximizes V, and its maximum value is $\frac{2000}{27}\pi\sqrt{3}$.

C03S06.038: Put the bases of the poles on the x-axis, one at the origin and the other at $x = 10$. Let the rope touch the ground at the point x. Then the rope reaches straight from $(0,\, 10)$ to $(x,\, 0)$ and straight from $(x,\, 0)$ to $(10,\, 10)$. In terms of x, its length is

$$\begin{aligned} L(x) &= \sqrt{100 + x^2} + \sqrt{100 + (10 - x)^2} \\ &= \sqrt{100 + x^2} + \sqrt{200 - 20x + x^2}, \qquad 0 \leqq x \leqq 10. \end{aligned}$$

So

$$L'(x) = \frac{x}{\sqrt{100 + x^2}} + \frac{x - 10}{\sqrt{200 - 20x + x^2}};$$

$L'(x) = 0$ when

$$\begin{aligned} x\sqrt{200 - 20x + x^2} &= (10 - x)\sqrt{x^2 + 100}\,; \\ x^2(x^2 - 20x + 200) &= \left(100 - 20x + x^2\right)\left(x^2 + 100\right); \\ x^4 - 20x^3 + 200x^2 &= x^4 - 20x^3 + 200x^2 - 2000x + 10000; \\ 2000x &= 10000; \end{aligned}$$

and thus when $x = 5$. Now $L(0) = L(10) = 10\left(1 + \sqrt{2}\,\right)$, which exceeds $L(5) = 10\sqrt{5}$. So the latter is the length of the shortest possible rope.

C03S06.039: Let x and y be the two numbers. Then $x \geqq 0$, $y \geqq 0$, and $x + y = 16$. We are to find both the maximum and minimum values of $x^{1/3} + y^{1/3}$. Because $y = 16 - x$, we seek the extrema of

$$f(x) = x^{1/3} + (16 - x)^{1/3}, \qquad 0 \leqq x \leqq 16.$$

Now

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3} - \frac{1}{3}(16 - x)^{-2/3} \\ &= \frac{1}{3x^{2/3}} - \frac{1}{3(16 - x)^{2/3}}; \end{aligned}$$

$f'(x) = 0$ when $(16 - x)^{2/3} = x^{2/3}$, so when $16 - x = x$, thus when $x = 8$. Now $f(0) = f(16) = 16^{1/3} \approx 2.52$, so $f(8) = 4$ maximizes f whereas $f(0)$ and $f(16)$ yield its minimum.

C03S06.040: If the base of the L has length x, then the vertical part has length $60 - x$. Place the L with its corner at the origin in the xy-plane, its base on the nonnegative x-axis, and the vertical part on the nonnegative y-axis. The two ends of the L have coordinates $(0,\, 60 - x)$ and $(x,\, 0)$, so they are at distance

$$d = d(x) = \sqrt{x^2 + (60 - x)^2}\,, \qquad 0 \leqq x \leqq 60.$$

A positive quantity is minimized when its square is minimal, so we minimize

$$f(x) = (d(x))^2 = x^2 + (60 - x)^2, \qquad 0 \leqq x \leqq 60.$$

Then $f'(x) = 2x - 2(60 - x) = 4x - 120$; $f'(x) = 0$ when $x = 30$. Now $f(0) = f(60) = 3600$, whereas $f(30) = 1800$. So $x = 30$ minimizes $f(x)$ and thus $d(x)$. The minimum possible distance between the two ends of the wire is therefore $d(30) = 30\sqrt{2}$.

C03S06.041: If $(x,\, x^2)$ is a point of the parabola, then its distance from $(0,\, 1)$ is

$$d(x) = \sqrt{x^2 + (x^2 - 1)^2}\,.$$

So we minimize

$$f(x) = (d(x))^2 = x^4 - x^2 + 1,$$

where the domain of f is the set of all real numbers. But because $f(x)$ is large positive when $|x|$ is large, we will not exclude a minimum if we restrict the domain of f to be an interval of the form $[-a,\, a]$ where a is a large positive number. On the interval $[-a,\, a]$, f is continuous and thus has a global minimum, which does not occur at $\pm a$ because $f(\pm a)$ is large positive. Because $f'(x)$ exists for all x, the minimum of f occurs at a point where $f'(x) = 0$:

$$4x^3 - 2x = 0; \qquad 2x(2x^2 - 1) = 0.$$

Hence $x = 0$ or $x = \pm\frac{1}{2}\sqrt{2}$. Now $f(0) = 1$ and $f\left(\pm\frac{1}{2}\sqrt{2}\,\right) = \frac{3}{4}$. So $x = 0$ yields a local maximum value for $f(x)$, and the minimum possible distance is $\sqrt{0.75} = \frac{1}{2}\sqrt{3}$.

C03S06.042: It suffices to minimize $x^2 + y^2$ given $y = (3x - 4)^{1/3}$. Let $f(x) = x^2 + (3x - 4)^{2/3}$. Then

$$f'(x) = 2x + 2(3x-4)^{-1/3}.$$

Then $f'(x) = 0$ when

$$\begin{aligned}
2x + \frac{2}{(3x-4)^{1/3}} &= 0;\\
2x(3x-4)^{1/3} &= -2;\\
x(3x-4)^{1/3} &= -1;\\
x^3(3x-4) &= -1;\\
x^3(3x-4)+1 &= 0;\\
3x^4 - 4x^3 + 1 &= 0;\\
(x-1)^2(3x^2+2x+1) &= 0.
\end{aligned}$$

Now $x = 1$ is the only real solution of the last equation, $f'(x)$ does not exist when $x = \frac{4}{3}$, and $f(1) = 2 > \frac{16}{9} = f\left(\frac{4}{3}\right)$. So the point closest to the origin is $\left(\frac{4}{3}, 0\right)$.

C03S06.043: Examine the plank on the right on Fig. 3.6.10. Let its height be $2y$ and its width (in the x-direction) be z. The total area of the four small rectangles in the figure is then $A = 4 \cdot z \cdot 2y = 8yz$. The circle has radius 1, and by Problem 35 the large inscribed square has dimensions $\sqrt{2}$ by $\sqrt{2}$. Thus

$$\left(\frac{1}{2}\sqrt{2} + z\right)^2 + y^2 = 1.$$

This implies that

$$y = \sqrt{\frac{1}{2} - z\sqrt{2} - z^2}\,.$$

Therefore

$$A(z) = 8z\sqrt{\frac{1}{2} - z\sqrt{2} - z^2}\,, \qquad 0 \leqq z \leqq 1 - \frac{1}{2}\sqrt{2}.$$

Now $A(z) = 0$ at each endpoint of its domain and

$$A'(z) = \frac{4\sqrt{2}\,\left(1 - 3z\sqrt{z} - 4z^2\right)}{\sqrt{1 - 2z\sqrt{z} - 2z^2}}.$$

So $A'(z) = 0$ when $z = \frac{1}{8}\left(-3\sqrt{2} \pm \sqrt{34}\right)$; we discard the negative solution, and find that when $A(z)$ is maximized,

$$z = \frac{-3\sqrt{2} + \sqrt{34}}{8} \approx 0.198539,$$

$$2y = \frac{\sqrt{7 - \sqrt{17}}}{2} \approx 0.848071, \quad \text{and}$$

$$A(z) = \frac{\sqrt{142 + 34\sqrt{17}}}{2} \approx 0.673500.$$

The four small planks use just under 59% of the wood that remains after the large plank is cut, a very efficient use of what might be scrap lumber.

C03S06.044: Place the base of the triangle on the x-axis and its upper vertex on the y-axis. Then its lower right vertex is at the point $\left(\frac{1}{2},\, 0\right)$ and its upper vertex is at $\left(0,\, \frac{1}{2}\sqrt{3}\,\right)$. It follows that the slope of the side of the triangle joining these two vertices is $-\sqrt{3}$. So this side lies on the straight line with equation

$$y = \sqrt{3}\left(\frac{1}{2} - x\right).$$

Let $(x,\, y)$ be the coordinates of the upper right-hand vertex of the rectangle. Then the rectangle has area $A = 2xy$, so

$$A(x) = \sqrt{3}\,(x - 2x^2), \qquad 0 \leqq x \leqq \frac{1}{2}.$$

Now $A'(x) = 0$ when $x = \frac{1}{4}$, and because $A(x) = 0$ at the endpoints of its domain, it follows that the maximum area of such a rectangle is $A\left(\frac{1}{4}\right) = \frac{1}{8}\sqrt{3}$.

C03S06.045: Set up a coordinate system in which the island is located at (0, 2) and the village at (6, 0), and let $(x,\, 0)$ be the point at which the boat lands. It is clear that $0 \leqq x \leqq 6$. The trip involves the land distance $6 - x$ traveled at 20 km/h and the water distance $(4 + x^2)^{1/2}$ traveled at 10 km/h. The total time of the trip is then given by

$$T(x) = \frac{1}{10}\sqrt{4 + x^2} + \frac{1}{20}(6 - x), \qquad 0 \leqq x \leqq 6.$$

Now

$$T'(x) = \frac{x}{10\sqrt{4 + x^2}} - \frac{1}{20}.$$

Thus $T'(x) = 0$ when $3x^2 = 4$; because $x \geqq 0$, we find that $x = \frac{2}{3}\sqrt{3}$. The value of T there is

$$\frac{1}{10}\left(3 + \sqrt{3}\right) \approx 0.473,$$

whereas $T(0) = 0.5$ and $T(6) \approx 0.632$. Therefore the boater should make landfall at $\frac{2}{3}\sqrt{3} \approx 1.155$ km from the point on the shore closest to the island.

C03S06.046: Set up a coordinate system in which the factory is located at the origin and the power station at $(L\,, W)$ in the xy-plane—$L = 4500$, $W = 2000$. Part of the path of the power cable will be straight along the river bank and part will be a diagonal running under water. It makes no difference whether the straight part is adjacent to the factory or to the power station, so we assume the former. Thus we suppose that the power cable runs straight from (0, 0) to $(x,\, 0)$, then straight from $(x,\, 0)$ to $(L,\, W)$, where $0 \leqq x \leqq L$. Let y be the length of the diagonal stretch of the cable. Then by the Pythagorean theorem,

$$W^2 + (L - x)^2 = y^2, \qquad \text{so} \qquad y = \sqrt{W^2 + (L - x)^2}\,.$$

The cost C of the cable is $C = kx + 3ky$ where k is the cost per unit distance of over-the-ground cable. Therefore the total cost of the cable is

$$C(x) = kx + 3k\sqrt{W^2 + (L - x)^2}\,, \qquad 0 \leqq x \leqq L.$$

It will not change the solution if we assume that $k = 1$, and in this case we have

$$C'(x) = 1 - \frac{3(L-x)}{\sqrt{W^2 + (L-x)^2}}.$$

Next, $C'(x) = 0$ when $x^2 + (L-x)^2 = 9(L-x)^2$, and this leads to the solution

$$x = L - \frac{1}{4}W\sqrt{2} \text{ and } y = \frac{3}{4}W\sqrt{2}.$$

It is not difficult to verify that the latter value of x yields a value of C smaller than either $C(0)$ or $C(L)$. Answer: Lay the cable $x = 4500 - 500\sqrt{2} \approx 3793$ meters along the bank and $y = 1500\sqrt{2} \approx 2121$ meters diagonally across the river.

C03S06.047: The distances involved are $|AP| = |BP| = \sqrt{x^2+1}$ and $|CP| = 3 - x$. Therefore we are to minimize

$$f(x) = 2\sqrt{x^2+1} + 3 - x, \qquad 0 \leqq x \leqq 3.$$

Now

$$f'(x) = \frac{2x}{\sqrt{x^2+1}} - 1; \quad f'(x) = 0 \quad \text{when} \quad \frac{2x}{\sqrt{x^2+1}} = 1.$$

This leads to the equation $3x^2 = 1$, so $x = \frac{1}{3}\sqrt{3}$. Now $f(0) = 5$, $f(3) \approx 6.32$, and at the critical point, $f(x) = 3 + \sqrt{3} \approx 4.732$. Answer: The distribution center should be located at the point $P(\frac{1}{3}\sqrt{3}, 0)$.

C03S06.048: **(a)** $T = \dfrac{1}{c}\sqrt{a^2+x^2} + \dfrac{1}{v}\sqrt{(s-x)^2+b^2}$.

(b) $T'(x) = \dfrac{x}{c\sqrt{a^2+x^2}} - \dfrac{s-x}{v\sqrt{(s-x)^2+b^2}}$.

$$T'(x) = 0 \quad \text{when} \quad \frac{x}{c\sqrt{a^2+x^2}} = \frac{s-x}{v\sqrt{(s-x)^2+b^2}};$$

$$\frac{x}{\sqrt{a^2+x^2}} \cdot \frac{\sqrt{(x-s)^2+b^2}}{s-x} = \frac{c}{v};$$

$$\sin\alpha\csc\beta = \frac{c}{v}$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{c}{v} = n.$$

C03S06.049: We are to minimize total cost

$$C = c_1\sqrt{a^2+x^2} + c_2\sqrt{(L-x)^2+b^2}.$$

$$C'(x) = \frac{c_1 x}{\sqrt{a^2+x^2}} - \frac{c_2(L-x)}{\sqrt{(L-x)^2+b^2}};$$

$$C'(x) = 0 \text{ when } \frac{c_1 x}{\sqrt{a^2+x^2}} = \frac{c_2(L-x)}{\sqrt{(L-x)^2+b^2}}.$$

The result in Part (a) is equivalent to the last equation. For Part (b), assume that $a = b = c_1 = 1$, $c_2 = 2$, and $L = 4$. Then we obtain

$$\frac{x}{\sqrt{1+x^2}} = \frac{2(4-x)}{\sqrt{(4-x)^2+1}};$$

$$\frac{x^2}{1+x^2} = \frac{4(16-8x+x^2)}{16-8x+x^2+1};$$

$$x^2(17-8x+x^2) = (4+4x^2)(16-8x+x^2);$$

$$17x^2 - 8x^3 + x^4 = 64 - 32x + 68x^2 - 32x^3 + 4x^4.$$

Therefore we wish to solve $f(x) = 0$ where

$$f(x) = 3x^4 - 24x^4 + 51x^2 - 32x + 64.$$

Now $f(0) = 64$, $f(1) = 62$, $f(2) = 60$, $f(3) = 22$, and $f(4) = -16$. Because $f(3) > 0 > f(4)$, we interpolate to estimate the zero of $f(x)$ between 3 and 4; it turns out that interpolation gives $x \approx 3.58$. Subsequent interpolation yields the more accurate estimate $x \approx 3.45$. (The equation $f(x) = 0$ has exactly two solutions, $x \approx 3.452462314$ and $x \approx 4.559682567$.)

C03S06.050: Because $x^3 + y^3 = 2000$, $y = (2000 - x^3)^{1/3}$. We want to maximize and minimize total surface area $A = 6x^2 + 6y^2$;

$$A = A(x) = 6x^2 + 6(2000 - x^3)^{2/3}, \qquad 0 \leqq x \leqq 10\sqrt[3]{2}.$$

$$A'(x) = \frac{-12[x^2 - x(2000 - x^3)^{1/3}]}{(2000 - x^3)^{1/3}}.$$

Now $A'(x) = 0$ at $x = 0$ and at $x = 10$; $A'(x)$ does not exist at $x = 10\sqrt[3]{2}$, the right-hand endpoint of the domain of A (at that point, the graph of A has a vertical tangent). Also $A(0) = 600 \cdot 2^{2/3} \approx 952.441$ and $A\left(10\sqrt[3]{2}\right)$ is the same; $A(10) = 1200$. So the maximum surface area is attained when each cube has edge length 10 and the minimum is attained when there is only one cube, of edge length $10\sqrt[3]{2} \approx 12.5992$.

C03S06.051: Let r be the radius of the sphere and x the edge length of the cube. We are to maximize and minimize total volume

$$V = \frac{4}{3}\pi r^3 + x^3 \quad \text{given} \quad 4\pi r^2 + 6x^2 = 1000.$$

The latter equation yields

$$x = \sqrt{\frac{1000 - 4\pi r^2}{6}},$$

so

$$V = V(r) = \frac{4}{3}\pi r^3 + \left(\frac{500 - 2\pi r^2}{3}\right)^{3/2}, \qquad 0 \leqq r \leqq r_1 = 5\sqrt{\frac{10}{\pi}}.$$

Next,

$$V'(r) = 4\pi r^2 - 2\pi r\sqrt{\frac{500 - 2\pi r^2}{3}},$$

and $V'(r) = 0$ when

$$4\pi r^2 = 2\pi r\sqrt{\frac{500 - 2\pi r^2}{3}}\,.$$

So $r = 0$ or

$$2r = \sqrt{\frac{500 - 2\pi r^2}{3}}\,.$$

The latter equation leads to

$$r = r_2 = 5\sqrt{\frac{10}{\pi + 6}}\,.$$

Now $V(0) \approx 2151.66$, $V(r_1) \approx 2973.54$, and $V(r_2) \approx 1743.16$. Therefore, to minimize the sum of the volumes, choose $r = r_2 \approx 5.229$ in. and $x = 2r_2 \approx 10.459$ in. To maximize the sum of their volumes, take $r = r_1 \approx 8.921$ in. and $x = 0$ in.

C03S06.052: Let the horizontal piece of wood have length $2x$ and the vertical piece have length $y + z$ where y is the length of the part above the horizontal piece and z the length of the part below it. Then

$$y = \sqrt{4 - x^2} \quad \text{and} \quad z = \sqrt{16 - x^2}\,.$$

Also the kite area is $A = x(y + z)$; $\dfrac{dA}{dx} = 0$ implies that

$$y + z = \frac{x^2}{y} + \frac{x^2}{z}.$$

Multiply each side of the last equation by yz to obtain

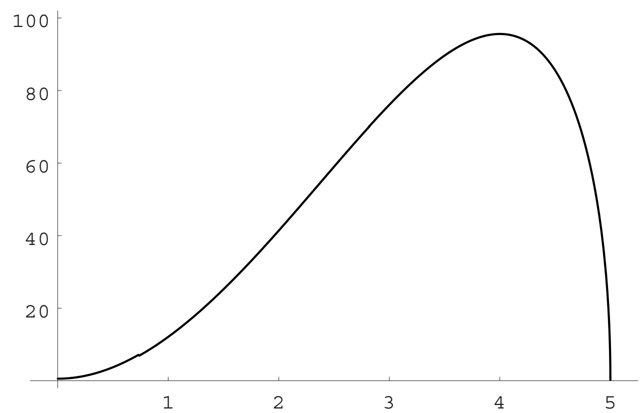
$$y^2z + yz^2 = x^2z + x^2y,$$

so that

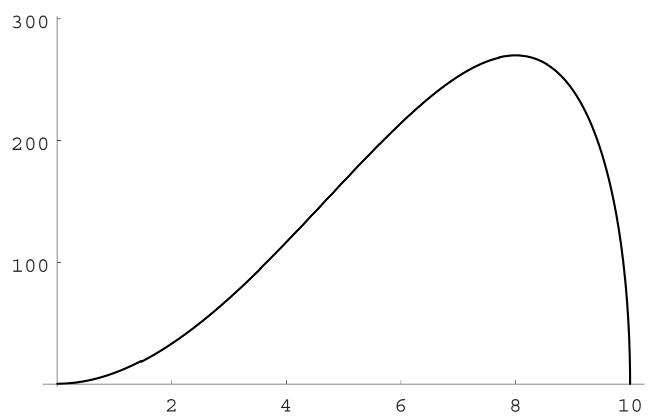
$$\begin{aligned}
&yz(y + z) = x^2(y + z);\\
&x^2 = yz;\\
&x^4 = y^2z^2 = (4 - x^2)(16 - x^2);\\
&x^4 = 64 - 20x^2 + x^4;\\
&20x^2 = 64;\\
&x = \frac{4}{5}\sqrt{5},\ y = \frac{2}{5}\sqrt{5},\ z = \frac{8}{5}\sqrt{5}.
\end{aligned}$$

Therefore $L_1 = \frac{8}{5}\sqrt{5} \approx 3.5777$ and $L_2 = 2\sqrt{5} \approx 4.47214$ for maximum area.

C03S06.053: The graph of $V(x)$ is shown next. The maximum volume seems to occur near the point $(4, V(4)) \approx (4, 95.406)$, so the maximum volume is approximately 95.406 cubic feet.



C03S06.054: The graph of $V(x)$ is shown next. The maximum volume seems to occur near the point $(8, V(8)) \approx (8, 269.848)$, so the maximum volume is approximately 269.848 cubic feet.



C03S06.055: Let V_1 and V_2 be the volume functions of problems 53 and 54, respectively. Then

$$V_1'(x) = \frac{20\sqrt{5}\,(4x - x^2)}{3\sqrt{5 - x}},$$

which is zero at $x = 0$ and at $x = 4$, and

$$V_2'(x) = \frac{10\sqrt{5}\,(8x - x^2)}{3\sqrt{10 - x}},$$

which is zero at $x = 0$ and at $x = 8$, as expected. Finally, $\dfrac{V_2(8)}{V_1(4)} = 2\sqrt{2}$.

C03S06.056: Let x denote the length of each edge of the base of the box; let y denote its height. If the box has total surface area A, then $2x^2 + 4xy = A$, and hence

$$y = \frac{A - 2x^2}{4x}. \tag{1}$$

The box has volume $V = x^2y$, so its volume can be expressed as a function of x alone:

$$V(x) = \frac{Ax - 2x^3}{4}, \quad 0 \leqq x \leqq \sqrt{A/2}\,.$$

Then

$$V'(x) = \frac{A - 6x^2}{4}; \qquad V'(x) = 0 \quad \text{when} \quad x = \sqrt{A/6}\,.$$

This critical point clearly lies in the interior of the domain of V, and (almost as clearly) $V'(x)$ is increasing to its left and decreasing to its right. Hence this critical point yields the box of maximal volume. Moreover, when $x = \sqrt{A/6}\,$, we have—by Eq. (1)—

$$y == \frac{A - (A/3)}{4\sqrt{A/6}} = \frac{2A}{3} \cdot \frac{\sqrt{6}}{4\sqrt{A}} = \frac{\sqrt{6}}{6} \cdot \sqrt{A} = \frac{\sqrt{A}}{\sqrt{6}}.$$

Therefore the closed box with square base, fixed surface area, and maximal volume is a cube.

C03S06.057: Let x denote the length of each edge of the square base of the box and let y denote its height. Given total surface area A, we have $x^2 + 4xy = A$, and hence

$$y = \frac{A - x^2}{4x}. \tag{1}$$

The volume of the box is $V = x^2 y$, and therefore

$$V(x) = \frac{Ax - x^2}{4}, \quad 0 \leqq x \leqq \sqrt{A}\,.$$

Next,

$$V'(x) = \frac{A - 3x^2}{4}; \qquad V'(x) = 0 \quad \text{when} \quad x = \sqrt{A/3}\,.$$

Because $V'(x) > 0$ to the left of this critical point and $V'(x) < 0$ to the right, it yields the global maximum value of $V(x)$. By Eq. (1), the corresponding height of the box is $\frac{1}{2}\sqrt{A/3}\,$. Therefore the open box with square base and maximal volume has height equal to half the length of the edge of its base.

C03S06.058: Let r denote the radius of the base of the closed cylindrical can, h its height, and A its total surface area. Then

$$2\pi r^2 + 2\pi rh = A, \quad \text{and hence} \quad h = \frac{A - 2\pi r^2}{2\pi r}. \tag{1}$$

The volume of the can is $V = \pi r^2 h$, and thus

$$V(r) = \frac{Ar - 2\pi r^3}{2}, \quad 0 \leqq r \leqq \left(\frac{A}{2\pi}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{A - 6\pi r^2}{2}; \qquad V'(r) = 0 \quad \text{when} \quad r = \left(\frac{A}{6\pi}\right)^{1/2}.$$

Because $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, it determines the global maximum value of $V(r)$. By Eq. (1), it follows that the can of maximum volume has equal height and diameter.

C03S06.059: Let r denote the radius of the base of the open cylindrical can and let h denote its height. Its total surface area A then satisfies the equation $\pi r^2 + 2\pi rh = A$, and therefore

$$h = \frac{A - \pi r^2}{2\pi r}. \qquad (1)$$

Thus the volume of the can is given by

$$V(r) = \frac{Ar - \pi r^3}{2}, \quad 0 \leqq r \leqq \left(\frac{A}{\pi}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{A - 3\pi r^2}{2}; \qquad V'(r) = 0 \quad \text{when} \quad r = \left(\frac{A}{3\pi}\right)^{1/2}.$$

Clearly $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, so it determines the global maximum value of $V(r)$. By Eq. (1) the corresponding value of h is the same, so the open cylindrical can of maximum volume has height equal to its radius.

C03S06.060: Let r denote the interior radius of the cylindrical can and h its interior height. Because the thickness t of the material of the can will be very small in comparison with r and h, the total amount of material M used to make the can will be very accurately approximated by multiplying the thickness of the bottom by its area, the thickness of the curved side by its area, and the thickness of the top by its area. That is,

$$\pi r^2 t + 2\pi rht + 3\pi r^2 t = M, \quad \text{so that} \quad h = \frac{M - 4\pi r^2 t}{2\pi rt}. \qquad (1)$$

Thus the volume of the can will be given by (the very accurate approximation)

$$V(r) = \frac{Mr - 4\pi r^3 t}{2t}, \quad 0 \leqq r \leqq \left(\frac{M}{4\pi t}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{M - 12\pi r^2 t}{2t}; \qquad V'(r) = 0 \quad \text{when} \quad r = \frac{1}{2} \cdot \left(\frac{M}{3\pi t}\right)^{1/2}.$$

Because $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, it yields the global maximum value of $V(r)$. By (1), the height of the corresponding can is four times as great, so the can of maximum volume has height twice its diameter (approximately, but quite accurately).

To solve this problem exactly, first establish that

$$4\pi t(r+t)^2 + \pi ht^2(2 * r + t) = M,$$

then that

$$V(r) = \frac{Mr^2 + 4\pi tr^2(r+t)^2}{t^2(2 * r + t)}.$$

Then show that $V'(r) = 0$ when

$$12\pi tr^4 + 24\pi t^2 r^3 + (16\pi t^3 - M)r^2 + (4\pi t^4 - Mt)r = 0,$$

and that the relevant critical point is

$$r = \frac{-3\pi t^2 + \sqrt{3\pi t(M - \pi t^3)}}{6\pi t}.$$

C03S06.061: Let

$$f(t) = \frac{1}{1+t^2}, \quad \text{so that} \quad f'(t) = -\frac{2t}{(1+t^2)^2}.$$

The line tangent to the graph of $y = f(t)$ at the point $(t, f(t))$ then has x-intercept and y-intercept

$$\frac{1+3t^2}{2t} \quad \text{and} \quad \frac{1+3t^2}{(1+t^2)^2},$$

respectively. The area of the triangle bounded by the part of the tangent line in the first quadrant and the coordinate axes is

$$A(t) = \frac{1}{2} \cdot \frac{1+3t^2}{2t} \cdot \frac{1+3t^2}{(1+t^2)^2}, \tag{1}$$

and

$$A'(t) = \frac{-9t^6 + 9t^4 + t^2 - 1}{4t^2(1+t^2)^3}.$$

Next, $A'(t) = 0$ when

$$(t-1)(t+1)(3t^2-1)(3t^2+1) = 0,$$

and the only two critical points of A in the interval $[0.5, 2]$ are

$$t_1 = 1 \quad \text{and} \quad t_2 = \frac{\sqrt{3}}{3} \approx 0.57735.$$

Significant values of $A(t)$ are then

$$A(0.5) = 0.98, \quad A(0.57735) \approx 0.97428, \quad A(1) = 1, \quad \text{and} \quad A(2) = 0.845.$$

Therefore $A(t)$ has a local maximum at $t = 0.5$, a local minimum at t_2, its global maximum at $t = 1$, and its global minimum at $t = 2$.

To answer the first question in Problem 61, Eq. (1) makes it clear that $A(t) \to +\infty$ as $t \to 0^+$ and, in addition, that $A(t) \to 0$ as $t \to +\infty$.

C03S06.062: If $0 \leqq x < 1$, then the cost of the power line will be

$$C(x) = 40x + 100\sqrt{1 + (1-x)^2}$$

(in thousand of dollars). If $x = 1$, then the cost will be 80 thousand dollars because there is no need to use underground cable. Next,

$$C'(x) = \frac{100x - 100 + 40\sqrt{x^2 - 2x + 2}}{\sqrt{x^2 - 2x + 2}},$$

and $C'(x) = 0$ when

$$x = x_0 = \frac{21 - 2\sqrt{21}}{21} \approx 0.563564219528.$$

The graph of $C(x)$ (using a computer algebra system) establishes that x_0 determines the global minimum for $C(x)$ on the interval $[0, 1)$, yielding the corresponding value $C(x_0) \approx 131.651513899117$. Hence the global minimum for $C(x)$ on $[0, 1]$ is $C(1) = 80$ (thousand dollars). It is neither necessary to cross the park nor to use underground cable.

Section 3.7

C03S07.001: If $f(x) = 3\sin^2 x = 3(\sin x)^2$, then $f'(x) = 6\sin x\cos x$.

C03S07.002: If $f(x) = 2\cos^4 x = 2(\cos x)^4$, then $f'(x) = 8(\cos x)^3(-\sin x) = -8\cos^3 x\sin x$.

C03S07.003: If $f(x) = x\cos x$, then $f'(x) = 1\cdot\cos x + x\cdot(-\sin x) = \cos x - x\sin x$.

C03S07.004: If $f(x) = x^{1/2}\sin x$, then $f'(x) = \frac{1}{2}x^{-1/2}\sin x + x^{1/2}\cos x = \dfrac{\sin x + 2x\cos x}{2\sqrt{x}}$.

C03S07.005: If $f(x) = \dfrac{\sin x}{x}$, then $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$.

C03S07.006: If $f(x) = \dfrac{\cos x}{x^{1/2}}$, then $f'(x) = \dfrac{x^{1/2}(-\sin x) - \frac{1}{2}x^{-1/2}\cos x}{x} = -\dfrac{2x\sin x + \cos x}{2x\sqrt{x}}$.

C03S07.007: If $f(x) = \sin x\cos^2 x$, then

$$f'(x) = \cos x\cos^2 x + (\sin x)(2\cos x)(-\sin x) = \cos^3 x - 2\sin^2 x\cos x.$$

C03S07.008: If $f(x) = \cos^3 x\sin^2 x$, then

$$f'(x) = (3\cos^2 x)(-\sin x)(\sin^2 x) + (2\sin x\cos x)(\cos^3 x) = -3\cos^2 x\sin^3 x + 2\sin x\cos^4 x.$$

C03S07.009: If $g(t) = (1+\sin t)^4$, then $g'(t) = 4(1+\sin t)^3\cdot\cos t$.

C03S07.010: If $g(t) = (2-\cos^2 t)^3$, then $g'(t) = 3(2-\cos^2 t)^2\cdot(2\cos t\sin t) = 6(2-\cos^2 t)^2(\sin t\cos t)$.

C03S07.011: If $g(t) = \dfrac{1}{\sin t + \cos t}$, then (by the reciprocal rule) $g'(t) = -\dfrac{\cos t - \sin t}{(\sin t + \cos t)^2} = \dfrac{\sin t - \cos t}{(\sin t + \cos t)^2}$.

C03S07.012: If $g(t) = \dfrac{\sin t}{1+\cos t}$, then (by the quotient rule)

$$g'(t) = \frac{(1+\cos t)(\cos t) - (\sin t)(-\sin t)}{(1+\cos t)^2} = \frac{\sin^2 t + \cos^2 t + \cos t}{(1+\cos t)^2} = \frac{1+\cos t}{(1+\cos t)^2} = \frac{1}{1+\cos t}.$$

C03S07.013: If $f(x) = 2x\sin x - 3x^2\cos x$, then (by the product rule)

$$f'(x) = 2\sin x + 2x\cos x - 6x\cos x + 3x^2\sin x = 3x^2\sin x - 4x\cos x + 2\sin x.$$

C03S07.014: If $f(x) = x^{1/2}\cos x - x^{-1/2}\sin x$,

$$f'(x) = \frac{1}{2}x^{-1/2}\cos x - x^{1/2}\sin x + \frac{1}{2}x^{-3/2}\sin x - x^{-1/2}\cos x = \frac{(1-2x^2)\sin x - x\cos x}{2x\sqrt{x}}.$$

C03S07.015: If $f(x) = \cos 2x\sin 3x$, then $f'(x) = -2\sin 2x\sin 3x + 3\cos 2x\cos 3x$.

C03S07.016: If $f(x) = \cos 5x\sin 7x$, then $f'(x) = -5\sin 5x\sin 7x + 7\cos 5x\cos 7x$.

C03S07.017: If $g(t) = t^3 \sin^2 2t = t^3(\sin 2t)^2$, then

$$g'(t) = 3t^2(\sin 2t)^2 + t^3 \cdot (2\sin 2t) \cdot (\cos 2t) \cdot 2 = 3t^2 \sin^2 2t + 4t^3 \sin 2t \cos 2t.$$

C03S07.018: If $g(t) = \sqrt{t}\, \cos^3 3t = t^{1/2}(\cos 3t)^3$, then

$$g'(t) = \frac{1}{2}t^{-1/2}(\cos 3t)^3 + t^{1/2} \cdot 3(\cos 3t)^2 \cdot (-3\sin 3t) = \frac{\cos^3 3t}{2\sqrt{t}} - 9\sqrt{t}\, \cos^2 3t\, \sin 3t.$$

C03S07.019: If $g(t) = (\cos 3t + \cos 5t)^{5/2}$, then $g'(t) = \frac{5}{2}(\cos 3t + \cos 5t)^{3/2}(-3\sin 3t - 5\sin 5t)$.

C03S07.020: If $g(t) = \dfrac{1}{\sqrt{\sin^2 t + \sin^2 3t}} = (\sin^2 t + \sin^2 3t)^{-1/2}$, then

$$g'(t) = -\frac{1}{2}(\sin^2 t + \sin^2 3t)^{-3/2}(2\sin t \cos t + 6\sin 3t \cos 3t) = -\frac{\sin t \cos t + 3\sin 3t \cos 3t}{(\sin^2 2t + \sin^2 3t)^{3/2}}.$$

C03S07.021: If $y = y(x) = \sin^2 \sqrt{x} = (\sin x^{1/2})^2$, then

$$\frac{dy}{dx} = 2(\sin x^{1/2})(\cos x^{1/2}) \cdot \frac{1}{2}x^{-1/2} = \frac{\sin\sqrt{x}\, \cos\sqrt{x}}{\sqrt{x}}.$$

C03S07.022: If $y = y(x) = \dfrac{\cos 2x}{x}$, then $\dfrac{dy}{dx} = \dfrac{x \cdot (-2\sin 2x) - 1 \cdot \cos 2x}{x^2} = -\dfrac{2x \sin 2x + \cos 2x}{x^2}$.

C03S07.023: If $y = y(x) = x^2 \cos(3x^2 - 1)$, then

$$\frac{dy}{dx} = 2x\cos(3x^2 - 1) - x^2 \cdot 6x \cdot \sin(3x^2 - 1) = 2x\cos(3x^2 - 1) - 6x^3 \sin(3x^2 - 1).$$

C03S07.024: If $y = y(x) = \sin^3 x^4 = (\sin x^4)^3$, then

$$\frac{dy}{dx} = 3(\sin x^4)^2 \cdot D_x(\sin x^4) = 3(\sin x^4)^2 \cdot (\cos x^4) \cdot D_x(x^4) = 12x^3 \sin^2 x^4\, \cos x^4.$$

C03S07.025: If $y = y(x) = \sin 2x \cos 3x$, then

$$\frac{dy}{dx} = (\sin 2x) \cdot D_x(\cos 3x) + (\cos 3x) \cdot D_x(\sin 2x) = -3\sin 2x \sin 3x + 2\cos 3x \cos 2x.$$

C03S07.026: If $y = y(x) = \dfrac{x}{\sin 3x}$, then $\dfrac{dy}{dx} = \dfrac{(\sin 3x) \cdot 1 - x \cdot D_x(\sin 3x)}{(\sin 3x)^2} = \dfrac{\sin 3x - 3x\cos 3x}{\sin^2 3x}$.

C03S07.027: If $y = y(x) = \dfrac{\cos 3x}{\sin 5x}$, then

$$\frac{dy}{dx} = \frac{(\sin 5x)(-3\sin 3x) - (\cos 3x)(5\cos 5x)}{(\sin 5x)^2} = -\frac{3\sin 3x \sin 5x + 5\cos 3x \cos 5x}{\sin^2 5x}.$$

C03S07.028: If $y = y(x) = \sqrt{\cos\sqrt{x}} = (\cos x^{1/2})^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}(\cos x^{1/2})^{-1/2}(-\sin x^{1/2}) \cdot \frac{1}{2}x^{-1/2} = -\frac{\sin\sqrt{x}}{4\sqrt{x}\,\sqrt{\cos\sqrt{x}}}.$$

C03S07.029: If $y = y(x) = \sin^2 x^2 = (\sin x^2)^2$, then

$$\frac{dy}{dx} = 2(\sin x^2) \cdot D_x(\sin x^2) = 2(\sin x^2) \cdot (\cos x^2) \cdot D_x(x^2) = 4x \sin x^2 \ \cos x^2.$$

C03S07.030: If $y = y(x) = \cos^3 x^3 = (\cos x^3)^3$, then

$$\frac{dy}{dx} = 3(\cos x^3)^2 \cdot D_x(\cos x^3) = 3(\cos x^3)^2 \cdot (-\sin x^3) \cdot D_x(x^3) = -9x^2 \cos^2 x^3 \ \sin x^3.$$

C03S07.031: If $y = y(x) = \sin 2\sqrt{x} = \sin(2x^{1/2})$, then

$$\frac{dy}{dx} = \Big[\cos(2x^{1/2})\Big] \cdot D_x(2x^{1/2}) = x^{-1/2}\cos(2x^{1/2}) = \frac{\cos 2\sqrt{x}}{\sqrt{x}}.$$

C03S07.032: If $y = y(x) = \cos 3\sqrt[3]{x} = \cos(3x^{1/3})$, then

$$\frac{dy}{dx} = \Big[-\sin(3x^{1/3})\Big] \cdot D_x(3x^{1/3}) = (-\sin 3x^{1/3}) \cdot (x^{-2/3}) = -\frac{\sin 3\sqrt[3]{x}}{\sqrt[3]{x^2}}.$$

C03S07.033: If $y = y(x) = x \sin x^2$, then $\dfrac{dy}{dx} = 1 \cdot \sin x^2 + x \cdot (\cos x^2) \cdot 2x = \sin x^2 + 2x^2 \cos x^2$.

C03S07.034: If $y = y(x) = x^2 \cos\left(\dfrac{1}{x}\right)$, then

$$\frac{dy}{dx} = 2x\cos\left(\frac{1}{x}\right) + x^2\left[-\sin\left(\frac{1}{x}\right)\right] \cdot D_x\left(\frac{1}{x}\right)$$

$$= 2x\cos\left(\frac{1}{x}\right) - x^2\left[\sin\left(\frac{1}{x}\right)\right] \cdot \left(-\frac{1}{x^2}\right) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

C03S07.035: If $y = y(x) = \sqrt{x}\,\sin\sqrt{x} = x^{1/2}\sin x^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\sin x^{1/2} + x^{1/2}(\cos x^{1/2}) \cdot \frac{1}{2}x^{-1/2} = \frac{\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos\sqrt{x}}{2} = \frac{\sin\sqrt{x} + \sqrt{x}\,\cos\sqrt{x}}{2\sqrt{x}}.$$

C03S07.036: If $y = y(x) = (\sin x - \cos x)^2$, then

$$\frac{dy}{dx} = 2(\sin x - \cos x)(\cos x + \sin x) = 2(\sin^2 x - \cos^2 x) = -2\cos 2x.$$

C03S07.037: If $y = y(x) = \sqrt{x}\,(x - \cos x)^3 = x^{1/2}(x - \cos x)^3$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}(x-\cos x)^3 + 3x^{1/2}(x-\cos x)^2(1+\sin x)$$
$$= \frac{(x-\cos x)^3}{2\sqrt{x}} + 3\sqrt{x}\,(x-\cos x)^2(1+\sin x) = \frac{(x-\cos x)^3 + 6x(x-\cos x)^2(1+\sin x)}{2\sqrt{x}}.$$

C03S07.038: If $y = y(x) = \sqrt{x}\,\sin\sqrt{x+\sqrt{x}} = x^{1/2}\sin(x+x^{1/2})^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\sin(x+x^{1/2})^{1/2} + x^{1/2}\left[\cos(x+x^{1/2})^{1/2}\right]\cdot\frac{1}{2}(x+x^{1/2})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right).$$

The symbolic algebra program *Mathematica* simplifies this to

$$\frac{dy}{dx} = \frac{(2x+\sqrt{x}\,)\cos\sqrt{x+\sqrt{x}} + 2\left(\sqrt{x+\sqrt{x}}\,\right)\sin\sqrt{x+\sqrt{x}}}{4\sqrt{x}\,\sqrt{x+\sqrt{x}}}.$$

C03S07.039: If $y = y(x) = \cos(\sin x^2)$, then $\dfrac{dy}{dx} = \left[-\sin(\sin x^2)\right]\cdot(\cos x^2)\cdot 2x = -2x\left[\sin(\sin x^2)\right]\cos x^2$.

C03S07.040: If $y = y(x) = \sin\left(1+\sqrt{\sin x}\right)$, then

$$\frac{dy}{dx} = \left[\cos\left(1+\sqrt{\sin x}\right)\right]\cdot\frac{1}{2}(\sin x)^{-1/2}\cdot\cos x = \frac{(\cos x)\cos\left(1+\sqrt{\sin x}\right)}{2\sqrt{\sin x}}.$$

C03S07.041: If $x = x(t) = \tan t^7 = \tan(t^7)$, then $\dfrac{dx}{dt} = \left(\sec t^7\right)^2\cdot D_t(t^7) = 7t^6\sec^2 t^7$.

C03S07.042: If $x = x(t) = \sec t^7 = \sec(t^7)$, then $\dfrac{dx}{dt} = (\sec t^7\,\tan t^7)\cdot D_t(t^7) = 7t^6\sec t^7\,\tan t^7$.

C03S07.043: If $x = x(t) = (\tan t)^7 = \tan^7 t$, then

$$\frac{dx}{dt} = 7(\tan t)^6\cdot D_t\tan t = 7(\tan t)^6\,\sec^2 t = 7\tan^6 t\,\sec^2 t.$$

C03S07.044: If $x = x(t) = (\sec 2t)^7 = \sec^7 2t$, then

$$\frac{dx}{dt} = 7(\sec 2t)^6\cdot D_t(\sec 2t) = 7(\sec 2t)^6(\sec 2t\tan 2t)\cdot D_t(2t) = 14\sec^7 2t\,\tan 2t.$$

C03S07.045: If $x = x(t) = t^7\tan 5t$, then $\dfrac{dx}{dt} = 7t^6\tan 5t + 5t^7\sec^2 5t$.

C03S07.046: If $x = x(t) = \dfrac{\sec t^5}{t}$, then

$$\frac{dx}{dt} = \frac{t\cdot(\sec t^5\,\tan t^5)\cdot 5t^4 - \sec t^5}{t^2} = \frac{5t^5\sec t^5\,\tan t^5 - \sec t^5}{t^2}.$$

C03S07.047: If $x = x(t) = \sqrt{t}\sec\sqrt{t} = t^{1/2}\sec(t^{1/2})$, then

$$\frac{dx}{dt} = \frac{1}{2}t^{-1/2}\sec(t^{1/2}) + t^{1/2}\left[\sec(t^{1/2})\tan(t^{1/2})\right]\cdot\frac{1}{2}t^{-1/2} = \frac{\sec\sqrt{t} + \sqrt{t}\,\sec\sqrt{t}\,\tan\sqrt{t}}{2\sqrt{t}}.$$

C03S07.048: If $x = x(t) = \sec\sqrt{t}\,\tan\sqrt{t} = \sec t^{1/2}\,\tan t^{1/2}$, then

$$\frac{dx}{dt} = \left(\sec t^{1/2}\right)\left(\frac{1}{2}t^{-1/2}\sec^2 t^{1/2}\right) + \left(\frac{1}{2}t^{-1/2}\sec t^{1/2}\,\tan t^{1/2}\right)\left(\tan t^{1/2}\right) = \frac{\sec^3\sqrt{t} + \sec\sqrt{t}\,\tan^2\sqrt{t}}{2\sqrt{t}}.$$

C03S07.049: If $x = x(t) = \csc\left(\dfrac{1}{t^2}\right)$, then

$$\frac{dx}{dt} = \left[-\csc\left(\frac{1}{t^2}\right)\cot\left(\frac{1}{t^2}\right)\right]\cdot\left(-\frac{2}{t^3}\right) = \frac{2}{t^3}\csc\left(\frac{1}{t^2}\right)\cot\left(\frac{1}{t^2}\right).$$

C03S07.050: If $x = x(t) = \cot\left(\frac{1}{\sqrt{t}}\right) = \cot t^{-1/2}$, then

$$\frac{dx}{dt} = -\left(\csc t^{-1/2}\right)^2\cdot D_t\left(t^{-1/2}\right) = \frac{1}{2}t^{-3/2}\csc^2 t^{-1/2} = \frac{2}{t\sqrt{t}}\csc^2\left(\frac{1}{\sqrt{t}}\right).$$

C03S07.051: If $x = x(t) = \dfrac{\sec 5t}{\tan 3t}$, then

$$\frac{dx}{dt} = \frac{5\tan 3t\sec 5t\tan 5t - 3\sec 5t\sec^2 3t}{(\tan 3t)^2} = 5\cot 3t\sec 5t\tan 5t - 3\csc^2 3t\sec 5t.$$

C03S07.052: If $x = x(t) = \sec^2 t - \tan^2 t$, then $\dfrac{dx}{dt} = (2\sec t)(\sec t\tan t) - (2\tan t)(\sec^2 t) \equiv 0.$

C03S07.053: If $x = x(t) = t\sec t\csc t$, then

$$\frac{dx}{dt} = \sec t\,\csc t + t\sec t\,\tan t\,\csc t - t\sec t\,\csc t\,\cot t = t\sec^2 t + \sec t\,\csc t - t\csc^2 t.$$

C03S07.054: If $x = x(t) = t^3\tan^3 t^3 = t^3(\tan t^3)^3$, then

$$\frac{dx}{dt} = 3t^2(\tan t^3)^3 + t^3\cdot 3(\tan t^3)^2(\sec t^3)^2\cdot 3t^2 = 3t^2\tan^3 t^3 + 9t^5\sec^2 t^3\tan^2 t^3.$$

C03S07.055: If $x = x(t) = \sec(\sin t)$, then $\dfrac{dx}{dt} = [\sec(\sin t)\tan(\sin t)]\cdot\cos t.$

C03S07.056: If $x = x(t) = \cot(\sec 7t)$, then $\dfrac{dx}{dt} = \left[-\csc^2(\sec 7t)\right]\cdot 7\sec 7t\tan 7t.$

C03S07.057: If $x = x(t) = \dfrac{\sin t}{\sec t} = \sin t\cos t$, then $\dfrac{dx}{dt} = \cos^2 t - \sin^2 t = \cos 2t.$

C03S07.058: If $x = x(t) = \dfrac{\sec t}{1+\tan t}$, then

$$\frac{dx}{dt} = \frac{(1+\tan t)\sec t\tan t - \sec t\sec^2 t}{(1+\tan t)^2} = \frac{\sec t\tan t + \sec t\tan^2 t - (1+\tan^2 t)\sec t}{(1+\tan t)^2} = \frac{\sec t\tan t - \sec t}{(1+\tan t)^2}.$$

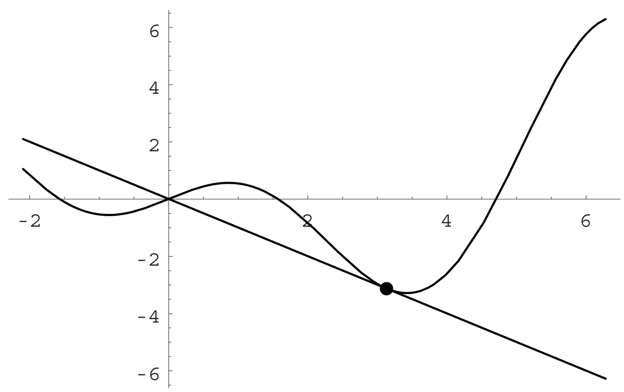
C03S07.059: If $x = x(t) = \sqrt{1 + \cot 5t} = (1 + \cot 5t)^{1/2}$, then

$$\frac{dx}{dt} = \frac{1}{2}(1 + \cot 5t)^{-1/2}(-5\csc^2 5t) = -\frac{5\csc^2 5t}{2\sqrt{1 + \cot 5t}}.$$

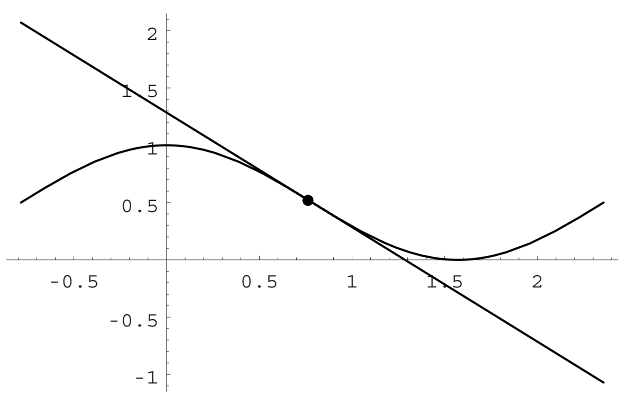
C03S07.060: If $x = x(t) = \sqrt{\csc \sqrt{t}} = (\csc t^{1/2})^{1/2}$, then

$$\frac{dx}{dt} = \frac{1}{2}(\csc t^{1/2})^{-1/2}(-\csc t^{1/2} \cot t^{1/2}) \cdot \frac{1}{2}t^{-1/2} = -\frac{\left(\cot \sqrt{t}\right)\sqrt{\csc \sqrt{t}}}{4\sqrt{t}} = -\frac{\left(\csc \sqrt{t}\right)^{3/2} \cos \sqrt{t}}{4\sqrt{t}}.$$

C03S07.061: If $f(x) = x \cos x$, then $f'(x) = -x \sin x + \cos x$, so the slope of the tangent at $x = \pi$ is $f'(\pi) = -\pi \sin \pi + \cos \pi = -1$. Because $f(\pi) = -\pi$, an equation of the tangent line is $y + \pi = -(x - \pi)$; that is, $y = -x$. The graph of f and this tangent line are shown next.

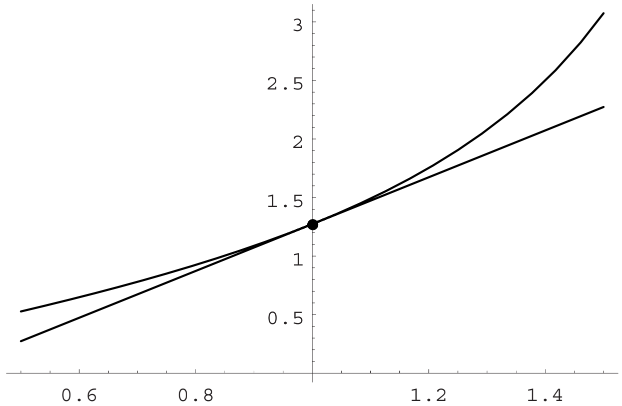


C03S07.062: If $f(x) = \cos^2 x$ then $f'(x) = -2 \cos x \sin x$, so the slope of the tangent at $x = \pi/4$ is $f'(\pi/4) = -2\cos(\pi/4)\sin(\pi/4) = -1$. Because $f(\pi/4) = \frac{1}{2}$, an equation of the tangent line is $y - \frac{1}{2} = -(x - \pi/4)$; that is, $4y = -4x + 2 + \pi$. The graph of f and this line are shown next.

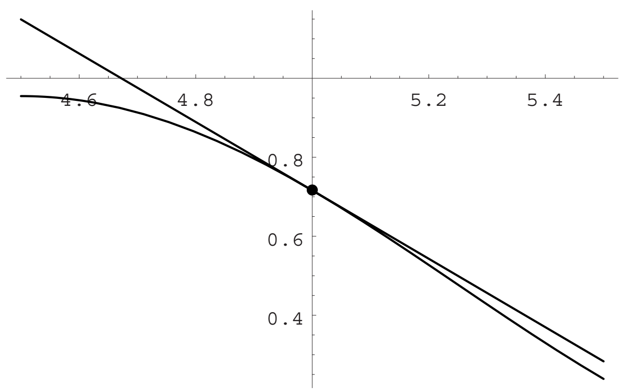


C03S07.063: If $f(x) = \dfrac{4}{\pi}\tan\left(\dfrac{\pi x}{4}\right)$, then $f'(x) = \sec^2\left(\dfrac{\pi x}{4}\right)$, so the slope of the tangent at $x = 1$ is $f'(1) = \sec^2\left(\dfrac{\pi}{4}\right) = 2$. Because $f(1) = \dfrac{4}{\pi}$, an equation of the tangent line is $y - \dfrac{4}{\pi} = 2\,(x - 1)$; that is,

$y = 2x - 2 + \dfrac{4}{\pi}$. The graph of f and this tangent line are shown next.



C03S07.064: If $f(x) = \dfrac{3}{\pi}\sin^2\left(\dfrac{\pi x}{3}\right)$, then $f'(x) = 2\sin\left(\dfrac{\pi x}{3}\right)\cos\left(\dfrac{\pi x}{3}\right)$, so the slope of the tangent at $x = 5$ is $f'(5) = 2\sin\dfrac{5\pi}{3}\cos\dfrac{5\pi}{3} = -\frac{1}{2}\sqrt{3}$. Because $f(5) = \dfrac{9}{4\pi}$, an equation of the tangent line is $y - \dfrac{9}{4\pi} = -\frac{1}{2}\sqrt{3}\,(x-5)$; that is, $y = -\dfrac{x\sqrt{3}}{2} + \dfrac{9+10\pi\sqrt{3}}{4\pi}$. The graph of f and this tangent line are shown next.



C03S07.065: $\dfrac{dy}{dx} = -2\sin 2x$. This derivative is zero at all values of x for which $\sin 2x = 0$; i.e., values of x for which $2x = 0$, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, ... Therefore the tangent line is horizontal at points with x-coordinate an integral multiple of $\frac{1}{2}\pi$. These are points of the form $(n\pi,\ 1)$ for any integer n and $\left(\frac{1}{2}m\pi,\ -1\right)$ for any odd integer m.

C03S07.066: $\dfrac{dy}{dx} = 1 - 2\cos x$, which is zero for $x = \left(\frac{1}{3}\pi\right) + 2k\pi$ and for $x = -\left(\frac{1}{3}\pi\right) + 2k\pi$ for any integer k. The tangent line is horizontal at all points of the form $\Big(\pm\frac{1}{3}\pi + 2k\pi, y\left(\pm\frac{1}{3}\pi + 2k\pi\right)\Big)$ where k is an integer.

C03S07.067: If $f(x) = \sin x\cos x$, then $f'(x) = \cos^2 x - \sin^2 x$. This derivative is zero at $x = \frac{1}{4}\pi + n\pi$ and at $x = \frac{3}{4}\pi + n\pi$ for any integer n. The tangent line is horizontal at all points of the form $\left(n\pi + \frac{1}{4}\pi,\ \frac{1}{2}\right)$ and at all points of the form $\left(n\pi + \frac{3}{4}\pi,\ -\frac{1}{2}\right)$ where n is an integer.

C03S07.068: If

$$f(x) = \frac{1}{3\sin^2 x + 2\cos^2 x}, \qquad \text{then} \qquad f'(x) = -\frac{\sin 2x}{\left(2 + \sin^2 x\right)^2}.$$

This derivative is zero at all values of x for which $\sin 2x = 0$; i.e., values of x for which $2x = 0$, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, $\ldots$. Therefore the tangent line is horizontal at points with x-coordinate an integral multiple of $\frac{1}{2}\pi$. These are points of the form $\left(n\pi, \frac{1}{2}\right)$ for any integer n and $\left(\frac{1}{2}m\pi, \frac{1}{3}\right)$ for any odd integer m.

C03S07.069: Let $f(x) = x - 2\cos x$. Then $f'(x) = 1 + 2\sin x$, so $f'(x) = 1$ when $2\sin x = 0$; that is, when $x = n\pi$ for some integer n. Moreover, if n is an integer then $f(n\pi) = n\pi - 2\cos n\pi$, so $f(n\pi) = n\pi + 2$ if n is even and $f(n\pi) = n\pi - 2$ if n is odd. In particular, $f(0) = 2$ and $f(\pi) = \pi - 2$. So the two lines have equations $y = x + 2$ and $y = x - 2$, respectively.

C03S07.070: If

$$f(x) = \frac{16 + \sin x}{3 + \sin x}, \qquad \text{then} \qquad f'(x) = -\frac{13\cos x}{(3 + \sin x)^2},$$

so that $f'(x) = 0$ when $\cos x = 0$; that is, when x is an odd integral multiple of $\frac{1}{2}\pi$. In particular,

$$f\left(\tfrac{1}{2}\pi\right) = \frac{16 + 1}{3 + 1} = \frac{17}{4};$$

similarly, $f\left(\frac{3}{4}\pi\right) = \frac{15}{2}$. Hence equations of the two lines are $y \equiv \frac{17}{4}$ and $y \equiv \frac{15}{2}$.

C03S07.071: To derive the formulas for the derivatives of the cotangent, secant, and cosecant functions, express each in terms of sines and cosines and apply the quotient rule (or the reciprocal rule) and various trigonometric identities (see Appendix C). Thus

$$D_x \cot x = D_x \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} - -\frac{1}{\sin^2 x} = -\csc^2 x,$$

$$D_x \sec x = D_x \frac{1}{\cos x} = -\frac{-\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \, \tan x, \qquad \text{and}$$

$$D_x \csc x = D_x \frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \, \cot x.$$

C03S07.072: If $g(x) = \cos x$, then

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos x \, \cos h - \sin x \, \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \frac{1 - \cos h}{h}(-\cos x) - \lim_{h \to 0} \frac{\sin h}{h}(\sin x) \\
&= 0 \cdot (-\cos x) - 1 \cdot \sin x = -\sin x.
\end{aligned}$$

C03S07.073: Write $R = R(\alpha) = \dfrac{1}{32} v^2 \sin 2\alpha$. Then

$$R'(\alpha) = \frac{1}{16} v^2 \cos 2\alpha,$$

which is zero when $\alpha = \pi/4$ (we assume $0 \leqq \alpha \leqq \pi/2$). Because R is zero at the endpoints of its domain, we conclude that $\alpha = \pi/4$ maximizes the range R.

C03S07.074: Let h be the altitude of the balloon (in feet) at time t (in seconds) and let θ be its angle of elevation with respect to the observer. From the obvious figure, $h = 300 \tan\theta$, so

$$\frac{dh}{dt} = (300 \sec^2\theta)\frac{d\theta}{dt}.$$

When $\theta = \pi/4$ and $\dfrac{d\theta}{dt} = \pi/180$, we have

$$\frac{dh}{dt} = 300 \cdot 2 \cdot \frac{\pi}{180} = \frac{10\pi}{3} \approx 10.47 \text{ (ft/s)}$$

as the rate of the balloon's ascent then.

C03S07.075: Let h be the altitude of the rocket (in miles) at time t (in seconds) and let α be its angle of elevation then. From the obvious figure, $h = 2\tan\alpha$, so

$$\frac{dh}{dt} = (2\sec^2\alpha)\frac{d\alpha}{dt}.$$

When $\alpha = 5\pi/18$ and $d\alpha/dt = 5\pi/180$, we have $dh/dt \approx 0.4224$ (mi/s; about 1521 mi/h).

C03S07.076: Draw a figure in which the airplane is located at $(0, 25000)$ and the fixed point on the ground is located at $(x, 0)$. A line connecting the two produces a triangle with angle θ at $(x, 0)$. This angle is also the angle of depression of the pilot's line of sight, and when $\theta = 65°$, $d\theta/dt = 1.5°/\text{s}$. Now

$$\tan\theta = \frac{25000}{x}, \text{ so } x = 25000\frac{\cos\theta}{\sin\theta},$$

thus

$$\frac{dx}{d\theta} = -\frac{25000}{\sin^2\theta}.$$

The speed of the airplane is

$$-\frac{dx}{dt} = \frac{25000}{\sin^2\theta}\cdot\frac{d\theta}{dt}.$$

When $\theta = \dfrac{13}{36}\pi$, $\dfrac{d\theta}{dt} = \dfrac{\pi}{120}$. So the ground speed of the airplane is

$$\frac{25000}{\sin^2\left(\dfrac{13\pi}{36}\right)}\cdot\frac{\pi}{120} \approx 796.81 \text{ (ft/s)}.$$

Answer: About 543.28 mi/h.

C03S07.077: Draw a figure in which the observer is located at the origin, the x-axis corresponds to the ground, and the airplane is located at $(x, 20000)$. The observer's line of sight corrects the origin to the point $(x, 20000)$ and makes an angle θ with the ground. Then

$$\tan\theta = \frac{20000}{x},$$

so that $x = 20000\cot\theta$. Thus

$$\frac{dx}{dt} = (-20000 \csc^2 \theta)\frac{d\theta}{dt}.$$

When $\theta = 60°$, we are given $\dfrac{d\theta}{dt} = 0.5°/\text{s}$; that is, $\dfrac{d\theta}{dt} = \pi/360$ radians per second when $\theta = \pi/3$. We evaluate dx/dt at this time with these values to obtain

$$\frac{dx}{dt} = (-20000)\frac{1}{\sin^2\left(\dfrac{\pi}{3}\right)} \cdot \frac{\pi}{360} = -\frac{2000\pi}{27},$$

approximately -232.71 ft/s. Answer: About 158.67 mi/h.

C03S07.078: The area of the rectangle is $A = 4xy$, but $x = \cos\theta$ and $y = \sin\theta$, so

$$A = A(\theta) = 4\sin\theta\,\cos\theta, \qquad 0 \leqq \theta \leqq \pi/2.$$

Now $A'(\theta) = 4(\cos^2\theta - \sin^2\theta) = 4\cos 2\theta$, so $A'(\theta) = 0$ when $\cos 2\theta = 0$. Because $0 \leqq 2\theta \leqq \pi$, it follows that $2\theta = \pi/2$, so $\theta = \pi/4$. But $A(0) = 0 = A(\pi/2)$ and $A(\pi/4) = 2$, so the latter is the largest possible area of a rectangle inscribed in the unit circle.

C03S07.079: The cross section of the trough is a trapezoid with short base 2, long base $2 + 4\cos\theta$, and height $2\sin\theta$. Thus its cross-sectional area is

$$\begin{aligned} A(\theta) &= \frac{2 + (2 + 4\cos\theta)}{2} \cdot 2\sin\theta \\ &= 4(\sin\theta + \sin\theta\cos\theta), \qquad 0 \leqq \theta \leqq \pi/2 \end{aligned}$$

(the real upper bound on θ is $2\pi/3$, but the maximum value of A clearly occurs in the interval $[0, \pi/2]$).

$$\begin{aligned} A'(\theta) &= 4(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= 4(2\cos^2\theta + \cos\theta - 1) \\ &= 4(2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The only solution of $A'(\theta) = 0$ in the given domain occurs when $\cos\theta = \frac{1}{2}$, so that $\theta = \frac{1}{3}\pi$. It is easy to verify that this value of θ maximizes the function A.

C03S07.080: In the situation described in the problem, we have $D = 20\sec\theta$. The illumination of the walkway is then

$$I = I(\theta) = \frac{k}{400}\sin\theta\,\cos^2\theta, \qquad 0 \leqq \theta \leqq \pi/2.$$

$$\frac{dI}{d\theta} = \frac{k\cos\theta}{400}(\cos^2\theta - 2\sin^2\theta);$$

$dI/d\theta = 0$ when $\theta = \pi/2$ and when $\cos^2\theta = 2\sin^2\theta$. The solution θ in the domain of I of the latter equation has the property that $\sin\theta = \sqrt{3}/3$ and $\cos\theta = \sqrt{6}/3$. But $I(0) = 0$ and $I(\theta) \to 0$ as $\theta \to (\pi/2)^-$, so the optimal height of the lamp post occurs when $\sin\theta = \sqrt{3}/3$. This implies that the optimal height is $10\sqrt{2} \approx 14.14$ m.

C03S07.081: The following figure shows a cross section of the sphere-with-cone through the axis of the cone and a diameter of the sphere. Note that $h = r\tan\theta$ and that

$$\cos\theta = \frac{R}{h-R}.$$

Therefore

$$h = R + R\sec\theta, \qquad \text{and thus} \qquad r = \frac{R + R\sec\theta}{\tan\theta}.$$

Now $V = \frac{1}{3}\pi r^2 h$, so for θ in the interval $(0,\, \pi/2)$, we have

$$V = V(\theta) = \frac{1}{3}\pi R^3 \cdot \frac{(1+\sec\theta)^3}{\tan^2\theta}.$$

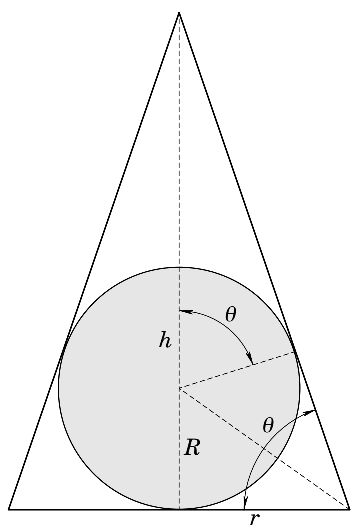
Therefore

$$V'(\theta) = \frac{\pi R^3}{3\tan^4\theta}\left[3(\tan^2\theta)(1+\sec\theta)^2\sec\theta\,\tan\theta - (1+\sec\theta)^3(2\tan\theta\,\sec^2\theta)\right].$$

If $V'(\theta) = 0$ then either $\sec\theta = -1$ (so $\theta = \pi$, which we reject), or $\sec\theta = 0$ (which has no solutions), or $\tan\theta = 0$ (so either $\theta = 0$ or $\theta = \pi$, which we also reject), or (after replacement of $\tan^2\theta$ with $\sec^2\theta - 1$)

$$\sec^2\theta - 2\sec\theta - 3 = 0.$$

It follows that $\sec\theta = 3$ or $\sec\theta = -1$. We reject the latter as before, and find that $\sec\theta = 3$, so $\theta \approx 1.23095$ (radians). The resulting minimum volume of the cone is $\frac{8}{3}\pi R^3$, twice the volume of the sphere!



C03S07.082: Let L be the length of the crease. Then the right triangle of which L is the hypotenuse has sides $L\cos\theta$ and $L\sin\theta$. Now $20 = L\sin\theta + L\sin\theta\,\cos 2\theta$, so

$$L = L(\theta) = \frac{20}{(\sin\theta)(1+\cos 2\theta)}, \qquad 0 < \theta \leqq \frac{\pi}{4}.$$

Next, $\dfrac{dL}{d\theta} = 0$ when

$$(\cos\theta)(1+\cos 2\theta) = (\sin\theta)(2\sin 2\theta);$$

$$(\cos\theta)(2\cos^2\theta) = 4\sin^2\theta\,\cos\theta;$$

so $\cos\theta = 0$ (which is impossible given the domain of L) or

$$\cos^2\theta = 2\sin^2\theta = 2 - 2\cos^2\theta; \qquad \cos^2\theta = \frac{2}{3}.$$

This implies that $\cos\theta = \frac{1}{3}\sqrt{6}$ and $\sin\theta = \frac{1}{3}\sqrt{3}$. Because $L \to +\infty$ as $\theta \to 0^+$, we have a minimum either at the horizontal tangent just found or at the endpoint $\theta = \frac{1}{4}\pi$. The value of L at $\frac{1}{4}\pi$ is $20\sqrt{2} \approx 28.28$ and at the horizontal tangent we have $L = 15\sqrt{3} \approx 25.98$. So the shortest crease is obtained when $\cos\theta = \frac{1}{3}\sqrt{6}$; that is, for θ approximately $35^\circ 15'\ 52''$. The bottom of the crease should be one-quarter of the way across the page from the lower left-hand corner.

C03S07.083: Set up coordinates so the diameter is on the x-axis and the equation of the circle is $x^2+y^2 = 1$; let $(x\,,y)$ denote the northwest corner of the trapezoid. The chord from $(1,\,0)$ to $(x,\,y)$ forms a right triangle with hypotenuse 2, side z opposite angle θ, and side w; moreover, $z = 2\sin\theta$ and $w = 2\cos\theta$. It follows that

$$y = w\sin\theta = 2\sin\theta\,\cos\theta \qquad \text{and}$$

$$-x = 1 - w\cos\theta = -\cos 2\theta.$$

Now

$$A = y(1-x) = (2\sin\theta\cos\theta)(1-\cos 2\theta) = 4\sin\theta\cos\theta\sin^2\theta,$$

and therefore

$$A = A(\theta) = 4\sin^3\theta\,\cos\theta, \qquad \frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{2}.$$

$$\begin{aligned} A'(\theta) &= 12\sin^2\theta\,\cos^2\theta - 4\sin^4\theta \\ &= (4\sin^2\theta)(3\cos^2\theta - \sin^2\theta). \end{aligned}$$

To solve $A'(\theta) = 0$, we note that $\sin\theta \neq 0$, so we must have $3\cos^2\theta = \sin^2\theta$; that is $\tan^2\theta = 3$. It follows that $\theta = \frac{1}{3}\pi$. The value of A here exceeds its value at the endpoints, so we have found the maximum value of the area—it is $\frac{3}{4}\sqrt{3}$.

C03S07.084: Let $\theta = \alpha/2$ (see Fig. 3.7.18 of the text) and denote the radius of the circular log by r. Using the technique of the solution of Problem 82, we find that the area of the hexagon is

$$A = A(\theta) = 8r^2\sin^3\theta\,\cos\theta, \qquad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

After some simplifications we also find that

$$\frac{dA}{d\theta} = 8r^2(\sin^2\theta)(4\cos^2\theta - 1).$$

Now $dA/d\theta = 0$ when $\sin\theta = 0$ and when $\cos\theta = \frac{1}{2}$. When $\sin\theta = 0$, $A = 0$; also, $A(0) = 0 = A\left(\frac{1}{2}\pi\right)$. Therefore A is maximal when $\cos\theta = \frac{1}{2}$: $\theta = \frac{1}{3}\pi$. When this happens, we find that $\alpha = \frac{2}{3}\pi$ and that $\beta = \pi - \theta = \frac{2}{3}\pi$. Therefore the figure of maximal area is a regular hexagon.

C03S07.085: The area in question is the area of the sector minus the area of the triangle in Fig. 3.7.19 and turns out to be

$$\begin{aligned} A &= \frac{1}{2}r^2\theta - r^2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2} \\ &= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{s^2(\theta - \sin\theta)}{2\theta^2} \end{aligned}$$

because $s = r\theta$. Now

$$\frac{dA}{d\theta} = \frac{s^2(2\sin\theta - \theta\cos\theta - \theta)}{2\theta^3},$$

so $dA/d\theta = 0$ when $\theta(1 + \cos\theta) = 2\sin\theta$. Let $\theta = 2x$; note that $0 < x \leqq \pi$ because $0 < \theta \leqq 2\pi$. So the condition that $dA/d\theta = 0$ becomes

$$x = \frac{\sin\theta}{1 + \cos\theta} = \tan x.$$

But this equation has no solution in the interval $(0,\,\pi]$. So the only possible maximum of A must occur at an endpoint of its domain, or where x is undefined because the denominator $1 + \cos\theta$ is zero—and this occurs when $\theta = \pi$. Finally,

$$A(2\pi) = \frac{s^2}{4\pi} \qquad \text{and} \qquad A(\pi) = \frac{s^2}{2\pi},$$

so the maximum area is attained when the arc is a semicircle.

C03S07.086: The length of the forest path is $2\csc\theta$. So the length of the part of the trip along the road is $3 - 2\csc\theta\,\cos\theta$. Thus the total time for the trip is given by

$$T = T(\theta) = \frac{2}{3\sin\theta} + \frac{3 - \dfrac{2\cos\theta}{\sin\theta}}{8}.$$

Note that the range of values of θ is determined by the condition

$$\frac{3\sqrt{13}}{13} \geqq \cos\theta \geqq 0.$$

After simplifications, we find that

$$T'(\theta) = \frac{3 - 8\cos\theta}{12\sin^2\theta}.$$

Now $T'(\theta) = 0$ when $\cos\theta = \frac{3}{8}$; that is, when θ is approximately $67°58'\,32''$. For this value of θ, we find that $\sin\theta = \frac{1}{8}\sqrt{55}$. There's no problem in verifying that we have found the minimum. Answer: The distance to walk down the road is

$$\left(3 - 2\frac{\cos\theta}{\sin\theta}\right)\bigg|_{\sin\theta = \frac{1}{8}\sqrt{55}} = 3 - \frac{6\sqrt{55}}{55} \approx 2.19096 \text{ (km)}.$$

C03S07.087: Following the *Suggestion*, we note that if n is a positive integer and

$$h = \frac{2}{(4n+1)\pi},$$

then

$$\frac{f(h) - f(0)}{h} = \frac{(4n+1)\pi \sin \frac{1}{2}(4n+1)\pi}{(4n+1)\pi} = 1,$$

and that if

$$h = \frac{2}{(4n-1)\pi},$$

then

$$\frac{f(h) - f(0)}{h} = \frac{(4n-1)\pi \sin \frac{1}{2}(4n-1)\pi}{(4n-1)\pi} = -1.$$

Therefore there are values of h arbitrarily close to zero for which

$$\frac{f(0+h) - f(0)}{h} = +1$$

and values of h arbitrarily close to zero for which

$$\frac{f(0+h) - f(0)}{h} = -1.$$

It follows that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist;}$$

that is, $f'(0)$ does not exist, and so f is not differentiable at $x = 0$.

C03S07.088: $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h^2 \sin \dfrac{1}{h}}{h} = \lim_{h \to 0} h \sin \dfrac{1}{h}.$

It now follows from the Squeeze Law of Section 2.3 (page 79) that

$$f'(0) = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

because $-|h| \leqq h \sin \dfrac{1}{h} \leqq |h|$ if $h \neq 0$.

Section 3.8

Note: Your answers may differ from ours in the last one or two decimal places because of differences in hardware or in the way the problem was solved. We used *Mathematica* 3.0 and carried 40 decimal digits throughout all calculations, and our answers are correct or correctly rounded to the number of digits shown here. In most of the first 20 problems the initial guess x_0 was obtained by linear interpolation. Finally, the equals mark is used in this section to mean "equal or approximately equal."

C03S08.001: With $f(x) = x^2 - 5$, $a = 2$, $b = 3$, and

$$x_0 = a - \frac{(b-a)f(a)}{f(b) - f(a)} = 2.2,$$

we used the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for} \quad n \geqq 0.$$

Thus we obtained $x_1 = 2.236363636$, $x_2 = 2.236067997$, and $x_3 = x_4 = 2.236067977$. Answer: 2.2361.

C03S08.002: $x_0 = 1.142857143$; we use $f(x) = x^3 - 2$. Then $x_1 = 1.272321429$, $x_2 = 1.260041515$, $x_3 = 1.259921061$, and $x_4 = x_5 = 1.259921050$. Answer: 1.2599.

C03S08.003: $x_0 = 2.322274882$; we use $f(x) = x^5 - 100$. Then $x_1 = 2.545482651$, $x_2 = 2.512761634$, $x_3 = 2.511887041$, and $x_4 = x_5 = 2.511886432$. Answer: 2.5119.

C03S08.004: Let $f(x) = x^{3/2} - 10$. Then $x_0 = 4.628863603$. From the iterative formula

$$x \longleftarrow x - \frac{x^{3/2} - 10}{\frac{3}{2}x^{1/2}}$$

we obtain $x_1 = 4.641597575$, $x_2 = 4.641588834 = x_3$. Answer: 4.6416.

C03S08.005: 0.25, 0.3035714286, 0.3027758133, 0.3027756377. Answer: 0.3028.

C03S08.006: 0.2, 0.2466019417, 0.2462661921, 0.2462661722. Answer: 0.2463.

C03S08.007: $x_0 = -0.5$, $x_1 = -0.8108695652$, $x_2 = -0.7449619516$, $x_3 = -0.7402438226$, $x_4 = -0.7402217826 = x_5$. Answer: 0.7402.

C03S08.008: Let $f(x) = x^3 + 2x^2 + 2x - 10$. With initial guess $x_0 = 1.5$ (the midpoint of the interval), we obtain $x_1 = 1.323943661972$, $x_2 = 1.309010783652$, $x_3 = 1.308907324710$, $x_4 = 1.308907319765$, and $x_5 = 1.308907319765$. Answer: 1.3089.

C03S08.009: With $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, and calculator set in *radian* mode, we obtain $x_0 = 0.5854549279$, $x_1 = 0.7451929664$, $x_2 = 0.7390933178$, $x_3 = 0.7390851332$, and $x_4 = x_3$. Answer: 0.7391.

C03S08.010: Let $f(x) = x^2 - \sin x$. Then $f'(x) = 2x - \cos x$. The linear interpolation formula yields $x_0 = 0.7956861008$, and the iterative formula

$$x \longleftarrow x - \frac{x^2 - \sin x}{2x - \cos x}$$

(with calculator in *radian* mode) yields the following results: $x_1 = 0.8867915207$, $x_2 = 0.8768492470$, $x_3 = 0.8767262342$, and $x_4 = 0.8767262154 = x_5$. Answer: 0.8767.

C03S08.011: With $f(x) = 4x - \sin x - 4$ and calculator in *radian* mode, we get the following results: $x_0 = 1.213996400$, $x_1 = 1.236193029$, $x_2 = 1.236129989 = x_3$. Answer: 1.2361.

C03S08.012: $x_0 = 0.8809986055$, $x_1 = 0.8712142932$, $x_3 = 0.8712215145$, and $x_4 = x_3$. Answer: 0.8712.

C03S08.013: With $x_0 = 2.188405797$ and the iterative formula

$$x \longleftarrow x - \frac{x^4(x+1) - 100}{x^3(5x+4)},$$

we obtain $x_1 = 2.360000254$, $x_2 = 2.339638357$, $x_3 = 2.339301099$, and $x_4 = 2.339301008 = x_5$. Answer: 2.3393.

C03S08.014: $x_0 = 0.7142857143$, $x_1 = 0.8890827860$, $x_2 = 0.8607185590$, $x_3 = 0.8596255544$, and $x_4 = 0.8596240119 = x_5$. Answer: 0.8596.

C03S08.015: The nearest discontinuities of $f(x) = x - \tan x$ are at $\pi/2$ and at $3\pi/2$, approximately 1.571 and 4.712. Therefore the function $f(x) = x - \tan x$ has the intermediate value property on the interval $[2, 3]$. Results: $x_0 = 2.060818495$, $x_1 = 2.027969226$, $x_2 = 2.028752991$, and $x_3 = 2.028757838 = x_4$. Answer: 2.0288.

C03S08.016: As $\frac{7}{2}\pi \approx 10.9956$ and $\frac{9}{2}pi \approx 14.1372$ are the nearest discontinuities of $f(x) = x - \tan x$, this function has the intermediate value property on the interval $[11, 12]$. Because $f(11) \approx -214.95$ and $f(12) \approx 11.364$, the equation $f(x) = 0$ has a solution in $[11, 12]$. We obtain $x_0 = 11.94978618$ by interpolation, and the iteration

$$x \longleftarrow x - \frac{x + \tan x}{1 + \sec^2 x}$$

of Newton's method yields the successive approximations

$$x_1 = 7.457596948, \;\; x_2 = 6.180210620, \;\; x_3 = 3.157913273, \;\; x_4 = 1.571006986;$$

after many more iterations we arrive at the answer 2.028757838 of Problem 15. The difficulty is caused by the fact that $f(x)$ is generally a very large number, so the iteration of Newton's method tends to alter the value of x excessively. A little experimentation yields the fact that $f(11.08) \approx -0.736577$ and $f(11.09) \approx 0.531158$. We begin anew on the better interval $[11.08, 11.09]$ and obtain $x_0 = 11.08581018$, $x_1 = 11.08553759$, $x_2 = 11.08553841$, and $x_3 = x_2$. Answer: 11.0855.

C03S08.017: $x_0 = 2.105263158$, $x_1 = 2.155592105$, $x_2 = 2.154435311$, $x_3 = 2.154434690 = x_4$. Answer: 2.1544.

C03S08.018: $x_0 = 2.058823529$, $x_1 = 2.095291459$, $x_2 = 2.094551790$, $x_3 = 2.094551482 = x_4$. Answer: 2.0946.

C03S08.019: We find that $x_0 = 1.538461538$, $x_1 = 1.932798309$, $x_2 = 1.819962709$, $x_3 = 1.802569044$, $x_4 = 1.802191039$, $x_5 = 1.802190864$, and $x_6 = 1.802190864 = x_5$. The convergence is slow because $|f'(x)|$ is large when x is near 1.8.

C03S08.020: $x_0 = 0.0512$, $x_1 = 931322.6156$, $x_2 = 745058.0925$, ... —better improve the initial guess! Maybe the (naïve) choice of the midpoint of the interval would be better? Then we obtain $x_0 = 2.5$, $x_1 = 2.16384$, $x_2 = 2.023002449$, $x_3 = 2.000517182$, $x_4 = 2.000000267$, $x_5 = 2.000000000 = x_6$.

C03S08.021: Let $f(x) = x^3 - a$. Then the iteration of Newton's method in Eq. (6) takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^3 - a}{3(x_n)^2} = \frac{2(x_n)^3 + a}{3(x_n)^2} = \frac{1}{3}\left(2x_n + \frac{a}{(x_n)^2}\right).$$

Because $1 < \sqrt[3]{2} < 2$, we begin with $x_0 = 1.5$ and apply this formula with $a = 2$ to obtain $x_1 = 1.296296296$, $x_2 = 1.260932225$, $x_3 = 1.259921861$, and $x_4 = 1.259921050 = x_5$. Answer: 1.25992.

C03S08.022: The formula in Eq. (6) of the text, with $f(x) = x^k - a$, takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^k - a}{k(x_n)^{k-1}} = \frac{(k-1)(x_n)^k + a}{k(x_n)^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{(x_n)^{k-1}}\right].$$

We take $a = 100$, $k = 10$, and $x_0 = 1.5$ and obtain $x_1 = 1.610122949$, $x_2 = 1.586599871$, $x_3 = 1.584901430$, $x_4 = 1.584893193$, and $x_5 = 1.584893192 = x_6$. Answer: 1.58489.

C03S08.023: We get $x_0 = 0.5$, $x_1 = 0.4387912809$, $x_2 = 0.4526329217$, $x_3 = 0.4496493762$, ..., $x_{14} = 0.4501836113 = x_{15}$. The method of repeated substitution tends to converge much more slowly than Newton's method, has the advantage of not requiring that you compute a derivative or even that the functions involved are differentiable, and has the disadvantage of more frequent failure than Newton's method when both are applicable (see Problems 24 and 25).

C03S08.024: Our results using the first formula: $x_0 = 1.5$, $x_1 = 1.257433430$, $x_2 = 1.225755182$, $x_3 = 1.221432153$, ..., $x_{10} = 1.220745085 = x_{11}$. When we use the second formula, we obtain $x_1 = 4.0625$, $x_2 = 271.3789215$, $x_3 = 5423829645$, and x_4 has 39 digits to the left of the decimal point. It frequently requires some ingenuity to find a suitable way to put the equation $f(x) = 0$ into the form $x = G(x)$.

C03S08.025: Beginning with $x_0 = 0.5$, the first formula yields $x_0 = 0.5$, $x_1 = -1$, $x_2 = 2$, $x_3 = 2.75$, $x_4 = 2.867768595$, ..., $x_{12} = 2.879385242 = x_{13}$. Wrong root! At least the method converged. If your calculator or computer balks at computing the cube root of a negative number, then you can rewrite the second formula in Problem 25 in the form

$$x = \operatorname{Sgn}(3x^2 - 1) \cdot |3x^2 - 1|^{1/3}.$$

The results, again with $x_0 = 0.5$, are $x_1 = -0.629960525$, $x_2 = 0.575444686$, $x_3 = -0.187485243$, $x_4 = -0.963535808$, ..., $x_{25} = 2.877296053$, $x_{26} = 2.877933902$, ..., and $x_{62} = 2.879385240 = x_{63}$. Not only is convergence extremely slow, the method of repeated substitution again leads to the wrong root. Finally, the given equation can also be written in the form

$$x = \frac{1}{\sqrt{3 - x}},$$

and in this case, again with $x_0 = 0.5$, we obtain $x_1 = 0.632455532$, $x_2 = 0.649906570$, $x_3 = 0.652315106$, $x_4 = 0.652649632$, ..., and $x_{12} = 0.652703645 = x_{13}$.

C03S08.026: If $f(x) = \dfrac{1}{x} - a$, then Newton's method uses the iteration

$$x \longleftarrow x - \frac{\dfrac{1}{x} - a}{-\dfrac{1}{x^2}} = x + x^2\left(\frac{1}{x} - a\right) = 2x - ax^2.$$

C03S08.027: Let $f(x) = x^5 + x - 1$. Then $f(x)$ is a polynomial, thus is continuous everywhere, and thus has the intermediate value property on every interval. Also $f(0) = -1$ and $f(1) = 1$, so $f(x)$ must assume the intermediate value 0 somewhere in the interval $[0, 1]$. Thus the equation $f(x) = 0$ has *at least* one solution. Next, $f'(x) = 5x^4 + 1$ is positive for all x, so f is an increasing function. Because f is continuous, its graph can therefore cross the x-axis at most once, and so the equation $f(x) = 0$ has *at most* one solution. Thus it has exactly one solution. Incidentally, Newton's method yields the approximate solution 0.75487766624669276. To four places, 0.7549.

C03S08.028: Let $f(x) = x^2 - \cos x$. The graph of f on $[-1, 1]$ shows that there are two solutions, one near -0.8 and the other near 0.8. With $x_0 = 0.8$, Newton's method yields $x_1 = 0.824470434$, $x_2 = 0.824132377$, and $x_3 = 0.824132312 = x_4$. Because $f(-x) = f(x)$, the other solution is -0.824132312. Answer: ± 0.8241.

C03S08.029: Let $f(x) = x - 2\sin x$. The graph of f on $[-2, 2]$ shows that there are exactly three solutions, the largest of which is approximately $x_0 = 1.9$. With Newton's method we obtain $x_1 = 1.895505940$, and $x_2 = 1.895494267 = x_3$. Because $f(-x) = -f(x)$, the other two solutions are 0 and -1.895494267. Answer: ± 1.8955 and 0.

C03S08.030: Let $f(x) = x + 5\cos x$. The graph of f on the interval $[-5, 5]$ shows that there are exactly three solutions, approximately -1.3, 2.0, and 3.9. Newton's method then yields

n	First x_n	Second x_n	Third x_n
1	-1.306444739	1.977235450	3.839096917
2	-1.306440008	1.977383023	3.837468316
3	-1.306440008	1.977383029	3.837467106
4	-1.306440008	1.977383029	3.837467106

Answers: -1.3064, 1.9774, and 3.8375.

C03S08.031: Let $f(x) = x^7 - 3x^3 + 1$. Then $f(x)$ is a polynomial, so f is continuous on every interval of real numbers, including the intervals $[-2, -1]$, $[0, 1]$, and $[1, 2]$. Also $f(-2) = -103 < 0 < 3 = f(-1)$, $f(0) = 1 > 0 > -1 = f(1)$, and $f(1) = -1 < 0 < 105 = f(2)$. Therefore the equation $f(x) = 0$ has one solution in $(-2, -1)$, another in $(0, 1)$, and a third in $(1, 2)$. (It has no other real solutions.) The graph of f shows that the first solution is near -1.4, the second is near 0.7, and the third is near 1.2. Then Newton's method yields

n	First x_n	Second x_n	Third x_n
1	-1.362661201	0.714876604	1.275651936
2	-1.357920265	0.714714327	1.258289744
3	-1.357849569	0.714714308	1.256999591
4	-1.357849553	0.714714308	1.256992779
5	-1.357849553	0.714714308	1.256992779

Answers: -1.3578, 0.7147, and 1.2570.

C03S08.032: Let $f(x) = x^3 - 5$. Use the iteration

$$x \longleftarrow x - \frac{x^3 - 5}{3x^2}.$$

With $x_0 = 2$, we obtain the sequence of approximations 1.75, 1.710884354, 1.709976429, 1.709975947, and 1.709975947. Answer: 1.7100.

C03S08.033: There is only one solution of $x^3 = \cos x$ for the following reasons: $x^3 < -1 \leqq \cos x$ if $x < -1$, $x^3 < 0 < \cos x$ if $-1 < x < 0$, x^3 is increasing on $[0, 1]$ whereas $\cos x$ is decreasing there (and their graphs cross in this interval as a consequence of the intermediate value property of continuous functions), and $x^3 > 1 \geqq \cos x$ for $x > 1$. The graph of $f(x) = x^3 - \cos x$ crosses the x-axis near $x_0 = 0.9$, and Newton's method yields $x_1 = 0.866579799$, $x_2 = 0.865475218$, and $x_3 = 0.865474033 = x_4$. Answer: Approximately 0.8654740331016145.

C03S08.034: The graphs of $y = x$ and $y = \tan x$ show that the smallest positive solution of the equation $f(x) = x - \tan x = 0$ is between π and $3\pi/2$. With initial guess $x = 4.5$ we obtain 4.493613903, 4.493409655, 4.493409458, and 4.493409458. Answer: Approximately 4.493409457909064.

C03S08.035: With $x_0 = 3.5$, we obtain the sequence $x_1 = 3.451450588$, $x_2 = 3.452461938$, and finally $x_3 = 3.452462314 = x_4$. Answer: Approximately 3.452462314057969.

C03S08.036: To find a zero of $f(\theta) = \theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi$, we use the iteration

$$\theta \longleftarrow \theta - \frac{\theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi}{1 - \frac{1}{2}\cos\theta}.$$

The results, with $\theta_0 = 1.5$ $(86^\circ 56'\, 37'')$, are: $\theta_1 = 1.569342$ $(89^\circ 55'\, 00'')$, $\theta_2 = 1.568140$ $(89^\circ 50'\, 52'')$, $\theta_3 = 1.568140$.

C03S08.037: If the plane cuts the sphere at distance x from its center, then the smaller spherical segment has height $h = a - x = 1 - x$ and the larger has height $h = a + x = 1 + x$. So the smaller has volume

$$V_1 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 - x)^2(2 + x)$$

and the larger has volume

$$V_2 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 + x)^2(2 - x) = 2V_1.$$

These equations leads to

$$\begin{aligned}
&(1 + x)^2(2 - x) = 2(1 - x)^2(2 + x);\\
&(x^2 + 2x + 1)(x - 2) + 2(x^2 - 2x + 1)(x + 2) = 0;\\
&x^3 - 3x - 2 + 2x^3 - 6x + 4 = 0;\\
&3x^3 - 9x + 2 = 0.
\end{aligned}$$

The last of these equations has three solutions, one near -1.83 (out of range), one near 1.61 (also out of range), and one near $x_0 = 0.2$. Newton's method yields $x_1 = 0.225925926$, $x_2 = 0.226073709$, and $x_3 = 0.226073714 = x_4$. Answer: 0.2261.

C03S08.038: This table shows that the equation $f(x) = 0$ has solutions in each of the intervals $(-3, -2)$, $(0, 1)$, and $(1, 2)$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

The next table shows the results of the iteration of Newton's method:

n	x_n	x_n	x_n
0	1.5	0.5	-2.5
1	2.090909091	0.2307692308	-2.186440678
2	1.895903734	0.2540002371	-2.118117688
3	1.861832371	0.2541016863	-2.114914461
4	1.860806773	0.2541016884	-2.114907542
5	1.860805853	0.2541016884	-2.114907541
6	1.860805853		-2.114907541

Answer: -2.1149, 0.2541, and 1.8608.

C03S08.039: We iterate using the formula

$$x \longleftarrow x - \frac{x + \tan x}{1 + \sec^2 x}.$$

Here is a sequence of simple *Mathematica* commands to find approximations to the four least positive solutions of the given equation, together with the results. (The command **list=g[list]** was executed repeatedly, but deleted from the output to save space.)

list={2.0, 5.0, 8.0, 11.0}
f[x_]:=x+Tan[x]
g[x_]:=N[x−f[x]/f′[x], 10]
list=g[list]
2.027314579, 4.879393859, 7.975116372, 11.00421012
2.028754298, 4.907699753, 7.978566616, 11.01202429
2.028757838, 4.913038110, 7.978665635, 11.02548807
2.028757838, 4.913180344, 7.978665712, 11.04550306
2.028757838, 4.913180439, 7.978665712, 11.06778114
2.028757838, 4.913180439, 7.978665712, 11.08205766
2.028757838, 4.913180439, 7.978665712, 11.08540507
2.028757838, 4.913180439, 7.978665712, 11.08553821
2.028757838, 4.913180439, 7.978665712, 11.08553841

Answer: 2.029 and 4.913.

C03S08.040: Plot the graph of $f(x) = 4x^3 - 42x^2 - 19x - 28$ on $[-3, 12]$ to see that the equation $f(x) = 0$ has exactly one real solution neear $x = 11$. The initial guess $x_0 = 0$ yields the solution $x = 10.9902$ after

20 iterations. The initial guess $x_0 = 10$ yields the solution after three iterations. The initial guess $x_0 = 100$ yields the solution after ten iterations.

C03S08.041: Similar triangles show that

$$\frac{x}{u+v} = \frac{5}{v} \qquad \text{and} \qquad \frac{y}{u+v} = \frac{5}{u},$$

so that

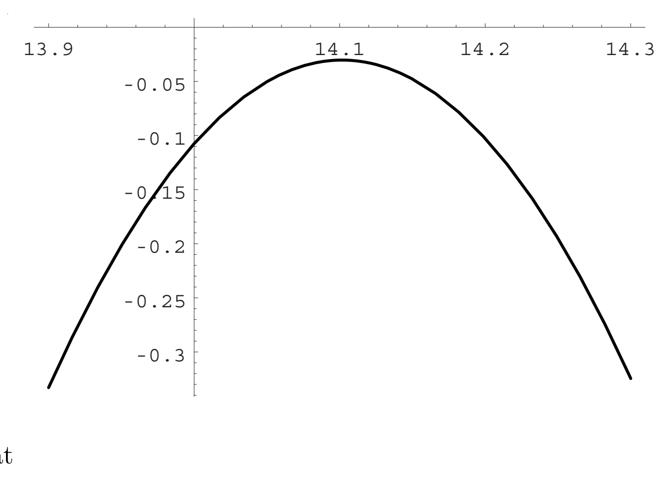
$$x = 5 \cdot \frac{u+v}{v} = 5(1+t) \qquad \text{and} \qquad y = 5 \cdot \frac{u+v}{u} = 5\left(1 + \frac{1}{t}\right).$$

Next, $w^2 + y^2 = 400$ and $w^2 + x^2 = 225$, so that:

$$\begin{aligned}
&400 - y^2 = 225 - x^2;\\
&175 + x^2 = y^2;\\
&175 + 25(1+t)^2 = 25\left(1 + \frac{1}{t}\right)^2;\\
&175t^2 + 25t^2(1+t)^2 = 25(1+t)^2;\\
&7t^2 + t^4 + 2t^3 + t^2 = t^2 + 2t + 1;\\
&t^4 + 2t^3 + 7t^2 - 2t - 1 = 0.
\end{aligned}$$

The graph of $f(t) = t^4 + 2t^3 + 7t^2 - 2t - 1$ shows a solution of $f(t) = 0$ near $x_0 = 0.5$. Newton's method yields $x_1 = 0.491071429$, $x_2 = 0.490936940$, and $x_3 = 0.490936909 = x_4$. It now follows that $x = 7.454684547$, that $y = 15.184608052$, that $w = 13.016438772$, that $u = 4.286063469$, and that $v = 8.730375303$. Answers: $t = 0.4909$ and $w = 13.0164$.

C03S008.042: We let $f(x) = 15\sin x - 4x^{1/2}$. The graph of f on the interval $[0, 16]$ does not make it clear whether there are no solutions of $f(x) = 0$ between 13 and 15, or one solution, or two. But the graph on $[13.9, 14.9]$ (shown next) makes it clear that there is no solution there.



Alternatively, we found that

$$f'(x) = 15\cos x - \frac{2}{\sqrt{x}},$$

then used Newton's method to obtain a good approximation of the solution of $f'(x) = 0$ near $x = 14$. Beginning with the initial guess $x_0 = 14.1$, *Mathematica* 3.0 produced the following (rounded) results: $x_1 = 14.1016533657$, $x_2 = 14.1016533154 = x_3$. Then we found that $f(14.1015433143) \approx -0.03032539145$, and we can rest assured that the equation $f(x) = 0$ has no solution near $x = 14$.

Next, we implemented (in *Mathematica* 3.0) the iteration of Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

to approximate the three positive solutions of $f(x) = 0$. After defining $f(x)$ as we did here, we let

```
g[x_] := N[ x − f[x]/f′[x], 20 ]
```

so that $g(x_0) = x_1$, $g(x_1) = x_2$, etc. Beginning with three good initial *guesses*, the resulting values of x_1 are

```
{ g[27/10], g[71/10], g[85/10] }
```

{2.6890398011110983881, 7.0709795391423543651, 8.53246163932340}

Then the repeated command `g[%]` resulted in the triple sequence of improved approximations shown next (rounded, of course):

n	x_n	x_n	x_n
1	2.689014344905159	7.071441437179106	8.5318440104615885
2	2.689014344765676	7.071441552066110	8.5318437904702344
3	2.689014344765676	7.071441552066117	8.5318437904702065
4	2.689014344765676	7.071441552066117	8.5318437904702065

The last line in the table gives the three positive solutions to the accuracy shown (with the last digit rounded).

C03S008.043: Let $f(\theta) = (100 + \theta)\cos\theta - 100$. The iterative formula of Newton's method is

$$\theta_{i+1} = \theta_i - \frac{f(\theta_i)}{f'(\theta_i)} \tag{1}$$

where, of course, $f'(\theta) = \cos\theta - (100 + \theta)\sin\theta$. Beginning with $\theta_0 = 1$, iteration of the formula in (1) yields

0.4620438212,	0.2325523723,	0.1211226155,	0.0659741863,
0.0388772442,	0.0261688780,	0.0211747166,	0.0200587600,
0.0199968594,	0.0199966678,	0.0199966678,	0.0199966678.

We take the last value of θ_i to be sufficiently accurate. The corresponding radius of the asteriod is thus approximately $1000/\theta_{12} \approx 50008.3319$ ft, about 9.47 mi.

C03S008.044: The length of the circular arc is $2R\theta = 5281$; the length of its chord is $2R\sin\theta = 5280$ (units are radians and feet). Division of the second of these equations by the first yields

$$\frac{\sin\theta}{\theta} = \frac{5280}{5281}.$$

To solve for θ by means of Newton's method, we let $f(\theta) = 5281\sin\theta - 5280\theta$. The iterative formula of Newton's method is

$$\theta_{i+1} = \theta_i - \frac{5281\sin\theta - 5280\theta}{5281\cos\theta - 5280}. \tag{1}$$

Beginning with the [poor] initial guess $\theta_0 = 1$, iteration of the formula in (1) yields these results:

0.655415,	0.434163,	0.289117,	0.193357,	0.130147,
0.0887267,	0.0621344,	0.0459270,	0.0373185,	0.0341721,
0.0337171,	0.0337078,	0.0337078,	0.0337078,	0.0337078.

Hence the radius of the circular arc is

$$R \approx \frac{5281}{2\theta_{15}} \approx 78335.1,$$

and its height at its center is

$$x = R(1 - \cos\theta) \approx 44.4985.$$

That is, the maximum height is about 44.5 feet! Surprising to almost everyone.

Section 3.9

3.9.1: $2x - 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{x}{y}$. Also, $y = \pm\sqrt{x^2 - 1}$, so $\dfrac{dy}{dx} = \pm\dfrac{x}{\sqrt{x^2-1}} = \dfrac{x}{\pm\sqrt{x^2-1}} = \dfrac{x}{y}$.

3.9.2: $x\dfrac{dy}{dx} + y = 0$, so $\dfrac{dy}{dx} = -\dfrac{y}{x}$. By substituting $y = x^{-1}$ here, we get $\dfrac{dy}{dx} = -\dfrac{x^{-1}}{x} = -x^{-2}$, which is the result obtained by explicit differentiation.

3.9.3: $32x + 50y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{16x}{25y}$. Substituting $y = \pm\frac{1}{5}\sqrt{400 - 16x^2}$ into the derivative, we get $\dfrac{dy}{dx} = \mp\dfrac{16x}{5\sqrt{400-16x^2}}$, which is the result obtained by explicit differentiation.

3.9.4: $3x^2 + 3y^2\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$. $y = \sqrt[3]{1-x^3}$, so substitution results in $\dfrac{dy}{dx} = -\dfrac{x^2}{(1-x^3)^{2/3}}$. Explicit differentiation yields the same answer.

3.9.5: $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$.

3.9.6: $4x^3 + 2x^2y\dfrac{dy}{dx} + 2xy^2 + 4y^3\dfrac{dy}{dx} = 0$: $\quad (2x^2y + 4y^3)\dfrac{dy}{dx} = -(4x^3 + 2xy^2)$; $\quad \dfrac{dy}{dx} = -\dfrac{4x^3 + 2xy^2}{2x^2y + 4y^3}$.

3.9.7: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{-1/3} = -\left(\dfrac{y}{x}\right)^{1/3}$.

3.9.8: $y^2 + 2(x-1)y\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1-y^2}{2y(x-1)}$.

3.9.9: Given: $\quad x^3 - x^2y = xy^2 + y^3$:

$$3x^2 - x^2\frac{dy}{dx} - 2xy = y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx};$$

$$3x^2 - 2xy - y^2 = (2xy + 3y^2 + x^2)\frac{dy}{dx};$$

$$\frac{dy}{dx} = \frac{3x^2 - 2xy - y^2}{3y^2 + 2xy + x^2}.$$

3.9.10: Given: $\quad x^5 + y^5 = 5x^2y^2$:

$$5x^4 + 5y^4\frac{dy}{dx} = 10x^2y\frac{dy}{dx} + 10xy^2;$$

$$\frac{dy}{dx} = \frac{10xy^2 - 5x^4}{5y^4 - 10x^2y}.$$

3.9.11: Given: $\quad x\sin y + y\sin x = 1$:

$$x\cos y\frac{dy}{dx} + \sin y + y\cos x + \sin x\frac{dy}{dx} = 0;$$

$$\frac{dy}{dx} = -\frac{\sin y + y\cos x}{x\cos y + \sin x}.$$

3.9.12: Given: $\cos(x+y) = \sin x \, \sin y$:

$$-\sin(x+y)(1+\frac{dy}{dx}) = \sin x \, \cos y \, \frac{dy}{dx} + \sin y \, \cos x;$$

$$\frac{dy}{dx} = -\frac{\sin y \, \cos x + \sin(x+y)}{\sin(x+y) + \sin x \, \cos y}.$$

3.9.13: Given: $2x + 3e^y = e^{x+y}$. Differentiation of both sides of this equation (actually, an *identity*) with respect to x yields

$$2 + 3e^y \frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right), \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{e^{x+y} - 2}{3e^y - e^{x+y}} = \frac{3e^y + 2x - 2}{(3 - e^x)e^y}.$$

3.9.14: Given: $xy = e^{-xy}$. Differentiation of both sides with respect to x yields

$$x\frac{dy}{dx} + y = -e^{xy}\left(x\frac{dy}{dx} + y\right), \qquad \text{and so} \qquad (1 + e^{-xy})x\frac{dy}{dx} = -(1 + e^{-xy})y.$$

Because $1 + e^{-xy} > 0$ for all x and y, it follows that

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Another way to solve this problem is to observe that the equation $e^{-z} = z$ has exactly one real solution $a \approx 0.5671432904$. Hence if $e^{-xy} = xy$, then $xy = a$, so that $y = a/x$. Hence

$$\frac{dy}{dx} = -\frac{a}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}.$$

3.9.15: $2x + 2y\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -\dfrac{x}{y}$. At $(3, -4)$ the tangent has slope $\frac{3}{4}$ and thus equation $y + 4 = \frac{3}{4}(x - 3)$.

3.9.16: $x\dfrac{dy}{dx} + y = 0$: $\dfrac{dy}{dx} = -\dfrac{y}{x}$. At $(4, -2)$ the tangent has slope $\frac{1}{2}$ and thus equation $y + 2 = \frac{1}{2}(x - 4)$.

3.9.17: $x^2\dfrac{dy}{dx} + 2xy = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy}{x^2}$. At $(2, 1)$ the tangent has slope $-\frac{3}{4}$ and thus equation $3x + 4y = 10$.

3.9.18: $\frac{1}{4}x^{-3/4} + \frac{1}{4}y^{-3/4}\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -(y/x)^{3/4}$. At $(16, 16)$ the tangent has slope -1 and thus equation $x + y = 32$.

3.9.19: $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{2xy + x^2}$. At $(1, -2)$ the slope is zero, so an equation of the tangent there is $y = -2$.

3.9.20: $-\dfrac{1}{(x+1)^2} - \dfrac{1}{(y+1)^2} \cdot \dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{(y+1)^2}{(x+1)^2}$. At $(1, 1)$ the tangent line has slope -1 and thus equation $y - 1 = -(x - 1)$.

3.9.21: $24x + 24y\dfrac{dy}{dx} = 25y + 25x\dfrac{dy}{dx}$: $\quad \dfrac{dy}{dx} = \dfrac{25y - 24x}{24y - 25x}$. At (3, 4) the tangent line has slope $\frac{4}{3}$ and thus equation $4x = 3y$.

3.9.22: $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{2x + y}{x + 2y}$. At $(3, -2)$ the tangent line has slope 4 and thus equation $y + 2 = 4(x - 3)$.

3.9.23: $\dfrac{dy}{dx} = \dfrac{3e^{2x} + 2e^{y}}{3e^{2x} + e^{x+2y}}$, so the tangent line at (0, 0) has slope $\frac{5}{4}$ and equation $4y = 5x$.

3.9.24: $\dfrac{dy}{dx} = \dfrac{12e^{2x} - ye^{3y}}{18e^{2x} + xe^{3y}}$, so the tangent line at (3, 2) has slope $\frac{10}{21}$ and equation $10x + 12 = 21y$.

3.9.25: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$. At (8, 1) the tangent line has slope $-\frac{1}{2}$ and thus equation $y - 1 = -\frac{1}{2}(x - 8)$; that is, $x + 2y = 10$.

3.9.26: $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$. At $(3, -2)$ the tangent line has slope $\frac{8}{7}$ and thus equation $y + 2 = \frac{8}{7}(x - 3)$; that is, $7y = 8x - 38$.

3.9.27: $2\left(x^2 + y^2\right)\left(2x + 2y\dfrac{dy}{dx}\right) = 50x\dfrac{dy}{dx} + 50y$:

$$\frac{dy}{dx} = -\frac{2x^3 - 25y + 2xy^2}{-25x + 2x^2y + 2y^3}.$$

At (2, 4) the tangent line has slope $\frac{2}{11}$ and thus equation $y - 4 = \frac{2}{11}(x - 2)$; that is, $11y = 2x + 40$.

3.9.28: $2y\dfrac{dy}{dx} = 3x^2 + 14x$: $\quad \dfrac{dy}{dx} = \dfrac{3x^2 + 14x}{2y}$. At $(-3, 6)$ the tangent line has slope $-\frac{5}{4}$ and thus equation $y - 6 = -\frac{5}{4}(x + 3)$; alternatively, $4y = 9 - 5x$.

3.9.29: $3x^2 + 3y^2\dfrac{dy}{dx} = 9x\dfrac{dy}{dx} + 9y$: $\quad \dfrac{dy}{dx} = \dfrac{3y - x^2}{y^2 - 3x}$.

(a): At (2, 4) the tangent line has slope $\frac{4}{5}$ and thus equation $y - 4 = \frac{4}{5}(x - 2)$; that is, $5y = 4x + 12$.

(b): At a point on the curve at which $\dfrac{dy}{dx} = -1$, $3y - x^2 = -y^2 - 3x$ and $x^3 + y^3 = 9xy$. This pair of simultaneous equations has solutions $x = 0$, $y = 0$ and $x = \frac{9}{2}$, $y = \frac{9}{2}$, but the derivative does not exist at the point (0, 0). Therefore the tangent line with slope -1 has equation $y - \frac{9}{2} = -\left(x - \frac{9}{2}\right)$.

3.9.30: First, $2x^2 - 5xy + 2y^2 = (y - 2x)(2y - x)$.

(a): Hence if $2x^2 - 5xy + 2y^2 = 0$, then $y - 2x = 0$ or $2y - x = 0$. This is a pair of lines through the origin; the first has slope 2 and the second has slope $\frac{1}{2}$.

(b): Differentiating implicitly, we obtain $4x - 5x\dfrac{dy}{dx} - 5y + 4y\dfrac{dy}{dx} = 0$, which gives $\dfrac{dy}{dx} = \dfrac{5y - 4x}{4y - 5x}$, which is 2 if $y = 2x$ and $-\frac{1}{2}$ if $y = -\frac{1}{2}x$.

3.9.31: Here $\dfrac{dy}{dx} = \dfrac{2-x}{y-2}$, so horizontal tangents can occur only if $x = 2$ and $y \neq 2$. When $x = 2$, the original equation yields $y^2 - 4y - 4 = 0$, so that $y = 2 \pm \sqrt{8}$. Thus there are two points at which the tangent line is horizontal: $\left(2,\ 2-\sqrt{8}\right)$ and $\left(2,\ 2+\sqrt{8}\right)$.

3.9.32: First, $\dfrac{dy}{dx} = \dfrac{y-x^2}{y^2-x}$ and $\dfrac{dx}{dy} = \dfrac{y^2-x}{y-x^2}$. Horizontal tangents require $y = x^2$, and the equation $x^3 + y^3 = 3xy$ of the folium yields $x^3(x^3 - 2) = 0$, so either $x = 0$ or $x = \sqrt[3]{2}$. But dy/dx is not defined at $(0,\ 0)$, so only at $\left(\sqrt[3]{2},\ \sqrt[3]{4}\right)$ is there a horizontal tangent. By symmetry or by a similar argument, there is a vertical tangent at $\left(\sqrt[3]{4},\ \sqrt[3]{2}\right)$ and nowhere else.

3.9.33: By direct differentiation, $dx/dy = (1+y)e^y$. By implicit differentiation, $\dfrac{dy}{dx} = \dfrac{1}{(1+y)e^y}$, and the results are equivalent.

(a): At $(0, 0)$, $dy/dx = 1$, so an equation of the line tangent to the curve at $(0,\ 0)$ is $y = x$.

(b): At $(e,\ 1)$, $dy/dx = 1/(2e)$, so an equation of the line tangent to the curve at $(e,\ 1)$ is $x + e = 2ey$.

3.9.34: (a): By direct differentiation, $dx/dy = (1+y)e^y$, so there is only one point on the curve where the tangent line is vertical ($dx/dy = 0$): $(-1/e,\ -1)$.

(b): Because $\dfrac{dy}{dx} = \dfrac{1}{(1+y)e^y}$ is never zero, the graph has no horizontal tangents.

3.9.35: From $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$ it follows that

$$\frac{dy}{dx} = \frac{x[1 - 2(x^2+y^2)]}{y[1 + 2(x^2+y^2)]}.$$

So $dy/dx = 0$ when $x^2 + y^2 = \frac{1}{2}$, but is undefined when $x = 0$, for then $y = 0$ as well. If $x^2 + y^2 = \frac{1}{2}$, then $x^2 - y^2 = \frac{1}{4}$, so that $x^2 = \frac{3}{8}$, and it follows that $y^2 = \frac{1}{8}$. Consequently there are horizontal tangents at all four points where $|x| = \frac{1}{4}\sqrt{6}$ and $|y| = \frac{1}{4}\sqrt{2}$.

Also $dx/dy = 0$ only when $y = 0$, and if so, then $x^4 = x^2$, so that $x = \pm 1$ (dx/dy is undefined when $x = 0$). So there are vertical tangents at the two points $(-1,\ 0)$ and $(1,\ 0)$.

3.9.36: Base edge of block: x. Height: y. Volume: $V = x^2 y$. We are given $dx/dt = -2$ and $dy/dt = -3$. Implicit differentiation yields

$$\frac{dV}{dt} = x^2\frac{dy}{dt} + 2xy\frac{dx}{dt}.$$

When $x = 20$ and $y = 15$, $dV/dt = (400)(-3) + (600)(-2) = -2400$. So the rate of flow at the time given is 2400 in.^{3}/h.

3.9.37: Suppose that the pile has height $h = h(t)$ at time t (seconds) and radius $r = r(t)$ then. We are given $h = 2r$ and we know that the volume of the pile at time t is

$$V = V(t) = \frac{\pi}{3}r^2 h = \frac{2}{3}\pi r^3. \quad \text{Now} \quad \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}, \quad \text{so} \quad 10 = 2\pi r^2\frac{dr}{dt}.$$

When $h = 5$, $r = 2.5$; at that time $\dfrac{dr}{dt} = \dfrac{10}{2\pi(2.5)^2} = \dfrac{4}{5\pi} \approx 0.25645$ (ft/s).

3.9.38: Draw a vertical cross section through the center of the tank. Let r denote the radius of the (circular) water surface when the depth of water in the tank is y. From the drawing and the Pythagorean theorem derive the relationship $r^2 + (10 - y)^2 = 100$. Therefore

$$2r\frac{dr}{dt} - 2(10 - y)\frac{dy}{dt} = 0, \quad \text{and so} \quad r\frac{dr}{dt} = (10 - y)\frac{dy}{dt}.$$

We are to find dr/dt when $y = 5$, given $dy/dt = -3$. At that time, $r^2 = 100 - 25$, so $r = 5\sqrt{3}$. Thus

$$\left.\frac{dr}{dt}\right|_{y=5} = \frac{10 - y}{r} \cdot \left.\frac{dy}{dt}\right|_{y=5} = \frac{5}{5\sqrt{3}}(-3) = -\sqrt{3}.$$

Answer: The radius of the top surface is decreasing at $\sqrt{3}$ ft/s then.

3.9.39: We assume that the oil slick forms a solid right circular cylinder of height (thickness) h and radius r. Then its volume is $V = \pi r^2 h$, and we are given $V = 1$ (constant) and $dh/dt = -0.001$. Therefore $0 = \pi r^2 \dfrac{dh}{dt} + 2\pi r h \dfrac{dr}{dt}$. Consequently $2h\dfrac{dr}{dt} = \dfrac{r}{1000}$, and so $\dfrac{dr}{dt} = \dfrac{r}{2000h}$. When $r = 8$, $h = \dfrac{1}{\pi r^2} = \dfrac{1}{64\pi}$. At that time, $\dfrac{dr}{dt} = \dfrac{8 \cdot 64\pi}{2000} = \dfrac{32\pi}{125} \approx 0.80425$ (m/h).

3.9.40: Let x be the distance from the ostrich to the street light and u the distance from the base of the light pole to the tip of the ostrich's shadow. Draw a figure and so label it; by similar triangles you find that $\dfrac{u}{10} = \dfrac{u - x}{5}$, and it follows that $u = 2x$. We are to find du/dt and $D_t(u - x) = du/dt - dx/dt$. But $u = 2x$, so

$$\frac{du}{dt} = 2\frac{dx}{dt} = (2)(-4) = -8; \qquad \frac{du}{dt} - \frac{dx}{dt} = -8 - (-4) = -4.$$

Answers: (a): $+8$ ft/s; (b): $+4$ ft/s.

3.9.41: Let x denote the width of the rectangle; then its length is $2x$ and its area is $A = 2x^2$. Thus $\dfrac{dA}{dt} = 4x\dfrac{dx}{dt}$. When $x = 10$ and $dx/dt = 0.5$, we have

$$\left.\frac{dA}{dt}\right|_{x=10} = (4)(10)(0.5) = 20 \ (\text{cm}^2/\text{s}).$$

3.9.42: Let x denote the length of each edge of the triangle. Then the triangle's area is $A(x) = \left(\frac{1}{4}\sqrt{3}\right)x^2$, and therefore $\dfrac{dA}{dt} = \left(\frac{1}{2}\sqrt{3}\right)x\dfrac{dx}{dt}$. Given $x = 10$ and $\dfrac{dx}{dt} = 0.5$, we find that

$$\left.\frac{dA}{dt}\right|_{x=10} = \frac{\sqrt{3}}{2} \cdot 10 \cdot (0.5) = \frac{5\sqrt{3}}{2} \ (\text{cm}^2/\text{s}).$$

3.9.43: Let r denote the radius of the balloon and V its volume at time t (in seconds). Then

$$V = \frac{4}{3}\pi r^3, \quad \text{so} \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are to find dr/dt when $r = 10$, and we are given the information that $dV/dt = 100\pi$. Therefore

$$100\pi = 4\pi(10)^2 \left.\frac{dr}{dt}\right|_{r=10},$$

and so at the time in question the radius is increasing at the rate of $dr/dt = \frac{1}{4} = 0.25$ (cm/s).

3.9.44: Because $pV = 1000$, $V = 10$ when $p = 100$. Moreover, $p\dfrac{dV}{dt} + V\dfrac{dp}{dt} = 0$. With $p = 100$, $V = 10$, and $dp/dt = 2$, we find that

$$\left.\frac{dV}{dt}\right|_{p=100} = -\frac{V}{p} \cdot \left.\frac{dp}{dt}\right|_{p=100} = -\frac{10}{100} \cdot 2 = -\frac{1}{5}.$$

Therefore the volume is decreasing at 0.2 in.^{3}/s.

3.9.45: Place the person at the origin and the kite in the first quadrant at $(x,\,400)$ at time t, where $x = x(t)$ and we are given $dx/dt = 10$. Then the length $L = L(t)$ of the string satisfies the equation $L^2 = x^2 + 160000$, and therefore $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Moreover, when $L = 500$, $x = 300$. So

$$1000\left.\frac{dL}{dt}\right|_{L=500} = 600 \cdot 10,$$

which implies that the string is being let out at 6 ft/s.

3.9.46: Locate the observer at the origin and the balloon in the first quadrant at $(300,\,y)$, where $y = y(t)$ is the balloon's altitude at time t. Let θ be the angle of elevation of the balloon (in radians) from the observer's point of view. Then $\tan\theta = y/300$. We are given $d\theta/dt = \pi/180$ rad/s. Hence we are to find dy/dt when $\theta = \pi/4$. But $y = 300\tan\theta$, so

$$\frac{dy}{dt} = (300\sec^2\theta)\frac{d\theta}{dt}.$$

Substitution of the given values of θ and $d\theta/dt$ yields the answer

$$\left.\frac{dy}{dt}\right|_{\theta=45^\circ} = 300 \cdot 2 \cdot \frac{\pi}{180} = \frac{10\pi}{3} \approx 10.472 \text{ (ft/s)}.$$

3.9.47: Locate the observer at the origin and the airplane at $(x,\,3)$, with $x > 0$. We are given dx/dt where the units are in miles, hours, and miles per hour. The distance z between the observer and the airplane satisfies the identity $z^2 = x^2 + 9$, and because the airplane is traveling at 8 mi/min, we find that $x = 4$, and therefore that $z = 5$, at the time 30 seconds after the airplane has passed over the observer. Also $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$, so at the time in question, $10\dfrac{dz}{dt} = 8 \cdot 480$. Therefore the distance between the airplane and the observer is increasing at 384 mi/h at the time in question.

3.9.48: In this problem we have $V = \frac{1}{3}\pi y^2(15 - y)$ and $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(10y - y^2)\dfrac{dy}{dt}$. Therefore $\dfrac{dy}{dt} = -\dfrac{13 \cdot 37}{\pi y(10 - y)}$. Answers: (a): Approximately 0.2027 ft/min; (b): The same.

3.9.49: We use $a = 10$ in the formula given in Problem 42. Then

$$V = \frac{1}{3}\pi y^2(30 - y).$$

Hence $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(20y - y^2)\dfrac{dy}{dt}$. Thus $\dfrac{dy}{dt} = -\dfrac{13 \cdot 37}{\pi y(20 - y)}$. Substitution of $y = 7$ and $y = 3$ now yields the two answers:

$$\text{(a):} \quad -\frac{191}{1300\pi} \approx -0.047 \ \ \text{(ft/min)}; \qquad \text{(b):} \quad -\frac{1337}{5100\pi} \approx -0.083 \ \ \text{(ft/min)}.$$

3.9.50: When the height of the water at the deep end of the pool is 10 ft, the length of the water surface is 50 ft. So by similar triangles, if the height of the water at the deep end is y feet ($y \geqq 10$), then the length of the water surface is $x = 5y$ feet. A cross section of the water perpendicular to the width of the pool thus forms a right triangle of area $5y^2/2$. Hence the volume of the pool is $V(y) = 50y^2$. Now $133.7 = \dfrac{dV}{dt} = 100y\dfrac{dy}{dt}$, so when $y = 6$ we have

$$\left.\frac{dy}{dt}\right|_{y=6} = \frac{133.7}{600} \approx 0.2228 \text{ (ft/min)}.$$

3.9.51: Let the positive y-axis represent the wall and the positive x-axis the ground, with the top of the ladder at $(0, y)$ and its lower end at $(x, 0)$ at time t. Given: $dx/dt = 4$, with units in feet, seconds, and feet per second. Also $x^2 + y^2 = 41^2$, and it follows that $y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Finally, when $y = 9$, we have $x = 40$, so at that time $9\dfrac{dy}{dt} = -40 \cdot 4$. Therefore the top of the ladder is moving downward at $\frac{160}{9} \approx 17.78$ ft/s.

3.9.52: Let x be the length of the base of the rectangle and y its height. We are given $dx/dt = +4$ and $dy/dt = -3$, with units in centimeters and seconds. The area of the rectangle is $A = xy$, so

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = -3x + 4y.$$

Therefore when $x = 20$ and $y = 12$, we have $dA/dt = -12$, so the area of the rectangle is decreasing at the rate of 12 cm^2/s then.

3.9.53: Let r be the radius of the cone, h its height. We are given $dh/dt = -3$ and $dr/dt = +2$, with units in centimeters and seconds. The volume of the cone at time t is $V = \frac{1}{3}\pi r^2 h$, so

$$\frac{dV}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}.$$

When $r = 4$ and $h = 6$, $\dfrac{dV}{dt} = \dfrac{2}{3} \cdot 24\pi \cdot 2 + \dfrac{1}{3} \cdot 16\pi \cdot (-3) = 16\pi$, so the volume of the cone is increasing at the rate of 16π cm^3/s then.

3.9.54: Let x be the edge length of the square and $A = x^2$ its area. Given: $\dfrac{dA}{dt} = 120$ when $x = 10$. But $dA/dt = 2x(dx/dt)$, so $dx/dt = 6$ when $x = 10$. Answer: At 6 in./s.

3.9.55: Locate the radar station at the origin and the rocket at $(4, y)$ in the first quadrant at time t, with y in miles and t in hours. The distance z between the station and the rocket satisfies the equation $y^2+16 = z^2$, so $2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$. When $z = 5$, we have $y = 3$, and because $dz/dt = 3600$ it follows that $dy/dt = 6000$ mi/h.

3.9.56: Locate the car at $(x, 0)$, the truck at $(0, y)$ $\quad (x, y > 0)$. Then at 1 P.M. we have $x = 90$ and $y = 80$. We are given that data $dx/dt = 30$ and $dy/dt = 40$, with units in miles, hours, and miles per hour. The distance z between the vehicles satisfies the equation $z^2 = x^2 + y^2$, so

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Finally, at 1 P.M. $z^2 = 8100 + 6400 = 14500$, so $z = 10\sqrt{145}$ then. So at 1 P.M.

$$\frac{dz}{dt} = \frac{2700 + 3200}{10\sqrt{145}} = \frac{590}{\sqrt{145}}$$

mi/h—approximately 49 mi/h.

3.9.57: Put the floor on the nonnegative x-axis and the wall on the nonnegative y-axis. Let x denote the distance from the wall to the foot of the ladder (measured along the floor) and let y be the distance from the floor to the top of the ladder (measured along the wall). By the Pythagorean theorem, $x^2 + y^2 = 100$, and we are given $dx/dt = \frac{22}{15}$ (because we will use units of feet and seconds rather than miles and hours). From the Pythagorean relation we find that

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

so that $\dfrac{dy}{dt} = -\dfrac{x}{y}\cdot\dfrac{dx}{dt} = -\dfrac{22x}{15y}$.

(a): If $y = 4$, then $x = \sqrt{84} = 2\sqrt{21}$. Hence when the top of the ladder is 4 feet above the ground, it is moving a a rate of

$$\left.\frac{dy}{dt}\right|_{y=4} = -\frac{44\sqrt{21}}{60} = -\frac{11\sqrt{21}}{15} \approx -3.36$$

feet per second, about 2.29 miles per hour downward.

(b): If $y = \frac{1}{12}$ (one inch), then

$$x^2 = 100 - \frac{1}{144} = \frac{14399}{144}, \quad \text{so that} \quad x \approx 9.99965.$$

In this case,

$$\left.\frac{dy}{dt}\right|_{y=1/12} = -\frac{22\cdot(9.99965)}{15\cdot\frac{1}{12}} = -\frac{88}{5}\cdot(9.99965) \approx -176$$

feet per second, about 120 miles per hour downward.

(c): If $y = 1$ mm, then $x \approx 10$ (ft), and so

$$\frac{dy}{dt} \approx -\frac{22}{15} \cdot (3048) \approx 4470$$

feet per second, about 3048 miles per hour.

The results in parts (b) and (c) are not plausible. This shows that the assumption that the top of the ladder never leaves the wall is invalid

3.9.58: Let x be the distance between the *Pinta* and the island at time t and y the distance between the *Niña* and the island then. We know that $x^2 + y^2 = z^2$ where $z = z(t)$ is the distance between the two ships, so

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \tag{1}$$

When $x = 30$ and $y = 40$, $z = 50$. It follows from Eq. (1) that $dz/dt = -25$ then. Answer: They are drawing closer at 25 mi/h at the time in question.

3.9.59: Locate the military jet at $(x, 0)$ with $x < 0$ and the other aircraft at $(0, y)$ with $y \geqq 0$. With units in miles, minutes, and miles per minute, we are given $dx/dt = +12$, $dy/dt = +8$, and when $t = 0$, $x = -208$ and $y = 0$. The distance z between the aircraft satisfies the equation $x^2 + y^2 = z^2$, so

$$\frac{dz}{dt} = \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{12x + 8y}{\sqrt{x^2 + y^2}}.$$

The closest approach will occur when $dz/dt = 0$: $y = -3x/2$. Now $x(t) = 12t - 208$ and $y(t) = 8t$. So at closest approach we have

$$8t = y(t) = -\frac{3}{2}x(t) = -\frac{3}{2}(12t - 208).$$

Hence at closest approach, $16t = 624 - 36t$, and thus $t = 12$. At this time, $x = -64$, $y = 96$, and $z = 32\sqrt{13} \approx 115.38$ (mi).

3.9.60: Let x be the distance from the anchor to the point on the seabed directly beneath the hawsehole; let L be the amount of anchor chain out. We must find dx/dt when $L = 13$ (fathoms), given $dL/dt = -10$. Now $x^2 + 144 = L^2$, so $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Consequently, $\dfrac{dx}{dt} = \dfrac{L}{x} \cdot \dfrac{dL}{dt}$. At the time in question in the problem, $x^2 = 13^2 - 12^2$, so $x = 5$. It follows that $dx/dt = -26$ then. Thus the ship is moving at 26 fathoms per minute—about 1.77 mi/h.

3.9.61: Let x be the radius of the water surface at time t and y the height of the water remaining at time t. If Q is the amount of water remaining in the tank at time t, then (because the water forms a cone) $Q = Q(t) = \frac{1}{3}\pi x^2 y$. But by similar triangles, $\dfrac{x}{y} = \dfrac{3}{5}$, so $x = \dfrac{3y}{5}$. So

$$Q(t) = \frac{1}{3}\pi\frac{9}{25}y^3 = \frac{3}{25}\pi y^3.$$

We are given $dQ/dt = -2$ when $y = 3$. This implies that when $y = 3$, $-2 = \dfrac{dQ}{dt} = \dfrac{9}{25}\pi y^2 \dfrac{dy}{dt}$. So at the time in question,

$$\left.\frac{dy}{dt}\right|_{y=3} = -\frac{50}{81\pi} \approx -0.1965 \text{ (ft/s)}.$$

3.9.62: Given $V = \frac{1}{3}\pi(30y^2 - y^3)$, find dy/dt given V, y, and dy/dt. First,

$$\frac{dV}{dt} = \frac{1}{3}\pi(60y - 3y^2)\frac{dy}{dt} = \pi(20y - y^2)\frac{dy}{dt}.$$

So $\dfrac{dy}{dt} = \dfrac{1}{\pi(20y - y^2)} \cdot \dfrac{dV}{dt}$. Therefore, when $y = 5$, we have

$$\left.\frac{dy}{dt}\right|_{y=5} = \frac{(200)(0.1337)}{\pi(100 - 25)} \approx 0.113488 \text{ (ft/min)}.$$

3.9.63: Let r be the radius of the water surface at time t, h the depth of water in the bucket then. By similar triangles we find that

$$\frac{r - 6}{h} = \frac{1}{4}, \quad \text{so } r = 6 + \frac{h}{4}.$$

The volume of water in the bucket then is

$$\begin{aligned} V &= \frac{1}{3}\pi h(36 + 6r + r^2) \\ &= \frac{1}{3}\pi\left(36 + 36 + \frac{3}{2}h + 36 + 3h + \frac{1}{16}h^2\right) \\ &= \frac{1}{3}\pi h\left(108 + \frac{9}{2}h + \frac{1}{16}h^2\right). \end{aligned}$$

Now $\dfrac{dV}{dt} = -10$; we are to find dh/dt when $h = 12$.

$$\frac{dV}{dt} = \frac{1}{3}\pi(108 + 9h + \frac{3}{16}h^2)\frac{dh}{dt}.$$

Therefore $\left.\dfrac{dh}{dt}\right|_{h=12} = \dfrac{3}{\pi} \cdot \dfrac{-10}{108 + 9 \cdot 12 + \dfrac{3 \cdot 12^2}{16}} = -\dfrac{10}{81\pi} \approx -0.0393$ (in./min).

3.9.64: Let x denote the distance between the ship and A, y the distance between the ship and B, h the perpendicular distance from the position of the ship to the line AB, u the distance from A to the foot of this perpendicular, and v the distance from B to the foot of the perpendicular. At the time in question, we know that $x = 10.4$, $dx/dt = 19.2$, $y = 5$, and $dy/dt = -0.6$. From the right triangles involved, we see that $u^2 + h^2 = x^2$ and $(12.6 - u)^2 + h^2 = y^2$. Therefore

$$x^2 - u^2 = y^2 - (12.6 - u)^2. \tag{1}$$

We take $x = 10.4$ and $y = 5$ in Eq. (1); it follows that $u = 9.6$ and that $v = 12.6 - u = 3$. From Eq. (1), we know that

$$x\frac{dx}{dt} - u\frac{du}{dt} = y\frac{dy}{dt} + (12.6 - u)\frac{du}{dt},$$

so

$$\frac{du}{dt} = \frac{1}{12.6}\left(x\frac{dx}{dt} - y\frac{dy}{dt}\right).$$

From the data given, $du/dt \approx 16.0857$. Also, because $h = \sqrt{x^2 - u^2}$, $h = 4$ when $x = 10.4$ and $y = 9.6$. Moreover, $h\dfrac{dh}{dt} = x\dfrac{dx}{dt} - u\dfrac{du}{dt}$, and therefore

$$\left.\frac{dh}{dt}\right|_{h=4} \approx \frac{1}{4}\left[(10.4)(19.2) - (9.6)(16.0857)\right] \approx 11.3143.$$

Finally, $\dfrac{dh/dt}{du/dt} \approx 0.7034$, so the ship is sailing a course about $35°7'$ north *or* south of east at a speed of $\sqrt{(du/dt)^2 + (dh/dt)^2} \approx 19.67$ mi/h. It is located 9.6 miles east and 4 miles north *or* south of A, or 10.4 miles from A at a bearing of either $67°22'\,48''$ or $112°37'\,12''$.

3.9.65: Set up a coordinate system in which the radar station is at the origin, the plane passes over it at the point (0, 1) (so units on the axes are in miles), and the plane is moving along the graph of the equation $y = x + 1$. Let s be the distance from (0, 1) to the plane and let u be the distance from the radar station to the plane. We are given $du/dt = +7$ mi/min. We may deduce from the law of cosines that $u^2 = s^2 + 1 + s\sqrt{2}$. Let v denote the speed of the plane, so that $v = ds/dt$. Then

$$2u\frac{du}{dt} = 2sv + v\sqrt{2} = v\left(2s + \sqrt{2}\right), \quad \text{and so} \quad v = \frac{2u}{2s + \sqrt{2}} \cdot \frac{du}{dt}.$$

When $u = 5$, $s^2 + s\sqrt{2} - 24 = 0$. The quadratic formula yields the solution $s = 3\sqrt{2}$, and it follows that $v = 5\sqrt{2}$ mi/min; alternatively, $v \approx 424.26$ mi/h.

3.9.66: $V(y) = \frac{1}{3}\pi(30y^2 - y^3)$ where the depth is y. Now $\dfrac{dV}{dt} = -k\sqrt{y} = \dfrac{dV}{dy} \cdot \dfrac{dy}{dt}$, and therefore

$$\frac{dy}{dt} = -\frac{k\sqrt{y}}{\dfrac{dV}{dy}} = -\frac{k\sqrt{y}}{\pi(20y - y^2)}.$$

To minimize dy/dt, write $F(y) = dy/dt$. It turns out (after simplifications) that

$$F'(y) = \frac{k}{2\pi} \cdot \frac{20y - 3y^2}{(20y - y^2)^2\sqrt{y}}.$$

So $F'(y) = 0$ when $y = 0$ and when $y = \frac{20}{3}$. When y is near 20, $F(y)$ is very large; the same is true for y near zero. So $y = \frac{20}{3}$ minimizes dy/dt, and therefore the answer to part (b) is 6 ft 8 in.

3.9.67: Place the pole at the origin in the plane, and let the horizontal strip $0 \leqq y \leqq 30$ represent the road. Suppose that the person is located at $(x, 30)$ with $x > 0$ and is walking to the right, so $dx/dt = +5$. Then the distance from the pole to the person will be $\sqrt{x^2 + 900}$. Let z be the length of the person's shadow. By similar triangles it follows that $2z = \sqrt{x^2 + 900}$, so $4z^2 = x^2 + 900$, and thus $8z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$. When $x = 40$, we find that $z = 25$, and therefore that

$$100\left.\frac{dz}{dt}\right|_{z=25} = 40 \cdot 5 = 200.$$

Therefore the person's shadow is lengthening at 2 ft/s at the time in question.

3.9.68: Set up a coordinate system in which the officer is at the origin and the van is moving in the positive direction along the line $y = 200$ (so units on the coordinate axes are in feet). When the van is at position $(x, 200)$, the distance from the officer to the van is z, where $x^2 + 200^2 = z^2$, so that $x\dfrac{dx}{dt} = z\dfrac{dz}{dt}$. When the van reaches the call box, $x = 200$,

Section 3.10

Note: Your answers may differ from ours in the last one or two decimal places because of differences in hardware or in the way the problem was solved. We used *Mathematica* and carried 40 decimal digits throughout all calculations, and our answers are correct or correctly rounded to the number of digits shown here. In most of the first 20 problems the initial guess x_0 was obtained by linear interpolation. Finally, the equals mark is used in this section to mean "equal or approximately equal."

3.10.1: With $f(x) = x^2 - 5$, $a = 2$, $b = 3$, and

$$x_0 = a - \frac{(b-a)f(a)}{f(b) - f(a)} = 2.2,$$

we used the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for} \quad n \geqq 0.$$

Thus we obtained $x_1 = 2.236363636$, $x_2 = 2.236067997$, and $x_3 = x_4 = 2.236067977$. Answer: 2.2361.

3.10.2: $x_0 = 1.142857143$; we use $f(x) = x^3 - 2$. Then $x_1 = 1.272321429$, $x_2 = 1.260041515$, $x_3 = 1.259921061$, and $x_4 = x_5 = 1.259921050$. Answer: 1.2599.

3.10.3: $x_0 = 2.322274882$; we use $f(x) = x^5 - 100$. Then $x_1 = 2.545482651$, $x_2 = 2.512761634$, $x_3 = 2.511887041$, and $x_4 = x_5 = 2.511886432$. Answer: 2.5119.

3.10.4: Let $f(x) = x^{3/2} - 10$. Then $x_0 = 4.628863603$. From the iterative formula

$$x - \frac{x^{3/2} - 10}{\frac{3}{2}x^{1/2}} \longrightarrow x$$

we obtain $x_1 = 4.641597575$, $x_2 = 4.641588834 = x_3$. Answer: 4.6416.

3.10.5: 0.25, 0.3035714286, 0.3027758133, 0.3027756377. Answer: 0.3028.

3.10.6: 0.2, 0.2466019417, 0.2462661921, 0.2462661722. Answer: 0.2463.

3.10.7: $x_0 = -0.5$, $x_1 = -0.8108695652$, $x_2 = -0.7449619516$, $x_3 = -0.7402438226$, $x_4 = -0.7402217826 = x_5$. Answer: 0.7402.

3.10.8: Let $f(x) = x^3 + 2x^2 + 2x - 10$. With initial guess $x_0 = 1.5$ (the midpoint of the interval), we obtain $x_1 = 1.323943661972$, $x_2 = 1.309010783652$, $x_3 = 1.308907324710$, $x_4 = 1.308907319765$, and $x_5 = 1.308907319765$. Answer: 1.3089.

3.10.9: With $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, and calculator set in *radian* mode, we obtain $x_0 = 0.5854549279$, $x_1 = 0.7451929664$, $x_2 = 0.7390933178$, $x_3 = 0.7390851332$, and $x_4 = x_3$. Answer: 0.7391.

3.10.10: Let $f(x) = x^2 - \sin x$. Then $f'(x) = 2x - \cos x$. The linear interpolation formula yields $x_0 = 0.7956861008$, and the iterative formula

$$x - \frac{x^2 - \sin x}{2x - \cos x} \longrightarrow x$$

(with calculator in *radian* mode) yields the following results: $x_1 = 0.8867915207$, $x_2 = 0.8768492470$, $x_3 = 0.8767262342$, and $x_4 = 0.8767262154 = x_5$. Answer: 0.8767.

3.10.11: With $f(x) = 4x - \sin x - 4$ and calculator in *radian* mode, we get the following results: $x_0 = 1.213996400$, $x_1 = 1.236193029$, $x_2 = 1.236129989 = x_3$. Answer: 1.2361.

3.10.12: $x_0 = 0.8809986055$, $x_1 = 0.8712142932$, $x_3 = 0.8712215145$, and $x_4 = x_3$. Answer: 0.8712.

3.10.13: With $x_0 = 2.188405797$ and the iterative formula

$$x - \frac{x^4(x+1) - 100}{x^3(5x+4)} \longrightarrow x,$$

we obtain $x_1 = 2.360000254$, $x_2 = 2.339638357$, $x_3 = 2.339301099$, and $x_4 = 2.339301008 = x_5$. Answer: 2.3393.

3.10.14: $x_0 = 0.7142857143$, $x_1 = 0.8890827860$, $x_2 = 0.8607185590$, $x_3 = 0.8596255544$, and $x_4 = 0.8596240119 = x_5$. Answer: 0.8596.

3.10.15: The nearest discontinuities of $f(x) = x - \tan x$ are at $\pi/2$ and at $3\pi/2$, approximately 1.571 and 4.712. Therefore the function $f(x) = x - \tan x$ has the intermediate value property on the interval $[2, 3]$. Results: $x_0 = 2.060818495$, $x_1 = 2.027969226$, $x_2 = 2.028752991$, and $x_3 = 2.028757838 = x_4$. Answer: 2.0288.

3.10.16: As $\frac{7}{2}\pi \approx 10.9956$ and $\frac{9}{2}pi \approx 14.1372$ are the nearest discontinuities of $f(x) = x - \tan x$, this function has the intermediate value property on the interval $[11, 12]$. Because $f(11) \approx -214.95$ and $f(12) \approx 11.364$, the equation $f(x) = 0$ has a solution in $[11, 12]$. We obtain $x_0 = 11.94978618$ by interpolation, and the iteration

$$x - \frac{x + \tan x}{1 + \sec^2 x} \longrightarrow x$$

of Newton's method yields the successive approximations

$$x_1 = 7.457596948,\ x_2 = 6.180210620,\ x_3 = 3.157913273,\ x_4 = 1.571006986;$$

after many more iterations we arrive at the answer 2.028757838 of Problem 15. The difficulty is caused by the fact that $f(x)$ is generally a very large number, so the iteration of Newton's method tends to alter the value of

x excessively. A little experimentation yields the fact that $f(11.08) \approx -0.736577$ and $f(11.09) \approx 0.531158$. We begin anew on the better interval $[11.08, 11.09]$ and obtain $x_0 = 11.08581018$, $x_1 = 11.08553759$, $x_2 = 11.08553841$, and $x_3 = x_2$. Answer: 11.0855.

3.10.17: $x - e^{-x} = 0$; $[0, 1]$: $x_0 = 0.5$, $x_1 \approx 0.5663$, $x_2 \approx 0.5671$, $x_3 \approx 0.5671$.

3.10.18: $x_0 = 2.058823529$, $x_1 = 2.095291459$, $x_2 = 2.094551790$, $x_3 = 2.094551482 = x_4$. Answer: 2.0946.

3.10.19: $e^x + x - 2 = 0$; $[0, 1]$: $x_0 = 0.5$, $x_1 \approx 0.4439$, $x_2 \approx 0.4429$, $x_3 \approx 0.4429$.

3.10.20: $e^{-x} - \ln x = 0$; $[1, 2]$: $x_0 = 1.5$, $x_1 \approx 1.2951$, $x_2 \approx 1.3097$, $x_3 \approx 1.3098 \approx x_4$.

3.10.21: Let $f(x) = x^3 - a$. Then the iteration of Newton's method in Eq. (6) takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^3 - a}{3(x_n)^2} = \frac{2(x_n)^3 + a}{3(x_n)^2} = \frac{1}{3}\left(2x_n + \frac{a}{(x_n)^2}\right).$$

Because $1 < \sqrt[3]{2} < 2$, we begin with $x_0 = 1.5$ and apply this formula with $a = 2$ to obtain $x_1 = 1.296296296$, $x_2 = 1.260932225$, $x_3 = 1.259921861$, and $x_4 = 1.259921050 = x_5$. Answer: 1.25992.

3.10.22: The formula in Eq. (6) of the text, with $f(x) = x^k - a$, takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^k - a}{k(x_n)^{k-1}} = \frac{(k-1)(x_n)^k + a}{k(x_n)^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{(x_n)^{k-1}}\right].$$

We take $a = 100$, $k = 10$, and $x_0 = 1.5$ and obtain $x_1 = 1.610122949$, $x_2 = 1.586599871$, $x_3 = 1.584901430$, $x_4 = 1.584893193$, and $x_5 = 1.584893192 = x_6$. Answer: 1.58489.

3.10.23: We get $x_0 = 0.5$, $x_1 = 0.4387912809$, $x_2 = 0.4526329217$, $x_3 = 0.4496493762$, $\dots$, $x_{14} = 0.4501836113 = x_{15}$. The method of repeated substitution tends to converge much more slowly than Newton's method, has the advantage of not requiring that you compute a derivative or even that the functions involved are differentiable, and has the disadvantage of more frequent failure than Newton's method when both are applicable (see Problems 24 and 25).

3.10.24: Our results using the first formula: $x_0 = 1.5$, $x_1 = 1.257433430$, $x_2 = 1.225755182$, $x_3 = 1.221432153$, $\dots$, $x_{10} = 1.220745085 = x_{11}$. When we use the second formula, we obtain $x_1 = 4.0625$, $x_2 = 271.3789215$, $x_3 = 5423829645$, and x_4 has 39 digits to the left of the decimal point. It frequently requires some ingenuity to find a suitable way to put the equation $f(x) = 0$ into the form $x = G(x)$.

3.10.25: Beginning with $x_0 = 0.5$, the first formula yields $x_0 = 0.5$, $x_1 = -1$, $x_2 = 2$, $x_3 = 2.75$, $x_4 = 2.867768595$, $\dots$, $x_{12} = 2.879385242 = x_{13}$. Wrong root! At least the method converged. If your

calculator or computer balks at computing the cube root of a negative number, then you can rewrite the second formula in Problem 25 in the form

$$x = \mathrm{Sgn}(3x^2 - 1) \cdot |3x^2 - 1|^{1/3}.$$

The results, again with $x_0 = 0.5$, are $x_1 = -0.629960525$, $x_2 = 0.575444686$, $x_3 = -0.187485243$, $x_4 = -0.963535808$, $\ldots$, $x_{25} = 2.877296053$, $x_{26} = 2.877933902$, $\ldots$, and $x_{62} = 2.879385240 = x_{63}$. Not only is convergence extremely slow, the method of repeated substitution again leads to the wrong root. Finally, the given equation can also be written in the form

$$x = \frac{1}{\sqrt{3-x}},$$

and in this case, again with $x_0 = 0.5$, we obtain $x_1 = 0.632455532$, $x_2 = 0.649906570$, $x_3 = 0.652315106$, $x_4 = 0.652649632$, $\ldots$, and $x_{12} = 0.652703645 = x_{13}$.

3.10.26: If $f(x) = \dfrac{1}{x} - a$, then Newton's method uses the iteration

$$x - \frac{\dfrac{1}{x} - a}{-\dfrac{1}{x^2}} = x + x^2\left(\frac{1}{x} - a\right) = 2x - ax^2 \longrightarrow x.$$

3.10.27: Let $f(x) = x^5 + x - 1$. Then $f(x)$ is a polynomial, thus is continuous everywhere, and thus has the intermediate value property on every interval. Also $f(0) = -1$ and $f(1) = 1$, so $f(x)$ must assume the intermediate value 0 somewhere in the interval $[0, 1]$. Thus the equation $f(x) = 0$ has *at least* one solution. Next, $f'(x) = 5x^4 + 1$ is positive for all x, so f is an increasing function. Because f is continuous, its graph can therefore cross the x-axis at most once, and so the equation $f(x) = 0$ has *at most* one solution. Thus it has exactly one solution. Incidentally, Newton's method yields the approximate solution 0.75487766624669276. To four places, 0.7549.

3.10.28: Let $f(x) = x^2 - \cos x$. The graph of f on $[-1, 1]$ shows that there are two solutions, one near -0.8 and the other near 0.8. With $x_0 = 0.8$, Newton's method yields $x_1 = 0.824470434$, $x_2 = 0.824132377$, and $x_3 = 0.824132312 = x_4$. Because $f(-x) = f(x)$, the other solution is -0.824132312. Answer: ± 0.8241.

3.10.29: Let $f(x) = x - 2\sin x$. The graph of f on $[-2, 2]$ shows that there are exactly three solutions, the largest of which is approximately $x_0 = 1.9$. With Newton's method we obtain $x_1 = 1.895505940$, and $x_2 = 1.895494267 = x_3$. Because $f(-x) = -f(x)$, the other two solutions are 0 and -1.895494267. Answer: ± 1.8955 and 0.

3.10.30: Let $f(x) = x + 5\cos x$. The graph of f on the interval $[-5, 5]$ shows that there are exactly three solutions, approximately -1.3, 2.0, and 3.9. Newton's method then yields

n	First x_n	Second x_n	Third x_n
1	-1.306444739	1.977235450	3.839096917
2	-1.306440008	1.977383023	3.837468316
3	-1.306440008	1.977383029	3.837467106
4	-1.306440008	1.977383029	3.837467106

Answers: -1.3064, 1.9774, and 3.8375.

3.10.31: Let $f(x) = x^7 - 3x^3 + 1$. Then $f(x)$ is a polynomial, so f is continuous on every interval of real numbers, including the intervals $[-2, -1]$, $[0, 1]$, and $[1, 2]$. Also $f(-2) = -103 < 0 < 3 = f(-1)$, $f(0) = 1 > 0 > -1 = f(1)$, and $f(1) = -1 < 0 < 105 = f(2)$. Therefore the equation $f(x) = 0$ has one solution in $(-2, -1)$, another in $(0, 1)$, and a third in $(1, 2)$. (It has no other real solutions.) The graph of f shows that the first solution is near -1.4, the second is near 0.7, and the third is near 1.2. Then Newton's method yields

n	First x_n	Second x_n	Third x_n
1	-1.362661201	0.714876604	1.275651936
2	-1.357920265	0.714714327	1.258289744
3	-1.357849569	0.714714308	1.256999591
4	-1.357849553	0.714714308	1.256992779
5	-1.357849553	0.714714308	1.256992779

Answers: -1.3578, 0.7147, and 1.2570.

3.10.32: Let $f(x) = x^3 - 5$. Use the iteration

$$x - \frac{x^3 - 5}{3x^2} \longrightarrow x.$$

With $x_0 = 2$, we obtain the sequence of approximations 1.75, 1.710884354, 1.709976429, 1.709975947, and 1.709975947. Answer: 1.7100.

3.10.33: There is only one solution of $x^3 = \cos x$ for the following reasons: $x^3 < -1 \leqq \cos x$ if $x < -1$, $x^3 < 0 < \cos x$ if $-1 < x < 0$, x^3 is increasing on $[0, 1]$ whereas $\cos x$ is decreasing there (and their graphs cross in this interval as a consequence of the intermediate value property of continuous functions), and $x^3 > 1 \geqq \cos x$ for $x > 1$. The graph of $f(x) = x^3 - \cos x$ crosses the x-axis near $x_0 = 0.9$, and Newton's method yields $x_1 = 0.866579799$, $x_2 = 0.865475218$, and $x_3 = 0.865474033 = x_4$. Answer: Approximately 0.8654740331016145.

3.10.34: The graphs of $y = x$ and $y = \tan x$ show that the smallest positive solution of the equation $f(x) = x - \tan x = 0$ is between π and $3\pi/2$. With initial guess $x = 4.5$ we obtain 4.493613903, 4.493409655, 4.493409458, and 4.493409458. Answer: Approximately 4.493409457909064.

3.10.35: With $x_0 = 3.5$, we obtain the sequence $x_1 = 3.451450588$, $x_2 = 3.452461938$, and finally $x_3 = 3.452462314 = x_4$. Answer: Approximately 3.452462314057969.

3.10.36: To find a zero of $f(\theta) = \theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi$, we use the iteration

$$\theta - \frac{\theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi}{1 - \frac{1}{2}\cos\theta} \longrightarrow \theta.$$

The results, with $\theta_0 = 1.5$ $(86^\circ 56' \, 37'')$, are: $\theta_1 = 1.569342$ $(89^\circ 55' \, 00'')$, $\theta_2 = 1.568140$ $(89^\circ 50' \, 52'')$, $\theta_3 = 1.568140$.

3.10.37: If the plane cuts the sphere at distance x from its center, then the smaller spherical segment has height $h = a - x = 1 - x$ and the larger has height $h = a + x = 1 + x$. So the smaller has volume

$$V_1 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 - x)^2(2 + x)$$

and the larger has volume

$$V_2 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 + x)^2(2 - x) = 2V_1.$$

These equations leads to

$$(1 + x)^2(2 - x) = 2(1 - x)^2(2 + x);$$

$$(x^2 + 2x + 1)(x - 2) + 2(x^2 - 2x + 1)(x + 2) = 0;$$

$$x^3 - 3x - 2 + 2x^3 - 6x + 4 = 0;$$

$$3x^3 - 9x + 2 = 0.$$

The last of these equations has three solutions, one near -1.83 (out of range), one near 1.61 (also out of range), and one near $x_0 = 0.2$. Newton's method yields $x_1 = 0.225925926$, $x_2 = 0.226073709$, and $x_3 = 0.226073714 = x_4$. Answer: 0.2261.

3.10.38: This table shows that the equation $f(x) = 0$ has solutions in each of the intervals $(-3, -2)$, $(0, 1)$, and $(1, 2)$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

The next table shows the results of the iteration of Newton's method:

n	x_n	x_n	x_n
0	1.5	0.5	-2.5
1	2.090909091	0.2307692308	-2.186440678
2	1.895903734	0.2540002371	-2.118117688
3	1.861832371	0.2541016863	-2.114914461
4	1.860806773	0.2541016884	-2.114907542
5	1.860805853	0.2541016884	-2.114907541
6	1.860805853		-2.114907541

Answer: -2.1149, 0.2541, and 1.8608.

3.10.39: We iterate using the formula

$$x - \frac{x + \tan x}{1 + \sec^2 x} \longrightarrow x.$$

Here is a sequence of simple *Mathematica* commands to find approximations to the four least positive solutions of the given equation, together with the results. (The command **list=g[list]** was executed repeatedly, but deleted from the output to save space.)

```
list={2.0, 5.0, 8.0, 11.0}
f[x_]:=x+Tan[x]
g[x_]:=N[x−f[x]/f′[x], 10]
list=g[list]
2.027314579, 4.879393859, 7.975116372, 11.00421012
2.028754298, 4.907699753, 7.978566616, 11.01202429
2.028757838, 4.913038110, 7.978665635, 11.02548807
2.028757838, 4.913180344, 7.978665712, 11.04550306
2.028757838, 4.913180439, 7.978665712, 11.06778114
2.028757838, 4.913180439, 7.978665712, 11.08205766
2.028757838, 4.913180439, 7.978665712, 11.08540507
2.028757838, 4.913180439, 7.978665712, 11.08553821
2.028757838, 4.913180439, 7.978665712, 11.08553841
```

Answer: 2.029 and 4.913.

3.10.40: Plot the graph of $f(x) = 4x^3 - 42x^2 - 19x - 28$ on $[-3, 12]$ to see that the equation $f(x) = 0$ has exactly one real solution neear $x = 11$. The initial guess $x_0 = 0$ yields the solution $x = 10.9902$ after 20 iterations. The initial guess $x_0 = 10$ yields the solution after three iterations. The initial guess $x_0 = 100$ yields the solution after ten iterations.

3.10.41: Similar triangles show that

$$\frac{x}{u+v} = \frac{5}{v} \qquad \text{and} \qquad \frac{y}{u+v} = \frac{5}{u},$$

so that

$$x = 5 \cdot \frac{u+v}{v} = 5(1+t) \qquad \text{and} \qquad y = 5 \cdot \frac{u+v}{u} = 5\left(1 + \frac{1}{t}\right).$$

Next, $w^2 + y^2 = 400$ and $w^2 + x^2 = 225$, so that:

$$400 - y^2 = 225 - x^2;$$

$$175 + x^2 = y^2;$$

$$175 + 25(1+t)^2 = 25\left(1 + \frac{1}{t}\right)^2;$$

$$175t^2 + 25t^2(1+t)^2 = 25(1+t)^2;$$

$$7t^2 + t^4 + 2t^3 + t^2 = t^2 + 2t + 1;$$

$$t^4 + 2t^3 + 7t^2 - 2t - 1 = 0.$$

The graph of $f(t) = t^4 + 2t^3 + 7t^2 - 2t - 1$ shows a solution of $f(t) = 0$ near $x_0 = 0.5$. Newton's method yields $x_1 = 0.491071429$, $x_2 = 0.490936940$, and $x_3 = 0.490936909 = x_4$. It now follows that $x = 7.454684547$, that $y = 15.184608052$, that $w = 13.016438772$, that $u = 4.286063469$, and that $v = 8.730375303$. Answers: $t = 0.4909$ and $w = 13.0164$.

3.10.42: We let $f(x) = 3\sin x - \ln x$. The graph of f on the interval $[1, 22]$ does not make it clear whether there are no solutions of $f(x) = 0$ between 20 and 22, or one solution, or two. But the graph on $[20, 21]$ makes it quite clear that there is no solution there: The maximum value of $f(x)$ there is approximately -0.005 and occurs close to $x = 20.4$. The iteration of Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

beginning with the initial values $x_0 = 7,\ x_0 = 9,\ x_0 = 13.5,$ and $x_0 = 14.5$, yielded the following (rounded) results:

n	x_n	x_n	x_n	x_n
1	6.9881777136	8.6622012723	13.6118476398	14.6151381365
2	6.9882410659	8.6242919485	13.6226435579	14.6025252483
3	6.9882410677	8.6236121268	13.6227513693	14.6023754151
4	6.9882410677	8.6236119024	13.6227513801	14.6023753939
5	6.9882410677	8.6236119024	13.6227513801	14.6023753939

The last line in the table gives the other four solutions to ten-place accuracy.

3.10.43: Let $f(\theta) = (100+\theta)\cos\theta\ -\ 100$. The iterative formula of Newton's method is

$$\theta_{i+1}\ =\ \theta_i - \frac{f(\theta_i)}{f'(\theta_i)} \qquad (1)$$

where, of course, $f'(\theta) = \cos\theta - (100+\theta)\sin\theta$. Beginning with $\theta_0 = 1$, iteration of the formula in (1) yields

0.4620438212, 0.2325523723, 0.1211226155, 0.0659741863,

0.0388772442, 0.0261688780, 0.0211747166, 0.0200587600,

0.0199968594, 0.0199966678, 0.0199966678, 0.0199966678.

We take the last value of θ_i to be sufficiently accurate. The corresponding radius of the asteriod is thus approximately $1000/\theta_{12} \approx 50008.3319$ ft, about 9.47 mi.

3.10.44: The length of the circular arc is $2R\theta = 5281$; the length of its chord is $2R\sin\theta = 5280$ (units are radians and feet). Division of the second of these equations by the first yields

$$\frac{\sin\theta}{\theta}\ =\ \frac{5280}{5281}.$$

To solve for θ by means of Newton's method, we let $f(\theta)\ =\ 5281\sin\theta\ -\ 5280\theta$. The iterative formula of Newton's method is

$$\theta_{i+1}\ =\ \theta_i - \frac{5281\sin\theta\ -\ 5280\theta}{5281\cos\theta\ -\ 5280}. \qquad (1)$$

Beginning with the [poor] initial guess $\theta_0 = 1$, iteration of the formula in (1) yields these results:

$$\begin{array}{lllll} 0.655415, & 0.434163, & 0.289117, & 0.193357, & 0.130147, \\ 0.0887267, & 0.0621344, & 0.0459270, & 0.0373185, & 0.0341721, \\ 0.0337171, & 0.0337078, & 0.0337078, & 0.0337078, & 0.0337078. \end{array}$$

Hence the radius of the circular arc is

$$R \approx \frac{5281}{2}$$

Chapter 3 Miscellaneous Problems

C03S0M.001: If $y = y(x) = x^2 + 3x^{-2}$, then $\dfrac{dy}{dx} = 2x - 6x^{-3} = 2x - \dfrac{6}{x^3}$.

C03S0M.002: Given $y = \sqrt{x^3} = x^{3/2}$, we find immediately that $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2}$. Results using *Mathematica* 3.0 are more interesting:

```
D[ Sqrt[x∧3], x ]
```

$$\frac{3x^2}{2\sqrt{x^3}}$$

C03S0M.003: If $y = y(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x}} = x^{1/2} + x^{-1/3}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-4/3} = \frac{1}{2x^{1/2}} - \frac{1}{3x^{4/3}} = \frac{3x^{5/6} - 2}{6x^{4/3}}.$$

C03S0M.004: Given $y = y(x) = (x^2 + 4x)^{5/2}$, the chain rule yields $\dfrac{dy}{dx} = \dfrac{5}{2}(x^2 + 4x)^{3/2}(2x + 4)$.

C03S0M.005: Given $y = y(x) = (x - 1)^7(3x + 2)^9$, the product rule and the chain rule yield

$$\frac{dy}{dx} = 7(x-1)^6(3x+2)^9 + 27(x-1)^7(3x+2)^8 = (x-1)^6(3x+2)^8(48x - 13).$$

C03S0M.006: Given $y = y(x) = \dfrac{x^4 + x^2}{x^2 + x + 1}$, the quotient rule yields

$$\frac{dy}{dx} = \frac{(x^2 + x + 1)(4x^3 + 2x) - (x^4 + x^2)(2x + 1)}{(x^2 + x + 1)^2} = \frac{2x^5 + 3x^4 + 4x^3 + x^2 + 2x}{(x^2 + x + 1)^2}.$$

C03S0M.007: If $y = y(x) = \left(3x - \dfrac{1}{2x^2}\right)^4 = \left(3x - \frac{1}{2}x^{-2}\right)^4$, then

$$\frac{dy}{dx} = 4\left(3x - \tfrac{1}{2}x^{-2}\right)^3 \cdot \left(3 + x^{-3}\right) = 4\left(3x - \frac{1}{2x^2}\right)^3 \cdot \left(3 + \frac{1}{x^3}\right).$$

C03S0M.008: Given $y = y(x) = x^{10}\sin 10x$, the product rule and the chain rule yield

$$\frac{dy}{dx} = 10x^9 \sin 10x + 10x^{10}\cos 10x = 10x^9(\sin 10x + x\cos 10x).$$

C03S0M.009: Given $y = 9x^{-1}$, we find immediately that

$$\frac{dy}{dx} = (-1)\cdot 9x^{-2} = -9x^{-2} = -\frac{9}{x^2}.$$

It is also possible to find the derivative beginning with the equation $xy = 9$ and *without* first solving for y; this topic is addressed in Section 4.1.

C03S0M.010: $y = y(x) = (5x^6)^{-1/2}$: $\dfrac{dy}{dx} = -\dfrac{1}{2}(5x^6)^{-3/2}(30x^5) = -\dfrac{3}{x\sqrt{5x^6}} = -\dfrac{3\sqrt{5}}{5x^4}$.

C03S0M.011: Given $y = y(x) = \dfrac{1}{\sqrt{(x^3 - x)^3}} = (x^3 - x)^{-3/2}$,

$$\frac{dy}{dx} = -\frac{3}{2}(x^3 - x)^{-5/2}(3x^2 - 1) = -\frac{3(3x^2 - 1)}{2(x^3 - x)^{5/2}}.$$

C03S0M.012: Given $y = y(x) = (2x + 1)^{1/3}(3x - 2)^{1/5}$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{3}(2x + 1)^{-2/3}(3x - 2)^{1/5} + \frac{3}{5}(3x - 2)^{-4/5}(2x + 1)^{1/3}\\ &= \frac{10(3x - 2) + 9(2x + 1)}{15(2x + 1)^{2/3}(3x - 2)^{4/5}} = \frac{48x - 11}{15(2x + 1)^{2/3}(3x - 2)^{4/5}}.\end{aligned}$$

C03S0M.013: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{-2u}{(1 + u^2)^2} \cdot \dfrac{-2x}{(1 + x^2)^2}$. Now $1 + u^2 = 1 + \dfrac{1}{(1 + x^2)^2} = \dfrac{x^4 + 2x^2 + 2}{(1 + x^2)^2}$.

So $\dfrac{dy}{du} = \dfrac{-2u}{(1 + u^2)^2} = \dfrac{-2}{1 + x^2} \cdot \dfrac{(1 + x^2)^4}{(x^4 + 2x^2 + 2)^2} = \dfrac{-2(1 + x^2)^3}{(x^4 + 2x^2 + 2)^2}$.

Therefore $\dfrac{dy}{dx} = \dfrac{-2(1 + x^2)^3}{(x^4 + 2x^2 + 2)^2} \cdot \dfrac{-2x}{(1 + x^2)^2} = \dfrac{4x(1 + x^2)}{(x^4 + 2x^2 + 2)^2}$.

C03S0M.014: If $y = (\sin x)^{2/3}$, then the power rule and the chain rule yield

$$\frac{dy}{dx} = \frac{2}{3}(\sin x)^{-1/3}\cos x = \frac{2\cos x}{3(\sin x)^{1/3}}.$$

C03S0M.015: Given $y = y(x) = \left(x^{1/2} + 2^{1/3}x^{1/3}\right)^{7/3}$,

$$\frac{dy}{dx} = \frac{7}{3}\left(x^{1/2} + 2^{1/3}x^{1/3}\right)^{4/3} \cdot \left(\frac{1}{2}x^{-1/2} + \frac{2^{1/3}}{3}x^{-2/3}\right).$$

C03S0M.016: Given $y = y(x) = \sqrt{3x^5 - 4x^2} = (3x^5 - 4x^2)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(3x^5 - 4x^2)^{-1/2} \cdot (15x^4 - 8x) = \frac{15x^4 - 8x}{2\sqrt{3x^5 - 4x^2}}.$$

C03S0M.017: If $y = \dfrac{u + 1}{u - 1}$ and $u = (x + 1)^{1/2}$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{(u - 1) - (u + 1)}{(u - 1)^2} \cdot \frac{1}{2}(x + 1)^{-1/2} = -\frac{2}{(u - 1)^2} \cdot \frac{1}{2\sqrt{x + 1}} = -\frac{1}{\left(\sqrt{x + 1} - 1\right)^2\sqrt{x + 1}}.$$

C03S0M.018: Given $y = y(x) = \sin(2\cos 3x)$,

$$\frac{dy}{dx} = [\cos(2\cos 3x)] \cdot (-6\sin 3x) = -6(\sin 3x)\,\cos(2\cos 3x).$$

C03S0M.019: Given $y = \sqrt{x^6 + x^4} = (x^6 + x^4)^{1/2}$, the power rule, the chain rule, and a little care with algebra yield

$$\frac{dy}{dx} = \frac{6x^5 + 4x^3}{2(x^6 + x^4)^{1/2}} = \frac{3x^5 + 2x^3}{\sqrt{x^4}\,\sqrt{x^2+1}} = \frac{3x^5 + 2x^3}{x^2\sqrt{x^2+1}} = \frac{3x^3 + 2x}{\sqrt{x^2+1}} \qquad (x \neq 0).$$

C03S0M.020: Given $y = y(x) = \left(1 + \sin x^{1/2}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + \sin x^{1/2}\right)^{-1/2}\left(\cos x^{1/2}\right) \cdot \frac{1}{2}x^{-1/2} = \frac{\cos\sqrt{x}}{4\sqrt{x}\,\sqrt{1 + \sin\sqrt{x}}}.$$

C03S0M.021: Given $y = y(x) = \sqrt{x + \sqrt{2x + \sqrt{3x}}} = \left(x + \left[2x + (3x)^{1/2}\right]^{1/2}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \left[2x + (3x)^{1/2}\right]^{1/2}\right)^{-1/2} \cdot \left(1 + \frac{1}{2}\left[2x + (3x)^{1/2}\right]^{-1/2} \cdot \left[2 + \frac{3}{2}(3x)^{-1/2}\right]\right).$$

The symbolic algebra program *Mathematica* writes this answer without exponents as follows:

$$\frac{dy}{dx} = \frac{1 + \dfrac{2 + \dfrac{\sqrt{3}}{2\sqrt{x}}}{2\sqrt{2x + \sqrt{3x}}}}{2\sqrt{x + \sqrt{2x + \sqrt{3x}}}}.$$

C03S0M.022: $\dfrac{dy}{dx} = \dfrac{(x^2 + \cos x)(1 + \cos x) - (x + \sin x)(2x - \sin x)}{(x^2 + \cos x)^2} = \dfrac{1 - x^2 - x\sin x + \cos x + x^2\cos x}{(x^2 + \cos x)^2}.$

C03S0M.023: Given $y = (4 - x^{1/3})^3$, the power rule and chain rule yield

$$\frac{dy}{dx} = 3(4 - x^{1/3})^2 \cdot \left(-\frac{1}{3}x^{-2/3}\right) = -\frac{(4 - x^{1/3})^2}{x^{2/3}}.$$

C03S0M.024: Given $y = (x^4 + x^3)^{1/3}$, the power rule and chain rule yield

$$\frac{dy}{dx} = \frac{1}{3}(x^4 + x^3)^{-2/3}(4x^3 + 3x^2) = \frac{4x^3 + 3x^2}{3(x^3)^{2/3}(x+1)^{2/3}} = \frac{4x^3 + 3x^2}{3x(x+1)^{2/3}} = \frac{4x^2 + 3x}{3(x+1)^{2/3}}.$$

C03S0M.025: Given $y = (1 + 2u)^3$ where $u = (1 + x)^{-3}$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6(1 + 2u)^2 \cdot (-3)(1 + x)^{-4} = -\frac{18(1 + 2u)^2}{(1 + x)^4} = -\frac{18(1 + 2(1 + x)^{-3})^2}{(1 + x)^4}$$

$$= -\frac{18(1 + x)^6(1 + 2(1 + x)^{-3})^2}{(1 + x)^{10}} = -\frac{18((1 + x)^3 + 2)^2}{(1 + x)^{10}} = -18 \cdot \frac{(x^3 + 3x^2 + 3x + 3)^2}{(x + 1)^{10}}.$$

C03S0M.026: $\dfrac{dy}{dx} = \left(-2\cos(\sin^2 x)\sin(\sin^2 x)\right) \cdot \left(2\sin x\,\cos x\right).$

C03S0M.027: Given $y = y(x) = \left(\dfrac{\sin^2 x}{1 + \cos x}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\sin^2 x}{1+\cos x}\right)^{-1/2} \cdot \frac{(1+\cos x)(2\sin x\,\cos x) + \sin^3 x}{(1+\cos x)^2}$$

$$= \left(\frac{1+\cos x}{\sin^2 x}\right)^{1/2} \cdot \frac{2\sin x\,\cos x + 2\sin x\,\cos^2 x + \sin^3 x}{2(1+\cos x)^2}.$$

C03S0M.028: $\dfrac{dy}{dx} = \dfrac{3\,(1+\sqrt{x}\,)^2}{2\sqrt{x}}\,(1 - 2\sqrt[3]{x}\,)^4 + 4\,(1-2\sqrt[3]{x}\,)^3\left(-\frac{2}{3}x^{-2/3}\right)(1+\sqrt{x}\,)^3.$

C03S0M.029: Given: $y = y(x) = \dfrac{\cos 2x}{\sqrt{\sin 3x}} = (\cos 2x)(\sin 3x)^{-1/2},$

$$\frac{dy}{dx} = (-2\sin 2x)(\sin 3x)^{-1/2} + (\cos 2x)\left(-\frac{1}{2}(\sin 3x)^{-3/2}\right)(3\cos 3x)$$

$$= -\frac{2\sin 2x}{\sqrt{\sin 3x}} - \frac{3\cos 2x\,\cos 3x}{2(\sin 3x)^{3/2}} = -\frac{4\sin 2x\,\sin 3x + 3\cos 2x\,\cos 3x}{2(\sin 3x)^{3/2}}.$$

C03S0M.030: Given $y = \sqrt{4+x^2} - \sqrt{4-x^2}$, the power rule and chain rule yield

$$\frac{dy}{dx} = \frac{1}{2}(4+x^2)^{-1/2}\cdot 2x - \frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x) = \frac{x}{\sqrt{4+x^2}} + \frac{x}{\sqrt{4-x^2}} = \frac{x\sqrt{4-x^2} + x\sqrt{4+x^2}}{\sqrt{16-x^4}}.$$

C03S0M.031: If $y = y(x) = \sin^3 2x\,\cos^2 3x$, then

$$\frac{dy}{dx} = (\sin^3 2x)(2)(\cos 3x)(-\sin 3x)(3) + (\cos^2 3x)(3\sin^2 2x)(2\cos 2x)$$

$$= 6(\cos 3x\,\sin^2 2x)(\cos 3x\,\cos 2x - \sin 2x\,\sin 3x) = 6\cos 3x\,\cos 5x\,\sin^2 2x.$$

We obtained the last step with the aid of the trigonometric identities that immediately precede Problem 59 in Section 7.4. These identities are consequences of the sine and cosine addition formulas, which appear inside the front cover and are derived in Problems 41 and 42 of Appendix C.

C03S0M.032: $\dfrac{dy}{dx} = \dfrac{2}{3}\left[1+(2+3x)^{-3/2}\right]^{-1/3}\cdot\left[-\dfrac{9}{2}(2+3x)^{-5/2}\right] = -\dfrac{3}{\left[1+(2+3x)^{-3/2}\right]^{1/3}\,(2+3x)^{5/2}}.$

C03S0M.033: $\dfrac{dy}{dx} = 5\left[\sin^4\left(x+\dfrac{1}{x}\right)\right]\left[\cos\left(x+\dfrac{1}{x}\right)\right]\left[1-\dfrac{1}{x^2}\right].$

C03S0M.034: Given $y = \left(\dfrac{1+x}{1-x}\right)^{3/2}$, the power rule and quotient rule then yield

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\cdot\frac{1-x+1+x}{(1-x)^2} = \frac{3x(1+x)^{1/2}}{(1-x)^{5/2}}.$$

C03S0M.035: First write $y = y(x) = \left[\cos\left(x^4+1\right)^{1/3}\right]^3$. Then

$$\frac{dy}{dx} = 3\left[\cos\left(x^4+1\right)^{1/3}\right]^2\left[-\sin\left(x^4+1\right)^{1/3}\right]\cdot\frac{1}{3}(x^4+1)^{-2/3}\cdot 4x^3.$$

C07S0M.036: $f'(x) = (3x^2 + 4x^3)\sec(x^3 + x^4)\tan(x^3 + x^4)$.

C03S0M.037: $f'(x) = 2(3x^2 + 4x^3)\sec^2(x^3 + x^4)\tan(x^3 + x^4)$.

C03S0M.038: $f'(x) = 2\sec^5 2x + 6\sec^3 2x\tan^2 2x$.

C03S0M.039: $f'(x) = \dfrac{\sec^3\sqrt{x} + \sec\sqrt{x}\tan^2\sqrt{x}}{2\sqrt{x}}$.

C03S0M.040: $f'(x) = 3x^2\sec 2x + 2x^3\sec 2x\tan 2x$.

C03S0M.041: $g'(t) = \dfrac{2t\sec 2t\tan 2t - \sec 2t}{t^2}$.

C03S0M.042: $g'(t) = -6t\sec(1 - t^2)\tan(1 - t^2)$.

C03S0M.043: $g'(t) = \dfrac{3\sec^2\sqrt{t}\tan^2\sqrt{t}}{2\sqrt{t}}$.

C03S0M.044: If $f(x) = \dfrac{1 + \tan x}{\sec x}$, then

$$f'(x) = \frac{\sec^3 x - \sec x\tan x - \sec x\tan^2 x}{\sec^2 x} = \sec x - \sin x - \sin x\tan x.$$

Alternatively, $f(x) = \cos x + \sin x$, and hence $f'(x) = \cos x - \sin x$.

C03S0M.045: If $g(t) = \dfrac{1 - \sec t}{1 + \sec t}$, then

$$g'(t) = \frac{-(1 + \sec t)\sec t\tan t - (1 - \sec t)\sec t\tan t}{(1 + \sec t)^2} = -\frac{2\sec t\tan t}{(1 + \sec t)^2}.$$

C03S0M.046: $f'(x) = (\sec x\tan x)\cos(\sec x)$.

C03S0M.047: Because $f(x) = \sin x\sec x = \dfrac{\sin x}{\cos x} = \tan x$, it follows immediately that $f'(x) = \sec^2 x$.

C03S0M.048: $g'(t) = [\sec(\sin t)]^2 \cdot \cos t$.

C03S0M.049: $g'(t) = \sin t + \sec t\tan t$.

C03S0M.050: Given $h(x) = \dfrac{1}{\cos(\tan x)}$, the reciprocal rule and chain rule yield

$$h'(x) = -\frac{(-[\sin(\tan x)]\sec^2 x}{[\cos(\tan x)]^2} = \sec^2 x\sec(\tan x)\tan(\tan x).$$

C03S0M.051: Because $h(x) = \dfrac{1}{\cos x\tan x} = \dfrac{1}{\sin x} = \csc x$, we see that $h'(x) = -\csc x\cot x$.

C03S0M.052: Given $\phi(t) = \cot(\csc t)$, the chain rule yields $\phi'(t) = \csc t\cot t\csc^2(\csc t)$.

C03S0M.053: Given $\phi(t) = \csc t\cot t$, the product rule yields $\phi'(t) = -\csc t\cot^2 t - \csc^3 t$.

C03S0M.054: If $f(x) = (1+\sin^2 3x)^{1/2}$, then

$$f'(x) = \frac{1}{2}(1+\sin^2 3x)^{-1/2}(2\sin 3x\cos 3x)\cdot 3 = \frac{3\sin 3x\cos 3x}{\sqrt{1+\sin^2 3x}}.$$

C03S0M.055: If $g(x) = \sqrt{\sin(1+x^3)} = \left[\sin(1+x^3)\right]^{1/2}$, then

$$g'(x) = \frac{1}{2}\left[\sin(1+x^3)\right]^{-1/2}\cdot 3x^2\cos(1+x^3) = \frac{3x^2\cos(1+x^3)}{2\sqrt{\sin(1+x^3)}}.$$

C03S0M.056: If $h(t) = \left(1+\sec^2 6t\right)^{1/3}$, then

$$h'(t) = \frac{4\sec^2 6t\tan 6t}{(1+\sec^2 6t)^{2/3}}.$$

C03S0M.057: Given $\phi(t) = (t-\tan t)^{3/5}$, the chain rule yields

$$\phi'(t) = \frac{3}{5}(t-\tan t)^{-2/5}(1-\sec^2 t) = \frac{3(1-\sec^2 t)}{5(t-\tan t)^{2/5}}.$$

C03S0M.058: $\dfrac{dy}{dx} = \dfrac{(x-1)-(x+1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$; the slope of the line tangent at $(0,\,-1)$ is -2; an equation of the tangent line is $y+1=-2x$; that is, $2x+y+1=0$.

C03S0M.059: If $y = f(x) = \sin 3x$, then $f'(x) = 3\cos 3x$. So the slope of the line tangent to the graph of $y=f(x)$ at the point $P(\pi/6, 1)$ is $m = f'(\pi/6) = 0$. Hence an equation of the line tangent to the graph of $y = f(x)$ at the point P is

$$y - f\left(\frac{\pi}{6}\right) = m\cdot\left(x-\frac{\pi}{6}\right); \quad \text{that is,} \quad y \equiv 1.$$

To automate this calculation with *Mathematica* 3.0, simply enter the formula for $f(x)$, then the command

```
Solve[ y − f[ Pi/6 ] == f′[ Pi/6 ]∗(x − Pi/6), y ]
```

to receive the immediate response $\{\{\, y \to 1 \,\}\}$.

C03S0M.060: If

$$y = f(x) = \frac{1}{1+\sqrt{x}}, \quad \text{then} \quad \frac{dy}{dx} = f'(x) = -\frac{1}{2\sqrt{x}\,\left(1+\sqrt{x}\right)^2}.$$

Hence the slope of the line tangent to the graph of $y = f(x)$ at the point $\left(4,\,\frac{1}{3}\right)$ is $m = f'(4) = -\frac{1}{36}$. Therefore an equation of that line is

$$y - f(4) = m(x-4); \quad \text{that is,} \quad y = \frac{16-x}{36}.$$

C03S0M.061: If

$$y = f(x) = \frac{2x}{(x+1)^{1/3}}, \quad \text{then} \quad \frac{dy}{dx} = f'(x) = \frac{2(2x+3)}{3(x+1)^{4/3}}.$$

Hence the slope of the line L tangent to the graph of $y = f(x)$ at the point $P(7, f(7)) = P(7, 7)$ is $m = f'(7) = \frac{17}{24}$. Therefore an equation of L is

$$y - f(7) = m \cdot (x - 7); \quad \text{that is,} \quad y = \frac{17x + 49}{24}.$$

C03S0M.062: If $f(x) = \sec x \tan x$, then $f(0) = \sec x \tan x - \sec^2 x$, so that $f'(0) = -1$. Hence an equation of the straight line tangent to the graph of $y = f(x)$ at the point $(0, 1)$ is

$$y - 1 = -1 \cdot (x - 0); \quad \text{that is,} \quad x + y = 1.$$

C03S0M.063: If $f(x) = (x^2 + 2x)^{1/3}$, then

$$f'(x) = \frac{1}{3}(x^2 + 2x)^{-2/3}(2x + 2) = \frac{2x + 2}{3(x^2 + 2x)^{2/3}}.$$

Thus the slope of the straight line tangent to the graph of $y = f(x)$ at the point $(2, 2)$ is $f'(2) = \frac{1}{2}$. Hence an equation of that line is

$$y - 2 = \tfrac{1}{2}(x - 2); \quad \text{that is,} \quad 2y = x + 2.$$

C03S0M.064: Divide each term in the numerator and denominator by $\sin x$ to obtain

$$\lim_{x \to 0} \frac{x}{\sin x} - \lim_{x \to 0} \frac{1}{\cos x} = 1 - 1 = 0.$$

C03S0M.065: $x \cot 3x = \dfrac{1}{3} \cdot \dfrac{3x}{\sin 3x} \to \dfrac{1}{3} \cdot 1 = \dfrac{1}{3}$ as $x \to 0$.

C03S0M.066: $\dfrac{\sin 2x}{\sin 5x} = \dfrac{2}{5} \cdot \dfrac{\sin 2x}{2x} \cdot \dfrac{5x}{\sin 5x} \to \dfrac{2}{5}$ as $x \to 0$.

C03S0M.067: $x^2 \csc 2x \cot 2x = \dfrac{1}{4} \cdot \dfrac{2x}{\sin 2x} \cdot \dfrac{2x}{\sin 2x} \cdot \cos 2x \to \dfrac{1}{4} \cdot 1 \cdot 1 \cdot 1 = \dfrac{1}{4}$ as $x \to 0$.

C03S0M.068: $-1 \leqq \sin u \leqq 1$ for all u. So

$$-x^2 \leqq x^2 \sin \frac{1}{x^2} \leqq x^2$$

for all $x \neq 0$. But $x^2 \to 0$ as $x \to 0$, so the limit of the expression caught in the squeeze is also zero.

C03S0M.069: $-1 \leqq \sin u \leqq 1$ for all u. So

$$-\sqrt{x} \leqq \sqrt{x} \sin \frac{1}{x} \leqq \sqrt{x}$$

for all $x > 0$. But $\sqrt{x} \to 0$ as $x \to 0^+$, so the limit is zero.

C03S0M.070: $h(x) = (x + x^4)^{1/3} = f(g(x))$ where $f(x) = x^{1/3}$ and $g(x) = x + x^4$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{3}(x + x^4)^{-2/3} \cdot (1 + 4x^3)$.

C03S0M.071: $h(x) = (x^2 + 25)^{-1/2} = f(g(x))$ where $f(x) = x^{-1/2}$ and $g(x) = x^2 + 25$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = -\frac{1}{2}(x^2 + 25)^{-3/2} \cdot 2x$.

C03S0M.072: First,

$$h(x) = \sqrt{\frac{x}{x^2+1}} = \left(\frac{x}{x^2+1}\right)^{1/2} = f(g(x)) \quad \text{where} \quad f(x) = x^{1/2} \quad \text{and} \quad g(x) = \frac{x}{x^2+1}.$$

Therefore

$$h'(x) = \frac{1}{2}\left(\frac{x}{x^2+1}\right)^{-1/2} \cdot \frac{x^2+1-2x^2}{(x^2+1)^2} = \left(\frac{x^2+1}{x}\right)^{1/2} \cdot \frac{1-x^2}{2(x^2+1)^2} = \frac{1-x^2}{2x^{1/2}(x^2+1)^{3/2}}.$$

C03S0M.073: One solution: $h(x) = (x-1)^{5/3} = f(g(x))$ where $f(x) = x^{5/3}$ and $g(x) = x-1$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = \frac{5}{3}(x-1)^{2/3} \cdot 1 = \frac{5}{3}(x-1)^{2/3}$. You might alternatively choose $f(x) = x^{1/3}$ and $g(x) = (x-1)^5$.

C03S0M.074: If

$$h(x) = \frac{(x+1)^{10}}{(x-1)^{10}}, \quad \text{then} \quad h(x) = f(g(x)) \quad \text{where} \quad f(x) = x^{10} \quad \text{and} \quad g(x) = \frac{x+1}{x-1}.$$

Hence

$$h'(x) = f'(g(x)) \cdot g'(x) = 10\left(\frac{x+1}{x-1}\right)^9 \cdot \frac{(x+1)-(x-1)}{(x-1)^2} = 10\left(\frac{x+1}{x-1}\right)^9 \cdot \frac{2}{(x-1)^2} = \frac{20(x+1)^9}{(x-1)^{11}}.$$

C03S0M.075: $h(x) = \cos(x^2+1) = f(g(x))$ where $f(x) = \cos x$ and $g(x) = x^2+1$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = -2x\sin(x^2+1)$.

C03S0M.076: If $h(x) = (1+\sin x)^3 = f(g(x))$, then natural choices for f and g are $f(x) = x^3$ and $g(x) = 1+\sin x$. Then

$$h'(x) = f'(g(x)) \cdot g'(x) = 3(1+\sin x)^2 \cdot \cos x.$$

C03S0M.077: If $h(x) = \sec 6x = f(g(x))$, then natural choices for f and g are $f(x) = \sec x$ and $g(x) = 6x$. Then

$$h'(x) = f'(g(x)) \cdot g'(x) = [\sec g(x) \tan g(x)] \cdot g'(x) = 6\sec 6x \tan 6x.$$

C03S0M.078: Let (a, b) denote the point of tangency; note that

$$b = a + \frac{1}{a}, \quad a > 0, \quad \text{and} \quad h'(x) = 1 - \frac{1}{x^2}.$$

The slope of the tangent line can be computed using the two-point formula for slope and by using the derivative. We equate the results to obtain

$$\frac{a + \frac{1}{a} - 0}{a-1} = 1 - \frac{1}{a^2} = \frac{a^2-1}{a^2}.$$

It follows that $a^3 + a = (a-1)(a^2-1) = a^3 - a^2 - a + 1$. Thus $a^2 + 2a - 1 = 0$, and so $a = -1 + \sqrt{2}$ (the positive root because $a > 0$). Consequently the tangent line has slope $-2\left(1+\sqrt{2}\right)$ and thus equation

$$y = -2\left(1+\sqrt{2}\right)(x-1).$$

C03S0M.079: If (x, y) is the coordinate of the corner point of the rectangle in the first quadrant, then $y = \cos x$. The area of the rectangle is $A = 2xy$, so we are to maximize $A(x) = 2x\cos x$, $0 \leqq x \leqq \pi/2$. Because $A(x)$ is clearly minimal, not maximal, at the endpoints of its domain, we have (by the usual argument) a global maximum where $A'(x) = 0$; that is, where

$$2\cos x - 2x\sin x = 0; \qquad x\sin x = \cos x; \qquad x = \cot x.$$

To solve the third equation, which is a transcendental equation, we turn to approximate methods, but applied to the second equation: We let $f(x) = \cos x - x\sin x$ and apply Newton's method to the equation $f(x) = 0$, beginning with the initial guess $x_0 = 0.8$. Results: $x_1 \approx 0.861655$, $x_2 \approx 0.860334$, $x_3 \approx 0.860334 \approx x_4$. The graph of $h(x) = x - \cot x$, $0 \leqq x \leqq \pi/2$ makes it clear that we have located the global maximum; the maximum possible area is therefore $A(x_4) \approx 1.1222$.

C03S0M.080: Current production per well: 200 (bbl/day). Number of new wells: x ($x \geqq 0$). Production per well: $200 - 5x$. Total production:

$$T = T(x) = (20+x)(200-5x), \qquad 0 \leqq x \leqq 40.$$

Now $T(x) = 4000 + 100x - 5x^2$, so $T'(x) = 100 - 10x$. $T'(x) = 0$ when $x = 10$. $T(0) = 4000$, $T(40) = 0$, and $T(10) = 4500$. So $x = 10$ maximizes $T(x)$. Answer: Ten new wells should be drilled, thereby increasing total production from 4000 bbl/day to 4500 bbl/day.

C03S0M.081: Let the circle be the one with equation $x^2 + y^2 = R^2$ and let the base of the triangle lie on the x-axis; denote the opposite vertex of the triangle by (x, y). The area of the triangle $A = Ry$ is clearly maximal when y is maximal; that is, when $y = R$. To solve this problem using calculus, let θ be the angle of the triangle at $(-R, 0)$. Because the triangle has a right angle at (x, y), its two short sides are $2R\cos\theta$ and $2R\sin\theta$, so its area is

$$A(\theta) = 2R^2\sin\theta\,\cos\theta = R^2\sin 2\theta, \qquad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Then $A'(\theta) = 2R^2\cos 2\theta$; $A'(\theta) = 0$ when $\cos 2\theta = 0$; because θ lies in the first quadrant, $\theta = \frac{1}{4}\pi$. Finally, $A(0) = 0 = A(\pi/2)$, but $A(\pi/4) = R^2 > 0$. Hence the maximum possible area of such a triangle is R^2.

C03S0M.082: Let x be the length of the edges of each of the 20 small squares. The first five boxes measure $210 - 2x$ by $336 - 2x$ by x. The total volume is then

$$V(x) = 5x(210-2x)(336-2x) + 8x^3, \qquad 0 \leqq x \leqq 105.$$

Thus $V(x) = 28x^3 - 5460x^2 + 352800x$, and so

$$V'(x) = 84x^2 - 10290x + 352800 = 84(x^2 - 130x + 4200) = 84(x-60)(x-70).$$

So $V'(x) = 0$ when $x = 60$ and when $x = 70$. But $V(0) = 0$, $V(60) = 7{,}560{,}000$, $V(70) = 7{,}546{,}000$, and $V(105) = 9{,}261{,}000$. Answer: For maximal volume, make x as large as possible: 105 cm. This yields the maximum volume, 9,261,000 cm^3. Note that it is attained by constructing one large cubical box and that some material is wasted.

C03S0M.083: Let one sphere have radius r; the other, s. We seek the extrema of $A = 4\pi(r^2 + s^2)$ given $\frac{4}{3}\pi(r^3 + s^3) = V$, a constant. We illustrate here the **method of auxiliary variables:**

$$\frac{dA}{dr} = 4\pi\left(2r + 2s\frac{ds}{dr}\right);$$

the condition $dA/dr = 0$ yields $ds/dr = -r/s$. But we also know that $\frac{4}{3}\pi(r^3 + s^3) = V$; differentiation of both sides of this *identity* with respect to r yields

$$\frac{4}{3}\pi\left(3r^2 + 3s^2\frac{ds}{dr}\right) = 0, \quad \text{and so}$$
$$3r^2 + 3s^2\left(-\frac{r}{s}\right) = 0;$$
$$r^2 - rs = 0.$$

Therefore $r = 0$ or $r = s$. Also, ds/dr is undefined when $s = 0$. So we test these three critical points. If $r = 0$ or if $s = 0$, there is only one sphere, with radius $(3V/4\pi)^{1/3}$ and surface area $(36\pi V^2)^{1/3}$. If $r = s$, then there are two spheres of equal size, both with radius $\frac{1}{2}(3V/\pi)^{1/3}$ and surface area $(72\pi V^2)^{1/3}$. Therefore, for maximum surface area, make two equal spheres. For minimum surface area, make only one sphere.

C03S0M.084: Let x be the length of the edge of the rectangle on the side of length 4 and y the length of the adjacent edges. By similar triangles, $3/4 = (3 - y)/x$, so $x = 4 - \frac{4}{3}y$. We are to maximize $A = xy$; that is,

$$A = A(y) = 4y - \frac{4}{3}y^2, \qquad 0 \leqq y \leqq 3.$$

Now $dA/dy = 4 - \frac{8}{3}y$; $dA/dy = 0$ when $y = \frac{3}{2}$. Because $A(0) = A(3) = 0$, the maximum is $A(2) = 3$ (m^2).

C03S0M.085: Let r be the radius of the cone; let its height be $h = R + y$ where $0 \leqq y \leqq R$. (Actually, $-R \leqq y \leqq R$, but the cone will have maximal volume if $y \geqq 0$.) A central vertical cross section of the figure (*draw it!*) shows a right triangle from which we read the relation $y^2 = R^2 - r^2$. We are to maximize $V = \frac{1}{3}\pi r^2 h$, so we write

$$V = V(r) = \frac{1}{3}\pi\left[r^2\left(R + \sqrt{R^2 - r^2}\right)\right], \qquad 0 \leqq r \leqq R.$$

The condition $V'(r) = 0$ leads to the equation $r\left(2R^2 - 3r^2 + 2R\sqrt{R^2 - r^2}\,\right) = 0$, which has the two solutions $r = 0$ and $r = \frac{2}{3}R\sqrt{2}$. Now $V(0) = 0$, $V(R) = \frac{1}{3}\pi R^3$ (which is one-fourth the volume of the sphere), and $V\left(\frac{2}{3}R\sqrt{2}\,\right) = \frac{32}{81}\pi R^3$ (which is 8/27 of the volume of the sphere). Answer: The maximum volume is $\frac{32}{81}\pi R^3$.

C03S0M.086: Let x denote the length of the two sides of the corral that are perpendicular to the wall. There are two cases to consider.

Case 1: Part of the wall is used. Let y be the length of the side of the corral parallel to the wall. Then $y = 400 - 2x$, and we are to maximize the area

$$A = xy = x(400 - 2x), \qquad 150 \leqq x \leqq 200.$$

Then $A'(x) = 400 - 4x$; $A'(x) = 0$ when $x = 100$, but that value of x is not in the domain of A. Note that $A(150) = 15000$ and that $A(200) = 0$.

Case 2: All of the wall is used. Let y be the length of fence added to one end of the wall, so that the side parallel to the wall has length $100 + y$. Then $100 + 2y + 2x = 400$, so $y = 150 - x$. We are to maximize the area

$$A = x(100 + y) = x(250 - x), \qquad 0 \leqq x \leqq 150.$$

In this case $A'(x) = 0$ when $x = 125$. And in this case $A(150) = 15000$, $A(0) = 0$, and $A(125) = 15625$.

Answer: The maximum area is 15625 ft^2; to attain it, use all the existing wall and build a square corral.

C03S0M.087: First, $R'(x) = kM - 2kx$; because $k \neq 0$, $R'(x) = 0$ when $x = M/2$. Moreover, because $R(0) = 0 = R(M)$ and $R(M/2) > 0$, the latter is the maximum value of $R(x)$. Therefore the incidence of the disease is the highest when half the susceptible individuals are infected.

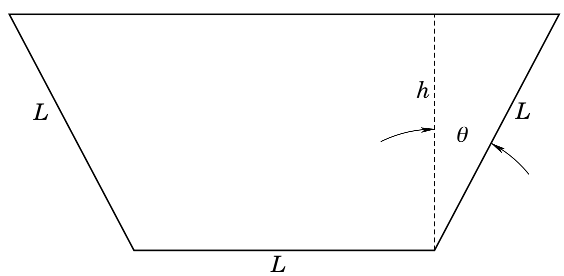
C03S0M.088: The trapezoid is shown next. It has altitude $h = L\cos\theta$ and the length of its longer base is $L + 2L\sin\theta$, so its area is

$$A(\theta) = L^2(1 + \sin\theta)\cos\theta, \qquad -\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{2}.$$

Now $dA/d\theta = 0$ when

$$1 - \sin\theta - 2\sin^2\theta = 0;$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0;$$

the only solution is $\theta = \pi/6$ because $\sin\theta$ cannot equal -1 in the range of A. Finally, $A(\pi/2) = 0$, $A(-\pi/6) = \frac{1}{4}L^2\sqrt{3}$, and $A(\pi/6) = \frac{3}{4}L^2\sqrt{3}$. The latter maximizes $A(\theta)$, and the fourth side of the trapezoid then has length $2L$.



C03S0M.089: Let x be the width of the base of the box, so that the base has length $2x$; let y be the height of the box. Then the volume of the box is $V = 2x^2y$, and for its total surface area to be 54 ft^2, we require $2x^2 + 6xy = 54$. Therefore the volume of the box is given by

$$V = V(x) = 2x^2\left(\frac{27 - x^2}{3x}\right) = \frac{2}{3}(27x - x^3), \qquad 0 < x \leqq 3\sqrt{3}.$$

Now $V'(x) = 0$ when $x^2 = 9$, so that $x = 3$. Also $V(x) \to 0$ as $x \to 0^+$ and $V\big(3\sqrt{3}\,\big) = 0$, so $V(3) = 36$ (ft^3) is the maximum possible volume of the box.

C03S0M.090: Suppose that the small cone has radius x and height y. Similar triangles that appear in a vertical cross section of the cones (*draw it!*) show that $\dfrac{x}{H - y} = \dfrac{R}{H}$. Hence $y = H - \dfrac{H}{R}x$, and we seek to maximize the volume $V = \frac{1}{3}\pi x^2 y$. Now

$$V = V(x) = \frac{\pi H}{3R}(Rx^2 - x^3), \qquad 0 \leqq x \leqq R.$$

So $V'(x) = \dfrac{\pi H}{3R}x(2R - 3x)$. $V'(x) = 0$ when $x = 0$ and when $x = \frac{2}{3}R$ (in this case, $y = H/3$). But $V(0) = 0$ and $V(R) = 0$, so $x = \frac{2}{3}R$ maximizes V. Finally, it is easy to find that $V_{\max} = \dfrac{4}{27} \cdot \dfrac{\pi}{3}R^2H$, so the largest fraction of the large cone that the small cone can occupy is $4/27$.

C03S0M.091: Let (x, y) be the coordinates of the vertex of the trapezoid lying properly in the first quadrant and let θ be the angle that the radius of the circle to (x, y) makes with the x-axis. The bases of the trapezoid have lengths 4 and $4\cos\theta$ and its altitude is $2\sin\theta$, so its area is

$$A(\theta) = \frac{1}{2}(4 + 4\cos\theta)(2\sin\theta) = 4(1 + \cos\theta)\sin\theta, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Now

$$\begin{aligned} A'(\theta) &= 4(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= 4(2\cos^2\theta + \cos\theta - 1) \\ &= 4(2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The only zero of A' in its domain occurs at $\theta = \pi/3$. At the endpoints, we have $A(0) = 0$ and $A(\pi/2) = 4$. But $A(\pi/3) = 3\sqrt{3} \approx 5.196$, so the latter is the maximum possible area of such a trapezoid.

C03S0M.092: The square of the length of PQ is a function of x, $G(x) = (x - x_0)^2 + (y - y_0)^2$, which we are to maximize given the constraint $C(x) = y - f(x) = 0$. Now

$$\frac{dG}{dx} = 2(x - x_0) + 2(y - y_0)\frac{dy}{dx} \quad \text{and} \quad \frac{dC}{dx} = \frac{dy}{dx} - f'(x).$$

When both vanish, $f'(x) = \dfrac{dy}{dx} = -\dfrac{x - x_0}{y - y_0}$. The line containing P and Q has slope

$$\frac{y - y_0}{x - x_0} = -\frac{1}{f'(x)},$$

and therefore this line is normal to the graph at Q.

C03S0M.093: If $Ax + By + C = 0$ is an equation of a straight line L, then not both A and B can be zero.

Case 1: $A = 0$ and $B \neq 0$. Then L has equation $y = -C/B$ and thus is a horizontal line. So the shortest segment from $P(x_0, y_0)$ to Q on L is a vertical segment that therefore meets L in the point $Q(x_0, -C/B)$. Therefore, because $A = 0$, the distance from P to Q is

$$\left| y_0 + \frac{C}{B} \right| = \frac{|By_0 + C|}{|B|} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Case 2: $A \neq 0$ and $B = 0$. Then L has equation $x = -C/A$ and thus is a vertical line. So the shortest segment from $P(x_0, y_0)$ to Q on L is a horizontal segment that therefore meets L in the point $Q(-C/A, y_0)$. Therefore, because $B = 0$, the distance from P to Q is

$$\left| x_0 + \frac{C}{A} \right| = \frac{|Ax_0 + C|}{|A|} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Case 3: $A \neq 0$ and $B \neq 0$. Then L is neither horizontal nor vertical, and the segment joining $P(x_0, y_0)$ to the nearest point $Q(u, v)$ on L is also neither horizontal nor vertical. The equation of L may be written in the form

$$y = -\frac{A}{B} - \frac{C}{B},$$

so L has slope $-A/B$. Thus the slope of PQ is B/A (by the result in Problem 70), and therefore PQ lies on the line K with equation

$$y - y_0 = \frac{B}{A}(x - x_0).$$

Consequently $A(v - y_0) = B(u - x_0)$. But $Q(u, v)$ also lies on L, and so $Au + Bv = -C$. Thus we have the simultaneous equations

$$\begin{aligned} Au + Bv &= -C; \\ Bu - Av &= Bx_0 - Ay_0. \end{aligned}$$

These equations may be solved for

$$u = \frac{-AC + B^2x_0 - ABy_0}{A^2 + B^2} \qquad \text{and} \qquad v = \frac{-BC - ABx_0 + A^2y_0}{A^2 + B^2},$$

and it follows that

$$u - x_0 = \frac{A(-C - Ax_0 - By_0)}{A^2 + B^2} \qquad \text{and} \qquad v - y_0 = \frac{B(-C - Ax_0 - By_0)}{A^2 + B^2}.$$

Therefore

$$\begin{aligned} (u - x_0)^2 + (v - y_0)^2 &= \frac{A^2(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2} + \frac{B^2(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2} \\ &= \frac{(A^2 + B^2)(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2} = \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}. \end{aligned}$$

The square root of this expression then gives the distance from P to Q as

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}},$$

and the proof is complete.

C03S0M.094: Let r be the radius of each semicircle and x the length of the straightaway. We wish to maximize $A = 2rx$ given $C = 2\pi r + 2x - 4 = 0$. We use the method of auxiliary variables (as in the solution of Problem 83):

$$\frac{dA}{dx} = 2r + 2x\frac{dr}{dx} \quad \text{and} \quad \frac{dC}{dx} = 2\pi\frac{dr}{dx} + 2.$$

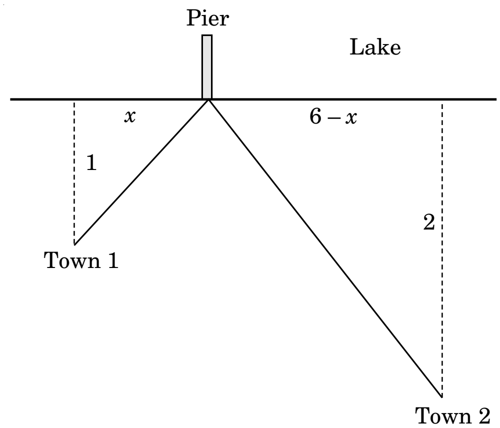
When both derivatives are zero, $-r/x = dr/dx = -1/\pi$, and so $x = \pi r$. Also $2\pi r + 2x = 4$, and it follows that $r = \dfrac{1}{\pi}$ and that $x = 1$. Answer: Design the straightaway 1 km long with semicircles of radius $\dfrac{1}{\pi}$ at each end.

C03S0M.095: As the following diagram suggests, we are to minimize the sum of the lengths of the two diagonals. Fermat's principle of least time may be used here, so we know that the angles at which the roads

meet the shore are equal, and thus so are the tangents of those angles: $\dfrac{x}{1} = \dfrac{6-x}{2}$. It follows that the pier should be built two miles from the point on the shore nearest the first town. To be sure that we have found a minimum, consider the function that gives the total length of the two diagonals:

$$f(x) = \sqrt{x^2+1} + \sqrt{(6-x)^2+4}, \qquad 0 \leqq x \leqq 6.$$

(The domain certainly contains the global minimum value of f.) Moreover, $f(0) = 1 + \sqrt{40} \approx 7.32$, $f(6) = 2 + \sqrt{37} \approx 8.08$, and $f(2) = \sqrt{5} + \sqrt{20} \approx 6.71$. This establishes that $x = 2$ yields the global minimum of $f(x)$.



C03S0M.096: The length of each angled path is $\dfrac{2}{\sin\theta}$. The length of the roadway path is $10 - \dfrac{4\cos\theta}{\sin\theta}$. So the total time of the trip will be

$$T = T(\theta) = \frac{5}{4} + \frac{32 - 12\cos\theta}{24\sin\theta}.$$

Note that $\cos\theta$ varies in the range $0 \leqq \cos\theta \leqq \frac{5}{29}\sqrt{29}$, so $21.80^\circ \leqq \theta^\circ \leqq 90^\circ$. After simplifications,

$$T'(\theta) = \frac{12 - 32\cos\theta}{24\sin^2\theta};$$

$T'(\theta) = 0$ when $\cos\theta = \frac{3}{8}$, so $\theta^\circ \approx 67.98^\circ$. With this value of θ, we find that the time of the trip is

$$T = \frac{2\sqrt{55} + 15}{12} \approx 2.486 \text{ (hours)}.$$

Because $T \approx 3.590$ with $\theta^\circ \approx 21.80^\circ$ and $T \approx 2.583$ when $\theta^\circ \approx 90^\circ$, the value $\theta^\circ \approx 69.98^\circ$ minimizes T, and the time saved is about 50.8 minutes.

C03S0M.097: Denote the initial velocity of the arrow by v. First, we have

$$\frac{dy}{dx} = m - \frac{32x}{v^2}(m^2+1);$$

$dy/dx = 0$ when $mv^2 = 32x(m^2+1)$, so that $x = \dfrac{mv^2}{32(m^2+1)}$. Substitution of this value of x in the formula given for y in the problem yields the maximum height

$$y_{\text{max}} = \frac{m^2v^2}{64(m^2+1)}.$$

For part (b), we set $y = 0$ and solve for x to obtain the range

$$R = \frac{mv^2}{16(m^2+1)}.$$

Now R is a continuous function of the slope m of the arrow's path at time $t = 0$, with domain $0 \leqq m < +\infty$. Because $R(m) = 0$ and $R(m) \to 0$ as $m \to +\infty$, the function R has a global maximum; because R is differentiable, this maximum occurs at a point where $R'(m) = 0$. But

$$\frac{dR}{dm} = \frac{v^2}{16} \cdot \frac{(m^2+1) - 2m^2}{(m^2+1)^2},$$

so $dR/dm = 0$ when $m = 1$ and only then. So the maximum range occurs when $\tan\alpha = 1$; that is, when $\alpha = \frac{1}{4}\pi$.

C03S0M.098: Here we have

$$R = R(\theta) = \frac{v^2\sqrt{2}}{16}(\cos\theta\,\sin\theta - \cos^2\theta) \quad \text{for} \quad \tfrac{1}{4}\pi \leqq \theta \leqq \tfrac{1}{2}\pi.$$

Now

$$R'(\theta) = \frac{v^2\sqrt{2}}{16}(\cos^2\theta - \sin^2\theta + 2\sin\theta\,\cos\theta);$$

$R'(\theta) = 0$ when $\cos 2\theta + \sin 2\theta = 0$, so that $\tan 2\theta = -1$. It follows that $\theta = 3\pi/8$ (67.5°). This yields the maximum range because $R(\pi/4) = 0 = R(\pi/2)$.

C03S0M.099: With initial guess $x_0 = 2.5$ (the midpoint of the given interval $[2, 3]$), the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^2 - 7}{2x_n}$$

of Newton's method yields $x_1 = 2.65$, $x_2 = 2.645754717$, and $x_3 = 2.645751311$. Answer: 2.6458.

C03S0M.100: We get $x_0 = 1.5$, $x_1 = 1.444444444$, $x_2 = 1.442252904$, and $x_3 = 1.442249570$. Answer: 1.4422.

C03S0M.101: With $x_0 = 2.5$, we obtain $x_1 = 2.384$, $x_2 = 2.371572245$, $x_3 = 2.371440624$, and $x_4 = 2.371440610$. Answer: 2.3714.

C03S0M.102: With $x_0 = 5.5$ we get $x_1 = 5.623872512$, $x_2 = 5.623413258$, and $x_3 = 5.623413252$. Answer: 5.6234. If your calculator won't raise numbers to fractional powers, you could solve instead the equation $x^4 - 1000 = 0$. With $x_0 = 5.5$ the results should be $x_1 = 5.627629602$, $x_2 = 5.623417988$, and $x_3 = 5.623413252$.

C03S0M.103: With $x_0 = -0.5$ we obtain $x_1 = -0.333333333$, $x_2 = -0.347222222$, $x_3 = -0.347296353$, and $x_4 = -0.347296355$. Answer: -0.3473.

C03S0M.104: With $x_0 = -0.5$ we obtain $x_1 = -0.230769231$, $x_2 = -0.254000237$, $x_3 = -0.254101686$, and $x_4 = -0.254101688$. Answer: -0.2541.

C03S0M.105: Given the equation $x^6 + 7x^2 - 4 = 0$ on the interval $[0, 1]$, we first let $f(x) = x^6 + 7x^2 - 4$. Then the *Mathematica* 3.0 command

```
Plot[ f[x], { x, 0, 1 } ];
```

produces a graph that shows a solution of $f(x) = 0$ just a little less than 0.75. Hence we let

```
g[x_] := N[ x - f[x]/f'[x], 20 ]
```

in order to partially automate the iterative formula of Newton's method. Then we obtain the usual sequence of improving approximations to the solution we seek as follows:

```
x0 = 3/4;

x1 = g[x0]

    0.74031531531531531532

x2 = g[x1]

    0.74022179070829104327

x3 = g[x2]

    0.74022178210472572446

x4 = g[x3]

    0.74022178210472565166

x5 = g[x4]

    0.74022178210472565166
```

To four places, the solution in question is 0.7402. Because the equation $f(x) = 0$ is actually a cubic (in x^2) polynomial equation, it can be solved exactly. The *Mathematica* 3.0 command

```
Solve[ x^6 + 7*x^2 - 4 == 0, x ]
```

produces all six solutions, two pairs of which are complex conjugates; the fifth is the negative of the last, and the last is the one we seek; it is

$$x = \sqrt{\frac{\left(18 + \sqrt{1353}\,\right)^{1/3}}{3^{2/3}} - \frac{7}{\left(54 + 3\sqrt{1353}\,\right)^{1/3}}}$$

$$\approx 0.74022178210472565166386996526789486492539893517531331614726616056063377 5.$$

C03S0M.106: With $x_0 = -1.5$, we obtain $x_1 = -1.323943662$, $x_2 = -1.309010784$, $x_3 = -1.308907325$, and $x_4 = -1.308907320$. Answer: -1.3089.

C03S0M.107: With $x_0 = -1.0$ we obtain $x_1 = -0.750363868$, $x_2 = -0.739112891$, $x_3 = -0.739085133$, and $x_4 = -0.739085133$. Answer: -0.7391.

C03S0M.108: With $x_0 = -0.75$, we obtain $x_1 = -0.905065774$, $x_2 = -0.877662556$, $x_3 = -0.876727303$, $x_4 = -0.876726215$, and $x_5 = -0.876726215$. Answer: -0.8767.

C03S0M.109: With $x_0 = -1.5$, we obtain $x_1 = -1.244861806$, $x_2 = -1.236139793$, $x_3 = -1.236129989$, and $x_4 = -1.236129989$. Answer: -1.2361.

C03S0M.110: With $x_0 = -0.5$ we obtain $x_1 = -0.858896298$, $x_2 = -0.871209876$, $x_3 = -0.871221514$, and $x_4 = -0.871221514$. Answer: -0.8712.

C03S0M.111: The volume of a spherical segment of height h is

$$V = \frac{1}{3}\pi h^2(3r - h)$$

if the sphere has radius r. If ρ is the density of water and the ball sinks to the depth h, then the weight of the water that the ball displaces is equal to the total weight of the ball, so

$$\frac{1}{3}\pi\rho h^2(3r - h) = \frac{4}{3^2}\pi\rho r^3.$$

Because $r = 2$, this leads to the equation $p(h) = 3h^3 - 18h^2 + 32 = 0$. This equation has at most three [real] solutions because $p(h)$ is a polynomial of degree 3, and it turns out to have exactly three solutions because $p(-2) = -64$, $p(-1) = 11$, $p(2) = -16$, and $p(6) = 32$. Newton's method yields the three approximate solutions $h = -1.215825766$, $h = 1.547852572$, and $h = 5.667973193$. Only one is plausible, so the answer is that the ball sinks to a depth of approximately 1.54785 ft, about 39% of the way up a diameter.

C03S0M.112: The iteration is

$$x \longleftarrow x - \frac{x^2 + 1}{2x} = \frac{x^2 - 1}{2x}.$$

With $x_0 = 2.0$, the sequence obtained by iteration of Newton's method is 0.75, -0.2917, 1.5685, 0.4654, -0.8415, 0.1734, -2.7970, -1.2197, -0.1999, 2.4009, 0.9922, -0.0078, 63.7100, $\dots$.

C03S0M.113: Let $f(x) = x^5 - 3x^3 + x^2 - 23x + 19$. Then $f(-3) = -65$, $f(0) = 19$, $f(1) = -5$, and $f(3) = 121$. So there are at least three, and at most five, real solutions. Newton's method produces three real solutions, specifically $r_1 = -2.722493355$, $r_2 = 0.8012614801$, and $r_3 = 2.309976541$. If one divides the polynomial $f(x)$ by $(x - r_1)(x - r_2)(x - r_3)$, one obtains the quotient polynomial $x^2 + (0.38874466)x + 3.770552031$, which has no real roots—the quadratic formula yields the two complex roots $-0.194372333 \pm (1.932038153)i$. Consequently we have found all three real solutions.

C03S0M.114: Let $f(x) = \tan x - \dfrac{1}{x}$. We iterate

$$x \longleftarrow x - \frac{\tan x - \dfrac{1}{x}}{\sec^2 x + \dfrac{1}{x^2}}.$$

The results are shown in the following table. The instability in the last one or two digits is caused by machine rounding and is common. Answers: To three places, $\alpha_1 = 0.860$ and $\alpha_2 = 3.426$.

```
f[x_]:=Tan[x]−1/x
g[x_]:=N[x−f[x]/f′[x], 20]
list={1.0,4.0};
g[list]
0.8740469203219249386,    3.622221245370322529
0.8604001629909660496,    3.440232462677783381
0.8603335904117901655,    3.425673797668214504
0.8603335890193797612,    3.425618460245614115
0.8603335890193797636,    3.425618459481728148
0.8603335890193797608,    3.425618459481728146
0.8603335890193797634,    3.425618459481728148
```

C03S0M.115: The number of summands on the right is variable, and we have no formula for finding its derivative. One thing is certain: Its derivative is *not* $2x^2$.

C03S0M.116: We factor:

$$z^{3/2} - x^{3/2} = (z^{1/2})^3 - (x^{1/2})^3 = (z^{1/2} - x^{1/2})(z + z^{1/2}x^{1/2} + x)$$

and $z - x = (z^{1/2})^2 - (x^{1/2})^2 = (z^{1/2} - x^{1/2})(z^{1/2} + x^{1/2})$. Therefore

$$\frac{z^{3/2} - x^{3/2}}{z - x} = \frac{z + z^{1/2}x^{1/2} + x}{z^{1/2} + x^{1/2}} \to \frac{3x}{2x^{1/2}} = \frac{3}{2}x^{1/2} \quad \text{as} \quad z \to x.$$

C03S0M.117: We factor:

$$z^{2/3} - x^{2/3} = (z^{1/3})^2 - (x^{1/3})^2 = (z^{1/3} - x^{1/3})(z^{1/3} + x^{1/3}) \qquad \text{and}$$

$$z - x = (z^{1/3})^3 - (x^{1/3})^3 = (z^{1/3} - x^{1/3})(z^{2/3} + z^{1/3}x^{1/3} + x^{2/3}).$$

Therefore

$$\frac{z^{2/3} - x^{2/3}}{z - x} = \frac{z^{1/3} + x^{1/3}}{z^{2/3} + z^{1/3}x^{1/3} + z^{2/3}} \ \to\ \frac{2x^{1/3}}{3x^{2/3}} = \frac{2}{3}x^{-1/3} \quad \text{as} \quad x \to x.$$

C03S0M.118: Given $f(x) = x^4 - 2x^3 - 3x^2 + 6x + 7$, suppose that the straight line L is tangent to the graph of $y = f(x)$ at the *two* points $P(a, f(a))$ and $Q(b, f(b))$. Then $b \neq a$. The condition that L is tangent to the graph at these two points leads to the (two) equations

$$f(b) \;=\; \frac{f(b) - f(a)}{b - a} \;=\; f'(a),$$

which can be simplified to

$$4b^3 - 2b^2 - 6b \;=\; b^3 + ab^2 + a^2b + a^3 - 2b^2 - 2ab - 2a^2 - 3a - 3b \;=\; 4a^3 - 2a^2 - 6a.$$

These equations can be solved by hand by a careful and determined person with some skill at algebra, but we follow the suggestion in the problem and use a computer algebra system. After entering the definition of $f(x)$, the *Mathematica* 3.0 command

```
Solve[ {(f[b] − f[a])/(b − a) == f′[b], (f[b] − f[a])/(b − a) == f′[a]}, {a, b} ]
```

produces a number of solutions in which $a = b$ (which we reject) and the pair of solutions

$$\{\{\, a \to -1,\ b \to 2\,\},\ \{\, a \to 2,\ b \to -1\,\}\}$$

—which represent the unique solution: One of the two points is $(-1, 1)$ and the other is $(2, 7)$.

C03S0M.119: The straight line through $P(x_0, y_0)$ and $Q(a, a^2)$ has slope $\dfrac{a^2 - y_0}{a - x_0} = 2a$, a consequence of the two-point formula for slope and the fact that the line is tangent to the parabola at Q. Hence $a^2 - 2ax_0 + y_0 = 0$. Think of this as a quadratic equation in the unknown a. It has two real solutions when the discriminant is positive: $(x_0)^2 - y_0 > 0$, and this establishes the conclusion in part (b). There are no real solutions when $(x_0)^2 - y_0 < 0$, and this establishes the conclusion in part (c). What if $(x_0)^2 - y_0 = 0$?

C03S0M.120: To solve this problem by hand, we make as many simplifications as possible. First we suppose that $f(x) = ax^3 + bx^2 + cx + d$ for some constants a, b, c, and d. By adjusting the position of the origin on the y-axis, we may suppose that $d = 0$. By adjusting the scale on the y-axis, we may further assume that $a = 1$. Thus we work with $f(x) = x^3 + bx^2 + cx$. Suppose that there are two points of tangency, $P(p, f(p))$ and $Q(q, f(q))$. Then the resulting (two) equations are

$$f'(q) = \frac{f(q) - f(p)}{q - p} = f'(p). \tag{1}$$

After defining $f(x)$, the *Mathematica* 3.0 command

```
Solve[ {(f[q] − f[p])/(q − p) == f′[p], (f[q] − f[p])/(q − p) == f′[q]}, { p, q } ]
```

produces first the warning message

```
Solve::svars:  Equations may not give solutions for all "solve" variables.
```

and then the report $q \to p$, meaning that the only solution it found was the family of solutions in which $q = p$. We doubted that solutions had been missed, but nevertheless we determined to solve the equations in (1) by hand to be sure. The first of these equations leads to

$$\begin{aligned}
\frac{q^3 - p^3 + bq^2 - bp^2 + cq - cp}{q - p} &= 3p^2 + 2bp + c;\\
q^2 + pq + p^2 + b(q + p) + c &= 3p^2 + 2bp + c;\\
b(q + p) - 2bp &= 3p^2 - q^2 - pq - p^2;\\
b(q + p - 2p) &= 2p^2 - pq - q^2;\\
b(q - p) &= (2p + q)(p - q);\\
b &= -2p - q \qquad \text{(because } p \neq q\text{)}.
\end{aligned}$$

Then we substituted this value of b in the second equation in (1) and found that

$$\begin{aligned}
q^2 + pq + p^2 - (2p + q)(q + p) &= 3q^2 - 2(2p + q)q;\\
q^2 + pq + p^2 - 2pq - 2p^2 - q^2 - pq &= 3q^2 - 4pq - 2q^2;
\end{aligned}$$

$$
\begin{aligned}
q^2 - 2pq + p^2 &= 0; \\
(q-p)^2 &= 0.
\end{aligned}
$$

Thus $q = p$ is, indeed, the only solution of the equations in (1). We thus conclude that no straight line can be tangent to the graph of a cubic polynomial at more than a single point.

Section 4.1

C04S01.001: $2x - 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{x}{y}$. Also, $y = \pm\sqrt{x^2-1}$, so $\dfrac{dy}{dx} = \pm\dfrac{x}{\sqrt{x^2-1}} = \dfrac{x}{\pm\sqrt{x^2-1}} = \dfrac{x}{y}$.

C04S01.002: $x\dfrac{dy}{dx} + y = 0$, so $\dfrac{dy}{dx} = -\dfrac{y}{x}$. By substituting $y = x^{-1}$ here, we get $\dfrac{dy}{dx} = -\dfrac{x^{-1}}{x} = -x^{-2}$, which is the result obtained by explicit differentiation.

C04S01.003: $32x + 50y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{16x}{25y}$. Substituting $y = \pm\frac{1}{5}\sqrt{400-16x^2}$ into the derivative, we get $\dfrac{dy}{dx} = \mp\dfrac{16x}{5\sqrt{400-16x^2}}$, which is the result obtained by explicit differentiation.

C04S01.004: $3x^2 + 3y^2\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$. $y = \sqrt[3]{1-x^3}$, so substitution results in $\dfrac{dy}{dx} = -\dfrac{x^2}{(1-x^3)^{2/3}}$. Explicit differentiation yields the same answer.

C04S01.005: $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$.

C04S01.006: $4x^3 + 2x^2y\dfrac{dy}{dx} + 2xy^2 + 4y^3\dfrac{dy}{dx} = 0$: $\quad (2x^2y + 4y^3)\dfrac{dy}{dx} = -(4x^3 + 2xy^2)$; $\quad \dfrac{dy}{dx} = -\dfrac{4x^3 + 2xy^2}{2x^2y + 4y^3}$.

C04S01.007: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{-1/3} = -\left(\dfrac{y}{x}\right)^{1/3}$.

C04S01.008: $y^2 + 2(x-1)y\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1-y^2}{2y(x-1)}$.

C04S01.009: Given: $\quad x^3 - x^2y = xy^2 + y^3$:

$$3x^2 - x^2\frac{dy}{dx} - 2xy = y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx};$$

$$3x^2 - 2xy - y^2 = (2xy + 3y^2 + x^2)\frac{dy}{dx};$$

$$\frac{dy}{dx} = \frac{3x^2 - 2xy - y^2}{3y^2 + 2xy + x^2}.$$

C04S01.010: Given: $\quad x^5 + y^5 = 5x^2y^2$:

$$5x^4 + 5y^4\frac{dy}{dx} = 10x^2y\frac{dy}{dx} + 10xy^2;$$

$$\frac{dy}{dx} = \frac{10xy^2 - 5x^4}{5y^4 - 10x^2y}.$$

C04S01.011: Given: $\quad x\sin y + y\sin x = 1$:

$$x\cos y\frac{dy}{dx} + \sin y + y\cos x + \sin x\frac{dy}{dx} = 0;$$

$$\frac{dy}{dx} = -\frac{\sin y + y\cos x}{x\cos y + \sin x}.$$

C04S01.012: Given: $\cos(x+y) = \sin x \sin y$:

$$-\sin(x+y)\left(1+\frac{dy}{dx}\right) = \sin x \cos y \frac{dy}{dx} + \sin y \cos x;$$

$$\frac{dy}{dx} = -\frac{\sin y \cos x + \sin(x+y)}{\sin(x+y) + \sin x \cos y}.$$

C04S01.013: Given: $\cos^3 x + \cos^3 y = \sin(x+y)$:

$$3\left(\cos^2 x\right)(-\sin x) + 3\left(\cos^2 y\right)(-\sin y)\frac{dy}{dx} = \cos(x+y)\left(1+\frac{dy}{dx}\right);$$

$$\frac{dy}{dx} = -\frac{\cos(x+y) + 3\cos^2 x \sin x}{\cos(x+y) + 3\cos^2 y \sin y}.$$

C04S01.014: Given: $xy = \tan xy$:

$$x\frac{dy}{dx} + y = \left(\sec^2 xy\right)\left(x\frac{dy}{dx} + y\right);$$

$$\frac{dy}{dx} = \frac{y \sec^2 xy - y}{x - x\sec^2 xy} = -\frac{y}{x}.$$

Note: The original equation is of the form $\tan u = u$, which is true only for some isolated constant values of u; under the assumption that $xy = k$ for some constant k, we also obtain $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

C04S01.015: $2x + 2y\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -\dfrac{x}{y}$. At $(3, -4)$ the tangent has slope $\frac{3}{4}$ and thus equation $y + 4 = \frac{3}{4}(x-3)$.

C04S01.016: $x\dfrac{dy}{dx} + y = 0$: $\dfrac{dy}{dx} = -\dfrac{y}{x}$. At $(4, -2)$ the tangent has slope $\frac{1}{2}$ and thus equation $y + 2 = \frac{1}{2}(x-4)$.

C04S01.017: $x^2\dfrac{dy}{dx} + 2xy = 1$, so $\dfrac{dy}{dx} = \dfrac{1-2xy}{x^2}$. At $(2, 1)$ the tangent has slope $-\frac{3}{4}$ and thus equation $3x + 4y = 10$.

C04S01.018: $\frac{1}{4}x^{-3/4} + \frac{1}{4}y^{-3/4}\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -(y/x)^{3/4}$. At $(16, 16)$ the tangent has slope -1 and thus equation $x + y = 32$.

C04S01.019: $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{2xy + x^2}$. At $(1, -2)$ the slope is zero, so an equation of the tangent there is $y = -2$.

C04S01.020: $-\dfrac{1}{(x+1)^2} - \dfrac{1}{(y+1)^2}\cdot\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{(y+1)^2}{(x+1)^2}$. At $(1, 1)$ the tangent line has slope -1 and thus equation $y - 1 = -(x-1)$.

C04S01.021: $24x + 24y\dfrac{dy}{dx} = 25y + 25x\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = \dfrac{25y - 24x}{24y - 25x}$. At $(3, 4)$ the tangent line has slope $\frac{4}{3}$ and thus equation $4x = 3y$.

C04S01.022: $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At $(3, -2)$ the tangent line has slope 4 and thus equation $y + 2 = 4(x-3)$.

C04S01.023: $-3x^{-4} - 3y^{-4}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{y^4}{x^4}$. At $(1, 1)$ the tangent has slope -1 and thus equation $y - 1 = -(x-1)$.

C04S01.024: $3(x^2+y^2)^2\left(2x + 2y\dfrac{dy}{dx}\right) = 16xy^2 + 16x^2y\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{16xy^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 16x^2y}.$$

At $(1, -1)$ the tangent line has slope 1 and thus equation $y + 1 = x - 1$.

C04S01.025: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$. At $(8, 1)$ the tangent line has slope $-\frac{1}{2}$ and thus equation $y - 1 = -\frac{1}{2}(x-8)$; that is, $x + 2y = 10$.

C04S01.026: $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. At $(3, -2)$ the tangent line has slope $\frac{8}{7}$ and thus equation $y + 2 = \frac{8}{7}(x-3)$; that is, $7y = 8x - 38$.

C04S01.027: $2\left(x^2+y^2\right)\left(2x + 2y\dfrac{dy}{dx}\right) = 50x\dfrac{dy}{dx} + 50y$:

$$\frac{dy}{dx} = -\frac{2x^3 - 25y + 2xy^2}{-25x + 2x^2y + 2y^3}.$$

At $(2, 4)$ the tangent line has slope $\frac{2}{11}$ and thus equation $y - 4 = \frac{2}{11}(x-2)$; that is, $11y = 2x + 40$.

C04S01.028: $2y\dfrac{dy}{dx} = 3x^2 + 14x$: $\quad \dfrac{dy}{dx} = \dfrac{3x^2+14x}{2y}$. At $(-3, 6)$ the tangent line has slope $-\frac{5}{4}$ and thus equation $y - 6 = -\frac{5}{4}(x+3)$; alternatively, $4y = 9 - 5x$.

C04S01.029: $3x^2 + 3y^2\dfrac{dy}{dx} = 9x\dfrac{dy}{dx} + 9y$: $\quad \dfrac{dy}{dx} = \dfrac{3y - x^2}{y^2 - 3x}$.

(a): At $(2, 4)$ the tangent line has slope $\frac{4}{5}$ and thus equation $y - 4 = \frac{4}{5}(x-2)$; that is, $5y = 4x + 12$.

(b): At a point on the curve at which $\dfrac{dy}{dx} = -1$, $3y - x^2 = -y^2 - 3x$ and $x^3 + y^3 = 9xy$. This pair of simultaneous equations has solutions $x = 0$, $y = 0$ and $x = \frac{9}{2}$, $y = \frac{9}{2}$, but the derivative does not exist at the point $(0, 0)$. Therefore the tangent line with slope -1 has equation $y - \frac{9}{2} = -\left(x - \frac{9}{2}\right)$.

C04S01.030: First, $2x^2 - 5xy + 2y^2 = (y-2x)(2y-x)$.

(a): Hence if $2x^2 - 5xy + 2y^2 = 0$, then $y - 2x = 0$ or $2y - x = 0$. This is a pair of lines through the origin; the first has slope 2 and the second has slope $\frac{1}{2}$.

(b): Differentiating implicitly, we obtain $4x - 5x\dfrac{dy}{dx} - 5y + 4y\dfrac{dy}{dx} = 0$, which gives $\dfrac{dy}{dx} = \dfrac{5y-4x}{4y-5x}$, which is 2 if $y = 2x$ and $-\frac{1}{2}$ if $y = -\frac{1}{2}x$.

C04S01.031: Here $\dfrac{dy}{dx} = \dfrac{2-x}{y-2}$, so horizontal tangents can occur only if $x = 2$ and $y \neq 2$. When $x = 2$, the original equation yields $y^2 - 4y - 4 = 0$, so that $y = 2 \pm \sqrt{8}$. Thus there are two points at which the tangent line is horizontal: $\left(2,\ 2-\sqrt{8}\right)$ and $\left(2,\ 2+\sqrt{8}\right)$.

C04S01.032: First, $\dfrac{dy}{dx} = \dfrac{y-x^2}{y^2-x}$ and $\dfrac{dx}{dy} = \dfrac{y^2-x}{y-x^2}$. Horizontal tangents require $y = x^2$, and the equation $x^3 + y^3 = 3xy$ of the folium yields $x^3(x^3 - 2) = 0$, so either $x = 0$ or $x = \sqrt[3]{2}$. But dy/dx is not defined at $(0,\ 0)$, so only at $\left(\sqrt[3]{2},\ \sqrt[3]{4}\right)$ is there a horizontal tangent. By symmetry or by a similar argument, there is a vertical tangent at $\left(\sqrt[3]{4},\ \sqrt[3]{2}\right)$ and nowhere else.

C04S01.033: Given: $x^2 - xy + y^2 = 9$. The x-intercepts are $(3,\ 0)$ and $(-3,\ 0)$. Next, $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. The slope of the ellipse at both x-intercepts is 2, and this shows that the tangent lines are parallel. Their equations may be written in the form $y = 2(x-3)$ and $y = 2(x+3)$.

C04S01.034: Here, $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$ and $\dfrac{dx}{dy} = \dfrac{2y-x}{y-2x}$. For horizontal tangents, $y = 2x$, and from the original equation $x^2 - xy + y^2 = 9$; it follows upon substitution of $2x$ for y that $x^2 = 3$. So the tangent line is horizontal at $\left(\sqrt{3},\ 2\sqrt{3}\right)$ and at $\left(-\sqrt{3},\ -2\sqrt{3}\right)$. Where there are vertical tangents we must have $x = 2y$, and (as above) it turns out that $y^2 = 3$; there are vertical tangents at $\left(2\sqrt{3},\ \sqrt{3}\right)$ and at $\left(-2\sqrt{3},\ -\sqrt{3}\right)$.

C04S01.035: From $2(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$ it follows that

$$\frac{dy}{dx} = \frac{x[1-2(x^2+y^2)]}{y[1+2(x^2+y^2)]}.$$

So $dy/dx = 0$ when $x^2 + y^2 = \frac{1}{2}$, but is undefined when $x = 0$, for then $y = 0$ as well. If $x^2 + y^2 = \frac{1}{2}$, then $x^2 - y^2 = \frac{1}{4}$, so that $x^2 = \frac{3}{8}$, and it follows that $y^2 = \frac{1}{8}$. Consequently there are horizontal tangents at all four points where $|x| = \frac{1}{4}\sqrt{6}$ and $|y| = \frac{1}{4}\sqrt{2}$.

Also $dx/dy = 0$ only when $y = 0$, and if so, then $x^4 = x^2$, so that $x = \pm 1$ (dx/dy is undefined when $x = 0$). So there are vertical tangents at the two points $(-1,\ 0)$ and $(1,\ 0)$.

C04S01.036: Base edge of block: x. Height: y. Volume: $V = x^2 y$. We are given $dx/dt = -2$ and $dy/dt = -3$. Implicit differentiation yields

$$\frac{dV}{dt} = x^2\frac{dy}{dt} + 2xy\frac{dx}{dt}.$$

When $x = 20$ and $y = 15$, $dV/dt = (400)(-3) + (600)(-2) = -2400$. So the rate of flow at the time given is 2400 in.^{3}/h.

C04S01.037: Suppose that the pile has height $h = h(t)$ at time t (seconds) and radius $r = r(t)$ then. We are given $h = 2r$ and we know that the volume of the pile at time t is

$$V = V(t) = \frac{\pi}{3}r^2 h = \frac{2}{3}\pi r^3. \quad \text{Now} \quad \frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}, \quad \text{so} \quad 10 = 2\pi r^2\frac{dr}{dt}.$$

When $h = 5$, $r = 2.5$; at that time $\dfrac{dr}{dt} = \dfrac{10}{2\pi(2.5)^2} = \dfrac{4}{5\pi} \approx 0.25645$ (ft/s).

C04S01.038: Draw a vertical cross section through the center of the tank. Let r denote the radius of the (circular) water surface when the depth of water in the tank is y. From the drawing and the Pythagorean theorem derive the relationship $r^2 + (10-y)^2 = 100$. Therefore

$$2r\frac{dr}{dt} - 2(10-y)\frac{dy}{dt} = 0, \quad \text{and so} \quad r\frac{dr}{dt} = (10-y)\frac{dy}{dt}.$$

We are to find dr/dt when $y = 5$, given $dy/dt = -3$. At that time, $r^2 = 100 - 25$, so $r = 5\sqrt{3}$. Thus

$$\left.\frac{dr}{dt}\right|_{y=5} = \frac{10-y}{r} \cdot \left.\frac{dy}{dt}\right|_{y=5} = \frac{5}{5\sqrt{3}}(-3) = -\sqrt{3}.$$

Answer: The radius of the top surface is decreasing at $\sqrt{3}$ ft/s then.

C04S01.039: We assume that the oil slick forms a solid right circular cylinder of height (thickness) h and radius r. Then its volume is $V = \pi r^2 h$, and we are given $V = 1$ (constant) and $dh/dt = -0.001$. Therefore $0 = \pi r^2 \dfrac{dh}{dt} + 2\pi r h \dfrac{dr}{dt}$. Consequently $2h\dfrac{dr}{dt} = \dfrac{r}{1000}$, and so $\dfrac{dr}{dt} = \dfrac{r}{2000h}$. When $r = 8$, $h = \dfrac{1}{\pi r^2} = \dfrac{1}{64\pi}$. At that time, $\dfrac{dr}{dt} = \dfrac{8 \cdot 64\pi}{2000} = \dfrac{32\pi}{125} \approx 0.80425$ (m/h).

C04S01.040: Let x be the distance from the ostrich to the street light and u the distance from the base of the light pole to the tip of the ostrich's shadow. Draw a figure and so label it; by similar triangles you find that $\dfrac{u}{10} = \dfrac{u-x}{5}$, and it follows that $u = 2x$. We are to find du/dt and $D_t(u-x) = du/dt - dx/dt$. But $u = 2x$, so

$$\frac{du}{dt} = 2\frac{dx}{dt} = (2)(-4) = -8; \qquad \frac{du}{dt} - \frac{dx}{dt} = -8 - (-4) = -4.$$

Answers: (a): $+8$ ft/s; (b): $+4$ ft/s.

C04S01.041: Let x denote the width of the rectangle; then its length is $2x$ and its area is $A = 2x^2$. Thus $\dfrac{dA}{dt} = 4x\dfrac{dx}{dt}$. When $x = 10$ and $dx/dt = 0.5$, we have

$$\left.\frac{dA}{dt}\right|_{x=10} = (4)(10)(0.5) = 20 \text{ (cm}^2\text{/s)}.$$

C04S01.042: Let x denote the length of each edge of the triangle. Then the triangle's area is $A(x) = \left(\frac{1}{4}\sqrt{3}\right)x^2$, and therefore $\dfrac{dA}{dt} = \left(\frac{1}{2}\sqrt{3}\right)x\dfrac{dx}{dt}$. Given $x = 10$ and $\dfrac{dx}{dt} = 0.5$, we find that

$$\left.\frac{dA}{dt}\right|_{x=10} = \frac{\sqrt{3}}{2} \cdot 10 \cdot (0.5) = \frac{5\sqrt{3}}{2} \text{ (cm}^2\text{/s)}.$$

C04S01.043: Let r denote the radius of the balloon and V its volume at time t (in seconds). Then

$$V = \frac{4}{3}\pi r^3, \quad \text{so} \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are to find dr/dt when $r = 10$, and we are given the information that $dV/dt = 100\pi$. Therefore

$$100\pi = 4\pi(10)^2 \left.\frac{dr}{dt}\right|_{r=10},$$

and so at the time in question the radius is increasing at the rate of $dr/dt = \frac{1}{4} = 0.25$ (cm/s).

C04S01.044: Because $pV = 1000$, $V = 10$ when $p = 100$. Moreover, $p\dfrac{dV}{dt} + V\dfrac{dp}{dt} = 0$. With $p = 100$, $V = 10$, and $dp/dt = 2$, we find that

$$\left.\frac{dV}{dt}\right|_{p=100} = -\frac{V}{p}\cdot\left.\frac{dp}{dt}\right|_{p=100} = -\frac{10}{100}\cdot 2 = -\frac{1}{5}.$$

Therefore the volume is decreasing at 0.2 in.^{3}/s.

C04S01.045: Place the person at the origin and the kite in the first quadrant at $(x,\,400)$ at time t, where $x = x(t)$ and we are given $dx/dt = 10$. Then the length $L = L(t)$ of the string satisfies the equation $L^2 = x^2 + 160000$, and therefore $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Moreover, when $L = 500$, $x = 300$. So

$$1000\left.\frac{dL}{dt}\right|_{L=500} = 600\cdot 10,$$

which implies that the string is being let out at 6 ft/s.

C04S01.046: Locate the observer at the origin and the balloon in the first quadrant at $(300,\,y)$, where $y = y(t)$ is the balloon's altitude at time t. Let θ be the angle of elevation of the balloon (in radians) from the observer's point of view. Then $\tan\theta = y/300$. We are given $d\theta/dt = \pi/180$ rad/s. Hence we are to find dy/dt when $\theta = \pi/4$. But $y = 300\tan\theta$, so

$$\frac{dy}{dt} = (300\sec^2\theta)\frac{d\theta}{dt}.$$

Substitution of the given values of θ and $d\theta/dt$ yields the answer

$$\left.\frac{dy}{dt}\right|_{\theta=45^\circ} = 300\cdot 2\cdot\frac{\pi}{180} = \frac{10\pi}{3} \approx 10.472 \text{ (ft/s)}.$$

C04S01.047: Locate the observer at the origin and the airplane at $(x,\,3)$, with $x > 0$. We are given dx/dt where the units are in miles, hours, and miles per hour. The distance z between the observer and the airplane satisfies the identity $z^2 = x^2 + 9$, and because the airplane is traveling at 8 mi/min, we find that $x = 4$, and therefore that $z = 5$, at the time 30 seconds after the airplane has passed over the observer. Also $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$, so at the time in question, $10\dfrac{dz}{dt} = 8\cdot 480$. Therefore the distance between the airplane and the observer is increasing at 384 mi/h at the time in question.

C04S01.048: In this problem we have $V = \frac{1}{3}\pi y^2(15 - y)$ and $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(10y - y^2)\dfrac{dy}{dt}$. Therefore $\dfrac{dy}{dt} = -\dfrac{13\cdot 37}{\pi y(10 - y)}$. Answers: (a): Approximately 0.2027 ft/min; (b): The same.

C04S01.049: We use $a = 10$ in the formula given in Problem 42. Then

$$V = \frac{1}{3}\pi y^2(30 - y).$$

Hence $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(20y - y^2)\dfrac{dy}{dt}$. Thus $\dfrac{dy}{dt} = -\dfrac{13\cdot 37}{\pi y(20 - y)}$. Substitution of $y = 7$ and $y = 3$ now yields the two answers:

$$\text{(a):}\quad -\frac{191}{1300\pi} \approx -0.047 \text{ (ft/min)};\qquad \text{(b):}\quad -\frac{1337}{5100\pi} \approx -0.083 \text{ (ft/min)}.$$

C04S01.050: When the height of the water at the deep end of the pool is 10 ft, the length of the water surface is 50 ft. So by similar triangles, if the height of the water at the deep end is y feet ($y \geqq 10$), then the length of the water surface is $x = 5y$ feet. A cross section of the water perpendicular to the width of the pool thus forms a right triangle of area $5y^2/2$. Hence the volume of the pool is $V(y) = 50y^2$. Now $133.7 = \dfrac{dV}{dt} = 100y\dfrac{dy}{dt}$, so when $y = 6$ we have

$$\left.\frac{dy}{dt}\right|_{y=6} = \frac{133.7}{600} \approx 0.2228 \text{ (ft/min)}.$$

C04S01.051: Let the positive y-axis represent the wall and the positive x-axis the ground, with the top of the ladder at $(0, y)$ and its lower end at $(x, 0)$ at time t. Given: $dx/dt = 4$, with units in feet, seconds, and feet per second. Also $x^2 + y^2 = 41^2$, and it follows that $y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Finally, when $y = 9$, we have $x = 40$, so at that time $9\dfrac{dy}{dt} = -40 \cdot 4$. Therefore the top of the ladder is moving downward at $\frac{160}{9} \approx 17.78$ ft/s.

C04S01.052: Let x be the length of the base of the rectangle and y its height. We are given $dx/dt = +4$ and $dy/dt = -3$, with units in centimeters and seconds. The area of the rectangle is $A = xy$, so

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = -3x + 4y.$$

Therefore when $x = 20$ and $y = 12$, we have $dA/dt = -12$, so the area of the rectangle is decreasing at the rate of 12 cm^2/s then.

C04S01.053: Let r be the radius of the cone, h its height. We are given $dh/dt = -3$ and $dr/dt = +2$, with units in centimeters and seconds. The volume of the cone at time t is $V = \frac{1}{3}\pi r^2 h$, so

$$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}.$$

When $r = 4$ and $h = 6$, $\dfrac{dV}{dt} = \dfrac{2}{3} \cdot 24\pi \cdot 2 + \dfrac{1}{3} \cdot 16\pi \cdot (-3) = 16\pi$, so the volume of the cone is increasing at the rate of 16π cm^3/s then.

C04S01.054: Let x be the edge length of the square and $A = x^2$ its area. Given: $\dfrac{dA}{dt} = 120$ when $x = 10$. But $dA/dt = 2x(dx/dt)$, so $dx/dt = 6$ when $x = 10$. Answer: At 6 in./s.

C04S01.055: Locate the radar station at the origin and the rocket at $(4,\, y)$ in the first quadrant at time t, with y in miles and t in hours. The distance z between the station and the rocket satisfies the equation $y^2 + 16 = z^2$, so $2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$. When $z = 5$, we have $y = 3$, and because $dz/dt = 3600$ it follows that $dy/dt = 6000$ mi/h.

C04S01.056: Locate the car at $(x,\, 0)$, the truck at $(0,\, y)$ $\quad (x, y > 0)$. Then at 1 P.M. we have $x = 90$ and $y = 80$. We are given that data $dx/dt = 30$ and $dy/dt = 40$, with units in miles, hours, and miles per hour. The distance z between the vehicles satisfies the equation $z^2 = x^2 + y^2$, so

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Finally, at 1 P.M. $z^2 = 8100 + 6400 = 14500$, so $z = 10\sqrt{145}$ then. So at 1 P.M.

$$\frac{dz}{dt} = \frac{2700 + 3200}{10\sqrt{145}} = \frac{590}{\sqrt{145}}$$

mi/h—approximately 49 mi/h.

C04S01.057: Put the floor on the nonnegative x-axis and the wall on the nonnegative y-axis. Let x denote the distance from the wall to the foot of the ladder (measured along the floor) and let y be the distance from the floor to the top of the ladder (measured along the wall). By the Pythagorean theorem, $x^2 + y^2 = 100$, and we are given $dx/dt = \frac{22}{15}$ (because we will use units of feet and seconds rather than miles and hours). From the Pythagorean relation we find that

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

so that $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt} = -\dfrac{22x}{15y}$.

(a): If $y = 4$, then $x = \sqrt{84} = 2\sqrt{21}$. Hence when the top of the ladder is 4 feet above the ground, it is moving a a rate of

$$\left.\frac{dy}{dt}\right|_{y=4} = -\frac{44\sqrt{21}}{60} = -\frac{11\sqrt{21}}{15} \approx -3.36$$

feet per second, about 2.29 miles per hour downward.

(b): If $y = \frac{1}{12}$ (one inch), then

$$x^2 = 100 - \frac{1}{144} = \frac{14399}{144}, \qquad \text{so that} \qquad x \approx 9.99965.$$

In this case,

$$\left.\frac{dy}{dt}\right|_{y=1/12} = -\frac{22 \cdot (9.99965)}{15 \cdot \frac{1}{12}} = -\frac{88}{5} \cdot (9.99965) \approx -176$$

feet per second, about 120 miles per hour downward.

(c): If $y = 1$ mm, then $x \approx 10$ (ft), and so

$$\frac{dy}{dt} \approx -\frac{22}{15} \cdot (3048) \approx 4470$$

feet per second, about 3048 miles per hour.

The results in parts (b) and (c) are not plausible. This shows that the assumption that the top of the ladder never leaves the wall is invalid

C04S01.058: Let x be the distance between the *Pinta* and the island at time t and y the distance between the *Niña* and the island then. We know that $x^2 + y^2 = z^2$ where $z = z(t)$ is the distance between the two ships, so

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \tag{1}$$

When $x = 30$ and $y = 40$, $z = 50$. It follows from Eq. (1) that $dz/dt = -25$ then. Answer: They are drawing closer at 25 mi/h at the time in question.

C04S01.059: Locate the military jet at $(x,\,0)$ with $x < 0$ and the other aircraft at $(0,\,y)$ with $y \geqq 0$. With units in miles, minutes, and miles per minute, we are given $dx/dt = +12$, $dy/dt = +8$, and when $t = 0$, $x = -208$ and $y = 0$. The distance z between the aircraft satisfies the equation $x^2 + y^2 = z^2$, so

$$\frac{dz}{dt} = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{12x+8y}{\sqrt{x^2+y^2}}.$$

The closest approach will occur when $dz/dt = 0$: $y = -3x/2$. Now $x(t) = 12t - 208$ and $y(t) = 8t$. So at closest approach we have

$$8t = y(t) = -\frac{3}{2}x(t) = -\frac{3}{2}(12t - 208).$$

Hence at closest approach, $16t = 624 - 36t$, and thus $t = 12$. At this time, $x = -64$, $y = 96$, and $z = 32\sqrt{13} \approx 115.38$ (mi).

C04S01.060: Let x be the distance from the anchor to the point on the seabed directly beneath the hawsehole; let L be the amount of anchor chain out. We must find dx/dt when $L = 13$ (fathoms), given $dL/dt = -10$. Now $x^2 + 144 = L^2$, so $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Consequently, $\dfrac{dx}{dt} = \dfrac{L}{x}\cdot\dfrac{dL}{dt}$. At the time in question in the problem, $x^2 = 13^2 - 12^2$, so $x = 5$. It follows that $dx/dt = -26$ then. Thus the ship is moving at 26 fathoms per minute—about 1.77 mi/h.

C04S01.061: Let x be the radius of the water surface at time t and y the height of the water remaining at time t. If Q is the amount of water remaining in the tank at time t, then (because the water forms a cone) $Q = Q(t) = \frac{1}{3}\pi x^2 y$. But by similar triangles, $\dfrac{x}{y} = \dfrac{3}{5}$, so $x = \dfrac{3y}{5}$. So

$$Q(t) = \frac{1}{3}\pi\frac{9}{25}y^3 = \frac{3}{25}\pi y^3.$$

We are given $dQ/dt = -2$ when $y = 3$. This implies that when $y = 3$, $-2 = \dfrac{dQ}{dt} = \dfrac{9}{25}\pi y^2\dfrac{dy}{dt}$. So at the time in question,

$$\left.\frac{dy}{dt}\right|_{y=3} = -\frac{50}{81\pi} \approx -0.1965 \text{ (ft/s)}.$$

C04S01.062: Given $V = \frac{1}{3}\pi(30y^2 - y^3)$, find dy/dt given V, y, and dy/dt. First,

$$\frac{dV}{dt} = \frac{1}{3}\pi(60y - 3y^2)\frac{dy}{dt} = \pi(20y - y^2)\frac{dy}{dt}.$$

So $\dfrac{dy}{dt} = \dfrac{1}{\pi(20y - y^2)}\cdot\dfrac{dV}{dt}$. Therefore, when $y = 5$, we have

$$\left.\frac{dy}{dt}\right|_{y=5} = \frac{(200)(0.1337)}{\pi(100-25)} \approx 0.113488 \text{ (ft/min)}.$$

C04S01.063: Let r be the radius of the water surface at time t, h the depth of water in the bucket then. By similar triangles we find that

$$\frac{r-6}{h} = \frac{1}{4}, \text{ so } r = 6 + \frac{h}{4}.$$

The volume of water in the bucket then is

$$\begin{aligned} V &= \frac{1}{3}\pi h(36 + 6r + r^2) \\ &= \frac{1}{3}\pi\left(36 + 36 + \frac{3}{2}h + 36 + 3h + \frac{1}{16}h^2\right) \\ &= \frac{1}{3}\pi h\left(108 + \frac{9}{2}h + \frac{1}{16}h^2\right). \end{aligned}$$

Now $\dfrac{dV}{dt} = -10$; we are to find dh/dt when $h = 12$.

$$\frac{dV}{dt} = \frac{1}{3}\pi(108 + 9h + \frac{3}{16}h^2)\frac{dh}{dt}.$$

Therefore $\left.\dfrac{dh}{dt}\right|_{h=12} = \dfrac{3}{\pi} \cdot \dfrac{-10}{108 + 9 \cdot 12 + \dfrac{3 \cdot 12^2}{16}} = -\dfrac{10}{81\pi} \approx -0.0393$ (in./min).

C04S01.064: Let x denote the distance between the ship and A, y the distance between the ship and B, h the perpendicular distance from the position of the ship to the line AB, u the distance from A to the foot of this perpendicular, and v the distance from B to the foot of the perpendicular. At the time in question, we know that $x = 10.4$, $dx/dt = 19.2$, $y = 5$, and $dy/dt = -0.6$. From the right triangles involved, we see that $u^2 + h^2 = x^2$ and $(12.6 - u)^2 + h^2 = y^2$. Therefore

$$x^2 - u^2 = y^2 - (12.6 - u)^2. \tag{1}$$

We take $x = 10.4$ and $y = 5$ in Eq. (1); it follows that $u = 9.6$ and that $v = 12.6 - u = 3$. From Eq. (1), we know that

$$x\frac{dx}{dt} - u\frac{du}{dt} = y\frac{dy}{dt} + (12.6 - u)\frac{du}{dt},$$

so

$$\frac{du}{dt} = \frac{1}{12.6}\left(x\frac{dx}{dt} - y\frac{dy}{dt}\right).$$

From the data given, $du/dt \approx 16.0857$. Also, because $h = \sqrt{x^2 - u^2}$, $h = 4$ when $x = 10.4$ and $y = 9.6$. Moreover, $h\dfrac{dh}{dt} = x\dfrac{dx}{dt} - u\dfrac{du}{dt}$, and therefore

$$\left.\frac{dh}{dt}\right|_{h=4} \approx \frac{1}{4}\left[(10.4)(19.2) - (9.6)(16.0857)\right] \approx 11.3143.$$

Finally, $\dfrac{dh/dt}{du/dt} \approx 0.7034$, so the ship is sailing a course about $35°7'$ north *or* south of east at a speed of $\sqrt{(du/dt)^2 + (dh/dt)^2} \approx 19.67$ mi/h. It is located 9.6 miles east and 4 miles north *or* south of A, or 10.4 miles from A at a bearing of either $67°22'\,48''$ or $112°37'\,12''$.

C04S01.065: Set up a coordinate system in which the radar station is at the origin, the plane passes over it at the point (0, 1) (so units on the axes are in miles), and the plane is moving along the graph of the equation $y = x + 1$. Let s be the distance from (0, 1) to the plane and let u be the distance from the radar station to the plane. We are given $du/dt = +7$ mi/min. We may deduce from the law of cosines that $u^2 = s^2 + 1 + s\sqrt{2}$. Let v denote the speed of the plane, so that $v = ds/dt$. Then

$$2u\frac{du}{dt} = 2sv + v\sqrt{2} = v\left(2s + \sqrt{2}\right), \quad \text{and so} \quad v = \frac{2u}{2s+\sqrt{2}} \cdot \frac{du}{dt}.$$

When $u = 5$, $s^2 + s\sqrt{2} - 24 = 0$. The quadratic formula yields the solution $s = 3\sqrt{2}$, and it follows that $v = 5\sqrt{2}$ mi/min; alternatively, $v \approx 424.26$ mi/h.

C04S01.066: $V(y) = \frac{1}{3}\pi(30y^2 - y^3)$ where the depth is y. Now $\dfrac{dV}{dt} = -k\sqrt{y} = \dfrac{dV}{dy} \cdot \dfrac{dy}{dt}$, and therefore

$$\frac{dy}{dt} = -\frac{k\sqrt{y}}{\dfrac{dV}{dy}} = -\frac{k\sqrt{y}}{\pi(20y - y^2)}.$$

To minimize dy/dt, write $F(y) = dy/dt$. It turns out (after simplifications) that

$$F'(y) = \frac{k}{2\pi} \cdot \frac{20y - 3y^2}{(20y - y^2)^2\sqrt{y}}.$$

So $F'(y) = 0$ when $y = 0$ and when $y = \frac{20}{3}$. When y is near 20, $F(y)$ is very large; the same is true for y near zero. So $y = \frac{20}{3}$ minimizes dy/dt, and therefore the answer to part (b) is 6 ft 8 in.

C04S01.067: Place the pole at the origin in the plane, and let the horizontal strip $0 \leqq y \leqq 30$ represent the road. Suppose that the person is located at $(x, 30)$ with $x > 0$ and is walking to the right, so $dx/dt = +5$. Then the distance from the pole to the person will be $\sqrt{x^2 + 900}$. Let z be the length of the person's shadow. By similar triangles it follows that $2z = \sqrt{x^2 + 900}$, so $4z^2 = x^2 + 900$, and thus $8z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$. When $x = 40$, we find that $z = 25$, and therefore that

$$100\frac{dz}{dt}\bigg|_{z=25} = 40 \cdot 5 = 200.$$

Therefore the person's shadow is lengthening at 2 ft/s at the time in question.

C04S01.068: Set up a coordinate system in which the officer is at the origin and the van is moving in the positive direction along the line $y = 200$ (so units on the coordinate axes are in feet). When the van is at position $(x, 200)$, the distance from the officer to the van is z, where $x^2 + 200^2 = z^2$, so that $x\dfrac{dx}{dt} = z\dfrac{dz}{dt}$. When the van reaches the call box, $x = 200$, $z = 200\sqrt{2}$, and $dz/dt = 66$. It follows that

$$\frac{dx}{dt}\bigg|_{x=200} = 66\sqrt{2},$$

which translates to about 63.6 mi/h.

Section 4.2

C04S02.001: $y = y(x) = 3x^2 - 4x^{-2}$: $\dfrac{dy}{dx} = 6x + 8x^{-3}$, so $dy = \left(6x + \dfrac{8}{x^3}\right) dx.$

C04S02.002: $y = y(x) = 2x^{1/2} - 3x^{-1/3}$: $\dfrac{dy}{dx} = x^{-1/2} + x^{-4/3}$, so

$$dy = \left(x^{-1/2} + x^{-4/3}\right) dx = \frac{1 + x^{5/6}}{x^{4/3}}\ dx.$$

C04S02.003: $y = y(x) = x - (4 - x^3)^{1/2}$: $\dfrac{dy}{dx} = 1 - \dfrac{1}{2}(4 - x^3)^{-1/2} \cdot (-3x^2)$, so

$$dy = \left(1 + \frac{3x^2}{2\sqrt{4 - x^3}}\right) dx = \frac{3x^2 + 2\sqrt{4 - x^3}}{2\sqrt{4 - x^3}}\ dx.$$

C04S02.004: $y = y(x) = \dfrac{1}{x - \sqrt{x}}$: $dy = -\dfrac{1 - \frac{1}{2}x^{-1/2}}{\left(x - \sqrt{x}\,\right)^2}\ dx = \dfrac{1 - 2\sqrt{x}}{2\sqrt{x}\ \left(x - \sqrt{x}\,\right)^2}\ dx.$

C04S02.005: $y = y(x) = 3x^2(x - 3)^{3/2}$, so

$$dy = \left[6x(x - 3)^{3/2} + \frac{9}{2}x^2(x - 3)^{1/2}\right] dx = \frac{3}{2}(7x^2 - 12x)\sqrt{x - 3}\ \ dx.$$

C04S02.006: $y = y(x) = \dfrac{x}{x^2 - 4}$, so $dy = \dfrac{(x^2 - 4) - 2x^2}{(x^2 - 4)^2}\ dx = -\dfrac{x^2 + 4}{(x^2 - 4)^2}\ dx.$

C04S02.007: $y = y(x) = x(x^2 + 25)^{1/4}$, so

$$dy = (x^2 + 25)^{1/4} + \frac{1}{4}x(x^2 + 25)^{-3/4} \cdot 2x\ \ dx = \frac{3x^2 + 50}{2(x^2 + 25)^{3/4}}\ \ dx.$$

C04S02.008: $y = y(x) = (x^2 - 1)^{-4/3}$, so $dy = -\dfrac{8x}{3(x^2 - 1)^{7/3}}\ \ dx.$

C04S02.009: $y = y(x) = \cos\sqrt{x}$, so $dy = -\dfrac{\sin\sqrt{x}}{2\sqrt{x}}\ \ dx.$

C04S02.010: $y = y(x) = x^2 \sin x$, so $dy = (x^2 \cos x + 2x \sin x)\ dx.$

C04S02.011: $y = y(x) = \sin 2x\ \cos 2x$, so $dy = (2\cos^2 2x - 2\sin^2 2x)\ dx.$

C04S02.012: $y = y(x) = (\cos 3x)^3$, so $dy = -9\cos^2 3x\ \sin 3x\ dx.$

C04S02.013: $y = y(x) = \dfrac{\sin 2x}{3x}$, so $dy = \dfrac{2x\cos 2x - \sin 2x}{3x^2}\ dx.$

C04S02.014: $y = y(x) = \dfrac{\cos x}{\sqrt{x}}$, so $dy = -\dfrac{\cos x + 2x\sin x}{2x^{3/2}}\ \ dx.$

C04S02.015: $y = y(x) = \dfrac{1}{1 - x\sin x}$, so $dy = \dfrac{x\cos x + \sin x}{(1 - x\sin x)^2}\ \ dx.$

C04S02.016: If $y = y(x) = (1 + \cos 2x)^{3/2}$, then $dy = -3(\sin 2x)\sqrt{1 + \cos 2x}\ dx$.

C04S02.017: $f'(x) = \dfrac{1}{(1-x)^2}$, so $f'(0) = 1$. Therefore

$$f(x) = \frac{1}{1-x} \approx f(0) + f'(0)(x-0) = 1 + 1 \cdot x = 1 + x.$$

C04S02.018: $f'(x) = -\dfrac{1}{2(1+x)^{3/2}}$, so $f'(0) = -\dfrac{1}{2}$. Therefore

$$f(x) = \frac{1}{\sqrt{1+x}} \approx f(0) + f'(0)(x-0) = 1 - \frac{1}{2}x.$$

C04S02.019: $f'(x) = 2(1+x)$, so $f'(0) = 2$. Therefore $f(x) = (1+x)^2 \approx f(0) + f'(0)(x-0) = 1 + 2x$.

C04S02.020: $f'(x) = -3(1-x)^2$, so $f'(0) = -3$. Therefore $f(x) = (1-x)^3 \approx f(0)+f'(0)(x-0) = 1-3x$.

C04S02.021: $f'(x) = -3\sqrt{1-2x}$, so $f'(0) = -3$; $f(x) = (1-2x)^{3/2} \approx f(0) + f'(0)(x-0) = 1 - 3x$.

C04S02.022: If

$$f(x) = \frac{1}{(1+3x)^{2/3}} = (1+3x)^{-2/3}, \quad \text{then} \quad f'(x) = -\frac{2}{3}(1+3x)^{-5/3} \cdot 3 = -\frac{2}{(1+3x)^{5/3}}.$$

Hence $f'(0) = -2$, and so $f(x) \approx f(0) + f'(0)(x-0) = 1 - 2x$.

C04S02.023: If $f(x) = \sin x$, then $f'(x) = \cos x$, so that $f'(0) = 1$. Therefore

$$f(x) = \sin x \approx f(0) + f'(0)(x-0) = 0 + 1 \cdot x = x.$$

C04S02.024: If $f(x) = \cos x$, then $f'(x) = -\sin x$, so that $f'(0) = 0$. Therefore

$$f(x) = \cos x \approx f(0) + f'(0)(x-0) = 1 + 0 \cdot x \equiv 1.$$

C04S02.025: Choose $f(x) = x^{1/3}$ and $a = 27$. Then $f'(x) = \dfrac{1}{3x^{2/3}}$, so that $f'(a) = \dfrac{1}{27}$. So the linear approximation to $f(x)$ near $a = 27$ is $L(x) = 2 + \dfrac{1}{27}x$. Hence

$$\sqrt[3]{25} = f(25) \approx L(25) = \frac{79}{27} \approx 2.9259.$$

A calculator reports that $f(25)$ is actually closer to 2.9240, but the linear approximation is fairly accurate, with an error of only about -0.0019.

C04S02.026: Choose $f(x) = \sqrt{x}$ and $a = 100$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$, so that $f'(a) = \dfrac{1}{20}$. So the linear approximation to $f(x)$ near $a = 100$ is $L(x) = 5 + \dfrac{1}{20}x$. Hence

$$\sqrt{102} = f(102) \approx L(102) = \frac{101}{10} = 10.1000.$$

A calculator reports that $f(25)$ is actually closer to 10.0995, but the linear approximation is quite accurate, with an error of only about -0.0005.

C04S02.027: Choose $f(x) = x^{1/4}$ and $a = 16$. Then $f'(x) = \dfrac{1}{4x^{3/4}}$, so that $f'(a) = \dfrac{1}{32}$. So the linear approximation to $f(x)$ near $a = 16$ is $L(x) = \dfrac{3}{2} + \dfrac{1}{32}x$. Hence

$$\sqrt[4]{15} = f(15) \approx L(15) = \frac{63}{32} = 1.96875.$$

A calculator reports that $f(15)$ is actually closer to 1.96799.

C04S02.028: Choose $f(x) = \sqrt{x}$ and $a = 81$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$, so that $f'(a) = \dfrac{1}{18}$. So the linear approximation to $f(x)$ near $a = 81$ is $L(x) = \dfrac{9}{2} + \dfrac{1}{18}x$. Hence

$$\sqrt{80} = f(80) \approx L(80) = \frac{161}{18} \approx 8.9444.$$

A calculator reports that $f(15)$ is actually closer to 8.9443.

C04S02.029: Choose $f(x) = x^{-2/3}$ and $a = 64$. Then $f'(x) = -\dfrac{2}{3x^{5/3}}$, so that $f'(a) = -\dfrac{1}{1536}$. So the linear approximation to $f(x)$ near $a = 64$ is $L(x) = \dfrac{5}{48} - \dfrac{1}{1536}x$. Hence

$$65^{-2/3} = f(65) \approx L(65) = \frac{95}{1536} \approx 0.06185.$$

A calculator reports that $f(65)$ is actually closer to 0.06186.

C04S02.030: Choose $f(x) = x^{3/4}$ and $a = 81$. Then $f'(x) = \dfrac{3}{4x^{1/4}}$, so that $f'(a) = \dfrac{1}{4}$. So the linear approximation to $f(x)$ near $a = 81$ is $L(x) = \dfrac{27}{4} + \dfrac{1}{4}x$. Hence

$$80^{3/4} = f(80) \approx L(80) = \frac{107}{4} = 26.7500.$$

A calculator reports that $f(80)$ is actually closer to 26.7496.

C04S02.031: Choose $f(x) = \cos x$ and $a = \dfrac{45}{180}\pi = \dfrac{1}{4}\pi$. Then $f'(x) = -\sin x$, so that $f'(a) = -\dfrac{1}{2}\sqrt{2}$. So the linear approximation to $f(x)$ near a is $L(x) = \dfrac{1}{2}\sqrt{2}\left(\dfrac{1}{4}\pi + 1\right) - \dfrac{1}{2}x\sqrt{2}$. Hence

$$\cos 43^\circ = f\left(\frac{43}{180}\pi\right) \approx L\left(\frac{43}{180}\pi\right) = \frac{\pi + 90}{90\sqrt{2}} \approx 0.7318.$$

A calculator reports that $\cos 43^\circ$ is actually closer to 0.7314.

C04S02.032: Choose $f(x) = \sin x$ and $a = \dfrac{30}{180}\pi = \dfrac{1}{6}\pi$. Then $f'(x) = \cos x$, so that $f'(a) = \dfrac{1}{2}\sqrt{3}$. So the linear approximation to $f(x)$ near a is

$$L(x) = \frac{6 - \pi\sqrt{3}}{12} + \frac{1}{2}x\sqrt{3}.$$

Hence

$$\sin 32^\circ = f\left(\frac{32}{180}\pi\right) \approx L\left(\frac{32}{180}\pi\right) = \frac{90 + \pi\sqrt{3}}{180} \approx 0.5302.$$

C04S02.033: Choose $f(x) = \sin x$ and $a = \dfrac{90}{180}\pi = \dfrac{1}{2}\pi$. Then $f'(x) = \cos x$, so that $f'(a) = 0$. So the linear approximation to $f(x)$ near a is $L(x) \equiv 1$. Hence

$$\sin 88^\circ = f\left(\frac{88}{180}\pi\right) \approx L\left(\frac{88}{180}\pi\right) = 1.$$

A calculator reports that $\sin 88^\circ \approx 0.9994$.

C04S02.034: Choose $f(x) = \cos x$ and $a = \dfrac{60}{180}\pi = \dfrac{1}{3}\pi$. Then $f'(x) = -\sin x$, so that $f'(a) = -\dfrac{1}{2}\sqrt{3}$. So the linear approximation to $f(x)$ near a is

$$L(x) = \frac{3 + \pi\sqrt{3}}{6} - \frac{1}{2}x\sqrt{3}.$$

Hence

$$\cos 62^\circ = f\left(\frac{62}{180}\pi\right) \approx L\left(\frac{62}{180}\pi\right) = \frac{90 - \pi\sqrt{3}}{180} \approx 0.4698.$$

C04S02.035: Given $x^2 + y^2 = 1$, we compute the differential of both sides and obtain

$$2x\ dx + 2y\ dy = 0;$$

$$y\ dy = -x\ dx;$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

C04S02.036: Given: $x \sin y = 1$. We first compute the differential of each side of this equation (actually, an identity), then solve for dy/dx:

$$(x\cos y)\ dy + (\sin y)\ dx = 0;$$

$$x\cos y\ dy = -\sin y\ dx;$$

$$\frac{dy}{dx} = -\frac{\sin y}{x\cos y} = -\frac{x\sin y}{x^2\cos y} = -\frac{1}{x^2\cos y}.$$

C04S02.037: Given $x^3 + y^3 = 3xy$, we compute the differential of each side and obtain

$$3x^2\ dx + 3y^2\ dy = 3y\ dx + 3x\ dy;$$

$$(y^2 - x)\ dy = (y - x^2)\ dx;$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

C04S02.038: Given: $x \sec y = y$. We first compute the differential of each side, then solve for dy/dx:

$$(x \sec y \, \tan y) \, dy \; + \; (\sec y) \, dx \; = \; dy;$$

$$(1 - x \sec y \, \tan y) \, dy \; = \; \sec y \, dx;$$

$$\frac{dy}{dx} \; = \; \frac{\sec y}{1 - x \sec y \, \tan y} \; = \; \frac{\sec y}{1 - y \tan y} \; = \; \frac{x \sec y}{x(1 - y \tan y)} \; = \; \frac{y}{x(1 - y \tan y)}.$$

The extensive simplifications in the last line are not really necessary. Choose for the answer the form most suitable for your applications thereof.

C04S02.039: If $f(x) = (1 + x)^k$, then $f'(x) = k(1 + x)^{k-1}$, and so $f'(0) = k$. Hence the linear approximation to $f(x)$ near zero is $L(x) = 1 + kx$.

C04S02.040: If C is the circumference of the circle and r its radius, then $C = 2\pi r$. Thus $dC = 2\pi \, dr$, and so $\Delta C \approx 2\pi \Delta r$. With $r = 10$ and $\Delta r = 0.5$, we obtain $\Delta C \approx 2\pi(0.5) = \pi \approx 3.1416$. This happens to be the exact value as well (because C is a linear function of r).

C04S02.041: If the square has edge length x and area A, then $A = x^2$. Therefore $dA = 2x \; dx$, and so $\Delta A \approx 2x \; \Delta x$. With $x = 10$ and $\Delta x = -0.2$, we obtain $\Delta A \approx 2 \cdot 10 \cdot (-0.2) = -4$. So the area of the square decreases by 4 in.^{2}.

C04S02.042: The relationship between the surface area A and the radius r of the sphere is $A = 4\pi r^2$, and hence $dA = 8\pi r \, dr$. Thus $\Delta A \approx 8\pi r \, \Delta r$. With $r = 5$ and $\Delta r = 0.2$ we obtain $\Delta A \approx 8\pi(5)(0.2) = 8\pi \approx$ 25.1327 square inches. The true value is approximately 25.6354 square inches.

C04S02.043: A [right circular] cylinder of base radius r and height h has volume $V = \pi r^2 h$, and hence $dV = \pi r^2 \, dh + 2\pi r h \, dr$. Therefore $\Delta V \approx \pi r^2 \, \Delta h + 2\pi r h \, \Delta r$. With $r = h = 15$ and $\Delta r = \Delta h = -0.3$ we find that $\Delta V \approx (225\pi)(-0.3) + (450\pi)(-0.3) = \frac{405}{2}\pi$, so the volume of the cylinder decreases by approximately 636.17 cm^3.

C04S02.044: With volume V, height h, and radius r, we have $V = \frac{1}{3}\pi r^2 h$. Because $r = 14$ is constant, we may think of V as a function of r alone, so that

$$dV = \frac{1}{3}\pi r^2 \, dh.$$

With $r = 14$, $h = 7$, and $dh = 0.1$, we find that

$$dV = \frac{1}{3}\pi(196)(0.1) \approx 20.5251.$$

The true value is exactly the same because V is a linear function of h.

C04S02.045: Because $\theta = 45°$, the range R of the shell is a function of its initial velocity v alone, and $R = \frac{1}{16}v^2$. Hence $dR = \frac{1}{8}v \, dv$. With $v = 80$ and $dv = 1$, we find that $dR = \frac{1}{8} \cdot 80 = 10$, so the range is increased by approximately 10 ft.

C04S02.046: Because v is constant, R is a function of the angle of inclination θ alone, and hence

$$dR = \frac{1}{8}v^2(\cos 2\theta) \, d\theta.$$

With $\theta = \pi/4$, $d\theta = \pi/180$ $(1°)$, and $v = 80$, we obtain

$$\Delta R \approx \frac{1}{8}(6400)(0)\frac{\pi}{180} = 0.$$

The true value of ΔR is approximately -0.2437 (ft).

C04S02.047: Technically, if $W = RI^2$, then $dW = I^2\ dR + 2IR\ dI$. But in this problem, R remains constant, so that $dR = 0$ and hence $dW = 2IR\ dI$. We take $R = 10$, $I = 3$, and $dI = 0.1$, and find that $dW = 6$. So the wattage increases by approximately 6 watts.

C04S02.048: With circumference C and radius r, we have $C = 2\pi r$. Therefore, given $\Delta r = +10$, $\Delta C = 2\pi\ \Delta r \approx 20\pi$ (feet). Thus the wire should be lengthened by approximately 63 feet.

C04S02.049: Let V be the volume of the ball and let r be its radius, so that $V = \frac{4}{3}\pi r^3$. Then the calculated value of the volume is $V_{\text{calc}} = \frac{4}{3}(1000\pi) \approx 4188.7902$ in.^{3}, whereas $\Delta V \approx 4\pi(10)^2\frac{1}{16} = 25\pi \approx 78.5398$ in.^{3} (the true value of ΔV is approximately 79.0317).

C04S02.050: With volume V and radius r, we have $V = \frac{4}{3}\pi r^3$, and thus $dV = 4\pi r^2\ dr$. For $|\Delta V| \leqq 1$, we require that $4\pi r^2\,|\Delta r| \leqq 1$, so

$$|\Delta r| \leqq \frac{1}{4\pi(10)^2} \approx 0.0008$$

inches. Thus the radius must be measured with error not exceeding 0.0008 inches.

C04S02.051: With surface area S and radius r, we have $S = 2\pi r^2$ (*half* the surface area of a sphere of radius r), so that $dS = 4\pi r\ dr \approx 4\pi(100)(0.01) = 4\pi$. That is, $\Delta S \approx 12.57$ square meters.

C04S02.052: With the notation of the preceding solution, we now require that

$$\frac{|dS|}{S} \leqq 0.0001;$$

thus, at least approximately,

$$\frac{|4\pi r\ dr|}{|2\pi r^2|} \leqq 0.0001.$$

Hence

$$2\left|\frac{dr}{r}\right| \leqq 0.0001,$$

which implies that $\dfrac{|dr|}{r} \leqq 0.00005$. Answer: With percentage error not exceeding 0.005%.

C04S02.053: We plotted $f(x) = x^2$ and its linear approximation $L(x) = 1 + 2(x - 1)$ on the interval $[0.5,\ 1.5]$, and it was clear that the interval $I = (0.58,\ 1.42)$ would be an adequate answer to this problem. We then used Newton's nethod to find a "better" interval, which turns out to be $I = (0.5528,\ 1.4472)$.

C04S02.054: We plotted $f(x) = \sqrt{x}$ and its linear approximation $L(x) = 1 + \frac{1}{2}(x - 1)$ on the interval $[0.3,\ 2.15]$, and it was clear that the interval $I = (0.32,\ 1.98)$ would be adequate. We then used Newton's method to "improve" the answer to $I = (0.3056,\ 2.0944)$.

C04S02.055: We plotted $f(x) = 1/x$ and its linear approximation $L(x) = \frac{1}{2} + \frac{1}{4}(2 - x)$ on the interval $[1.73,\ 2.32]$, and it was clear that the interval was a little too large to be a correct answer. We used Newton's

method to find more accurate endpoints, and came up with the answer $I = (1.7365,\ 2.3035)$. Of course, any open subinterval of this interval that contains $a = 2$ is also a correct answer.

C04S02.056: We plotted $f(x) = x^{1/3}$ and its linear approximation $L(x) = \frac{1}{12}(x + 16)$ on the interval $[6.3,\ 9.9]$ and it was thereby clear that the interval $(6.5,\ 9.5)$ would suffice. We then used Newton's method to find "better" endpoints and found $(6.4023,\ 9.7976)$.

C04S02.057: We plotted $f(x) = \sin x$ and its linear approximation $L(x) = x$ on the interval $[-2.8,\ 2.8]$ and it was clear that the interval $(-0.5,\ 0.5)$ would suffice. We then used Newton's method to find "better" endpoints and found that $(-0.6746,\ 0.6745)$ would suffice.

C04S02.058: Given $f(x) = \tan x$, its linear approximation $L(x)$ near $a = 0$ can be obtained as follows:

$$f'(x) = \sec^2 x; \qquad f'(0) = 1;$$

$$L(x) = f(0) + f'(0)(x - 0) = 0 + 1 \cdot x = x.$$

Our goal is to find an interval containing $a = 0$ on which $|f(x) - L(x)| < \epsilon$; that is, to find an interval on which $|\tan x - x| < 0.01$. We used the *Mathematica* 3.0 command

```
Plot[ { Tan[x] − x, −0.01, 0.01 }, { x, −0.306, 0.306 },
         PlotRange → {{ −0.36, 0.36 }, { −0.012, 0.012 }} ];
```

to plot the graph of $f(x) - L(x)$ and the horizontal lines $y = \pm 0.01$ on an interval containing $a = 0$; we found the numerical values in the command options by trial-and-error. The graph showed that the desired inequality was satisfied on the interval $(-0.306, 0.306)$. To find a more precise bound on the endpoints of the interval, we applied Newton's method to approximate the least positive solution of the equation $\tan x - x = 0.01$. We found after three or four iterations that the solution was approximately

$$c = 0.306772547185126073907,$$

rounded *down*, so that the inequality in question is satisfied on the interval $(-c, c)$ (by symmetry of $f(x)$ and $L(x)$ around the origin) but not on a significantly larger interval.

C04S02.059: We plotted $f(x) = \sin x$ and its linear approximation

$$L(x) = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4} + x\right)$$

on the interval $[0.5,\ 1.1]$, and it was clear that the interval $(0.6,\ 0.95)$ would be adequate. We then used Newton's method to "improve" the endpoints and found that the interval $(0.5364,\ 1.0151)$ would suffice.

C04S02.060: We plotted $f(x) = \tan x$ and its linear approximation $L(x) = \frac{1}{2}(2 - \pi + 4x)$ on the interval $[0.7,\ 0.9]$, from which it was clear that the interval $(0.7,\ 0.85)$ would suffice. We then used Newton's method to find "better" endpoints, with the result that the interval $(0.6785,\ 0.8789)$ was nearly the best possible result.

Section 4.3

C04S03.001: $f'(x) = -2x$; f is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$. Matching graph: (c).

C04S03.002: $f'(x) = 2x - 2$; f is increasing on $(1, +\infty)$, decreasing on $(-\infty, 1)$. Matching graph: (b).

C04S03.003: $f'(x) = 2x + 4$; f is increasing on $(-2, +\infty)$, decreasing on $(-\infty, -2)$. Matching graph: (f).

C04S03.004: $f'(x) = \frac{3}{4}x^2 - 3$; $f'(x) = 0$ when $x = \pm 2$; f is increasing on $(-\infty, -2)$ and on $(2, \infty)$, decreasing on $(-2, +2)$. Matching graph: (a).

C04S03.005: $f'(x) = x^2 - x - 2 = (x+1)(x-2)$; $f'(x) = 0$ when $x = -1$ and when $x = 2$; f is increasing on $(-\infty, -1)$ and on $(2, +\infty)$, decreasing on $(-1, 2)$. Matching graph: (d).

C04S03.006: $f'(x) = 2 - \frac{1}{3}x - \frac{1}{3}x^2 = -\frac{1}{3}(x^2 + x - 6) = -\frac{1}{3}(x+3)(x-2)$; $f'(x) = 0$ when $x = -3$ and when $x = 2$. f is increasing on $(-3, 2)$ and decreasing on $(-\infty, -3)$ and on $(2, +\infty)$. Matching graph: (e).

C04S03.007: $f(x) = 2x^2 + C$; $5 = f(0) = C$: $f(x) = 2x^2 + 5$.

C04S03.008: $f(x) = 2x^{3/2} + C$; $4 = f(0) = C$: $f(x) = 2x^{3/2} + 4$.

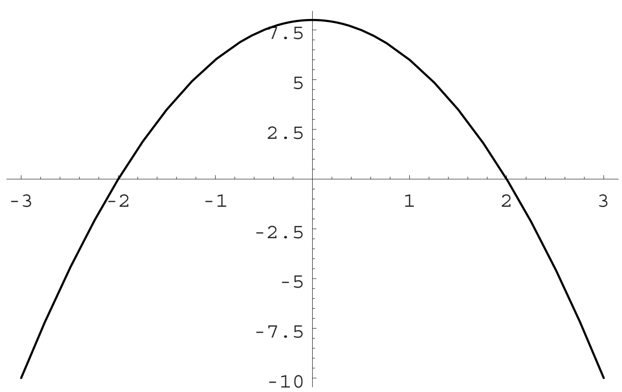
C04S03.009: $f(x) = -\dfrac{1}{x} + C$; $1 = f(1) = C - 1$: $f(x) = -\dfrac{1}{x} + 2$.

C04S03.010: $f(x) = 4\sqrt{x} + C$; $3 = f(0) = C$: $f(x) = 4\sqrt{x} + 3$.

C04S03.011: $f'(x) \equiv 3 > 0$ for all x, so f is increasing for all x.

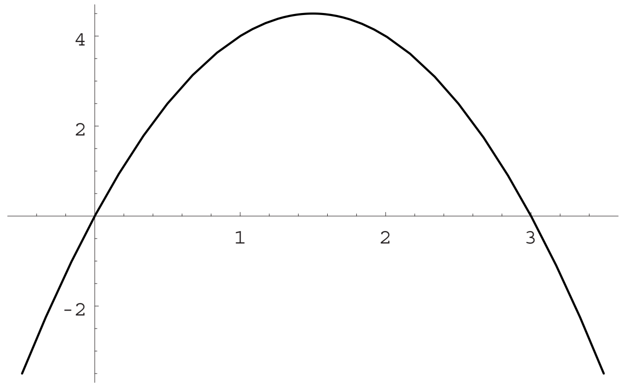
C04S03.012: $f'(x) \equiv -5 < 0$ for all x, so f is decreasing for all x.

C04S03.013: $f'(x) = -4x$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$. The graph of $y = f(x)$ is shown next.



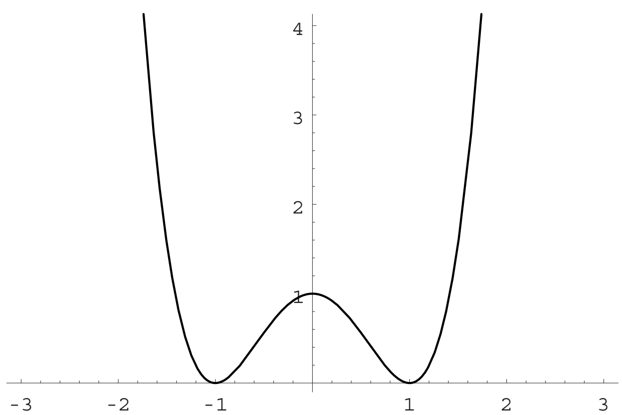
C04S03.014: $f'(x) = 8x + 8 = 8(x + 1)$. Therefore f is increasing for $x > -1$ and decreasing for $x < -1$.

C04S03.015: $f'(x) = 6 - 4x$. Therefore f is increasing for $x < \frac{3}{2}$ and decreasing for $x > \frac{3}{2}$. The graph of $y = f(x)$ is shown next.



C04S03.016: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$. Hence f is increasing for $x > 2$ and for $x < -2$, decreasing for x in the interval $(-2, 2)$.

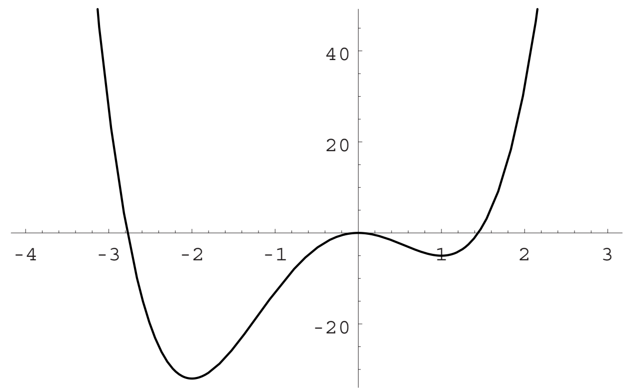
C04S03.017: $f'(x) = 4x^3 - 4x = 4x(x + 1)(x - 1)$. The intervals on which $f'(x)$ cannot change sign are $x < -1$, $-1 < x < 0$, $0 < x < 1$, and $1 < x$. Because $f'(-2) = -24$, $f'(-0.5) = 1.5$, $f'(0.5) = -1.5$, and $f'(2) = 24$, we may conclude that f is increasing if $-1 < x < 0$ or if $x > 1$, decreasing for $x < -1$ and for $0 < x < 1$. The graph of $y = f(x)$ is shown next.



C04S03.018: Because $f'(x) = \dfrac{1}{(x + 1)^2}$, $f'(x) > 0$ for all x other than $x = -1$. Hence f is increasing for $x > -1$ and for $x < -1$.

C04S03.019: $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x + 2)(x - 1)$, so the only points where $f'(x)$ can change sign are -2, 0, and 1. If $0 < x < 1$ then $12x > 0$, $x + 2 > 0$, and $x - 1 < 0$, and therefore $f'(x) < 0$ if $0 < x < 1$. Therefore f is decreasing on the interval $(0, 1)$. A similar analysis shows that f is also decreasing

on $(-\infty, -2)$ and is increasing on $(-2, 0)$ and on $(1, +\infty)$. The graph of $y = f(x)$ is shown next.

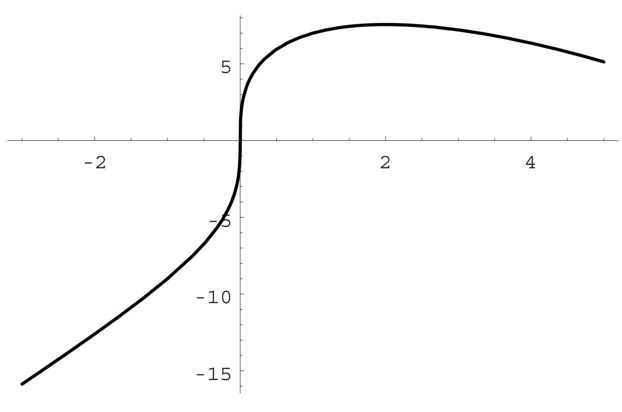


C04S03.020: Note that f is continuous for all x, as is $f'(x) = \dfrac{2x^2+1}{(x^2+1)^{1/2}}$; also, $f'(x) > 0$ for all x. Hence f is increasing for all x.

C04S03.021: Given $f(x) = 8x^{1/3} - x^{4/3}$, we find that

$$f'(x) = \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{8}{3x^{2/3}} - \frac{4x^{1/3}}{3} = \frac{8-4x}{3x^{2/3}} = \frac{4(2-x)}{3x^{2/3}}.$$

Hence $f'(x) > 0$, and the graph of f is increasing, if $x < 2$ and $x \neq 0$; $f'(x) < 0$, and the graph of f is decreasing, if $x > 2$. The graph of f, shown next, strongly (and correctly) suggests that f is continuous and increasing in any small open interval containing zero, so it is correct (and simpler) to say that f is increasing for $x < 2$ and decreasing for $x > 2$.

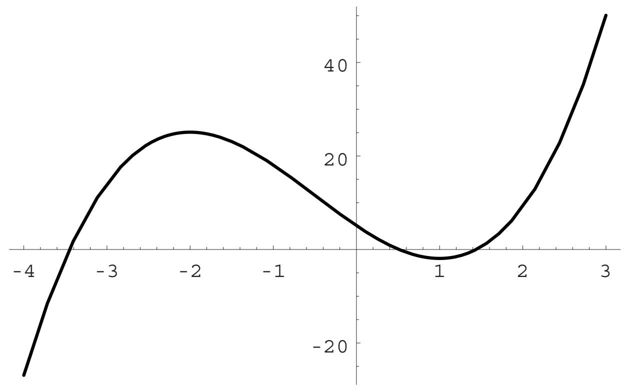


C04S03.022: If $f(x) = 2x^3 + 3x^2 - 12x + 5$, then

$$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1).$$

Therefore f is increasing if $x < -2$ and if $x > 1$; f is decreasing on the interval $(-2, 1)$. The graph of f is

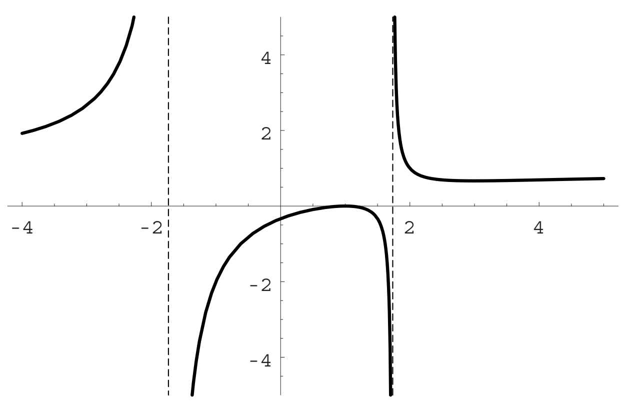
shown next.



C04S03.023: After simplifications,

$$f'(x) = \frac{2(x-1)(x-3)}{(x^2-3)^2}.$$

The intervals on which $f'(x)$ cannot change sign are $x < -\sqrt{3}$, $-\sqrt{3} < x < 1$, $1 < x < \sqrt{3}$, $\sqrt{3} < x < 3$, and $3 < x$. Now $f'(-2) = 30$, $f'(0) = 2/3$, $f'(1.5) = -8/3$, $f'(2) = -2$, and $f'(4) = 6/169$. Therefore f is increasing for $x < -\sqrt{3}$, for $-\sqrt{3} < x < 1$, and for $x > 3$; decreasing for $1 < x < \sqrt{3}$ and for $\sqrt{3} < x < 3$. The graph of $y = f(x)$ is shown next. There are vertical asymptotes at $x = \pm\sqrt{3}$ and (not visible because of the scale of the graph) a horizontal asymptote at $y = 1$.

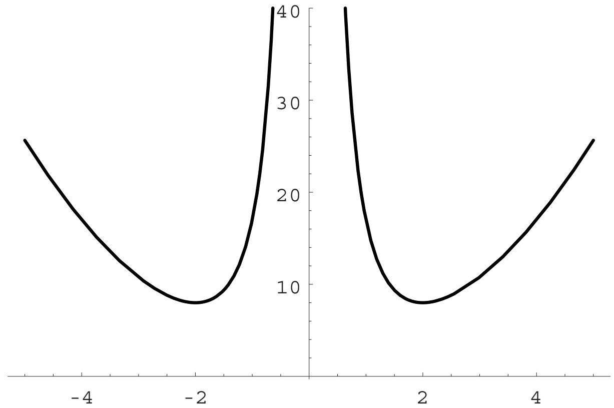


C04S03.024: If $f(x) = x^2 + \dfrac{16}{x^2}$, then

$$f'(x) = 2x - \frac{32}{x^3} = \frac{2(x^4-16)}{x^3} = \frac{2(x-2)(x+2)(x^2+4)}{x^3}.$$

Note that the sign of $f'(x)$ *does* change at $x = 0$. Hence we find that $f'(x) > 0$, and f is increasing, where $-2 < x < 0$ and where $2 < x$; $f'(x) < 0$, and f is decreasing, where $x < -2$ and $0 < x < 2$. The graph o4

$y = f(x)$ is next.



C04S03.025: $f(0) = 0$, $f(2) = 0$, f is continuous for $0 \leqq x \leqq 2$, and $f'(x) = 2x - 2$ exists for $0 < x < 2$. To find the numbers c satisfying the conclusion of Rolle's theorem, we solve $f'(c) = 0$ to find that $c = 1$ is the only such number.

C04S03.026: $f(-3) = 81 - 81 = 0 = f(3)$, f is continuous everywhere, and $f'(x) = 18x - 4x^3$ exists for all x, including all x in the interval $(-3, 3)$. Thus f satisfies the hypotheses of Rolle's theorem. To find what value or values c might assume, we solve the equation $f'(x) = 0$ to obtain the three values $c = 0$, $c = \frac{3}{2}\sqrt{2}$, and $c = -\frac{3}{2}\sqrt{2}$. All three of these numbers lie in the interval $(-3, 3)$, so these are the three possible values for the number c whose existence is guaranteed by Rolle's theorem.

C04S03.027: Given: $f(x) = 2\sin x \cos x$ on the interval $I = [0, \pi]$. Then

$$f'(x) = 2\cos^2 x - 2\sin^2 x = 2\cos 2x,$$

and it is clear that all the hypotheses of Rolle's theorem are satisfied by f on the interval I. Also $f'(x) = 0$ when $2x = \pi/2$ and when $2x = 3\pi/2$, so the numbers whose existence is guaranteed by Rolle's theorem are $c = \pi/4$ and $c = 3\pi/4$.

C04S03.028: Here,

$$f'(x) = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{10 - 5x}{3x^{1/3}};$$

$f'(x)$ exists for all x in $(0, 5)$ and f is continuous on the interval $0 \leqq x \leqq 5$ (the only point that might cause trouble is $x = 0$, but the limit of f and its value there are the same). Because $f(0) = 0 = f(5)$, there is a solution c of $f(x) = 0$ in $(0, 5)$, and clearly $c = 2$.

C04S03.029: On the interval $(-1, 0)$, $f'(x) = 1$; on the interval $(0, 1)$, we have $f'(x) = -1$. Because f is not differentiable at $x = 0$, it does not satisfy the hypotheses of Rolle's theorem, so there is no guarantee that the equation $f'(x) = 0$ has a solution—and, indeed, it has no solution in $(-1, 1)$.

C04S03.030: Because

$$f'(x) = \frac{2}{3(2 - x)^{1/3}} \quad \text{for} \quad x \neq 2,$$

$f'(x)$ can never be zero, so the conclusion of Rolle's theorem does not hold here. The reason is that f is not differentiable at every point of the interval $1 \leqq x \leqq 3$; specifically, $f'(x)$ is undefined at $x = 2$.

C04S03.031: Because $f(1) = 2 \neq 0$, this function does not satisfy the hypotheses of Rolle's theorem. Nor does the conclusion hold, for $f'(x) = 4x^3 + 2x = 2x(2x^2 + 1)$ is never zero on $(0, 1)$.

C04S03.032: It is clear that f satisfies the hypotheses of the mean value theorem, because every polynomial is continuous and differentiable everywhere. To find c, we solve

$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}; \qquad \text{that is,} \qquad 3c^2 = \frac{1+1}{1+1}.$$

Thus $3c^2 = 1$, with the two solutions $c = \frac{1}{3}\sqrt{3}$ and $c = -\frac{1}{3}\sqrt{3}$. Both these numbers lie in the interval $-1 \leqq x \leqq 1$, so each is an answer to this problem.

C04S03.033: Here, $f'(x) = 6x + 6$; $f(-2) = -5$ and $f(1) = 4$. So we are to solve the equation

$$6(c+1) = \frac{4 - (-5)}{1 - (-2)} = 3. \qquad \text{It follows that} \quad c = -\frac{1}{2}.$$

C04S03.034: Here we note that

$$f'(x) = \frac{1}{2(x-1)^{1/2}}$$

exists for all $x > 1$, so f satisfies the hypotheses of the mean value theorem for $2 \leqq x \leqq 5$. To find c, we solve

$$\frac{1}{2(c-1)^{1/2}} = \frac{(4)^{1/2} - (1)^{1/2}}{5-2};$$

thus $2(c-1)^{1/2} = 3$, and so $c = \frac{13}{4} = 3.25$. Note that $2 < c < 5$.

C04S03.035: First, $f'(x) = \frac{2}{3}(x-1)^{-1/3}$ is defined on $(1, 2)$; moreover, f is continuous for $1 \leqq x \leqq 2$ (the only "problem point" is $x = 1$, but the limit of f there is equal to its value there). To find c, we solve

$$f'(c) = \frac{2}{3(c-1)^{1/3}} = \frac{f(2) - f(1)}{2-1} = \frac{1-0}{1} = 1.$$

This leads to the equation $(c-1)^{1/3} = \frac{2}{3}$, and thereby $c = \frac{35}{27}$. Note that c does lie in the interval $(1, 2)$.

C04S03.036: Because $f(x)$ can be expressed as a rational function with denominator never zero if $x \neq 0$, it is both continuous and differentiable on $(2, 3)$. Next,

$$f'(c) = 1 - \frac{1}{c^2} = \frac{3 + \frac{1}{3} - \left(2 + \frac{1}{2}\right)}{3-2} = \frac{5}{6}$$

yields the information that $c^2 = 6$, and thus that $c = +\sqrt{6}$ (not $-\sqrt{6}$; it's not in the interval $(2, 3)$).

C04S03.037: First, $f(x) = |x - 2|$ is not differentiable at $x = 2$, so does not satisfy the hypotheses of the mean value theorem on the given interval $1 \leqq x \leqq 4$. Wherever $f'(x)$ is defined, its value is 1 or -1, but

$$\frac{f(4) - f(1)}{4-1} = \frac{2-1}{3} = \frac{1}{3}$$

is never a value of $f'(x)$. So f satisfies neither the hypotheses nor the conclusion of the mean value theorem on the interval $1 \leqq x \leqq 4$.

C04S03.038: Because f is not differentiable at $x = 1$, the hypotheses of the mean value theorem do not hold. The only values of $f'(x)$ are 1 (for $x > 1$) and -1 (for $x < 1$). Neither of these is equal to the everage slope of f on the interval $0 \leqq x \leqq 3$:

$$\frac{f(3) - f(0)}{3 - 0} = \frac{3 - 2}{3} = \frac{1}{3},$$

so the conclusion of the mean value theorem also fails to hold.

C04S03.039: The greatest integer function is continuous at x if and only if x is not an integer. Consequently all the hypotheses of the mean value theorem fail here: f is discontinuous at -1, 0, and 1, and also $f'(0)$ does not exist because (for one reason) f is not continuous at $x = 0$. Finally, the average slope of the graph of f is 1, but $f'(x) = 0$ wherever it is defined. Thus the conclusion of the mean value theorem also fails to hold.

C04S03.040: The function $f(x) = 3x^{2/3}$ is continuous everywhere, but its derivative $f'(x) = 2x^{-1/3}$ does not exist at $x = 0$. Because $f'(x)$ does not exist for *all* x in $(-1, 1)$, an essential hypothesis of the mean value theorem is not satisfied. Moreover, $f'(x)$ is never zero (the average slope of the graph of f on the interval $-1 \leqq x \leqq 1$), so the conclusion of the mean value theorem also fails to hold.

C04S03.041: Let $f(x) = x^5 + 2x - 3$. Then $f'(x) = 5x^4 + 2$, so $f'(x) > 0$ for all x. This implies that f is an increasing continuous function, and therefore $f(x)$ can have at most one zero in any interval. To show that f has at least one zero in the interval $0 \leqq x \leqq 1$, it is sufficient to notice that $f(1) = 0$. Therefore the equation $f(x) = 0$ has exactly one solution in $[0, 1]$.

C04S03.042: Let $f(x) = x^{10} - 1000$. Then $f'(x) = 10x^9$, so $f(x)$ is increasing and continuous on the interval $1 \leqq x \leqq 2$. Therefore the equation $f(x) = 0$ has at most one solution in that interval. But $f(1) = -999 < 0$ and $f(2) = 1024 - 1000 > 0$. Because f is continuous, it has the intermediate value property, so $f(x)$ has at least one zero in $[1, 2]$. Consequently the equation $x^{10} = 1000$ has exactly one solution in that interval.

Newton's method, applied to the equation $f(x) = 0$, yield the approximate solution

$$1.9952623149688796013524553967395355558.$$

C04S03.043: Let $f(x) = x^4 - 3x - 20$. Then

$$f(2) = -10 < 0 \quad \text{and} \quad f(3) = 52 > 0.$$

Because $f(x)$ is a polynomial, f is continuous. Therefore, by the intermediate value property, the equation $f(x) = 0$ has at least one solution in the interval $(2, 3)$. Next, $f'(x) = 4x^3 - 3$, so if $2 \leqq x \leqq 3$ then

$$f'(x) > f'(2) = 29 > 0.$$

Hence f is increasing on $[2, 3]$, and therefore the equation $f(x) = 0$ has at most one solution in that interval. Consequently the equation $x^4 - 3x = 20$ has exactly one solution in $[2, 3]$. (By Newton's method, that solution is approximately 2.27585905881042301934162491364923.)

C04S03.044: Let $f(x) = \sin x - 3x + 1$. Because $f'(x) = -3 + \cos x$ is always negative, the graph of f is decreasing on every interval of real numbers, and in particular is decreasing on $[-1, 1]$. Hence the equation $f(x) = 0$ can have *at most* one solution in that interval. Moreover, $f(-1) \approx 3.16 > 0$ and $f(1) \approx -1.16 < 0$. Because f is continuous on $[-1, 1]$, the intermediate value property of continuous functions guarantees that

the equation $f(x) = 0$ has *at least* one solution in $[-1, 1]$. So the equation $f(x) = 0$ has exactly one solution there.

C04S03.045: The car traveled 35 miles in 18 minutes, which is an average speed of $\frac{250}{3} \approx 83.33$ miles per hour. By the mean value theorem, the car must have been traveling over 83 miles per hour at some time between 3:00 P.M. and 3:18 P.M.

C04S03.046: A change of 15 miles per hour in 10 minutes is an average change of 1.5 miles per hour per minute, which is an average change of 90 miles per hour per hour. By the mean value theorem, the instantaneous rate of change of velocity must have been exactly 90 miles per hour per hour at some time in the given 10-minute interval.

C04S03.047: Let $f(t)$ be the distance that the first car has traveled from point A on its way to point B at time t, with t measured in hours and with $t = 0$ corresponding to 9:00 A.M. (so that $t = 1$ corresponds to 10:00 A.M.). Let $g(t)$ be the corresponding function for the second car. Let $h(t) = f(t) - g(t)$. We make the very plausible assumption that the functions f and g are differentiable on $(0, 1)$ and continuous on $[0, 1]$, so h has the same properties. In addition, $h(0) = f(0) - g(0) = 0$ and $h(1) = 0$ as well. By Rolle's theorem, $h'(c) = 0$ for some c in $(0, 1)$. But this implies that $f'(c) = g'(c)$. That is, the velocity of the first car is exactly the same as that of the second car at time $t = c$.

C04S03.048: Because $f'(0)$ does not exist, the function $f(x) = x^{2/3}$ does not satisfy the hypotheses of the mean value theorem on the given interval. But consider the equation

$$f'(c) = \frac{f(27) - f(-1)}{27 - (-1)}; \tag{1}$$

that is,

$$\frac{2}{3c^{1/3}} = \frac{9 - 1}{28} = \frac{2}{7},$$

which leads to $c^{1/3} = 7/3$, and thus to $c = \frac{343}{27} \approx 12.7$. Because $-1 < c < 27$, there is indeed a number c satisfying Eq. (1).

C04S03.049: Because

$$f'(x) = \frac{3}{2}(1 + x)^{1/2} - \frac{3}{2} = \frac{3}{2}\left(\sqrt{1 + x} - 1\right),$$

it is clear that $f'(x) > 0$ for $x > 0$. Also $f(0) = 0$; it follows that $f(x) > 0$ for all $x > 0$. That is,

$$(1 + x)^{3/2} > 1 + \frac{3}{2}x \quad \text{for} \quad x > 0.$$

C04S03.050: Proof: Suppose that $f'(x)$ is the constant K on the interval $a \leqq x \leqq b$. Let

$$g(x) = Kx + f(a) - Ka.$$

Then the graph of g is a straight line, and $g'(x) = K$ for all x. Consequently f and g differ by a constant on the interval $a \leqq x \leqq b$. But $g(a) = Ka + f(a) - Ka = f(a)$, so $g(x) = f(x)$ for all x in that interval. Therefore the graph of f is a straight line.

C04S03.051: Proof: Suppose that $f'(x)$ is a polynomial of degree $n - 1$ on the interval $I = [a, b]$. Then $f'(x)$ has the form

$$f'(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where $a_{n-1} \neq 0$. Note that $f'(x)$ is the derivative of the function

$$g(x) = \frac{1}{n}a_{n-1}x^n + \frac{1}{n-1}a_{n-2}x^{n-1} + \ldots + \frac{1}{3}a_2x^3 + \frac{1}{2}a_1x^2 + a_0x.$$

By Corollary 2, $f(x)$ and $g(x)$ can differ only by a constant, and this is sufficient to establish that $f(x)$ must also be a polynomial, and one of degree n because the coefficient of x^n in $f(x)$ is the same as the coefficient of x^n in $g(x)$, and that coefficient is nonzero.

C04S03.052: Suppose that $f(x) = 0$ for $x = x_1,\ x_2,\ \ldots,\ x_k$ in the interval $[a,\ b]$. By Rolle's theorem, $f'(x) = 0$ for some c_1 in $(x_1,\ x_2)$, some c_2 in $(x_2,\ x_3)$, $\ldots$, and some c_{k-1} in $(x_{k-1},\ x_k)$. The numbers $c_1,\ c_2,\ \ldots,\ c_{k-1}$ are distinct because they lie in nonoverlapping intervals, and this proves the desired result.

C04S03.053: First note that $f'(x) = \frac{1}{2}x^{-1/2}$, and that the hypotheses of the mean value theorem are all satisfied for the given function f on the given interval [100, 101]. Thus there does exist a number c between 100 and 101 such that

$$\frac{1}{2c^{1/2}} = \frac{f(101) - f(100)}{101 - 100} = \sqrt{101} - \sqrt{100}.$$

Therefore $1/(2\sqrt{c}\,) = \sqrt{101} - 10$, and thus we have shown that $\sqrt{101} = 10 + 1/(2\sqrt{c}\,)$ for some number c in (100, 101).

Proof for part (b): If $0 \leqq \sqrt{c} \leqq 10$, then $0 \leqq c \leqq 100$; because $c > 100$, we see that $0 \leqq \sqrt{c} \leqq 10$ is impossible. If $10.5 \leqq \sqrt{c}$ then $110.25 \leqq c$, which is also impossible because $c < 110$. So we may conclude that $10 < \sqrt{c} < 10.5$. Finally,

$$10 < \sqrt{c} < 10.5 \quad \text{implies that} \quad 20 < 2\sqrt{c} < 21.$$

Consequently

$$\frac{1}{21} < \frac{1}{2c^{1/2}} < \frac{1}{20}, \qquad \text{so} \qquad 10 + \frac{1}{21} < \sqrt{101} < 10 + \frac{1}{20}.$$

The decimal expansion of 1/21 begins $0.047619047619\ldots$, and therefore $10.0476 < \sqrt{101} < 10.05$.

C04S03.054: Let $f(x) = x^7 + x^5 + x^3 + 1$. Then $f(-1) = -2$, $f(1) = 4$, and $f'(x) = 7x^6 + 5x^4 + 3x^2$. Now $f'(x) > 0$ for all x except that $f'(0) = 0$, so f is increasing on the set of all real numbers. This information together with the fact that f (continuous) has the intermediate value property establishes that the equation $f(x) = 0$ has exactly one [real] solution (approximately -0.79130272).

C04S03.055: Let $f(x) = (\tan x)^2$ and let $g(x) = (\sec x)^2$. Then

$$f'(x) = 2(\tan x)(\sec^2 x) \quad \text{and} \quad g'(x) = 2(\sec x)(\sec x\, \tan x) = f'(x) \quad \text{on} \quad (-\pi/2,\ \pi/2).$$

Therefore there exists a constant C such that $f(x) = g(x) + C$ for all x in $(-\pi/2,\ \pi/2)$. Finally, $f(0) = 0$ and $g(0) = 1$, so $C = f(0) - g(0) = -1$.

C04S03.056: The mean value theorem does not apply here because $f'(0)$ does not exist.

C04S03.057: The average slope of the graph of f on the given interval $[-1,\ 2]$ is

$$\frac{f(2) - f(-1)}{2 - (-1)} = \frac{5 - (-1)}{3} = 2$$

and f satisfies the hypotheses of the mean value theorem there. Therefore $f'(c) = 2$ for some number c, $-1 < c < 2$. This implies that the tangent line to the graph of f at the point $(c, f(c))$ has slope 2 and is therefore parallel to the line with equation $y = 2x$ because the latter line also has slope 2.

C04S03.058: To show that the graph of $f(x) = x^4 - x^3 + 7x^2 + 3x - 11$ has a horizontal tangent line, we must show that its derivative $f'(x) = 4x^3 - 3x^2 + 14x + 3$ has the value zero at some number c. Now $f'(x)$ is a polynomial, thus is continuous everywhere, and so has the intermediate value property; moreover, $f'(-1) = -18$ and $f'(0) = 3$, so $f'(c) = 0$ for some number c in $(-1, 0)$. (The value of c is approximately -0.203058.)

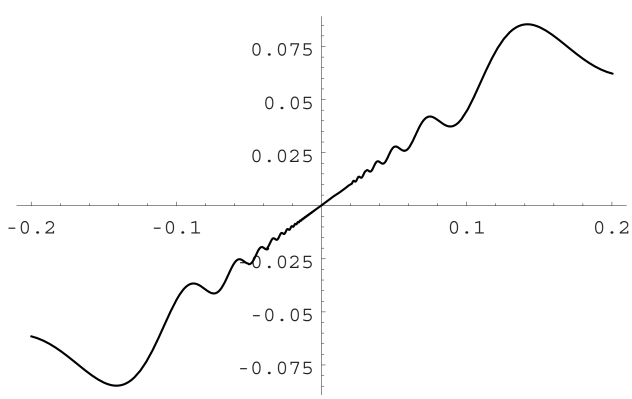
C04S03.059: Use the definition of the derivative:

$$\begin{aligned} g'(0) &= \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} \\ &= \lim_{h \to 0} \left[\frac{1}{2} + \frac{1}{h} h^2 \sin\left(\frac{1}{h}\right) \right] \\ &= \lim_{h \to 0} \left[\frac{1}{2} + h \sin\left(\frac{1}{h}\right) \right] \\ &= \frac{1}{2} + 0 \qquad \text{(by the squeeze law)} \\ &= \frac{1}{2} > 0. \end{aligned}$$

If $x \neq 0$ then

$$g'(x) = \frac{1}{2} + 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Because $\cos(1/x)$ oscillates between $+1$ and -1 near $x = 0$ and $2x\sin(1/x)$ is near zero for x close to zero, it follows that every interval about $x = 0$ contains subintervals on which $g'(x) > 0$ and subintervals on which $g'(x) < 0$. They are not clearly visible near $x = 0$ in the graph of $y = g(x)$ (shown next) because they are very short intervals.



C04S03.060: To prove a mathematical result it is frequently very helpful to restate exactly what it is that you must prove. In this case, to show that f is increasing on the unbounded open interval $(2, +\infty)$, we need to show that if $2 < x_1 < x_2$, then $f(x_1) < f(x_2)$. Suppose that $2 < x_1 < x_2$. Let

$$a = \frac{2 + x_1}{2} \qquad \text{and} \qquad b = 1 + x_2.$$

Then $[a, b]$ is a closed interval, and by hypothesis f is increasing there. Moreover, $a < x_1$ (because a is the midpoint of the interval $(2, x_1)$), $x_1 < x_2$, and $x_2 < b$. So x_1 and x_2 are two numbers in $[a, b]$ for which $x_1 < x_2$. Therefore $f(x_1) < f(x_2)$. This is what we have agreed that it means for f to be increasing on the unbounded interval $(2, +\infty)$. This concludes the proof.

C04S03.061: Let $h(x) = 1 - \frac{1}{2}x^2 - \cos x$. Then $h'(x) = -x + \sin x$. By Example 8, $\sin x < x$ for all $x > 0$, so $h'(x) < 0$ for all $x > 0$. If $x > 0$, then $\dfrac{h(x) - 0}{x - 0} = h'(c)$ for some $c > 0$, so $h(x) < 0$ for all $x > 0$; that is, $\cos x > 1 - \frac{1}{2}x^2$ for all $x > 0$.

C04S03.062: (a): Let $j(x) = \sin x - x + \frac{1}{6}x^3$. Then

$$j'(x) = \cos x - 1 + \frac{1}{2}x^2.$$

By the result in Problem 61, $j'(x) > 0$ for all $x > 0$. Also, if $x > 0$, then $\dfrac{j(x) - 0}{x - 0} = j'(c)$ for some $c > 0$. Hence $j(x) > 0$ for all $x > 0$; that is, $\sin x > x - \frac{1}{6}x^3$ for all $x > 0$.

(b) By part (a) and Example 8 of the text,

$$x - \frac{1}{6}x^3 < \sin x < x$$

for all $x > 0$. So

$$\frac{\pi}{36} - \frac{1}{6}\left(\frac{\pi}{36}\right)^3 < \sin\frac{\pi}{36} < \frac{\pi}{36};$$
$$0.0871557 < \sin 5^\circ < 0.0872665;$$
$$\sin 5^\circ \approx 0.087.$$

C04S03.063: (a): Let $K(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cos x$. Then

$$K'(x) = -x + \frac{1}{6}x^3 + \sin x = \sin x - (x - \frac{1}{6}x^3).$$

By Problem 62, part (a), $K'(x) > 0$ for all $x > 0$. So if $x > 0$, $\dfrac{K(x) - 0}{x - 0} = K'(c)$ for some $c > 0$. Therefore $K(x) > 0$ for all $x > 0$. That is,

$$\cos x < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \quad \text{for all} \quad x > 0.$$

(b): By Problem 61 and part (a),

$$1 - \frac{1}{2}x^2 < \cos x < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

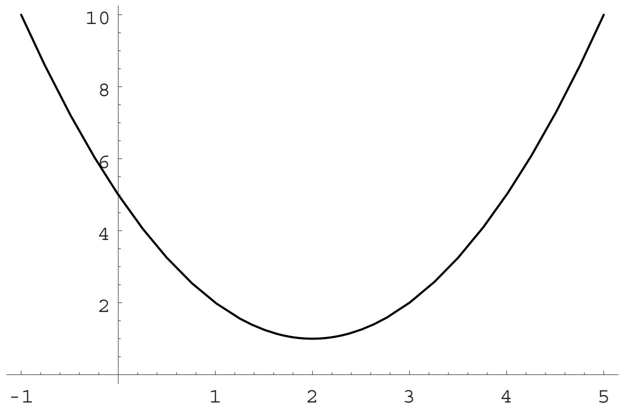
for all $x > 0$. In particular,

$$1-\frac{1}{2}\left(\frac{\pi}{18}\right)^2<\cos\frac{\pi}{18}<1-\frac{1}{2}\left(\frac{\pi}{18}\right)^2+\frac{1}{24}\left(\frac{\pi}{18}\right)^4;$$

hence $0.984769<\cos 10^\circ<0.984808$. So $\cos 10^\circ\approx 0.985$.

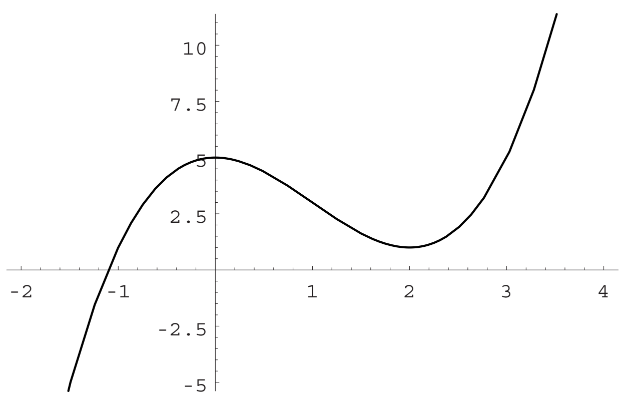
Section 4.4

C04S04.001: $f'(x) = 2x - 4$; $x = 2$ is the only critical point. Because $f'(x) > 0$ for $x > 2$ and $f'(x) < 0$ for $x < 2$, it follows that $f(2) = 1$ is the global minimum value of $f(x)$. The graph of $y = f(x)$ is shown next.



C04S04.002: $f'(x) = 6 - 2x$, so $x = 3$ is the only critical point. If $x < 3$ then f is increasing, whereas f is decreasing for $x > 3$, so $f(3) = 9$ is the global maximum value of $f(x)$.

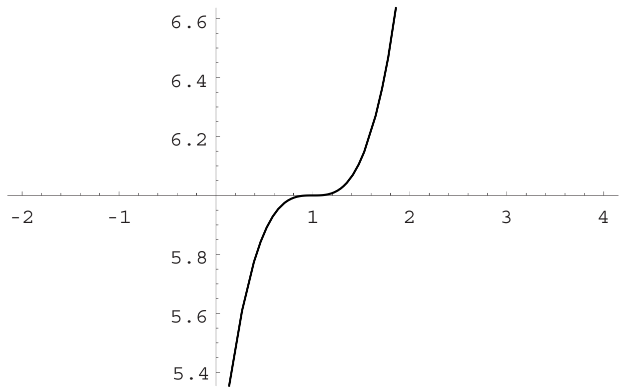
C04S04.003: $f'(x) = 3x^2 - 6x = 3x(x - 2)$, so $x = 0$ and $x = 2$ are the only critical points. If $x < 0$ or if $x > 2$ then $f'(x)$ is positive, but $f'(x) < 0$ for $0 < x < 2$. So $f(0) = 5$ is a local maximum and $f(2) = 1$ is a local minimum. The graph of $y = f(x)$ is shown next.



C04S04.004: $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$, so $x = 1$ and $x = -1$ are the only critical points. If $x < -1$ or if $x > 1$, then $f'(x) > 0$, whereas $f'(x) < 0$ on $(-1, 1)$. So $f(-1) = 7$ is a local maximum value and $f(1) = 3$ is a local minimum value.

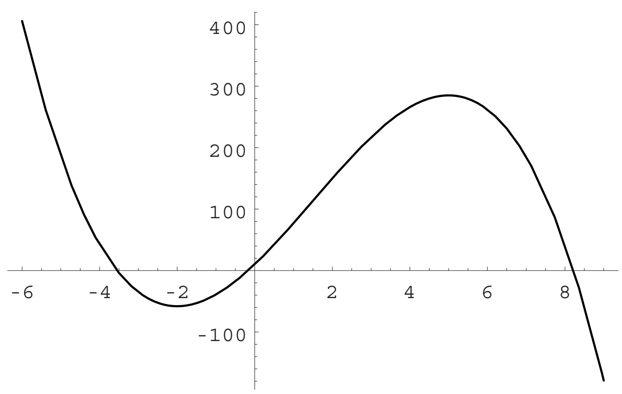
C04S04.005: $f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$, so $x = 1$ is the only critical point of f. $f'(x) > 0$ if $x \neq 1$, so the graph of f is increasing for *all* x; so f has no extrema of any sort. The graph of $y = f(x)$ is shown

next.



C04S04.006: $f'(x) = 6x^2 + 6x - 36 = 6(x+3)(x-2)$, so $x = -3$ and $x = 2$ are the only critical points. If $x < -3$ or if $x > 2$ then $f'(x) > 0$, but $f'(x) < 0$ on the interval $(-3, 2)$. So $f(-3) = 98$ is a local maximum value of $f(x)$ and $f(2) = -27$ is a local minimum value.

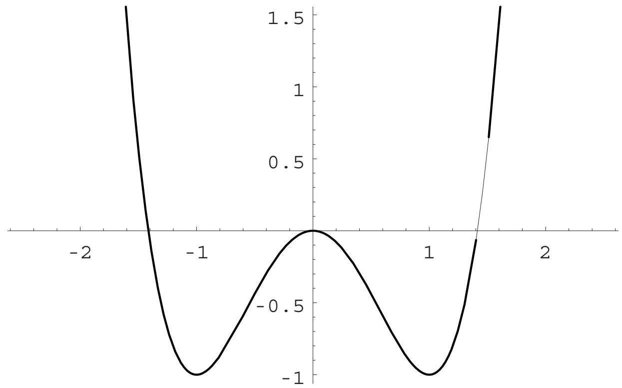
C04S04.007: $f'(x) = -6(x-5)(x+2)$; $f'(x) < 0$ if $x < -2$ and if $x > 5$, but $f'(x) > 0$ for $-2 < x < 5$. Hence $f(-2) = -58$ is a local minimum value of f and $f(5) = 285$ is a local maximum value. The graph of $y = f(x)$ is shown next.



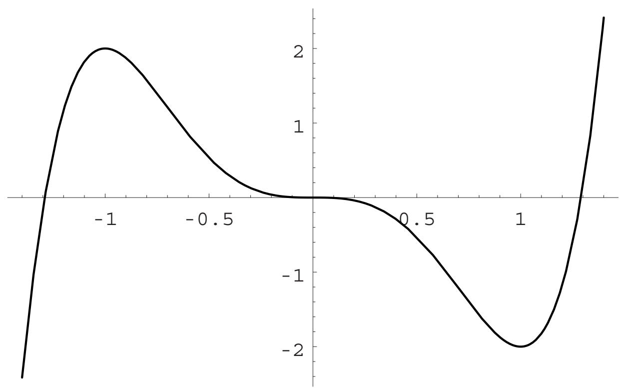
C04S04.008: $f'(x) = -3x^2$, so $x = 0$ is the only critical point. But $f'(x) < 0$ if $x \neq 0$, so f is decreasing everywhere. Therefore there are no extrema.

C04S04.009: $f'(x) = 4x(x-1)(x+1)$; $f'(x) < 0$ for $x < -1$ and on the interval $(0, 1)$, whereas $f'(x) > 0$ for $x > 1$ and on the interval $(-1, 0)$. Consequently, $f(-1) = -1 = f(1)$ is the global minimum value of $f(x)$ and $f(0) = 0$ is a local maximum value. Note that the [unique] global minimum *value* occurs at two

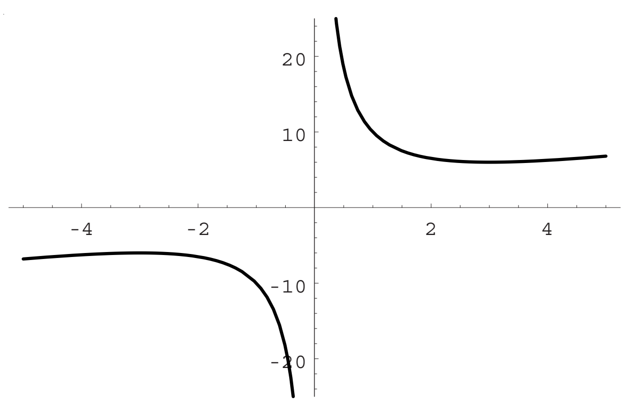
different *points* on the graph of f, which is shown next.



C04S04.010: $f'(x) = 15x^2(x+1)(x-1)$, so $f'(x) > 0$ if $x < -1$ and if $x > 1$, but $f'(x) < 0$ on $(-1, 0)$ and on $(0, 1)$. Therefore $f(0) = 0$ is not an extremum of $f(x)$, but $f(-1) = 2$ is a local maximum value and $f(1) = -2$ is a local minimum value. The graph of $y = f(x)$ is shown next.



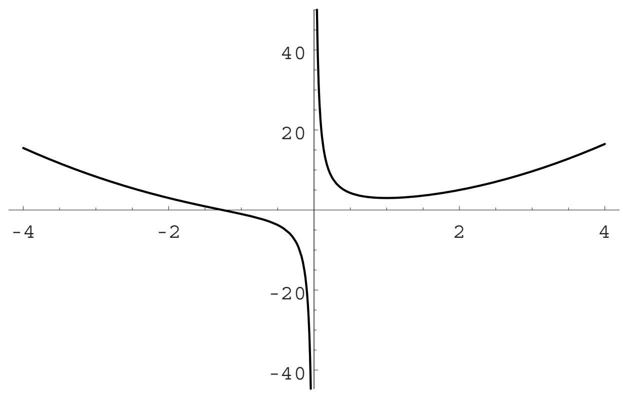
C04S04.011: $f'(x) = 1 - 9x^{-2}$, so the critical points occur where $x = -3$ and $x = 3$ (horizontal tangents); note that f is not defined at $x = 0$. If $x^2 > 9$ then $f'(x) > 0$, so f is increasing if $x > 3$ and if $x < -3$. If $x^2 < 9$ then $f'(x) < 0$, so f is decreasing on $(-3, 0)$ and on $(0, 3)$. Therefore $f(-3) = -6$ is a local maximum value and $f(3) = 6$ is a local minimum value for $f(x)$. The graph of $y = f(x)$ is next.



C04S04.012: Here,

$$f'(x) = 2x - \frac{2}{x^2} = \frac{2(x^3 - 1)}{x^2} = \frac{2(x-1)(x^2+x+1)}{x^2}.$$

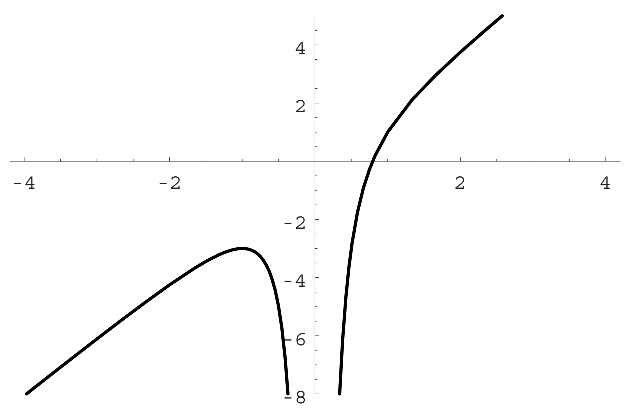
Because $x^2 + x + 1 > 0$ for all x, the only critical point is $x = 1$; note that f is not defined at $x = 0$. Also $f'(x)$ has the sign of $x - 1$, so $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $0 < x < 1$ and for $x < 0$. Consequently $f(1) = 3$ is a local minimum value of $f(x)$. It is not a global minimum because $f(x) \to -\infty$ as $x \to 0^-$. The graph of $y = f(x)$ is shown next.



C04S04.013: If $f(x) = 2x - \dfrac{1}{x^2}$, then

$$f'(x) = 2 + \frac{2}{x^3} = \frac{2(x^3+1)}{x^3} = \frac{2(x+1)(x^2-x+1)}{x^3}.$$

Hence the graph of f is increasing if $x < -1$ and if $0 < x$; the graph is decreasing if $-1 < x < 0$. Thus there is a local maximum where $x = -1$. Because $f(x) \to +\infty$ as $x \to +\infty$, the local maximum at $(-1, -3)$ is not global. There are no other extrema. The graph of f is next.

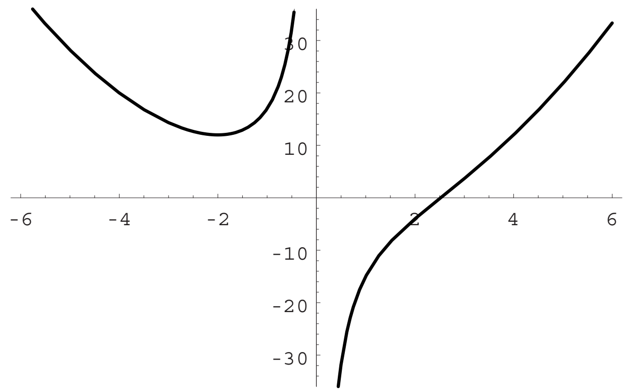


C04S04.014: If $f(x) = x^2 - \dfrac{16}{x}$, then

$$f'(x) = 2x + \frac{16}{x^2} = \frac{2(x^3+8)}{x^2} = \frac{2(x+2)(x^2-2x+4)}{x^2}.$$

Hence the graph of f is decreasing if $x < -2$ but increasing if $-2 < x < 0$ and if $0 < x$. Thus there is a local minimum at $x = -2$. Because $f(x) \to -\infty$ as $x \to 0^+$, the local minimum at $(-2, 12)$ is not global.

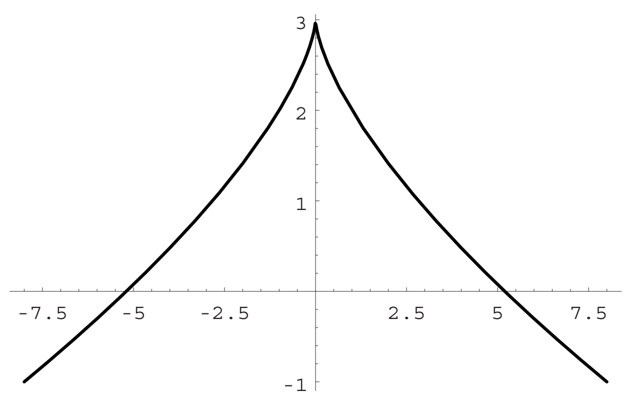
There are no other extrema. The graph of f is next.



C04S04.015: If $f(x) = 3 - x^{2/3}$, then

$$f'(x) = -\frac{2}{3}x^{-1/3} = -\frac{2}{3x^{1/3}}.$$

Hence the graph of f is increasing if $x < 0$ and decreasing if $x > 0$. Because f is continuous at $x = 0$ (indeed, f is continuous everywhere), there is a global maximum at the point $(0, 3)$. The graph of f is next.

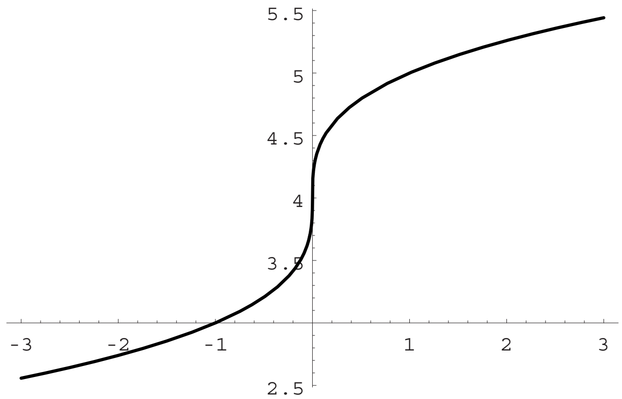


C04S04.016: If $f(x) = 4 + x^{1/3}$, then

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$

Hence the graph of f is increasing if $x < 0$ and if $x > 0$. Because f is continuous everywhere, including $x = 0$, it is simpler and correct to state that f is increasing everywhere. Consequently f has no extrema of

any kind. The graph of f is next.



C04S04.017: $f'(x) = 2\sin x\cos x$; $f'(x) = 0$ when x is any integral multiple of $\pi/2$. In $(0, 3)$, $f'(x) = 0$ when $x = \pi/2$. Because $f'(x) > 0$ if $0 < x < \pi/2$ and $f'(x) < 0$ if $\pi/2 < x < 3$, $f(x)$ has the global maximum value $f(\pi/2) = 1$.

C04S04.018: $f'(x) = -2\sin x\cos x$; $f'(x) = 0$ when $x = 0$ and when $x = \pi/2$. f is increasing on $(-1, 0)$ and on $(\pi/2, 3)$, whereas f is decreasing on $(0, \pi/2)$. So f has a global maximum at $(0, 1)$ and a global minimum at $(\pi/2, 0)$.

C04S04.019: $f'(x) = 3\sin^2 x\cos x$; $f'(x) = 0$ when $x = -\pi/2$, 0, or $\pi/2$. f is decreasing on $(-3, -\pi/2)$ and on $(\pi/2, 3)$, but increasing on $(-\pi/2, \pi/2)$. So f has a global minimum at $(-\pi/2, -1)$ and a global maximum at $(\pi/2, 1)$; there is no extremum at the critical point $(0, 0)$.

C04S04.020: $f'(x) = -4\cos^3 x\sin x$ vanishes at $\pi/2$ and at π. f is decreasing on $(0, \pi/2)$ and on $(\pi, 4)$, but increasing on $(\pi/2, \pi)$. Hence there is a global minimum at $(\pi/2, 0)$ and a global maximum at $(\pi, 1)$.

C04S04.021: $f'(x) = x\sin x$; $f'(x) = 0$ at $-\pi$, 0, and π. $f'(x) > 0$ on $(-\pi, \pi)$, $f'(x) < 0$ on $(-5, -\pi)$ and on $(\pi, 5)$. So f has a global maximum at (π, π) and a global minimum at $(-\pi, -\pi)$. Note that the critical point $(0, 0)$ is not an extremum.

C04S04.022: Given: $f(x) = \cos x + x\sin x$ on $I = (-5, 5)$. First, $f'(x) = x\cos x$, so $f'(x) = 0$ (for x in I) when

$$x = 0, \quad x = \pm\frac{\pi}{2}, \quad x = \pm\frac{3\pi}{2}.$$

Hence $f'(x) < 0$ on $(-5, -3\pi/2)$, $(-\pi/2, 0)$, and $(\pi/2, 3\pi/2)$; $f'(x) > 0$ on $(-3\pi/2, -\pi/2)$, $(0, \pi/2)$, and on $(3\pi/2, 5)$. Therefore there are global minima at $(-3\pi/2, -3\pi/2)$ and $(3\pi/2, -3\pi/2)$, global maxima at $(-\pi/2, \pi/2)$ and $(\pi/2, \pi/2)$, and a local minimum at $(0, 1)$. The global minima are global rather than local because $f(\pm 5) \approx -4.510959$.

C04S04.023: If $f(x) = \tan^2 x = (\tan x)^2$, $-1 < x < 1$, then $f'(x) = 2\sec^2 x\tan x$, so the only critical point of f is $x = 0$. Because $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, the graph of f has a global minimum at $(0, 0)$. There are no other critical points in the given domain.

C04S04.024: If $f(x) = \tan^3 x$, $-1 < x < 1$, then $f'(x) = 3\tan^2 x\sec^2 x$, so the only critical point of f in the given domain occurs where $x = 0$. But $f'(x) > 0$ if $-1 < x < 0$ and if $0 < x < 1$, so the graph of f is increasing on $(-1, 1)$. Hence f has no extremum at $(0, 0)$.

C04S04.025: If $f(x) = 2\tan x - \tan^2 x$, then

$$f'(x) = 2\sec^2 x - 2\sec^2 x\,\tan x = 2(1-\tan x)\sec^2 x.$$

Hence $f'(x) = 0$ when $x = \pi/4$; $f'(x) > 0$ if $0 < x < \pi/4$ and $f'(x) < 0$ if $\pi/4 < x < 1$. Therefore the graph of $y = f(x)$ has a global maximum at $(\pi/4, 1)$ and there are no other extrema.

C04S04.026: Given: $f(x) = (1-2\sin x)^2$ for $0 < x < 2$. Then $f'(x) = -4(1-2\sin x)\cos x$, so $f'(x) = 0$ when $x = \pi/6$ and when $x = \pi/2$. Also $f'(x) < 0$ if $0 < x < \pi/6$ and if $\pi/2 < x < 2$, whereas $f'(x) > 0$ if $\pi/6 < x < \pi/2$. Hence there is a local minimum where $x = \pi/6$ and a local maximum where $x = \pi/2$. Because $f(\pi/6) = 0$ and $f(x) > 0$ if $x \neq \pi/6$, the graph of f has a global minimum at $(\pi/6, 0)$. Because $f(\pi/2) = 1$ and $f(x) < 1$ if $x \neq \pi/2$, the graph of f has a global maximum at $(\pi/2, 1)$.

C04S04.027: Let x be the smaller of the two numbers; then the other is $x + 20$ and their product is $f(x) = x^2 + 20x$. Consequently $f'(x) = 2x + 20$, so $x = -10$ is the only critical point of f. The graph of f is decreasing for $x < -10$ and increasing for $x > -10$. Therefore $(-10, -100)$ is the lowest point on the graph of f. Answer: The two numbers are -10 and 10.

C04S04.028: We assume that the length turned upward is the same on each side—call it y. If the width of the gutter is x, then we have the constraint $xy = 18$, and we are to minimize the width $x + 2y$ of the strip. Its width is given by the function

$$f(x) = x + \frac{36}{x}, \qquad x > 0,$$

for which

$$f'(x) = 1 - \frac{36}{x^2}.$$

The only critical point in the domain of f is $x = 6$, and if $0 < x < 6$ then $f'(x) < 0$, whereas $f'(x) > 0$ for $x > 6$. Thus $x = 6$ yields the global minimum value $f(6) = 12$ of the function f. Answer: The minimum possible width of the strip is 12 inches.

C04S04.029: Let us minimize

$$g(x) = (x-3)^2 + (3-2x-2)^2 = (x-3)^2 + (1-2x)^2,$$

the square of the distance from (x, y) on the line $2x + y = 3$ to the point $(3, 2)$. We have

$$g'(x) = 2(x-3) - 4(1-2x) = 10x - 10,$$

so $x = 1$ is the only critical point of $g(x)$. If $x > 1$ then $g'(x) > 0$, but $g'(x) < 0$ for $x < 1$. Thus $x = 1$ minimizes $g(x)$, and so the point on the line $2x+y = 3$ closest to the point $(3, 2)$ is $(1, 1)$. As an independent check, note that the slope of the line segment joining $(3, 2)$ and $(1, 1)$ is $\frac{1}{2}$, whereas the slope of the line $2x + y = 3$ is -2, so the segment and the line are perpendicular; see Miscellaneous Problem 70 of Chapter 3.

C04S04.030: Base of box: x wide, $2x$ long. Height: y. Then the box has volume $2x^2y = 576$, so $y = 288x^{-2}$. Its total surface area is $A = 4x^2 + 6xy$, so we minimize

$$A = A(x) = 4x^2 + \frac{1728}{x}, \qquad x > 0.$$

Now

$$A'(x) = 8x - \frac{1728}{x^2},$$

so the only critical point of $A(x)$ occurs when $8x^3 = 1728$; that is, when $x = 6$. It is easy to verify that $A'(x) < 0$ for $0 < x < 6$ and $A'(x) > 0$ for $x > 6$. Therefore $A(6)$ is the global minimum value of $A(x)$. Also, when $x = 6$ we have $y = 8$. Answer: The dimensions of the box of minimal surface area are 6 inches wide by 12 inches long by 8 inches high.

C04S04.031: Base of box: x wide, $2x$ long. Height: y. Then the box has volume $2x^2y = 972$, so $y = 486x^{-2}$. Its total surface area is $A = 2x^2 + 6xy$, so we minimize

$$A = A(x) = 2x^2 + \frac{2916}{x}, \qquad x > 0.$$

Now

$$A'(x) = 4x - \frac{2916}{x^2},$$

so the only critical point of $A(x)$ occurs when $4x^3 = 2916$; that is, when $x = 9$. It is easy to verify that $A'(x) < 0$ for $0 < x < 9$ and that $A'(x) > 0$ for $x > 9$. Therefore $A(9)$ is the global minimum value of $A(x)$. Answer: The dimensions of the box are 9 inches wide, 18 inches long, and 6 inches high.

C04S04.032: If the radius of the base of the pot is r and its height is h (inches), then we are to minimize the total surface area A given the constraint $\pi r^2 h = 125$. Thus $h = 125/(\pi r^2)$, and so

$$A = \pi r^2 + 2\pi rh = A(r) = \pi r^2 + \frac{250}{r}, \qquad r > 0.$$

Now

$$A'(r) = 2\pi r - \frac{250}{r^2};$$

$A'(r) = 0$ when $r^3 = 125/\pi$, so that $r = 5/\sqrt[3]{\pi}$. The latter point is the only value of r at which $A'(r)$ can change sign (for $r > 0$), and it is easy to see that $A'(r)$ is positive when r is large positive, whereas $A'(r)$ is negative when r is near zero. Therefore we have located the global minimum of $A(r)$, and it occurs when the pot has radius $r = 5/\sqrt[3]{\pi}$ inches and height $h = 5/\sqrt[3]{\pi}$ inches. Thus the pot will have its radius equal to its height, each approximately 3.414 inches.

C04S04.033: Let r denote the radius of the pot and h its height. We are given the constraint $\pi r^2 h = 250$, so $h = 250/(\pi r^2)$. Now the bottom of the pot has area πr^2, and thus costs $4\pi r^2$ cents. The curved side of the pot has area $2\pi rh$, and thus costs $4\pi rh$ cents. So the total cost of the pot is

$$C = 4\pi r^2 + 4\pi rh; \qquad \text{thus} \qquad C = C(r) = 4\pi r^2 + \frac{1000}{r}, \qquad r > 0.$$

Now

$$C'(r) = 8\pi r - \frac{1000}{r^2};$$

$C'(r) = 0$ when $8\pi r^3 = 1000$, so that $r = 5/\sqrt[3]{\pi}$. It is clear that this is the only (positive) value of r at which $C'(r)$ can change sign, and that $C'(r) < 0$ for r positive and near zero, but $C'(r) > 0$ for r large positive. Therefore we have found the value of r that minimizes $C(r)$. The corresponding value of h is $10/\sqrt[3]{\pi}$, so the pot of minimal cost has height equal to its diameter, each approximately 6.828 centimeters.

C04S04.034: If $(x, y) = (x, 4 - x^2)$ is a point on the parabola $y = 4 - x^2$, then the square of its distance from the point $(3, 4)$ is

$$h(x) = (x-3)^2 + (4 - x^2 - 4)^2 = (x-3)^2 + x^4.$$

We minimize the distance by minimizing its square:

$$h'(x) = 2(x-3) + 4x^3;$$

$h'(x) = 0$ when $2x^3 + x - 3 = 0$. It is clear that $h'(1) = 0$, so $x - 1$ is a factor of $h'(x)$; $h'(x) = 0$ is equivalent to $(x-1)(2x^2 + 2x + 3) = 0$. The quadratic factor in the last equation is always positive, so $x = 1$ is the only critical point of $h(x)$. Also $h'(x) < 0$ if $x < 1$, whereas $h'(x) > 0$ for $x > 1$, so $x = 1$ yields the global minimum value $h(1) = 5$ for $h(x)$. When $x = 1$ we have $y = 3$, so the point on the parabola $y = 4 - x^2$ closest to $(3, 4)$ is $(1, 3)$, at distance $\sqrt{5}$ from it.

C04S04.035: If the sides of the rectangle are x and y, then $xy = 100$, so that $y = \dfrac{100}{x}$. Therefore the perimeter of the rectangle is

$$P = P(x) = 2x + \frac{200}{x}, \qquad x > 0.$$

Then

$$P'(x) = 2 - \frac{200}{x^2};$$

$P'(x) = 0$ when $x = 10$ (-10 is not in the domain of $P(x)$). Now $P'(x) < 0$ on $(0, 10)$ and $P'(x) > 0$ for $x > 10$, and so $x = 10$ minimizes $P(x)$. A little thought about the behavior of $P(x)$ for x near zero and for x large makes it clear that we have found the global minimum value for P: $P(10) = 40$. When $x = 10$, also $y = 10$, so the rectangle of minimal perimeter is indeed a square.

C04S04.036: Let x denote the length of each side of the square base of the solid and let y denote its height. Then its total volume is $x^2y = 1000$. We are to minimize its total surface area $A = 2x^2 + 4xy$. Now $y = \dfrac{1000}{x^2}$, so

$$A = A(x) = 2x^2 + \frac{4000}{x}, \qquad x > 0.$$

Therefore

$$\frac{dA}{dx} = 4x - \frac{4000}{x^2}.$$

The derivative is zero when $4x^3 = 4000$; that is, when $x = 10$. Also $A(x)$ is decreasing on $(0, 10)$ and increasing for $x > 10$. So $x = 10$ yields the global minimum value of $A(x)$. In this case, $y = 10$ as well, so the solid is indeed a cube.

C04S04.037: Let the square base of the box have edge length x and let its height be y, so that its total volume is $x^2y = 62.5$ and the surface area of this box-without-top will be $A = x^2 + 4xy$. So

$$A = A(x) = x^2 + \frac{250}{x}, \qquad x > 0.$$

Now

$$A'(x) = 2x - \frac{250}{x^2},$$

so $A'(x) = 0$ when $x^3 = 125$: $x = 5$. In this case, $y = 2.5$. Also $A'(x) < 0$ if $0 < x < 5$ and $A'(x) > 0$ if $x > 5$, so we have found the global minimum for $A(x)$. Answer: Square base of edge length 5 inches, height 2.5 inches.

C04S04.038: Let r denote the radius of the can and h its height (in centimeters). We are to minimize its total surface area $A = 2\pi r^2 + 2\pi rh$ given the constraint $\pi r^2 h = V = 16\pi$. First we note that $h = V/(\pi r^2)$, so we minimize

$$A = A(r) = 2\pi r^2 + \frac{2V}{r}, \qquad r > 0.$$

Now

$$A'(r) = 4\pi r - \frac{2V}{r^2};$$

$A'(r) = 0$ when $4\pi r^3 = 2V = 32\pi$—that is, when $r = 2$. Now $A(r)$ is decreasing on $(0, 2)$ and increasing for $r > 2$, so the global minimum of $A(r)$ occurs when $r = 2$, for which $h = 4$.

C04S04.039: Let x denote the radius and y the height of the cylinder (in inches). Then its cost (in cents) is $C = 8\pi x^2 + 4\pi xy$, and we also have the constraint $\pi x^2 y = 100$. So

$$C = C(x) = 8\pi x^2 + \frac{400}{x}, \qquad x > 0.$$

Now $dC/dx = 16\pi x - 400/(x^2)$; $dC/dx = 0$ when $x = (25/\pi)^{1/3}$ (about 1.9965 inches) and consequently, when $y = (1600/\pi)^{1/3}$ (about 7.9859 inches). Because $C'(x) < 0$ if $x^3 < 25/\pi$ and $C'(x) > 0$ if $x^3 > 25/\pi$, we have indeed found the dimensions that minimize the total cost of the can. For simplicity, note that $y = 4x$ at minimum: The height of the can is twice its diameter.

C04S04.040: If the print width is x and its height is y (in inches), then the page area is $A = (x+2)(y+4)$. We are to minimize A given $xy = 30$. Because $y = 30/x$,

$$A = A(x) = 4x + 38 + \frac{60}{x}, \quad x > 0.$$

Now

$$A'(x) = 4 - \frac{60}{x^2};$$

$A'(x) = 0$ when $x = \sqrt{15}$. But $A'(x) > 0$ for $x > \sqrt{15}$ whereas $A'(x) < 0$ for $0 < x < \sqrt{15}$. Therefore $x = \sqrt{15}$ yields the global minimum value of $A(x)$, which is $38 + 8\sqrt{15}$, approximately 68.98 square inches.

C04S04.041: Let $(x, y) = (x,\, x^2)$ denote an arbitrary point on the curve. The square of its distance from $(0,\, 2)$ is then

$$f(x) = x^2 + (x^2 - 2)^2.$$

Now $f'(x) = 2x(2x^2 - 3)$, and therefore $f'(x) = 0$ when $x = 0$, when $x = -\sqrt{3/2}$, and when $x = +\sqrt{3/2}$. Now $f'(x) < 0$ if $x < -\sqrt{3/2}$ and if $0 < x < \sqrt{3/2}$; $f'(x) > 0$ if $-\sqrt{3/2} < x < 0$ and if $x > \sqrt{3/2}$. Therefore

$x = 0$ yields a local maximum for f; the other two zeros of $f'(x)$ yield its global minimum. Answer: There are exactly two points on the curve that are nearest $(0, 2)$; they are $(+\sqrt{3/2}, 3/2)$ and $(-\sqrt{3/2}, 3/2)$.

C04S04.042: Let $(a, 0)$ and $(0, b)$ denote the endpoints of the segment and denote its point of tangency by $(c, 1/c)$. The segment then has slope $-1/c^2$, and therefore

$$\frac{b-0}{0-a} = -\frac{1}{c^2} = \frac{(1/c)-0}{c-a}.$$

It follows that $b = a/c^2$ and that $c = a - c$, so $a = 2c$ and $b = 2/c$. The square of the length of the segment is then

$$f(c) = 4c^2 + \frac{4}{c^2}, \qquad c > 0.$$

Now

$$f'(c) = 8c - \frac{8}{c^3};$$

$f'(c) = 0$ when $c = -1$ and when $c = 1$. We reject the negative solution. On the interval $(0, 1)$, f is decreasing; f is increasing for $c > 1$. Therefore $c = 1$ gives the segment of minimal length, which is $L = \sqrt{f(1)} = 2\sqrt{2}$.

C04S04.043: If the dimensions of the rectangle are x by y, and the line segment bisects the side of length x, then the square of the length of the segment is

$$f(x) = \left(\frac{x}{2}\right)^2 + y^2 = \frac{x^2}{4} + \frac{4096}{x^2}, \qquad x > 0,$$

because $y = 64/x$. Now

$$f'(x) = \frac{x}{2} - \frac{8192}{x^3}.$$

When $f'(x) = 0$, we must have $x = +8\sqrt{2}$, so that $y = 4\sqrt{2}$. We have found the minimum of f because if $0 < x < 8\sqrt{2}$ then $f'(x) < 0$, and $f'(x) > 0$ if $x > 8\sqrt{2}$. The minimum length satisfies $L^2 = f\left(8\sqrt{2}\right)$, so that $L = 8$ centimeters.

C04S04.044: Let y be the height of the cylindrical part and x the length of the radii of both the cylinder and the hemisphere. The total surface area is

$$A = \pi x^2 + 2\pi xy + 2\pi x^2 = 3\pi x^2 + 2\pi xy.$$

But the can must have volume $V = \pi x^2 y + \frac{2}{3}\pi x^3$, so

$$y = \frac{1000 - \frac{2}{3}\pi x^3}{\pi x^2}.$$

Therefore

$$A = A(x) = \frac{5}{3}\pi x^2 + \frac{2000}{x}, \qquad x > 0.$$

Thus

$$\frac{dA}{dx} = \frac{10}{3}\pi x - \frac{2000}{x^2}.$$

Now $dA/dx = 0$ when $x = (600/\pi)^{1/3} \approx 5.7588$. Because $dA/dx < 0$ for smaller values of x and $dA/dx > 0$ for larger values, we have found the point at which $A(x)$ attains its global minimum value. After a little arithmetic, we find that $y = x$, so the radius of the hemisphere and the radius and height of the cylinder should all be equal to $(600/\pi)^{1/3}$ to attain minimal surface area.

This argument contains the implicit assumption that $y > 0$. If $y = 0$, then

$$x = (1500/\pi)^{1/3} \approx 7.8159, \text{ for which}$$
$$A = (150)(18\pi)^{1/3} \approx 575.747 \text{ cubic inches.}$$

But with $x = y = (600/\pi)^{1/3}$, we have

$$A = (100)(45\pi)^{1/3} \approx 520.940 \text{ cubic inches.}$$

So the solution in the first paragraph indeed yields the dimensions of the can requiring the least amount of material.

C04S04.045: If the end of the rod projects the distance y into the narrower hall, then we have the proportion $y/2 = 4/x$ by similar triangles. So $y = 8/x$. The square of the length of the rod is then

$$f(x) = (x+2)^2 + \left(4 + \frac{8}{x}\right)^2, \qquad x > 0.$$

It follows that

$$f'(x) = 4 + 2x - \frac{64}{x^2} - \frac{128}{x^3},$$

and that $f'(x) = 0$ when $(x+2)(x^3 - 32) = 0$. The only admissible solution is $x = \sqrt[3]{32}$, which indeed minimizes $f(x)$ by the usual argument ($f(x)$ is very large positive if x is either large positive or positive and very close to zero). The minimum length is

$$L = \left(20 + 12\sqrt[3]{4} + 12\sqrt[3]{16}\right)^{1/2} \approx 8.323876 \text{ (meters).}$$

C04S04.046: By similar triangles, $y/1 = 8/x$, and

$$L_1 + L_2 = L = \left[(x+1)^2 + (y+8)^2\right]^{1/2}.$$

We minimize L by minimizing

$$f(x) = L^2 = (x+1)^2 + \left(8 + \frac{8}{x}\right)^2, \qquad x > 0.$$

$$f'(x) = 2 + 2x - \frac{128}{x^3} - \frac{128}{x^2};$$

$f'(x) = 0$ when $2x^3 + 2x^4 - 128 - 128x = 0$, which leads to the equation

$$(x+1)(x-4)(x^2 + 4x + 16) = 0.$$

The only relevant solution is $x = 4$. Because $f'(x) < 0$ for x in the interval $(-1, 4)$ and $f'(x) > 0$ if $x > 4$, we have indeed found the global minimum of f. The corresponding value of y is 2, and the length of the shortest ladder is $L = 5\sqrt{5}$ feet, approximately 11 ft 2 in.

C04S04.047: If the pyramid has base edge length x and altitude y, then its volume is $V = \frac{1}{3}x^2y$. From Fig. 4.4.30 we see also that

$$\frac{2y}{x} = \tan\theta \quad \text{and} \quad \frac{a}{y-a} = \cos\theta$$

where θ is the angle that each side of the pyramid makes with its base. It follows, successively, that

$$\begin{aligned}
&\left(\frac{a}{y-a}\right)^2 = \cos^2\theta;\\
&\sin^2\theta = 1 - \cos^2\theta = \frac{(y-a)^2 - a^2}{(y-a)^2}\\
&\qquad\quad = \frac{y^2 - 2ay}{(y-a)^2} = \frac{y(y-2a)}{(y-a)^2}.\\
&\sin\theta = \frac{\big(y(y-2a)\big)^{1/2}}{y-a}.\\
&y = \frac{x}{2}\frac{\sin\theta}{\cos\theta} = \left(\frac{x}{2}\right)\left(\frac{\big(y(y-2a)\big)^{1/2}}{y-a}\right)\left(\frac{y-a}{a}\right);\\
&2y = \frac{x}{a}\sqrt{y(y-2a)}\,;\\
&x^2 = \frac{4a^2y^2}{y(y-2a)}.
\end{aligned}$$

Therefore

$$V = \frac{1}{3}x^2y = V(y) = \frac{4a^2y^2}{3(y-2a)}, \qquad y > 2a.$$

Now

$$\frac{dV}{dy} = \frac{24a^2y(y-2a) - 12a^2y^2}{9(y-2a)^2}.$$

The condition $dV/dy = 0$ then implies that $2(y - 2a) = y$, and thus that $y = 4a$. Consequently the minimum volume of the pyramid is

$$V(4a) = \frac{(4a^2)(16a^2)}{(3)(2a)} = \frac{32}{3}a^3.$$

The ratio of the volume of the smallest pyramid to that of the sphere is then

$$\frac{32/3}{4\pi/3} = \frac{32}{4\pi} = \frac{8}{\pi}.$$

C04S04.048: Let x denote the distance from the noisier of the two discos. Let K be the "noise proportionality" constant. The noise level at x is then

$$N(x) = \frac{4K}{x^2} + \frac{K}{(1000 - x)^2}.$$

$$N'(x) = -\frac{8K}{x^3} + \frac{2K}{(1000 - x)^3}.$$

Now $N'(x) = 0$ when $4(1000 - x)^3 = x^3$; if so, it follows that

$$x = \frac{(1000)4^{1/3}}{1 + 4^{1/3}} \approx 613.512.$$

Because the noise level is very high when x is near zero and when x is near 1000, the last value of x minimizes the noise level—the quietest point is about 613.5 feet from the noisier of the two discos.

C04S04.049: Let z be the length of the segment from the top of the tent to the midpoint of one side of its base. Then $x^2 + y^2 = z^2$. The total surface area of the tent is

$$A = 4x^2 + (4)(\tfrac{1}{2})(2x)(z) = 4x^2 + 4xz = 4x^2 + 4x(x^2 + y^2)^{1/2}.$$

Because the [fixed] volume V of the tent is given by

$$V = \frac{1}{3}(4x^2)(y) = \frac{4}{3}x^2 y,$$

we have $y = 3V/(4x^2)$, so

$$A = A(x) = 4x^2 + \frac{1}{x}(16x^6 + 9V^2)^{1/2}.$$

After simplifications, the condition $dA/dx = 0$ takes the form

$$8x\big(16x^6 + 9V^2\big)^{1/2} - \frac{1}{x^2}\left(16x^6 + 9V^2\right) + 48x^4 = 0,$$

which has solution $x = 2^{-7/6}\sqrt[3]{3V}$. Because this is the only positive solution of the equation, and because it is clear that neither large values of x nor values of x near zero will yield small values of the surface area, this is the desired value of x.

C04S04.050: By similar triangles in Fig. 4.4.28, we have $y/a = b/x$, and thus $y = ab/x$. If L denotes the length of the ladder, then we minimize

$$L^2 = f(x) = (x + a)^2 + (y + b)^2 = (x + a)^2 + b^2\left(1 + \frac{a}{x}\right)^2, \qquad x > 0.$$

Now

$$f'(x) = 2(x + a) + 2b^2\left(1 + \frac{a}{x}\right)\left(-\frac{a}{x^2}\right);$$

$f'(x) = 0$ when $x^3 = ab^2$, so that $x = a^{1/3}b^{2/3}$ and $y = a^{2/3}b^{1/3}$. It's clear that f is differentiable on its domain and that $f(x) \to +\infty$ as $x \to 0^+$ and as $x \to +\infty$. Therefore we have minimized L. With these values of x and y, we find that

$$\begin{aligned}L &= \left(a^2+y^2\right)^{1/2} + \left(x^2+b^2\right)^{1/2} = \left(a^2 + a^{4/3}b^{2/3}\right)^{1/2} + \left(a^{2/3}b^{4/3} + b^2\right)^{1/2}\\ &= a^{2/3}\left(a^{2/3}+b^{2/3}\right)^{1/2} + b^{2/3}\left(a^{2/3}+b^{2/3}\right)^{1/2} = \left(a^{2/3}+b^{2/3}\right)\left(a^{2/3}+b^{2/3}\right)^{1/2}\\ &= \left(a^{2/3}+b^{2/3}\right)^{3/2}.\end{aligned}$$

Note that the answer is dimensionally correct.

C04S04.051: Let x denote the length of each edge of the square base of the box and let y denote the height of the box. Then $x^2y = V$ where V is the fixed volume of the box. The surface total area of this closed box is

$$A = 2x^2 + 4xy, \quad \text{and hence} \quad A(x) = 2x^2 + \frac{4V}{x}, \quad 0 < x < +\infty.$$

Then

$$A'(x) = 4x - \frac{4V}{x^2} = \frac{4(x^3 - V)}{x^2},$$

so $A'(x) = 0$ when $x = V^{1/3}$. This is the only critical point of A, and $A'(x) < 0$ if x is near zero while $A'(x) > 0$ if x is large positive. Thus the global minimum value of the surface area occurs at this critical point. And if $x = V^{1/3}$, then the height of the box is

$$y = \frac{V}{x^2} = \frac{V}{V^{2/3}} = V^{1/3} = x,$$

and hence the closed box with square base, fixed volume, and minimal surface area is a cube.

C04S04.052: Let x denote the length of each edge of the square base of the box and let y denote its height. Then the box has fixed volume $V = x^2y$. The total surface area of the open box is $A = x^2 + 4xy$, and hence

$$A(x) = x^2 + \frac{4V}{x}, \quad 0 < x < +\infty.$$

Next,

$$A'(x) = 2x - \frac{4V}{x^2} = \frac{2(x^3 - 2V)}{x^2}.$$

Thus $A'(x) = 0$ when $x = (2V)^{1/3}$. This critical point yields the global minimum value of $A(x)$ because $A'(x) < 0$ when x is small positive and $A'(x) > 0$ when x is large positive. And at this critical point, we have

$$y = \frac{V}{x^2} = \frac{2V}{2 \cdot (2V)^{2/3}} = \frac{1}{2}(2V)^{1/3} = \frac{1}{2}x.$$

Therefore the box with square base, no top, and fixed volume has minimal surface area when its height is half the edge length of its base.

C04S04.053: Let r denote the radius of the base (and top) of the closed cylindrical can and let h denote its height. Then its fixed volume is $V = \pi r^2 h$ and its total surface area is $A = 2\pi r^2 + 2\pi rh$. Hence

$$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}, \quad 0 < r < +\infty.$$

Then

$$A'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2};$$

$A'(r) = 0$ when $r = (V/2\pi)^{1/3}$. This critical point minimizes total surface area $A(r)$ because $A'(r) < 0$ when r is small positive and $A'(r) > 0$ when r is large positive. And at this critical point we have

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi (V/2\pi)^{2/3}} = \frac{2 \cdot (V/2\pi)}{(V/2\pi)^{2/3}} = 2(V/2\pi)^{1/3} = 2r.$$

Therefore the closed cylindrical can with fixed volume and minimal total surface area has height equal to the diameter of its base.

C04S04.054: Let r denote the radius of the circular base of the cylindrical can and let h denote its height. Then its fixed volume is $V = \pi r^2 h$ and the total surface area of the open cylindrical can is $A = \pi r^2 + 2\pi r h$. Therefore

$$A(r) = \pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = \pi r^2 + \frac{2V}{r}, \quad 0 < r < +\infty.$$

Now

$$A'(r) = 2\pi r - \frac{2V}{r^2} = \frac{2\pi r^3 - 2V}{r^2},$$

so $A'(r) = 0$ when $r = (V/\pi)^{1/3}$. This critical point yields the global minimum value of $A(r)$ because $A'(r) < 0$ when r is small positive and $A'(r) > 0$ when r is large positive. Moreover, at this critical point we have

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi (V/\pi)^{2/3}} = \frac{V/\pi}{(V/\pi)^{2/3}} = (V/\pi)^{1/3} = r.$$

Therefore the open cylindrical can with fixed volume and minimal total surface area has height equal to the radius of its base.

C04S04.055: Finding the exact solution of this problem is quite challenging. In the spirit of mathematical modeling we accept the very good approximation that—if the thickness of the material of the can is small in comparison with its other dimensions—the total volume of material used to make the can may be approximated sufficiently accurately by multiplying the area of the bottom by its thickness, the area of the curved side by its thickness, the area of the top by its thickness, then adding these three products. Thus let r denote the radius of the inside of the cylindrical can, let h denote the height of the inside, and let t denote the thickness of its bottom and curved side; $3t$ will be the thickness of its top. The total (inner) volume of the can is the fixed number $V = \pi r^2 h$. The amount of material to make the can will (approximately, but accurately)

$$M = \pi r^2 t + 2\pi r h t + 3\pi r^2 t = 4\pi r^2 t + 2\pi r h t, \tag{1}$$

so that

$$M(r) = 4\pi r^2 t + 2\pi r t \cdot \frac{V}{\pi r^2} = \left(4\pi r^2 + \frac{2V}{r}\right) \cdot t, \quad 0 < r < L,$$

where L is some rather large positive number that we don't actually need to evaluate. (You can find L from Eq. (1) by setting $h = 0$ there and solving for r in terms of M and t.) Next,

$$M'(r) = \left(8\pi r - \frac{2V}{r^2}\right)\cdot t = \frac{2t(4\pi r^3 - V)}{r^2};$$

$M'(r) = 0$ when $r = (V/4\pi)^{1/3}$. This critical point yields the global minimum value of $M(r)$ because $M'(r) < 0$ when r is small positive and $M'(r) > 0$ when $(V/2\pi)^{1/3} < r < L$. (You need to verify that, under reasonable assumptions about the relative sizes of the linear measurements, $L > (V/4\pi)^{1/3}$.) And at this critical point, we have

$$h = \frac{V}{\pi r^2} = \frac{4\cdot(V/4\pi)}{(V/4\pi)^{2/3}} = 4\cdot(V/4\pi)^{1/3} = 4r.$$

Therefore the pop-top soft drink can of fixed internal volume V, with thickness as described in the problem, and using the minimal total volume of material for its top, bottom, and curved side, will have height (approximately) twice the diameter of its base.

In support of this conclusion, the smallest commonly available pop-top can of a popular blend of eight vegetable juices has height about 9 cm and base diameter about 5 cm. Most of the 12-oz pop-top soft drink cans we measured had height about 12.5 cm and base diameter about 6.5 cm.

C04S04.056: Let each edge of the square base of the box have length x and let y denote the height of the box. Then the fixed volume of the box is $V = x^2y$. The cost of its six faces is then $(2x^2 + 4xy)\cdot a$ and the cost to glue the edges together is $(8x + 4y)\cdot b$. Hence the total cost of material and construction will be

$$C = (2x^2 + 4xy)\cdot a + (8x + 4y)\cdot b.$$

Because $y = V/x^2$, we have

$$C(x) = \left(2x^2 + 4x\cdot\frac{V}{x^2}\right)\cdot a + \left(8x + 4\cdot\frac{V}{x^2}\right)\cdot b = \left(2x^2 + \frac{4V}{x}\right)\cdot a + \left(8x + \frac{4V}{x^2}\right)\cdot b, \quad 0 < x < +\infty.$$

Next,

$$C'(x) = \left(4x - \frac{4V}{x^2}\right)\cdot a + \left(8 - \frac{8V}{x^3}\right)\cdot b = \frac{4ax(x^3 - V) + 8b(x^3 - V)}{x^3} = \frac{4(ax + 2b)(x^3 - V)}{x^3}.$$

Because $x > 0$, the only significant critical point of $C(x)$ occurs when $x = V^{1/3}$. Clearly $C'(x) < 0$ when x is small positive and $C'(x) > 0$ if $x > V^{1/3}$. Therefore we have found the value of x that yields the global minimum value of the cost of the box. The corresponding value of the height of the box is

$$y = \frac{V}{x^2} = \frac{V}{V^{2/3}} = V^{1/3} = x.$$

Therefore the box of Problem 56 of minimal cost is a cube. It is remarkable and quite unexpected that the (positive) values of a and b do not affect the answer.

Section 4.5

C04S05.001: $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$. Matching graph: 4.5.13(c).

C04S05.002: $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$. Matching graph: 4.5.13(a).

C04S05.003: $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$. Matching graph: 4.5.13(d).

C04S05.004: $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$. Matching graph: 4.5.13(b).

C04S05.005: $y'(x) = 4x - 10$, so the only critical point is $\left(\frac{5}{2}, -\frac{39}{2}\right)$, the lowest point on the graph of y because $y'(x) < 0$ if $x < \frac{5}{2}$ and $y'(x) > 0$ if $x > \frac{5}{2}$.

C04S05.006: The only critical point occurs where $x = \frac{3}{2}$. The graph is increasing if $x < \frac{3}{2}$, decreasing if $x > \frac{3}{2}$.

C04S05.007: $y'(x) = 12x^2 - 6x - 90$ is zero when $x = -\frac{5}{2}$ and when $x = 3$. The graph of y is increasing if $x < -\frac{5}{2}$, decreasing if $-\frac{5}{2} < x < 3$, and increasing if $x > 3$. Consequently there is a local maximum at $(-2.5, 166.75)$ and a local minimum at $(3, -166)$.

C04S05.008: The only critical points occur where $x = -\frac{7}{2}$ and where $x = \frac{5}{3}$. The graph is increasing between them and decreasing otherwise.

C04S05.009: $y'(x) = 12x^3 + 12x^2 - 72x = 12(x-2)x(x+3)$, so there are critical points at $P(-3, -149)$, at $Q(0, 40)$, and $R(2, -24)$. The graph is decreasing to the left of P and between Q and R; it is increasing otherwise.

C04S05.010: The critical points occur where $x = -\frac{8}{3}$, where $x = 0$, and where $x = \frac{5}{2}$. The graph is increasing on $\left(-\infty, -\frac{8}{3}\right)$ and on $\left(0, \frac{5}{2}\right)$. The graph is decreasing on $\left(-\frac{8}{3}, 0\right)$ and on $\left(\frac{5}{2}, +\infty\right)$.

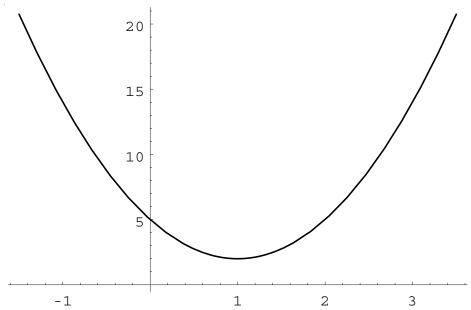
C04S05.011: $y'(x) = 15x^4 - 300x^2 + 960 = 15(x+4)(x+2)(x-2)(x-4)$, so there are critical points at $P(-4, -512)$, $Q(-2, -1216)$, $R(2, 1216)$, and $S(4, 512)$. The graph is increasing to the left of P, between Q and R, and to the right of S; it is decreasing otherwise.

C04S05.012: The critical points occur at $x = -5$, $x = -2$, $x = 0$, $x = 2$, and $x = 5$. The graph of y is decreasing on $(-\infty, -5)$, on $(-2, 0)$, and on $(2, 5)$. The graph is increasing on $(-5, -2)$, on $(0, 2)$, and on $(5, +\infty)$.

C04S05.013: $y'(x) = 21x^6 - 420x^4 + 1344x^2 = 21(x+4)(x+2)x^2(x-2)(x-4)$, so there are critical points at $P(-4, 8192)$, $Q(-2, -1280)$, $R(0, 0)$, $S(2, 1280)$, and $T(4, -8192)$. The graph is increasing to the left of P, between Q and S (there is no extremum at R), and to the right of T; it is decreasing otherwise.

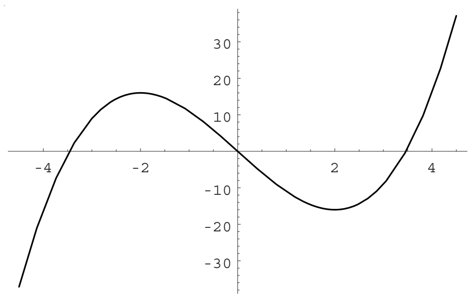
C04S05.014: The critical points occur where $x = -3$, $x = -2$, $x = 0$, $x = 2$, and $x = 3$. The graph of y is decreasing on $(-\infty, -3)$, on $(-2, 0)$, and on $(2, 3)$. It is increasing on $(-3, -2)$, on $(0, 2)$, and on $(3, +\infty)$.

C04S05.015: $f'(x) = 6x - 6$, so there is a critical point at $(1, 2)$. Because $f'(x) < 0$ if $x < 1$ and $f'(x) > 0$ if $x > 1$, there is a global minimum at the critical point. The graph of $y = f(x)$ is shown next.



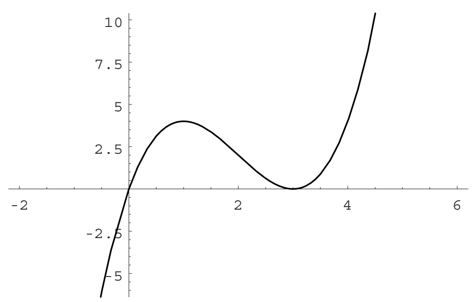
C04S05.016: $f'(x) = -8 - 4x$ is positive for $x < -2$, negative for $x > -2$. The graph is a parabola opening downward, with vertical axis, and vertex (and global maximum) at $(-2, 13)$.

C04S05.017: $f'(x) = 3(x^2 - 4)$. There is a local maximum at $(-2, 16)$ and a local minimum at $(2, -16)$; neither is global. The graph of $y = f(x)$ is shown next.



C04S05.018: The function f is increasing on the set of all real numbers because $f'(x) = 3x^2 + 3$ is positive for all x. Thus f has no extrema of any kind.

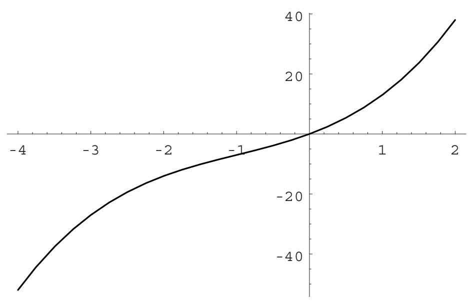
C04S05.19: $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, so f is increasing for $x < 1$ and for $x > 3$, decreasing for $1 < x < 3$. It has a local maximum at $(1, 4)$ and a local minimum at $(3, 0)$. Its graph is shown next.



C04S05.020: $f'(x) = 3x^2 + 12x + 9 = 3(x + 1)(x + 3)$ is positive for $x > -1$ and for $x < -3$, negative for $-3 < x < -1$. So there is a local maximum at $(-3, 0)$ and a local minimum at $(-1, -4)$. There are intercepts at $(-3, 0)$ and $(0, 0)$.

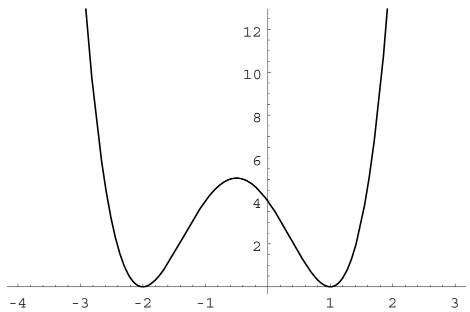
C04S05.021: $f'(x) = 3(x^2 + 2x + 3)$ is positive for all x, so the graph of f is increasing on the set of all

real numbers; there are no extrema. The graph of f is shown next.



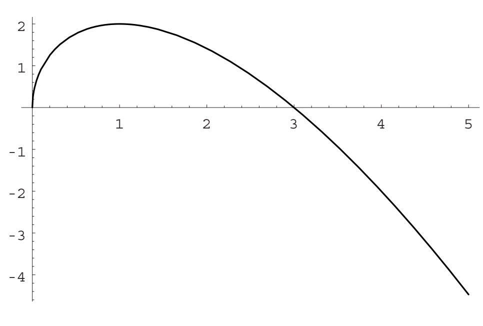
C04S05.022: $f'(x) = 3(x+3)(x-3)$: Local maximum at $(-3,\ 54)$, local minimum at $(3,\ -54)$.

C04S05.023: $f'(x) = 2(x-1)(x+2)(2x+1)$; there are global minima at $(-2,\ 0)$ and $(1,\ 0)$ and a local maximum at $\left(-\frac{1}{2},\ \frac{81}{16}\right)$. The minimum value 0 is global because [clearly] $f(x) = (x-1)^2(x+2)^2 \geqq 0$ for all x. The maximum value is local because $f(x) \to +\infty$ as $x \to \pm\infty$. The graph of $y = f(x)$ is next.



C04S05.024: $f'(x) = 2(x-2)(2x+3)(4x-1)$: Global minimum value 0 at $x = -1.5$ and at $x = 2$, local maximum at $(0.25, 37.515625)$.

C04S05.025: $f'(x) = \dfrac{3(1-x)}{2\sqrt{x}}$, so $f'(x) > 0$ if $0 < x < 1$ and $f'(x) < 0$ if $x > 1$. Therefore there is a local minimum at $(0,\ 0)$ and a global maximum at $(1,\ 2)$. The minimum is only local because $f(x) \to -\infty$ as $x \to +\infty$. The graph of $y = f(x)$ is shown next.



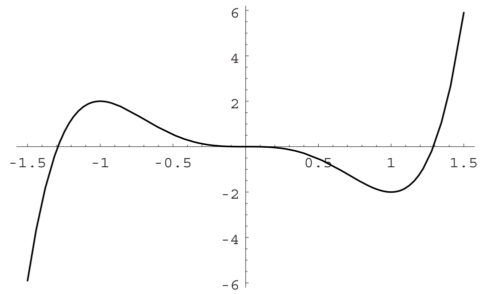
C04S05.026: Given: $f(x) = x^{2/3}(5-x)$:

$$f'(x) = \frac{2}{3}x^{-1/3}(5-x) - x^{2/3} = \frac{2(5-x)}{3x^{1/3}} - x^{2/3} = \frac{10-2x-3x}{3x^{1/3}} = \frac{5(2-x)}{3x^{1/3}}.$$

Hence $f'(x)$ can change sign only at $x = 2$ and at $x = 0$. It's clear that f is increasing for $0 < x < 2$, decreasing for $x < 0$ and for $x > 2$. Thus there is a local maximum at $(2,\ f(2))$ and a local minimum at

(0, 0). Note that $f'(0)$ does not exist, but that f is continuous at $x = 0$. Neither extremum is global because $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

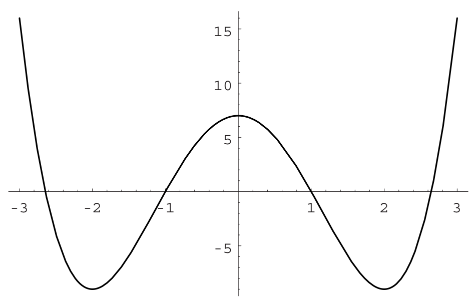
C04S05.027: $f'(x) = 15x^2(x-1)(x+1)$, so f is increasing for $x < -1$ and for $x > 1$, decreasing for $-1 < x < 1$. Hence there is a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$. The critical point at $(0, 0)$ is not an extremum. The graph of $y = f(x)$ is shown next.



C04S05.028: $f'(x) = 4x^2(x+3)$ is positive for $x > -3$ and negative for $x < -3$; there is a horizontal tangent but no extremum at $x = 0$. There is a minimum at $(-3, -27)$; it is global because

$$f(x) = x^4 + 4x^3 = x^4\left(1 + \frac{4}{x}\right) \to +\infty \qquad \text{as} \qquad x \to \pm\infty.$$

C04S05.029: $f'(x) = 4x(x-2)(x+2)$, so the graph of f is decreasing for $x < -2$ and for $0 < x < 2$; it is increasing if $-2 < x < 0$ and if $x > 2$. Therefore the global minimum value -9 of $f(x)$ occurs at $x = \pm 2$ and the extremum at $(0, 7)$ is a maximum, but not global because $f(x) \to +\infty$ as $x \to \pm\infty$. The graph of $y = f(x)$ is shown next.



C04S05.030: Given

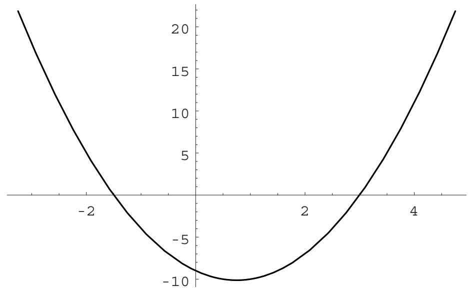
$$f(x) = \frac{1}{x} = x^{-1}, \qquad \text{we see that} \qquad f'(x) = -x^{-2} = -\frac{1}{x^2}.$$

Therefore $f'(x)$ is negative for all $x \neq 0$, so $f(x)$ is decreasing for all $x \neq 0$; there is an infinite discontinuity at $x = 0$. There are no extrema and no intercepts. Note that as x increases without bound, $f(x)$ approaches zero. In sketching the graph of f it is very helpful to note that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \text{and that} \qquad \lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

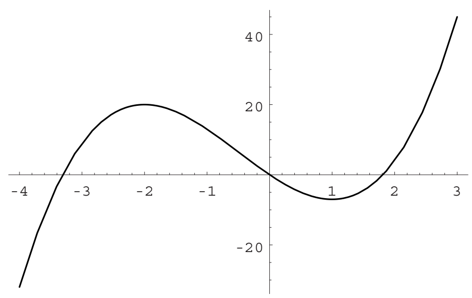
C04S05.031: Because $f'(x) = 4x - 3$, there is a critical point at $\left(\frac{3}{4}, -\frac{81}{8}\right) = (0.75, -10.125)$. The graph of f is decreasing to the left of this point and increasing to its right, so there is a global minimum at this

critical point and no other extrema. The graph of $y = f(x)$ is shown next.



C04S05.032: The graph is a parabola, opening downward, vertical axis, vertex at $\left(-\frac{5}{12}, \frac{169}{24}\right)$, which is thus the highest point on the graph.

C04S05.033: $f'(x) = 6(x-1)(x+2)$, so the graph of f is increasing for $x < -2$ and for $x > 1$, decreasing if $-2 < x < 1$. Hence there is a local maximum at $(-2, 20)$ and a local minimum at $(1, -7)$. The first is not a global maximum because $f(10) = 2180 > 20$; the second is not a global minimum because $f(-10) = -1580 < -7$. The graph of $y = f(x)$ is shown next.

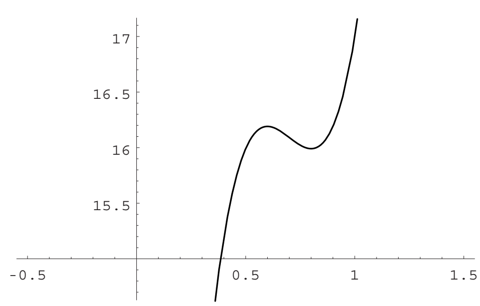


C04S05.034: $f'(x) = 3x^2 + 4$ is positive for all x, so $f(x)$ is increasing for all x; there are no extrema and $(0, 0)$ is the only intercept.

C04S05.035: $f'(x) = 6(5x-3)(5x-4)$, so there are critical points at $(0.6, 16.2)$ and $(0.8, 16)$. The graph of f is increasing for $x < 0.6$ and for $x > 0.8$; it is decreasing between these two points. Hence there is a local maximum at the first and a local minimum at the second. Neither is global because

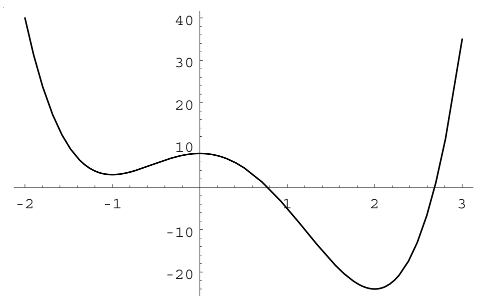
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3\left(50 - \frac{105}{x} + \frac{72}{x^2}\right) = +\infty$$

and, similarly, $f(x) \to -\infty$ as $x \to -\infty$. A graph of $y = f(x)$ is shown next.



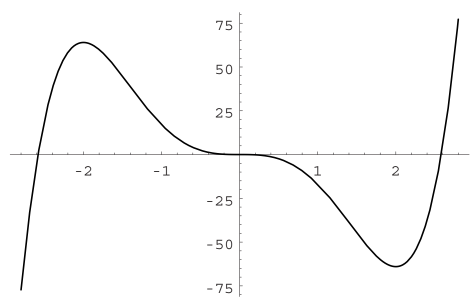
C04S05.036: $f'(x) = 3(x-1)^2$ is positive except at $x = 1$, so the graph is increasing for all x; there are no extrema, and the intercepts are at $(0, -1)$ and $(1, 0)$.

C04S05.037: $f'(x) = 12x(x-2)(x+1)$, so f is decreasing for $x < -1$ and for $0 < x < 2$, increasing for $-1 < x < 0$ and for $x > 2$. There is a local minimum at $(-1, 3)$, a local maximum at $(0, 8)$, and a global minimum at $(2, -24)$. The latter is global rather than local because $f(x) \to +\infty$ as $x \to \pm\infty$. The graph of f is shown next.



C04S05.038: $f(x) = (x^2 - 1)^2$; $f'(x) = 4x(x+1)(x-1)$. So $f(x)$ is increasing for $-1 < x < 0$ and for $x > 1$, decreasing if $x < -1$ or if $0 < x < 1$. The global minimum value is $0 = f(-1) = f(1)$ and there is a local maximum at $(0, 1)$.

C04S05.039: $f'(x) = 15x^2(x-2)(x+2)$, so f is increasing if $|x| > 2$ and decreasing if $|x| < 2$.There is a local maximum at $(-2, 64)$ and a local minimum at $(2, -64)$; the critical point at $(0, 0)$ is not an extremum. The graph of $y = f(x)$ appears next.



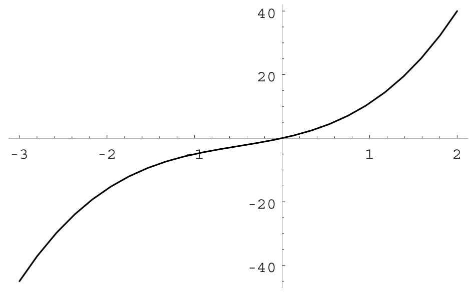
C04S05.040: $f'(x) = 15(x+1)(x-1)(x+2)(x-2)$, so f is increasing if $x < -2$, if $-1 < x < 1$, and if $x > 1$, decreasing if $-2 < x < -1$ and if $1 < x < 2$. So there are local maxima at $(-2, -16)$ and $(1, 38)$ and local minima at $(-1, -38)$ and $(2, 16)$. The only intercept is $(0, 0)$. None of the extrema is global because

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (3x^5 - 25x^3 + 60x) = \lim_{x\to\infty} x^5\left(3 - \frac{25}{x^2} + \frac{60}{x^4}\right) = +\infty$$

and, similarly, $f(x) \to -\infty$ as $x \to -\infty$.

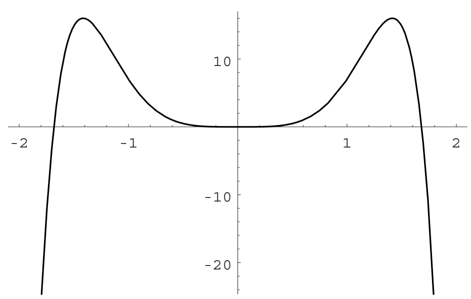
C04S05.041: $f'(x) = 6(x^2 + x + 1)$ is positive for all x, so the graph of f is increasing everywhere, with

no critical points and thus no extrema. The only intercept is (0, 0). The graph of f is shown next.



C04S05.042: $f'(x) = 4x^2(x-3)$, so f is increasing for $x > 3$ and decreasing for $x < 3$. Therefore there is a local (and global) minimum at $(3, -27)$; (0, 0) and (4, 0) are the only intercepts. There is a horizontal tangent at (0, 0) but no extremum there.

C04S05.043: $f'(x) = 32x^3 - 8x^7 = -8x^3(x^2+2)(x^2-2)$, so the graph of f is increasing if $x < -\sqrt{2}$ and if $0 < x < \sqrt{2}$ but decreasing if $-\sqrt{2} < x < 0$ and if $x > \sqrt{2}$. The global maximum value of $f(x)$ is $16 = f(\sqrt{2}) = f(-\sqrt{2})$ and there is a local minimum at (0, 0). The graph of f is shown next.



C04S05.044: Here we have

$$f'(x) = -\frac{1}{3x^{2/3}},$$

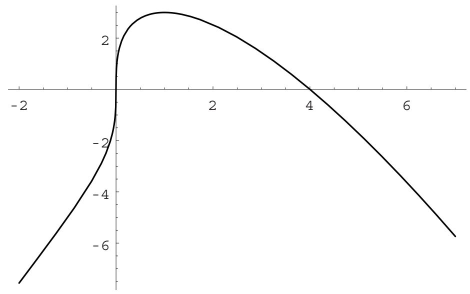
which is negative for all $x \neq 0$. Though $f'(0)$ is not defined, f is continuous at $x = 0$; careful examination of the behavior of f and f' near zero shows that the graph has a vertical tangent at (0, 1); there are no extrema.

C04S05.045: Given $f(x) = x^{1/3}(4-x)$, we find that

$$f'(x) = \frac{1}{3}x^{-2/3}(4-x) - x^{1/3} = \frac{4(1-x)}{3x^{2/3}}.$$

Therefore f is increasing if $x < 0$ and if $0 < x < 1$, decreasing if $x > 1$. Because f is continuous everywhere, including the point $x = 0$, it is also correct to say that f is increasing on $(-\infty, 1)$. The point (1, 3) is the highest point on the graph and there are no other extrema. Careful examination of the behavior of $f(x)$ and $f'(x)$ for x near zero shows that there is a vertical tangent at the critical point (0, 0). The point (4, 0) is an

x-intercept. The graph of $y = f(x)$ appears next.



C04S05.046: In this case

$$f'(x) = \frac{8(x+2)(x-2)}{3x^{1/3}},$$

which is positive for $x > 2$ and for $-2 < x < 0$, negative for $x < -2$ and for $0 < x < 2$. Note that f is continuous at $x = 0$ even though $f'(0)$ does not exist. Moreover, for x near zero, we have

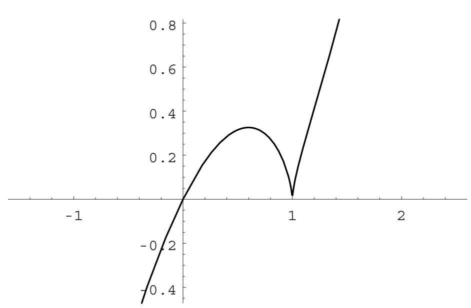
$$f(x) \approx -16x^{2/3} \qquad \text{and} \qquad f'(x) \approx -\frac{32}{3x^{1/3}}.$$

Consequently $f'(x) \to -\infty$ as $x \to 0^+$, whereas $f'(x) \to +\infty$ as $x \to 0^-$. This is consistent with the observation that $f(x) < 0$ for all x near (but not equal to) zero. The origin is a local maximum and there are global minima where $|x| = 2$.

C04S05.047: Given $f(x) = x(x-1)^{2/3}$, we find that

$$f'(x) = (x-1)^{2/3} + \frac{2}{3}x(x-1)^{-1/3} = \frac{5x-3}{3(x-1)^{1/3}}.$$

Thus $f'(x) = 0$ when $x = \frac{3}{5}$ and $f'(x)$ does not exist when $x = 1$. So f is increasing for $x < \frac{3}{5}$ and for $x > 1$, decreasing if $\frac{3}{5} < x < 1$. Thus there is a local maximum at $\left(\frac{3}{5}, 0.3257\right)$ (y-coordinate approximate) and a local minimum at $(1, 0)$. Examination of $f(x)$ and $f'(x)$ for x near 1 shows that there is a vertical tangent at $(1, 0)$. There are no global extrema because $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$. The graph of f is shown next.



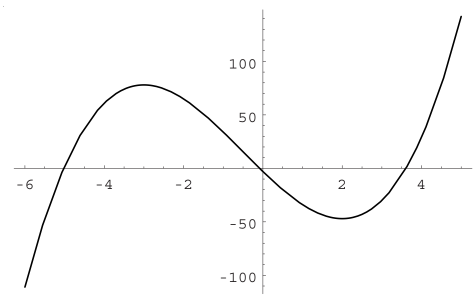
C04S05.048: After simplifications, we find that

$$f'(x) = \frac{2-3x}{3x^{2/3}(2-x)^{1/3}}.$$

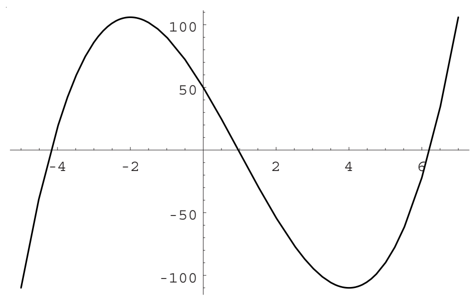
So $f'(x) = 0$ when $x = \frac{2}{3}$ and $f'(x)$ does not exist at $x = 0$ and at $x = 2$. Nevertheless, f is continuous everywhere. Its graph is increasing for $x < \frac{2}{3}$ and for $x > 2$, decreasing for $\frac{2}{3} < x < 2$. There is a

vertical tangent at (0, 0), which is not an extremum. There's a horizontal tangent at $\left(\frac{2}{3}, 1.058\right)$ (ordinate approximate), which is a local maximum. There is a cusp at (2, 0), which is also a local minimum. Note that $f(x) \approx x$ for $|x|$ large; this aids in constructing the global sketch of the graph.

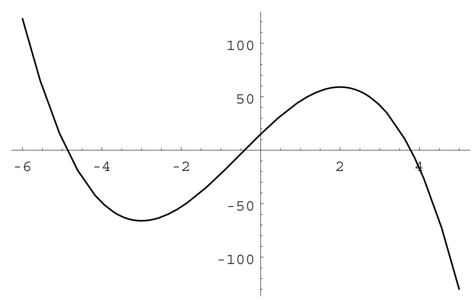
C04S05.049: The graph of $f(x) = 2x^3 + 3x^2 - 36x - 3$ is shown next.



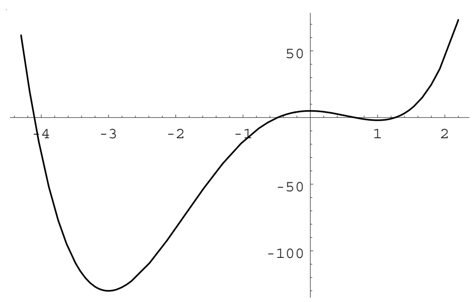
C04S05.050: The graph of $f(x) = 2x^3 - 6x^2 - 48x + 50$ is shown next.



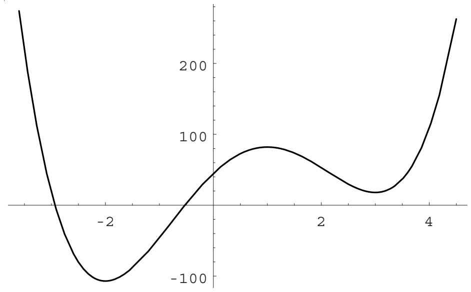
C04S05.051: The graph of $f(x) = -2x^3 - 3x^2 + 36x + 15$ is shown next.



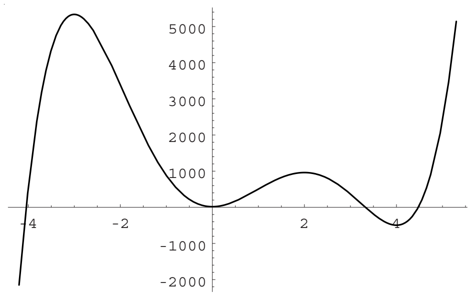
C04S05.052: The graph of $f(x) = 3x^4 + 8x^3 - 18x^2 + 5$ appears next.



C04S05.053: The graph of $f(x) = 3x^4 - 8x^3 - 30x^2 + 72x + 45$ is next.



C04S05.054: The graph of $f(x) = 12x^5 - 45x^4 - 200x^3 + 720x^2 + 17$ is shown next.



C04S05.055: Let $f(x) = x^3 - 3x + 3$. For part (a), we find that if we let $x = -2.1038034027$, then $f(x) \approx 7.58 \times 10^{-9}$. For part (b), we find that

$$f(x) \approx (x + 2.1038034027)(x^2 - (2.1038034027)x + 1.4259887573).$$

And in part (c), we find by the quadratic formula that the complex conjugate roots of $f(x) = 0$ are approximately $1.0519017014 + 0.5652358517i$ and $1.0519017014 - 0.5652358517i$.

C04S05.056: Let $f(x) = x^3 - 3x + q$. Then $f'(x) = 3(x^2 - 1)$, so the graph of $y = f(x)$ will always have a local maximum at $(-1, f(-1)) = (-1, q + 2)$ and a local minimum at $(1, f(1)) = (1, q - 2)$. If the ordinates of these points have the same sign then the equation $f(x) = 0$ will have only one [real] solution—see Figs. 4.5.9 through 4.5.11. And this situation is equivalent to $q + 2 < 0$ or $q - 2 > 0$; that is, $q < -2$ or $q > 2$. If the ordinates have opposite signs, then the equation $f(x) = 0$ will have three real solutions, and this will occur if $q - 2 < 0 < q + 2$; that is, if $-2 < q < 2$. If $q = \pm 2$, then there will be exactly two real solutions because

$$x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)^2(x + 2) \quad \text{and} \quad x^3 - 3x - 2 = (x + 1)(x^2 - x - 2) = (x + 1)^2(x - 2).$$

C04S05.057: If $f(x) = [x(x - 1)(2x - 1)]^2$, then

$$f'(x) = 2x(x - 1)(2x - 1)(6x^2 - 6x + 1),$$

so the critical points of the graph of f will be $(0, 0)$, $\left(\frac{1}{2}, 0\right)$, $(1, 0)$, $\left(\frac{1}{6}\left[3 - \sqrt{3}\right], 0.009259259\right)$ (ordinate approximate), and $\left(\frac{1}{6}\left[3 + \sqrt{3}\right], 0.009259259\right)$. The graph of f will be

$$
\begin{aligned}
&\text{decreasing for} && x < 0,\\
&\text{increasing for} && 0 < x < \tfrac{1}{6}\left(3-\sqrt{3}\right),\\
&\text{decreasing for} && \tfrac{1}{6}\left(3-\sqrt{3}\right) < x < \tfrac{1}{2},\\
&\text{increasing for} && \tfrac{1}{2} < x < \tfrac{1}{6}\left(3+\sqrt{3}\right),\\
&\text{decreasing for} && \tfrac{1}{6}\left(3+\sqrt{3}\right) < x < 1, \quad \text{and}\\
&\text{increasing for} && 1 < x.
\end{aligned}
$$

There will be global minima at $x = 0$, $x = \frac{1}{2}$, and $x = 1$ and [equal] local maxima at $x = \frac{1}{6}\left(3-\sqrt{3}\right)$ and $x = \frac{1}{6}\left(3+\sqrt{3}\right)$.

C04S05.058: A *Mathematica* solution:

```
poly = x∧3 − 3∗x + 1;

r = (−1 + Sqrt[3])/2;

x1 = r∧(−1/3) + r∧(1/3);
x2 = r∧(4/3) + r∧(5/3);
x3 = r∧(2/3) + r∧(7/3);

(poly /.  x → x1) // Expand // Simplify
(poly /.  x → x2) // Expand // Simplify
(poly /.  x → x3) // Expand // Simplify

0
0
0

N[x1] // chop
N[x2] // chop
N[x3] // chop
```

1.53209

−1.87939

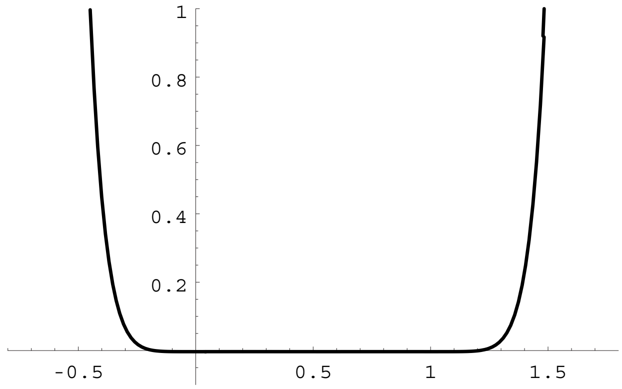
0.347296

Thus we see three distinct real roots.

C04S05.059: Given $f(x) = \left[\frac{1}{6}x(9x-5)(x-1)\right]^4$, the *Mathematica* command

```
Plot[ f[x], { x, −1, 2 }, PlotPoints → 97,
       PlotRange → {{ −0.8, 1.8 }, { −0.1, 1.0 }} ];
```

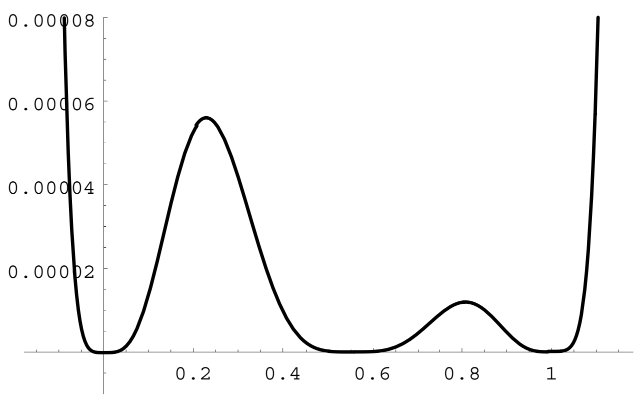
generated the graph shown next. As predicted, the graph seems to have a "flat spot" on the interval [0, 1].



Then we modified the Plot command to restrict the range of y-values to the interval $[-0.00001, 0.00008]$:

```
Plot[ f[x], { x, −1, 2 }, PlotPoints → 197,
      PlotRange → {{ −0.18, 1.18 }, { −0.00001, 0.00008 }} ];
```

the graph generated by this command is next.



Then we used *Mathematica* to identify the extrema:

soln = Solve[f′[x] == 0, x]

$$\{\{\ x \to 0\ \},\ \{\ x \to 0\ \},\ \{\ x \to 0\ \},\ \{\ x \to \frac{5}{9}\ \},\ \{\ x \to \frac{5}{9}\ \},$$

$$\{\ x \to \frac{5}{9}\ \},\ \{\ x \to 1\ \},\ \{\ x \to 1\ \},\ \{\ x \to 1\ \},$$

$$\{\ x \to \frac{14 - \mathtt{Sqrt}[61]}{27}\ \},\ \{\ x \to \frac{14 + \mathtt{Sqrt}[61]}{27}\ \}\}$$

```
{ x1 = soln[[1,1,2]], x2 = soln[[4,1,2]], x3 = soln[[7,1,2]],
x4 = N[soln[[10,1,2]], 20], x5 = N[soln[[11,1,2]], 20] }
```

$$\{\ 0,\ \frac{5}{9},\ 1,\ 0.22925001200345724466,\ 0.80778702503357979 24\ \}$$

```
{ y1 = f[x1], y2 = f[x2], y3 = f[x3], y4 = f[x4], y5 = f[x5] }
```

{ 0, 0, 0, 0.000055944116435930 3138, 0.000011909140281097 8625 }

The second graph makes it clear that $(x_1, 0)$, $(x_2, 0)$, and $(x_3, 0)$ are local (indeed, global) minima, while (x_4, y_4) and (x_5, y_5) are local (not global) maxima.

C04S05.060: After constructing the functions

$$f(x) = x^4 - 55x^3 + 505x^2 + 11000x - 110000$$

and

$$g(x) = f(x) + ex^2$$

where $e = 1$, we used *Mathematica* to find the zeros of these polynomials:

```
NSolve[ f[x] == 0, x ]
    {{ x → −13.4468 }, { x → 9.38408 }, { x → 28.6527 }, { x → 30.4143 }}

NSolve[ g[x] == 0, x ]
    {{ x → −13.4468 }, { x → 9.38459 }, { x → 29.5361 − 0.480808 I },
    { x → 29.5371 + 0.480808 I }}
```

In part (b), by changing the value of e, working up from $e = 0$ and down from $e = 1$, we bracketed the transition point.

```
e = 0.7703;
h[x_] := f[x] + e∗x∧2;
NSolve[ h[x] == 0, x ]
    {{ x → −13.4478 }, { x → 9.37677 }, { x → 29.528 }, { x → 29.543 }

e = 0.7704;
h[x_] := f[x] + e∗x∧2;
NSolve[ h[x] == 0, x ]
    {{ x → −13.4478 }, { x → 9.37676 }, { x → 29.5355 − 0.00665912 I },
    { x → 29.5355 + 0.00665912 I }}
```

Section 4.6

C04S06.001: $f'(x) = 8x^3 - 9x^2 + 6,\ \ f''(x) = 24x^2 - 18x,\ \ f'''(x) = 48x - 18.$

C04S06.002: $f'(x) = 10x^4 + \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-2},\ \ f''(x) = 40x^3 + \frac{3}{4}x^{-1/2} - x^{-3},\ \ f'''(x) = 120x^2 - \frac{3}{8}x^{-3/2} + 3x^{-4}.$

C04S06.003: $f'(x) = -8(2x-1)^{-3},\ \ f''(x) = 48(2x-1)^{-4},\ \ f'''(x) = -384(2x-1)^{-5}.$

C04S06.004: $g'(t) = 2t + \frac{1}{2}(t+1)^{-1/2},\ \ g''(t) = 2 - \frac{1}{4}(t+1)^{-3/2},\ \ g'''(t) = \frac{3}{8}(t+1)^{-5/2}.$

C04S06.005: $g'(t) = 4(3t-2)^{1/3},\ \ g''(t) = 4(3t-2)^{-2/3},\ \ g'''(t) = -8(3t-2)^{-5/3}.$

C04S06.006: $f'(x) = (x+1)^{1/2} + \frac{1}{2}x(x+1)^{-1/2},\ \ f''(x) = (x+1)^{-1/2} - \frac{1}{4}x(x+1)^{-3/2},$
$f'''(x) = -\frac{3}{4}(x+1)^{-3/2} + \frac{3}{8}x(x+1)^{-5/2}.$

C04S06.007: $h'(y) = (y+1)^{-2},\ \ h''(y) = -2(y+1)^{-3},\ \ h'''(y) = 6(y+1)^{-4}.$

C04S06.008: $f'(x) = \frac{3}{2}x^{-1/2} + 3 + \frac{3}{2}x^{1/2},\ \ f''(x) = -\frac{3}{4}x^{-3/2} + \frac{3}{4}x^{-1/2},\ \ f'''(x) = \frac{9}{8}x^{-5/2} - \frac{3}{8}x^{-3/2}.$

C04S06.009: $g'(t) = -\dfrac{1}{(1-t)^{4/3}} - \dfrac{1}{4t^{3/2}},\ \ g''(t) = -\dfrac{4}{3(1-t)^{7/3}} + \dfrac{3}{8t^{5/2}},\ \ g'''(t) = -\dfrac{28}{9(1-t)^{10/3}} - \dfrac{15}{16t^{7/2}}.$

C04S06.010: $h'(z) = \dfrac{8z}{(z^2+4)^2},\ \ h''(z) = \dfrac{32 - 24z^2}{(z^2+4)^3},\ \ h'''(z) = \dfrac{96z^3 - 384z}{(z^2+4)^4}.$

C04S06.011: $f'(x) = 3\cos 3x,\ \ f''(x) = -9\sin 3x,\ \ f'''(x) = -27\cos 3x.$

C04S06.012: $f'(x) = -4\sin 2x\ \cos 2x,\ \ f''(x) = 8\sin^2 2x - 8\cos^2 2x,\ \ f'''(x) = 64\sin 2x\ \cos 2x.$

C04S06.013: $f'(x) = \cos^2 x - \sin^2 x,\ \ f''(x) = -4\sin x\ \cos x,\ \ f'''(x) = 4\sin^2 x - 4\cos^2 x.$

C04S06.014: $f'(x) = 2x\cos x - x^2\sin x,\ \ f''(x) = 2\cos x - 4x\sin x - x^2\cos x,$
$f'''(x) = -2\sin x - 4\sin x - 6x\cos x + x^2\sin x.$

C04S06.015: $f'(x) = \dfrac{x\cos x - \sin x}{x^2},\ \ f''(x) = \dfrac{(2-x^2)\sin x - 2x\cos x}{x^3},$

$f'''(x) = \dfrac{(6-x^2)x\cos x + (3x^2-6)\sin x}{x^4}.$

C04S06.016: Given: $x^2 + y^2 = 4.$

$$2x + 2yy'(x) = 0, \qquad \text{so} \qquad y'(x) = -\frac{x}{y}.$$

$$y''(x) = -\frac{y - xy'(x)}{y^2} = -\frac{y + \dfrac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{4}{y^3}.$$

C04S06.017: Given: $x^2 + xy + y^2 = 3.$

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0, \qquad \text{so} \qquad y'(x) = -\frac{2x+y}{x+2y}.$$

$$y''(x) = -\frac{2(1 + y'(x) + [y'(x)]^2)}{x+2y} = -\frac{6(x^2+xy+y^2)}{(x+2y)^3} = -\frac{18}{(x+2y)^3}.$$

C04S06.018: Given: $x^{1/3} + y^{1/3} = 1$.

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0, \qquad \text{so} \qquad y'(x) = -\left(\frac{y}{x}\right)^{2/3}.$$

$$y''(x) = -\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3} \cdot \frac{xy'(x) - y}{x^2} = \frac{2}{3}\left(\frac{y}{x^5}\right)^{1/3}.$$

C04S06.019: Given: $y^3 + x^2 + x = 5$.

$$3y^2\frac{dy}{dx} + 2x + 1 = 0, \qquad \text{so} \qquad y'(x) = -\frac{2x+1}{3y^2}.$$

$$y''(x) = -\frac{2(1 + 3y(x)[y'(x)]^2)}{3[y(x)]^2} = -\frac{2[(2x+1)^2 + 3y^3]}{9y^5}.$$

C04S06.020: Given: $\dfrac{1}{x} + \dfrac{1}{y} = 1$.

$$-\frac{1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} = 0, \qquad \text{so} \qquad y'(x) = -\left(\frac{y}{x}\right)^2.$$

$$y''(x) = -2\left(\frac{y}{x}\right) \cdot \frac{xy'(x) - y(x)}{x^2} = \frac{2y^2(x+y)}{x^4} = 2\left(\frac{y}{x}\right)^3.$$

The last step is a consequence of the fact that, by the original equation, $x + y = xy$.

C04S06.021: Given: $\sin y = xy$.

$$(\cos y)\frac{dy}{dx} = y + x\frac{dy}{dx}, \qquad \text{so} \qquad y'(x) = -\frac{y}{x - \cos y}.$$

$$y''(x) = -\frac{[x - \cos y(x)]y'(x) - y(x)[1 + y'(x)\sin y]}{(x - \cos y)^2} = -\frac{(y\sin y + 2\cos y - 2x)y}{(x - \cos y)^3}.$$

C04S06.022: $\sin^2 x + \cos^2 y = 1$: $2\sin x\,\cos x - 2y'(x)\sin y\,\cos y = 0$; $y'(x) = \dfrac{\sin x\,\cos x}{\sin y\,\cos y}$.

$\dfrac{d^2y}{dx^2}$ can be simplified (with the aid of the original equation) to

$$\frac{d^2y}{dx^2} = \frac{\cos^2 x\,\sin^2 y - \sin^2 x\,\cos^2 y}{\sin^3 y\,\cos^3 y} \equiv 0 \quad \text{if } y \text{ is not an integral multiple of } \pi/2.$$

C04S06.023: $f'(x) = 3x^2 - 6x - 45 = 3(x+3)(x-5)$, so there are critical points at $(-3,\ 81)$ and $(5, -175)$. $f''(x) = 6(x-1)$, so the inflection point is located at $(1,\ -47)$.

C04S06.024: Critical points: $(-3,\ 389)$ and $(6,\ -340)$; inflection point: $(1.5,\ 24.5)$.

C04S06.025: $f'(x) = 12x^2 - 12x - 189 = 3(2x+7)(2x-9)$, so there are critical points at $(-3.5, 553.5)$ and $(4.5, -470.5)$. $f''(x) = 24x - 12$, so the inflection point is located at $(0.5, 41.5)$.

C04S06.026: Critical points: $(-6.25, -8701.56)$ and $(3.4, 9271.08)$; inflection point: $(-1.425, 284.76)$.

C04S06.027: $f'(x) = 4x^3 - 108x = 4x(x^2 - 27)$, so there are critical points at $(0, 237)$, $\left(-3\sqrt{3}, -492\right)$, and $\left(3\sqrt{3}, -492\right)$. Next, $f''(x) = 12x^2 - 108 = 12(x-3)(x+3)$, so the inflection points are at $(-3, -168)$ and $(3, -168)$.

C04S06.028: Critical point: $(7.5, -1304.69)$; inflection points: $(0, -250)$ and $(5, -875)$.

C04S06.029: $f'(x) = 15x^4 - 80x^3 = 5x^3(3x-16)$, so there are critical points at $(0, 1000)$ and $\left(\frac{16}{3}, -\frac{181144}{81}\right)$ (approximately $(5.333333, -2236.345679)$). $f''(x) = 60x^3 - 240x^2 = 60x^2(x-4)$, so the inflection points are at $(0, 1000)$ and $(4, -1048)$.

C04S06.030: Critical points: $\left(-4\sqrt{2}, 8192\sqrt{2}\right)$ and $\left(4\sqrt{2}, -8192\sqrt{2}\right)$; inflection points: $(0, 0)$, $(-4, 7168)$, and $(4, -7168)$.

C04S06.031: $f'(x) = 2x - 4$ and $f''(x) \equiv 2$, so there is a critical point at $(2, -1)$ and no inflection points. Because $f''(2) = 2 > 0$, there is a local minimum at the critical point. The first derivative test shows that it is in fact a global minimum.

C04S06.032: $f'(x) = -6 - 2x$; $f''(x) = -2$. The only critical point is at $(-3, 14)$, and it is a local maximum point because $f''(-3) = -2 < 0$. There are no inflection points because $f''(x)$ never changes sign. The critical point is actually a global maximum by the first derivative test.

C04S06.033: $f'(x) = 3(x+1)(x-1)$ and $f''(x) = 6x$. $f(-1) = 3$ and $f''(-1) = -6$, so the critical point at $(-1, 3)$ is a local maximum. Similarly, the critical point at $(1, -1)$ is a local minimum. The point $(0, 1)$ is an inflection point because $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. The extrema are not global because $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

C04S06.034: $f'(x) = 3x(x-2)$; $f''(x) = 6(x-1)$. There is a critical point at $(0, 0)$ and one at $(2, -4)$. Now $f''(0) = -6 < 0$, so there is a local maximum at $(0, 0)$; $f''(2) = 6 > 0$, so there is a local minimum at $(2, -4)$. The only possible inflection point is $(1, -2)$, and it is indeed an inflection point because f'' changes sign there. The extrema are not global.

C04S06.035: $f'(x = 3x^2$ and $f''(x) = 6x$, so there is a critical point and possible inflection point at $(0, 0)$. But $f''(0) = 0$, so the second derivative test fails; the first derivative test shows that there is no extremum at $(0, 0)$. Because $f''(x) > 0$ if $x > 0$ and $f''(x) < 0$ if $x < 0$, there is an inflection point at $(0, 0)$.

C04S06.036: $f'(x) = 4x^3$; $f''(x) = 12x^2$. The only critical point and the only possible inflection point is $(0, 0)$. The second derivative does not identify this point as a local maximum or minimum, and it is not an inflection point because f'' does not change sign there. (It is, of course, the location of the global minimum of f by the first derivative test.)

C04S06.037: $f'(x) = 5x^4 + 2$ and $f''(x) = 20x^3$, so there are no critical points ($f'(x) > 0$ for all x) and $(0, 0)$ is the only possible inflection point. And it is an inflection point because $f''(x)$ changes sign at $x = 0$.

C04S06.038: $f'(x) = 4x(x+2)(x-2)$; $f''(x) = 12x^2 - 16$. The critical points are located at $(-2, -16)$, $(0, 0)$, and $(2, -16)$. Now $f''(-2) > 0$ and $f''(2) > 0$, so $(-2, -16)$ and $(2, -16)$ are local minimum points. But $f''(0) < 0$, so $(0, 0)$ is a local maximum point. The only possible inflection points are where $3x^2 - 4 = 0$;

$x = \frac{2}{3}\sqrt{3}$ and $x = -\frac{2}{3}\sqrt{3}$. Because $f''(x) = 4(3x^2 - 4)$, it is clear that $f''(x)$ changes sign at each of these two points, so the corresponding points on the graph are inflection points. The local minima are actually global by a careful application of the first derivative test.

C04S06.039: $f'(x) = 2x(x-1)(2x-1)$ and $f''(x) = 2(6x^2 - 6x + 1)$. So the critical points are $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{16}\right)$, and $(1, 0)$. The second derivative test indicates that the first and third are local minima and that the second is a local maximum. The only possible inflection points are $\left(\frac{1}{6}\left(3 - \sqrt{3}\right), \frac{1}{36}\right)$ and $\left(\frac{1}{6}\left(3 + \sqrt{3}\right), \frac{1}{36}\right)$. Why *are* they both inflection points? The graph of $f''(x) = 2(6x^2 - 6x + 1)$ is a parabola opening upward and with its vertex below the x-axis (because $f''(x) = 0$ has two real solutions). The possible inflection points are located at the zeros of $f''(x)$, so it should now be clear that $f''(x)$ changes sign at each of the two possible inflection points. Because $f(x)$ is never negative, the two local minima are actually global, but the local maximum is not (because $f(x) \to +\infty$ as $x \to \pm\infty$).

C04S06.040: $f'(x) = x^2(x+2)(5x+6)$ and $f''(x) = 4x(5x^2 + 12x + 6)$. So the critical points occur where $x = 0$, $x = -2$, and $x = -\frac{6}{5}$. Now $f''(0) = 0$, so the second derivative test fails here, but $f'(x) > 0$ for x near zero but $x \neq 0$, so $(0, 0)$ is not an extremum. Next, $f''(-2) = -16 < 0$, so $(-2, 0)$ is a local maximum point; $f''(-1.2) = 5.76 > 0$, so $\left(-\frac{6}{5}, \frac{3456}{3125}\right)$ is a local minimum point. The possible inflection points occur at

$$x = 0, \qquad x = \tfrac{1}{5}\left(-6 + \sqrt{6}\right), \qquad \text{and} \qquad x = \tfrac{1}{5}\left(-6 - \sqrt{6}\right).$$

In decimal form these are $x = 0$, $x \approx -0.710$, and $x \approx -1.690$. Because $f''(-2) = -16 < 0$, $f''(-1) = 4 > 0$, $f''(-0.5) = -2.5 < 0$, and $f''(1) = 92 > 0$, each of the three numbers displayed above is the abscissa of an inflection point of the graph of f. None of the extrema is global because $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

C04S06.041: $f'(x) = \cos x$ and $f''(x) = -\sin x$. $f'(x) = 0$ when $x = \pi/2$ and when $x = 3\pi/2$; $f''(\pi/2) = -1 < 0$ and $f''(3\pi/2) = 1 > 0$, so the first of these critical points is a local maximum and the second is a local minimum. Because $|\sin x| \leqq 1$ for all x, these extrema are in fact global. $f''(x) = 0$ when $x = \pi$ and clearly changes sign there, so there is an inflection point at $(\pi, 0)$.

C04S06.042: Local (indeed, global) maximum point: $(0, 1)$; no inflection points.

C04S06.043: $f'(x) = \sec^2 x \geqq 1$ for $-\pi/2 < x < \pi/2$, so there are no extrema. $f''(x) = 2\sec^2 x \tan x$, so there is a possible inflection point at $(0, 0)$. Because $\tan x$ changes sign at $x = 0$ (and $2\sec^2 x$ does not), $(0, 0)$ is indeed an inflection point.

C04S06.044: $f'(x) = \sec x \tan x$ and $f''(x) = \sec^3 x + \sec x \tan^2 x = (\sec x)(\sec^2 x + \tan^2 x)$. So $(0, 1)$ is the only critical point and there are no possible inflection points. $f''(0) = 1$, so the second derivative test shows that $(0, 1)$ is a local minimum point. The first derivative test identifies $(0, 1)$ as a global minimum point.

C04S06.045: $f'(x) = -2\sin x \cos x$ and $f''(x) = 2\sin^2 x - 2\cos^2 x$. Hence $(0, 1)$, $(\pi/2, 0)$, and $(\pi, 1)$ are critical points. By the second derivative test the first and third are local maxima and the second is a local minimum. (Because $0 \leqq \cos^2 x \leqq 1$ for all x, these extrema are all global.) There are possible inflection points at $(-\pi/4, 1/2)$, $(\pi/4, 1/2)$, $(3\pi/4, 1/2)$, and $(5\pi/4, 1/2)$. A close examination of $f''(x)$ reveals that it changes sign at all four of these points, so each is an inflection point.

C04S06.046: $f(x) = \sin^3 x$, $-\pi < x < \pi$:

$$f'(x) = 3\sin^2 x \cos x, \qquad f''(x) = 6\sin x \cos^2 x - 3\sin^3 x = 3(2\cos^2 x - \sin^2 x)\sin x.$$

Global minimum at $(-\pi/2, -1)$; global maximum at $(\pi/2, 1)$; inflection points at $x = 0$ and at the four solutions of $\tan^2 x = 2$ in $(-\pi, \pi)$: approximately $(-2.186276, -0.544331)$, $(-0.955317, -0.544331)$, $(0, 0)$, $(0.955317, 0.544331)$, and $(2.186276, 0.544331)$.

C04S06.047: $f'(x) = \cos x - \sin x$ and $f''(x) = -\cos x - \sin x$. So $f'(x) = 0$ when $\tan x = 1$ for $0 < x < 2\pi$; thus there are critical points at $(\pi/4, \sqrt{2})$ and $(5\pi/4, -\sqrt{2})$. The first is a local (in fact, global) maximum because $f''(\pi/4) = -\sqrt{2} < 0$, the second is a local (in fact, global) minimum because $f''(5\pi/4) = \sqrt{2} > 0$. Next, $f''(x) = 0$ when $\tan x = -1$, thus there are possible inflection points at $(3\pi/4, 0)$ and $(7\pi/4, 0)$. If x is near $3\pi/4$ but $x < 3\pi/4$, then $\cos x$ is slightly greater than $-\frac{1}{2}\sqrt{2}$ and $\sin x$ is slightly greater than $\frac{1}{2}\sqrt{2}$, so $f''(x) = -\cos x - \sin x < \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} = 0$. If x is near $3\pi/4$ but $x > 3\pi/4$, then $\cos x$ is slightly less than $-\frac{1}{2}\sqrt{2}$ and $\sin x$ is slightly less than $\frac{1}{2}\sqrt{2}$, so $f''(x) = -\cos x - \sin x > \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} = 0$. Therefore the graph of f is concave down to the left of $(3\pi/2, 0)$ and concave upward to the right. Hence $(3\pi/4, 0)$ is an inflection point. A similar analysis shows that $(7\pi/4, 0)$ is also an inflection point.

C04S06.048: If $f(x) = \cos x - \sin x$, then

$$f'(x) = -\sin x - \cos x \qquad \text{and} \qquad f''(x) = \sin x - \cos x.$$

Now $f'(x) = 0$ when $\tan x = -1$ and $0 < x < 2\pi$. So there are critical points at $(3\pi/4, -\sqrt{2})$ and $(7\pi/4, \sqrt{2})$. The first is a local minimum because $f(3\pi/4) = \sqrt{2} > 0$, the second is a local maximum because $f(7\pi/4) = -\sqrt{2} < 0$. Next, $f''(x) = 0$ when $x = \pi/4$ and when $x = 5\pi/4$. By an analysis similar to that in the solution of Problem 47, there are inflection points at both $(\pi/4, 0)$ and $5\pi/4, 0)$.

C04S06.049: Given: $f(x) = \sin x + 2\cos x$, $0 < x < 2\pi$. First,

$$f'(x) = \cos x - 2\sin x \qquad \text{and} \qquad f''(x) = -\sin x - 2\cos x.$$

Therefore $f'(x) = 0$ when $\tan x = \frac{1}{2}$, thus when $x = a = \arctan\left(\frac{1}{2}\right)$ and when $x = b = a + \pi$. Now $\sin a = \frac{1}{5}\sqrt{5}$, $\cos a = \frac{2}{5}\sqrt{5}$, $\sin b = -\frac{1}{5}\sqrt{5}$, and $\cos b = -\frac{2}{5}\sqrt{5}$. So $f(a) = \sqrt{5}$ and $f(b) = -\sqrt{5}$. Also $f''(a) = -2\cos a - \sin a = -\sqrt{5} < 0$ and $f''(b) = \sqrt{5} > 0$, so there is a local (in fact, global) maximum at $(a, \sqrt{5})$ and a local (in fact, global) minimum at $(b, -\sqrt{5})$. Next, $f''(x) = 0$ when $x = p = \pi - \arctan 2$ and when $x = q = 2\pi - \arctan 2$, and—by an analysis similar to that in the solution of Problem 47—there are inflection points at $(p, 0)$ and $(q, 0)$.

C04S06.050: Given: $f(x) = 3\sin x - 4\cos x$, $0 < x < 2\pi$. Then

$$f'(x) = 3\cos x + 4\sin x \qquad \text{and} \qquad f''(x) = 4\cos x - 3\sin x.$$

Then $f'(x) = 0$ when $x = a = \arctan\left(-\frac{3}{4}\right) \approx -0.6435$, which is not in the domain of f. So the solutions of $f'(x) = 0$ in the domain are $a + \pi$ and $a + 2\pi$. The second derivative test shows that there is a local (indeed, global) maximum at $(a + \pi, 5)$ and a local (indeed, global) minimum at $(a + 2\pi, -5)$. Next, $f''(x) = 0$ when $x = b = \arctan\left(\frac{4}{3}\right)$, and by an analysis similar to that in the solution of Problem 47 it can be shown that there are inflection points at $(b, 0)$ and $(b + \pi, 0)$.

C04S06.051: We are to minimize the product of two numbers whose difference is 20; thus if x is the smaller, we are to minimize $f(x) = x(x + 20)$. Now $f'(x) = 20 + 2x$, so $f'(x) = 0$ when $x = -10$. But $f''(x) \equiv 2$ is positive when $x = -10$, so there is a local minimum at $(-10, -100)$. Because the graph of f is a parabola opening upward, this local minimum is in fact the global minimum. Answer: The two numbers are -10 and 10.

C04S06.052: We assume that the length turned upward is the same on each side—call it y. If the width of the gutter is x, then we have the constraint $xy = 18$, and we are to minimize the width $x + 2y$ of the strip. Its width is given by the function

$$f(x) = x + \frac{36}{x}, \qquad x > 0,$$

for which

$$f'(x) = 1 - \frac{36}{x^2} \qquad \text{and} \qquad f''(x) = \frac{72}{x^3}.$$

The only critical point in the domain of f is $x = 6$, and $f''(x) > 0$ on the entire domain of f. Consequently the graph of f is concave upward for all $x > 0$. Because f is continuous for such x, $f(6) = 12$ is the global minimum of f.

C04S06.053: Let us minimize

$$g(x) = (x-3)^2 + (3-2x-2)^2 = (x-3)^2 + (1-2x)^2,$$

the square of the distance from the point (x, y) on the line $2x + y = 3$ to the point $(3, 2)$. We have $g'(x) = 2(x-3) - 4(1-2x) = 10x - 10$; $g''(x) \equiv 10$. So $x = 1$ is the only critical point of g. Because $g''(x)$ is always positive, the graph of g is concave upward on the set $\mathbb{R}$ of all real numbers, and therefore $(1, g(1)) = (1, 1)$ yields the global minimum for g. So the point on the given line closest to $(3, 2)$ is $(1, 1)$.

C04S06.054: Base of box: x wide, $2x$ long. Height: y. Then the box has volume $2x^2y = 576$, so $y = 288x^{-2}$. Its total surface area is $A = 4x^2 + 6xy$, so we minimize

$$A = A(x) = 4x^2 + \frac{1728}{z}, \qquad x > 0.$$

Now

$$A'(x) = 8x - \frac{1728}{x^2} \qquad \text{and} \qquad A''(x) = 8 + \frac{3456}{x^3}.$$

The only critical point of $A(x)$ occurs when $8x^3 = 1728$; that is, when $x = 6$. But $A''(x) > 0$ for all $x > 0$, so the graph of $y = A(x)$ is concave upward for all $x > 0$. Therefore $A(6)$ is the global minimum value of $A(x)$. Also, when $x = 6$ we have $y = 8$. Answer: The dimensions of the box of minimal surface area are 6 inches wide by 12 inches long by 8 inches high.

C04S06.055: Base of box: x wide, $2x$ long. Height: y. Then the box has volume $2x^2y = 972$, so $y = 486x^{-2}$. Its total surface area is $A = 2x^2 + 6xy$, so we minimize

$$A = A(x) = 2x^2 + \frac{2916}{x}, \qquad x > 0.$$

Now

$$A'(x) = 4x - \frac{2916}{x^2} \qquad \text{and} \qquad A''(x) = 4 + \frac{5832}{x^3}.$$

The only critical point of A occurs when $x = 9$, and $A''(x)$ is always positive. So the graph of $y = A(x)$ is concave upward for all $x > 0$; consequently, $(9, A(9))$ is the lowest point on the graph of A. Answer: The dimensions of the box are 9 inches wide, 18 inches long, and 6 inches high.

C04S06.056: If the radius of the base of the pot is r and its height is h (inches), then we are to minimize the total surface area A given the constraint $\pi r^2 h = 125$. Thus $h = 125/(\pi r^2)$, and so

$$A = \pi r^2 + 2\pi rh = A(r) = \pi r^2 + \frac{250}{r}, \qquad r > 0.$$

Hence

$$A'(r) = 2\pi r - \frac{250}{r^2} \qquad \text{and} \qquad A''(r) = 2\pi + \frac{500}{r^3}.$$

Now $A'(r) = 0$ when $r^3 = 125/\pi$, so that $r = 5/\sqrt[3]{\pi}$. This is the only critical point of A, and $A''(r) > 0$ for all r, so the graph of $y = A(r)$ is concave upward for all r in the domain of A. Consequently we have located the global minimum, and it occurs when the pot has radius $r = 5/\sqrt[3]{\pi}$ inches and height $h = 5/\sqrt[3]{\pi}$ inches. Thus the pot will have its radius equal to its height, each approximately 3.414 inches.

C04S06.057: Let r denote the radius of the pot and h its height. We are given the constraint $\pi r^2 h = 250$, so $h = 250/(\pi r^2)$. Now the bottom of the pot has area πr^2, and thus costs $4\pi r^2$ cents. The curved side of the pot has area $2\pi rh$, and thus costs $4\pi rh$ cents. So the total cost of the pot is

$$C = 4\pi r^2 + 4\pi rh = C(r) = 4\pi r^2 + \frac{1000}{r}, \qquad r > 0.$$

Now

$$C'(r) = 8\pi r - \frac{1000}{r^2} \qquad \text{and} \qquad C''(r) = 8\pi + \frac{2000}{r^3}.$$

$C'(r) = 0$ when $8\pi r^3 = 1000$, so that $r = 5/\sqrt[3]{\pi}$. Because $C''(r) > 0$ for all $r > 0$, the graph of $y = C(r)$ is concave upward on the domain of C. Therefore we have found the value of r that minimizes $C(r)$. The corresponding value of h is $10/\sqrt[3]{\pi}$, so the pot of minimal cost has height equal to its diameter, each approximately 6.828 centimeters.

C04S06.058: Let x denote the length of each side of the square base of the solid and let y denote its height. Then its total volume is $x^2 y = 1000$. We are to minimize its total surface area $A = 2x^2 + 4xy$. Now $y = 1000/(x^2)$, so

$$A = A(x) = 2x^2 + \frac{4000}{x}, \qquad x > 0.$$

Therefore

$$A'(x) = 4x - \frac{4000}{x^2} \qquad \text{and} \qquad A''(x) = 4 + \frac{8000}{x^3}.$$

The only critical point occurs when $x = 10$, and $A''(x) > 0$ for all x in the domain of A, so $x = 10$ yields the global minimum value of $A(x)$. In this case, $y = 10$ as well, so the solid is indeed a cube.

C04S06.059: Let the square base of the box have edge length x and let its height be y, so that its total volume is $x^2 y = 62.5$ and the surface area of this box-without-top will be $A = x^2 + 4xy$. So

$$A = A(x) = x^2 + \frac{250}{x}, \qquad x > 0.$$

Now

$$A'(x) = 2x - \frac{250}{x^2} \qquad \text{and} \qquad A''(x) = 2 + \frac{500}{x^3}.$$

The only critical point occurs when $x = 5$, and $A''(x) > 0$ for all x in the domain of A, so $x = 5$ yields the global minimum for A. Answer: Square base of edge length $x = 5$ inches, height $y = 2.5$ inches.

C04S06.060: Let r denote the radius of the can and h its height (in centimeters). We are to minimize its total surface area $A = 2\pi r^2 + 2\pi rh$ given the constraint $\pi r^2 h = V = 16\pi$. First we note that $h = V/(\pi r^2)$, so we minimize

$$A = A(r) = 2\pi r^2 + \frac{2V}{r}, \qquad r > 0.$$

Now

$$A'(r) = 4\pi r - \frac{2V}{r^2} \qquad \text{and} \qquad A''(r) = 4\pi + \frac{4V}{r^3}.$$

The only critical point of A occurs when $4\pi r^3 = 2V = 32\pi$—that is, when $r = 2$. Now $A''(r) > 0$ for all $r > 0$, so the graph of $y = A(r)$ is concave upward for all $r > 0$. Thus the global minimum occurs when $r = 2$ centimeters, for which $h = 4$ centimeters.

C04S06.061: Let x denote the radius and y the height of the cylinder (in inches). Then its cost (in cents) is $C = 8\pi x^2 + 4\pi xy$, and we also have the constraint $\pi x^2 y = 100$. So

$$C = C(x) = 8\pi x^2 + \frac{400}{x}, \qquad x > 0.$$

Now

$$C'(x) = 16\pi x - \frac{400}{x^2} \qquad \text{and} \qquad C''(x) = 16\pi + \frac{800}{x^3}.$$

The only critical point in the domain of C is $x = \sqrt[3]{25/\pi}$ (about 1.9965 inches) and, consequently, when $y = \sqrt[3]{1600/\pi}$ (about 7.9859 inches). Because $C''(x) > 0$ for all x in the domain of C, we have indeed found the dimensions that minimize the cost of the can. For simplicity, note that $y = 4x$ at the minimum: The height of the can is twice its diameter.

C04S06.062: Let x denote the width of the print. Then $30/x$ is the height of the print, $x + 2$ is the width of the page, and $(30/x) + 4$ is the height of the page. We minimize the area A of the page, where

$$A = A(x) = (x+2)\left(\frac{30}{x} + 4\right) = 4x + 38 + \frac{60}{x}, \qquad 0 < x < \infty.$$

Now

$$A'(x) = 4 - \frac{60}{x^2} \qquad \text{and} \qquad A''(x) = \frac{120}{x^3}.$$

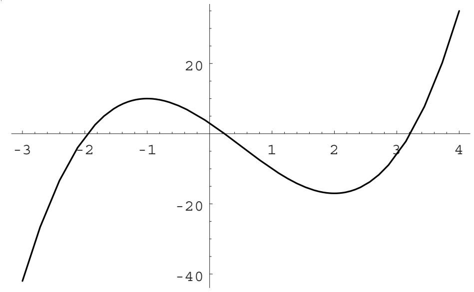
$A'(x) = 0$ when $x = \sqrt{15}$ and $A''\left(\sqrt{15}\right) = 120/\left(15\sqrt{15}\right) > 0$, so $x = \sqrt{15}$ yields a local minimum of $A(x)$. In fact, $A'(x) < 0$ if $0 < x < \sqrt{15}$ and $A'(x) > 0$ if $\sqrt{15} > x$, so $x = \sqrt{15}$ yields the global minimum value of $A(x)$; this minimum value is $4\sqrt{15} + 38 + 60/\sqrt{15} = 8\sqrt{15} + 38 \approx 68.983867$ in.^{2}

C04S06.063: Given: $f(x) = 2x^3 - 3x^2 - 12x + 3$. We have

$$f(x) = 6(x-2)(x+1) \qquad \text{and} \qquad f''(x) = 12x - 6.$$

Hence $(-1, 10)$ and $(2, -17)$ are critical points and $(0.5, -3.5)$ is a possible inflection point. Because $f''(x) > 0$ if $x > 0.5$ and $f''(x) < 0$ if $x < 0.5$, the possible inflection point is an actual inflection point,

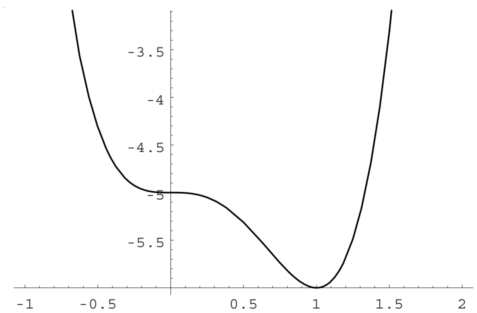
there is a local maximum at $(-1, 10)$, and a local minimum at $(2, -17)$. The extrema are not global because $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$. The graph of f is next.



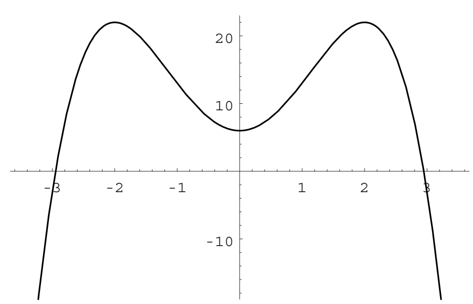
C04S06.064: Given: $f(x) = 3x^4 - 4x^3 - 5$. Then

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1) \qquad \text{and} \qquad f''(x) = 36x^2 - 24x = 12x(3x-2).$$

So the graph of f is increasing for $x > 1$ and decreasing for $x < 1$ (even though there's a horizontal tangent at $x = 0$), concave upward for $x < 0$ and $x > \frac{2}{3}$, concave downward on $\left(0, \frac{2}{3}\right)$. There is a global minimum at $(1, -6)$, inflection points at $(0, -5)$ and at $\left(\frac{2}{3}, -\frac{151}{27}\right)$. The x-intercepts are approximately -0.906212 and 1.682971. The graph of $y = f(x)$ is next.



C04S06.065: If $f(x) = 6 + 8x^2 - x^4$, then $f'(x) = -4x(x+2)(x-2)$ and $f''(x) = 16 - 12x^2$. So f is increasing for $x < -2$ and for $0 < x < 2$, decreasing otherwise; its graph is concave upward on $\left(-\frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{3}\right)$ and concave downward otherwise. Therefore the global maximum value of f is $f(-2) = f(2) = 22$ and there is a local minimum at $f(0) = 6$. There are inflection points at $\left(-\frac{2}{3}\sqrt{3}, \frac{134}{9}\right)$ and at $\left(\frac{2}{3}\sqrt{3}, \frac{134}{9}\right)$. The graph of f is next.

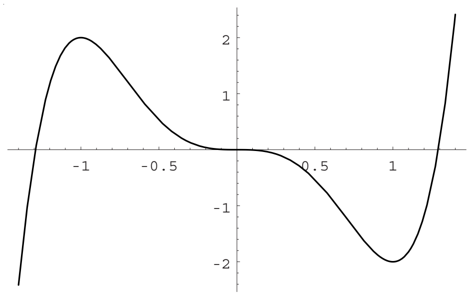


C04S06.066: Given: $f(x) = 3x^5 - 5x^3$. Then

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1) \qquad \text{and}$$

$$f''(x) = 60x^3 - 30x = 60x(x+r)(x-r) \quad \text{where} \quad r = \frac{1}{2}\sqrt{2}.$$

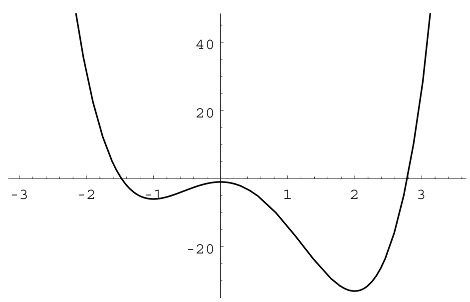
The graph is increasing for $x < -1$ and for $x > 1$, decreasing for $-1 < x < 1$ (although there is a horizontal tangent at the origin). It is concave upward on $(-r, 0)$ and on $(r, +\infty)$, concave downward on $(-\infty, -r)$ and on $(0, r)$. Thus there is a local maximum at $(-1, 2)$, a local minimum at $(1, -2)$, and inflection points at $(-r, 7r/4)$, $(0, 0)$, and $(r, -7r/4)$ (the last ordinate is approximately -1.237437). Finally, the x-intercepts are 0, $-\sqrt{5/3}$, and $\sqrt{5/3} \approx 1.29099$. The graph of $y = f(x)$ is shown next.



C04S06.067: If $f(x) = 3x^4 - 4x^3 - 12x^2 - 1$, then

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1) \quad \text{and} \quad f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2).$$

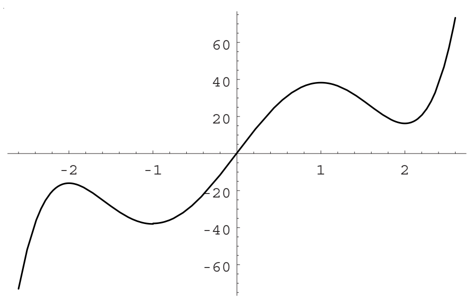
So the graph of f is decreasing for $x < -1$ and for $0 < x < 2$ and increasing otherwise; it is concave upward for $x < \frac{1}{3}\left(1 - \sqrt{7}\right)$ and for $x > \frac{1}{3}\left(1 + \sqrt{7}\right)$ and concave downward otherwise. So there is a local minimum at $(-1, -6)$, a local maximum at $(0, -1)$, and a global minimum at $(2, -33)$. There are inflection points at $\left(\frac{1}{3}\left(1 - \sqrt{7}\right), \frac{1}{27}\left(-311 + 80\sqrt{7}\right)\right)$ and at $\left(\frac{1}{3}\left(1 + \sqrt{7}\right), \frac{1}{27}\left(-311 - 80\sqrt{7}\right)\right)$. The graph of $y = f(x)$ is next.



C04S06.068: Given: $f(x) = 3x^5 - 25x^3 + 60x$. Then

$$f'(x) = 15x^4 - 75x^2 + 60 = 15(x^2 - 4)(x^2 - 1) \quad \text{and} \quad f''(x) = 60x^3 - 150x = 30x(2x^2 - 5).$$

There are local maxima where $x = -2$ and $x = 1$, local minima where $x = -1$ and $x = 2$. Inflection points occur where $x = 0$, $x = -\sqrt{5/2}$, and $x = \sqrt{5/2}$. The graph is next.

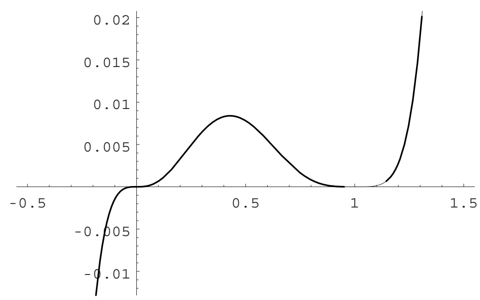


C04S06.069: If $f(x) = x^3(x-1)^4$, then

$$f'(x) = 3x^2(x-1)^4 + 4x^3(x-1)^3 = x^2(x-1)^3(7x-3) \qquad \text{and}$$

$$f''(x) = 6x(x-1)^4 + 24x^2(x-1)^3 + 12x^3(x-1)^2 = 6x(x-1)^2(7x^2 - 6x + 1).$$

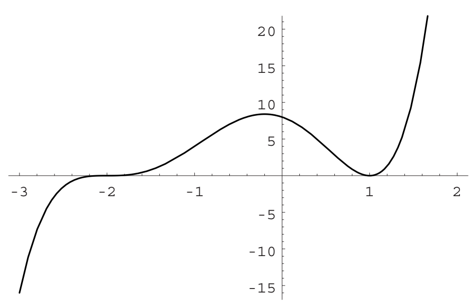
Hence the graph of f is increasing for $x < \frac{3}{7}$ and for $x > 1$, decreasing otherwise; concave upward for $0 < x < \frac{1}{7}\left(3 - \sqrt{2}\right)$ and for $x > \frac{1}{7}\left(3 + \sqrt{2}\right)$. So there is a local maximum at $\left(\frac{3}{7}, \frac{6912}{823543}\right)$ and a local minimum at $(1, 0)$. Also there are inflection points at $(0, 0)$ and at the two points with x-coordinates $\frac{1}{7}\left(3 \pm \sqrt{2}\right)$. The graph of $y = f(x)$ is shown next. Note the scale on the y-axis.



C04S06.070: Given: $f(x) = (x-1)^2(x+2)^3$. Then

$$f'(x) = (x-1)(x+2)^2(5x+1) \quad \text{and} \quad f''(x) = 2(x+2)(10x^2 + 4x - 5).$$

The zeros of $f''(x)$ are $x = -2$, $x \approx 0.535$, and $x \approx -0.935$. It follows that $(1, 0)$ is a local minimum (from the second derivative test), that $(-0.2, 8.39808)$ is a local maximum, and that $(-2, 0)$ is not an extremum. Also, the second derivative changes sign at each of its zeros, so each of these three zeros is the abscissa of an inflection point on the graph. The graph is next.

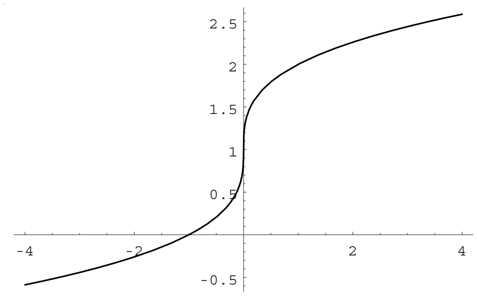


C04S06.071: If $f(x) = 1 + x^{1/3}$ then

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} \qquad \text{and} \qquad f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}.$$

Therefore $f'(x) > 0$ for all $x \neq 0$; because f is continuous even at $x = 0$, the graph of f is increasing for all x, but $(0, 1)$ is a critical point. Because $f''(x)$ has the sign of $-x$, the graph of f is concave upward for $x < 0$ and concave downward for $x > 0$. Thus there is an inflection point at $(0, 1)$. Careful examination of

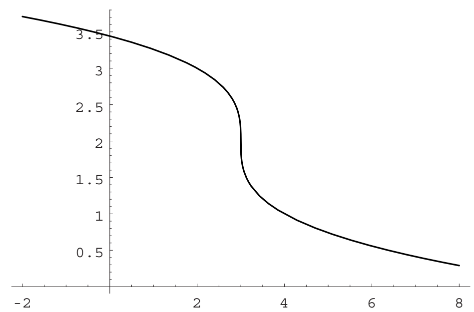
the first derivative shows also that there is a vertical tangent at (0, 1). The graph is next.



C04S06.072: Given: $f(x) = 2 - (x-3)^{1/3}$. Then

$$f'(x) = -\frac{1}{3(x-3)^{2/3}} \qquad \text{and} \qquad f''(x) = \frac{2}{9(x-3)^{5/3}}.$$

There is a vertical tangent at (3, 2) but there are no other critical points. The graph is decreasing for all x, concave down for $x < 3$, and concave up for $x > 3$. Because f is continuous for all x, there is an inflection point at (3, 2). The y-intercept is at (0, 3.44225) (ordinate approximate) and the x-intercept is at (11, 0).

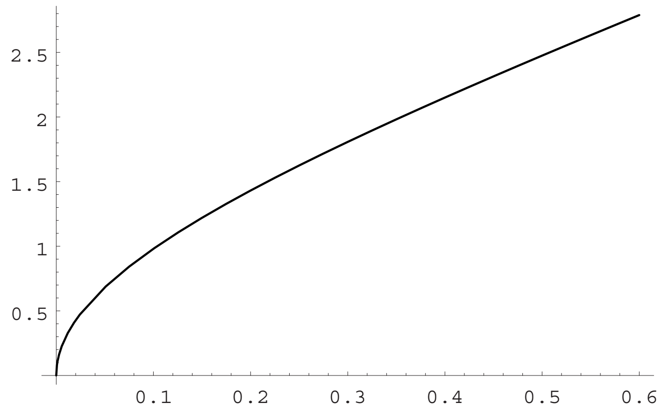


C04S06.073: Given $f(x) = (x+3)\sqrt{x}$,

$$f'(x) = \frac{3(x+1)}{2\sqrt{x}} \qquad \text{and} \qquad f''(x) = \frac{3(x-1)}{4x\sqrt{x}}.$$

Note that $f(x)$ has domain $x \geqq 0$. Hence $f'(x) > 0$ for all $x > 0$; in fact, because f is continuous (from the right) at $x = 0$, f is increasing on $[0, +\infty)$. It now follows that (1, 4) is an inflection point, but it's not shown on the following figure for two reasons: First, it's not detectable; second, the behavior of the graph near $x = 0$ is of more interest, and that behavior is not clearly visible when f is graphed on a larger interval. The point (0, 0) is, of course, the location of the global minimum of f and is of particular interest because

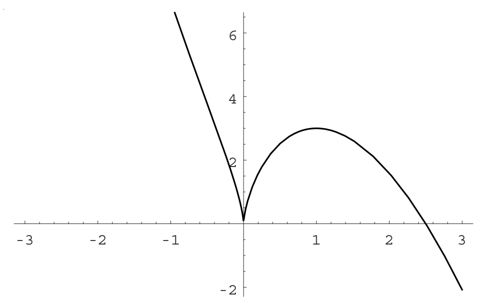
$f'(x) \to +\infty$ as $x \to 0^+$. The graph of $y = f(x)$ is next.



C04S06.074: Given: $f(x) = x^{2/3}(5 - 2x)$. Then

$$f'(x) = \frac{10 - 10x}{3x^{1/3}} \qquad \text{and} \qquad f''(x) = -\frac{20x + 10}{9x^{4/3}}.$$

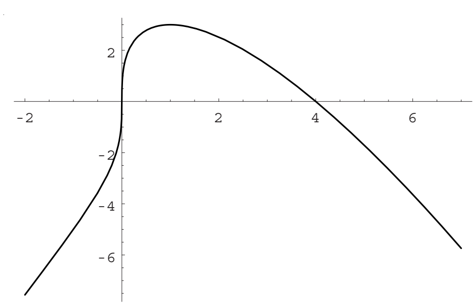
If $|x|$ is large, then $f(x) \approx -2x^{5/3}$, which (because the exponent $5/3$ has odd numerator and odd denominator) acts rather like $-2x^3$ for $|x|$ large (at least qualitatively). This aids in determining the behavior of $f(x)$ for $|x|$ large. The graph is decreasing for $x < 0$ and for $x > 1$, increasing on the interval $(0, 1)$. It is concave upward for $x < -0.5$, concave downward for $x > 0$ and on the interval $(-0.5, 0)$. There is a vertical tangent and a local minimum at the origin, a local maximum at $(1, 3)$, an inflection point where $x = -0.5$, a dual intercept at $(0, 0)$, and an x-intercept at $x = 2.5$. The graph is next.



C04S06.075: Given $f(x) = (4 - x)x^{1/3}$, we have

$$f'(x) = \frac{4(1 - x)}{3x^{2/3}} \qquad \text{and} \qquad f''(x) = -\frac{4(x + 2)}{9x^{5/3}}.$$

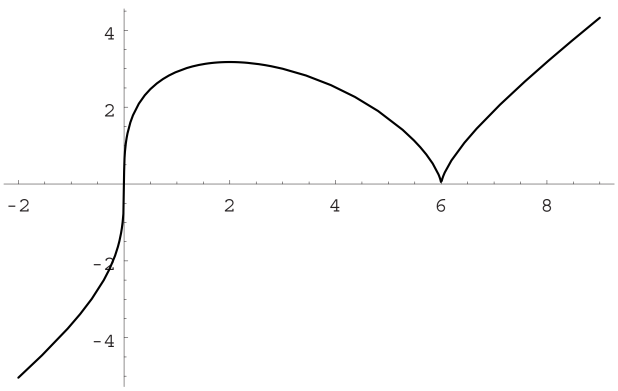
There is a global maximum at $(1, 3)$, a vertical tangent, dual intercept, and inflection point at $(0, 0)$, an x-intercept at $(4, 0)$, and an inflection point at $\left(-2, -6\sqrt[3]{2}\right)$. The graph of f is next.

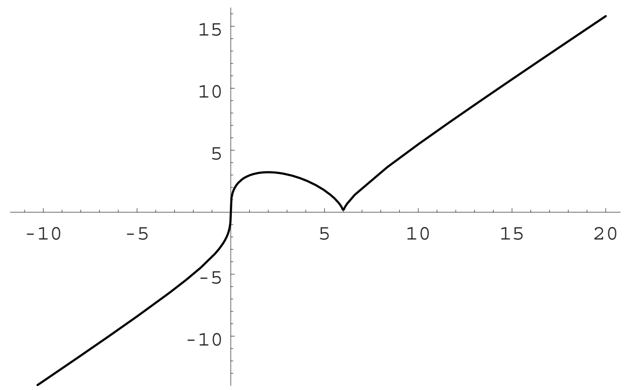


C04S06.076: Given: $f(x) = x^{1/3}(6-x)^{2/3}$. Then

$$f'(x) = \frac{2-x}{x^{2/3}(6-x)^{1/3}} \qquad \text{and} \qquad f''(x) = -\frac{8}{x^{5/3}(6-x)^{4/3}}.$$

If $|x|$ is large, then $(6-x)^{2/3} \approx x^{2/3}$, so $f(x) \approx x$ for such x. This aids in sketching the graph, which has a local maximum where $x = 2$, a local minimum at (6, 0), vertical tangents at (6, 0) and at the origin. It is increasing for $x < 2$ and for $x > 6$, decreasing on the interval (2, 6), concave upward for $x < 0$, and concave downward on (0, 6) and for $x > 6$. All the intercepts have been mentioned, too. The figure on the left shows a "close-up" of the graph and the figure on the right gives a more distant view.





C04S06.077: Figure 4.6.34 shows a graph that is concave downward, then concave upward, so its second derivative is negative, then zero, then positive. This matches the graph in (c).

C04S06.078: Figure 4.6.35 shows a graph that is concave upward, then downward, so the second derivative will be positive, then zero, then negative. This matches the graph in (e).

C04S06.079: Figure 4.6.36 shows a graph that is concave upward, then downward, then upward again, so the second derivative will be positive, then negative, then positive again. This matches the graph in (b).

C04S06.080: Figure 4.6.37 shows a graph that is concave downward, then upward, then downward, so the second derivative is negative, then positive, then negative. This matches the graph in (f).

C04S06.081: Figure 4.6.38 shows a graph that is concave upward, then almost straight, then strongly concave downward, so the second derivative must be positive, then close to zero, then large negative. This matches the graph in (d).

C04S06.082: Figure 4.6.39 shows a graph that is concave upward, then downward, then upward, then downward. So the second derivative must be positive, then negative, then positive, and then negative. This matches the graph in (a).

C04S06.083: (a): Proof: The result holds when $n = 1$. Suppose that it holds for $n = k$ where $k \geqq 1$. Then $f^{(k)}(x) = k!$ if $f(x) = x^k$. Now if $g(x) = x^{k+1}$, then $g(x) = xf(x)$. So by the product rule,

$$g'(x) = xf'(x) + f(x) = x\left(kx^{k-1}\right) + x^k = (k+1)x^k.$$

Thus

$$g^{(k+1)}(x) = (k+1)D_x^{\,k}(x^k) = (k+1)f^{(k)}(x) = (k+1)(k!) = (k+1)!.$$

That is, whenever the result holds for $n = k$, it follows for $n = k + 1$. Therefore, by induction, it holds for all integers $n \geqq 1$.

(b): Because the nth derivative of x^n is constant, any higher order derivative of x^n is zero. The result now follows immediately.

C04S06.084: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)} = -\cos x$, and $f^{(4)} = \sin x = f(x)$. It is now clear that

$$f^{(n+4)}(x) = f^{(n)}(x) \quad \text{for all } n \geqq 0$$

(we interpret $f^{(0)}(x)$ to mean $f(x)$).

C04S06.085: $\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}$. So $\dfrac{d^2z}{dx^2} = \dfrac{dz}{dy} \cdot \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} \cdot \dfrac{d^2z}{dy^2} \cdot \dfrac{dy}{dx}$.

C04S06.086: If $f(x) = Ax^2 + Bx + C$, then $A \neq 0$. So $f''(x) = 2A \neq 0$. Because $f''(x)$ never changes sign, the graph of $f(x)$ can have no inflection points.

C04S06.087: If $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$, then both $f'(x)$ and $f''(x)$ exist for all x and $f''(x) = 6ax + 2b$. The latter is zero when and only when $x = -b/(3a)$, and this is the abscissa of an inflection point because $f''(x)$ changes sign at $x = -b/(3a)$. Therefore the graph of a cubic polynomial has exactly one inflection point.

C04S06.088: If $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$, then both $f'(x)$ and $f''(x)$ are continuous for all x, and $f''(x) = 12Ax^2 + 6Bx + 2C$. In order for $f''(x)$ to change sign, we must have $f''(x) = 0$. If so, then (because $f''(x)$ is a quadratic polynomial) either the graph of $f''(x)$ crosses the x-axis in two places or is tangent to it at a single point. In the first case, $f''(x)$ changes sign twice, so there are two points of inflection on the graph of f. In the second case, $f''(x)$ does not change sign, so f has no inflection points. Therefore the graph of a polynomial of degree four has either exactly two inflection points or else none at all.

C04S06.089: First, $p = p(V) = \dfrac{RT}{V-b} - \dfrac{a}{V^2}$, so

$$p'(V) = \frac{2a}{V^3} - \frac{RT}{(V-b)^2} \quad \text{and} \quad p''(V) = \frac{2RT}{(V-b)^3} - \frac{6a}{V^4}.$$

From now on, use the constant values $p = 72.8$, $V = 128.1$, and $T = 304$; we already have $n = 1$. Then

$$p = \frac{RT}{V-b} - \frac{a}{V^2}, \quad \frac{2a}{V^3} = \frac{RT}{(V-b)^2}, \quad \text{and} \quad \frac{3a}{V^4} = \frac{RT}{(V-b)^3}.$$

The last two equations yield

$$\frac{RTV^3}{a(V-b)^2} = 2 = \frac{3(V-b)}{V},$$

and thus $b = \frac{1}{3}V$ and $V - b = \frac{2}{3}V$. Next,

$$a = \frac{V^3RT}{2(V-b)^2} = \frac{V^3RT}{2(2V/3)^2} = \frac{9V^3RT}{8V^2} = \frac{9}{8}VRT.$$

Finally, $\dfrac{RT}{V-b} = p + \dfrac{a}{V^2}$, so

$$R = \frac{2V}{3T}\left(p + \frac{a}{V^2}\right) = \frac{2V}{3T}\left(p + \frac{9RT}{8V}\right) = \frac{2Vp}{3T} + \frac{3R}{4}.$$

Therefore $R = \dfrac{8Vp}{3T}$. We substitute this into the earlier formula for a, in order to determine that $a = \frac{9}{8}VRT = 3V^2p$. In summary, and using the values given in the problem, we find that

$$b = \frac{1}{3}V = 42.7, \quad a = 3V^2p \approx 3{,}583{,}859, \quad \text{and} \quad R = \frac{8Vp}{3T} \approx 81.8.$$

C04S06.090: (a): If $f''(c) > 0$ and $h > 0$, then

$$\frac{f'(c+h)}{h} > 0$$

if h is close to zero. Thus $f'(c+h) > 0$ for such h. Similarly, if $f''(c) > 0$ and $h < 0$, then again

$$\frac{f'(c+h)}{h} > 0$$

if h is near zero, so $f'(c+h) < 0$ for such h. So $f'(x) > 0$ for $x > c$ (but close to c) and $f'(x) < 0$ for $x < c$ (but close to c). By the first derivative test, $f(c)$ is a local minimum for f. The proof in part (b) is very similar.

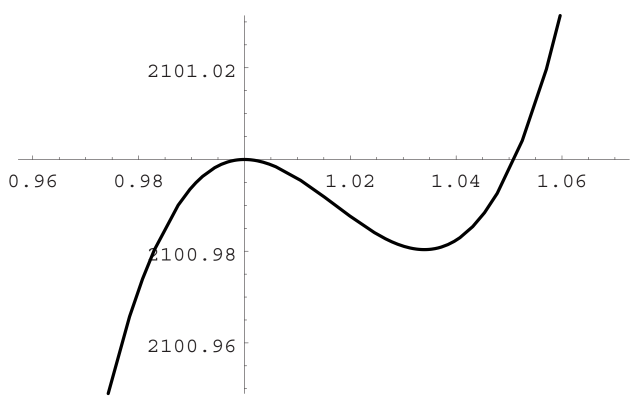
C04S06.091: If $f(x) = 1000x^3 - 3051x^2 + 3102x + 1050$, then

$$f'(x) = 3000x^2 - 6102x + 3102 \quad \text{and} \quad f''(x) = 6000x - 6102.$$

So the graph of f has horizontal tangents at the two points (1, 2101) and (1.034, 2100.980348) (coordinates exact) and there is a possible inflection point at (1.017, 2100.990174) (coordinates exact). Indeed, the usual tests show that the first of these is a local maximum, the second is a local minimum, and the third is an inflection point. The *Mathematica* command

```
Plot[ f[x], { x, 0.96, 1.07 } ];
```

produces a graph that shows all three points clearly; it's next.



C04S06.092: If $f(x) = [x(1-x)(9x-7)(4x-1)]^4$, then

$$f'(x) \;=\; 4x^3(4x-1)^3(x-1)^3(9x-7)^3(144x^3 - 219x^2 + 88x - 7)$$

and

$$f''(x) = 4x^2(4x-1)^2(x-1)^2(9x-7)^2(77760x^6 - 236520x^5 + 274065x^4 - 150400x^3 + 39368x^2 - 4312x + 147).$$

Next, the graph of $y = f(x)$ has horizontal tangents at the points

$$(0, 0), \quad (0.1052, 0.0119), \quad \left(\tfrac{1}{4}, 0\right), \quad (0.5109, 0.1539), \quad \left(\tfrac{7}{9}, 0\right), \quad (0.9048, 0.0044), \quad \text{and} \quad (1, 0)$$

(all four-place decimals shown here are rounded approximations). The usual tests show that the first, third, fifth, and seventh of these are global minima and the other three are local maxima. Moreover, there are possible inflection points at

$$(0, 0),\ (0.0609, 0.0061),\ (0.1499, 0.0069),\ \left(\tfrac{1}{4}, 0\right),\ (0.4246, 0.0865),$$

$$(0.5971, 0.0870),\ \left(\tfrac{7}{9}, 0\right),\ (0.8646, 0.0026),\ (0.9446, 0.0023),\ \text{ and }\ (1, 0)$$

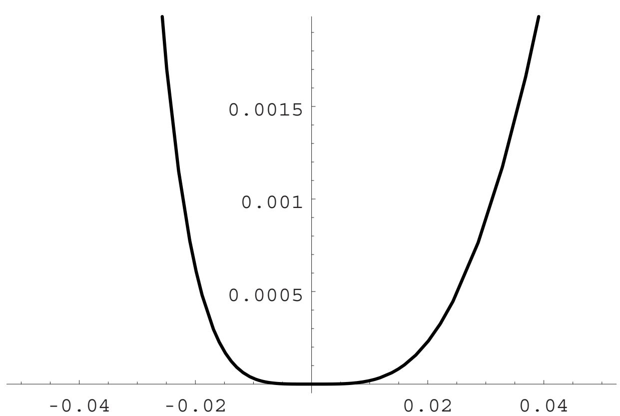
(again, all four-place decimals are rounded approximations). The usual tests show that the six of these points not extrema are indeed inflection points. The *Mathematica* command

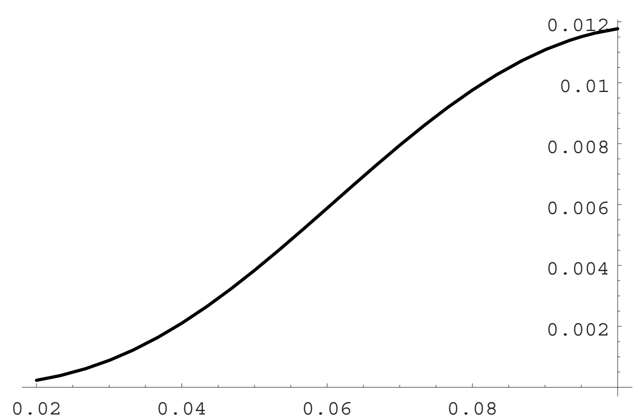
```
Plot[ f[x], { x, −0.05, 0.05 } ];
```

clearly shows the global minimum at (0, 0). The command

```
Plot[ f[x], { x, 0.02, 0.1 } ];
```

clearly shows the inflection point near (0.0609, 0.0061). These graphs are shown next.





To see the other extrema and inflection points, plot $y = f(x)$ on the intervals [0.08, 0.14], [0.12, 0.2], [0.23, 0.27], [0.3, 0.5], [0.48, 0.56], [0.54, 0.68], [0.76, 0.8], [0.8, 0.9], [0.87, 0.94], [0.92, 0.98], and (finally) [0.98, 1.04].

Section 4.7

C04S07.001: It is almost always a good idea, when evaluating the limit of a *rational* function of x as $x \to \pm\infty$, to divide each term in numerator and denominator by the highest power of x that appears there. This technique occasionally succeeds with more complicated functions. Here we obtain

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{\dfrac{x}{x}}{\dfrac{x}{x}+\dfrac{1}{x}} = \lim_{x\to\infty} \frac{1}{1+\dfrac{1}{x}} = \frac{1}{1+0} = 1.$$

C04S07.002: $\dfrac{x^2+1}{x^2-1} = \dfrac{1+(1/x^2)}{1-(1/x^2)} \to \dfrac{1+0}{1-0} = 1$ as $x \to -\infty$.

C04S07.003: $\dfrac{x^2+x-2}{x-1} = \dfrac{(x+2)(x-1)}{x-1} = x+2$ for $x \neq 1$, so $\dfrac{x^2+x-2}{x-1} \to 1+2 = 3$ as $x \to 1$.

C04S07.004: The numerator approaches -2 as $x \to -1$, whereas the denominator approaches zero. Therefore this limit does not exist.

C04S07.005: $\dfrac{2x^2-1}{x^2-3x} = \dfrac{2-(1/x^2)}{1-(3/x)} \to \dfrac{2-0}{1-0} = 2$ as $x \to +\infty$.

C04S07.006: Divide each term in numerator and denominator by x^3 to obtain

$$\lim_{x\to-\infty} \frac{x^2+3x}{x^3-5} = \lim_{x\to-\infty} \frac{\dfrac{1}{x}-\dfrac{3}{x^2}}{1-\dfrac{5}{x^3}} = \frac{0-0}{1-0} = \frac{0}{1} = 0.$$

C04S07.007: The numerator is equal to the denominator for all $x \neq -1$, so the limit is 1. If you'd prefer to write the solution more symbolically, try this:

$$\lim_{x\to-1} \frac{x^2+2x+1}{(x+1)^2} = \lim_{x\to-1} \frac{x^2+2x+1}{x^2+2x+1} = \lim_{x\to-1} 1 = 1.$$

C04S07.008: $\dfrac{5x^3-2x+1}{7x^3+4x^2-2} = \dfrac{5-(2/x^2)+(1/x^3)}{7+(4/x)-(2/x^3)} \to \dfrac{5-0+0}{7+0-0} = \dfrac{5}{7}$ as $x \to +\infty$.

C04S07.009: Factor the numerator: $x-4 = (\sqrt{x}+2)(\sqrt{x}-2)$. Thus the fraction is equal to $\sqrt{x}+2$ if $x \neq 4$. Therefore as $x \to 4$, the fraction approaches $\sqrt{4}+2 = 4$. Alternatively, for a more symbolic solution, write

$$\lim_{x\to4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to4} \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2} = \lim_{x\to4} \left(\sqrt{x}+2\right) = \sqrt{4}+2 = 2+2 = 4.$$

For another approach, use the conjugate (as discussed in Chapter 2):

$$\lim_{x\to4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to4} \frac{(x-4)\left(\sqrt{x}+2\right)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to4} \frac{(x-4)\left(\sqrt{x}+2\right)}{x-4} = \lim_{x\to4} \left(\sqrt{x}+2\right) = \sqrt{4}+2 = 4.$$

C04S07.010: Divide each term in numerator and denominator by $x^{3/2}$, the highest power of x that appears in any term. (This works well for evaluating limits of rational functions as $x \to \pm\infty$, and sometimes works for

more complicated functions.) The numerator then becomes $2x^{-1/2}+x^{-3/2}$, which approaches 0 as $x \to +\infty$; the denominator becomes $x^{-1/2}-1$, which approaches -1 as $x \to +\infty$. Therefore the limit is 0.

C04S07.011: Divide each term in numerator and denominator by x to obtain

$$\frac{\left(\dfrac{8}{x}\right)-\left(\dfrac{1}{\sqrt[3]{x^2}}\right)}{\left(\dfrac{2}{x}\right)+1} \to \frac{0-0}{0+1}=0 \quad \text{as} \quad x \to -\infty.$$

C04S07.012: Divide each term in numerator and denominator by x^3. Then

$$\lim_{x\to\infty} \frac{2x^2-17}{x^3-2x+27} = \lim_{x\to\infty} \frac{\dfrac{2}{x}-\dfrac{17}{x^3}}{1-\dfrac{2}{x^2}+\dfrac{27}{x^3}} = \frac{0-0}{1-0+0} = \frac{0}{1} = 0.$$

C04S07.013: Ignoring the radical for a moment, we divide each term in numerator and denominator by x^2, the higest power of x that appears in either. (Remember that this technique is effective with limits as $x \to \pm\infty$ but is not likely to be productive in other cases.) Result:

$$\lim_{x\to+\infty} \sqrt{\frac{4x^2-x}{x^2+9}} = \lim_{x\to+\infty} \sqrt{\frac{4-(1/x)}{1+(9/x^2)}} = \sqrt{\frac{4-0}{1+0}} = \sqrt{4} = 2.$$

C04S07.014: Divide each term in numerator and denominator by x, because in effect the "term of largest degree" in the numerator is x. Of course in the numerator we must divide each term under the radical by x^3; the result is that

$$\frac{\sqrt[3]{1-\dfrac{8}{x^2}+\dfrac{1}{x^3}}}{3-\dfrac{4}{x}} \to \frac{1}{3} \quad \text{as} \quad x \to -\infty.$$

C04S07.015: As $x \to -\infty$, $x^2+2x = x(x+2)$ is the product of two very large negative numbers, so $x^2+2x \to +\infty$. Therefore

$$\lim_{x\to-\infty} \sqrt{x^2+2x} = +\infty$$

as well. If the large *positive* number $-x$ is added to $\sqrt{x^2+2x}$, then the resulting sum also approaches $+\infty$. But arguments such as this are sometimes misleading (this subject will be taken up in detail in Sections 4.8 and 4.9), so it is more reliable to reason analytically, as follows:

$$\lim_{x\to-\infty} \left(\sqrt{x^2+2x}\ -x\right) = \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+2x}\ -x\right)\left(\sqrt{x^2+2x}\ +x\right)}{\sqrt{x^2+2x}\ +x} = \lim_{x\to-\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}\ +x}$$

$$= \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+2x}\ +x} = \lim_{x\to-\infty} \frac{\dfrac{2x}{x}}{\dfrac{\sqrt{x^2+2x}\ +x}{x}} = \lim_{x\to-\infty} \frac{2}{\dfrac{x}{x}-\sqrt{\dfrac{x^2+2x}{x^2}}} \qquad \text{(see Note 1)}$$

$$= \lim_{x\to-\infty} \frac{2}{1-\sqrt{1+\dfrac{2}{x}}} = +\infty$$

because, if x is large negative, then $1 + \dfrac{2}{x}$ is slightly smaller than 1, so that

$$1 - \sqrt{1 + \frac{2}{x}}$$

is a very small positive number, approaching zero through positive values as $x \to -\infty$.

Note 1: The minus sign is necessary because $x < 0$, and therefore $\sqrt{x^2} = -x$, not x. It's important not to miss this detail because the other sign will give the incorrect limit 1.

Note 2: There are so many dangers associated with minus signs and negative numbers in this problem that it would probably be better to let $u = -x$ and recast the problem in the form

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + 2x} \ - x \right) = \lim_{u \to +\infty} \left(\sqrt{u^2 - 2u} \ + u \right),$$

then multiply numerator and denominator by the conjugate of the numerator as in the previous calculation.

C04S07.016: The following computation is correct:

$$\lim_{x \to -\infty} \left(2x - \sqrt{4x^2 - 5x} \right) = \lim_{x \to -\infty} \frac{4x^2 - (4x^2 - 5x)}{2x + \sqrt{4x^2 - 5x}} = \lim_{x \to -\infty} \frac{5x}{2x + \sqrt{4x^2 - 5x}}$$

$$= \lim_{x \to -\infty} \frac{5}{2 + \left(\dfrac{\sqrt{4x^2 - 5x}}{-\sqrt{x^2}} \right)} = \lim_{x \to -\infty} \frac{5}{2 - \sqrt{4 - \dfrac{5}{x}}} = -\infty.$$

See the Notes for the solution of Problem 15. Following the second note, we let $u = -x$ and proceed as follows:

$$\lim_{x \to -\infty} \left(2x - \sqrt{4x^2 - 5x} \right) = \lim_{u \to +\infty} \left(-2u - \sqrt{4u^2 + 5u} \right) = -\left[\lim_{u \to +\infty} \left(2u + \sqrt{4u^2 + 5u} \right) \right] = -\infty$$

with little difficulty with negative numbers or minus signs.

C04S07.017: Matches Fig. 4.7.20(g), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.018: Matches Fig. 4.7.20(i), because $f(x) \to -\infty$ as $x \to 1^+$, $f(x) \to +\infty$ as $x \to 1^-$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.019: Matches Fig. 4.7.20(a), because $f(x) \to +\infty$ as $x \to 1$ and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.020: Matches Fig. 4.7.20(d), because $f(x) \to -\infty$ as $x \to 1$ and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.021: Matches Fig. 4.7.20(f), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, $f(x) \to +\infty$ as $x \to -1^-$, $f(x) \to -\infty$ as $x \to -1^+$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.022: Matches Fig. 4.7.20(c), because $f(x) \to -\infty$ as $x \to 1^+$, $f(x) \to +\infty$ as $x \to 1^-$, $f(x) \to -\infty$ as $x \to -1^-$, $f(x) \to +\infty$ as $x \to -1^+$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.023: Matches Fig. 4.7.20(j), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, $f(x) \to -\infty$ as $x \to -1^-$, $f(x) \to +\infty$ as $x \to -1^+$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.024: Matches Fig. 4.7.20(h), because $f(x) \to -\infty$ as $x \to 1^+$, $f(x) \to +\infty$ as $x \to 1^-$, $f(x) \to +\infty$ as $x \to -1^-$, $f(x) \to -\infty$ as $x \to -1^+$, and $f(x) \to 0$ as $x \to \pm\infty$.

C04S07.025: Matches Fig. 4.7.20(l), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, and $f(x) \to 1$ as $x \to \pm\infty$.

C04S07.026: Matches Fig. 4.7.20(b), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, $f(x) \to +\infty$ as $x \to -1^-$, $f(x) \to -\infty$ as $x \to -1^+$, and $f(x) \to 1$ as $x \to \pm\infty$.

C04S07.027: Matches Fig. 4.7.20(k), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, and because

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1} \qquad \text{if} \quad x \neq 1,$$

the graph of f has the slant asymptote with equation $y = x + 1$.

C04S07.028: Matches Fig. 4.7.20(e), because $f(x) \to +\infty$ as $x \to 1^+$, $f(x) \to -\infty$ as $x \to 1^-$, $f(x) \to -\infty$ as $x \to -1^-$, $f(x) \to +\infty$ as $x \to -1^+$, and because

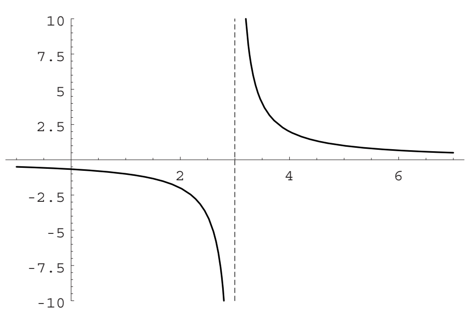
$$f(x) = \frac{x^3}{x^2-1} = x + \frac{x}{x^2-1},$$

the graph of f has the slant asymptote $y = x$.

C04S07.029: Given $f(x) = \dfrac{2}{x-3}$, we find that

$$f'(x) = -\frac{2}{(x-3)^2} \qquad \text{and} \qquad f''(x) = \frac{4}{(x-3)^3}.$$

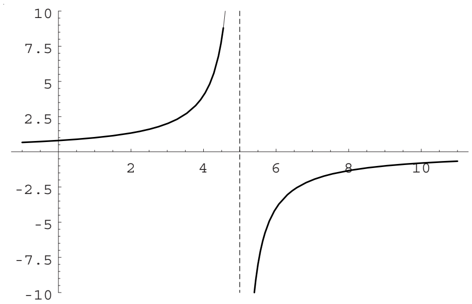
So there are no extrema or inflection points, the only intercept is $\left(0, -\frac{2}{3}\right)$, and $f(x) \to +\infty$ as $x \to 3^+$ whereas $f(x) \to -\infty$ as $x \to 3^-$. So the line $x = 3$ is a vertical asymptote. Also $f(x) \to 0$ as $x \to \pm\infty$, so the line $y = 0$ is a [two-way] horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.030: Given $f(x) = \dfrac{4}{5-x}$, we find that

$$f'(x) = \frac{4}{(5-x)^2} \qquad \text{and} \qquad f''(x) = \frac{8}{(5-x)^3}.$$

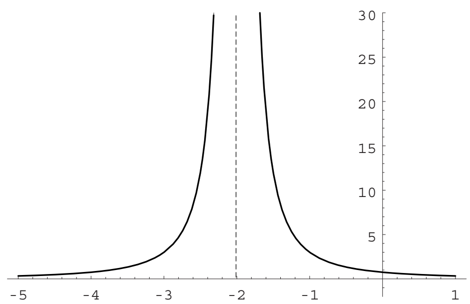
So there are no extrema or inflection points, the only intercept is $\left(0, \frac{4}{5}\right)$, $x = 5$ is a vertical asymptote, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.031: Given $f(x) = \dfrac{3}{(x+2)^2}$, we find that

$$f'(x) = -\frac{6}{(x+2)^3} \qquad \text{and} \qquad f''(x) = \frac{18}{(x+2)^4}.$$

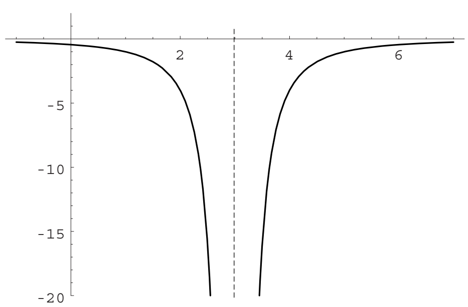
So there are no extrema or inflection points, the only intercept is $\left(0, \frac{3}{4}\right)$, the line $x = -2$ is a vertical asymptote, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.



C04S07.032: Given $f(x) = -\dfrac{4}{(3-x)^2}$, we find that

$$f'(x) = -\frac{8}{(3-x)^3} \qquad \text{and} \qquad f''(x) = -\frac{24}{(3-x)^4}.$$

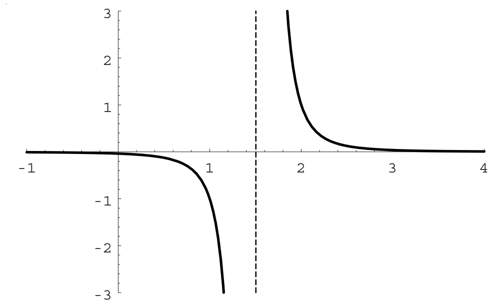
So there are no extrema or inflection points, the only intercept is $\left(0, -\frac{4}{9}\right)$, the line $x = 3$ is a vertical asymptote, and the line $y = 0$ is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.033: If $f(x) = \dfrac{1}{(2x-3)^3}$, it follows that

$$f'(x) = -\frac{6}{(2x-3)^4} \qquad \text{and} \qquad f''(x) = \frac{48}{(2x-3)^5}.$$

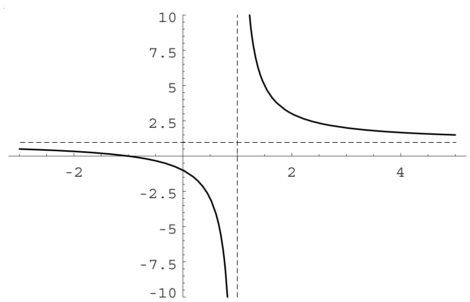
So there are no extrema or inflection points, the only intercept is $\left(0,\, -\frac{1}{27}\right)$, the line $x = \frac{3}{2}$ is a vertical asymptote, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.034: Given: $f(x) = \dfrac{x+1}{x-1}$. Then

$$f'(x) = -\frac{2}{(x-1)^2} \qquad \text{and} \qquad f''(x) = \frac{4}{(x-1)^3}.$$

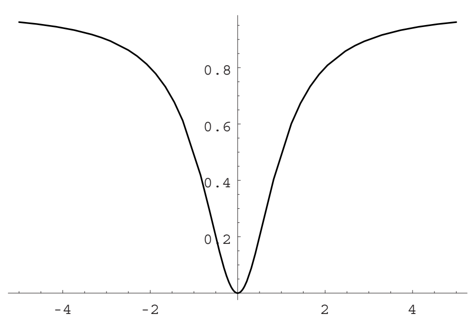
Thus there are no extrema or inflection points, the only intercepts are $(-1,\, 0)$ and $(0,\, -1)$, the line $x = 1$ is a vertical asymptote, and the line $y = 1$ is a [two-way] horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.035: Given: $f(x) = \dfrac{x^2}{x^2+1}$. Then

$$f'(x) = \frac{2x}{(x^2+1)^2} \qquad \text{and} \qquad f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}.$$

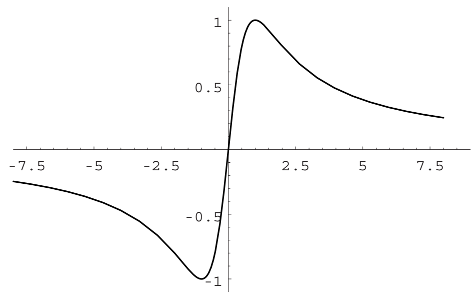
Thus the only intercept is $(0,\, 0)$, where there is a global minimum, there are inflection points at $\left(\pm\frac{1}{3}\sqrt{3},\, \frac{1}{4}\right)$, and the line $y = 1$ is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next (without the asymptote).



C04S07.036: Given: $f(x) = \dfrac{2x}{x^2+1}$, we find that

$$f'(x) = \frac{2(1 - x^2)}{(x^2 + 1)^2} \qquad \text{and} \qquad f''(x) = \frac{4x(x^2 - 3)}{(x^2 + 1)^3}.$$

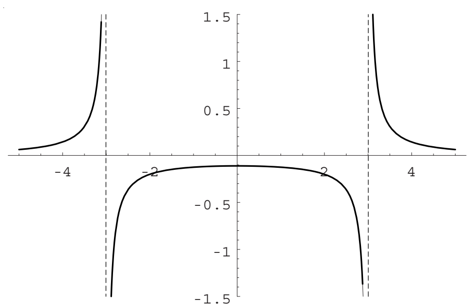
There is a global minimum at $(-1, -1)$, a global maximum at $(1, 1)$, and the three points $\left(-\sqrt{3}, -\frac{1}{2}\sqrt{3}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{1}{2}\sqrt{3}\right)$ are inflection points. Because $f(x) \to 0$ as $x \to \pm\infty$, the y-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.



C04S07.037: If $f(x) = \dfrac{1}{x^2 - 9}$, then

$$f'(x) = -\frac{2x}{(x^2 - 9)^2} \qquad \text{and} \qquad f''(x) = \frac{6(x^2 + 3)}{(x^2 - 9)^3}.$$

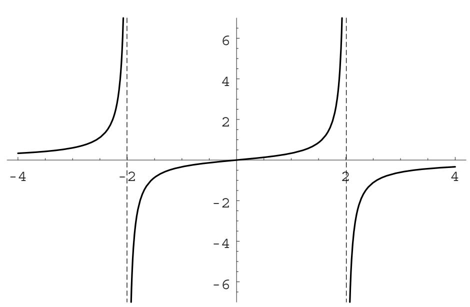
So there is a local maximum at $\left(0, -\frac{1}{9}\right)$, which is also the only intercept; there are vertical asymptotes at $x = -3$ and at $x = 3$, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next



C04S07.038: Given: $f(x) = \dfrac{x}{4 - x^2}$. Then

$$f'(x) = \frac{x^2 + 4}{(x^2 - 4)^2} \qquad \text{and} \qquad f''(x) = -\frac{2x(x^2 + 12)}{(x^2 - 4)^3}.$$

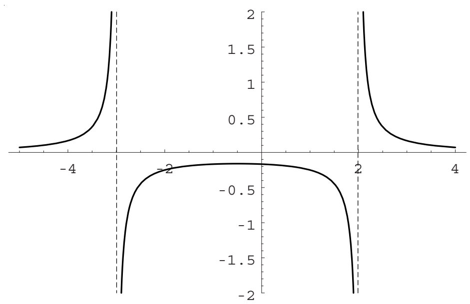
Thus there are no extrema, the origin is the only intercept and the only inflection point, there are vertical asymptotes at $x = \pm 2$, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.039: Given $f(x) = \dfrac{1}{x^2 + x - 6} = \dfrac{1}{(x-2)(x+3)}$, we find that

$$f'(x) = -\frac{2x+1}{(x^2+x-6)^2} \qquad \text{and} \qquad f''(x) = \frac{2(x^2+3x+7)}{(x^2+x-6)^3}.$$

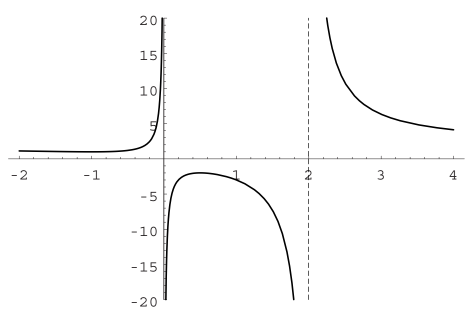
Thus there is a local maximum at $\left(-\frac{1}{2},\, -\frac{4}{25}\right)$ and the only intercept is at $\left(0,\, -\frac{1}{6}\right)$. The lines $x = -3$ and $x = 2$ are vertical asymptotes and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.



C04S07.040: If $f(x) = \dfrac{2x^2+1}{x^2-2x} = \dfrac{2x^2+1}{x(x-2)}$, then

$$f'(x) = -\frac{2(2x^2+x-1)}{x^2(x-2)^2} \qquad \text{and} \qquad f''(x) = \frac{2(4x^3+3x^2-6x+4)}{x^3(x-2)^3}.$$

So there is a local minimum at $(-1,\, 1)$, a local maximum at $\left(\frac{1}{2},\, -2\right)$, and an inflection point close to $(-1.851708, 1.101708)$. There are no intercepts, the lines $x = 0$ and $x = 2$ are vertical asymptotes, and the line $y = 2$ is a horizontal asymptote (not shown in the figure). A *Mathematica*-generated graph of $y = f(x)$ is next.

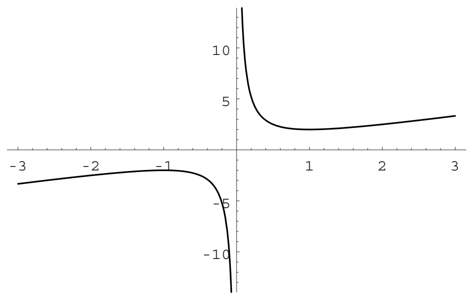


C04S07.041: Given: $f(x) = x + \dfrac{1}{x} = \dfrac{x^2+1}{x}$, we find that

$$f'(x) = \frac{x^2-1}{x^2} \qquad \text{and that} \qquad f''(x) = \frac{2}{x^3}.$$

Hence there is a local maximum at $(-1,\, -2)$, a local minimum at $(1,\, 2)$, and no inflection points or intercepts. The y-axis is a vertical asymptote and the line $y = x$ is a slant asymptote (not shown in the figure). A

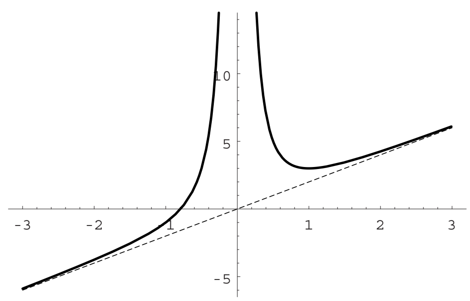
Mathematica-generated graph of $y = f(x)$ is next.



C04S07.042: If $f(x) = 2x + \dfrac{1}{x^2} = \dfrac{2x^2+1}{x^2}$, then

$$f'(x) = \frac{2(x^3-1)}{x^3} \qquad \text{and} \qquad f''(x) = \frac{6}{x^4}.$$

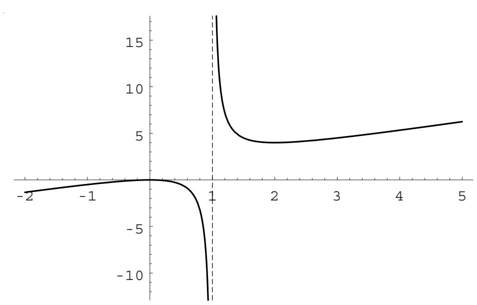
So there are no inflection points, a local minimum at $(1, 3)$; the only intercept is $\left(-\frac{1}{2}\sqrt[3]{4},\, 0\right)$, the y-axis is a vertical asymptote, and the line $y = 2x$ is a slant asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.043: If $f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$, then

$$f'(x) = \frac{x(x-2)}{(x-1)^2} \qquad \text{and} \qquad f''(x) = \frac{2}{(x-1)^3}.$$

So $(0, 0)$ is a local maximum and the only intercept, there is a local minimum at $(2, 4)$, the line $x = 1$ is a vertical asymptote, and the line $y = x + 1$ is a slant asymptote (not shown in the figure). A *Mathematica*-generated graph of $y = f(x)$ is shown next.



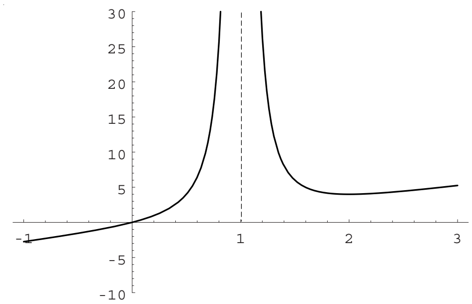
C04S07.044: Given:

$$f(x) = \frac{2x^3 - 5x^2 + 4x}{x^2 - 2x + 1} = \frac{x(2x^2 - 5x + 4)}{(x-1)^2} = 2x - 1 + \frac{1}{(x-1)^2},$$

we first compute

$$f'(x) = \frac{2(x-2)(x^2-x+1)}{(x-1)^3} \qquad \text{and} \qquad f''(x) = \frac{6}{(x-1)^4}.$$

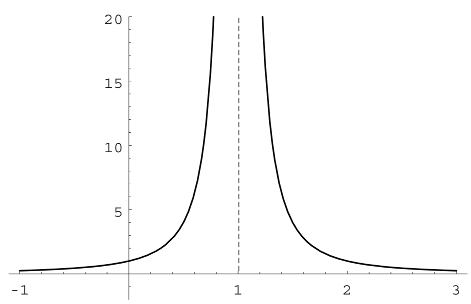
So there is a local minimum at (2, 4), (0, 0) is the only intercept, there are no inflection points, the line $x = 1$ is a vertical asymptote, and the line $y = 2x - 1$ is a slant asymptote (not shown in the figure). A *Mathematica*-generated graph of $y = x(f)$ is next.



C04S07.045: If $f(x) = \dfrac{1}{(x-1)^2}$, then

$$f'(x) = -\frac{2}{(x-1)^3} \qquad \text{and} \qquad f''(x) = \frac{6}{(x-1)^4}.$$

Hence there are no extrema or inflection points, the only intercept is (0, 1), the line $x = 1$ is a vertical asymptote, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.

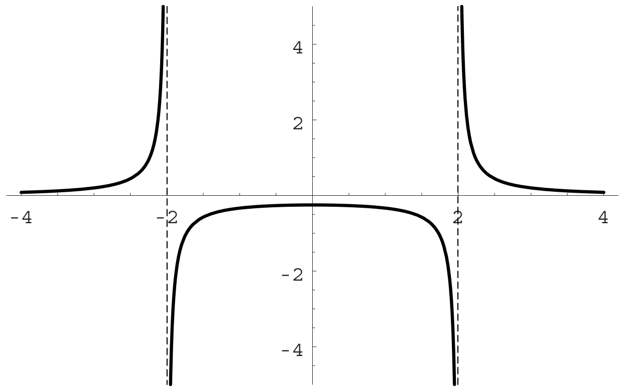


C04S07.046: Given: $f(x) = \dfrac{1}{x^2-4}$. Then

$$f'(x) = -\frac{2x}{(x^2-4)^2} \qquad \text{and} \qquad f''(x) = \frac{2(3x^2+4)}{(x^2-4)^3}.$$

The only critical point is at $\left(0, -\frac{1}{4}\right)$, which is a local maximum. There are no inflection points and no x-intercepts; the only y-intercept is the local maximum. The x-axis is a horizontal asymptote and the lines

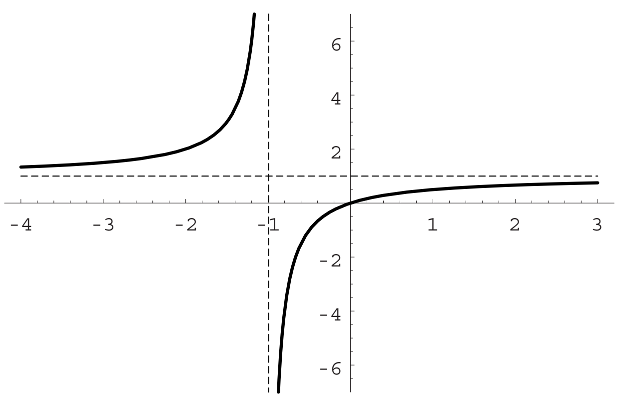
$x = \pm 2$ are vertical asymptotes. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.047: If $f(x) = \dfrac{x}{x+1} = 1 - \dfrac{1}{x+1}$, then

$$f'(x) = \frac{1}{(x+1)^2} \qquad \text{and} \qquad f''(x) = -\frac{2}{(x+1)^3}.$$

Therefore (0, 0) is the only intercept, there are no extrema or inflection points, the line $x = -1$ is a vertical asymptote, and the line $y = 1$ is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.

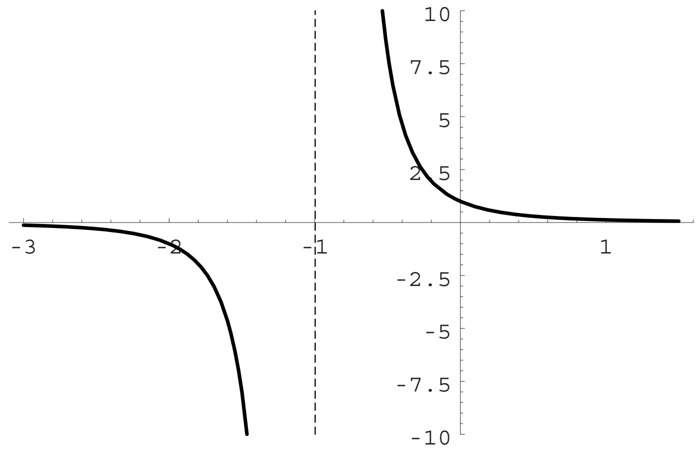


C04S07.048: If $f(x) = \dfrac{1}{(x+1)^3}$, then

$$f'(x) = -\frac{3}{(x+1)^4} \qquad \text{and} \qquad f''(x) = \frac{12}{(x+1)^5}.$$

Therefore there are no extrema or inflection points, the only intercept is (0, 1), the line $x = -1$ is a vertical

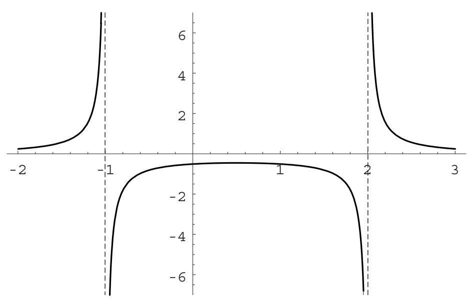
asymptote, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.049: If $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x-2)(x+1)}$, then

$$f'(x) = \frac{1 - 2x}{(x^2 - x - 2)^2} \qquad \text{and} \qquad f''(x) = \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}.$$

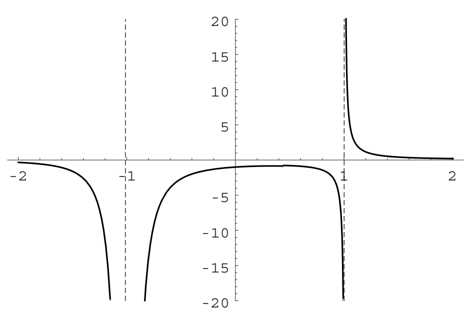
Therefore $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a local maximum and the only extremum, $\left(0, -\frac{1}{2}\right)$ is the only intercept, the lines $x = -1$ and $x = 2$ are vertical asymptotes, and the x-axis is a horizonal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.050: If $f(x) = \dfrac{1}{(x-1)(x+1)^2}$, then

$$f'(x) = \frac{1 - 3x}{(x-1)^2(x+1)^3} \qquad \text{and} \qquad f''(x) = \frac{4(3x^2 - 2x + 1)}{(x-1)^3(x+1)^4}.$$

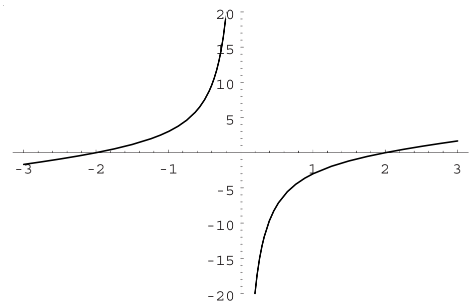
Therefore $(0, -1)$ is the only intercept, $\left(\frac{1}{3}, -\frac{27}{32}\right)$ is a local maximum and the only extremum, the lines $x = -1$ and $x = 1$ are vertical asymptotes, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.



C04S07.051: Given: $f(x) = \dfrac{x^2 - 4}{x} = x - \dfrac{4}{x}$, we find that

$$f'(x) = \frac{x^2 + 4}{x^2} \qquad \text{and that} \qquad f''(x) = -\frac{8}{x^3}.$$

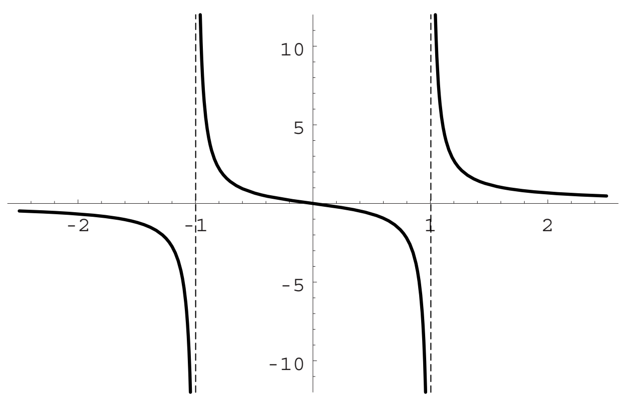
Therefore $(-2, 0)$ and $(2, 0)$ are the only intercepts, there are no inflection points or extrema, the y-axis is a vertical asymptote, and the line $y = x$ is a slant asymptote (not shown in the figure). A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.052: If $f(x) = \dfrac{x}{x^2 - 1}$, then

$$f'(x = -\frac{x^2 + 1}{(x^2 - 1)^2} \qquad \text{and} \qquad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

So $(0, 0)$ is an inflection point and the only intercept, there are no extrema, the lines $x = -1$ and $x = 1$ are vertical asymptotes, and the x-axis is a horizontal asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.

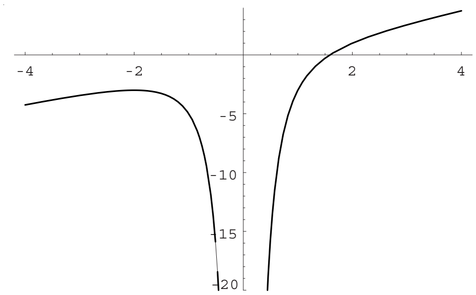


C04S07.053: If $f(x) = \dfrac{x^3 - 4}{x^2} = x - \dfrac{4}{x^2}$, then

$$f'(x) = \frac{x^3 + 8}{x^3} \qquad \text{and} \qquad f''(x) = -\frac{24}{x^4}.$$

Thus $\left(\sqrt[3]{4},\, 0\right)$ is the only intercept, there is a local maximum at $(-2, -3)$ and no other extrema, and there are no inflection points. The y-axis is a vertical asymptote and the line $y = x$ is a slant asymptote (not

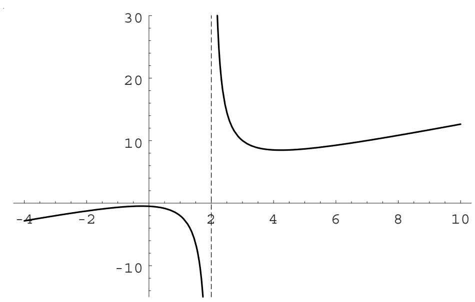
shown in the figure). A *Mathematica*-generated graph of $y = f(x)$ is next.



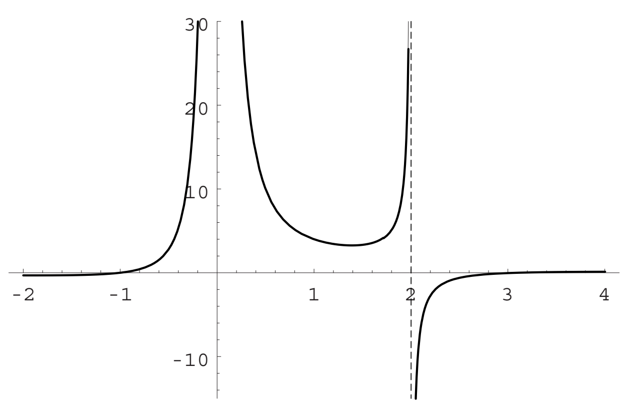
C04S07.054: If $f(x) = \dfrac{x^2+1}{x-2} = x + 2 + \dfrac{5}{x-2}$, then

$$f'(x) = \frac{x^2 - 4x - 1}{(x-2)^2} \qquad \text{and} \qquad f''(x) = \frac{10}{(x-2)^3}.$$

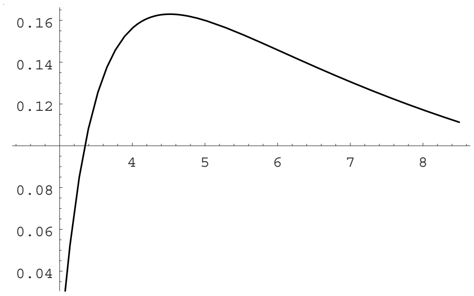
Therefore the only intercept is $\left(0, -\frac{1}{2}\right)$ and there are no inflection points. There is a local maximum at $\left(2-\sqrt{5},\, 4-2\sqrt{5}\right)$ and a local minimum at $\left(2+\sqrt{5},\, 4+2\sqrt{5}\right)$. The line $x = 2$ is a vertical asymptote and the line $y = x + 2$ is a slant asymptote (not shown in the figure). A *Mathematica*-generated graph of $y = f(x)$ is next.

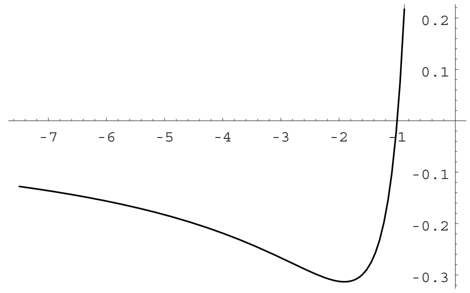


C04S07.055: The x-axis is a horizontal asymptote, and there are vertical asymptotes at $x = 0$ and $x = 2$. There are local minima at $(-1.9095, -0.3132)$ and $(1.3907, 3.2649)$ and a local maximum at $(4.5188, 0.1630)$ (all coordinates approximate, of course), and inflection points at $(-2.8119, -0.2768)$ and $(6.0623, 0.1449)$. A *Mathematica*-generated graph of $y = f(x)$ is next.

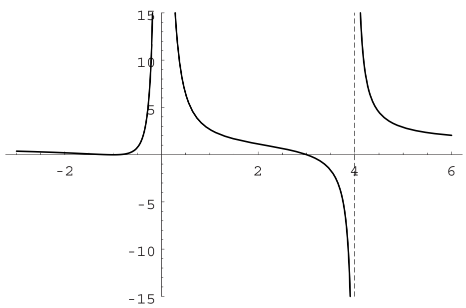


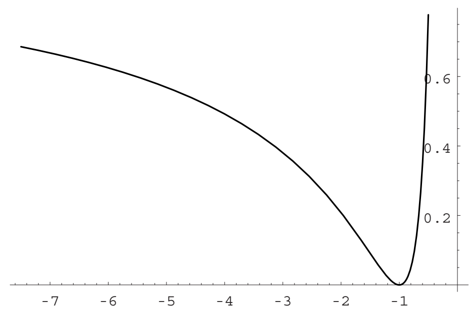
A "close-up" of the graph for $3 \leqq x \leqq 8$ is next, on the left, and another for $-7 \leqq x \leqq -1$ is on the right.



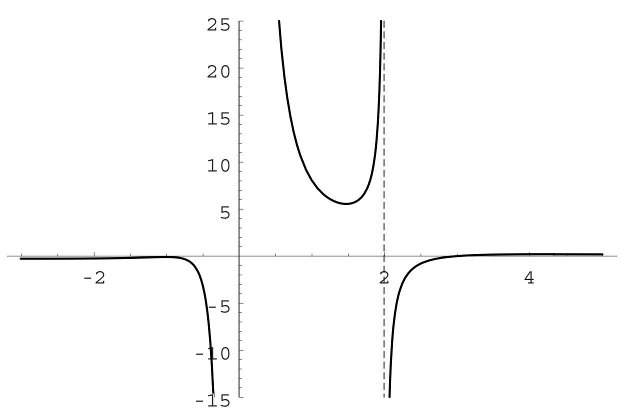


C04S07.056: The line $y = 1$ is a horizontal asymptote (not shown in the figures) and the lines $x = 0$ and $x = 4$ are vertical asymptotes. There is a local minimum at $(-1,\ 0)$ and inflection points at $(-1.5300,\ 0.0983)$ and $(2.1540,\ 0.9826)$ (numbers with decimal points are approximations). A *Mathematica*-generated graph of $y = f(x)$ is next, on the left; a close-up of the graph for $-7.5 \leqq x \leqq -0.5$ is on the right.



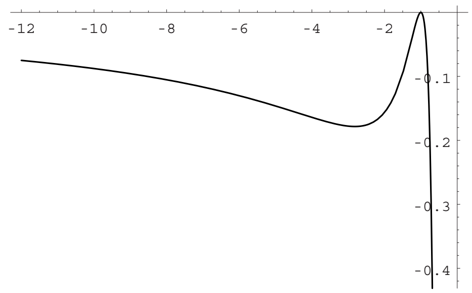


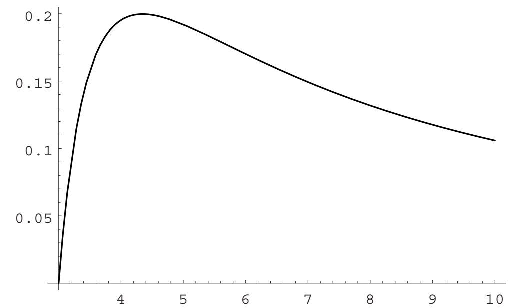
C04S07.057: The x-axis is a horizontal asymptote and there are vertical asymptotes at $x = 0$ and $x = 2$. There are local minima at $(-2.8173,\ -0.1783)$ and $(1.4695,\ 5.5444)$ and local maxima at $(-1,\ 0)$ and $(4.3478,\ 0.1998)$. There are inflection points at the three points $(-4.3611,\ -0.1576)$, $(-1.2569,\ -0.0434)$, and $(5.7008,\ 0.1769)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next.



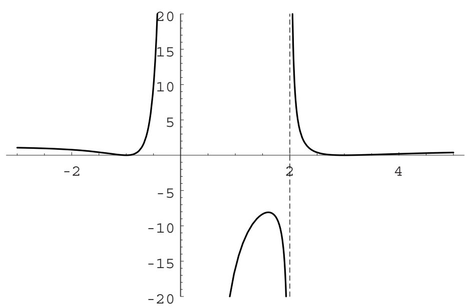
A "close-up" of the graph for $-12 \leqq x \leqq -0.5$ is shown next, on the left; the graph for $3 \leqq x \leqq 10$ is on the

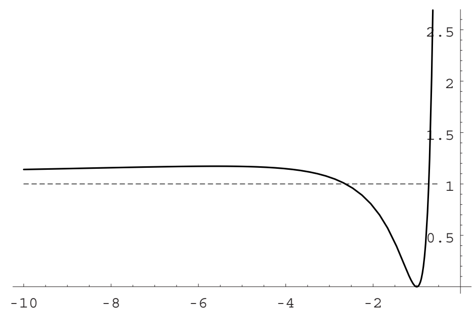
right.



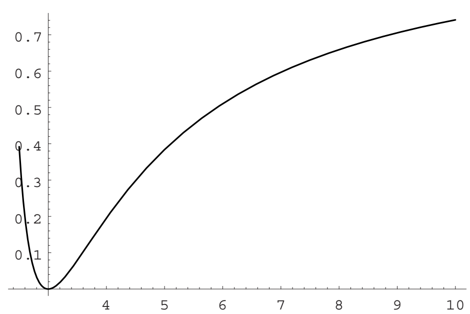


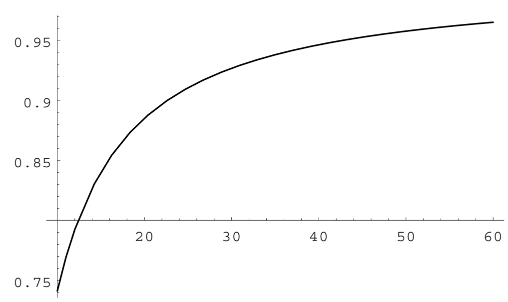
C04S07.058: The horizontal line $y = 1$ is an asymptote, as are the vertical lines $x = 0$ and $x = 2$. There are local maxima at $(-5.6056, 1.1726)$ and $(1.6056, -8.0861)$, local minima at $(-1, 0)$, and $(3, 0)$. (Numbers with decimal points are approximations.) There are inflection points at $(-8.54627, 1.15324)$, $(-1.29941, 0.228917)$, and $(3.67765, 0.120408)$. A *Mathematica*-generated graph of $y = f(x)$ is shown next, on the left; the graph for $-10 \leqq x \leqq -0.5$ is on the right along with the horizontal asymptote.





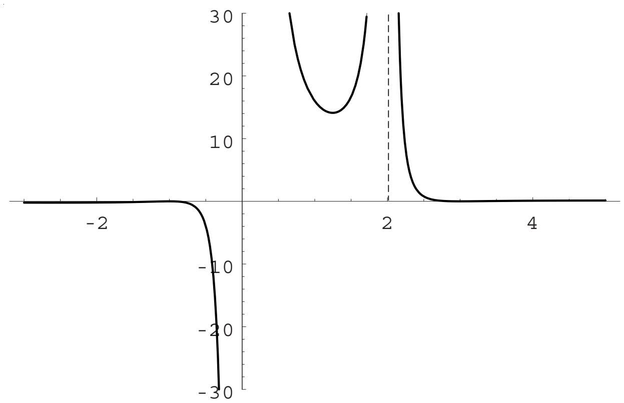
The graph for $2.5 \leqq x \leqq 10$ is next, on the left; the graph for $10 \leqq x \leqq 60$ is on the right.



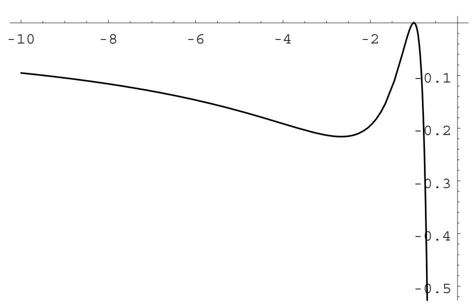


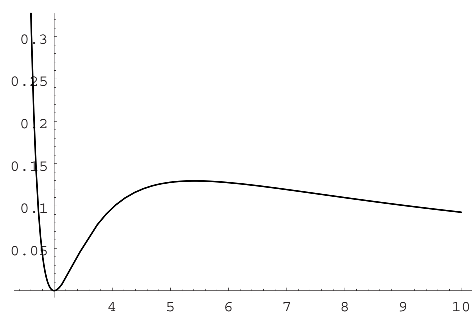
C04S07.059: The horizontal line $y = 0$ is an asymptote, as are the vertical lines $x = 0$ and $x = 2$. There are local minima at $(-2.6643, -0.2160)$, $(1.2471, 14.1117)$, and $(3, 0)$; there are local maxima at $(-1, 0)$ and $(5.4172, 0.1296)$. There are inflection points at $(-4.0562, -0.1900)$, $(-1.2469, -0.0538)$, $(3.3264, 0.0308)$, and $(7.4969, 0.1147)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph

of $y = f(x)$ is shown next.

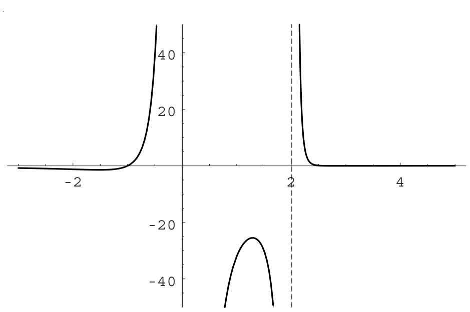


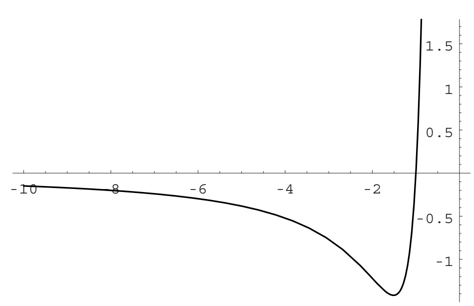
The graph for $-10 \leqq x \leqq -0.5$ is next, on the left; the graph for $2.5 \leqq x \leqq 10$ is on the right.



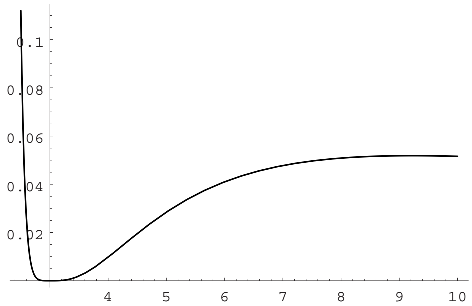


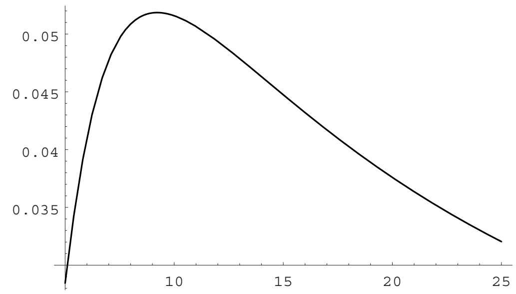
C04S07.060: The horizontal line $y = 0$ is an asymptote, as are the vertical lines $x = 0$ and $x = 2$. There are local minima at $(-1.5125,\ -1.4172)$ and $(3,\ 0)$, local maxima at $(1.2904,\ -25.4845)$ and $(9.2221,\ 0.0519)$ and inflection points at $(-2.0145,\ -1.2127)$, $(4.2422,\ 0.0145)$, and $(14.2106,\ 0.0460)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next, on the left; the graph for $-10 \leqq x \leqq -0.75$ is on the right.



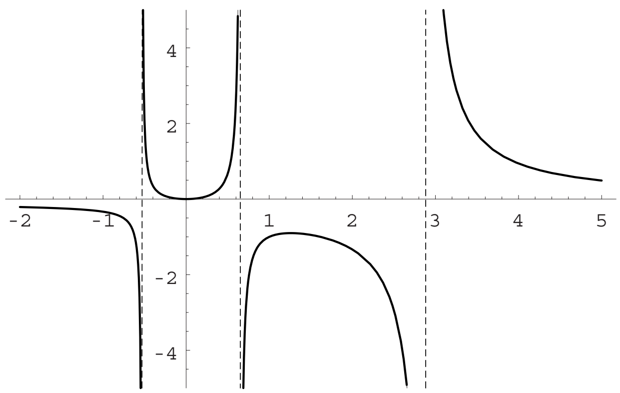


The graph for $2.5 \leqq x \leqq 10$ is next, on the left; the graph for $5 \leqq x \leqq 25$ is on the right.

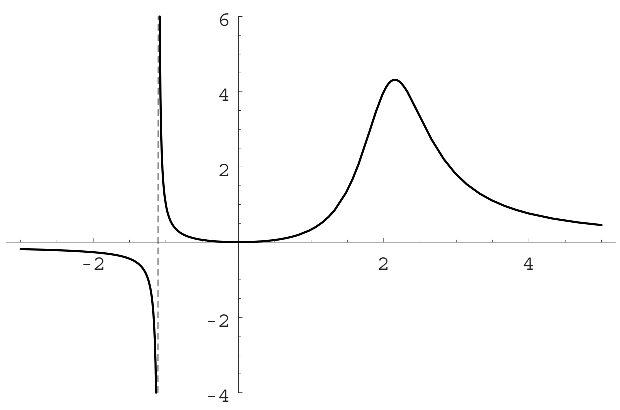




C04S07.061: The x-axis is a horizontal asymptote; there are vertical asymptotes at $x = -0.5321$, $x = 0.6527$, and $x = 2.8794$. There is a local minimum at $(0,\,0)$ and a local maximum at $(\sqrt[3]{2},\,-0.9008)$. There are no inflection points (Numbers with decimal points are approximations.) A *Mathematics*-generated graph of $y = f(x)$ is next.

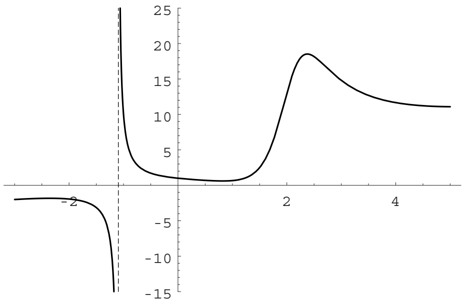


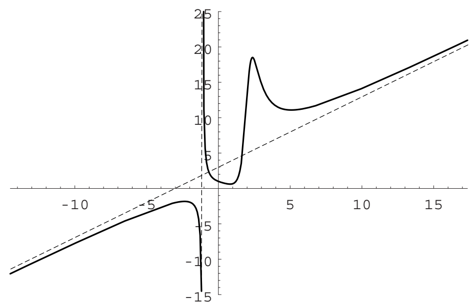
C04S07.062: The x-axis is a horizontal asymptote; there is a vertical asymptote at $x = -1.1038$. There is a local minimum at $(0,\,0)$ and a local maximum at $(2.1544,\,4.3168)$. There are inflection points at $(1.8107,\,2.9787)$ and $(2.4759,\,3.4299)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next.



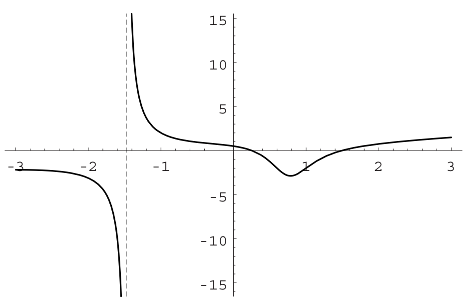
C04S07.063: The line $y = x + 3$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There is a vertical asymptote at $x = -1.1038$. There are local maxima at $(-2.3562,\,-1.8292)$ and $(2.3761,\,18.5247)$, local minima at $(0.8212,\,0.6146)$ and $(5.0827,\,11.0886)$. There are inflection points at $(1.9433,\,11.3790)$ and $(2.7040,\,16.8013)$. (Numbers with decimal points are

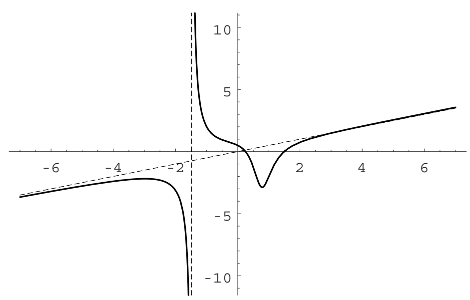
approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next, on the left; on the right the graph is shown on a wider scale, together with its slant asymptote.





C04S07.064: The line $2y = x$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There is a vertical asymptote at $x = -1.4757$. There is a local maximum at $(-2.9821, -2.1859)$ and a local minimum at $(0.7868, -2.8741)$. There are inflection points at $(-0.2971, 0.7736)$, $(0.5713, 0.5566)$, $(1, -2)$, and $(9.1960, 4.6515)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is shown next, on the left; the graph is also shown on the right, on a wider scale, along with its slant asymptote.





C04S07.065: Given $f(x) = \dfrac{x^5 - 4x^2 + 1}{2x^4 - 3x + 2}$, we first find that

$$f'(x) = \frac{2x^8 + 4x^5 + 10x^4 - 8x^3 + 12x^2 - 16x + 3}{(2x^4 - 3x + 2)^2} \qquad \text{and}$$

$$f''(x) = -\frac{2(30x^8 + 24x^7 - 40x^6 + 90x^5 - 102x^4 - 28x^3 + 24x^2 + 7)}{(2x^4 - 3x + 2)^3}.$$

The line $2y = x$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There also are no vertical asymptotes. There is

a local maximum at $(0.2200976580, 0.6000775882)$,

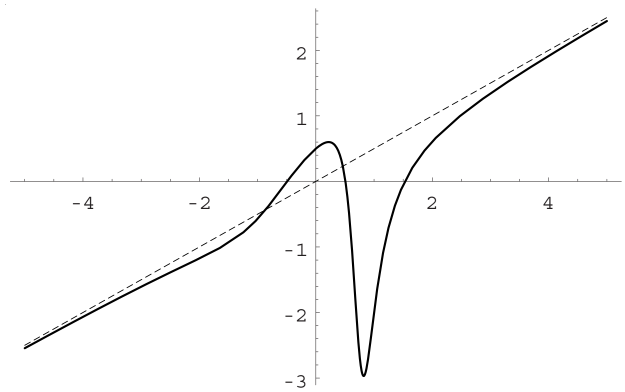
a local minimum at $(0.8221567934, -2.9690453671)$,

and inflection points at

$(-2.2416918017, -1.2782199626)$, $(-0.5946286318, -0.1211409770)$,

$(0.6700908810, -1.6820255735)$, and $(0.96490314661, -2.2501145861)$.

(Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next.



C04S07.066: The line $2y = x$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There also are no vertical asymptotes (the denominator in $f(x)$ is never zero). There are x-intercepts where

$$x = -2.05667157818, \quad x = 0.847885655376, \quad \text{and} \quad x = 1.929095045219$$

and $\left(0, \frac{2}{5}\right)$ is the y-intercept. (Numbers with decimal points are approximations throughout.) There are local maximum at

$$(-1.137867740647, 0.426255896993) \quad \text{and} \quad (0.472659729564, 0.585125167363)$$

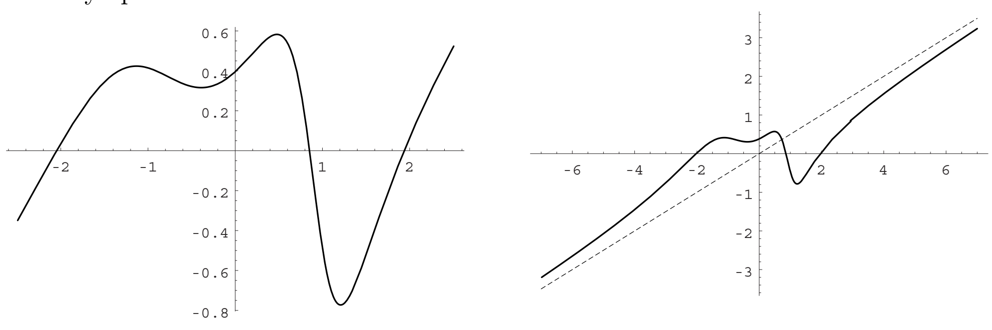
and local minima at

$$(-0.394835802615, 0.318479939692) \quad \text{and} \quad (1.203561740743, -0.770172527937).$$

There are inflection points at

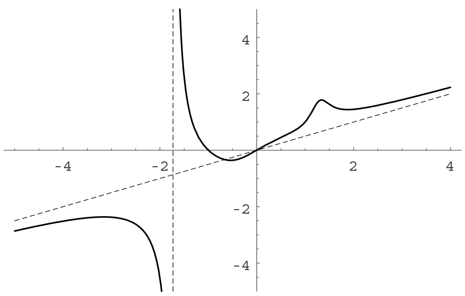
$$\begin{aligned} (-2.381297805169, -0.253127890429), &\qquad (-0.775152255017, 0.373332211612), \\ (0.189601709800, 0.486606037763), &\qquad (0.890253166310, -0.145751465654), \\ \text{and} &\qquad (1.553505928225, -0.444592872637). \end{aligned}$$

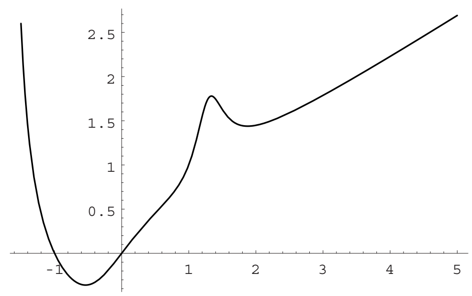
A *Mathematica*-generated graph of $y = f(x)$ is shown next, on the left; a wider view is on the right, along with the slant asymptote.



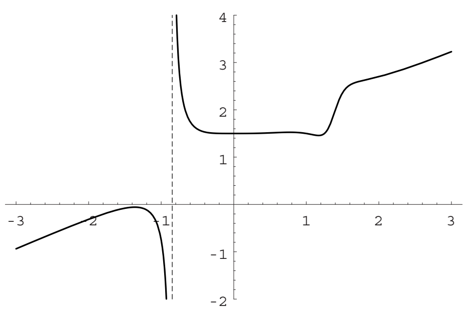
C04S07.067: The line $2y = x$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There is a vertical asymptote at $x = -1.7277$. There are local maxima at

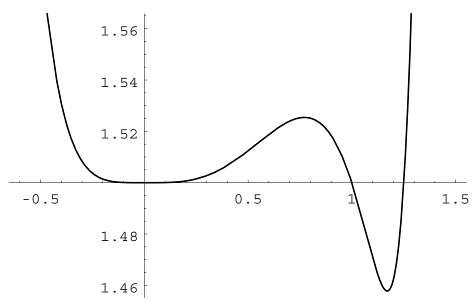
$(-3.1594, -2.3665)$ and $(1.3381, 1.7792)$, local minima at $(-0.5379, -0.3591)$ and $(1.8786, 1.4388)$. There are inflection points at $(0, 0)$, $(0.5324, 0.4805)$, $(1.1607, 1.4294)$, and $(1.4627, 1.6727)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is next, on the left; the figure on the right shows the graph for $-1.5 \leqq x \leqq 5$.





C04S07.068: The line $6x + 10 = 9y$ is a slant asymptote in both the positive and negative directions; thus there is no horizontal asymptote. There is a vertical asymptote at $x = -0.8529$. There are local maxima at $(-1.3637, -0.0573)$ and $(0.7710, 1.5254)$, local minima at $\left(0, \frac{3}{2}\right)$ and $(1.1703, 1.4578)$. There are inflection points at $(0.5460, 1.5154)$, $(1.0725, 1.4793)$, $(1.3880, 1.9432)$, and $(1.8247, 2.6353)$. (Numbers with decimal points are approximations.) A *Mathematica*-generated graph of $y = f(x)$ is shown next, on the left (with the vertical asymptote but without the slant asymptote). The figure on the right shows the graph of f for $-0.5 \leqq x \leqq 1.5$.



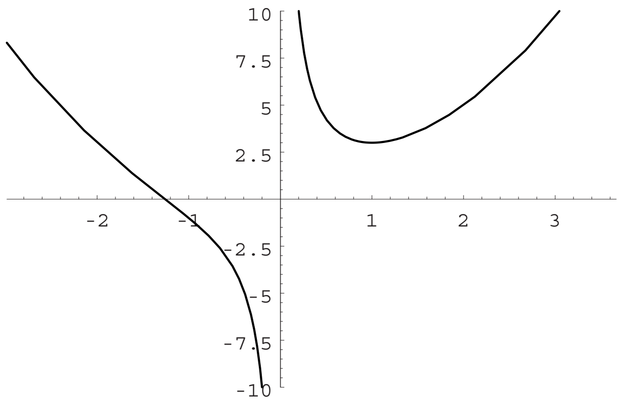


C04S07.069: Sketch the parabola $y = x^2$, but modify it by changing its behavior near $x = 0$: Let $y \to +\infty$ as $x \to 0^+$ and let $y \to -\infty$ as $x \to 0^-$. Using calculus, we compute

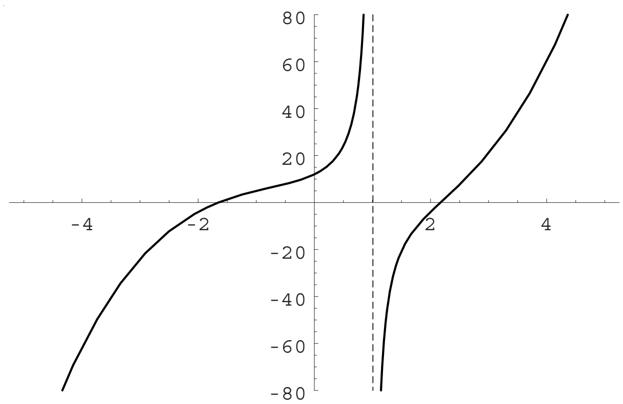
$$f'(x) = \frac{2(x^3 - 1)}{x^2} \qquad \text{and} \qquad f''(x) = \frac{2(x^3 + 2)}{x^3}.$$

It follows that the graph of f is decreasing for $0 < x < 1$ and for $x < 0$, increasing for $x > 1$. It is concave upward for $x < -\sqrt[3]{2}$ and also for $x > 0$, concave downward for $-\sqrt[3]{2} < x < 0$. The only intercept is at $\left(-\sqrt[3]{2}, 0\right)$; this is also the only inflection point. There is a local minimum at $(1, 3)$. The y-axis is a vertical

asymptote. A *Mathematica*-generated graph of $y = f(x)$ is shown next.



C04S07.070: Because $f(x) \approx x^3$ when $|x|$ is large, we obtain the graph of f by making "modifications" in the graph of $y = x^3$ at and near the discontinuity of $f(x)$ at $x = 1$. We are aided in sketching the graph of f by finding its x-intercepts—these are approximately -1.654 and 2.172—as well as its y-intercept 12 and its inflection points $(2.22,\ 1.04)$ and $(-0.75,\ 6.44)$ (also approximations). A *Mathematica*-generated graph of f appears next.



Chapter 4 Miscellaneous Problems

C04S0M.001: Given: $x^3 = \sin^2 y$:

$$3x^2 = (2\sin y\cos y)\frac{dy}{dx}, \qquad \text{and thus} \qquad \frac{dy}{dx} = \frac{3x^2}{2\sin y\cos y}.$$

C04S0M.002: Given: $x^2y^2 = x + y$. Then

$$2xy^2 + 2x^2y\frac{dy}{dx} = 1 + \frac{dy}{dx}, \qquad \text{and hence} \qquad \frac{dy}{dx} = \frac{1-2xy^2}{2x^2y-1}.$$

C04S0M.003: Given: $x^{1/3} + y^{1/3} = 4$:

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0, \qquad \text{and so} \qquad \frac{dy}{dx} = -\frac{x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3}.$$

C04S0M.004: Given: $x^3 + y^3 = xy$:

$$\begin{aligned}
3x^2 + 3y^2\frac{dy}{dx} &= y + x\frac{dy}{dx};\\
(3y^2 - x)\frac{dy}{dx} &= y - 3x^2;\\
\frac{dy}{dx} &= \frac{y-3x^2}{3y^2-x}.
\end{aligned}$$

C04S0M.005: Given: $x^2 - x^2y + xy^2 - y^2 = 4$:

$$\begin{aligned}
3x^2 - 2xy - x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= 0;\\
(x^2 - 2xy + 3y^2)\frac{dy}{dx} &= 3x^2 - 2xy + y^2;\\
\frac{dy}{dx} &= \frac{3x^2-2xy+y^2}{x^2-2xy+3y^2}.
\end{aligned}$$

C04S0M.006: Given: $(x+y)^{1/2} = (x-y)^{1/3}$.
First solution: Raise both sides to the 6th power to eliminate fractional exponents. Thus

$$\begin{aligned}
(x+y)^3 &= (x-y)^2;\\
3(x+y)^2\left(1+\frac{dy}{dx}\right) &= 2(x-y)\left(1-\frac{dy}{dx}\right);\\
3(x+y)^2\frac{dy}{dx} + 2(x-y)\frac{dy}{dx} &= 2(x-y) - 3(x+y)^2;\\
\frac{dy}{dx} &= \frac{2(x-y)-3(x+y)^2}{3(x+y)^2+2(x-y)};
\end{aligned}$$

$$\frac{dy}{dx} = \frac{2(x+y)^{3/2} - 3(x+y)^2}{3(x+y)^2 + 2(x+y)^{3/2}} = \frac{2 - 3\sqrt{x+y}}{2 + 3\sqrt{x+y}}.$$

Second solution: Differentiate both sides immediately. Thus

$$\begin{aligned}
&\frac{1}{2}(x+y)^{-1/2}\left(1 + \frac{dy}{dx}\right) = \frac{1}{3}(x-y)^{-2/3}\left(1 - \frac{dy}{dx}\right);\\
&\frac{1}{2}(x+y)^{-1/2}\frac{dy}{dx} + \frac{1}{3}(x-y)^{-2/3}\frac{dy}{dx} = \frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2};\\
&\frac{dy}{dx} = \frac{\frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}}.
\end{aligned}$$

You can further simplify the second answer to obtain the first.

C04S0M.007: Given: $xy - x - y = 1$ and the point $P(0, -1)$. First,

$$x\frac{dy}{dx} + y - 1 - \frac{dy}{dx} = 0, \quad \text{and so} \quad \frac{dy}{dx} = \frac{1-y}{x-1}.$$

Hence the slope of the line L tangent to the graph of the given equation at the point P is

$$m = \frac{1-(-1)}{0-1} = -2.$$

Therefore an equation of L is

$$y - (-1) = -2(x-0); \quad \text{that is,} \quad 2x + y + 1 = 0.$$

C04S0M.008: Given: The equation $x = \sin 2y$ and the point $P(1, \pi/4)$. First,

$$1 = (2\cos 2y)\frac{dy}{dx}, \quad \text{and hence} \quad \frac{dy}{dx} = \frac{1}{2\cos 2y}.$$

More to the point in this problem, we see also that $dx/dy = 2\cos 2y$, so that at the point P we have $dx/dy = 0$. Thus the line tangent to the graph of the given equation at the point P is vertical; its equation is $x = 1$.

C04S0M.009: Given: The equation $x^2 - 3xy + 2y^2 = 0$ and the point $P(2, 1)$. First,

$$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0, \quad \text{so that} \quad \frac{dy}{dx} = \frac{3y-2x}{4y-3x}.$$

Hence the straight line L tangent to the graph of the equation at P has slope

$$m = \frac{3-4}{4-6} = \frac{1}{2},$$

and thus equation $y - 1 = m(x-2)$; that is, $2y = x$.

C04S0M.010: Given: The equation $y^3 = x^2 + x$ and the point $P(0, 0)$. A *Mathematica* 3.0 command for obtaining the derivative dy/dx by implicit differentiation is

```
Solve[ D[ (y[x])∧3 == x∧2 + x, x ], y′[x] ]
```

and, in response, the computer returns

$$y'(x) = \frac{-(-1-2x)}{3y^2}; \qquad \text{that is,} \qquad \frac{dy}{dx} = \frac{2x+1}{3y^2}.$$

Although dy/dx is undefined at the point P, there we have

$$\left.\frac{dx}{dy}\right|_{(0,0)} = \frac{3 \cdot 0}{2 \cdot 0 + 1} = 0.$$

Hence the straight line tangent to the graph of the given equation at the point P is vertical; it has equation $x = 0$.

C04S0M.011: $dy = \frac{3}{2}(4x - x^2)^{1/2}(4 - 2x)\,dx.$

C04S0M.012: $dy = [\,24x^2(x^2+9)^{1/2} + 4x^3(x^2+9)^{-1/2}(2x)\,]\,dx.$

C04S0M.013: $dy = -\dfrac{2}{(x-1)^2}\,dx.$

C04S0M.014: $dy = 2x\cos(x^2)\,dx = 2x\cos x^2\,dx.$

C04S0M.015: $dy = \left(2x\cos\sqrt{x} - \frac{1}{2}x^{3/2}\sin\sqrt{x}\right)\,dx.$

C04S0M.016: $dy = \dfrac{\sin 2x - 2x\cos 2x}{\sin^2 2x}\,dx.$

C04S0M.017: Let $f(x) = x^{1/2}$; $f'(x) = \frac{1}{2}x^{-1/2}$. Then

$$\begin{aligned}\sqrt{6401} = f(6400+1) &\approx f(6400) + 1 \cdot f'(6400)\\ &= 80 + \frac{1}{160} = \frac{12801}{160} = 80.00625.\end{aligned}$$

(A calculator reports that $\sqrt{6401} \approx 80.00624976$.)

C04S0M.018: Choose $f(x) = \dfrac{1}{x}$; $f'(x) = -\dfrac{1}{x^2}$. Choose $x = 1$ and $\Delta x = 0.000007$. Then

$$\begin{aligned}\frac{1}{1.000007} = f(x+\Delta x) &\approx f(x) + f'(x)\,\Delta x\\ &= 1 - 0.000007 = 0.999993.\end{aligned}$$

C04S0M.019: Let $f(x) = x^{10}$; then $f'(x) = 10x^9$. Choose $x = 2$ and $\Delta x = 0.0003$. Then

$$\begin{aligned}(2.0003)^{10} = f(x+\Delta x) &\approx f(x) + f'(x)\,\Delta x\\ &= 2^{10} + 10 \cdot 2^9 \cdot (0.0003) = 1024 + (5120)(0.0003) = 1024 + 1.536 = 1025.536.\end{aligned}$$

A calculator reports that $(2.0003)^{10} \approx 1025.537$.

C04S0M.020: Let $f(x) = x^{1/3}$; then $f'(x) = \frac{1}{3}x^{-2/3}$. Choose $x = 1000$ and $\Delta x = -1$. Then

$$\begin{aligned}\sqrt[3]{999} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (1000)^{1/3} + \frac{1}{3\cdot(1000)^{2/3}}\cdot(-1) = 10 - \frac{1}{3\cdot 100} = \frac{2999}{300} \approx 9.996667.\end{aligned}$$

A calculator reports that $\sqrt[3]{999} \approx 9.996665$.

C04S0M.021: Let $f(x) = x^{1/3}$; then $f'(x) = \frac{1}{3}x^{-2/3}$. Choose $x = 1000$ and $\Delta x = 5$. Then

$$\begin{aligned}\sqrt[3]{1005} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (1000)^{1/3} + \frac{1}{3(1000)^{2/3}}\cdot 5 = 10 + \frac{5}{3\cdot 100} = \frac{601}{60} \approx 10.0167.\end{aligned}$$

A calculator reports that $\sqrt[3]{1005} \approx 10.0166$.

C04S0M.022: Take $f(x) = x^{1/3}$; then $f'(x) = \frac{1}{3}x^{-2/3}$. Take $x = 64$ and $\Delta x = -2$. Thus

$$\begin{aligned}(62)^{1/3} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (64)^{1/3} + (-2)\left(\frac{1}{3}\right)(64)^{-2/3} = \frac{95}{24} \approx 3.958.\end{aligned}$$

C04S0M.023: Let $f(x) = x^{3/2}$; then $f'(x) = \frac{3}{2}x^{1/2}$. Let $x = 25$ and let $\Delta x = 1$. Then

$$\begin{aligned}26^{3/2} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (25)^{3/2} + \frac{3}{2}\cdot(25)^{1/2}\cdot 1 = 125 + 7.5 = 132.5.\end{aligned}$$

C04S0M.024: Take $f(x) = x^{1/5}$; then $f'(x) = \frac{1}{5}x^{-4/5}$. Take $x = 32$ and $\Delta x = -2$. Then

$$\begin{aligned}(30)^{1/5} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (32)^{1/5} + (-2)\left(\frac{1}{5}\right)(32)^{-4/5} = 2 - \frac{1}{40} = \frac{79}{40} = 1.975.\end{aligned}$$

C04S0M.025: Let $f(x) = x^{1/4}$; then $f'(x) = \frac{1}{4}x^{-3/4}$. Let $x = 16$ and $\Delta x = 1$. Then

$$\begin{aligned}\sqrt[4]{17} = (17)^{1/4} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (16)^{1/4} + \frac{1}{4\cdot(16)^{3/4}}\cdot 1 = 2 + \frac{1}{4\cdot 8} = \frac{65}{32} = 2.03125.\end{aligned}$$

C04S0M.026: With $f(x) = x^{1/10}$, $f'(x) = \frac{1}{10}x^{-9/10}$, $x = 1024$, and $\Delta x = -24$, we obtain

$$\begin{aligned}(1000)^{1/10} &= f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x\\ &= (1024)^{1/10} + (-24)\left(\frac{1}{10}\right)(1024)^{-9/10} = 2 - \frac{3}{640} = \frac{1277}{640} \approx 1.9953.\end{aligned}$$

C04S0M.027: The volume V of a cube of edge s is given by $V(s) = s^3$. So $dV = 3s^2\,ds$, and thus with $s = 5$ and $\Delta s = 0.1$ we obtain $\Delta V \approx 3(5)^2(0.1) = 7.5$ (cubic inches).

C04S0M.028: With radius r and area $A = \pi r^2$, we have $dA = 2\pi r\,dr$. We take $r = 10$ and $\Delta r = -0.2$ to obtain $\Delta A \approx (2\pi)(10)(-0.2) = -4\pi\ \ (\text{cm}^2)$.

C04S0M.029: The volume V of a sphere of radius r is given by $V(r) = \frac{4}{3}\pi r^3$. Hence $dV = 4\pi r^2\ dr$, so with $r = 5$ and $Deltar = \frac{1}{10}$ we obtain

$$\Delta V \approx 4\pi \cdot 25 \cdot \frac{1}{10} = 10\pi \quad (\text{cm}^3).$$

C04S0M.030: Given $V = \dfrac{1000}{p}$, it follows that $dV = -\dfrac{1000}{p^2}\ dp$. Therefore, with $p = 100$ and $\Delta p = -1$, we obtain

$$\Delta V \approx -\frac{1000}{(100)^2} \cdot (-1) = 0.1 \quad (\text{cubic inches}).$$

C04S0M.031: If

$$T = 2\pi\sqrt{\frac{L}{32}} = 2\pi\left(\frac{L}{32}\right)^{1/2}, \qquad \text{then} \qquad dT = \pi\left(\frac{L}{32}\right)^{-1/2} \cdot \frac{1}{32}\,dL = \frac{\pi}{32}\left(\frac{32}{L}\right)^{1/2}\,dL.$$

Therefore if $L = 2$ and $\Delta L = \frac{1}{12}$, we obtain

$$\Delta T \approx dT = \frac{\pi}{32}\left(\frac{32}{2}\right)^{1/2} \cdot \frac{1}{12} = \frac{\pi}{32} \cdot \frac{4}{12} = \frac{\pi}{96} \approx 0.0327 \quad (\text{seconds}).$$

C04S0M.032: Here, $dL = (-13)(10^{30})E^{-14}\,dE$. We take $E = 110$ and $\Delta E = +1$ and obtain

$$\Delta L \approx (-13)(10^{30})\left(110^{-14}\right)(+1) \approx -342 \ \ (\text{hours}).$$

The actual decrease is $L(110) - L(111) \approx 2896.6 - 2575.1 \approx 321.5$ (hours).

C04S0M.033: First, $f'(x) = 1 + \dfrac{1}{x^2}$, so $f'(x)$ exists for $1 < x < 3$ and f is continuous for $1 \leqq x \leqq 3$. So we are to solve

$$\frac{f(3) - f(1)}{3 - 1} = f'(c);$$

that is,

$$\frac{3 - \frac{1}{3} - 1 + 1}{2} = 1 + \frac{1}{c^2}.$$

After simplifications we find that $c^2 = 3$. Therefore, because $1 < c < 3$, $c = +\sqrt{3}$.

C04S0M.034: Every polynomial is continuous and differentiable everywhere, so all hypotheses are met.

$$\frac{f(3) - f(-2)}{3 - (-2)} = \frac{26 - (-14)}{5} = 8 = f'(c) = 3c^2 + 1,$$

so $c^2 = 7/3$. Both roots lie in $(-2, 3)$, so both $+\sqrt{7/3}$ and $-\sqrt{7/3}$ are solutions.

C04S0M.035: Every polynomial is continuous and differentiable everywhere, so all hypotheses of the mean value theorem are satisfied. Then

$$\frac{f(2)-f(-1)}{2-(-1)} = \frac{8+1}{3} = 3 = f'(c) = 3c^2,$$

so $c^2 = 1$. But -1 does not lie in the interval $(-1, 2)$, so the number whose existence is guaranteed by the mean value theorem is $c = 1$.

C04S0M.036: Every polynomial is continuous and differentiable everywhere, so all hypotheses of the mean value theorem are satisfied. Then

$$\frac{f(1)-f(-2)}{1-(-2)} = \frac{1+8}{3} = 3 = f'(c) = 3c^2,$$

so $c^2 = 1$. But 1 does not lie in the interval $(-2, 1)$, so the number whose existence is guaranteed by the mean value theorem is $c = -1$.

C04S0M.037: Given: $f(x) = \frac{11}{5}x^5$ on the interval $[-1, 2]$. Because $f(x)$ is a polynomial, it is continuous and differentiable everywhere, so the hypotheses of the mean value theorem are satisfied on the interval $[-1, 2]$. Moreover,

$$\frac{f(2)-f(-1)}{2-(-1)} = \frac{\frac{11}{5}\cdot 32 + \frac{11}{5}}{3} = \frac{11\cdot 33}{5\cdot 3} = \frac{121}{5} = f'(c) = 11c^4.$$

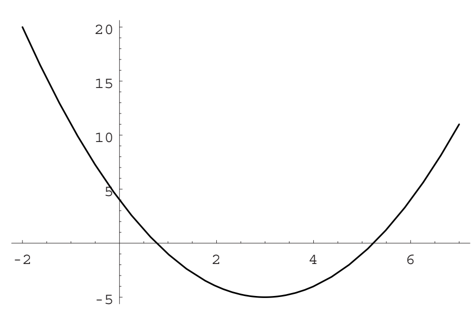
It follows that $c^4 = \frac{11}{5}$, so that $c = \pm\left(\frac{11}{5}\right)^{1/4}$. Only the positive root lies in the interval $[-1, 2]$, so the number whose existence is guaranteed by the mean value theorem is $c = \left(\frac{11}{5}\right)^{1/4}$.

C04S0M.38: Because $f(x) = \sqrt{x}$ is differentiable on $(0, 4)$ and continuous on $[0, 4]$, the hypotheses of the mean value theorem are satisfied. Moreover,

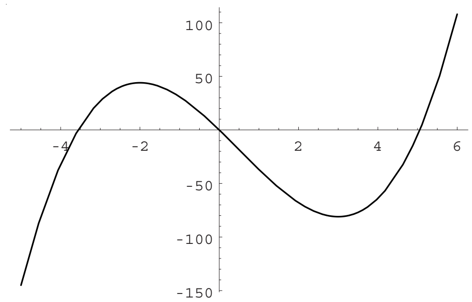
$$\frac{f(4)-f(0)}{4-0} = \frac{2}{4} = \frac{1}{2} = f'(c) = \frac{1}{2c^{1/2}},$$

and it follows that $c = 1$.

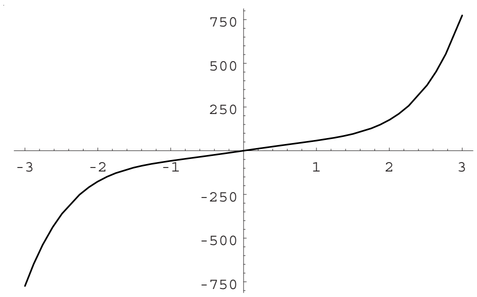
C04S0M.039: $f'(x) = 2x - 6$, so $f'(x) = 0$ when $x = 3$; $f''(x) \equiv 2$ is always positive, so there are no inflection points and there is a global minimum at $(3, -5)$. The y-intercept is $(0, 4)$ and the x-intercepts are $\left(3 \pm \sqrt{5}, 0\right)$. The graph of f is shown next.



C04S0M.040: $f'(x) = 6(x-3)(x+2)$, so there are critical points at $(-2, 44)$ and $(3, -81)$. $f''(x) = 12x-6$, so there is an inflection point at $\left(\frac{1}{2}, -\frac{37}{2}\right)$. The origin $(0, 0)$ is a dual intercept and there are two other x-intercepts. The graph of $y = f(x)$ is next.



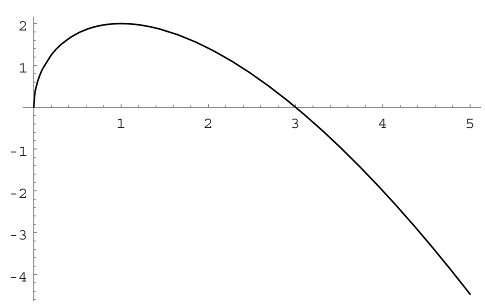
C04S0M.041: $f(x) = (3x^4 - 5x^2 + 60)x$ and $f'(x) = 15(x^4 - x^2 + 4)$, so $f'(x) > 0$ for all x and hence $(0, 0)$ is the only intercept. $f''(x) = 30x(2x^2 - 1)$, so there are inflection points at $\left(-\frac{1}{2}\sqrt{2}, -\frac{233}{8}\sqrt{2}\right)$, $(0, 0)$, and $\left(\frac{1}{2}\sqrt{2}, \frac{233}{8}\sqrt{2}\right)$. The graph is actually concave upward between the first and second of these inflection points and concave downward between the second and third, but so slightly that this is not visible on the graph of f that is shown next.



C04S0M.042: Given $f(x) = (3 - x)\sqrt{x}$, we have

$$f'(x) = \frac{3(1-x)}{2\sqrt{x}} \qquad \text{and} \qquad f''(x) = -\frac{3(x+1)}{4x\sqrt{x}}.$$

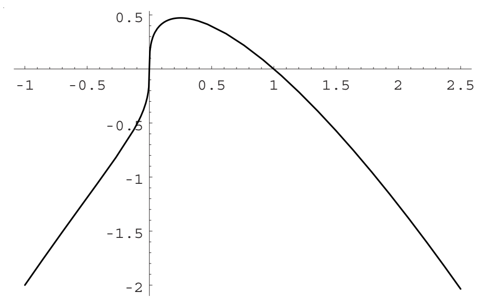
So there is a critical point at $(1, 2)$, intercepts at $(0, 0)$ and $(3, 0)$, and the graph of f is concave down on the entire domain $[0, +\infty)$ of f. Thus the critical point is a global maximum, there are no inflection points, and there is a local minimum at $(0, 0)$. The graph of f is shown next.



C04S0M.043: Given $f(x) = (1 - x)x^{1/3}$, we find that

$$f'(x) = \frac{1-4x}{3x^{2/3}} \qquad \text{and} \qquad f''(x) = -\frac{2(2x+1)}{9x^{5/3}}.$$

So there are intercepts at (0, 0) and (1, 0) and a critical point at $\left(\frac{1}{4}, \frac{3}{8}\sqrt[3]{2}\right)$. There is an inflection point at $\left(-\frac{1}{2}, -\frac{3}{4}\sqrt[3]{4}\right)$; the graph of f is actually concave upward between this point and (0, 0) and concave downward to its left, but the latter is not visible in the scale of the accompanying figure. The origin is also an inflection point and there is a vertical tangent there too.



C04S0M.044: Let $g(x) = x^5 + x - 5$. Then $g(2) = 29 > 0$ while $g(1) = -3 < 0$. Because $g(x)$ is a polynomial, it has the intermediate value property. Therefore the equation $g(x) = 0$ has at least one solution in the interval $1 \leqq x \leqq 2$. Moreover, $g'(x) = 5x^4 + 1$, so $g'(x) > 0$ for all x. Consequently g is increasing on the set of all real numbers, and so takes on each value—including zero—at most once. We may conclude that the equation $g(x) = 0$ has exactly one solution, and hence that the equation $x^5 + x = 5$ has exactly one solution. (The solution is approximately 1.299152792.)

C04S0M.045: $f'(x) = 3x^2 - 2$, $f''(x) = 6x$, and $f'''(x) \equiv 6$.

C04S0M.046: $f'(x) = 100(x+1)^{99}$, $f''(x) = 9900(x+1)^{98}$, and $f'''(x) = 970200(x+1)^{97}$.

C04S0M.047: Given $g(t) = \dfrac{1}{t} - \dfrac{1}{2t+1}$,

$$g'(t) = \frac{2}{(2t+1)^2} - \frac{1}{t^2}, \qquad g''(t) = \frac{2}{t^3} - \frac{8}{(2t+1)^3}, \qquad \text{and} \qquad g'''(t) = \frac{48}{(2t+1)^4} - \frac{6}{t^4}.$$

C04S0M.048: $h'(y) = \frac{3}{2}(3y-1)^{-1/2}$, $h''(y) = -\frac{9}{4}(3y-1)^{-3/2}$, and $h'''(y) = \frac{81}{8}(3y-1)^{-5/2}$.

C04S0M.049: $f'(t) = 3t^{1/2} - 4t^{1/3}$, $f''(t) = \frac{3}{2}t^{-1/2} - \frac{4}{3}t^{-2/3}$, and $f'''(t) = \frac{8}{9}t^{-5/3} - \frac{3}{4}t^{-3/2}$.

C04S0M.050: $g'(x) = -\dfrac{2x}{(x^2+9)^2}$, $g''(x) = \dfrac{6x^2 - 18}{(x^2+9)^3}$, and $g'''(x) = \dfrac{216x - 24x^3}{(x^2+9)^4}$.

C04S0M.051: $h'(t) = -\dfrac{4}{(t-2)^2}$, $h''(t) = \dfrac{8}{(t-2)^3}$, and $h'''(t) = -\dfrac{24}{(t-2)^4}$.

C04S0M.052: $f'(z) = \frac{1}{3}z^{-2/3} - \frac{3}{5}z^{-6/5}$, $f''(z) = -\frac{2}{9}z^{-5/3} + \frac{18}{25}z^{-11/5}$, and $f'''(z) = \frac{10}{27}z^{-8/3} - \frac{198}{125}z^{-16/5}$.

C04S0M.053: $g'(x) = -\dfrac{4}{3(5-4x)^{2/3}}$, $g''(x) = -\dfrac{32}{9(5-4x)^{5/3}}$, and $g'''(x) = -\dfrac{640}{27(5-4x)^{8/3}}$.

C04S0M.054: $g'(t) = 12(3-t)^{-5/2}$, $g''(t) = 30(3-t)^{-7/2}$, and $g'''(t) = 105(3-t)^{-9/2}$.

C04S0M.055: $x^{-2/3} + y^{-2/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -(y/x)^{2/3}$.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{2}{3}(y/x)^{-1/3}\cdot\frac{x(dy/dx)-y}{x^2} = -\frac{2}{3}\cdot\frac{x(dy/dx)-y}{x^{5/3}y^{1/3}}\\
&= \frac{2}{3}\cdot\frac{y+x^{1/3}y^{2/3}}{x^{5/3}y^{1/3}} = \frac{2}{3}y^{2/3}\cdot\frac{x^{1/3}+y^{1/3}}{x^{5/3}y^{1/3}} = \frac{2}{3}(y/x^5)^{1/3}.
\end{aligned}
$$

C04S0M.056: Given $2x^2 - 3xy + 5y^2 = 25$, we differentiate both sides with respect to x and obtain

$$4x - 3y - 3x\,\frac{dy}{dx} + 10y\,\frac{dy}{dx} = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{3y-4x}{10y-3x}.$$

We differentiate both sides of the second of these equations, again with respect to x, and find that

$$\frac{d^2y}{dx^2} = \frac{(10y-3x)(3y'(x)-4)-(3y-4x)(10y'(x)-3)}{(10y-3x)^2}.$$

Then replacement of $y'(x)$ with $\dfrac{3y-4x}{10y-3x}$ yields

$$\frac{d^2y}{dx^2} = -\frac{1550}{(10y-3x)^3}$$

after simplifications that use the original equation.

C04S0M.057: Given $y^5 - 4y + 1 = x^{1/2}$, we differentiate both sides of this equation (actually, an *identity*) with respect to x and obtain

$$5y^4\,\frac{dy}{dx} - 4\,\frac{dy}{dx} = \frac{1}{2x^{1/2}}, \qquad \text{so that} \qquad \frac{dy}{dx} = y'(x) = \frac{1}{2(5y^4-4)\sqrt{x}}. \tag{1}$$

Another differentiation yields

$$\frac{d^2y}{dx^2} = -\frac{1+80x^{3/2}y^3[y'(x)]^3}{4(5y^4-4)x^{3/2}},$$

then substitution of $y'(x)$ from Eq. (1) yields

$$y''(x) = \frac{40y^4 - 25y^8 - 20x^{1/3}y^3 - 16}{4x^{3/2}(5y^4-4)^3}.$$

C04S0M.058: Given: $\sin(xy) = xy$. The only solution of $\sin z = z$ is $z = 0$. Therefore $xy = 0$. Thus $x = 0$ or $y = 0$. This means that the graph of the equation $\sin(xy) = xy$ consists of the coordinate axes. The y-axis is not the graph of a function, so the derivative is defined only for $x \neq 0$, and $dy/dx = 0$ for $x \neq 0$. Therefore also $d^2y/dx^2 = 0$ for $x \neq 0$.

C04S0M.059: Given $x^2 + y^2 = 5xy + 5$, we differentiate both sides with respect to x to obtain

$$2x + 2y\,\frac{dy}{dx} = 5y + 5x\,\frac{dy}{dx}, \qquad \text{so that} \qquad \frac{dy}{dx} = y'(x) = \frac{2x-5y}{5x-2y}. \tag{1}$$

Another differentiation yields

$$y''(x) = \frac{2[(y'(x))^2 - 5y'(x) + 1]}{5x-2y},$$

then substitution of $y'(x)$ from Eq. (1) yields

$$y''(x) = -\frac{42(x^2 - 5xy + y^2)}{(5x-2y)^3} = -\frac{210}{(5x-2y)^3}.$$

In the last step we used the original equation in which y is defined implicitly as a function of x.

C04S0M.060: $x^5 + xy^4 = 1$: $5x^4 + y^4 + 4xy^3\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{5x^4+y^4}{4xy^3}$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{4xy^3(20x^3 + 4y^3\frac{dy}{dx}) - (5x^4+y^4)(4y^3 + 12xy^2\frac{dy}{dx})}{16x^2y^6} \\
&= \frac{20x^4y^3 + 4y^7 + (60x^5y^2 + 12xy^6)\frac{dy}{dx} - 80x^4y^3 - 16xy^6\frac{dy}{dx}}{16x^2y^6} \\
&= \frac{4y^7 - 60x^4y^3 + (60x^5y^2 - 4xy^6)\frac{dy}{dx}}{16x^2y^6} \\
&= \frac{4y^5 - 60x^4y + (60x^5 - 4xy^4)\frac{dy}{dx}}{16x^2y^4} \\
&= \frac{4y^5 - 60x^4y + (4xy^4 - 60x^5)\cdot\left(\dfrac{5x^4+y^4}{4xy^3}\right)}{16x^2y^4} \\
&= \frac{16xy^8 - 240x^5y^4 + 20x^5y^4 + 4xy^8 - 300x^9 - 60x^5y^4}{64x^3y^7} \\
&= \frac{20xy^8 - 280x^5y^4 - 300x^9}{64x^3y^7} = \frac{5y^8 - 70x^4y^4 - 75x^8}{16x^2y^7} \\
&= \frac{5(y^8 - 14x^4y^4 - 15x^8)}{16x^2y^7} = \frac{5(y^4+x^4)(y^4-15x^4)}{16x^2y^7}.
\end{aligned}$$

But $x^4 + y^4 = \dfrac{1}{x}$, so $\dfrac{d^2y}{dx^2} = \dfrac{5(y^4 - 15x^4)}{16x^3y^7}$.

C04S0M.061: Given: $y^3 - y = x^2y$:

$$\frac{dy}{dx} = y'(x) = -\frac{2xy}{x^2+1-3y^2}. \tag{1}$$

Then

$$y''(x) = -\frac{2[y + 2xy'(x) - 3y(y'(x))^2]}{x^2+1-3y^2}. \tag{2}$$

Substitution of $y'(x)$ from Eq. (1) in the right-hand side of Eq. (2) then yields

$$y''(x) = \frac{2y\,[3x^4 - 9y^4 + 6(x^2+1)y^2 + 2x^2 - 1]}{(x^2+1-3y^2)^3}.$$

C04S0M.062: $(x^2-y^2)^2 = 4xy$:

$$2(x^2-y^2)(2x - 2y\frac{dy}{dx}) = 4x\frac{dy}{dx} + 4y;$$

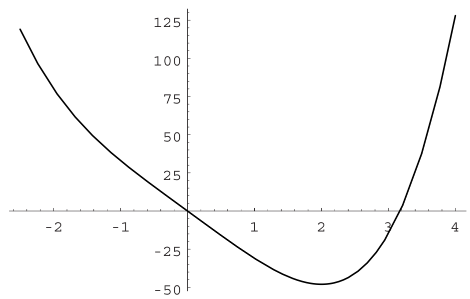
$$(x^2 - y^2) \cdot x - (x^2 - y^2) \cdot y\frac{dy}{dx} = x\frac{dy}{dx} + y;$$
$$(x + x^2y - y^3)\frac{dy}{dx} = x^3 - xy^2 - y;$$
$$\frac{dy}{dx} = \frac{x(x^2 - y^2) - y}{x + y(x^2 - y^2)} = \frac{x^3 - xy^2 - y}{x + x^2y - y^3};$$

$$\frac{d^2y}{dx^2} = \frac{\left(x + x^2y - y^3\right)\left(3x^2 - 2xy\dfrac{dy}{dx} - y^2 - \dfrac{dy}{dx}\right) - \left(x^3 - xy^2 - y\right)\left(1 + x^2\dfrac{dy}{dx} + 2xy - 3y^2\dfrac{dy}{dx}\right)}{\left(x + x^2y - y^3\right)^2},$$

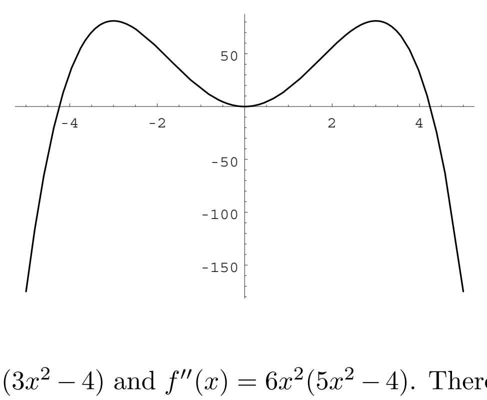
which upon simplification and substitution for dy/dx becomes

$$\frac{d^2y}{dx^2} = \frac{3xy(2 - xy)}{(x + x^2y - y^3)^3}.$$

C04S0M.063: $f'(x) = 4x^3 - 32$, so there is a critical point at $(2, -48)$. $f''(x) = 12x^2$, but there is no inflection point at $(0, 0)$ because the graph of f is concave upward for all x. But $(0, 0)$ is a dual intercept, and there is an x-intercept at $\left(\sqrt[3]{32}, 0\right)$. The graph of $y = f(x)$ is shown next.

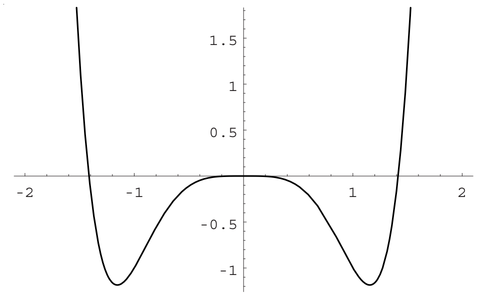


C04S0M.064: $f(x) = 18x^2 - x^4 = x^2(18 - x^2)$; $f'(x) = 36x - 4x^3 = 4x(3 + x)(3 - x)$; $f''(x) = 12(3 - x^2)$. There are global maxima at $(-3, 81)$ and $(3, 81)$ and a local minimum at $(0, 0)$. The other two x-intercepts are at $\left(-3\sqrt{2}, 0\right)$ and $\left(3\sqrt{2}, 0\right)$. There are inflection points at $\left(-\sqrt{3}, 45\right)$ and $\left(\sqrt{3}, 45\right)$. The graph of f is next.



C04S0M.065: $f'(x) = 2x^3(3x^2 - 4)$ and $f''(x) = 6x^2(5x^2 - 4)$. There are global minima at $\left(-\frac{2}{3}\sqrt{3}, -\frac{32}{27}\right)$ and $\left(\frac{2}{3}\sqrt{3}, -\frac{32}{27}\right)$ and a local maximum at $(0, 0)$, which is also a dual intercept. There are inflection points at $\left(-\frac{2}{5}\sqrt{5}, -\frac{95}{125}\right)$ and $\left(\frac{2}{5}\sqrt{5}, -\frac{95}{125}\right)$ but not at $(0, 0)$. There are x-intercepts at $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.

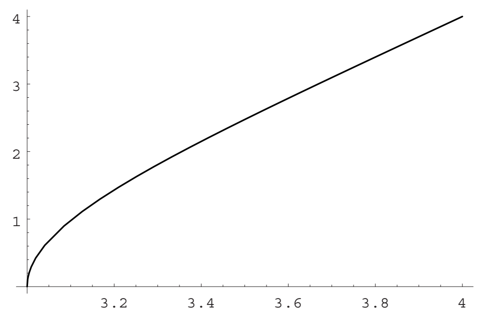
The graph of $y = f(x)$ is shown next.



C04S0M.066: If $f(x) = x\sqrt{x-3}$, then

$$f'(x) = \frac{3(x-2)}{2\sqrt{x-3}} \qquad \text{and} \qquad f''(x) = \frac{3(x-4)}{4(x-3)^{3/2}}.$$

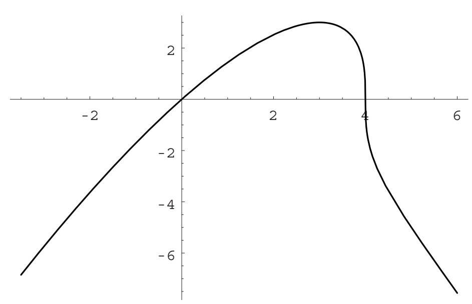
Therefore the graph of f is increasing for all x (the domain of f is $[3, +\infty)$), concave downward for $x < 4$, and concave upward for $x > 4$; there is an inflection point at $(4, 4)$. The only intercept is $(3, 0)$, which is also a global minimum and the location of a vertical tangent. The graph of f is next.



C04S0M.067: If $f(x) = x(4-x)^{1/3}$, then

$$f'(x) = \frac{4(3-x)}{3(4-x)^{2/3}} \qquad \text{and} \qquad f''(x) = \frac{4(x-6)}{9(4-x)^{5/3}}.$$

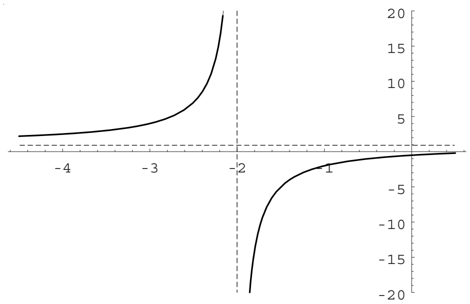
So there is a global maximum at $(3, 3)$ intercepts at $(0, 0)$ and $(4, 0)$, a vertical tangent and inflection point at the latter, and an inflection point at $\left(6, -6\sqrt[3]{2}\right)$. The graph is next.



C04S0M.068: If $(x) = \dfrac{x-1}{x+2} = 1 - \dfrac{3}{x+2}$, then

$$f'(x) = \frac{3}{(x+2)^2} \qquad \text{and} \qquad f''(x) = -\frac{6}{(x+2)^3}.$$

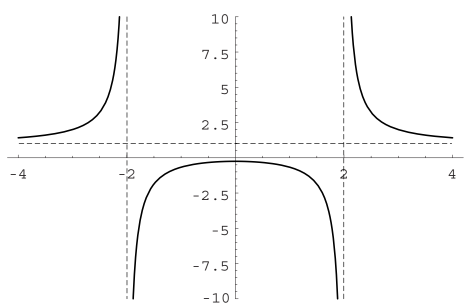
There are no critical points and no inflection points. The graph is increasing except at the discontinuity at $x = -2$. It is concave upward for $x < -2$ and concave downward if $x > -2$. The vertical line $x = -2$ and the horizontal line $y = 1$ are asymptotes, and the intercepts are $\left(0, -\frac{1}{2}\right)$ and $(1, 0)$. The graph of $y = f(x)$ is next.



C04S0M.069: If $f(x) = \dfrac{x^2+1}{x^2-4} = 1 + \dfrac{5}{x^2-4}$, then

$$f'(x) = -\frac{10x}{(x^2-4)^2} \qquad \text{and} \qquad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}.$$

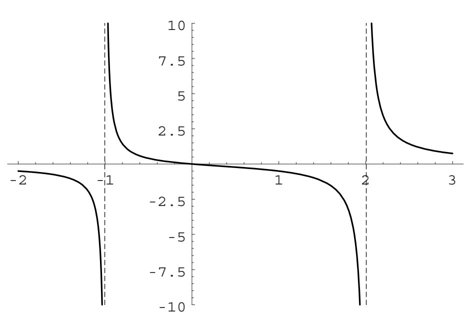
Thus there is a local maximum at $\left(0, -\frac{1}{4}\right)$, no other extrema, no inflection points, and no intercepts. The lines $x = -2$ and $x = 2$ are vertical asymptotes and $y = 1$ is a horizontal asymptote; the graph is next.



C04S0M.070: If $f(x) = \dfrac{x}{(x-2)(x+1)}$, then

$$f'(x) = -\frac{x^2+2}{(x^2-x-2)^2} \qquad \text{and} \qquad f''(x) = \frac{2(x^3+6x-2)}{(x+1)^3(x-2)^3}.$$

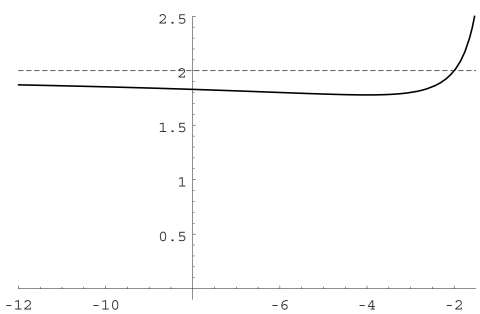
There are no critical points; there is an inflection point with the approximate coordinates $(0.3275, -0.1475)$. The graph is decreasing for all x other than -1 and 2, is concave upward on the intervals $(-1, 0.3275)$ and $(2, +\infty)$, and is concave downward on the intervals $(0.3275, 2)$ and $(-\infty, -1)$. The asymptotes are $y = 0$, $x = 2$, and $x = -1$ and $(0, 0)$ is the only intercept. The graph of $y = f(x)$ is shown next.

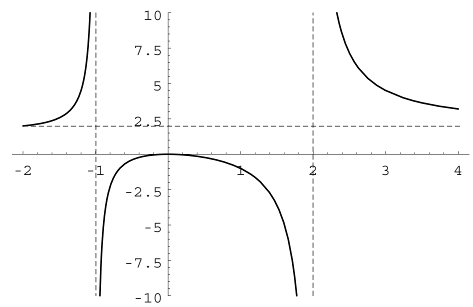


C04S0M.071: If $f(x) = \dfrac{2x^2}{(x-2)(x+1)}$, then

$$f'(x) = -\frac{2x(x+4)}{(x-2)^2(x+1)^2} \qquad \text{and} \qquad f''(x) = \frac{4(x^3+6x^2+4)}{(x^2-x-2)^3}.$$

Thus (0, 0) is a local maximum and the only intercept; there is a local minimum at $\left(-4, \frac{16}{9}\right)$. There is an inflection point with the approximate coordinates $(-6.107243, 1.801610)$. The lines $x = -1$ and $x = 2$ are vertical asymptotes and the line $y = 2$ is a horizontal asymptote. The graph of $y = f(x)$ for $-12 < x < -2$ is next, on the left; the graph for $-2 < x < 4$ is on the right.

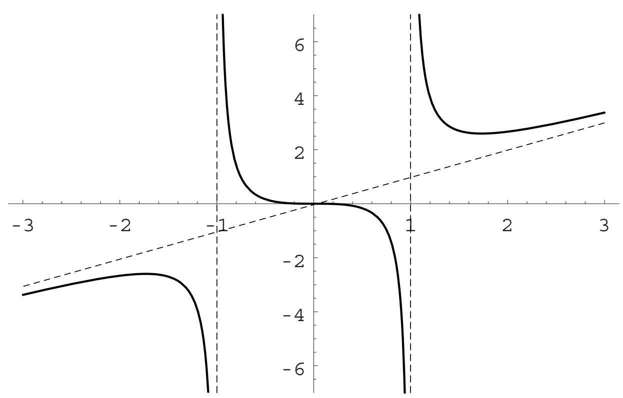




C04S0M.072: Given: $f(x) = \dfrac{x^3}{x^2-1} = x + \dfrac{1}{2(x+1)} + \dfrac{1}{2(x-1)}$. First,

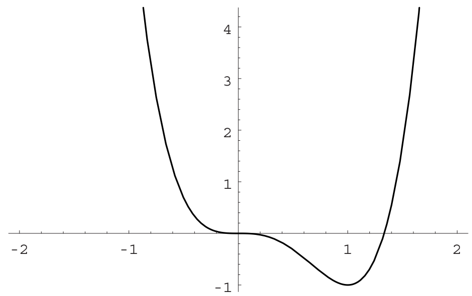
$$f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2} \qquad \text{and} \qquad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

Inflection point and sole intercept: (0, 0). There is a local minimum where $x = \sqrt{3}$ and a local maximum where $x = -\sqrt{3}$. The graph is concave downward on the intervals (0, 1) and $(-\infty, -1)$, concave upward on the intervals $(-1, 0)$ and $(1, +\infty)$. The graph is increasing if $x < -\sqrt{3}$ and if $x > \sqrt{3}$ and is decreasing otherwise. The lines $x = -1$, $x = 1$, and $y = x$ are asymptotes. The graph of $y = f(x)$ is next.

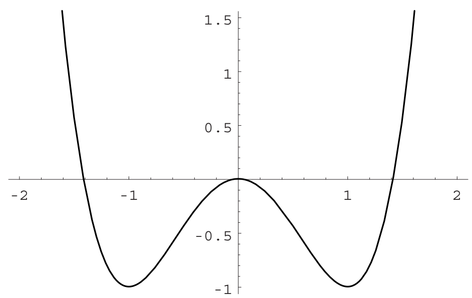


C04S0M.073: Here we have $f(x) = x^3(3x-4)$, $f'(x) = 12x^2(x-1)$, and $f''(x) = 12x(3x-2)$. Hence there are intercepts at (0, 0) and $\left(\frac{4}{3}, 0\right)$; the graph is increasing for $x > 1$ and decreasing for $x < 1$; it is concave upward for $x > \frac{2}{3}$ and for $x < 0$, concave downward on the interval $\left(0, \frac{2}{3}\right)$. Consequently there is a global minimum at $(1, -1)$ and inflection points at (0, 0) and $\left(\frac{2}{3}, -\frac{16}{27}\right)$. There are no asymptotes, no other

extrema, and $f(x) \to +\infty$ as $x \to +\infty$ and as $x \to -\infty$. The graph of f is shown next.



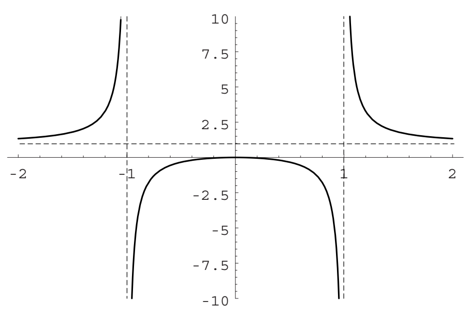
C04S0M.074: Here we have $f(x) = x^2(x^2 - 2)$, $f'(x) = 4x(x+1)(x-1)$, and $f''(x) = 4(3x^2 - 1)$. So there are intercepts at $\left(-\sqrt{2}, 0\right)$, $(0, 0)$, and $\left(\sqrt{2}, 0\right)$. The graph is increasing on the intervals $(1, +\infty)$ and $(-1, 0)$, decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$. It is concave upward where $x^2 > \frac{1}{3}$ and concave downward where $x^2 < \frac{1}{3}$. There are global minima at $(-1, -1)$ and $(1, -1)$ and a local maximum at the origin. There are inflection points at the two points where $x^2 = \frac{1}{3}$. The graph is next.



C04S0M.075: If $f(x) = \dfrac{x^2}{x^2 - 1}$, then

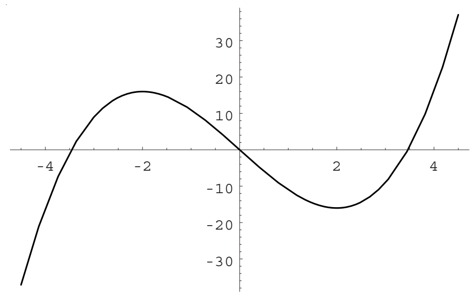
$$f'(x) = -\frac{2x}{(x^2-1)^2} \qquad \text{and} \qquad f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}.$$

Thus $(0, 0)$ is the only intercept and is a local maximum; there are no inflection points, the lines $x = -1$ and $x = 1$ are vertical asymptotes, and the line $y = 1$ is a horizontal asymptote. The graph of $y = f(x)$ is next.

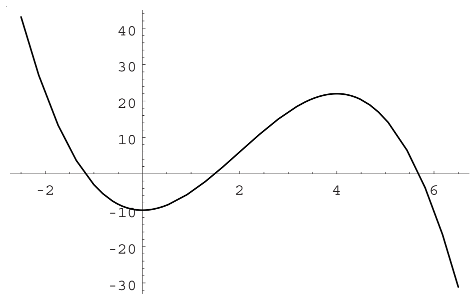


C04S0M.076: First, $f(x) = x(x^2 - 12)$, $f'(x) = 3(x+2)(x-2)$, and $f''(x) = 6x$. So there are intercepts at $\left(2\sqrt{3}, 0\right)$, $(0, 0)$, and $\left(-2\sqrt{3}, 0\right)$. The graph is increasing for $x > 2$ and for $x < -2$; it is decreasing on the interval $(-2, 2)$. It is concave upward for $x > 0$ and concave downward for $x < 0$. There is a local maximum at $(-2, 16)$, a local minimum at $(2, -16)$, and an inflection point at the origin. The graph of $y = f(x)$ is

shown next.



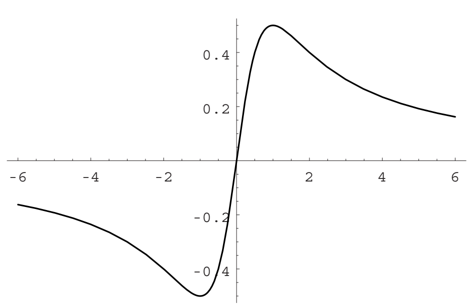
C04S0M.077: If $f(x) = -10 + 6x^2 - x^3$, then $f'(x) = 3x(4 - x)$ and $f''(x) = 6(2 - x)$. It follows that there is a local minimum at $(0, -10)$, a local maximum at $(4, 22)$, and an inflection point at $(2, 6)$. The intercepts are approximately $(-1.180140, 0)$, $(1.488872, 0)$, $(5.691268, 0)$, and [exactly] $(0, -10)$. The graph of f is next.



C04S0M.078: If $f(x) = \dfrac{x}{1 + x^2}$, then

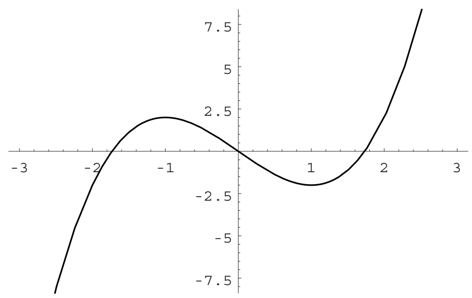
$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \qquad \text{and} \qquad f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}.$$

There is a global maximum at $\left(1, \frac{1}{2}\right)$ and a global minimum at $\left(-1, -\frac{1}{2}\right)$. The origin is the only intercept and an inflection point; there are also inflection points at $\left(\sqrt{3}, \frac{1}{4}\sqrt{3}\right)$ and $\left(-\sqrt{3}, -\frac{1}{4}\sqrt{3}\right)$. The x-axis is the only asymptote. The graph of $y = f(x)$ is shown next.

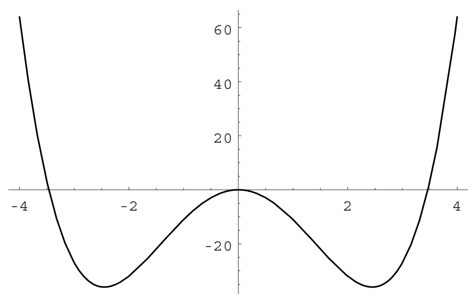


C04S0M.079: If $f(x) = x^3 - 3x$, then $f'(x) = 3(x + 1)(x - 1)$ and $f''(x) = 6x$. So there is a local maximum at $(-1, 2)$, a local minimum at $(1, -2)$, and an inflection point at $(0, 0)$. There are also intercepts

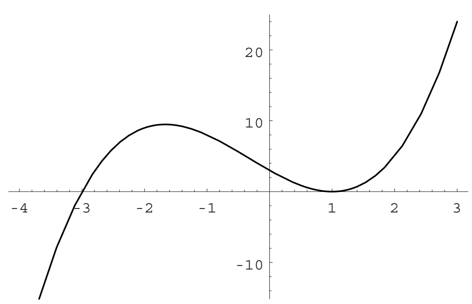
at $\left(-\sqrt{3},\, 0\right)$ and $\left(\sqrt{3},\, 0\right)$ but no asymptotes. The graph is next.



C04S0M.080: If $f(x) = x^4 - 12x^2 = x^2(x^2 - 12)$, then $f'(x) = 4x(x^2 - 6)$ and $f''(x) = 12(x^2 - 2)$. Hence there are global minima at $\left(-\sqrt{6},\, -36\right)$ and $\left(\sqrt{6},\, -36\right)$ and a local maximum at $(0,\, 0)$. There are also intercepts at $\left(-2\sqrt{3},\, 0\right)$ and $\left(2\sqrt{3},\, 0\right)$. There are inflection points at $\left(-\sqrt{2},\, -20\right)$ and $\left(\sqrt{2},\, -20\right)$ and no asymptotes. The graph of $y = f(x)$ is shown next.

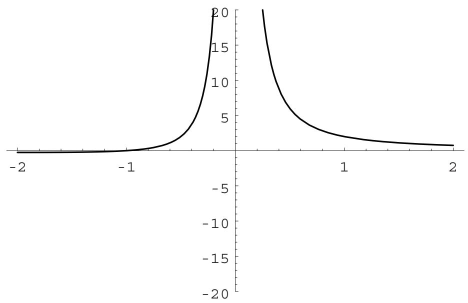


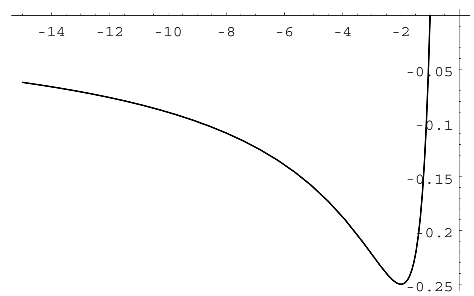
C04S0M.081: If $f(x) = x^3 + x^2 - 5x + 3 = (x-1)^2(x+3)$, then $f'(x) = (3x+5)(x-1)$ and $f''(x) = 6x+2$. So there is a local maximum at $\left(-\frac{5}{3},\, \frac{256}{27}\right)$ and a local minimum at $(1,\, 0)$. There is an inflection point at $\left(-\frac{1}{3},\, \frac{128}{27}\right)$. A second x-intercept is $(-3,\, 0)$ and there are no asymptotes. The graph of $y = f(x)$ is next.



C04S0M.082: $f(x) = \dfrac{x+1}{x^2}$, $f'(x) = -\dfrac{x+2}{x^3}$, $f''(x) = \dfrac{2(x+3)}{x^4}$. The graph is decreasing for $x > 0$ and for $x < -2$, increasing on the interval $(-2,\, 0)$. It is concave downward for $x < -3$, concave upward for $x > 0$ and on the interval $(-3,\, 0)$. The only intercept is $(-1,\, 0)$ and there is a discontinuity where $x = 0$. There is a global minimum at $(-2,\, -0.25)$ and an inflection point at $\left(-3,\, -\frac{2}{9}\right)$. As $x \to 0$, $f(x) \to +\infty$, so the y-axis is a vertical asymptote. As $|x| \to +\infty$, $f(x) \to 0$, so the x-axis is a horizontal asymptote. The graph

of $y = f(x)$ for $-2 < x < 2$ is next, on the left; the graph for $-15 < x < -1$ is on the right.





C04S0M.083: The given function $f(x)$ is expressed as a fraction with constant numerator, so we maximize $f(x)$ by minimizing its denominator $(x+1)^2 + 1$. It is clear that $x = -1$ does the trick, so the maximum value of $f(x)$ is $f(-1) = 1$.

C04S0M.084: Let k be the proportionality constant for cost; if the pot has radius r and height h, we are to minimize total cost

$$C = k\left[(5)(\pi r^2) + (1)(2\pi r h)\right]$$

subject to the constraint $\pi r^2 h = 1$. Then $h = 1/\left(\pi r^2\right)$, so

$$C = C(r) = k\left(5\pi r^2 + \frac{2}{r}\right), \quad r > 0.$$

Now

$$C'(r) = k\left(10\pi r - \frac{2}{r^2}\right);$$

$C'(r) = 0$ when

$$r = \left(\frac{1}{5\pi}\right)^{1/3} \qquad \text{and} \qquad h = \left(\frac{25}{\pi}\right)^{1/3}.$$

It's easy to establish in the usual way that these values minimize C, and it's worth noting that when C is minimized, we also have $h = 5r$.

C04S0M.085: Let x represent the width of the base of the box. Then the length of the base is $2x$ and, because the volume of the box is 4500, the height of the box is $4500/\left(2x^2\right)$. We minimize the surface area of the box, which is given by

$$f(x) = 2x^2 + 4x \cdot \frac{4500}{2x^2} + 2x \cdot \frac{4500}{2x^2} = 2x^2 + \frac{13500}{x}, \quad 0 < x < \infty.$$

Now $f'(x) = 4x - (13500/x^2)$, so $f'(x) = 0$ when $x = \sqrt[3]{3375} = 15$. Note that $f''(15) > 0$, so surface area is minimized when $x = 15$. The box of minimal surface area is 15 cm wide, 30 cm long, and 10 cm high.

C04S0M.086: Let x represent the edge length of the square base of the box. Because the volume of the box is 324, the box has height $324/x^2$. We minimize the cost C of materials to make the box, where

$$C = C(x) = 3x^2 + 4 \cdot x \cdot \frac{324}{x^2} = 3x^2 + \frac{1296}{x}, \quad 0 < x < \infty.$$

Now $C'(x) = 6x - (1296/x^2)$, so $C(x) = 0$ when $x = \sqrt[3]{1296/6} = 6$. Because $C''(6) > 0$, the cost C is minimized when $x = 6$. The box we seek has a square base 6 in. on a side and height 9 in.

C04S0M.087: Let x represent the width of the base of the box. Then the box has base of length $2x$ and height $200/x^2$. We minimize the cost C of the box, where

$$C = C(x) = 7 \cdot 2x^2 + 5 \cdot \left(6x \cdot \frac{200}{x^2}\right) + 5 \cdot 2x^2 = 24x^2 + \frac{6000}{x}, \quad 0 < x < \infty.$$

Now $C'(x) = 48x - (6000/x^2)$, so $C'(x) = 0$ when $x = \sqrt[3]{125} = 5$. Because $C''(5) > 0$, the cost C is minimized when $x = 5$. The box of minimal cost is 5 in. wide, 10 in. long, and 8 in. high.

C04S0M.088: If the zeros of $f(x)$ are at a, b, and c (with $a < b < c$), apply Rolle's theorem to f' on the two intervals $[a\,, b]$ and $[b,\, c]$.

C04S0M.089: If the speed of the truck is v, then the trip time is $T = 1000/v$. So the resulting cost is

$$C(v) = \frac{10000}{v} + (1000)\left(1 + (0.0003)v^{3/2}\right),$$

so that

$$\frac{C(v)}{1000} = \frac{10}{v} + 1 + (0.0003)v^{3/2}.$$

Thus

$$\frac{C'(v)}{1000} = -\frac{10}{v^2} + \frac{3}{2}(0.0003)\sqrt{v}.$$

Then $C'(v) = 0$ when $v = (200{,}000/9)^{2/5} \approx 54.79$ mi/h. This clearly minimizes the cost, because $C''(v) > 0$ for *all* $v > 0$.

C04S0M.090: The sum in question is

$$S(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

Now $S'(x) = 2(x - a_1) + 2(x - a_2) + \ldots + 2(x - a_n)$; $S'(x) = 0$ when $nx = a_1 + a_2 + \ldots + a_n$, so that

$$x = \frac{1}{n}(a_1 + a_2 + \ldots + a_n) \tag{1}$$

—the *average* of the n fixed numbers. It is clear that S is continuous and that $S(x) \to +\infty$ as $|x| \to +\infty$, so $S(x)$ must have a global minimum value. Therefore the value of x in Eq. (1) minimizes the sum of the squares of the distances.

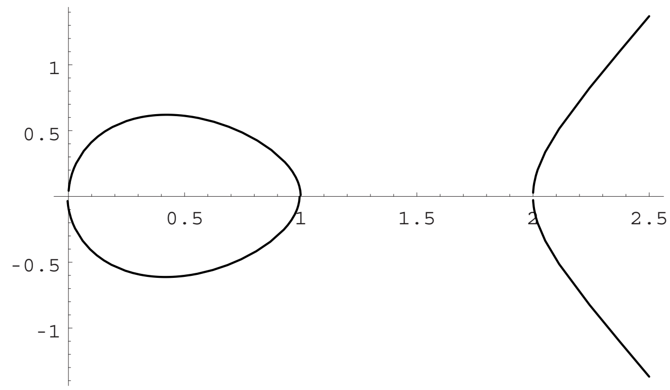
C04S0M.091: First, given $y^2 = x(x-1)(x-2)$, we differentiate implicitly and find that

$$2y\,\frac{dy}{dx} = 3x^2 - 6x + 2, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{3x^2 - 6x + 2}{2y}.$$

The only zero of dy/dx in the domain is $1 - \frac{1}{3}\sqrt{3}$, so there are two horizontal tangent lines (the y-coordinates are approximately ± 0.6204). Moreover, $dx/dy = 0$ when $y = 0$; that is, when $x = 0$, when $x = 1$, and when $x = 2$. So there are three vertical tangent lines. After lengthy simplifications, one can show that

$$\frac{d^2y}{dx^2} = \frac{3x^4 - 12x^3 + 12x^2 - 4}{4y^3}.$$

The only zero of $y''(x)$ in the domain is about 2.4679, and there the graph has the two values $y \approx \pm 1.3019$. These are the two inflection points. The graph of the given equation is shown next.



C04S0M.092: Let x represent the length of the internal divider. Then the field measures x by $2400/x$ ft. We minimize the total length of fencing, given by:

$$f(x) = 3x + \frac{4800}{x}, \quad 0 < x < \infty.$$

Now $f'(x) = 3 - 4800x^{-2}$, which is zero only when $x = \sqrt{1600} = 40$. Verification: $f'(x) > 0$ if $x > 40$ and $f'(x) < 0$ if $x < 40$, so $f(x)$ is minimized when $x = 40$. The minimum length of fencing required for this field is $f(40) = 240$ feet.

C04S0M.093: Let x represent the length of each of the dividers. Then the field measures x by $1800/x$ ft. We minimize the total length of fencing, given by:

$$f(x) = 4x + \frac{3600}{x}, \quad 0 < x < \infty.$$

Now $f'(x) = 4 - 3600x^{-2}$, which is zero only when $x = 30$. Verification: $f'(x) > 0$ if $x > 30$ and $f'(x) < 0$ if $x < 30$, so $f(x)$ is minimized when $x = 30$. The minimum length of fencing required for this field is $f(30) = 240$ ft.

C04S0M.094: Let x represent the length of each of the dividers. Then the field measures x by $2250/x$ meters. We minimize the total length of fencing, given by:

$$f(x) = 5x + \frac{4500}{x}, \quad 0 < x < \infty.$$

Now $f'(x) = 5 - 4500x^{-2}$, which is zero only when $x = 30$. Verification: $f'(x) > 0$ if $x > 30$ and $f'(x) < 0$ if $x < 30$, so $f(x)$ is minimized when $x = 30$. The minimum length of fencing required for this field is $f(30) = 300$ meters.

C04S0M.095: Let x represent the length of each of the dividers. Then the field measures x by A/x ft. We minimize the total length of fencing, given by:

$$f(x) = (n+2)x + \frac{2A}{x}, \quad 0 < x < \infty.$$

Now $f'(x) = n + 2 - 2Ax^{-2}$, which is zero only when $x = \sqrt{2A/(n+2)}$. Verification: $f'(x) > 0$ if $x > \sqrt{2A/(n+2)}$ and $f'(x) < 0$ if $x < \sqrt{2A/(n+2)}$, so $f(x)$ is minimized when $x = \sqrt{2A/(n+2)}$. The minimum length of fencing required for this field is

$$f\left(\sqrt{\frac{2A}{n+2}}\right) = (n+2)\sqrt{\frac{2A}{n+2}} + \frac{2A\sqrt{n+2}}{\sqrt{2A}}$$
$$= \sqrt{2A(n+2)} + \sqrt{2A(n+2)} = 2\sqrt{2A(n+2)} \quad \text{(ft)}.$$

C04S0M.096: Let L be the line segment with endpoints at $(0,\, c)$ and $(b,\, 0)$ on the coordinate axes and suppose that L is tangent to the graph of $y = 1/x^2$ at $(x,\, 1/x^2)$. We will minimize S, the square of the length of L, where $S = b^2 + c^2$. We compute the slope of L in several ways: as the value of dy/dx at the point of tangency, as the slope of the line segment between $(x,\, 1/x^2)$ and $(b,\, 0)$, and as the slope of the line segment between $(x,\, 1/x^2)$ and $(0,\, c)$:

$$-\frac{2}{x^3} = \frac{\dfrac{1}{x^2} - 0}{x - b}, \quad \text{so} \quad x = -2(x-b), \quad \text{hence} \quad b = \frac{3}{2}x.$$
$$-\frac{2}{x^3} = \frac{\dfrac{1}{x^2} - c}{x - 0}; \quad -2x = x^3\left(\frac{1}{x^2} - 3\right), \quad cx^2 = 3, \quad \text{hence} \quad c = \frac{3}{x^2}.$$

Therefore

$$S(x) = \frac{9}{x^4} + \frac{9x^2}{4}; \qquad S'(x) = -\frac{36}{x^5} + \frac{9}{2}x,$$

which is zero only when $x = \sqrt{2}$. Verification: $S'(x) < 0$ if $x < \sqrt{2}$ and $S'(x) > 0$ if $x > \sqrt{2}$. So S, hence the length of L, is minimized when $x = \sqrt{2}$. The length of this shortest line segment is $\frac{3}{2}\sqrt{3}$.

C04S0M.097: Let L be the line segment with endpoints at $(0,\, c)$ and $(b,\, 0)$ on the coordinate axes and suppose that L is tangent to the graph of $y = 1/x^2$ at $(x,\, 1/x^2)$. We compute the slope of L in several ways: as the value of dy/dx at the point of tangency, as the slope of the line segment between $(x,\, 1/x^2)$ and $(b,\, 0)$, and as the slope of the line segment between $(x,\, 1/x^2)$ and $(0,\, c)$:

$$-\frac{2}{x^3} = \frac{\dfrac{1}{x^2} - 0}{x - b}, \quad \text{so} \quad x = -2(x-b), \quad \text{hence} \quad b = \frac{3}{2}x.$$
$$-\frac{2}{x^3} = \frac{\dfrac{1}{x^2} - c}{x - 0}; \quad -2x = x^3\left(\frac{1}{x^2} - 3\right), \quad cx^2 = 3, \quad \text{hence} \quad c = \frac{3}{x^2}.$$

Now the length of the base of the right triangle is b and its height is c, so its area is given by

$$A(x) = \frac{1}{2} \cdot \frac{3x}{2} \cdot \frac{3}{x^2} = \frac{9}{4x}, \quad 0 < x < +\infty.$$

Clearly A is a strictly decreasing function of x, so it has neither a maximum nor a minimum—not even any local extrema.

C04S0M.098: Let L be the line segment in the first quadrant that is tangent to the graph of $y = 1/x$ at $(x,\, 1/x)$ and has endpoints $(0,\, c)$ and $(b,\, 0)$. Compute the slope of L in several ways: as the value of dy/dx at the point of tangency, as the slope of the line segment between $(x,\, 1/x)$ and $(b,\, 0)$, and as the slope of the line segment between $(x,\, 1/x)$ and $(0,\, c)$:

$$-\frac{1}{x^2} = \frac{\dfrac{1}{x} - 0}{x - b} \quad \text{so} \quad b = 2x.$$

$$-\frac{1}{x^2} = \frac{\dfrac{1}{x} - c}{x}: \quad c - \frac{1}{x} = \frac{1}{x}, \quad \text{so} \quad c = \frac{2}{x}.$$

Therefore the area A of the triangle is

$$A = A(x) = \frac{1}{2} \cdot 2x \cdot \frac{1}{x} \equiv 1.$$

Because A is a constant function, every such triangle has both maximal and minimal area.

C04S0M.099: Let x be the length of the shorter sides of the base and let y be the height of the box. Then its volume is $3x^2y$, so that $y = 96/x^2$. The total surface area of the box is $6x^2 + 8xy$, thus is given as a function of x by

$$A(x) = 6x^2 + 8x \cdot \frac{96}{x^2} = 6x^2 + \frac{768}{x}, \quad 0 < x < +\infty.$$

So $A'(x) = 12x - 768x^{-2}$; $A'(x) = 0$ when $x^3 = 64$, so that $x = 4$. Because $A''(x) > 0$ for all $x > 0$, we have found the value of x that yields the global minimum value of A, which is $A(4) = 288$ (in.^{2}). The fact that 288 is also the numerical value of the volume is a mere coincidence.

C04S0M.100: Let x be the length of the shorter sides of the base and let y be the height of the box. Then its volume is $4x^2y$, so that $y = 200/x^2$. The total surface area of the box is $8x^2 + 10xy$, thus is given as a function of x by

$$A(x) = 8x^2 + 10x \cdot \frac{200}{x^2} = 8x^2 + \frac{2000}{x}, \quad 0 < x < +\infty.$$

So $A'(x) = 16x - 2000x^{-2}$; $A'(x) = 0$ when $x^3 = 125$, so that $x = 5$. Because $A''(x) > 0$ for all $x > 0$, we have found the value of x that yields the global minimum value of A, which is $A(5) = 600$ (in.^{2}).

C04S0M.101: Let x be the length of the shorter sides of the base and let y be the height of the box. Then its volume is $5x^2y$, so that $y = 45/x^2$. The total surface area of the box is $10x^2 + 12xy$, thus is given as a function of x by

$$A(x) = 10x^2 + 12x \cdot \frac{45}{x^2} = 10x^2 + \frac{540}{x}, \quad 0 < x < +\infty.$$

So $A'(x) = 20x - 540x^{-2}$; $A'(x) = 0$ when $x^3 = 27$, so that $x = 3$. Because $A''(x) > 0$ for all $x > 0$, we have found the value of x that yields the global minimum value of A, which is $A(3) = 270$ (cm^2).

C04S0M.102: Let x represent the width of the box. Then the length of the box is nx and its height is $V/(nx^2)$. We minimize the surface area A of the box, where

$$A = A(x) = 2nx^2 + \frac{V}{nx^2} \cdot 2(n+1)x = 2nx^2 + \frac{2(n+1)V}{nx}, \qquad 0 < x < \infty.$$

Now

$$A'(x) = 4nx - \frac{2(n+1)V}{nx^2}, \qquad \text{and} \qquad A'(x) = 0 \qquad \text{when} \qquad x = \sqrt[3]{\frac{(n+1)V}{2n^2}}.$$

Verification: $A'(x) > 0$ for $x > \sqrt[3]{\dfrac{(n+1)V}{2n^2}}$ and $A'(x) < 0$ for $x < \sqrt[3]{\dfrac{(n+1)V}{2n^2}}$, so this critical point minimizes A. The value of $A(x)$, simplified, at this minimum point is $3\left(\dfrac{2(n+1)^2V^2}{n}\right)^{1/3}$.

C04S0M.103: First,

$$\begin{aligned} m = \lim_{x\to\pm\infty} \frac{f(x)}{x} &= \lim_{x\to\pm\infty} \frac{(1-x)^{2/3}}{x^{2/3}} \\ &= \lim_{x\to\pm\infty} \left(\frac{\frac{1}{x}-1}{1}\right)^{2/3} = +1. \end{aligned}$$

Then

$$\begin{aligned} b = \lim_{x\to\infty} [f(x) - mx] &= \lim_{x\to\infty} \left(x^{1/3}(1-x)^{2/3} - x\right) \\ &= \lim_{x\to\infty} \frac{\left(x^{1/3}(1-x)^{2/3} - x\right)\left(x^{2/3}(1-x)^{4/3} + x^{4/3}(1-x)^{2/3} + x^2\right)}{x^{2/3}(1-x)^{4/3} + x^{4/3}(1-x)^{2/3} + x^2} \\ &= \lim_{x\to\infty} \frac{x(1-x)^2 - x^3}{x^{2/3}(1-x)^{4/3} + x^{4/3}(1-x)^{2/3} + x^2} \\ &= \lim_{x\to\infty} \frac{x - 2x^2}{x^{2/3}(1-x)^{4/3} + x^{4/3}(1-x)^{2/3} + x^2} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x} - 2}{\left(\frac{1-x}{x}\right)^{4/3} + \left(\frac{1-x}{x}\right)^{2/3} + 1} \\ &= \frac{0-2}{1+1+1} = -\frac{2}{3}. \end{aligned}$$

The limit is the same as $x \to -\infty$. So the graph of $f(x) = x^{1/3}(1-x)^{2/3}$ has the oblique asymptote $y = x - \frac{2}{3}$.

C04S0M.104: Let θ be the angle between your initial path and due north, so that $0 \leqq \theta \leqq \pi/2$, and if $\theta = \pi/2$ then you plan to jog around a semicircle and not swim at all. Suppose that you can swim with speed v (in miles per hour). Then you will swim a length of $2\cos\theta$ miles at speed v and jog a length of 2θ miles at speed $2v$, for a total time of

$$T(\theta) = \frac{2\cos\theta}{v} + \frac{2\theta}{2v} = \frac{1}{v}(\theta + 2\cos\theta), \quad 0 \leqq \theta \leqq \pi/2.$$

It's easy to verify that this formula is correct even in the extreme case $\theta = \pi/2$. It turns out that although $T'(\theta) = 0$ when $\theta = \pi/6$, this value of θ actually *maximizes* $T(\theta)$; this function has an endpoint minimum not even at $\theta = 0$, but at $\theta = \pi/2$. Answer: Jog all the way.

C03S0M.105: The volume of the block is $V = x^2y$, and V is constant while x and y are functions of time t (in minutes). So

$$0 = \frac{dV}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt}. \tag{1}$$

We are given $dy/dt = -2$, $x = 30$, and $y = 20$, so by Eq. (1) $dx/dt = \frac{3}{2}$. Answer: At the time in question the edge of the base is increasing at 1.5 cm/min.

C03S0M.106: Let the nonnegative x-axis represent the ground and the nonnegative y-axis the wall. Let x be the distance from the base of the wall to the foot of the ladder; let y be the height of the top of the ladder above the ground. From the Pythagorean theorem we obtain $x^2 + y^2 = 100$, so

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

Thus $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt}$. We are given $\dfrac{dx}{dt} = \dfrac{5280}{3600} = \dfrac{22}{15}$ ft/s, and at the time when $y = 1$, we have

$$x = \sqrt{100 - (0.01)^2} = \sqrt{99.9999}\,.$$

At that time,

$$\left.\frac{dy}{dt}\right|_{y=0.01} = -\frac{\sqrt{99.9999}}{0.01} \cdot \frac{22}{15} \approx -1466.666 \text{ (ft/s)},$$

almost exactly 1000 mi/h. This shows that in reality, the top of the ladder cannot remain in contact with the wall. If it is forced to do so by some latching mechanism, then a downward force much greater than that caused by gravity will be needed to keep the bottom of the latter moving at the constant rate of 1 mi/h.

C03S0M.107: Let x denote the distance from plane A to the airport, y the distance from plane B to the airport, and z the distance between the two aircraft. Then

$$z^2 = x^2 + y^2 + (3-2)^2 = x^2 + y^2 + 1$$

and $dx/dt = -500$. Now

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

and when $x = 2$, $y = 2$. Therefore $z = 3$ at that time. Therefore,

$$3 \cdot (-600) = 2 \cdot (-500) + 2 \cdot \left.\frac{dy}{dt}\right|_{x=2},$$

and thus $\left.\dfrac{dy}{dt}\right|_{x=2} = -400$. Answer: Its speed is 400 mi/h.

Section 5.2

C05S02.001: $\displaystyle\int (3x^2 + 2x + 1)\,dx = x^3 + x^2 + x + C.$

C05S02.002: $\displaystyle\int (3t^4 + 5t - 6)\,dt = \tfrac{3}{5}t^5 + \tfrac{5}{2}t^2 - 6t + C.$

C05S02.003: $\displaystyle\int (1 - 2x^2 + 3x^3)\,dx = \tfrac{3}{4}x^4 - \tfrac{2}{3}x^3 + x + C.$

C05S02.004: $\displaystyle\int \left(-\frac{1}{t^2}\right) dt = \frac{1}{t} + C.$

C05S02.005: $\displaystyle\int (3x^{-3} + 2x^{3/2} - 1)\,dx = -\tfrac{3}{2}x^{-2} + \tfrac{4}{5}x^{5/2} - x + C.$

C05S02.006: $\displaystyle\int \left(x^{5/2} - \frac{5}{x^4} - \sqrt{x}\right) dx = \int (x^{5/2} - 5x^{-4} - x^{1/2})\,dx = \tfrac{2}{7}x^{7/2} + \tfrac{5}{3}x^{-3} - \tfrac{2}{3}x^{3/2} + C.$

C05S02.007: $\displaystyle\int \left(\tfrac{3}{2}t^{1/2} + 7\right) dt = t^{3/2} + 7t + C.$

C05S02.008: $\displaystyle\int \left(\frac{2}{x^{3/4}} - \frac{3}{x^{2/3}}\right) dx = \int (2x^{-3/4} - 3x^{-2/3})\,dx = 8x^{1/4} - 9x^{1/3} + C.$

C05S02.009: $\displaystyle\int (x^{2/3} + 4x^{-5/4})\,dx = \tfrac{3}{5}x^{5/3} - 16x^{-1/4} + C.$

C05S02.010: $\displaystyle\int \left(2x\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \int (2x^{3/2} - x^{-1/2})\,dx = \tfrac{4}{5}x^{5/2} - 2x^{1/2} + C.$

C05S02.011: $\displaystyle\int (4x^3 - 4x + 6)\,dx = x^4 - 2x^2 + 6x + C.$

C05S02.012: $\displaystyle\int \left(\frac{1}{4}t^5 - \frac{5}{t^2}\right) dt = \int \left(\tfrac{1}{4}t^5 - 5t^{-2}\right) dt = \tfrac{1}{24}t^6 + 5t^{-1} + C.$

C05S02.013: $\displaystyle\int 7\,dx = 7x + C.$

C05S02.014: $\displaystyle\int \left(4\sqrt[3]{x^2} - \frac{6}{\sqrt[3]{x}}\right) dx = \int \left(4x^{2/3} - 6x^{-1/3}\right) dx = \frac{12}{5}x^{5/3} - 9x^{2/3} + C.$

C05S02.015: $\displaystyle\int (x+1)^4\,dx = \tfrac{1}{5}(x+1)^5 + C.$ Note that many computer algebra systems give the answer

$$C + x + 2x^2 + 2x^3 + x^4 + \frac{x^5}{5}.$$

C05S02.016: $\displaystyle\int (t+1)^{10}\,dt = \tfrac{1}{11}(t+1)^{11} + C.$

C05S02.017: $\displaystyle\int \frac{1}{(x-10)^7}\,dx = \int (x-10)^{-7}\,dx = -\tfrac{1}{6}(x-10)^{-6} + C = -\frac{1}{6(x-10)^6} + C.$

C05S02.018: $\displaystyle\int \sqrt{z+1}\ dz = \int (z+1)^{1/2}\ dz = \tfrac{2}{3}(z+1)^{3/2} + C.$

C05S02.019: $\displaystyle\int \sqrt{x}\ (1-x)^2\ dx = \int (x^{1/2} - 2x^{3/2} + x^{5/2})\, dx = \tfrac{2}{3}x^{3/2} - \tfrac{4}{5}x^{5/2} + \tfrac{2}{7}x^{7/2} + C.$

C05S02.020: $\displaystyle\int \sqrt[3]{x}\,(x+1)^3\ dx = \int (x^{10/3}+3x^{7/3}+3x^{4/3}+x^{1/3})\, dx = \tfrac{3}{13}x^{13/3}+\tfrac{9}{10}x^{10/3}+\tfrac{9}{7}x^{7/3}+\tfrac{3}{4}x^{4/3}+C.$

C05S02.021: $\displaystyle\int \frac{2x^4-3x^3+5}{7x^2}\ dx = \int \left(\tfrac{2}{7}x^2 - \tfrac{3}{7}x + \tfrac{5}{7}x^{-2}\right) dx = \tfrac{2}{21}x^3 - \tfrac{3}{14}x^2 - \tfrac{5}{7}x^{-1} + C.$

C05S02.022: $\displaystyle\int \frac{(3x+4)^2}{\sqrt{x}}\ dx = \int x^{-1/2}\left(9x^2+24x+16\right)\ dx = \int \left(9x^{3/2}+24x^{1/2}+16x^{-1/2}\right)\ dx$

$= \tfrac{18}{5}x^{5/2} + 16x^{3/2} + 32x^{1/2} + C.$

C05S02.023: $\displaystyle\int (9t+11)^5\ dt = \tfrac{1}{54}(9t+11)^6 + C.$ *Mathematica* gives the answer

$$C + 161051t + \frac{658845t^2}{2} + 359370t^3 + \frac{441045t^4}{2} + 72171t^5 + \frac{19683t^6}{2}.$$

C05S02.024: $\displaystyle\int \frac{1}{(3z+10)^7}\ dz = \int (3z+10)^{-7}\, dz = -\tfrac{1}{18}(3z+10)^{-6} + C.$

C05S02.025: $\displaystyle\int \frac{7}{(x+77)^2}\ dx = 7\int (x+77)^{-2}\ dx = -7(x+77)^{-1} + C = -\frac{7}{x+77} + C.$

C05S02.026: $\displaystyle\int \frac{3}{\sqrt{(x-1)^3}}\ dx = \int 3(x-1)^{-3/2}\ dx = -6(x-1)^{-1/2} + C = -\frac{6}{\sqrt{x-1}} + C.$

C05S02.027: $\displaystyle\int (5\cos 10x - 10\sin 5x)\ dx = \tfrac{1}{2}\sin 10x + 2\cos 5x + C.$

C05S02.028: $\displaystyle\int (2\cos \pi x + 3\sin \pi x)\ dx = \frac{2}{\pi}\sin \pi x - \frac{3}{\pi}\cos \pi x + C.$

C05S02.029: $\displaystyle\int (3\cos \pi t + \cos 3\pi t)\ dt = \frac{3}{\pi}\sin \pi t + \frac{1}{3\pi}\sin 3\pi t + C.$

C05S02.030: $\displaystyle\int (4\sin 2\pi t - 2\sin 4\pi t)\ dt = -\frac{2}{\pi}\cos 2\pi t + \frac{1}{2\pi}\cos 4\pi t + C.$

C05S02.031: $D_x\left(\tfrac{1}{2}\sin^2 x + C_1\right) = \sin x\ \cos x = D_x\left(-\tfrac{1}{2}\cos^2 x + C_2\right).$ Because

$$\frac{1}{2}\sin^2 x + C_1 = -\frac{1}{2}\cos^2 x + C_2, \quad \text{it follows that} \qquad C_2 - C_1 = \frac{1}{2}\sin^2 x + \frac{1}{2}\cos^2 x = \frac{1}{2}.$$

C05S02.032: $F_1'(x) = \dfrac{1}{(1-x)^2}$, $F_2'(x) = \dfrac{1-x+x}{(1-x)^2} = \dfrac{1}{(1-x)^2}$. $F_1(x) - F_2(x) = C_1$ for some constant C_1 on $(-\infty, 1)$; $F_1(x) - F_2(x) = C_2$ for some constant C_2 on $(1, +\infty)$. On either interval, $F_1(x) - F_2(x) = \dfrac{1-x}{1-x} = 1.$

C05S02.033: $\displaystyle\int \sin^2 x\,dx = \int \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$ and

$$\int \cos^2 x\,dx = \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.$$

C05S02.034: (a): $D_x \tan x = \sec^2 x$; (b): $\displaystyle\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right) dx = (\tan x) - x + C.$

C05S02.035: $y(x) = x^2 + x + C$; $y(0) = 3$, so $y(x) = x^2 + x + 3$.

C05S02.036: $y(x) = \frac{1}{4}(x-2)^4 + C$ and $y(2) = 1$, so $y(x) = \frac{1}{4}(x-2)^4 + 1$.

C05S02.037: $y(x) = \frac{2}{3}x^{3/2} + C$ and $y(4) = 0$, so $y(x) = \frac{2}{3}x^{3/2} - \frac{16}{3}$.

C05S02.038: $y(x) = -\dfrac{1}{x} + C$ and $y(1) = 5$, so $y(x) = -\dfrac{1}{x} + 6$.

C05S02.039: $y(x) = 2\sqrt{x+2} + C$ and $y(2) = -1$, so $y(x) = 2\sqrt{x+2} - 5$.

C05S02.040: $\displaystyle y = \int \sqrt{x+9}\,dx = \tfrac{2}{3}(x+9)^{3/2} + C$; $0 = y(-4) = \frac{2}{3}(-4+9)^{3/2} + C = \frac{2}{3}\cdot 5\sqrt{5} + C$;

$y(x) = \frac{2}{3}(x+9)^{3/2} - \frac{10}{3}\sqrt{5}$.

C05S02.041: $y(x) = \frac{3}{4}x^4 - 2x^{-1} + C$; $y(1) = 1$, so $y(x) = \frac{3}{4}x^4 - 2x^{-1} + \frac{9}{4}$.

C05S02.042: $y(x) = \dfrac{1}{5}x^5 - \dfrac{3}{2}x^2 - \dfrac{3}{2x^2} + C = \dfrac{1}{5}x^5 - \dfrac{3}{2}x^2 - \dfrac{3}{2x^2} + \dfrac{9}{5}$.

C05S02.043: $y(x) = \frac{1}{4}(x-1)^4 + C$; $y(0) = 2 = \frac{1}{4} + C$, so $C = \frac{7}{4}$.

C05S02.044: $y(x) = \frac{2}{3}(x+5)^{3/2} + C$ and $y(4) = -3$, so $y(x) = \frac{2}{3}(x+5)^{3/2} - 21$.

C05S02.045: $y(x) = 2\sqrt{x-13} + C$; $y(17) = 2$, so $C = -2$.

C05S02.046: $\displaystyle y(x) = \int (2x+3)^{3/2}\,dx = \frac{1}{5}(2x+3)^{5/2} + C = \frac{1}{5}(2x+3)^{5/2} + \frac{257}{5}.$

C05S02.047: $v(t) = 6t^2 - 4t + C$; $v(0) = -10$, so $v(t) = 6t^2 - 4t - 10$. Next, $x(t) = 2t^3 - 2t^2 - 10t + K$; $x(0) = 0$, so $K = 0$.

C05S02.048: $v(t) = 10t - 15t^2 + C$; $v(0) = -5$, so $v(t) = 10t - 15t^2 - 5$. Next, $x(t) = 5t^2 - 5t^3 - 5t + K$; $x(0) = 5$, so $K = 5$.

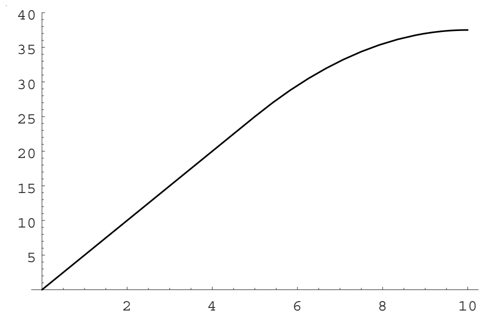
C05S02.049: $v(t) = \frac{2}{3}t^3 + C$; $v(0) = 3$, so $v(t) = \frac{2}{3}t^3 + 3$. Next, $x(t) = \frac{1}{6}t^4 + 3t + K$; $x(0) = -7$, so $K = -7$.

C05S02.050: $v(t) = 10t^{3/2} + C$; $v(0) = 7$, so $v(t) = 10t^{3/2} + 7$. Next, $x(t) = 4t^{5/2} + 7t + K$; $x(0) = 5$, so $K = 5$.

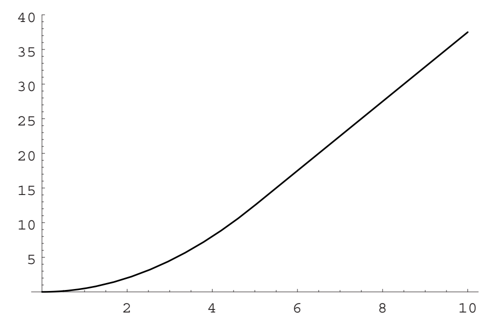
C05S02.051: $v(t) = C - \cos t$; $v(0) = 0$, so $v(t) = 1 - \cos t$. Next, $x(t) = t - \sin t + K$; $x(0) = 0$, so $K = 0$.

C05S02.052: $v(t) = 4\sin 2t + C$; $v(0) = 4$, so $v(t) = 4 + 4\sin 2t$. Next, $x(t) = 4t - 2\cos 2t + K$; $x(0) = -2$, so $K = 0$.

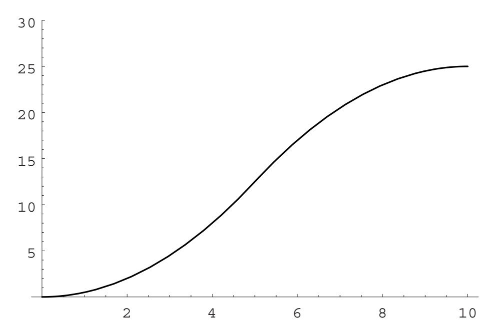
C05S02.053: Note that $v(t) = 5$ for $0 \leqq t \leqq 5$ and that $v(t) = 10 - t$ for $5 \leqq t \leqq 10$. Hence $x(t) = 5t + C_1$ for $0 \leqq t \leqq 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ for $5 \leqq t \leqq 10$. Also $C_1 = 0$ because $x(0) = 0$ and continuity of $x(t)$ requires that $5t + C_1$ and $10t - \frac{1}{2}t^2 + C_2$ agree when $t = 5$. This implies that $C_2 = -\frac{25}{2}$. The graph of x is next.



C05S02.054: First note that $v(t) = t$ for $0 \leqq t \leqq 5$ and that $v(t) = 5$ for $5 \leqq t \leqq 10$. Hence $x(t) = \frac{1}{2}t^2 + C_1$ for $0 \leqq t \leqq 5$ and $x(t) = 5t + C_2$ for $5 \leqq t \leqq 10$. The condition $x(0) = 0$ implies that $C_1 = 0$; continuity of $x(t)$ implies that $\frac{1}{2}t^2 + C_1$ and $5t + C_2$ agree when $t = 5$, and this implies that $C_2 = -\frac{25}{2}$. The graph of $x(t)$ is next.

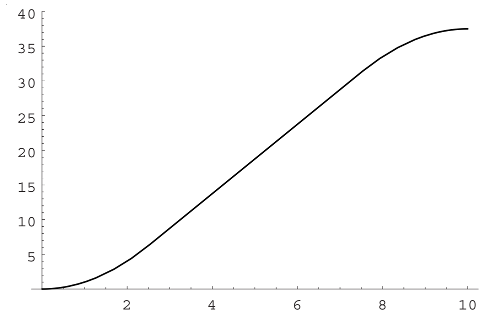


C05S02.055: First, $v(t) = t$ if $0 \leqq t \leqq 5$ and $v(t) = 10 - t$ if $5 \leqq t \leqq 10$. Hence $x(t) = \frac{1}{2}t^2 + C_1$ if $0 \leqq t \leqq 5$ and $x(t) = 10t - \frac{1}{2}t^2 + C_2$ if $5 \leqq t \leqq 10$. Finally, $C_1 = 0$ because $x(0) = 0$ and continuity of $x(t)$ requires that $\frac{1}{2}t^2 + C_1 = 10t - \frac{1}{2}t^2 + C_2$ when $t = 5$, so that $C_2 = -25$. The graph of $x(t)$ is next.



C05S02.056: The graph indicates that $v(t) = 2t$ if $0 \leqq t \leqq 2.5$, $v(t) = 5$ if $2.5 \leqq t \leqq 7.5$, and $v(t) = 20 - 2t$ if $7.5 \leqq t \leqq 10$. Thus $x(t) = t^2 + C_1$ if $0 \leqq t \leqq 2.5$, $x(t) = 5t + C_2$ if $2.5 \leqq t \leqq 7.5$, and $x(t) = 20t - t^2 + C_3$ if $7.5 \leqq t \leqq 10$. Next, $C_1 = 0$ because $x(0) = 0$, $C_2 = 6.25$ because $x(t)$ must be continuous when $t = 2.5$,

and $C_3 = -62.5$ because $x(t)$ must be continuous at $t = 7.5$. The graph of $x(t)$ is next.



In the solutions for Problems 57–78, unless otherwise indicated, we will take the upward direction to be the positive direction, $s = s(t)$ for position (in feet) at time t (in seconds) with $s = 0$ corresponding to ground level, and $v(t)$ velocity at time t in feet per second, $a = a(t)$ acceleration at time t in feet per second per second. The initial position will be denoted by s_0 and the initial velocity by v_0.

C05S02.057: Here, $a = -32$, $v(t) = -32t + 96$, $s(t) = -16t^2 + 96t$. The maximum height is reached when $v = 0$, thus when $t = 3$. The maximum height is therefore $s(3) = 144$. The ball remains aloft until $s(t) = 0$ for $t > 0$; $t = 6$. So it remains aloft for six seconds.

C05S02.058: With initial velocity v_0, here we have

$$a(t) = -32, \quad v(t) = -32t + v_0, \quad \text{and} \quad s(t) = -16t^2 + v_0 t$$

(because $s_0 = 0$). The maximum altitude is attained when $v = 0$, which occurs when $t = v_0/32$. Therefore

$$400 = s(v_0/32) = (-16)(v_0/32)^2 + (v_0)^2/32.$$

It follows that $\frac{1}{64}(v_0)^2 = 400$, and therefore that $v_0 = 160$ (ft/s).

C05S02.059: Here it is more convenient to take the downward direction as the positive direction. Thus

$$a(t) = +32, \quad v(t) = +32t \quad (\text{because } v_0 = 0), \quad \text{and} \quad s(t) = 16t^2 \quad (\text{because} \quad s_0 = 0).$$

The stone hits the bottom when $t = 3$, and at that time we have $s = s(3) = 144$. Answer: The well is 144 feet deep.

C05S02.060: We have $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t$. Also $0 = s(4)$, so $4v_0 = 256$: $v_0 = 64$. Thus

$$v(t) = -32t + 64 \quad \text{and} \quad s(t) = -16t^2 + 64t.$$

The height of the tree is the maximum value of $s(t)$, which occurs when $v(t) = 0$; that is, when $t = 2$. Therefore the height of the tree is $s(2) = 64$ (feet).

C05S02.061: Here, $v(t) = -32t + 48$ and $s(t) = -16t^2 + 48t + 160$. The ball strikes the ground at that value of $t > 0$ for which $s(t) = 0$:

$$0 = s(t) = -16(t - 5)(t + 2), \quad \text{so} \quad t = 5.$$

Therefore the ball remains aloft for 5 seconds. Its velocity at impact is $v(5) = -112$ (ft/s), so the ball strikes the ground at a speed of 112 ft/s.

C05S02.062: First ball: $v_0 = 0$, $s_0 = 576$. So $s(t) = -16t^2 + 576$. The first ball strikes the ground at that $t > 0$ for which $s(t) = 0$; $t = 6$. Second ball: The second ball must remain aloft from time $t = 3$ until time $t = 6$, thus for 3 seconds. Reset $t = 0$ as the time it is thrown downward. Then with initial velocity v_0, the second ball has velocity and position

$$v(t) = -32t + v_0 \quad \text{and} \quad s(t) = -16t^2 + v_0 t + 576$$

at time t. We require that $s(3) = 0$; that is, $0 = s(3) = -144 + 3v_0 + 576$, so that $v_0 = -144$. Answer: The second ball should be thrown straight downward with an initial velocity of 144 ft/s.

C05S02.063: One solution: Take $s_0 = 960$, $v_0 = 0$. Then

$$v(t) = -32t \quad \text{and} \quad s(t) = -16t^2 + 960.$$

The ball hits the street for that value of $t > 0$ for which $s(t) = 0$—that is, when $t = 2\sqrt{15}$. It therefore takes the ball approximately 7.746 seconds to reach the street. Its velocity then is $v(2\sqrt{15}) = -64\sqrt{15}$ (ft/s)—approximately 247.87 ft/s (downward), almost exactly 169 miles per hour.

C05S02.064: Here we have $v(t) = -32t + 320$ and $s(t) = -16t^2 + 320t$. After three seconds have elapsed the height of the arrow will be $s(3) = 816$ (ft). The height of the arrow will be 1200 feet when $s(t) = 1200$:

$$16t^2 - 320t + 1200 = 0;$$
$$16(t-5)(t-15) = 0;$$

$t = 5$ and $t = 15$ are both solutions. So the height of the arrow will be 1200 feet both when $t = 5$ (the arrow is still rising) and when $t = 15$ (the arrow is falling). The arrow strikes the ground at that value of $t > 0$ for which $s(t) = 0$: $t = 20$. So the arrow will strike the ground 20 seconds after it is released.

C05S02.065: With $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t$, we have the maximum altitude $s = 225$ occurring when $v(t) = 0$; that is, when $t = v_0/32$. So

$$225 = s(v_0/32) = (-16)(v_0/32)^2 + (v_0)^2/32 = (v_0)^2/64.$$

It follows that $v_0 = +120$. So the initial velocity of the ball was 120 ft/s.

C05S02.066: Note that the units are meters and seconds. It is also more convenient to take the downward direction as the positive direction here. Thus

$$v(t) = 9.8t \quad \text{and} \quad s(t) = 4.9t^2.$$

The rock reaches the water when $s(t) = 98$: $t = +2\sqrt{5}$. So it takes the rock $2\sqrt{5}$ seconds to reach the water. Its velocity as it penetrates the water surface is $v(2\sqrt{5}) \approx 43.8$ (m/s).

C05S02.067: In this problem, $v(t) = -32t + v_0 = -32t$ and $s(t) = -16t^2 + s_0 = -16t^2 + 400$. The ball reaches the ground when $s = 0$, thus when $16t^2 = 400$: $t = +5$. Therefore the impact velocity is $v(5) = (-32)(5) = -160$ (ft/s).

C05S02.068: Here s_0 will be the height of the building, so

$$v(t) = -32t - 25 \quad \text{and} \quad s(t) = -16t^2 - 25t + s_0.$$

The velocity at impact is -153 ft/s, so we can obtain the time of impact t by solving $v(t) = -153$: $t = 4$. At this time we also have $s = 0$:

$$0 = s(4) = (-16)(16) - (25)(4) + s_0,$$

so that $s_0 = 356$. Answer: The building is 356 feet high.

C05S02.069: For this problem, we take $s(t) = -16t^2 + 160t$ and $v(t) = -32t + 160$. Because $s = 0$ when $t = 0$ and when $t = 10$, the time aloft is 10 seconds. The velocity is zero at maximum altitude, and that occurs when $32t = 160$: $t = 5$. So the maximum altitude is $s(5) = 400$ (ft).

C05S02.070: Let $f(t)$ be the altitude of the sandbag at time t. Then $f(t) = -16t^2 + h$, so the altitude of the sandbag will be $h/2$ when $f(t) = h/2$:

$$t^2 = \frac{h}{32}, \quad \text{so} \quad t = \frac{1}{8}\sqrt{2h}.$$

Let the ball have initial velocity v_0 and altitude $s(t)$ at time t. Then $s(t) = -16t^2 + v_0 t$. We require that $s(t) = h/2$ at the preceding value of t. That is,

$$\frac{h}{2} = (-16)\left(\frac{h}{32}\right) + (v_0)\left(\frac{1}{8}\sqrt{2h}\right).$$

Solution of this equation yields $v_0 = 4\sqrt{2h}$.

C05S02.071: Because $v_0 = -40$, $v(t) = -32t - 40$. Thus $s(t) = -16t^2 - 40t + 555$. Now $s(t) = 0$ when $t = \frac{1}{4}\left(-5 + 2\sqrt{145}\,\right) \approx 4.77$ (s). The speed at impact is $|v(t)|$ for that value of t; that is, $16\sqrt{145} \approx 192.6655$ (ft/s), over 131 miles per hour.

C05S02.072: In this problem, $v(t) = -gt$ and $s(t) = -\frac{1}{2}gt^2 + h$. The rock strikes the ground when $s(t) = 0$, so that $t = \sqrt{2h/g}$. The speed of the rock then is $|-g\sqrt{2h/g}\,| = \sqrt{2gh}$.

C05S02.073: Bomb equations: $a = -32$, $v = -32t$, $s_B = s = -16t^2 + 800$. Here we have $t = 0$ at the time the bomb is released. Projectile equations: $a = -32$, $v = -32(t-2) + v_0$, and

$$s_P = s = 16(t-2)^2 + v_0(t-2), \quad t \geqq 2.$$

We require $s_B = s_P = 400$ at the same time. The equation $s_B = 400$ leads to $t = 5$, and for $s_P(5) = 400$, we must have $v_0 = 544/3 \approx 181.33$ (ft/s).

C05S02.074: Let $x(t)$ denote the distance the car has traveled t seconds after the brakes are applied; let $v(t)$ denote its velocity and $a(t)$ its acceleration at time t during the braking. Then we are given $a = -40$, so $v(t) = -40t + 88$ (because 60 mi/h is the same speed as 88 ft/s). The car comes to a stop when $v(t) = 0$; that is, when $t = 2.2$. The car travels the distance $x(2.2) \approx 96.8$. Answer: 96.8 feet.

C05S02.075: The deceleration $a = k > 0$ is unknown at first. But the velocity of the car is $v(t) = -kt + 88$, and so the distance it travels after the brakes are applied at time $t = 0$ is

$$x(t) = -\frac{1}{2}kt^2 + 88t.$$

But $x = 176$ when $v = 0$, so the stopping time t_1 is $88/k$ because that is the time at which $v = 0$. Therefore

$$176 = -\frac{1}{2}k\left(\frac{88}{k}\right)^2 + (88)\left(\frac{88}{k}\right) = \frac{3872}{k}.$$

It follows that $k = 22$ (ft/s^2), about $0.69g$.

C05S02.076: Let $x(t)$ be the altitude (in miles) of the spacecraft at time t (hours), with $t = 0$ corresponding to the time at which the retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time t. Then $v_0 = -1000$ and x_0 is unknown. But the acceleration constant is $a = +20000$, so

$$v(t) = (20000)t - 1000 \quad \text{and} \quad x(t) = (10000)t^2 - 1000t + x_0.$$

We want $v = 0$ exactly when $x = 0$—call the time then t_1. Then $0 = (20000)t_1 - 1000$, so $t_1 = 1/20$. Also $x(t_1) = 0$, so

$$0 = (10000)\left(\frac{1}{400}\right) - (1000)\left(\frac{1}{20}\right) + x_0.$$

Therefore $x_0 = 50 - 25 = 25$ miles. (Also $t_1 = 1/20$ of an hour; that is, exactly three minutes.)

C05S02.077: (a): With the usual coordinate system, the ball has velocity $v(t) = -32t + v_0$ (ft/s) at time t (seconds) and altitude $y(t) = -16t^2 + v_0 t$ (ft). We require $y(T) = 144$ when $v(T) = 0$: $v_0 = 32T$, so

$$144 = -16T^2 + 32T^2 = 16T^2,$$

and thus $T = 3$ and $v_0 = 96$. Answer: 96 ft/s.

(b): Now $v(t) = -\frac{26}{5}t + 96$, $y(t) = -\frac{13}{5}t^2 + 96t$. Maximum height occurs when $v(t) = 0$: $t = \frac{240}{13}$. The maximum height is

$$y\left(\frac{240}{13}\right) = -\frac{13}{5}\cdot\frac{240^2}{13^2} + 96\cdot\frac{240}{13} = \frac{11520}{13} \approx 886 \text{ ft.}$$

C05S02.078: Set up a coordinate system in which the Diana moves along the x-axis in the positive direction, with initial position $x_0 = 0$ and initial velocity $v_0 = 0$. Then $\dfrac{dv}{dt} = +0.032$ feet per second per second, so

$$v(t) = (0.032)t + v_0 = (0.032)t.$$

Thus

$$x(t) = (0.016)t^2 + x_0 = (0.016)t^2.$$

The units are in feet, seconds, and feet per second. After one minute we take $t = 60$ to find that $x(60) = 57.6$ (feet). After one hour we take $t = 3600$ to find that $x(3600) = 207360$ (ft)—over 39 miles. After one day we take $t = 86400$; $x(86400) = 119{,}439{,}360$ (feet), approximately 22621 miles! At this point the speed of the Diana would be $v(86400) = 2764.8$ feet per second, approximately 1885 miles per hour! After 30 days, the Diana will have traveled well over 20 million miles and will be speeding along in excess of 56000 miles per hour.

C05S02.079: Let a denote the deceleration constant of the car when braking. In the police experiment, we have the distance the car travels from $x_0 = 0$ at time t to be

$$x(t) = -\frac{1}{2}at^2 + 25 \cdot \frac{22}{15}t$$

(the factor $\frac{22}{15}$ converts 25 miles per hour to feet per second). When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$, we find that $a = \frac{1210}{81} \approx 14.938$. When we use this value of a and substitute data from the accident, we find the position function of the car to be

$$x(t) = -\frac{1}{2} \cdot \frac{1210}{81}t^2 + v_0 t$$

where v_0 is its initial velocity. Now when we solve simultaneously $x(t) = 210$ and $x'(t) = 0$, we find that $v_0 = \frac{110}{9}\sqrt{42} \approx 79.209$ feet per second, almost exactly 54 miles per hour.

Section 5.3

C05S03.001: $\displaystyle\sum_{i=1}^{5} 3^i = 3^1 + 3^2 + 3^3 + 3^4 + 3^5 = 3 + 9 + 27 + 81 + 243.$

C05S03.002: $\displaystyle\sum_{i=1}^{6} \sqrt{2i} = \sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{8} + \sqrt{10} + \sqrt{12}.$

C05S03.003: $\displaystyle\sum_{j=1}^{5} \frac{1}{j+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$

C05S03.004: $\displaystyle\sum_{j=1}^{6}(2j-1) = 1 + 3 + 5 + 7 + 9 + 11.$

C05S03.005: $\displaystyle\sum_{k=1}^{6} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}.$

C05S03.006: $\displaystyle\sum_{k=1}^{6} \frac{(-1)^{k+1}}{j^2} = \frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}.$

C05S03.007: $\displaystyle\sum_{n=1}^{5} x^n = x + x^2 + x^3 + x^4 + x^5.$

C05S03.008: $\displaystyle\sum_{n=1}^{5} (-1)^{n+1} x^{2n-1} = x - x^3 + x^5 - x^7 + x^9.$

C05S03.009: $\displaystyle 1 + 4 + 9 + 16 + 25 = \sum_{n=1}^{5} n^2.$

C05S03.010: $\displaystyle 1 - 2 + 3 - 4 + 5 - 6 = \sum_{n=1}^{6} n \cdot (-1)^{n+1}.$

C05S03.011: $\displaystyle 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \sum_{k=1}^{5} \frac{1}{k}.$

C05S03.012: $\displaystyle 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \sum_{i=1}^{5} \frac{1}{i^2}.$

C05S03.013: $\displaystyle \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \sum_{m=1}^{6} \frac{1}{2^m}.$

C05S03.014: $\displaystyle \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} = \sum_{n=1}^{5} \frac{(-1)^{n+1}}{3^n}.$

C05S03.015: $\dfrac{2}{3} + \dfrac{4}{9} + \dfrac{8}{27} + \dfrac{16}{81} + \dfrac{32}{243} = \displaystyle\sum_{n=1}^{5} \left(\frac{2}{3}\right)^{n}.$

C05S03.016: $1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \sqrt{6} + \sqrt{7} + 2\sqrt{2} + 3 = \displaystyle\sum_{j=1}^{9} \sqrt{j}.$

C05S03.017: $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots + \dfrac{x^{10}}{10} = \displaystyle\sum_{n=1}^{10} \frac{1}{n}\, x^n.$

C05S03.018: $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots - \dfrac{x^{19}}{19} = \displaystyle\sum_{n=1}^{10} \frac{(-1)^{n+1}}{2n-1}\, x^{2n-1}.$

C05S03.019: Using Eqs. (3), (4), (6), and (7), we find that

$$\sum_{i=1}^{10}(4i-3) = 4\cdot\sum_{i=1}^{10} i - 3\cdot\sum_{i=1}^{10} 1 = 4\cdot\frac{10\cdot 11}{2} - 3\cdot 10 = 190.$$

C05S03.020: We use Eqs. (3), (4), (6), and (7) to obtain

$$\sum_{j=1}^{8}(5-2j) = 5\cdot\sum_{j=1}^{8} 1 - 2\cdot\sum_{j=1}^{8} j = 5\cdot 8 - 2\cdot\frac{8\cdot 9}{2} = -32.$$

C05S03.021: We use Eqs. (3), (4), (6), and (8) to obtain

$$\sum_{i=1}^{10}(3i^2+1) = 3\cdot\sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 1 = 3\cdot\frac{10\cdot 11\cdot 21}{6} + 10 = 1165.$$

C05S03.022: We use Eqs. (3), (4), (7), and (8) and find thereby that

$$\sum_{k=1}^{6}(2k-3k^2) = 2\cdot\sum_{k=1}^{6} k - 3\cdot\sum_{k=1}^{6} k^2 = 2\cdot\frac{6\cdot 7}{2} - 3\cdot\frac{6\cdot 7\cdot 13}{6} = -231.$$

C05S03.023: First expand, then use Eqs. (3), (4), (6), (7), and (8):

$$\sum_{r=1}^{8}(r-1)(r+2) = \sum_{r=1}^{8}(r^2+r-2) = \sum_{r=1}^{8} r^2 + \sum_{r=1}^{8} r - 2\cdot\sum_{r=1}^{8} 1 = \frac{8\cdot 9\cdot 17}{6} + \frac{8\cdot 9}{2} - 2\cdot 8 = 224.$$

C05S03.024: We use Eqs. (3), (4), (6), (7), and (9) to find that

$$\sum_{i=1}^{5}(i^3-3i+2) = \sum_{i=1}^{5} i^3 - 3\cdot\sum_{i=1}^{5} i + 2\cdot\sum_{i=1}^{5} 1 = \frac{25\cdot 36}{4} - 3\cdot\frac{5\cdot 6}{2} + 2\cdot 5 = 190.$$

C05S03.025: $\displaystyle\sum_{i=1}^{6} \left(i^3 - i^2\right) = \frac{6^2\cdot 7^2}{4} - \frac{6\cdot 7\cdot 13}{6} = 3^2\cdot 7^2 - 7\cdot 13 = 441 - 91 = 350.$

C05S03.026: $\displaystyle\sum_{k=1}^{10}(4k^2-4k+1) = 4\cdot\frac{10\cdot 11\cdot 21}{6} - 4\cdot\frac{10\cdot 11}{2} + 10 = 1330.$

C05S03.027: $\displaystyle\sum_{i=1}^{100} i^2 = \frac{100\cdot 101\cdot 201}{6} = 50\cdot 101\cdot 67 = 338350.$

C05S03.028: $\displaystyle\sum_{i=1}^{100} i^3 = \frac{10^8}{4} + \frac{10^6}{2} + \frac{10^4}{4} = 25502500.$

C05S03.029: $\displaystyle\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{1}{3}.$

C05S03.030: $\displaystyle\lim_{n\to\infty}\frac{1^3+2^3+\cdots+n^3}{n^4} = \lim_{n\to\infty}\frac{n^2(n+1)^2}{4n^4} = \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right) = \frac{1}{4}.$

C05S03.031: $\displaystyle\sum_{i=1}^{n}(2i-1) = 2\cdot\frac{n(n+1)}{2} - n = n^2.$

C05S03.032: $\displaystyle\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{n}(4i^2-4i+1) = 4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + n = \frac{n(2n-1)(2n+1)}{3}.$

C05S03.033: $\displaystyle\underline{A}_5 = \sum_{i=1}^{5}\frac{i-1}{5}\cdot\frac{1}{5} = \frac{2}{5}, \qquad \overline{A}_5 = \sum_{i=1}^{5}\frac{i}{5}\cdot\frac{1}{5} = \frac{3}{5}.$

C05S03.034: $\displaystyle\underline{A}_5 = \sum_{i=1}^{5}\frac{2i+3}{5}\cdot\frac{2}{5} = \frac{18}{5}, \qquad \overline{A}_5 = \sum_{i=1}^{5}\frac{2i+5}{5}\cdot\frac{2}{5} = \frac{22}{5}.$

C05S03.035: $\displaystyle\underline{A}_6 = \sum_{i=1}^{6}\left[2\left(\frac{i-1}{2}\right)+3\right]\cdot\frac{3}{6} = \frac{33}{2}, \qquad \overline{A}_6 = \sum_{i=1}^{6}\left[2\cdot\frac{i}{2}+3\right]\cdot\frac{3}{6} = \frac{39}{2}.$

C05S03.036: $\displaystyle\underline{A}_6 = \sum_{i=1}^{6}\left[13-3\cdot\frac{i}{2}\right]\cdot\frac{3}{6} = \frac{93}{4}, \qquad \overline{A}_6 = \sum_{i=1}^{6}\left[13-3\cdot\frac{i-1}{2}\right]\cdot\frac{3}{6} = \frac{111}{4}.$

C05S03.037: $\displaystyle\underline{A}_5 = \sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^2\cdot\frac{1}{5} = \frac{6}{25}, \qquad \overline{A}_5 = \sum_{i=1}^{5}\left(\frac{i}{5}\right)^2\cdot\frac{1}{5} = \frac{11}{25}.$

C05S03.038: $\displaystyle\underline{A}_5 = \sum_{i=1}^{5}\left(\frac{2i+3}{5}\right)^2\cdot\frac{2}{5} = \frac{178}{25}, \qquad \overline{A}_5 = \sum_{i=1}^{5}\left(\frac{2i+5}{5}\right)^2\cdot\frac{2}{5} = \frac{258}{25}.$

C05S03.039: $\displaystyle\underline{A}_5 = \sum_{i=1}^{5}\left[9-\left(\frac{3i}{5}\right)^2\right]\cdot\frac{3}{5} = \frac{378}{25}, \qquad \overline{A}_5 = \sum_{i=1}^{5}\left[9-\left(\frac{3i-3}{5}\right)^2\right]\cdot\frac{3}{5} = \frac{513}{25}.$

C05S03.040: $\displaystyle\underline{A}_8 = \sum_{i=1}^{8}\left[9-\left(\frac{i}{4}+1\right)^2\right]\cdot\frac{2}{8} = \frac{133}{16}, \qquad \overline{A}_8 = \sum_{i=1}^{8}\left[9-\left(\frac{i-1}{4}+1\right)^2\right]\cdot\frac{2}{8} = \frac{165}{16}.$

C05S03.041: $\underline{A}_{10} = \displaystyle\sum_{i=1}^{10} \left(\frac{i-1}{10}\right)^3 \cdot \frac{1}{10} = \frac{81}{400}, \qquad \overline{A}_{10} = \sum_{i=1}^{10} \left(\frac{i}{10}\right)^3 \cdot \frac{1}{10} = \frac{121}{400}.$

C05S03.042: $\underline{A}_{10} = \displaystyle\sum_{i=1}^{10} \sqrt{\frac{i-1}{10}} \cdot \frac{1}{10} \approx 0.610509, \qquad \overline{A}_{10} = \sum_{i=1}^{10} \sqrt{\frac{i}{10}} \cdot \frac{1}{10} \approx 0.710509.$

C05S03.043: When we add the two given equations, we obtain

$$\begin{aligned} 2 \cdot \sum_{i=1}^{n} i &= (n+1) + (n+1) + \cdots + (n+1) \qquad (n \text{ terms}) \\ &= n(n+1), \qquad \text{and therefore} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \end{aligned}$$

C05S03.044: Following the directions in the problem, we get

$$\begin{aligned} 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1, \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1, \\ 4^3 - 3^3 &= 3 \cdot 3^2 + 3 \cdot 3 + 1, \\ 5^3 - 4^3 &= 3 \cdot 4^2 + 3 \cdot 4 + 1, \\ &\vdots \\ n^3 - (n-1)^3 &= 3(n-1)^2 + 3(n-1) + 1, \\ (n+1)^3 - n^3 &= 3n^2 + 3n + 1. \end{aligned}$$

When we add these equations, we get

$$(n+1)^3 - 1 = 3 \cdot \sum_{k=1}^{n} k^2 + 3 \cdot \frac{n(n+1)}{2} + n,$$

so that

$$3 \cdot \sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - \frac{3}{2}n^2 - \frac{3}{2}n - n = n^3 + \frac{3}{2}n^2 + \frac{1}{2}n.$$

Therefore

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

C05S03.045: $\displaystyle\sum_{i=1}^{n} \frac{i}{n^2} = \frac{n(n+1)}{2n^2} \to \frac{1}{2}$ as $n \to \infty$.

C05S03.046: $\displaystyle\sum_{i=1}^{n} \left(\frac{2i}{n^2}\right)^2 \cdot \frac{2}{n} = \frac{8n(n+1)(2n+1)}{6n^3} \to \frac{8}{3}$ as $n \to \infty$.

C05S03.047: $\displaystyle\sum_{i=1}^{n}\left(\frac{3i}{n}\right)^3\cdot\frac{3}{n}=\frac{81n^2(n+1)^2}{4n^4}\to\frac{81}{4}$ as $n\to\infty$.

C05S03.048: $\displaystyle\sum_{i=1}^{n}\left(\frac{2i}{n}+2\right)\cdot\left(\frac{2}{n}\right)=\frac{4n(n+1)}{2n^2}+2n\cdot\frac{2}{n}\to 6$ as $n\to\infty$.

C05S03.049: $\displaystyle\sum_{i=1}^{n}\left(5-\frac{3i}{n}\right)\cdot\left(\frac{1}{n}\right)=5n\cdot\frac{1}{n}-\frac{3n(n+1)}{2n^2}\to\frac{7}{2}$ as $n\to\infty$.

C05S03.050: $\displaystyle\sum_{i=1}^{n}\left(9-\left(\frac{3i}{n}\right)^2\right)\cdot\frac{3}{n}=9n\cdot\frac{3}{n}-\frac{27n(n+1)(2n+1)}{6n^3}\to 27-9=18$ as $n\to\infty$.

C05S03.051: $\displaystyle\sum_{i=1}^{n}f(x_i)\,\Delta x=\sum_{i=1}^{n}\frac{h}{b}\cdot\frac{bi}{n}\cdot\frac{b}{n}=\frac{bh}{n^2}\cdot\frac{n(n+1)}{2}\to\frac{1}{2}bh$ as $n\to\infty$.

C05S03.052: Let y be the length of the top half of the side of the polygon shown in Fig. 5.3.20 and let x be the distance from the center of the circle to the midpoint of that side. Then

$$\frac{y}{r}=\sin\frac{\pi}{n}\qquad\text{and}\qquad\frac{x}{r}=\cos\frac{\pi}{n}.$$

Hence the area of the large triangle in the figure is

$$A=\frac{1}{2}(x)(2y)=xy=r^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right),$$

and therefore the total area of the regular n-sided polygon consisting of all n such triangles is

$$A_n=nr^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right).$$

The length of the side of the polygon shown in the figure is

$$2y=2r\sin\frac{\pi}{n},$$

and so the perimeter of the n-sided polygon is

$$C_n=2ny=2nr\sin\left(\frac{\pi}{n}\right).$$

C05S03.053: Using the formulas derived in the solution of Problem 52, we have

$$\lim_{n\to\infty}\frac{A_n}{C_n}=\lim_{n\to\infty}\frac{r}{2}\cos\frac{\pi}{n}=\frac{r}{2}.$$

Thus $A=\frac{1}{2}rC$. Hence if $A=\pi r^2$, then $C=2\pi r$.

Section 5.4

C05S04.001: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i - 1)\,\Delta x = \int_1^3 (2x-1)\,dx.$

C05S04.002: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (2 - 3x_{i-1})\,\Delta x = \int_{-3}^2 (2-3x)\,dx.$

C05S04.003: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2 + 4)\,\Delta x = \int_0^{10} (x^2+4)\,dx.$

C05S04.004: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 - 3x_i^2 + 1)\,\Delta x = \int_0^3 (x^3 - 3x^2 + 1)\,dx.$

C05S04.005: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{m_i}\,\Delta x = \int_4^9 \sqrt{x}\,dx.$

C05S04.006: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{25 - x_i^2}\,\Delta x = \int_0^5 \sqrt{25 - x^2}\,dx.$

C05S04.007: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{1+m_i}}\,\Delta x = \int_3^8 \frac{1}{\sqrt{1+x}}\,dx.$

C05S04.008: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (\cos 2x_{i-1})\,\Delta x = \int_0^{\pi/2} \cos 2x\,dx.$

C05S04.009: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (\sin 2\pi m_i)\,\Delta x = \int_0^{1/2} \sin 2\pi x\,dx.$

C05S04.010: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} (\tan x_i)\,\Delta x = \int_0^{\pi/4} \tan x\,dx.$

C05S04.011: $\displaystyle \sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5} \left(\frac{i}{5}\right)^2 \cdot \left(\frac{1}{5}\right) = \frac{1}{125} \cdot \frac{5}{6} \cdot 6 \cdot 11 = \frac{11}{25}.$

C05S04.012: $\displaystyle \sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5} \left(\frac{i}{5}\right)^3 \cdot \left(\frac{1}{5}\right) = \frac{1}{625} \cdot \frac{1}{4} \cdot 25 \cdot 36 = \frac{9}{25}.$

C05S04.013: $\displaystyle \sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5} \frac{1}{1+i} = \frac{29}{20}.$

C05S04.014: $\displaystyle \sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5} \sqrt{i} \approx 8.382332347442.$

C05S04.015: $\displaystyle \sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6} \left[2\left(1 + \frac{i}{2}\right) + 1\right] \cdot \frac{1}{2} = \frac{39}{2}.$

C05S04.016: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left[\left(1+\frac{i}{2}\right)^2 + 2\left(1+\frac{i}{2}\right)\right]\cdot\frac{1}{2} = \frac{331}{8}.$

C05S04.017: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[\left(1+\frac{3i}{5}\right)^3 - 3\left(1+\frac{3i}{5}\right)\right]\cdot\frac{3}{5} = \frac{294}{5}.$

C05S04.018: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[1+2\left(2+\frac{i}{5}\right)^{1/2}\right]\cdot\frac{1}{5} \approx 4.220102178480.$

C05S04.019: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left(\cos\frac{i\pi}{6}\right)\cdot\frac{\pi}{6} = -\frac{\pi}{6}.$

C05S04.020: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left(\sin\frac{i\pi}{6}\right)\cdot\frac{1}{6}$

$$= \frac{1}{6}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}+\frac{1}{2}+0\right) = \frac{2+\sqrt{3}}{6} \approx 0.622008467928146215587907723584312.$$

C05S04.021: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^2\cdot\left(\frac{1}{5}\right) = \frac{6}{25}.$

C05S04.022: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left(\frac{i-1}{5}\right)^3\cdot\left(\frac{1}{5}\right) = \frac{4}{25}.$

C05S04.023: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\frac{1}{1+(i-1)} = \frac{137}{60}.$

C05S04.024: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\sqrt{i-1} \approx 6.146264369942.$

C05S04.025: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left[2\left(1+\frac{i-1}{2}\right)+1\right]\cdot\frac{1}{2} = \frac{33}{2}.$

C05S04.026: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left[\left(1+\frac{i-1}{2}\right)^2 + 2\left(1+\frac{i-1}{2}\right)\right]\cdot\frac{1}{2} = \frac{247}{8}.$

C05S04.027: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[\left(1+\frac{3(i-1)}{5}\right)^3 - 3\left(1+\frac{3(i-1)}{5}\right)\right]\cdot\frac{3}{5} = \frac{132}{5}.$

C05S04.028: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[1+2\left(2+\frac{i-1}{5}\right)^{1/2}\right]\cdot\frac{1}{5} \approx 4.092967280401.$

C05S04.029: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left(\cos\frac{(i-1)\pi}{6}\right)\cdot\frac{\pi}{6} = \frac{\pi}{6}.$

C05S04.030: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x \;=\; \sum_{i=1}^{6}\left(\sin\frac{(i-1)\pi}{6}\right)\cdot\frac{1}{6} \;=\; \frac{2+\sqrt{3}}{6}.$

C05S04.031: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left(\frac{2i-1}{10}\right)^2\cdot\left(\frac{1}{5}\right) = \frac{33}{100}.$

C05S04.032: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left(\frac{2i-1}{10}\right)^3\cdot\left(\frac{1}{5}\right) = \frac{49}{200}.$

C05S04.033: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\frac{2}{1+2i} = \frac{6086}{3465}.$

C05S04.034: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left(\frac{2i-1}{2}\right)^{1/2} \approx 7.505139519609.$

C05S04.035: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left[2\left(1+\frac{2i-1}{4}\right)+1\right]\cdot\frac{1}{2} = 18.$

C05S04.036: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left[\left(1+\frac{2i-1}{4}\right)^2+2\left(1+\frac{2i-1}{4}\right)\right]\cdot\frac{1}{2} = \frac{575}{16}.$

C05S04.037: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[\left(1+\frac{6i-3}{10}\right)^3-3\left(1+\frac{6i-3}{10}\right)\right]\cdot\frac{3}{5} = \frac{1623}{40}.$

C05S04.038: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\left[1+2\left(2+\frac{2i-1}{10}\right)^{1/2}\right]\cdot\frac{1}{5} \;\approx\; 4.157183161049.$

C05S04.039: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{6}\left(\cos\frac{(2i-1)\pi}{12}\right)\cdot\frac{\pi}{6} = 0.$

C05S04.040: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x \;=\; \sum_{i=1}^{6}\left(\sin\frac{(2i-1)\pi}{12}\right)\cdot\frac{1}{6}$

$$= \frac{1}{6}\left(\frac{-1+\sqrt{3}}{2\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{1+\sqrt{3}}{2\sqrt{2}}+\frac{1+\sqrt{3}}{2\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{-1+\sqrt{3}}{2\sqrt{2}}\right) \;=\; \frac{\sqrt{2}\,+\sqrt{6}}{6}$$

$$\approx\; 0.64395055085937885783316213315259824508 9.$$

In Section 5.5 we will see that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(x_i^\star\right)\;\Delta x \;=\; \frac{2}{\pi} \qquad \text{(exactly).}$$

C05S04.041: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5}\frac{5}{5i+2} = \frac{259775}{141372} \;\approx\; 1.837527940469.$

C05S04.042: $\displaystyle\sum_{i=1}^{n} f(x_i^\star)\,\Delta x = \sum_{i=1}^{5} \sqrt{\frac{3i-1}{3}} \approx 7.815585306501.$

C05S04.043: $\displaystyle\sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} = \frac{8n(n+1)(2n+1)}{6n^3} \to \frac{8}{3}$ as $n \to \infty$.

C05S04.044: $\displaystyle\sum_{i=1}^{n} \left(\frac{4i}{n}\right)^3 \cdot \frac{4}{n} = \frac{256n^2(n+1)^2}{4n^4} \to 64$ as $n \to \infty$.

C05S04.045: $\displaystyle\sum_{i=1}^{n} \left(2 \cdot \frac{3i}{n} + 1\right) \cdot \frac{3}{n} = \frac{18n(n+1)}{2n^2} + n \cdot \frac{3}{n} = 12 + \frac{9}{n} \to 12$ as $n \to \infty$.

C05S04.046: $\displaystyle\sum_{i=1}^{n} \left[4 - 3\left(1 + \frac{4i}{n}\right)\right] \cdot \frac{4}{n} = \frac{16}{n} \cdot n - \frac{12}{n} \cdot n - \frac{48n(n+1)}{2n^2} \to 4 - 24 = -20$ as $n \to \infty$.

C05S04.047: $\displaystyle\sum_{i=1}^{n} \left[3 \cdot \left(\frac{3i}{n}\right)^2 + 1\right] \cdot \frac{3}{n} = \frac{81n(n+1)(2n+1)}{6n^3} + n \cdot \frac{3}{n} \to 27 + 3 = 30$ as $n \to \infty$.

C05S04.048: $\displaystyle\sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^3 - \frac{4i}{n}\right] \cdot \frac{4}{n} = \frac{256n^2(n+1)^2}{4n^4} - \frac{16n(n+1)}{2n^2} \to 64 - 8 = 56$ as $n \to \infty$.

C05S04.049: Choose $x_i^\star = x_i = \dfrac{bi}{n}$ and $\Delta x = \dfrac{b}{n}$. Then

$$\int_0^b x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^2 \cdot \frac{b}{n} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} b^3 = \frac{1}{3} b^3.$$

C05S04.050: Choose $x_i^\star = x_i = \dfrac{bi}{n}$ and $\Delta x = \dfrac{b}{n}$. Then

$$\int_0^b x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{bi}{n}\right)^3 \cdot \frac{b}{n} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4} \cdot b^4 = \frac{1}{4} b^4.$$

C05S04.051: If $x_i^\star = \frac{1}{2}(x_{i-1} + x_i)$ for each i, then $x_i^\star$ is the midpoint of each subinterval of the partition, and hence $\{x_i^\star\}$ is a selection for the partition. Moreover, $\Delta x_i = x_i - x_{i-1}$ for each i. So

$$\begin{aligned}
\sum_{i=1}^{n} x_i^\star\,\Delta x_i &= \sum_{i=1}^{n} \tfrac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) \\
&= \tfrac{1}{2} \sum_{i=1}^{n} \left(x_i^2 - x_{i-1}^2\right) \\
&= \tfrac{1}{2} \left(x_1^2 - x_0^2 + x_2^2 - x_1^2 + x_3^2 - x_2^2 + \cdots + x_n^2 - x_{n-1}^2\right) \\
&= \tfrac{1}{2} \left(x_n^2 - x_0^2\right) = \tfrac{1}{2}(b^2 - a^2).
\end{aligned}$$

Therefore $\displaystyle\int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{m} x_i^\star\,\Delta x_i = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2.$

C05S04.052: Let $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and let $\{x_i^\star\}$ be a selection for $\mathcal{P}$. Then

$$\begin{aligned}\int_a^b kf(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^n kf(x_i^\star)\ \Delta x_i \\ &= k\left(\lim_{n\to\infty} \sum_{i=1}^n f(x_i^\star)\ \Delta x_i\right) = k\int_a^b f(x)\,dx.\end{aligned}$$

C05S04.053: Suppose that $a < b$. Let $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and let $\{x_i^\star\}$ be a selection for $\mathcal{P}$. Then

$$\begin{aligned}\int_a^b c\,dx &= \lim_{n\to\infty} \sum_{i=1}^n c\,\Delta x_i \\ &= \lim_{n\to\infty} c\cdot(x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \cdots + x_n - x_{n-1}) \\ &= \lim_{n\to\infty} c(x_n - x_0) = c(b-a).\end{aligned}$$

The proof is similar in the case $b < a$.

C05S04.054: Given any partition $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[0, 1]$ and any positive integer M, there is a selection $\{x_i^\star\}$ for $\mathcal{P}$ in which $x_1^\star = 1/M$. The corresponding Riemann sum then satisfies the inequality

$$\sum_{i=1}^n f\left(x_i^\star\right)\ \Delta x_i \geqq f\left(\frac{1}{M}\right)\cdot(x_1 - x_0) = M(x_1 - x_0),$$

which can be made artibrarily large by choosing M sufficiently large. Hence some Riemann sums remain arbitrariliy large as $n \to +\infty$ and $|\mathcal{P}| \to 0$. Therefore the limit of Riemann sums for this function on this interval does not exist. There is no contradiction to Theorem 1 because f is not continuous on $[0, 1]$.

C05S04.055: Whatever partition $\mathcal{P}$ is given, a selection $\{x_i^\star\}$ with all $x_i^\star$ irrational can be made because irrational numbers can be found in any interval of the form $[x_{i-1}, x_i]$ with $x_{i-1} < x_i$. (An explanation of why this is possible is given after the solution of Problem C02S04.069 of this manual.) For such a selection, we have

$$\sum_{i=1}^n f\left(x_i^\star\right)\ \Delta x_i = \sum_{i=1}^n 1\cdot\Delta x_i = 1$$

regardless of the choice of $\mathcal{P}$ or n. Similarly, by choosing $x_i^\star$ rational for all i, we get

$$\sum_{i=1}^n f\left(x_i^\star\right)\ \Delta x_i = \sum_{i=1}^n 0\cdot\Delta x_i = 0$$

regardless of the choice of $\mathcal{P}$ or n. Therefore the limit of Riemann sums for f on $[0, 1]$ does not exist, and therefore

$$\int_0^1 f(x)\,dx \quad \text{does not exist.}$$

C05S04.056: Let $h = b - a$, $x_i^\star = x_i = a + \dfrac{ih}{n}$, and $\Delta x = \dfrac{h}{n}$. Then

$$\sum_{i=1}^{n} \sin\left(x_i^\star\right)\,\Delta x \;=\; \frac{h}{n}\sum_{i=1}^{n} \sin\left(a + \frac{ih}{n}\right).$$

According to *Mathematica* 3.0,

$$\sum_{i=1}^{n} \sin\left(a + \frac{ih}{n}\right) \;=\; \csc\frac{h}{2n}\,\sin\frac{h}{2}\,\sin\left(\frac{1}{2}\left[2a + h + \frac{h}{n}\right]\right) \;=\; \csc\frac{b-a}{2n}\,\sin\frac{b-a}{2}\,\sin\left(\frac{1}{2}\left[b + a + \frac{b-a}{n}\right]\right).$$

But then,

$$\begin{aligned}\frac{b-a}{n}\sum_{i=1}^{n} \sin\left(a + i\cdot\frac{b-a}{n}\right) &= \frac{b-a}{n}\,\csc\frac{b-a}{2n}\,\sin\frac{b-a}{2}\,\sin\left(\frac{1}{2}\left[b+a+\frac{b-a}{n}\right]\right)\\ &= \frac{\dfrac{2(b-a)}{2n}}{\sin\dfrac{b-a}{2n}}\cdot\left(\sin\frac{b-a}{2}\right)\cdot\sin\left(\frac{1}{2}\left[b+a+\frac{b-a}{n}\right]\right) \to 2\cdot\left(\sin\frac{b-a}{2}\right)\cdot\left(\sin\frac{b+a}{2}\right)\end{aligned}$$

as $n \to +\infty$. But using one of the trigonometric identities that immediately precede Problems 59 through 62 in Section 7.4, we find that

$$2\cdot\left(\sin\frac{b-a}{2}\right)\cdot\left(\sin\frac{b+a}{2}\right) \;=\; \cos\left(\frac{b+a}{2} - \frac{b-a}{2}\right) \;-\; \cos\left(\frac{b+a}{2} + \frac{b-a}{2}\right) \;=\; \cos a \;-\; \cos b.$$

Therefore

$$\int_a^b \sin x\,dx \;=\; \cos a \;-\; \cos b.$$

C05S04.057: Let $x_i^\star \;=\; x_i \;=\; a + i\cdot\dfrac{b-a}{n}$ and let $\Delta x = \dfrac{b-a}{n}$. Then

$$\int_a^b \cos x\,dx \;=\; \lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(x_i^\star\right)\,\Delta x \;=\; \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n}\cos\left(a + i\cdot\frac{b-a}{n}\right)\right].$$

According to *Mathematica* 3.0,

$$\begin{aligned}\frac{b-a}{n}\sum_{i=1}^{n}\cos\left(a + i\cdot\frac{b-a}{n}\right) &= \frac{b-a}{n}\,\csc\frac{a-b}{2n}\,\sin\frac{a-b}{2}\,\cos\frac{b-a+n(b+a)}{2n}\\ &= \frac{2\cdot\dfrac{b-a}{2n}}{\sin\dfrac{b-a}{2n}}\,\sin\frac{b-a}{2}\,\cos\frac{b-a+n(b+a)}{2n} \to 2\cdot\left(\sin\frac{b-a}{2}\right)\cdot\left(\cos\frac{b+a}{2}\right)\end{aligned}$$

as $n \to +\infty$. Next, using one of the trigonometric identities that precede Problems 59 through 62 in Section 7.4, we find that

$$2\cdot\left(\sin\frac{b-a}{2}\right)\cdot\left(\cos\frac{b+a}{2}\right) \;=\; \sin b \;+\; \sin(-a) \;=\; \sin b \;-\; \sin a.$$

Therefore

$$\int_a^b \cos x\,dx \;=\; \sin b \;-\; \sin a.$$

Section 5.5

C05S05.001: $\displaystyle\int_0^1 \left(3x^2 + 2\sqrt{x} + 3\sqrt[3]{x}\,\right) dx = \left[\,x^3 + \tfrac{4}{3}x^{3/2} + \tfrac{9}{4}x^{4/3}\,\right]_0^1 = \tfrac{55}{12}.$

C05S05.002: $\displaystyle\int_1^3 \frac{6}{x^2}\,dx = \left[-\frac{6}{x}\right]_1^3 = -2 + 6 = 4.$

C05S05.003: $\displaystyle\int_0^1 x^3(1+x)^2\,dx = \left[\,\tfrac{1}{4}x^4 + \tfrac{2}{5}x^5 + \tfrac{1}{6}x^6\,\right]_0^1 = \tfrac{49}{60}.$

C05S05.004: $\displaystyle\int_{-2}^{-1} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_{-2}^{-1} = \frac{7}{24}.$

C05S05.005: $\displaystyle\int_0^1 (x^4 - x^3)\,dx = \left[\tfrac{1}{5}x^5 - \tfrac{1}{4}x^4\right]_0^1 = -\tfrac{1}{20}.$

C05S05.006: $\displaystyle\int_1^2 (x^4 - x^3)\,dx = \left[\,\tfrac{1}{5}x^5 - \tfrac{1}{4}x^4\,\right]_1^2 = \tfrac{49}{20} = 2.45.$

C05S05.007: $\displaystyle\int_{-1}^0 (x+1)^3\,dx = \left[\,\tfrac{1}{4}(x+1)^4\,\right]_{-1}^0 = \tfrac{1}{4}.$

C05S05.008: $\displaystyle\int_1^3 \frac{x^4+1}{x^2}\,dx = \left[\frac{1}{3}x^3 - \frac{1}{x}\right]_1^3 = \frac{28}{3}.$

C05S05.009: $\displaystyle\int_0^4 \sqrt{x}\,dx = \left[\,\tfrac{2}{3}x^{3/2}\,\right]_0^4 = \tfrac{16}{3}.$

C05S05.010: $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx = \left[\,2\sqrt{x}\,\right]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2.$

C05S05.011: $\displaystyle\int_{-1}^2 (3x^2 + 2x + 4)\,dx = \left[x^3 + x^2 + 4x\right]_{-1}^2 = 20 - (-4) = 24.$

C05S05.012: $\displaystyle\int_0^1 x^{99}\,dx = \left[\,\tfrac{1}{100}x^{100}\,\right]_0^1 = \tfrac{1}{100}.$

C05S05.013: $\displaystyle\int_{-1}^1 x^{99}\,dx = \left[\,\tfrac{1}{100}x^{100}\,\right]_{-1}^1 = 0.$

C05S05.014: $\displaystyle\int_0^4 (7x^{5/2} - 5x^{3/2})\,dx = \left[\,2x^{7/2} - 2x^{5/2}\,\right]_0^4 = 192.$

C05S05.015: $\displaystyle\int_1^3 (x-1)^5\,dx = \left[\,\tfrac{1}{6}(x-1)^6\,\right]_1^3 = \tfrac{32}{3}.$

C05S05.016: $\displaystyle\int_1^2 (x^2+1)^3\,dx = \int_1^2 (x^6 + 3x^4 + 3x^2 + 1)\,dx = \left[\,\tfrac{1}{7}x^7 + \tfrac{3}{5}x^5 + x^3 + x\,\right]_1^2 = \tfrac{1566}{35} \approx 44.742857.$

C05S05.017: $\displaystyle\int_{-1}^0 (2x+1)^3\,dx = \int_{-1}^0 (8x^3 + 12x^2 + 6x + 1)\,dx = \left[\,2x^4 + 4x^3 + 3x^2 + x\,\right]_{-1}^0 = 0 - 0 = 0.$

C05S05.018: $\displaystyle\int_1^3 \frac{10}{(2x+3)^2}\,dx = \left[-\frac{5}{2x+3}\right]_1^3 = \frac{4}{9}.$

C05S05.019: $\displaystyle\int_1^8 x^{2/3}\,dx = \left[\tfrac{3}{5}x^{5/3}\right]_1^8 = \tfrac{93}{5}.$

C05S05.020: $\displaystyle\int_1^9 \left(1+\sqrt{x}\right)^2\,dx = \int_1^9 (1+2\sqrt{x}+x)\,dx = \left[x+\tfrac{4}{3}x^{3/2}+\tfrac{1}{2}x^2\right]_1^9 = \tfrac{248}{3}.$

C05S05.021: $\displaystyle\int_0^1 (x^2-3x+4)\,dx \;=\; \left[\frac{1}{3}x^3-\frac{3}{2}x^2+4x\right]_0^1 \;=\; \frac{17}{6} \;\approx\; 2.8333333333333333333.$

C05S05.022: $\displaystyle\int_0^4 \sqrt{3t}\,dt = \left[\tfrac{2}{3}t^{3/2}\sqrt{3}\right]_0^4 = \tfrac{16}{3}\sqrt{3}.$

C05S05.023: $\displaystyle\int_1^9 \left(x^{1/2}-2x^{-1/2}\right)dx \;=\; \left[\frac{2}{3}x^{3/2}-4x^{1/2}\right]_1^9 \;=\; 6-\left(-\frac{10}{3}\right) \;=\; \frac{28}{3} \;\approx\; 9.333333333333.$

C05S05.024: $\displaystyle\int_2^3 \frac{du}{u^2} = \left[-\frac{1}{u}\right]_2^3 = -\frac{1}{3}+\frac{1}{2} = \frac{1}{6}.$

C05S05.025: $\displaystyle\int_1^4 \frac{x^2-1}{\sqrt{x}}\,dx \;=\; \int_1^4 \left(x^{3/2}-x^{-1/2}\right)\,dx \;=\; \left[\frac{2}{5}x^{5/2}-2x^{1/2}\right]_1^4 \;=\; \frac{52}{5} \;=\; 10.4.$

C05S05.026: $\displaystyle\int_1^4 (t^2-2)\sqrt{t}\,dt \;=\; \left[\frac{2}{7}t^{7/2}-\frac{4}{3}t^{3/2}\right]_1^4 \;=\; \frac{566}{21} \;\approx\; 26.952380952380952380952380952381.$

C05S05.027: $\displaystyle\int_4^7 \sqrt{3x+4}\,dx \;=\; \left[\frac{2}{9}\,(3x+4)^{3/2}\right]_4^7 \;=\; \frac{122}{9} \;\approx\; 13.5555555555555556.$

C05S05.028: $\displaystyle\int_0^{\pi/2} \cos 2x\,dx \;=\; \left[\frac{1}{2}\sin 2x\right]_0^{\pi/2} \;=\; 0.$

C05S05.029: $\displaystyle\int_0^{\pi/4} \sin x\,\cos x\,dx = \left[\tfrac{1}{2}(\sin x)^2\right]_0^{\pi/4} = \tfrac{1}{4}.$

C05S05.030: $\displaystyle\int_0^{\pi} \sin^2 x\,\cos x\,dx = \left[\tfrac{1}{3}\sin^3 x\right]_0^{\pi} = 0.$

C05S05.031: $\displaystyle\int_0^{\pi} \sin 5x\,dx = \left[-\tfrac{1}{5}\cos x\right]_0^{\pi} = \tfrac{2}{5}.$

C05S05.032: $\displaystyle\int_0^2 \cos \pi t\,dt = \left[\frac{1}{\pi}\sin \pi t\right]_0^2 = 0.$

C05S05.033: $\displaystyle\int_0^{\pi/2} \cos 3x\,dx = \left[\tfrac{1}{3}\sin 3x\right]_0^{\pi/2} = -\tfrac{1}{3}.$

C05S05.034: $\displaystyle\int_0^5 \sin\frac{\pi x}{10}\,dx = \left[-\frac{10}{\pi}\cos\frac{\pi x}{10}\right]_0^5 = \frac{10}{\pi}.$

C05S05.035: $\displaystyle\int_0^2 \cos\frac{\pi x}{4}\,dx = \left[\frac{4}{\pi}\sin\frac{\pi x}{4}\right]_0^2 = \frac{4}{\pi}.$

C05S05.036: $\displaystyle\int_0^{\pi/8} \sec^2 2t\,dt = \left[\tfrac{1}{2}\tan 2t\right]_0^{\pi/8} = \tfrac{1}{2}.$

C05S05.037: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the limit in question is the limit of a Riemann sum for the function $f(x) = 2x - 1$ on the interval $0 \leqq x \leqq 1$, and its value is therefore

$$\int_0^1 (2x-1)\,dx = \Big[x^2 - x\Big]_0^1 = 1 - 1 = 0.$$

C05S05.038: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the limit in question is the limit of a Riemann sum for the function $f(x) = x^2$ on the interval $0 \leqq x \leqq 1$, and therefore

$$\lim_{n\to\infty}\sum_{i=1}^n \frac{i^2}{n^3} = \int_0^1 x^2\,dx = \frac{1}{3}.$$

C05S05.039: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the limit in question is the limit of a Riemann sum for the function $f(x) = x$ on the interval $0 \leqq x \leqq 1$, and therefore

$$\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{n^2} = \lim_{n\to\infty}\sum_{i=1}^n \frac{i}{n^2} = \int_0^1 x\,dx = \frac{1}{2}.$$

C05S05.040: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the limit in question is the limit of a Riemann sum for the function $f(x) = x^3$ on the interval $0 \leqq x \leqq 1$, and therefore

$$\lim_{n\to\infty}\frac{1^3+2^3+3^3+\cdots+n^3}{n^4} = \lim_{n\to\infty}\sum_{i=1}^n \frac{i^3}{n^4} = \int_0^1 x^3\,dx = \frac{1}{4}.$$

C05S05.041: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the given limit is the limit of a Riemann sum for the function $f(x) = \sqrt{x}$ on the interval $0 \leqq x \leqq 1$, and therefore

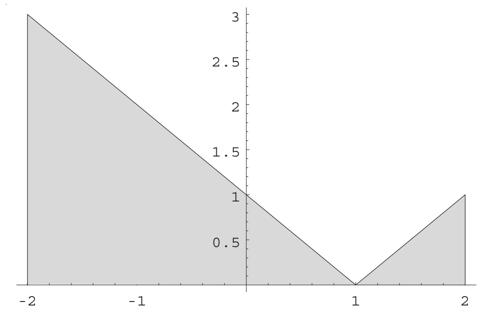
$$\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n\sqrt{n}} = \lim_{n\to\infty}\sum_{i=1}^n \frac{\sqrt{i}}{n\sqrt{n}} = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}.$$

C05S05.042: Choose $x_i = i/n$, $\Delta x = 1/n$, $x_0 = 0$, and $x_n = 1$. Then the given limit is the limit of a Riemann sum for $f(x) = \sin \pi x$ on the interval $0 \leqq x \leqq 1$, and therefore

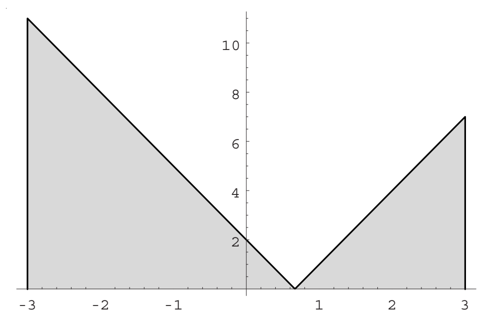
$$\lim_{n\to\infty}\sum_{i=1}^n \frac{1}{n}\sin\frac{\pi i}{n} = \int_0^1 \sin \pi x\,dx = \frac{2}{\pi}.$$

C05S05.043: The graph is shown next. The region represented by the integral consists of two triangles above the x-axis, one with base 3 and height 3, the other with base 1 and height 1. So the value of the

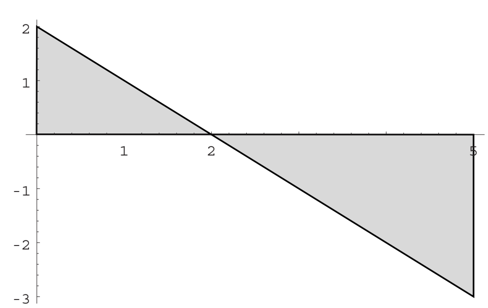
integral is the total area $\frac{9}{2} + \frac{1}{2} = 5$.



C05S05.044: The graph is next. The region represented by the integral consists of two triangles above the x-axis, one with base $\frac{11}{3}$ and height 11, the other with base $\frac{7}{3}$ and height 7, for a total area of $\frac{121}{6} + \frac{49}{6} = \frac{85}{3}$.

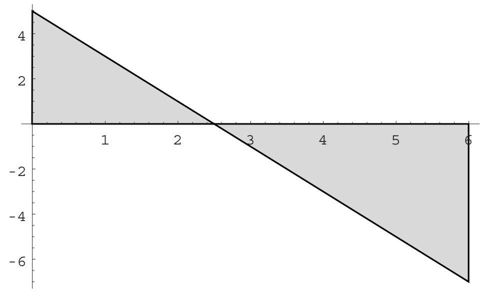


C05S05.045: The graph is next. The region represented by the integral consists of two triangles, one above the x-axis with base 2 and height 2, the other below the x-axis with base 3 and height 3, so the total value of the integral is $2 - \frac{9}{2} = -\frac{5}{2}$.



C05S05.046: The graph is next. The region represented by the integral consists of two triangles, one above the x-axis with base $\frac{5}{2}$ and height 5, the other below the x-axis with base $\frac{7}{2}$ and height 7, so the total value

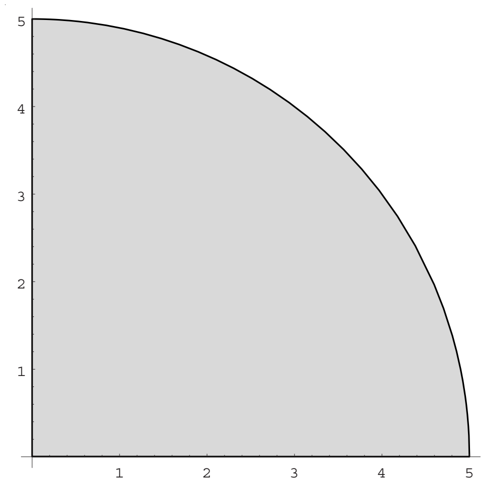
of the integral is $\frac{25}{4} - \frac{49}{4} = -6$.



C05S05.047: If $y = \sqrt{25 - x^2}$ for $0 \leqq x \leqq 5$, then

$$x^2 + y^2 = 25, \quad 0 \leqq x \leqq 5, \quad \text{and} \quad 0 \leqq y \leqq 5.$$

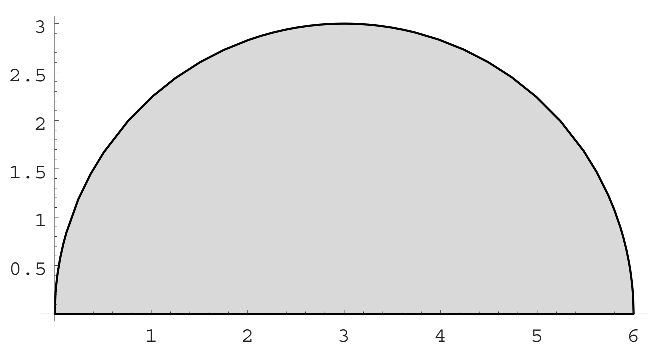
Therefore the region represented by the integral consists of the quarter of the circle $x^2 + y^2 = 25$ that lies in the first quadrant. This circle has radius 5, and therefore the value of the integral is $\frac{25}{4}\pi$. The region is shown next.



C05S05.048: If $y = \sqrt{6x - x^2}$ for $0 \leqq x \leqq 6$, then

$$x^2 - 6x + y^2 = 0; \quad \text{that is,} \quad (x - 3)^2 + y^2 = 9 \quad \text{where} \quad 0 \leqq x \leqq 6, \quad 0 \leqq y \leqq 3.$$

So the region represented by the integral consists of the half of the circle $(x - 3)^2 + y^2 = 9$ (radius 3, center $(3,\ 0)$) that lies on and above the x-axis. The circle has radius 3, and therefore the value of the integral is $\frac{9}{2}\pi$. The semicircle is shown next.



C05S05.049: $0 \leqq x^2 \leqq x$ if $0 \leqq x \leqq 1$. Hence $1 \leqq 1 + x^2 \leqq 1 + x$ for such x. Therefore

$$1 \leqq \sqrt{1+x^2} \leqq \sqrt{1+x}$$

if $0 \leqq x \leqq 1$. Hence, by the comparison property, the inequality in Problem 49 follows.

C05S05.050: $x \leqq x^3 \leqq 8$ if $1 \leqq x \leqq 2$. Hence $1 + x \leqq 1 + x^3 \leqq 9$ for such x. Therefore

$$\sqrt{1+x} \leqq \sqrt{1+x^3} \leqq 3$$

if $1 \leqq x \leqq 2$. The inequality in Problem 50 now follows from the comparison property.

C05S05.051: $x^2 \leqq x$ and $x \leqq \sqrt{x}$ if $0 \leqq x \leqq 1$. So $1 + x^2 \leqq 1 + \sqrt{x}$ for such x. Therefore

$$\frac{1}{1+\sqrt{x}} \leqq \frac{1}{1+x^2}$$

if $0 \leqq x \leqq 1$. The inequality in Problem 51 now follows from the comparison property for definite integrals.

C05S05.052: $x^2 \leqq x^5$ if $x \geqq 1$. So $1 + x^2 \leqq 1 + x^5$ if $2 \leqq x \leqq 5$. Therefore

$$\frac{1}{1+x^5} \leqq \frac{1}{1+x^2}$$

if $2 \leqq x \leqq 5$. The inequality in Problem 52 now follows from the comparison property.

C05S05.053: $\sin t \leqq 1$ for all t. Therefore $\displaystyle \int_0^2 \sin\left(\sqrt{x}\,\right)\,dx \leqq \int_0^2 1\,dx = 2.$

C05S05.054: If $0 \leqq x \leqq \frac{1}{4}\pi$, then

$$\frac{\sqrt{2}}{2} \leqq \cos x \leqq 1.$$

Thus $\frac{1}{2} \leqq \cos^2 x \leqq 1$ for such x, and therefore $\frac{3}{2} \leqq 1 + \cos^2 x \leqq 2$ if $0 \leqq x \leqq \frac{1}{4}\pi$. Hence

$$\frac{1}{2} \leqq \frac{1}{1+\cos^2 x} \leqq \frac{2}{3}$$

if $0 \leqq x \leqq \frac{1}{4}\pi$. So

$$\frac{1}{2} \cdot \frac{\pi}{4} \leqq \int_0^{\pi/4} \frac{1}{1+\cos^2 x}\,dx \leqq \frac{2}{3} \cdot \frac{\pi}{4},$$

and the inequality in Problem 54 follows immediately.

C05S05.055: If $0 \leqq x \leqq 1$, then

$$\begin{aligned}
&1 \leqq 1 + x \leqq 2;\\
&\frac{1}{2} \leqq \frac{1}{1+x} \leqq 1;\\
&\frac{1}{2} \cdot (1-0) \leqq \int_0^1 \frac{1}{1+x}\,dx \leqq 1 \cdot (1-0).
\end{aligned}$$

So the value of the integral lies between 0.5 and 1.0.

C05S05.056: If $4 \leqq x \leqq 9$, then

$$\begin{aligned}
&2 \leqq \sqrt{x} \leqq 3;\\
&3 \leqq 1+\sqrt{x} \leqq 4;\\
&\frac{1}{4} \leqq \frac{1}{1+\sqrt{x}} \leqq \frac{1}{3};\\
&\frac{1}{4}\cdot(9-4) \leqq \int_4^9 \frac{1}{1+\sqrt{x}}\,dx \leqq \frac{1}{3}\cdot(9-4).
\end{aligned}$$

Hence the value of the given integral lies between $\frac{5}{4}$ and $\frac{5}{3}$.

C05S05.057: If $0 \leqq x \leqq \frac{1}{6}\pi$, then

$$\begin{aligned}
&\frac{\sqrt{3}}{2} \leqq \cos x \leqq 1;\\
&\frac{3}{4} \leqq \cos^2 x \leqq 1;\\
&\frac{3}{4}\cdot\left(\frac{\pi}{6}-0\right) \leqq \int_0^{\pi/6} \cos^2 x\,dx \leqq 1\cdot\left(\frac{\pi}{6}-0\right).
\end{aligned}$$

Therefore

$$\frac{\pi}{8} \leqq \int_0^{\pi/6} \cos^2 x\,dx \leqq \frac{\pi}{6}.$$

C05S05.058: If $0 \leqq x \leqq \frac{1}{4}\pi$, then

$$\begin{aligned}
&0 \leqq \sin x \leqq \frac{\sqrt{2}}{2};\\
&0 \leqq \sin^2 x \leqq \frac{1}{2};\\
&0 \leqq 2\sin^2 x \leqq 1;\\
&16 \leqq 16+2\sin^2 x \leqq 17;\\
&4 \leqq \sqrt{16+2\sin^2 x} \leqq \sqrt{17};\\
&4\cdot\frac{\pi}{4} \leqq \int_0^{\pi/4} \sqrt{16+2\sin^2 x}\,dx \leqq \frac{\pi\sqrt{17}}{4}.
\end{aligned}$$

Therefore

$$3.14159 \leqq \int_0^{\pi/4} \sqrt{16+2\sin^2 x}\,dx \leqq 3.2384.$$

C05S05.059: Suppose that f is integrable on $[a, b]$ and that c is a constant. Then

$$\int_a^b cf(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} cf(x_i^\star)\,\Delta x = \lim_{\Delta x \to 0} c \cdot \sum_{i=1}^{n} f(x_i^\star)\,\Delta x$$

$$= c \cdot \left(\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^\star)\,\Delta x \right) = c \int_a^b f(x)\,dx.$$

C05S05.060: Suppose that f and g are integrable on $[a,\,b]$ and that $f(x) \leqq g(x)$ for all x in $[a,\,b]$. Suppose by way of contradiction that

$$I_1 = \int_a^b f(x)\,dx > \int_a^b g(x)\,dx = I_2.$$

Let $\epsilon = I_1 - I_2$. Choose n a positive integer so large that every Riemann sum for f based on a regular partition with n or more subintervals lies within $\epsilon/3$ of I_1 and every Riemann sum for g based on a regular partition with n or more subintervals lies within $\epsilon/3$ of I_2. (This can be done by determining n_1 for I_1, n_2 for I_2, and letting n be the maximum of n_1 and n_2.) Then every Riemann sum for f based on a regular partition with n or more subintervals exceeds every Riemann sum for g based on a regular partition with n or more subintervals. Let $\mathcal{P}$ be such a partition and let $\{x_i^\star\}$ be a selection for $\mathcal{P}$. Then

$$\sum_{i=1}^{n} f(x_i^\star)\,\Delta x > \sum_{i=1}^{n} g(x_i^\star)\,\Delta x.$$

But this is impossible because $f(x_i^\star) \leqq g(x_i^\star)$ for all i, $1 \leqq i \leqq n$. This contradiction shows that $I_1 \leqq I_2$ and establishes the first comparison property of the definite integral.

C05S05.061: Suppose that f is integrable on $[a,\,b]$ and that $f(x) \leqq M$ for all x in $[a,\,b]$. Let $g(x) \equiv M$. Then by the first comparison property,

$$\int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx = \Big[Mx \Big]_a^b = M(b-a).$$

The proof of the other inequality is similar.

C05S05.062: Suppose that $a < c < b$ and that f is integrable on $[a,\,b]$. Then f is integrable on $[a,\,c]$ and on $[c,\,b]$. Let $\{R_n\}$ and $\{S_n\}$ be sequences of Riemann sums, the former for f on $[a,\,c]$, the latter for f on $[c,\,b]$, such that, for each positive integer n, R_n and S_n each have norm less than $(b-a)/n$. Then

$$\lim_{n\to\infty} R_n = \int_a^c f(x)\,dx \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \int_c^b f(x)\,dx.$$

Then because $R_n + S_n$ is a Riemann sum of norm less than $(b-a)/n$ for f on $[a,\,b]$, we have

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \Big(\lim_{n\to\infty} R_n \Big) + \Big(\lim_{n\to\infty} S_n \Big) = \lim_{n\to\infty} (R_n + S_n) = \int_a^b f(x)\,dx.$$

C05S05.063: $1000 + \displaystyle\int_0^{30} V'(t)\,dt = 1000 + \Big[(0.4)t^2 - 40t \Big]_0^{30} = 160$ (gallons). Alternatively, the tank contains

$$V(t) = (0.4)t^2 - 40t + 1000$$

gallons at time $t \geqq 0$, so at time $t = 30$ it contains $V(30) = 160$ gallons.

C05S05.064: In 1990 the population in thousands was

$$125 + \int_{t=0}^{20} \left(8 + (0.5)t + (0.03)t^2\right)\,dt = 125 + \Big[\,8t + (0.25)t^2 + (0.01)t^3\,\Big]_0^{20}$$

$$= 125 + 160 + 100 + 80 = 465 \quad \text{(thousands).}$$

C05S05.065: In Fig. 5.5.11 of the text we see that

$$\frac{12 - 4x}{9} \leqq \frac{1}{x} \leqq \frac{3 - x}{2}.$$

Therefore

$$\frac{2}{3} \leqq \int_1^2 \frac{1}{x}\,dx \leqq \frac{3}{4}.$$

Another way to put it would be to write

$$\int_1^2 \frac{1}{x}\,dx = 0.708333 \pm 0.041667.$$

For a really sophisticated answer, you could point out that, from the figure, it appears that the low estimate of the integral is about twice as accurate as (has half the error of) the high estimate. So

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{2}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{3}{4} \approx 0.6944.$$

C05S05.066: Because $L(x) \leqq f(x) \leqq L(x) + 0.07$ if $0 \leqq x \leqq 1$, it follows that

$$\int_0^1 L(x)\,dx \leqq \int_0^1 f(x)\,dx \leqq \int_0^1 [L(x) + 0.07]\,dx.$$

That is,

$$\frac{3}{4} \leqq \int_0^1 \frac{1}{1+x^2}\,dx \leqq \frac{41}{50}.$$

Another way to put it would be to write

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785 \pm 0.035.$$

Or, for the reasons given in the solution of Problem 65,

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1}{3}\cdot\frac{41}{50} + \frac{2}{3}\cdot\frac{3}{4} \approx 0.7733.$$

Section 5.6

C05S06.001: $\dfrac{1}{2-0}\displaystyle\int_0^2 x^4\,dx = \frac{1}{2}\cdot\left[\frac{1}{5}x^5\right]_0^2 = \frac{16}{5}.$

C05S06.002: $\dfrac{1}{4-1}\displaystyle\int_1^4 \sqrt{x}\;dx = \frac{1}{3}\cdot\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{14}{9}.$

C05S06.003: $\dfrac{1}{2-0}\displaystyle\int_0^2 3x^2(x^3+1)^{1/2}\;dx = \frac{1}{2}\cdot\left[\frac{2}{3}(x^3+1)^{3/2}\right]_0^2 = \frac{26}{3}.$

C05S06.004: $\dfrac{1}{4-0}\displaystyle\int_0^4 8x\;dx = \frac{1}{4}\cdot\Big[4x^2\Big]_0^4 = 16.$

C05S06.005: $\dfrac{1}{4-(-4)}\displaystyle\int_{-4}^4 8x\;dx = \frac{1}{4}\cdot\Big[4x^2\Big]_{-4}^4 = 0.$

C05S06.006: $\dfrac{1}{4-(-4)}\displaystyle\int_{-4}^4 x^2\;dx = \frac{1}{8}\cdot\left[\frac{1}{3}x^3\right]_{-4}^4 = \frac{16}{3}.$

C05S06.007: $\dfrac{1}{5-0}\displaystyle\int_0^5 x^3\;dx = \frac{1}{5}\cdot\left[\frac{1}{4}x^4\right]_0^5 = \frac{125}{4}.$

C05S06.0008: $\dfrac{1}{4-1}\displaystyle\int_1^4 x^{-1/2}\;dx = \frac{1}{3}\cdot\Big[2x^{1/2}\Big]_1^4 = \frac{2}{3}.$

C05S06.009: $\dfrac{1}{3-0}\displaystyle\int_0^3 \sqrt{x+1}\;dx = \frac{1}{3}\cdot\left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = \frac{14}{9}.$

C05S06.010: $\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \sin 2x\;dx = \frac{2}{\pi}\cdot\left[-\frac{1}{2}\,\cos 2x\right]_0^{\pi/2} = \frac{2}{\pi}.$

C05S06.011: $\dfrac{1}{\pi-0}\displaystyle\int_0^{\pi} \sin 2x\;dx = \frac{1}{\pi}\cdot\left[-\frac{1}{2}\,\cos 2x\right]_0^{\pi} = 0.$

C05S06.012: $\dfrac{1}{1}\displaystyle\int_{-1/2}^{1/2} \cos 2\pi t\;dt \;=\; \left[\frac{1}{2\pi}\,\sin 2\pi t\right]_{-1/2}^{1/2} \;=\; 0.$

C05S06.013: $\displaystyle\int_{-1}^3 1\;dx = \Big[x\Big]_{-1}^3 = 3-(-1) = 4.$

C05S06.014: $\displaystyle\int_1^2 (y^5-1)\;dy = \left[\tfrac{1}{6}y^6 - y\right]_1^2 = \frac{19}{2}.$

C05S06.015: $\displaystyle\int_1^4 \frac{1}{\sqrt{9x^3}}\;dx = \int_1^4 \frac{1}{3}x^{-3/2}\;dx = \left[-\frac{2}{3}x^{-1/2}\right]_1^4 = -\frac{2}{3}\cdot\left(\frac{1}{2}-1\right) = \frac{1}{3}.$

C05S06.016: $\displaystyle\int_{-1}^1 (x^3+2)^2\;dx = \int_{-1}^1 (x^6+4x^3+4)\;dx = \left[\frac{1}{7}x^7 + x^4 + 4x\right]_{-1}^1 = \frac{58}{7}.$

C05S06.017: $\displaystyle\int_1^3 \frac{3t-5}{t^4}\,dt = \int_1^3 \left(3t^{-3} - 5t^{-4}\right)\,dt = \left[\frac{5}{3t^3} - \frac{3}{2t^2}\right]_1^3 = -\frac{22}{81}.$

C05S06.018: $\displaystyle\int_{-2}^{-1} \frac{x^2 - x + 3}{\sqrt[3]{x}}\,dx = \int_{-2}^{-1} \left(x^{5/3} - x^{2/3} + 3x^{-1/3}\right)\,dx = \left[\frac{3}{8}x^{8/3} - \frac{3}{5}x^{5/3} + \frac{9}{2}x^{2/3}\right]_{-2}^{-1}$

$$= \left(\frac{3}{8} + \frac{3}{5} + \frac{9}{2}\right) - \left(\frac{12}{8} + \frac{6}{5} + \frac{9}{2}\right)\cdot 2^{2/3} = \frac{219}{40} - \frac{36\sqrt[3]{4}}{5} = \frac{3}{40}\left(73 - 96\sqrt[3]{4}\right) \approx -5.95428758.$$

C05S06.019: $\displaystyle\int_0^{\pi} \sin x\,\cos x\,dx = \left[\frac{1}{2}\sin^2 x\right]_0^{\pi} = 0.$

C05S06.020: $\displaystyle\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} (-x)\,dx + \int_0^2 x\,dx = \frac{5}{2}.$ Alternatively, $\displaystyle\int_{-1}^{2} |x|\,dx = \left[\frac{1}{2}\,x\cdot|x|\right]_{-1}^{2} = \frac{5}{2}.$

C05S06.021: $\displaystyle\int_1^2 \left(t - \frac{1}{2}t^{-1}\right)^2 dt = \int_1^2 \left(t^2 - 1 + \frac{1}{4}t^{-2}\right)\,dt = \left[\frac{1}{3}t^3 - t - \frac{1}{4}t^{-1}\right]_1^2 = \frac{35}{24}.$

C05S06.022: $\displaystyle\int_{-1}^{1} \frac{x^2-4}{x+2}\,dx = \int_{-1}^{1} (x-2)\,dx = \left[\frac{1}{2}x^2 - 2x\right]_{-1}^{1} = -4.$

C05S06.023: $\displaystyle\int_0^{\sqrt{\pi}} x\cos x^2\,dx = \left[\frac{1}{2}\sin x^2\right]_0^{\sqrt{\pi}} = 0.$

C05S06.024: $\displaystyle\int_0^2 \left|x - \sqrt{x}\,\right|\,dx = \int_0^1 \left(\sqrt{x} - x\right)\,dx + \int_1^2 \left(x - \sqrt{x}\,\right)\,dx = \frac{7 - 4\sqrt{2}}{3} \approx 0.44771525.$

C05S06.025: Because $x^2 - 1 \geqq 0$ if $|x| \geqq 1$ and $x^2 - 1 < 0$ if $|x| < 1$, we split the integral into three:

$$\int_{-2}^{2} \left|x^2 - 1\right|\,dx = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4.$$

C05S06.026: $\displaystyle\int_0^{\pi/3} \sin 3x\,dx = \left[-\frac{1}{3}\cos 3x\right]_0^{\pi/3} = \frac{2}{3}.$

C05S06.027: $\displaystyle\int_2^7 (x+2)^{1/2}\,dx = \left[\frac{2}{3}(x+2)^{3/2}\right]_2^7 = \frac{38}{3} \approx 12.666666666666667.$

C05S06.028: $\displaystyle\int_5^{10} \frac{1}{\sqrt{x-1}}\,dx = \left[2\sqrt{x-1}\right]_5^{10} = 2.$

C05S06.029: $\displaystyle\int_{-1}^{0} \left(1 - x^4\right)dx + \int_0^1 \left(1 - x^3\right)dx = \left[x - \frac{1}{5}x^5\right]_{-1}^{0} + \left[x - \frac{1}{4}x^4\right]_0^1 = 1 - \frac{1}{5} + 1 - \frac{1}{4} = \frac{31}{20}.$

C05S06.030: $\displaystyle\int_0^{\pi/2} \frac{1}{4}\pi^2\sin x\,dx + \int_{\pi/2}^{\pi} \left(\pi x - x^2\right)\,dx = \left[-\frac{1}{4}\pi^2\cos x\right]_0^{\pi/2} + \left[\frac{1}{2}\pi x^2 - \frac{1}{3}x^3\right]_{\pi/2}^{\pi}$

$$= \frac{1}{4}\pi^2 + \frac{1}{2}\pi^3 - \frac{1}{3}\pi^3 - \frac{1}{8}\pi^3 + \frac{1}{24}\pi^3 = \frac{1}{4}\pi^2 + \frac{1}{12}\pi^3 = \frac{3\pi^2 + \pi^3}{12}.$$

C05S06.031: $\displaystyle\int_{-3}^{0}\left(x^3-9x\right)dx-\int_0^3\left(x^3-9x\right)dx=\left[\frac{1}{4}x^4-\frac{9}{2}x^2\right]_{-3}^{0}-\left[\frac{1}{4}x^4-\frac{9}{2}x^2\right]_0^3=\frac{81}{4}+\frac{81}{4}=\frac{81}{2}.$

C05S06.032: $\displaystyle\int_{-3}^{0}\left(x^3-2x^2-15x\right)dx-\int_0^5\left(x^3-2x^2-15x\right)dx$

$$=\left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{15}{2}x^2\right]_{-3}^{0}-\left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{15}{2}x^2\right]_0^5=\frac{117}{4}+\frac{1375}{12}=\frac{863}{6}.$$

C05S06.033: Height: $s(t)=400-16t^2$. Velocity: $v(t)=-32t$. Time T of impact occurs when $T^2=25$, so that $T=5$. Average height:

$$\frac{1}{5}\int_0^5 s(t)\,dt=\frac{1}{5}\left[400t-\frac{16}{3}t^3\right]_0^5=\frac{800}{3}\approx 266.666667\quad\text{(ft)}.$$

Average velocity:

$$\frac{1}{5}\int_0^5 v(t)\,dt=\frac{1}{5}\Big[-16t^2\Big]_0^5=-80\quad\text{(ft/s)}.$$

C05S06.034: The average value of $P(t)$ over the time interval $[0,\,10]$ is

$$\frac{1}{10}\int_0^{10} P(t)\,dt=\frac{1}{10}\left[\frac{1}{150}t^3+5t^2+100t\right]_0^{10}=\frac{452}{3}\approx 150.666667.$$

As an independent check,

$$\frac{P(0)+P(2)+P(4)+P(6)+P(8)+P(10)}{6}\approx\frac{100+120.08+140.32+160.72+181.28+202}{6}\approx 150.733.$$

C05S06.035: Clearly the tank empties itself in the time interval $[0,\,10]$. So the average amount of water in the tank during that time interval is

$$\frac{1}{10}\int_0^{10} V(t)\,dt=\frac{1}{10}\left[\frac{50}{3}t^3-500t^2+5000t\right]_0^{10}=\frac{5000}{3}\approx 1666.666667\quad\text{(L)}.$$

As an independent check,

$$\frac{V(0)+V(2)+V(4)+V(6)+V(8)+V(10)}{6}=\frac{5000+3200+1800+800+200+0}{6}\approx 1833.333.$$

C05S06.036: Noon corresponds to $t=12$ and 6 P.M. corresponds to $t=18$. So the average temperature over that time interval was

$$\frac{1}{6}\int_{12}^{18} T(t)\,dt=\frac{1}{6}\left[\frac{40}{\pi}\left(2\pi(t-10)-3\cos\frac{\pi(t-10)}{12}\right)\right]_{12}^{18}=\frac{10\left(1+\sqrt{3}+8\pi\right)}{\pi}\approx 88.69638782.$$

As an independent check,

$$\frac{T(12)+T(14)+T(16)+T(18)}{4}\approx\frac{85+88.6603+90+88.6603}{4}\approx 88.08.$$

C05S06.037: The average temperature of the rod is

$$\frac{1}{10}\int_0^{10} T(x)\,dx = \frac{1}{10}\left[20x^2 - \frac{4}{3}x^3\right]_0^{10} = \frac{200}{3} \approx 66.666667.$$

As an independent check,

$$\frac{T(0)+T(2)+T(4)+T(6)+T(8)+T(10)}{6} = \frac{0+64+96+96+64+0}{6} \approx 53.33.$$

C05S06.038: Because $r^2 + x^2 = 1$, the radius of the circular cross section at x is $r = \sqrt{1-x^2}$. Hence its area is $A(x) = \pi(1-x^2)$, so the average area of such a cross section is

$$\frac{1}{1}\int_0^1 A(x)\,dx = \pi\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}\pi \approx 2.0943951024.$$

C05S06.039: Similar triangles yield $y/r = 2/1$, so $r = y/2$. So the area of the cross section at y is $A(y) = \pi(y/2)^2$, and thus the average area of such a cross section is

$$\frac{1}{2}\int_0^2 A(y)\,dt = \frac{1}{2}\left[\frac{\pi}{12}y^3\right]_0^2 = \frac{\pi}{3}.$$

C05S06.040: The velocity of the sports car at time t is $v(t) = at$, so its final velocity is $v(T) = aT$ and its average velocity is

$$\frac{1}{T}\int_0^T v(t)\,dt = \frac{1}{T}\left[\frac{1}{2}at^2\right]_0^T = \frac{1}{2}aT.$$

The position of the sports car at time t is $x(t) = \frac{1}{2}at^2$, so its final position is $x(T) = \frac{1}{2}aT^2$ and its average position is

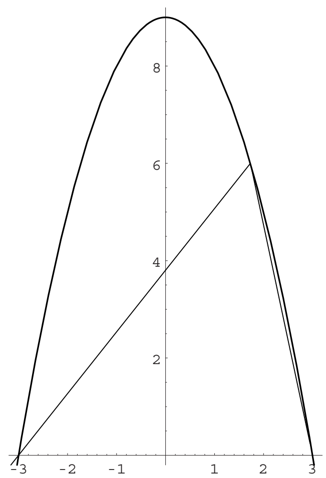
$$\frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\left[\frac{1}{6}at^3\right]_0^t = \frac{1}{6}aT^2.$$

C05S06.041: First, $A(x) = 3(9-x^2)$ for $-3 \leqq x \leqq 3$. Hence the average value of A on that interval is

$$\frac{1}{6}\int_{-3}^3 A(x)\,dx = \frac{1}{6}\Big[27x - x^3\Big]_{-3}^3 = 18.$$

Next, $A(x) = 18$ has the two solutions $x = \pm\sqrt{3}$ in the interval $[-3, 3]$, so there are two triangles having

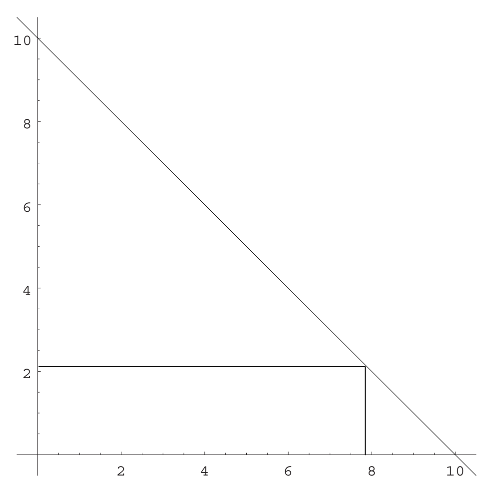
the same area as the average. One is shown next.



C05S06.042: First, $A(x) = x(10 - x)$ for $0 \leqq x \leqq 10$. So the average value of A on that interval is

$$\frac{1}{10}\int_0^{10} A(x)\,dx = \frac{1}{10}\left[5x^2 - \frac{1}{3}x^3\right]_0^{10} = \frac{50}{3}.$$

The equation $A(x) = \frac{50}{3}$ has the two solutions $\frac{5}{3}\left(3 \pm \sqrt{3}\right)$, so there are exactly two rectangles with the same area as the average rectangle. One of them is shown next.

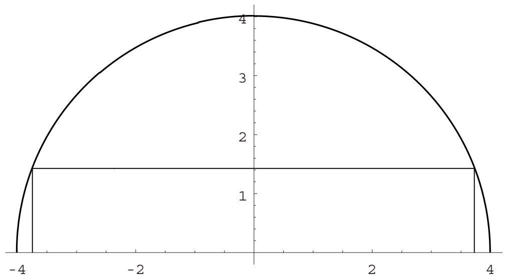


C05S06.043: The area function is $A(x) = 2x\sqrt{16 - x^2}$ for $0 \leqq x \leqq 4$, so the average value of A is

$$\frac{1}{4}\int_0^{4} A(x)\,dx = \frac{1}{4}\left[-\frac{2}{3}(16 - x^2)^{3/2}\right]_0^{4} = \frac{32}{3}.$$

The equation $A(x) = \frac{32}{3}$ has the two solutions $x = 2\sqrt{\frac{2}{3}\left(3 \pm \sqrt{5}\right)}$, so there are two rectangles having the

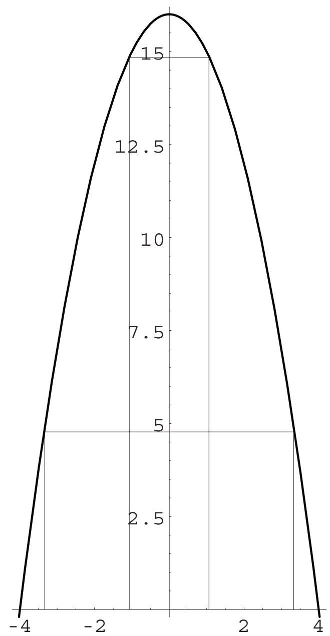
same area as the average. One is shown next.



C05S06.044: The area is $A(x) = 2x(16 - x^2)$ and its average value on $[0, 4]$ is

$$\frac{1}{4}\int_0^4 A(x)\,dx = \frac{1}{4}\left[16x^2 - \frac{1}{2}x^4\right]_0^4 = 32.$$

The equation $A(x) = 32$ has three solutions, but only two lie in the domain $0 \leqq x \leqq 4$ of A; they are approximately 1.07838 and 3.35026. So there are two rectangles having the same area as the average area. Both are shown next.



C05S06.045: $f'(x) = (x^2 + 1)^{17}$.

C05S06.046: $g'(t) = \sqrt{t^2 + 25}$.

C05S06.047: $h'(z) = (z - 1)^{1/3}$.

C05S06.048: $A'(x) = \dfrac{1}{x}$.

C05S06.049: Because

$$f(x) = -\int_{10}^{x}\left(t + \frac{1}{t}\right)\,dt,$$

part (1) of the fundamental theorem of calculus (Section 5.6) implies that

$$f'(x) \;=\; -\left(x + \frac{1}{x}\right).$$

C05S06.050: $G(x) = \displaystyle\int_2^x f(t)\,dt$, so $G'(x) = f(x) = \dfrac{x}{x^2+1}$.

C05S06.051: $G(x) = \displaystyle\int_0^x f(t)\,dt$, so $G'(x) = f(x) = \sqrt{x+4}\,$.

C05S06.052: $G(x) = \displaystyle\int_0^x f(t)\,dt$, so $G'(x) = f(x) = \sin^3 x$.

C05S06.053: $G(x) = \displaystyle\int_1^x f(t)\,dt$, so $G'(x) = f(x) = \sqrt{x^3+1}\,$.

C05S06.054: Let $u(x) = x^2$. Then

$$f(x) = \int_0^{x^2} \sqrt{1+t^3}\;dt = g(u) = \int_0^u \sqrt{1+t^3}\;dt.$$

Therefore

$$f'(x) = D_x\, g(u) = g'(u)\cdot u'(x) = 2x\sqrt{1+u^3} = 2x\sqrt{1+x^6}\,.$$

C05S06.055: Let $u(x) = 3x$. Then

$$f(x) = \int_2^{3x} \sin t^2\;dt = g(u) = \int_2^u \sin t^2\;dt.$$

Therefore

$$f'(x) = D_x\, g(u) = g'(u)\cdot u'(x) = 3\sin u^2 = 3\sin 9x^2.$$

C05S06.056: Let $u(x) = \sin x$. Then

$$f(x) = \int_0^{\sin x} \sqrt{1-t^2}\;dt = g(u) = \int_0^u \sqrt{1-t^2}\;dt.$$

Therefore

$$f'(x) = D_x\, g(u) = g'(u)\cdot u'(x) = (\cos x)\sqrt{1-u^2} = (\cos x)\sqrt{1-\sin^2 x} = (\cos x)|\cos x|.$$

C05S06.057: Let $u(x) = x^2$. Then

$$f(x) = \int_0^{x^2} \sin t\;dt = g(u) = \int_0^u \sin t\;dt.$$

Therefore

$$f'(x) = D_x\, g(u) = g'(u)\cdot u'(x) = 2x\sin u = 2x\sin x^2.$$

For an independent verification, note that

$$f(x) = \Big[-\cos t \Big]_0^{x^2} = 1 - \cos x^2,$$

and therefore that $f'(x) = 2x \sin x^2$.

C05S06.058: Let $u(x) = \sin x$. Then

$$f(x) = \int_1^{\sin x} (t^2+1)^3 \, dt = g(u) = \int_1^u (t^2+1)^3 \, dt.$$

Therefore

$$f'(x) = D_x \, g(u) = g'(u) \cdot u'(x) = (\cos x)(u^2+1)^3 = (\cos x)(1+\sin^2 x)^3.$$

C05S06.059: Let $u(x) = x^2+1$. Then

$$f(x) = \int_1^{x^2+1} \frac{1}{t} \, dt = g(u) = \int_1^u \frac{1}{t} \, dt.$$

Therefore

$$f'(x) = D_x \, g(u) = g'(u) \cdot u'(x) = 2x \cdot \frac{1}{u} = 2x \cdot \frac{1}{x^2+1}.$$

C05S06.060: Let $u(x) = x^5$. Then

$$f(x) = \int_1^{x^5} \sqrt{1+t^2} \, dt = g(u) = \int_1^u \sqrt{1+t^2} \, dt.$$

Therefore

$$f'(x) = D_x g(u) = g'(u) \cdot u'(x) = 5x^4 \sqrt{1+u^2} = 5x^4 \sqrt{1+x^{10}}\,.$$

C05S06.061: $y(x) = \displaystyle\int_1^x \frac{1}{t} \, dt.$

C05S06.062: $y(x) = \dfrac{\pi}{4} + \displaystyle\int_1^x \frac{1}{1+t^2} \, dt.$

C05S06.063: $y(x) = 10 + \displaystyle\int_5^x \sqrt{1+t^2} \, dt.$

C05S06.064: $y(x) = 2 + \displaystyle\int_1^x \tan t \, dt.$

C05S06.065: The fundamental theorem does not apply because the integrand is not continuous on $[-1, 1]$. We will see how to handle integrals such as the one in this problem in Section 9.8. One thing is certain: Its value is *not* -2.

C05S06.066: Suppose that f is differentiable on $[a, b]$. Then the average value of $f'(x)$ on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f'(x)\,dx = \frac{1}{b-a}\cdot\Big[f(x)\Big]_a^b = \frac{f(b)-f(a)}{b-a},$$

the average rate of change of f on $[a, b]$.

C05S06.067: If $0 \leqq x \leqq 2$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^x 2t\,dt = x^2.$$

Thus $g(0) = 0$ and $g(2) = 4$. If $2 \leqq x \leqq 6$, then continuity of g at $x = 2$ implies that

$$g(x) = g(2) + \int_2^x f(t)\,dt = 4 + \int_2^x (8-2t)\,dt = 4 + \Big[8t - t^2\Big]_2^x = 4 + 8x - x^2 - 16 + 4 = 8x - x^2 - 8.$$

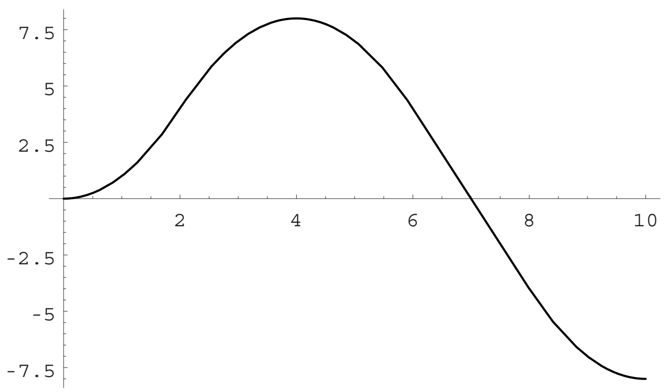
Therefore $g(4) = 8$ and $g(6) = 4$. If $6 \leqq x \leqq 8$, then continuity of g at $x = 6$ implies that

$$g(x) = g(6) + \int_6^x f(t)\,dt = 4 + \int_6^x (-4)\,dt = 4 - 4x + 24 = 28 - 4x.$$

Thus $g(8) = -4$. Finally, if $8 \leqq x \leqq 10$, then

$$g(x) = g(8) + \int_8^x (2t-20)\,dt = -4 + \Big[t^2 - 20t\Big]_8^x = -4 + x^2 - 20x - 64 + 160 = x^2 - 20x + 92,$$

and therefore $g(10) = -8$. Next, $g(x)$ is increasing where $f(x) > 0$ and decreasing where $f(x) < 0$, so g is increasing on $(0, 4)$ and decreasing on $(4, 10)$. The global maximum of g will therefore occur at $(4, 8)$ and its global minimum at $(10, -8)$. The graph of $y = g(x)$ is next.



C05S06.068: If $0 \leqq x \leqq 2$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^x (t+1)\,dt = \left[\frac{1}{2}t^2 + t\right]_0^x = \frac{1}{2}x^2 + x.$$

Therefore $g(0) = 0$ and $g(2) = 4$. Continuity of g at $x = 2$ then requires that if $2 \leqq x \leqq 4$, then

$$g(x) = g(2) + \int_2^x 3\,dt = 4 + \Big[3t\Big]_2^x = 3x - 2,$$

and therefore $g(4) = 10$. Then if $4 \leqq x \leqq 6$,

$$g(x) = g(4) + \int_4^x (15 - 3t)\ dt = 10 + \left[15t - \frac{3}{2}t^2\right]_4^x = 10 + 15x - \frac{3}{2}x^2 - 60 + 24 = 15x - \frac{3}{2}x^2 - 26.$$

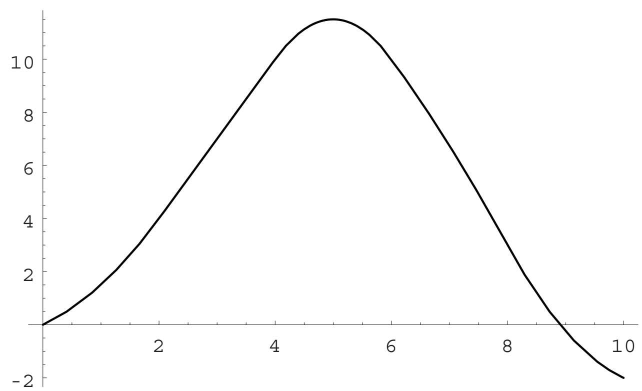
Thus $g(6) = 10$. If $6 \leqq x \leqq 8$, then

$$g(x) = 10 + \int_6^x -\frac{1}{2}t\ dt = 10 - \left[\frac{1}{4}t^2\right]_6^x = 10 - \frac{1}{4}x^2 + 9 = 19 - \frac{1}{4}x^2,$$

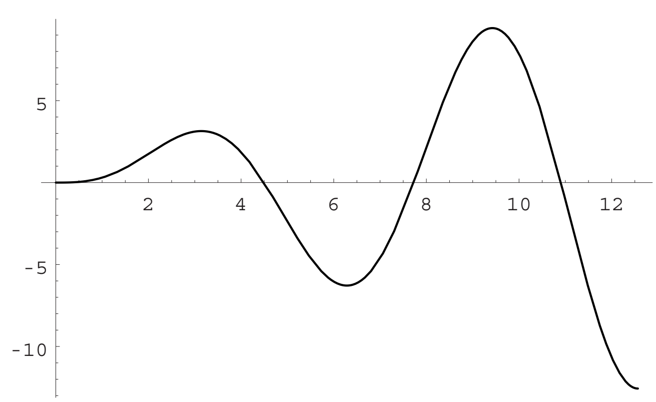
and so $g(8) = 3$. And if $8 \leqq x \leqq 10$, then

$$g(x) = 3 + \int_8^x \left(\frac{3}{2}t - 16\right)\ dt = \frac{3}{4}x^2 - 16x + 83,$$

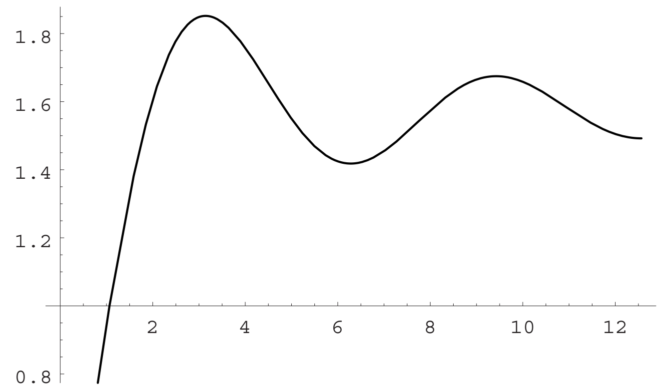
and it follows that $g(10) = -2$. Next, g is increasing where $f(x) > 0$ and decreasing where $f(x) < 0$, so g is increasing on $(0, 5)$ and decreasing on $(5, 10)$. The global maximum of g will therefore occur at $(5, 11.5)$ and its global minimum is at $(10, -2)$. The graph of $y = f(x)$ is shown next.



C05S06.069: The local extrema of $g(x)$ occur only where $g'(x) = f(x) = 0$, thus at π, 2π, and 3π, and at the endpoints 0 and 4π of the domain of g. Figure 5.6.19 shows us that g is increasing on $(0, \pi)$ and $(2\pi, 3\pi)$, decreasing on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$, so there are local minima at $(0, 0)$, $(2\pi, -2\pi)$, and $(4\pi, -4\pi)$; there are local maxima at (π, π) and $(3\pi, 3\pi)$. The global maximum occurs at $(3\pi, 3\pi)$ and the global minimum occurs at $(4\pi, -4\pi)$. To find the inflection points, we used Newton's method to solve the equation $f'(x) = 0$ and found the x-coordinates of the inflection points—it's clear from the graph that there are four of them—to be approximately 2.028758, 4.913180, 7.978666, and 11.085538. (The values of $f(x)$ at these four points are approximately 1.819706, −4.814470, 7.916727, and −11.040708.) The graph of $y = g(x)$ is shown next.



C05S06.070: By reasoning similar to that in the solution of Problem 69, g has a global minimum at (0, 0) (not shown on the following graph), a global maximum at (π, 1.851937) (numbers given in decimal form are approximations), a local minimum at (2π, 1.418152), a local maximum at (3π, 1.674762), and a local minimum at (4π, 1.492161). Newton's method gives the approximate coordinates of the points on the graph of f where g has inflection points to be (4.493409, 0.017435), (7.725252, 0.017400), and (10.904122, 0.017348). The graph of $y = g(x)$ is next.



Section 5.7

C05S07.001: Let $u = 3x - 5$. Then $du = 3\,dx$, so $dx = \frac{1}{3}\,du$. Thus

$$\int (3x-5)^{17}\ dx = \int \frac{1}{3}u^{17}\,du = \frac{1}{54}u^{18} + C = \frac{1}{54}(3x-5)^{18} + C.$$

C05S07.002: Let $u = 4x + 7$. Then $dx = \frac{1}{4}\,du$. Thus

$$\int \frac{1}{(4x+7)^6}\,dx = \frac{1}{4}\int u^{-6}\,du = -\frac{1}{20}u^{-5} + C = -\frac{1}{20(4x+7)^5} + C.$$

C05S07.003: The given substitution yields $\displaystyle\int \frac{1}{2}u^{1/2}\ du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(x^2+9\right)^{3/2} + C.$

C05S07.004: The given substitution yields $\displaystyle\int \frac{1}{6}u^{-1/3}\ du = \frac{1}{4}u^{2/3} + C = \frac{1}{4}\left(2x^3-1\right)^{2/3} + C.$

C05S07.005: The given substitution yields $\displaystyle\int \frac{1}{5}\sin u\ du = -\frac{1}{5}\cos u + C = -\frac{1}{5}\cos 5x + C.$

C05S07.006: The given substitution yields $\displaystyle\int \frac{1}{k}\cos u\,du = \frac{1}{k}\sin u + C = \frac{1}{k}\sin kx + C.$

C05S07.007: The given substitution yields $\displaystyle\int \frac{1}{4}\sin u\ du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos\left(2x^2\right) + C.$

C05S07.008: The given substitution yields

$$\int \frac{2}{3}\cos u\ du\ =\ \frac{2}{3}\sin u + C\ =\ \frac{2}{3}\sin\left(x^{3/2}\right) + C.$$

C05S07.009: The given substitution yields $\displaystyle\int u^5\ du = \frac{1}{6}u^6 + C = \frac{1}{6}(1-\cos x)^6 + C.$

C05S07.010: The given substitution yields $\displaystyle\int \frac{1}{6}u^{-1/2}\ du\ =\ \frac{1}{3}u^{1/2} + C = \frac{1}{3}\sqrt{5+2\sin 3x}\ + C.$

C05S07.011: If necessary, let $u = x + 1$. In any case, $\displaystyle\int (x+1)^6\ dx = \frac{1}{7}(x+1)^7 + C.$

C05S07.012: If necessary, let $u = 2 - x$. In any case, $\displaystyle\int (2-x)^5\ dx = -\frac{1}{6}(2-x)^6 + C.$

C05S07.013: If necessary, let $u = 4 - 3x$. In any case, $\displaystyle\int (4-3x)^7\ dx = -\frac{1}{24}(4-3x)^8 + C.$

C05S07.014: If necessary, let $u = 2x + 1$. In any case, $\displaystyle\int \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2} + C.$

C05S07.015: If necessary, let $u = 7x + 5$. In any case, $\displaystyle\int \frac{1}{\sqrt{7x+5}}\ dx = \frac{2}{7}(7x+5)^{1/2} + C.$

C05S07.016: If necessary, let $u = 3 - 5x$. In any case, $\displaystyle \int \frac{dx}{(3-5x)^2} = \frac{1}{5(3-5x)} + C.$

C05S07.017: If necessary, let $u = \pi x + 1$. In any case, $\displaystyle \int \sin(\pi x + 1)\,dx == -\frac{1}{\pi}\cos(\pi x + 1) + C.$

C05S07.018: If necessary, let $u = \dfrac{\pi t}{3}$. In any case, $\displaystyle \int \cos\frac{\pi t}{3}\,dt = \frac{3}{\pi}\sin\frac{\pi t}{3} + C.$

C05S07.019: If necessary, let $u = 2\theta$. In any case, $\displaystyle \int \sec 2\theta\,\tan 2\theta\,d\theta = \frac{1}{2}\sec 2\theta + C.$

C05S07.020: If necessary, let $u = 5x$. In any case, $\displaystyle \int \csc^2 5x\,dx = -\frac{1}{5}\cot 5x + C.$

C05S07.021: If necessary, let $u = x^2 - 1$, so that $du = 2x\,dx$; that is, $x\,dx = \frac{1}{2}\,du$. Then

$$\int x\sqrt{x^2-1}\ dx = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C.$$

Editorial Comment (DEP): In my opinion a much better, faster, shorter, and more reliable method runs as follows. Given

$$\int x\sqrt{x^2-1}\,dx,$$

it should be evident that the antiderivative involves $(x^2-1)^{3/2}$. Because

$$D_x\,(x^2-1)^{3/2} = \frac{3}{2}(x^2-1)^{1/2}\cdot 2x = 3x\sqrt{x^2-1},$$

the fact that if c is constant then $D_x\,cf(x) = cf'(x)$ now allows us to modify the initial guess $(x^2-1)^{3/2}$ for the antiderivative by multiplying it by $\frac{1}{3}$. Therefore

$$\int x\sqrt{x^2-1}\ dx = \frac{1}{3}(x^2-1)^{3/2} + C.$$

All that's necessary is to remember the constant of integration. Note that this technique is self-checking, requires less time and space, and is immune to the dangers of various illegal substitutions. The computation of the "correction factor" (in this case, $\frac{1}{3}$) can usually be done mentally.

C05S07.022: If necessary, let $u = 1 - 2t^2$. In any case, $\displaystyle \int 3t\left(1-2t^2\right)^{10}\,dt = -\frac{3}{44}\left(1-2t^2\right)^{11} + C.$

C05S07.023: If necessary, let $u = 2 - 3x^2$. In any case, $\displaystyle \int x(2-3x^2)^{1/2}\,dx = -\frac{1}{9}(2-3x^2)^{3/2} + C.$

C05S07.024: If necessary, let $u = 2t^2 + 1$. In any case, $\displaystyle \int \frac{t}{\sqrt{2t^2+1}}\,dt = \frac{1}{2}(2t^2+1)^{1/2} + C.$

C05S07.025: If necessary, let $u = x^4 + 1$. In any case, $\displaystyle \int x^3(x^4+1)^{1/2}\,dx = \frac{1}{6}(x^4+1)^{3/2} + C.$

C05S07.026: If necessary, let $u = x^3 + 1$. In any case, $\displaystyle \int x^2(x^3+1)^{-1/3}\,dx = \frac{1}{2}(x^3+1)^{2/3} + C.$

C05S07.027: If necessary, let $u = 2x^3$. In any case, $\displaystyle\int x^2 \cos(2x^3)\,dx = \frac{1}{6}\sin(2x^3) + C.$

C05S07.028: If necessary, let $u = t^2$. In any case, $\displaystyle\int t\sec^2 t^2\,dt = \frac{1}{2}\tan t^2 + C.$

C05S07.029: If necessary, let $u = x^3 + 5$. In any case, $\displaystyle\int x^2(x^3+5)^{-4}\,dx = -\frac{1}{9}(x^3+5)^{-3} + C.$

C05S07.030: If necessary, let $u = 2 - 4y^3$. In any case, $\displaystyle\int y^2\sqrt[3]{2-4y^3}\,dy = -\frac{1}{16}\left(2-4y^3\right)^{4/3} + C.$

C05S07.031: If necessary, let $u = \cos x$. In any case, $\displaystyle\int \cos^3 x\,\sin x\,dx = -\frac{1}{4}\cos^4 x + C.$

C05S07.032: There are two choices for the substitution. The substitution $u = 3z$ results in

$$\int \sin^5 3z\,\cos 3z\,dz = \int \frac{1}{3}\sin^5 u\,\cos u\,du = \frac{1}{18}\sin^6 u + C = \frac{1}{18}\sin^6 3z + C.$$

Alternatively, the substitution $u = \sin 3z$ yields $du = 3\cos 3z\,dz$, so that $\cos 3z\,dz = \frac{1}{3}\,du$. Then

$$\int \sin^5 3z\,\cos 3z\,dz = \int \frac{1}{3}u^5\,du = \frac{1}{18}u^6 + C = \frac{1}{18}\sin^6 3z + C.$$

C05S07.033: If necessary, let $u = \tan\theta$. In any case, $\displaystyle\int \tan^3\theta\,\sec^2\theta\,d\theta = \frac{1}{4}\tan^4\theta + C.$

C05S07.034: If necessary, let $u = \sec\theta$. Then $du = \sec\theta\,\tan\theta\,d\theta$, and hence

$$\int \sec^3\theta\,\tan\theta\,d\theta = \int(\sec^2\theta)\cdot\sec\theta\,\tan\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3\theta + C.$$

C05S07.035: If necessary, let $u = \sqrt{x}$. In any case, $\displaystyle\int x^{-1/2}\cos\left(x^{1/2}\right)\,dx \;=\; 2\sin\left(x^{1/2}\right) + C.$

C05S07.036: If necessary, let $u = 1 + \sqrt{x}$. In any case, $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x}\,)^2} = \frac{-2}{1+\sqrt{x}} + C.$

C05S07.047: If necessary, let $u = x^2 + 2x + 1$. This yields

$$\int (x^2+2x+1)^4(x+1)\,dx \;=\; \int \frac{1}{2}u^4\,du \;=\; \frac{1}{10}u^5 + C \;=\; \frac{1}{10}(x^2+2x+1)^5 + C.$$

Alternatively,

$$\int (x^2+2x+1)^4(x+1)\,dx \;=\; \int (x+1)^9\,dx \;=\; \frac{1}{10}(x+1)^{10} + C.$$

C05S07.038: If necessary, let $u = x^2 + 4x + 3$. In any case, $\displaystyle\int \frac{(x+2)\,dx}{(x^2+4x+3)^3} = -\frac{1}{4}(x^2+4x+3)^{-2} + C.$

C05S07.039: If necessary, let $u = 6t + t^3$. In any case, $\displaystyle\int (2+t^2)(6t+t^3)^{1/3}\,dt = \frac{1}{4}(6t+t^3)^{4/3} + C.$

C05S07.040: If necessary, let $u = x^3 - 6x + 1$. In any case, $\displaystyle\int \frac{2 - x^2}{\left(x^3 - 6x + 1\right)^5}\,dx = \frac{1}{12}\left(x^3 - 6x + 1\right)^{-4} + C.$

C05S07.041: $\displaystyle\int_1^2 \frac{dt}{(t+1)^3} = \left[-\frac{1}{2(t+1)^2}\right]_1^2 = -\frac{1}{18} - \left(-\frac{1}{8}\right) = \frac{5}{72} \approx 0.0694444444444444.$

C05S07.042: $\displaystyle\int_0^4 \frac{dx}{\sqrt{2x+1}} = \Big[(2x+1)^{1/2}\Big]_0^4 = 3 - 1 = 2.$

C05S07.053: $\displaystyle\int_0^4 x\sqrt{x^2+9}\,\,dx = \left[\frac{1}{3}(x^2+9)^{3/2}\right]_0^4 = \frac{125}{3} - 9 = \frac{98}{3}.$

C05S07.044: Given: $\displaystyle I = \int_1^4 \frac{\left(1+\sqrt{x}\right)^4}{\sqrt{x}}\,dx.$ Let $u = 1 + \sqrt{x}$. Then

$$du = \frac{1}{2}x^{-1/2}\,dx, \qquad \text{so} \qquad 2\,du = \frac{1}{\sqrt{x}}\,dx.$$

Thus

$$\int \frac{\left(1+\sqrt{x}\right)^4}{\sqrt{x}}\,dx = \int 2u^4\,du = \frac{2}{5}u^5 + C = \frac{2}{5}\left(1+\sqrt{x}\right)^5 + C.$$

Therefore $I = \dfrac{486}{5} - \dfrac{64}{5} = \dfrac{422}{5}.$

C05S07.045: Given: $\displaystyle J = \int_0^8 t\sqrt{t+1}\,\,dt.$ Let $u = t + 1$. Then $du = dt$, and so

$$J = \int_{t=0}^8 (u-1)u^{1/2}\,du = \int_{t=0}^8 \left(u^{3/2} - u^{1/2}\right)du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{t=0}^8$$

$$= \left[\frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2}\right]_0^8 = \frac{396}{5} - \left(-\frac{4}{15}\right) = \frac{1192}{15} \approx 79.4666666666666667.$$

Alternatively,

$$J = \int_{u=1}^9 (u-1)u^{1/2}\,du = \int_1^9 \left(u^{3/2} - u^{1/2}\right)du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^9 = \frac{396}{5} - \left(-\frac{4}{15}\right) = \frac{1192}{15}.$$

Thus you have at least two options:

1. Make the substitution for u in place of t, find the antiderivative, then express the antiderivative in terms of t before substituting the original limits of integration.

2. Make the substitution for u in place ot t, find the antiderivative, then substitute the new limits of integration in terms of u.

Whichever option you use, be sure to use the notation correctly (as shown here).

C05S07.046: $\displaystyle\int_0^{\pi/2} \sin x\,\cos x\,dx = \left[\frac{1}{2}\sin^2 x\right]_0^{\pi/2} = \frac{1}{2} - 0 = \frac{1}{2}.$

C05S07.047: $\displaystyle\int_0^{\pi/6} \sin 2x \cos^3 2x\,dx = \left[-\frac{1}{8}\cos^4 2x\right]_0^{\pi/6} = -\frac{1}{128} - \left(-\frac{1}{8}\right) = \frac{15}{128} = 0.1171875.$

C05S07.048: $\displaystyle\int_0^{\sqrt{\pi}} t \sin\frac{t^2}{2}\,dt = \left[-\cos\frac{t^2}{2}\right]_0^{\sqrt{\pi}} = 0 - (-1) = 1.$

C05S07.049: $\displaystyle\int_0^{\pi/2} (1 + 3\sin\theta)^{3/2} \cos\theta\,d\theta$

$$= \left[\frac{2}{15}(1 + 3\sin\theta)^{5/2}\right]_0^{\pi/2} = \frac{64}{15} - \frac{2}{15} = \frac{62}{15} \approx 4.1333333333333333.$$

If you prefer to use a substitution, try $u = 1 + 3\sin\theta$.

C05S07.050: $\displaystyle\int_0^{\pi/2} \sec^2\frac{x}{2}\,dx = \left[2\tan\frac{x}{2}\right]_0^{\pi/2} = 2 - 0 = 2.$

C05S07.051: Given: $\displaystyle K = \int_0^4 x\sqrt{4-x}\,dx$. Let $u = 4 - x$. Then $dx = -du$, so

$$K = \int_{x=0}^4 (u-4)u^{1/2}\,du = \left[\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right]_{u=4}^0 = \frac{128}{15} - 0 = \frac{128}{15} \approx 8.5333333333333333.$$

C05S07.052: $\displaystyle\int_0^{\pi/2} (\cos x)\sqrt{\sin x}\,dx = \left[\frac{2}{3}(\sin x)^{3/2}\right]_0^{\pi/2} = \frac{2}{3} - 0 = \frac{2}{3}.$

C05S07.053: $\displaystyle\int_0^1 t^3 \sin\pi t^4\,dt = \left[-\frac{1}{4\pi}\cos\pi t^4\right]_0^1 = \frac{1}{4\pi} - \left(-\frac{1}{4\pi}\right) = \frac{1}{2\pi} \approx 0.15915494309189533577.$

C05S07.054: $\displaystyle\int_{\pi^2/4}^{\pi^2} \frac{\sin\sqrt{x}\,\cos\sqrt{x}}{\sqrt{x}}\,dx = \left[\left(\sin\sqrt{x}\,\right)^2\right]_{\pi^2/4}^{\pi^2} = 0 - 1 = -1.$

C05S07.055: $\displaystyle\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C = \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C.$

(The last *two* answers are equally acceptable.)

C05S07.056: $\displaystyle\int \cos^2 x\,dx = \int \frac{1 + \cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C = \frac{1}{2}x + \frac{1}{2}\sin x\cos x + C.$

C05S07.057: $\displaystyle\int_0^{\pi} \sin^2 3t\,dt = \int_0^{\pi} \frac{1 - \cos 6t}{2}\,dt = \left[\frac{t}{2} - \frac{1}{12}\sin 6t\right]_0^{\pi} = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$

C05S07.058: $\displaystyle\int_0^1 \cos^2 \pi t\,dt = \int_0^1 \frac{1 + \cos 2\pi t}{2}\,dt = \left[\frac{t}{2} + \frac{1}{4\pi}\sin 2\pi t\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$

C05S07.059: $\displaystyle\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)dx = -x + \tan x + C.$

C05S07.060: $\displaystyle\int_0^{\pi/12} \tan^2 3t\,dt = \int_0^{\pi/12} \left(\sec^2 3t - 1\right)\,dt$

$$= \left[-t + \frac{1}{3}\tan 3t \right]_0^{\pi/12} = \frac{1}{3} - \frac{\pi}{12} = \frac{4-\pi}{12} \approx 0.0715339455341838967947797.$$

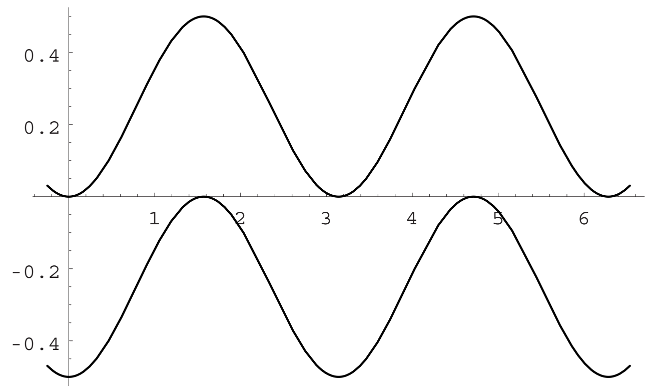
C05S07.061: $\displaystyle \int \sin^3 x \, dx = \int \big(\sin x - \cos^2 x \sin x\big)\, dx = -\cos x + \frac{1}{3}\cos^3 x + C.$

C05S07.062: $\displaystyle \int_0^{\pi/2} \cos^3 x \, dx = \int_0^{\pi/2} \big(\cos x - \sin^2 x \cos x\big)\, dx = \left[\sin x - \frac{1}{3}\sin^3 x \right]_0^{\pi/2} = \frac{2}{3} - 0 = \frac{2}{3}.$

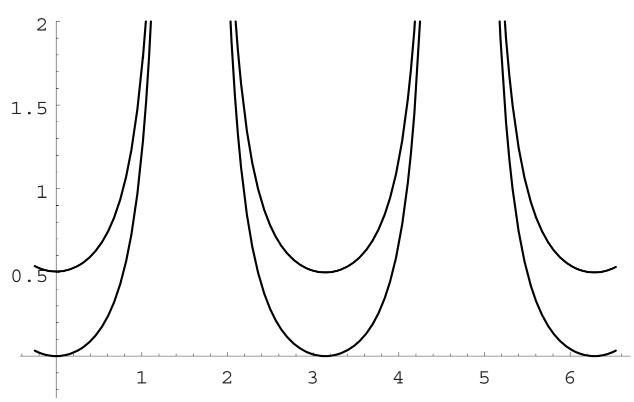
C05S07.063: If you solve the equation

$$\frac{1}{2}\sin^2\theta + C_1 = -\frac{1}{2}\cos^2\theta + C_2$$

for $C_2 - C_1$, you will find its value to be $\frac{1}{2}$. That is, $\cos^2\theta + \sin^2\theta = 1$. Two functions with the same derivative differ by a constant; in this case $\sin^2\theta - (-\cos^2\theta) = 1$. The graphs of $f(x) = \frac{1}{2}\sin^2 x$ and $g(x) = -\frac{1}{2}\cos^2 x$ are shown next.



C05S07.064: If you solve the equation $\frac{1}{2}\tan^2\theta + C_1 = \frac{1}{2}\sec^2\theta + C_2$ for $C_1 - C_2$, you will find its value to be $\frac{1}{2}$. That is, $\sec^2\theta = 1 + \tan^2\theta$. The graphs of $f(x) = \frac{1}{2}\tan^2 x$ and $g(x) = \frac{1}{2}\sec^2 x$ are shown next.



C05S07.065: First, if $f(x) = \dfrac{x}{1-x}$, then

$$f'(x) = \frac{(1-x)\cdot 1 - x\cdot(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

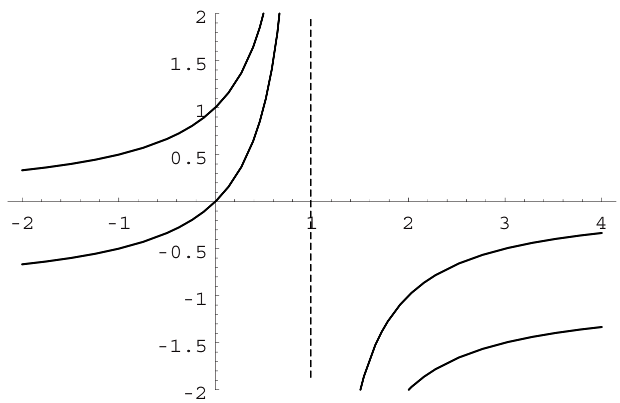
Next, if $u = 1 - x$ then $x = 1 - u$ and $dx = -\,du$, so

$$\int \frac{dx}{(1-x)^2} = \int \frac{-\,du}{u^2} = \frac{1}{u} + C_2 = \frac{1}{1-x} + C_2.$$

If $g(x) = \dfrac{1}{1-x}$, then

$$g(x) - f(x) = \frac{1}{1-x} - \frac{x}{1-x} = \frac{1-x}{1-x} \equiv 1.$$

As expected, because $g(x)$ and $f(x)$ have the same derivative on $(1, +\infty)$, they differ by a constant there. (This also holds on the interval $(-\infty, 1)$.) The graphs of $y = f(x)$ and $y = g(x)$ are shown next.



C05S07.066: The substitution $u = x^2$ requires $du = 2x\,dx$, so that $x\,dx = \frac{1}{2}\,du$. Thus

$$\int \frac{x\,dx}{(1-x^2)^2} = \int \frac{\frac{1}{2}\,du}{(1-u)^2} = \frac{1}{2} \cdot \frac{u}{1-u} + C_1 = \frac{x^2}{2(1-x^2)} + C_1.$$

Next, if we let $u = 1 - x^2$, then $du = -2x\,dx$, so that $x\,dx = -\frac{1}{2}\,du$. This yields

$$\int \frac{x\,dx}{(1-x^2)^2} = \int \frac{-\frac{1}{2}\,du}{u^2} = \frac{1}{2u} + C_2 = \frac{1}{2(1-x^2)} + C_2.$$

To reconcile these results, let

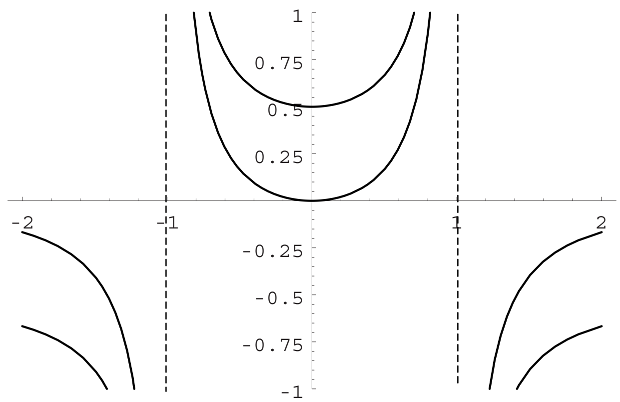
$$f(x) = \frac{x^2}{2(1-x^2)} \qquad \text{and} \qquad g(x) = \frac{1}{2(1-x^2)}.$$

Then

$$g(x) - f(x) = \frac{1}{2(1-x^2)} - \frac{x^2}{2(1-x^2)} = \frac{1-x^2}{2(1-x^2)} \equiv \frac{1}{2}.$$

That is, because $f'(x) = g'(x)$ on the interval $(-1, 1)$, $f(x)$ and $g(x)$ differ by a constant there. (This result also holds on the interval $(1, +\infty)$ and on the interval $(-\infty, -1)$.) The graphs of $y = f(x)$ and $y = g(x)$

are shown next.



C05S07.067: Suppose that f is continuous and odd. The substitution $u = -x$ requires $dx = -\,du$, so that

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} -f(-u)\,du = \int_{0}^{a} f(-u)\,du = -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx.$$

(The last equality follows because in the next-to-last integral, u is merely a "dummy" variable of integration, and can simply be replaced with x.) Therefore

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx \;+\; \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \;+\; \int_{0}^{a} f(x)\,dx = 0.$$

C05S07.068: If f is even and continuous, then the substitution $u = -x$ yields

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} -f(-u)\,du = \int_{0}^{a} f(-u)\,du = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx.$$

(The last equality is merely replacement of the dummy variable of integration by a different dummy variable.) Therefore

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx \;+\; \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx \;+\; \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

C05S07.069: Because the tangent function is continuous and odd on $[-1,\,1]$, it follows from the result in Problem 67 that

$$\int_{-1}^{1} \tan x\,dx = 0.$$

Next, $f(x) = x^{1/3}$ is continuous and odd on $[-1,\,1]$ while $g(x) = (1+x^2)^7$ is continuous and even there, so (Exercise!) their quotient is odd. Thus

$$\int_{-1}^{1} \frac{x^{1/3}}{(1+x^2)^7}\,dx = 0.$$

Finally, $h(x) = x^{17}$ is continuous and odd on $[-1,\,1]$ while $j(x) = \cos x$ is continuous and even there, so (Exercise!) their product is odd. Thus

$$\int_{-1}^{1} x^{17} \cos x \, dx = 0.$$

C05S07.070: Both $f(x) = x^{10} \sin x$ and $g(x) = x^5 \sqrt{1 + x^4}$ are continuous and odd on $[-5, 5]$, so

$$\int_{-5}^{5} \left(-x^{10} \sin x + x^5 \sqrt{1 + x^4} \right) dx = 0$$

by the result in Problem 67. Hence, using the result in Problem 68,

$$\int_{-5}^{5} \left(3x^2 - x^{10} \sin x + x^5 \sqrt{1 + x^4} \right) dx = \int_{-5}^{5} 3x^2 \, dx = 2 \int_{0}^{5} 3x^2 \, dx = 2 \Big[\, x^3 \, \Big]_0^5 = 2 \cdot 125 = 250.$$

C05S07.071: Given

$$I = \int_a^b f(x + k) \, dx,$$

let $u = x + k$. Then $x = u - k$, $dx = du$, $u = a + k$ when $x = a$, and $u = b + k$ when $x = b$. Hence

$$I = \int_{a+k}^{b+k} f(u) \, du = \int_{a+k}^{b+k} f(x) \, dx$$

(because it doesn't matter whether the variable of integration is called u or x).

C05S07.072: Given

$$J = k \int_a^b f(kx) \, dx,$$

let $u = kx$. Then $du = k \, dx$, $u = ka$ when $x = a$, and $u = kb$ when $x = b$. Hence

$$J = k \int_{ka}^{kb} \frac{1}{k} f(u) \, du = \int_{ka}^{kb} f(x) \, dx.$$

C05S07.073: (a) $D_u(\sin u - u \cos u) = \cos u - 1 \cdot \cos u + u \sin u = u \sin u$. (b) Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u \, du$, $u = 0$ when $x = 0$, and $u = \pi$ when $x = \pi^2$. Therefore

$$\int_0^{\pi^2} \sin \sqrt{x} \, dx = \int_0^{\pi} 2u \sin u \, du = 2 \cdot \Big[\sin u - u \cos u \Big]_0^{\pi} = 2 \cdot (-\pi \cos \pi) - 2 \cdot 0 = 2\pi.$$

Section 5.8

C05S08.001: To find the limits of integration, we solve $25 - x^2 = 9$ for $x = \pm 4$. Hence the area is

$$\int_{-4}^{4} (25 - x^2 - 9)\,dx = \left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \frac{128}{3} - \left(-\frac{128}{3}\right) = \frac{256}{3}.$$

C05S08.002: To find the limits of integration, we solve $16 - x^2 = -9$ for $x = \pm 5$. Thus the area is

$$\int_{-5}^{5} (16 - x^2 + 9)\,dx = \left[25x - \frac{1}{3}x^3\right]_{-5}^{5} = \frac{250}{3} - \left(-\frac{250}{3}\right) = \frac{500}{3}.$$

C05S08.003: To find the limits of integration, we solve $x^2 - 3x = 0$ for $x = 0$, $x = 3$. Hence the area is

$$\int_{0}^{3} (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{3} = \frac{9}{2} - 0 = \frac{9}{2}.$$

C05S08.004: To find the limits of integration, we solve $x^3 - 9x = 0$ for $x = -3$, $x = 0$, and $x = 3$. Therefore the area of the region shown in Fig. 5.8.18 is

$$\int_{0}^{3} (9x - x^3)\,dx = \left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{3} = \frac{81}{4} - 0 = \frac{81}{4}.$$

C05S08.005: To find the limits of integration, we solve $12 - 2x^2 = x^2$ for $x = \pm 2$. Therefore the area is

$$\int_{-2}^{2} (12 - 3x^2)\,dx = \left[12x - x^3\right]_{-2}^{2} = 16 - (-16) = 32.$$

C05S08.006: To find the limits of integration, we solve $2x - x^2 = 2x^2 - 4x$ for $x = 0$, $x = 2$. Therefore the area of the figure is

$$\int_{0}^{2} (6x - 3x^2)\,dx = \left[3x^2 - x^3\right]_{0}^{2} = 4 - 0 = 4.$$

C05S08.007: To find the limits of integration, we solve $4 - x^2 = 3x^2 - 12$ for $x = \pm 2$. So the area is

$$\int_{-2}^{2} (16 - 4x^2)\,dx = \left[16x - \frac{4}{3}x^3\right]_{-2}^{2} = \frac{64}{3} - \left(-\frac{64}{3}\right) = \frac{128}{3}.$$

C05S08.008: To find the limits of integration, we solve $12 - 3x^2 = 4 - x^2$ for $x = \pm 2$. So the area of the region is

$$\int_{-2}^{2} (8 - 2x^2)\,dx = \left[8x - \frac{2}{3}x^3\right]_{-2}^{2} = \frac{32}{3} - \left(-\frac{32}{3}\right) = \frac{64}{3}.$$

C05S08.009: To find the limits of integration, we solve $x^2 - 3x = 6$ for $x = a = \frac{1}{2}\left(3 - \sqrt{33}\right)$ and $x = b = \frac{1}{2}\left(3 + \sqrt{33}\right)$. So the area is

$$\int_a^b (6 - x^2 + 3x)\,dx = \left[6x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_a^b = \frac{45 + 11\sqrt{33}}{4} - \frac{45 - 11\sqrt{33}}{4} = \frac{11\sqrt{33}}{2}.$$

C05S08.010: To find the limits of integration, we solve $x^2 - 3x = x$ for $x = 0$, $x = 4$. So the area is

$$\int_0^4 (x - x^2 + 3x)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3} - 0 = \frac{32}{3}.$$

C05S08.011: $\displaystyle\int_0^1 (x - x^3)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$

C05S08.012: $\displaystyle\int_1^3 \frac{1}{(x+1)^2}\,dx = \left[\frac{-1}{x+1}\right]_1^3 = \frac{1}{4}.$

C05S08.013: $\displaystyle\int_0^1 (x^3 - x^4)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{20}.$

C05S08.014: $\displaystyle\int_{-1}^2 \big(x^2 - (-1)\big)\,dx = \left[\frac{1}{3}x^3 + x\right]_{-1}^2 = \frac{14}{3} - \left(-\frac{4}{3}\right) = 6.$

C05S08.015: $\displaystyle\int_0^2 \frac{1}{(x+1)^3}\,dx = \left[-\frac{1}{2(x+1)^2}\right]_0^2 = -\frac{1}{18} - \left(-\frac{1}{2}\right) = \frac{4}{9}.$

C05S08.016: To find the limits of integration, we first solve $4x - x^2 = 0$ for $x = 0$ and $x = 4$. Hence the area of R is

$$\int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^2\right]_0^4 = \frac{32}{3}.$$

C05S08.017: To find the limits of integration, we first solve $y^2 = 4$ for $y = \pm 2$. So the area of the region R is

$$\int_{-2}^2 (4 - y^2)\,dy = \left[4y - \frac{1}{3}y^3\right]_{-2}^2 = \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}.$$

C05S08.018: To find the limits of integration, we first solve $x^4 - 4 = 3x^2$ for $x = \pm 2$. So the area of R is

$$\int_{-2}^2 (3x^2 - x^4 + 4)\,dx = \left[4x + x^3 - \frac{1}{5}x^5\right]_{-2}^2 = \frac{48}{5} - \left(-\frac{48}{5}\right) = \frac{96}{5}.$$

C05S08.019: To find the limits of integration, we first solve $8 - y^2 = y^2 - 8$ for $y = a = -2\sqrt{2}$ and $y = b = 2\sqrt{2}$. So the area of R is

$$\int_a^b (16 - 2y^2)\,dy = \left[16y - \frac{2}{3}y^3\right]_a^b = \frac{64\sqrt{2}}{3} - \left(-\frac{64\sqrt{2}}{3}\right) = \frac{128\sqrt{2}}{3}.$$

C05S08.020: To find the limits of integration, we solve $x^{1/3} = x^3$ for $x = -1$, $x = 0$, and $x = 1$. Thus the region R comes in two parts, one in the first quadrant and one in the third. The area of the first is

$$\int_0^1 \left(x^{1/3} - x^3\right)\,dx = \left[\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right]_0^1 = \frac{1}{2}.$$

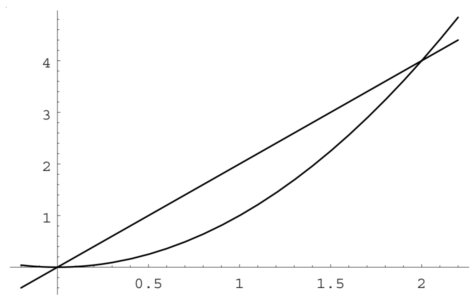
The area of the second is

$$\int_{-1}^0 \left(x^3 - x^{1/3}\right)\,dx = \left[\frac{1}{4}x^4 - \frac{3}{4}x^{4/3}\right]_{-1}^0 = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Therefore R has area 1.

C05S08.021: The area of the region—shown next—is

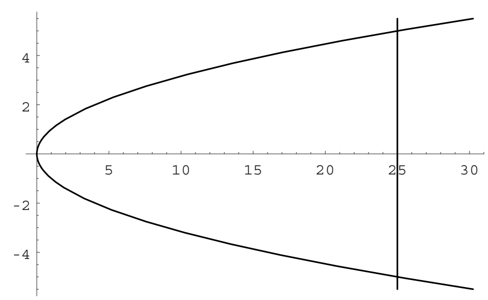
$$A = \int_0^2 \left(2x - x^2\right)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = \frac{4}{3}.$$



C05S08.022: $\displaystyle\int_{-2}^2 \left(8 - 2x^2\right)\,dx = \left[8x - \frac{2}{3}x^3\right]_{-2}^2 = \frac{32}{3} - \left(-\frac{32}{3}\right) = \frac{64}{3}.$

C05S08.023: We find the limits of integration by solving $y^2 = 25$ for $y = \pm 5$. The area of the region—shown next—is therefore

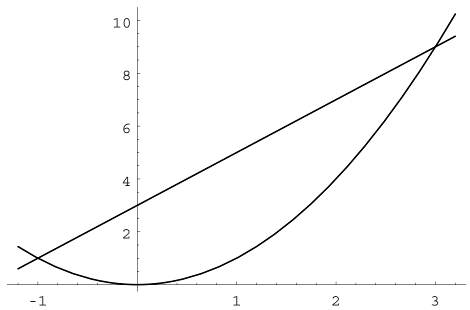
$$\int_{-5}^5 \left(25 - y^2\right)\,dy = \left[25y - \frac{1}{3}y^3\right]_{-5}^5 = \frac{250}{3} - \left(-\frac{250}{3}\right) = \frac{500}{3}.$$



C05S08.024: Area: $\displaystyle\int_{-4}^4 \left(32 - 2y^2\right)\,dy = \left[32y - \frac{2}{3}y^3\right]_{-4}^4 = \frac{256}{3} - \left(-\frac{256}{3}\right) = \frac{512}{3}.$

C05S08.025: We find the limits of integration by solving $x^2 = 2x + 3$ for $x = -1$, $x = 3$. Thus the area of the region—shown next—is

$$A = \int_{-1}^{3} \left(2x + 3 - x^2\right)\,dx = \left[3x + x^2 - \frac{1}{3}x^3\right]_{-1}^{3} = 9 - \left(-\frac{5}{3}\right) = \frac{32}{3}.$$

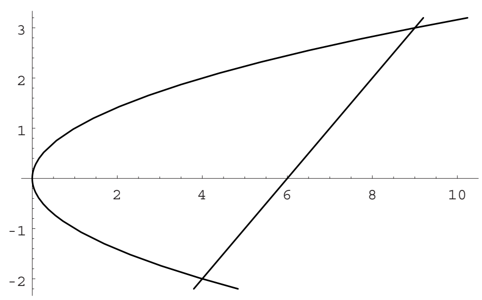


C05S08.026: We find the limits of integration by solving $x^2 = 2x + 8$ for $x = -2$, $x = 4$. The area is

$$\int_{-2}^{4} \left(2x + 8 - x^2\right)\,dx = \left[8x + x^2 - \frac{1}{3}x^3\right]_{-2}^{4} = \frac{80}{3} - \left(-\frac{28}{3}\right) = 36.$$

C05S08.027: We first find the limits of integration by solving $y^2 = y + 6$ for $y = -2$, $y = 3$. The area of the region—shown next—is therefore

$$A = \int_{-2}^{3} \left(y + 6 - y^2\right)\,dy = \left[6y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-2}^{3} = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}.$$

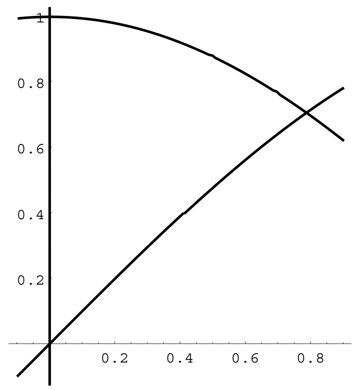


C05S08.028: Solving $y^2 = 8 - 2y$ for $y = -4$, $y = 2$ yields the limits of integration, and the area of the region is

$$A = \int_{-4}^{2} (8 - 2y - y^2)\,dy = \left[8y - y^2 - \frac{1}{3}y^3\right]_{-4}^{2} = \frac{28}{3} - \left(-\frac{80}{3}\right) = 36.$$

C05S08.029: The two graphs meet at the right-hand endpoint of the given interval, where $x = \pi/4$. Therefore the area of the region they bound—shown next—is

$$A = \int_{0}^{\pi/4} (\cos x - \sin x)\,dx = \left[\sin x + \cos x\right]_{0}^{\pi/4} = \sqrt{2} - 1 \approx 0.414213562373.$$

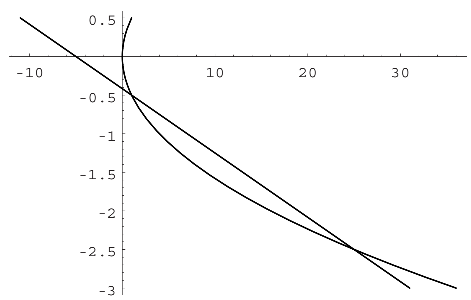


C05S08.030: The two graphs meet at the left-hand endpoint $x = -3\pi/4$ of the given interval. The area they bound over that interval is

$$A = \int_{-3\pi/4}^{0} (\cos x - \sin x)\,dx = \Big[\sin x + \cos x\Big]_{-3\pi/4}^{0} = 1 + \sqrt{2} \approx 2.414213562373.$$

C05S08.031: Solution of $4y^2 + 12y + 5 = 0$ yields the limits of integration $y = a = -\frac{5}{2}$ and $y = b = -\frac{1}{2}$. Hence the area of the region bounded by the two given curves—shown next—is

$$A = \int_a^b \left(-12y - 5 - 4y^2\right)\,dy = \left[-5y - 6y^2 - \frac{4}{3}y^3\right]_a^b = \frac{7}{6} - \left(-\frac{25}{6}\right) = \frac{16}{3}.$$

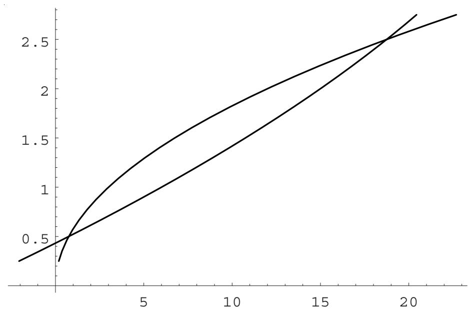


C05S08.032: Solution of $x^2 = 3 + 5x - x^2$ yields the limits of integration $x = a = -\frac{1}{2}$ and $x = b = 3$. Hence the area bounded by the gives curves is

$$A = \int_a^b (3 + 5x - 2x^2)\,dx = \left[3x + \frac{5}{2}x^2 - \frac{2}{3}x^3\right]_a^b = \frac{27}{2} - \left(-\frac{19}{24}\right) = \frac{343}{24}.$$

C05S08.033: Solution of $3y^2 = 12y - y^2 - 5$ yields the limits of integration $y = a = \frac{1}{2}$ and $y = b = \frac{5}{2}$. Hence the area of the region bounded by the given curves—shown next—is

$$A = \int_a^b (-5 + 12y - 4y^2)\,dy = \left[-5y + 6y^2 - \frac{4}{3}y^3\right]_a^b = \frac{25}{6} - \left(-\frac{7}{6}\right) = \frac{16}{3}.$$

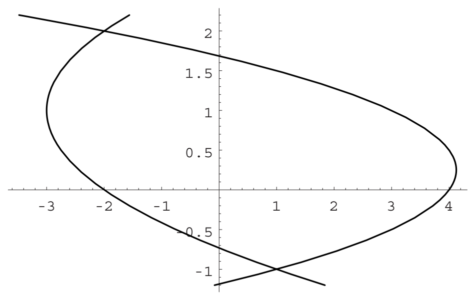


C05S08.034: We solve $x^2 = 4(x-1)^2$ for $x = a = \frac{2}{3}$ and $x = b = 2$ to find the limits of integration. The area of the region bounded by the two given curves is

$$A = \int_a^b (8x - 4 - 3x^2)\,dx = \Big[4x^2 - 4x - x^3\Big]_a^b = 0 - \left(-\frac{32}{27}\right) = \frac{32}{27}.$$

C05S08.035: We first solve $y^2 - 2y - 2 = 4 + y - 2y^2$ to find the limits of integration $y = -1$ and $y = 2$. The area of the region bounded by the two given curves—shown next—is

$$A = \int_{-1}^2 (6 + 3y - 3y^2)\,dy = \left[6y + \frac{3}{2}y^2 - y^3\right]_{-1}^2 = 10 - \left(-\frac{7}{2}\right) = \frac{27}{2}.$$

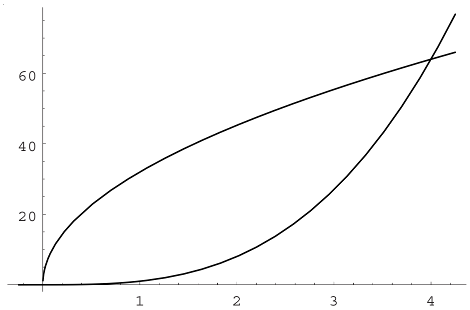


C05S08.036: Solving $x^4 = 32 - x^4$ yields the limits of integration $x = -2$ and $x = 2$, so the area bounded by the given curves is

$$A = \int_{-2}^2 (32 - 2x^4)\,dx = \left[32x - \frac{2}{5}x^5\right]_{-2}^2 = \frac{256}{5} - \left(-\frac{256}{5}\right) = \frac{512}{5}.$$

C05S08.037: We solve $x^3 = 32\sqrt{x}$ to find the limits of integration $x = 0$ and $x = 4$. Thus the area of the region bounded by the given curves—shown next—is

$$A = \int_0^4 \left(32x^{1/2} - x^3\right)\,dx = \left[\frac{64}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^4 = \frac{512}{3} - 64 = \frac{320}{3}.$$



C05S08.038: When we solve $x^3 = 2x - x^2$, we find that the two given curves meet at three points—where $x = -2$, $x = 0$, and $x = 1$. Thus there are two bounded regions bounded by the given curves. The area of the one on the left is

$$A_1 = \int_{-2}^{0} (x^3 - 2x + x^2)\, dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_{-2}^{0} = 0 - \frac{8}{3} = \frac{8}{3}.$$

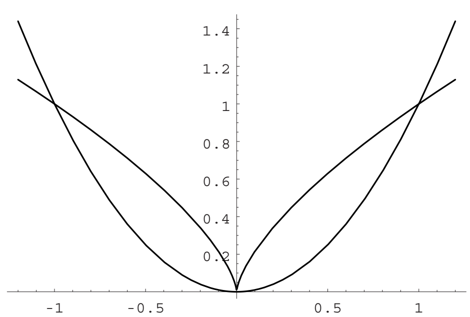
The area of the region on the right is

$$A_2 = \int_{0}^{1} (2x - x^2 - x^3)\, dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{1} = \frac{5}{12}.$$

Thus the total area of the bounded regions bounded by the given curves is $A_1 + A_2 = \dfrac{37}{12}$.

C05S08.039: Solution of $x^2 = x^{2/3}$ yields the three solutions $x = -1$, $x = 0$, and $x = 1$. The following figure shows that we can find the total area A bounded by the two curves with a single integral:

$$A = \int_{-1}^{1} \left(x^{2/3} - x^2\right) dx = \left[\frac{3}{5}x^{5/3} - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{15} - \left(-\frac{4}{15}\right) = \frac{8}{15}.$$

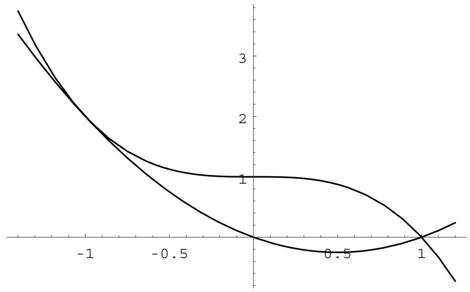


C05S08.040: Solving the simultaneous equations $y^2 = x$, $y^2 = 2(x-3)$ yields the two points of intersection $\left(6, -\sqrt{6}\right)$ and $\left(6, \sqrt{6}\right)$ of the given curves. So the limits of integration are $y = a = -\sqrt{6}$ and $y = b = \sqrt{6}$. Hence the area of the region they bound is

$$A = \int_{a}^{b} \left(3 - \frac{1}{2}y^2\right) dy = \left[3y - \frac{1}{6}y^3\right]_{a}^{b} = 2\sqrt{6} - \left(-2\sqrt{6}\right) = 4\sqrt{6} \approx 9.7979589711327123927891 36.$$

C05S08.041: The curves meet where $x = -1$ and where $x = 1$, so the area of the region they bound—shown next—is

$$\int_{-1}^{1} (1 - x^3 - x^2 + x)\ dx = \left[x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^{1} = \frac{11}{12} - \left(-\frac{5}{12} \right) = \frac{4}{3}.$$



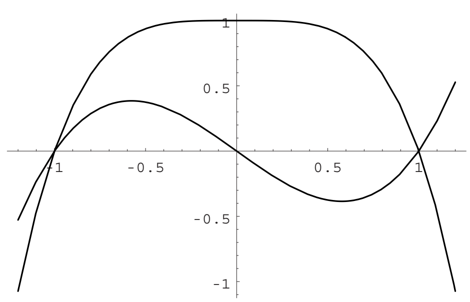
C05S08.042: First we solve

$$\begin{aligned} x^3 - x &= 1 - x^4: \\ x^4 + x^3 - x - 1 &= 0; \\ x^3(x+1) - (x+1) &= 0; \\ (x+1)(x^3 - 1) &= 0; \\ (x+1)(x-1)(x^2 - x + 1) &= 0; \\ x = -1 \quad &\text{or} \quad x = 1 \end{aligned}$$

($x^2 - x + 1 = 0$ has no real solutions). Hence the two curves meet at $(-1,\ 0)$ and $(1,\ 0)$. Therefore the area of the region they bound is

$$A = \int_{-1}^{1} (1 - x^4 - x^3 + x)\ dx = \left[x + \frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5 \right]_{-1}^{1} = \frac{21}{20} - \left(-\frac{11}{20} \right) = \frac{8}{5}.$$

The region itself is shown next.



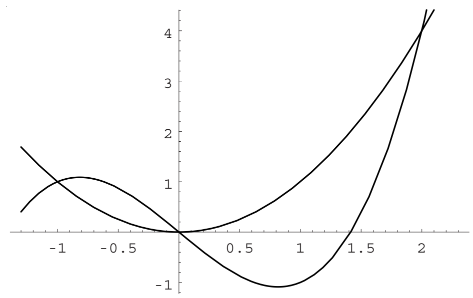
C05S08.043: We solve $x^2 = x^3 - 2x$ to find $x = -1$, $x = 0$, and $x = 2$. So the two given curves meet at $(-1,\ 1)$, $(0,\ 0)$, and $(2,\ 4)$, as shown in the following figure. The area of the region on the left is

$$A_1 = \int_{-1}^{0} (x^3 - 2x - x^2)\ dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 \right]_{-1}^{0} = 0 - \left(-\frac{5}{12} \right) = \frac{5}{12}.$$

The area of the region on the right is

$$A_2 = \int_0^2 (2x + x^2 - x^3)\ dx = \left[x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{8}{3}.$$

Therefore the total area bounded by the two regions is $A_1 + A_2 = \dfrac{37}{12}$.



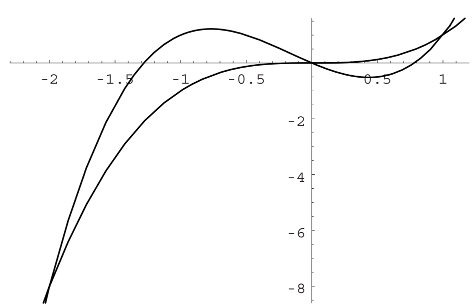
C05S08.044: We solve $x^3 = 2x^3 + x^2 - 2x$ for $x = -2$, $x = 0$, and $x = 1$, indicating that the two given curves meet at the three point $(-2, -8)$, $(0, 0)$, and $(1, 1)$, as shown in the following figure. The area of the region on the left is

$$A_1 = \int_{-2}^0 (x^3 + x^2 - 2x)\ dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 \right]_{-2}^0 = 0 - \left(-\frac{8}{3} \right) = \frac{8}{3}.$$

The area of the region on the right is

$$A_2 = \int_0^1 (2x - x^2 - x^3)\ dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{5}{12}.$$

Therefore the total area of the two regions is $A_1 + A_2 = \dfrac{37}{12}$.



C05S08.045: The first integral is

$$I_1 = \int_{-3}^3 4x(9 - x^2)^{1/2}\ dx = \left[-\frac{4}{3}(9 - x^2)^{3/2} \right]_{-3}^3 = 0.$$

The second is

$$I_2 = \int_{-3}^3 5\sqrt{9 - x^2}\ dx = 5 \int_{-3}^3 \sqrt{9 - x^2}\ dx.$$

Because the graph of $y = \sqrt{9 - x^2}$ is a semicircle of radius 3, centered at the origin, and lying in the first and second quadrants, I_2 is thus five times the area of such a semicircle, so that

$$I_2 = 5 \cdot \frac{1}{2} \cdot \pi \cdot 3^2 = \frac{45}{2}\pi,$$

and because $I_1 = 0$, this is also the value of the integral given in Problem 45.

C05S08.046: Let $u = x^2$, so that $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Then

$$I = \int_0^3 x(81 - x^4)^{1/2}\,dx = \frac{1}{2}\int_0^9 (81 - u^2)^{1/2}\,du = \frac{1}{2}\int_0^9 (81 - x^2)^{1/2}\,dx. \tag{1}$$

The graph of $y = \sqrt{81 - x^2}$ is a quarter circle of radius 9 centered at $(0,\,0)$ and lying in the first quadrant, so the last expression in Eq. (1) is half the area of that quarter circle. Therefore

$$I = \frac{1}{2} \cdot \frac{1}{4} \cdot \pi \cdot 9^2 = \frac{81}{8}\pi.$$

C05S08.047: We solve the equation of the ellipse for

$$y = \frac{b}{a}(a^2 - x^2)^{1/2},$$

and hence the area of the ellipse is given by

$$A = 4\int_0^a \frac{b}{a}(a^2 - x^2)^{1/2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,\,dx. \tag{1}$$

The last integral in Eq. (1) is the area of the quarter-circle in the first quadrant with center at the origin and radius a, and therefore

$$A = \frac{4b}{a} \cdot \frac{1}{4}\pi a^2 = \pi ab.$$

C05S08.048: The area bounded by the parabolic segment is

$$\int_{-1}^1 (1 - x^2)\,dx = \left[x - \frac{1}{3}x^3\right]_{-1}^1 = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}.$$

The triangle has base 2 and height 1, so its area is 1, and the area of the parabolic segment is indeed $\frac{4}{3}$ times the area of the triangle.

C05S08.049: By solving the equations of the line and the parabola simultaneously, we find that $A = (-1,\,1)$ and $B = (2,\,4)$. The slope of the tangent line at C is the same as the slope 1 of the line through A and B, and it follows that C has x-coordinate $\frac{1}{2}$ and thus y-coordinate $\frac{1}{4}$. It now follows that the distance from A to B is $AB = 3\sqrt{2}$, the distance from B to C is $BC = \frac{3}{4}\sqrt{29}$, and the distance from A to C is $AC = \frac{3}{4}\sqrt{5}$. Heron's formula then allows us to find the area of triangle ABC; it is the square root of the product of

$$\frac{3}{8}\left(4\sqrt{2} + \sqrt{5} + \sqrt{29}\right), \qquad -3\sqrt{2} + \frac{3}{8}\left(4\sqrt{2} + \sqrt{5} + \sqrt{29}\right),$$

$$-\frac{3}{4}\sqrt{5} + \frac{3}{8}\left(4\sqrt{2} + \sqrt{5} + \sqrt{29}\right), \qquad \text{and} \qquad -\frac{3}{4}\sqrt{29} + \frac{3}{8}\left(4\sqrt{2} + \sqrt{5} + \sqrt{29}\right).$$

The product can be simplified to $\frac{729}{64}$, so the area of triangle ABC is $\frac{27}{8}$. The area of the parabolic segment is

$$\int_{-1}^{2} (x+2-x^2)\,dx = \frac{9}{2} = \frac{4}{3}\cdot\frac{27}{8},$$

exactly as Archimedes proved in more general form over 2000 years ago. *Mathematica* did the arithmetic for us in this problem. If you prefer to do it by hand, show that the line through C perpendicular to the tangent line there has equation $y = \frac{3}{4} - x$. Show that this line meets the line through AB at the point $\left(-\frac{5}{8},\, \frac{11}{8}\right)$. Show that the perpendicular from C to that line has length $h = \frac{9}{8}\sqrt{2}$. Show that AB has length $3\sqrt{2}$. Then triangle ABC has base AB and height h, so its area is

$$\frac{1}{2}\cdot\left(\frac{9}{8}\sqrt{2}\right)\cdot 3\sqrt{2} = \frac{27}{8}.$$

C05S08.050: The area of the part of the region R lying over $[1,\, b]$ is

$$\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}.$$

When we evaluate the limit of this expression as $b \to +\infty$, we find that the area of R is 1. We will return to this and related topics in Section 9.8.

C05S08.051: The graph of the cubic $y = 2x^3 - 2x^2 - 12x$ meets the x-axis at $x = -2$, $x = 0$, and $x = 3$. The graph is above the x-axis for $-2 < x < 0$ and below it for $0 < x < 3$, so the graph of the cubic and the x-axis form two bounded plane regions. The area of the one on the left is

$$A_1 = \int_{-2}^{0} (2x^3 - 2x^2 - 12x)\,dx = \left[\frac{1}{2}x^4 - \frac{2}{3}x^3 - 6x^2\right]_{-2}^{0} = \frac{32}{3}$$

and the area of the one on the right is

$$A_2 = \int_0^3 (12x + 2x^2 - 2x^3)\,dx = \left[6x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^3 = \frac{63}{2}.$$

Therefore the total area required in Problem 51 is $A_1 + A_2 = \dfrac{253}{6}$.

C05S08.052: On the one hand, the area in question is

$$A = \int_{-h}^{h} (px^2 + qx + r)\,dx = \left[\frac{1}{3}px^3 + \frac{1}{2}qx^2 + rx\right]_{-h}^{h} = \frac{2}{3}ph^3 + 2rh.$$

But

$$\frac{1}{3}h\left[f(-h) + 4f(0) + f(h)\right] = \frac{1}{3}h\left[ph^2 - qh + r + 4r + ph^2 + qh + r\right] = \frac{1}{3}h(2ph^2 + 6r) = \frac{2}{3}ph^3 + 2rh,$$

and this establishes the result sought in Problem 52. This problem figures significantly in the subsection on parabolic approximations in Section 5.9.

C05S08.053: Given $y^2 = x(5-x)^2$, the loop lies above and below the interval $[0,\, 5]$, so by symmetry (around the x-axis) its area is

$$2\int_0^5 (5-x)x^{1/2}\,dx = 2\left[\frac{2}{15}(25x^{3/2} - 3x^{5/2})\right]_0^5 = 2\cdot\frac{20}{3}\sqrt{5} = \frac{40}{3}\sqrt{5}.$$

C05S08.054: Given $y^2 = x^2(x+3)$, the loop lies above and below the interval $[-3, 0]$, so by symmetry (around the x-axis) its area is

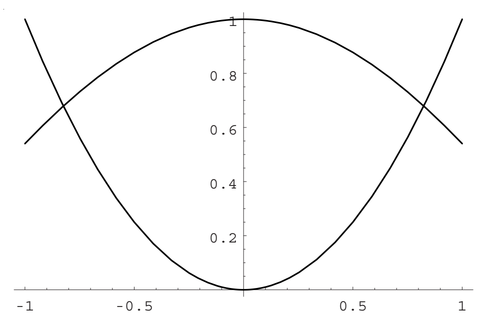
$$A = 2\int_{-3}^{0} -x\sqrt{x+3}\,dx.$$

The minus sign is required because $x < 0$ and $\sqrt{x+3} > 0$ for $-3 < x < 0$. Next, the substitution $u = x+3$ yields

$$A = 2\int_0^3 (3-u)u^{1/2}\,du = 2\left[2u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^3 = 2\cdot\left(2\cdot 3\sqrt{3} - \frac{2}{5}\cdot 9\sqrt{3}\right) = \frac{24}{5}\sqrt{3}.$$

C05S08.055: We applied Newton's method to the equation $x^2 - \cos x = 0$ to find that the two curves $y = x^2$ and $y = \cos x$ meet at the two points $(-0.824132, 0.679194)$ and $(0.824132, 0.679194)$ (numbers involving decimals are approximations). Let $a = -0.824132$ and $b = 0.824132$. The region between the two curves (shown next) has area

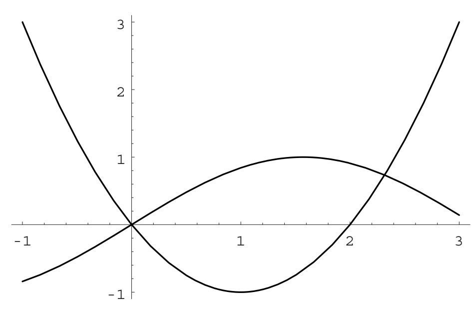
$$\int_a^b (\cos x \,-\, x^2)\,dx = \left[\sin x \,-\, \frac{1}{3}x^3\right]_a^b \approx 1.09475.$$



C05S08.056: We applied Newton's method to $f(x) = \sin x - x^2 + 2x$ to find the positive solution $b = 2.316934$ of $f(x) = 0$ (numbers involving decimals are approximations). Clearly $a = 0$ is the other solution; the points of intersection of the two graphs are $(0, 0)$ and $(2.316934, 0.734316)$. The area bounded by the two graphs is

$$\int_a^b (\sin x \,-\, x^2 + 2x)\,dx = \left[x^2 - \frac{1}{3}x^3 - \cos x\right]_a^b \approx 2.90108.$$

The region bounded by the two curves is shown next.



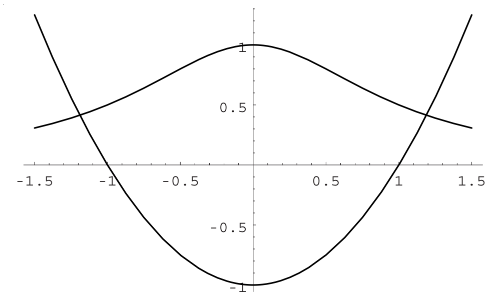
C05S08.057: We used Newton's method to solve $f(x) = 0$ where

$$f(x) = \frac{1}{1+x^2} - x^2 + 1,$$

and found the points at which the two curves intersect to be $(-1.189207, 0.414214)$ and $(1.189207, 0.414214)$ (numbers involving decimals are approximations). With $b = 1.189207$ and $a = -b$, the area bounded by the two curves (shown next) is

$$A = \int_a^b \left(\frac{1}{1+x^2} - x^2 + 1 \right) dx = \left[\arctan(x) - \frac{1}{3}x^3 + x \right]_a^b \approx 3.00044$$

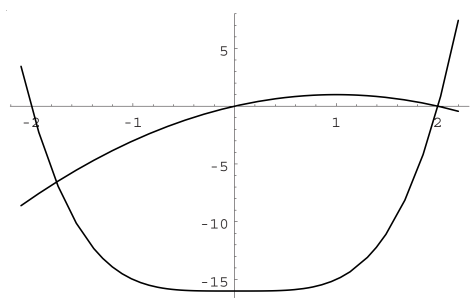
(the antiderivative was computed with the aid of Formula 17 of the endpapers).



C05S08.058: We used Newton's method with $f(x) = 2x - x^2 - x^4 + 16$ to find the negative solution $x = a$ of $f(x) = 0$; it turns out that the corresponding point where the two curves meet is $(-1.752172, -6.574449)$ (numbers involving decimals are approximations). The other solution is $b = 2$, so the curves also meet at $(2, 0)$. The area between them is thus

$$\int_a^b (2x - x^2 - x^4 + 16)\, dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{5}x^5 + 16x \right]_a^b \approx 46.8018.$$

The region bounded by the two curves is shown next.



C05S08.059: The curves $y = x^2$ and $y = k - x^2$ meet at the two points where $x = a = -(k/2)^{1/2}$ and $x = b = (k/2)^{1/2}$. So the area between the two curves is

$$\int_a^b (k - 2x^2)\, dx = \left[kx - \frac{2}{3}x^3 \right]_a^b = \frac{2\sqrt{2}}{3} k^{3/2}.$$

When we set the last expression equal to 72, we find that $k = 18$.

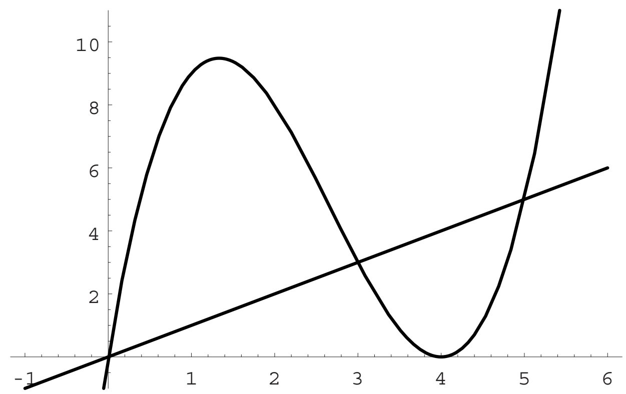
C05S08.060: By symmetry, it is sufficient to work in the first quadrant. Then the curves $y = k$ and $y = 100 - x^2$ cross at the point $x = b = \sqrt{100 - k}$. When we solve the equation

$$\int_0^b \left(100 - x^2 - k\right)\,dx = kb + \int_b^{10} \left(100 - x^2\right)\,dx$$

for k, we find that $k = 50\left(2 - 2^{1/3}\right)$.

C05S08.061: We are interested in the region (or regions) bounded by the graphs of the two functions $f(x) = x$ and $g(x) = x(x-4)^2$. First we plot the two curves, using *Mathematica* 3.0, to guide our future computations.

```
Plot[ { f[x], g[x] }, { x, −1, 6 }, PlotRange → { −1, 11 } ];
```



It is easy to see that the curves cross where $x = 0$, $x = 3$, and $x = 5$. Note that $g(x) \geqq f(x)$ if $0 \leqq x \leqq 3$ and that $f(x) \geqq g(x)$ if $3 \leqq x \leqq 5$. Hence we compute the values of two integrals and add the results to get the total area of the (bounded) regions bounded by the two curves.

```
a1 = Integrate[ g[x] - f[x], { x, 0, 3 } ]
```

$$\frac{63}{4}$$

```
a2 = Integrate[ g[x] - f[x], { x, 3, 5 } ]
```

$$\frac{16}{3}$$

```
a1 + a2
```

$$\frac{253}{12}$$

```
N[ %, 20 ]
```

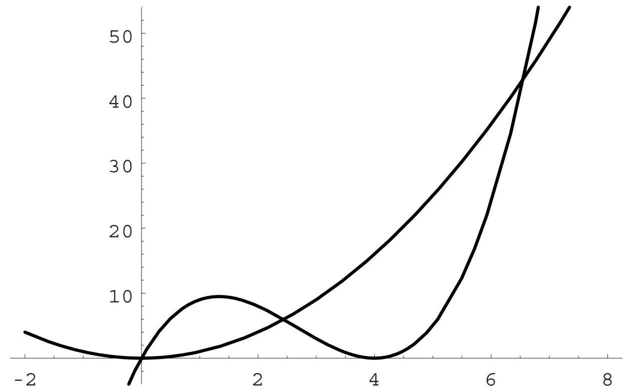
21.083333333333333333

C05S08.062: With $f(x) = x^2$ and $g(x) = x(x-4)^2$, we can easily use *Mathematica* 3.0 to find where the graphs cross and to plot the graphs.

```
Solve[ f[x] == g[x], x ]
```

$$\{\{\, x \to 0 \,\},\ \{\, x \to \frac{9 - \sqrt{17}}{2} \,\},\ \{\, x \to \frac{9 + \sqrt{17}}{2} \,\}\}$$

```
Plot[ { f[x], g[x] }, {x, −2, 8 }, PlotRange → { −4, 54 } ];
```



As in the solution of Problem 61, two integrals are required to find the total area bounded by the two curves.

```
r1 = 0; r2 = (9 − Sqrt[17])/2; r3 = (9 + Sqrt[17])/2;
```

```
a1 = Integrate[ g[x] - f[x], { x, r1, r2 } ]
```

$$\frac{51\sqrt{17}\ - 107}{8}$$

```
a2 = Integrate[ f[x] - g[x], { x, r2, r3 } ]
```

$$\frac{51\sqrt{17}}{4}$$

```
a1 + a2 // Together
```

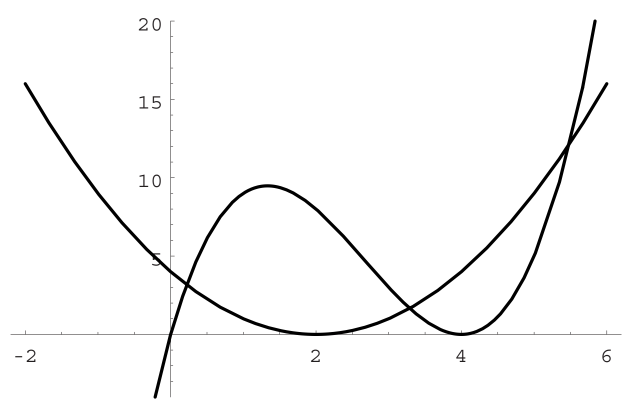
$$\frac{153\sqrt{17}\ - 107}{8}$$

```
N[ %, 20 ]
```

65.479395089937758902

C05S08.063: We are given $f(x) = (x-2)^2$ and $g(x) = x(x-4)^2$. We begin by plotting the graphs of both functions.

```
Plot[ { f[x], g[x] }, { x, −2, 6 }, PlotRange → { −4, 20 } ];
```



The `Solve` command in *Mathematica* returns exact solutions all of which include the number $i = \sqrt{-1}$, but their numerical values are numbers such as $5.48929 + 10^{-51}i$, so all solutions are pure real, as indicated in the preceding graph. We entered the numerical approximations to the roots:

```
r1 = 0.22154288174161126812;  r2 = 3.28916854644830996908;
r3 = 5.48928857181007876279;
```

Then we computed two integrals and added the results to find the total area bounded by the two curves.

```
a1 = Integrate[ g[x] − f[x], { x, r1, r2 } ]
```

$$17.96479813911499075801$$

```
a2 = Integrate[ f[x] − g[x], { x, r2, r3 } ]
```

$$7.39746346656350500268$$

```
a1 + a2
```

$$25.36226160567849576069$$

Section 5.9

C05S09.001: With $\Delta x = 1$, $f(x) = x$, $n = 4$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = 8,$$

which is also the true value of the given integral.

C05S09.002: With $\Delta x = 0.2$, $f(x) = x^2$, $n = 5$, and $x_i = 1 + i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{117}{50} = 2.34.$$

The exact value of the integral is $\dfrac{7}{3} \approx 2.333333$.

C05S09.003: With $\Delta x = 0.5$, $f(x) = \sqrt{x}$, $n = 5$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) \approx 0.6497385976 \approx 0.65.$$

The exact value of the integral is $\dfrac{2}{3} \approx 0.666667$.

C05S09.004: With $\Delta x = 0.5$, $f(x) = \dfrac{1}{x^2}$, $n = 4$, and $x_i = 1 + i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{141}{200} = 0.705000 \approx 0.71.$$

The exact value of the integral is $\dfrac{2}{3} \approx 0.666667$.

C05S09.005: With $\Delta x = \pi/6$, $f(x) = \cos x$, $n = 3$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{\pi}{12}\left(2 + \sqrt{3}\right) \approx 0.9770486167 \approx 0.98.$$

The exact value of the integral is 1.

C05S09.006: With $\Delta x = \pi/4$, $f(x) = \sin x$, $n = 4$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{\pi}{4}\left(1 + \sqrt{2}\right) \approx 1.8961188979 \approx 1.90.$$

The exact value of the integral is 2.

C05S09.007: With $\Delta x = 1$, $f(x) = x$, $n = 4$, and $m_i = \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) = 8.$$

The exact value of the integral is also 8.

C05S09.008: With $\Delta x = 0.2$, $f(x) = x^2$, $n = 5$, and $m_i = 1 + \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) = \frac{233}{100} = 2.33.$$

The exact value of the integral is $\dfrac{7}{3} \approx 2.333333$.

C05S09.009: With $\Delta x = 0.2$, $f(x) = \sqrt{x}$, $n = 5$, and $m_i = \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) \approx 0.6712800859 \approx 0.67.$$

The exact value of the integral is $\dfrac{2}{3} \approx 0.666667$.

C05S09.010: With $\Delta x = 0.5$, $f(x) = \dfrac{1}{x^2}$, $n = 4$, and $m_i = 1 + \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) = \frac{7781792}{12006225} \approx 0.65.$$

The exact value of the integral is $\dfrac{2}{3} \approx 0.666667$.

C05S09.011: With $\Delta x = \pi/6$, $f(x) = \cos x$, $n = 3$, and $m_i = \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) = \frac{\pi}{12}\left(\sqrt{2} + \sqrt{6}\right) \approx 1.01.$$

The exact value of the integral is 1.

C05S09.012: With $\Delta x = \pi/4$, $f(x) = \sin x$, $n = 4$, and $m_i = \left(i - \frac{1}{2}\right) \cdot \Delta x$, we have

$$M_n = (\Delta x) \cdot \sum_{i=1}^{n} f(m_i) = \frac{\pi}{4}\left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{5\pi}{8} + \sin\frac{7\pi}{8}\right) \approx 2.05.$$

The exact value of the integral is 2.

C05S09.013: With $\Delta x = 0.5$, $n = 4$, $f(x) = x^2$, and $x_i = 1 + i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i)\right) = \frac{35}{4} = 8.75$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\,] = \frac{26}{3} \approx 8.67.$$

The true value of the integral is also $\dfrac{26}{3}$ (see Problem 29).

C05S09.014: With $\Delta x = 0.75$, $n = 4$, $f(x) = x^3$, and $x_i = 1 + i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{4215}{64} = 65.859375 \approx 65.86$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\,] = \frac{255}{4} = 63.75.$$

The true value of the integral is also $\frac{255}{4}$ (see Problem 29).

C05S09.015: With $\Delta x = 0.5$, $n = 4$, $f(x) = \frac{1}{x^3}$, and $x_i = 2 + i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{28845889}{296352000} \approx 0.97$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\,] = \frac{41785153}{444528000} \approx 0.094$$

The true value of the integral is $\frac{3}{32} = 0.09375$.

C05S09.016: With $\Delta x = 0.25$, $n = 4$, $f(x) = \sqrt{1+x}$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) \approx 1.2181903242 \approx 1.22$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\,] \approx 1.2189451569 \approx 1.22$$

The true value of the integral is $\frac{2}{3}\left(2\sqrt{2} - 1\right) \approx 1.2189514165 \approx 1.22$.

C05S09.017: With $\Delta x = \frac{1}{3}$, $n = 6$, $f(x) = \sqrt{1+x^3}$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) \approx 3.2598849023 \approx 3.26$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\,] \approx 3.2410894400 \approx 3.24.$$

The true value of the integral is approximately 3.24131. The antiderivative of f is known to be a nonelementary function.

C05S09.018: With $\Delta x = 0.5$, $n = 6$, $f(x) = \frac{1}{1+x^4}$, and $x_i = i \cdot \Delta x$, we have

$$T_n = \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) = \frac{22392745}{20394056} \approx 1.0980035065 \approx 1.10$$

and

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n) \,] = \frac{192249939}{173349476} \approx 1.1090309786 \approx 1.11.$$

The true value of the integral is approximately 1.0984398680. The antiderivative of f is an elementary function but is not easy to find by hand; techniques of the later sections of Chapter 7 are required.

C05S09.019: With $\Delta x = 0.5$, $n = 8$, $f(x) = (1 + x^2)^{1/3}$, and $x_i = 1 + i \cdot \Delta x$, we have

$$T_n \;=\; \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) \;\approx\; 8.5498640075 \approx 8.55$$

and

$$S_n \;=\; \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n) \,] \;\approx\; 8.5508517478 \approx 8.55.$$

The true value of the integral is approximately 8.55073. The antiderivative of f is known to be a nonelementary function.

C05S09.020: With $\Delta x = 0.1$, $n = 10$, $f(x) = \dfrac{\tan x}{x}$, and $x_i = i \cdot \Delta x$, we have

$$T_n \;=\; \frac{\Delta x}{2} \cdot \left(f(x_0) + f(x_n) + 2 \cdot \sum_{i=1}^{n-1} f(x_i) \right) \;\approx\; 1.1507030742 \;\approx\; 1.15$$

and

$$S_n \;=\; \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n) \,] \;\approx\; 1.1491698791 \;\approx\; 1.15.$$

The *Mathematica* 3.0 command

```
NIntegrate[ (Tan[x])/x, { x, 0, 1 }, WorkingPrecision → 30 ]
```

quickly returns the approximation 1.14915123052353060405 to the true value of the integral. No computer algebra program we tried (*Mathematica* 3.0, *Derive* 2.56, or *Maple* V version 5.0) could antidifferentiate the integrand in closed form or evaluate the definite integral exactly.

C05S09.021: With $\Delta x = 0.25$ and $n = 6$, we have

$$T_n = \frac{\Delta x}{2}\,[3.43 + 2 \cdot (2.17 + 0.38 + 1.87 + 2.65 + 2.31) + 1.97\,] = 3.02$$

and

$$S_n = \frac{\Delta x}{3}\,[3.43 + 4 \cdot (2.17 + 1.87 + 2.31) + 2 \cdot (0.38 + 2.65) + 1.97\,] \approx 3.07167.$$

C05S09.022: With $\Delta x = 1$ and $n = 10$, we have

$$T_n = \frac{\Delta x}{2}\,[23 + 2 \cdot (8 - 4 + 12 + 35 + 47 + 53 + 50 + 39 + 29) + 5\,] = 283$$

and

$$S_n = \frac{\Delta x}{3}\,[\,23 + 4\cdot(8 + 12 + 47 + 50 + 29) + 2\cdot(-4 + 35 + 53 + 39) + 5\,] = 286.$$

C05S09.023: We read the following data from Fig. 5.9.16:

0	1	2	3	4	5	6	7	8	9	10
250	300	320	327	318	288	250	205	158	110	80

With $\Delta x = 1$ and $n = 10$, we have

$$T_n = \frac{\Delta x}{2}\,[\,250 + 2\cdot(300 + 320 + 327 + 318 + 288 + 250 + 205 + 158 + 110) + 80\,] = 2441$$

and

$$S_n = \frac{\Delta x}{3}\,[\,250 + 4\cdot(300 + 327 + 288 + 205 + 110) + 2\cdot(320 + 318 + 250 + 158) + 80\,] = \frac{7342}{3} \approx 2447.33.$$

C05S09.024: We read the following data from Fig. 5.9.17:

0	3	6	9	12	15	18	21	24	27	30
19	15	13	20	24	18	13	8	4	9	16

With $\Delta x = 3$ and $n = 10$, we have

$$T_n = \frac{\Delta x}{2}\,[\,19 + 2\cdot(15 + 13 + 20 + 24 + 18 + 13 + 8 + 4 + 9) + 16\,] = \frac{849}{2}.$$

Divide by 30 to obtain an approximation to the average value of the temperature over the 30-day period; the result is 14.15. Next,

$$S_n = \frac{\Delta x}{3}\,[\,19 + 4\cdot(15 + 20 + 18 + 8 + 9) + 2\cdot(13 + 24 + 13 + 4) + 16\,] = 423;$$

Divide by 30 to get an average temperature of 14.10.

C05S09.025: With $\Delta x = 50$ and $n = 12$, we find

$$T_n = \frac{\Delta x}{2}\,[\,0 + 2\cdot(165 + 192 + 146 + 63 + 42 + 84 + 155 + 224 + 270 + 267 + 215) + 0\,] = 91150.$$

This result is in square feet. Divide by 9 to convert to square yards, then divide by 4840 to convert to acres; the result is approximately 2.093 acres. Next,

$$S_n = \frac{\Delta x}{3}\,[\,0 + 4\cdot(165 + 146 + 42 + 155 + 270 + 215) + 2\cdot(192 + 63 + 84 + 224 + 267) + 0\,] = \frac{281600}{3}.$$

As before, divide by 9 and by 4840 to obtain the estimate 2.155 acres.

C05S09.026: With $a = 1$, $b = 2.7$, $n = 100$, $\Delta x = (b - a)/n$, $x_i = 1 + i\cdot\Delta x$, and $f(x) = \dfrac{1}{x}$, we find that

$$S_n = \frac{\Delta x}{3}\,[\,f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\,] \approx 0.993252 < 1.$$

The same computation with $b = 2.8$ yields $S_n \approx 1.02962 > 1$, and this is enough to show that $2.7 < e < 2.8$. We can be sure of both inequalities because Simpson's error estimate indicates that in each case the error is less than

$$\frac{24 \cdot (2.8-1)^5}{180 \cdot 100^4} \approx 2.52 \times 10^{-8}.$$

C05S09.027: We have $f''(x) = \dfrac{2}{x^3}$, so the constant in the error estimate is $K_2 = 2$. With $a = 1$, $b = 2$, and n subintervals, we require

$$\frac{K_2(b-a)^3}{12n^2} < 0.0005,$$

which implies that $n^2 > 333.33$ and thus that $n > 18.2$; $n = 19$ will suffice. (In fact, with only $n = 13$ subintervals, the trapezoidal estimate of 0.6935167303 differs from the true value of $\ln 2$ by less than 0.00037.)

C05S09.028: We have $f^{(4)}(x) = 24x^{-5}$, so the constant in Simpson's error estimate is $K_4 = 24$. With $a = 1$, $b = 2$, and n subintervals, we require

$$\frac{K_4(b-a)^5}{180n^4} < 0.000005,$$

which implies that $n^2 > 26666.7$ and thus that $n > 12.78$. Because n must be even, the answer is that n should be 14. (With $n = 14$ subintervals, the difference between $S_{14} \approx 0.6931479839$ and the true value of $\ln 2$ is less than 8.1×10^{-7}.)

C05S09.029: If $p(x) = ax^3 + bx^2 + cx + d$ is a polynomial of degree 3 or smaller, then $p^{(4)}(x) \equiv 0$, so the constant K_4 in Simpson's error estimate is zero, which implies that the error will also be zero no matter what the value of the even positive integer n may be.

C05S09.030: Let $f(x) = (4-2x)-(6x^2-7x) = 4+5x-6x^2$. Choose $a = -\frac{1}{2}$, $b = \frac{4}{3}$, $n = 2$, $\Delta x = (b-a)/n$, and $x_i = i \cdot \Delta x$. Then

$$S_2 = \frac{\Delta x}{3}\left[\, f(x_0) + 4f(x_1) + f(x_2)\,\right] = \frac{1331}{216}.$$

C05S09.031: With the usual meanings of the notation, we have

$$\begin{aligned}
M_n + T_n &= (\Delta x) \cdot \left[\, f(m_1) + f(m_2) + f(m_3) + \cdots + f(m_n)\,\right] \\
&\quad + \frac{\Delta x}{2} \cdot \left[\, f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\,\right] \\
&= \frac{\Delta x}{2} \cdot \left[\, f(x_0) + 2f(m_1) + 2f(x_1) + 2f(m_2) + 2f(x_2) + \cdots + 2f(x_{n-1}) + 2f(m_n) + f(x_n)\,\right] = 2T_{2n}.
\end{aligned}$$

The result in Problem 31 follows immediately.

C05S09.032: With $\alpha = 10°$ (which we convert to radians) and

$$f(x) = \frac{1}{\sqrt{1-(k\sin x)^2}},$$

we use $n = 10$, $a = 0$, $b = \pi/2$, $\Delta x = (b-a)/n$, and $x_i = i \cdot \Delta x$ and obtain

$$S_n = \frac{\Delta x}{3} \cdot [\, f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\,] \approx 1.5737921309.$$

We multiply by $4(L/g)^{1/2}$ to find the period T of the pendulum to be approximately 2.0109178213 (seconds). The same computation with $\alpha = 50°$ yields $T \approx 2.1070088018$ (seconds). The limiting value of T as $\alpha \to 0$ is approximately 2.00709.

C05S09.033: Suppose first that $f(x) > 0$ and $f''(x) > 0$ for $a \leqq x \leqq b$. Then the graph of f is concave upward on $[a,\, b]$. Now examine Fig. 5.9.11, where it is shown that the midpoint approximation is the same as the tangent approximation. The tangent line will lie under the graph of f because the graph is concave upward; the chord connecting $(x_{i-1},\, f(x_{i-1}))$ and $(x_i,\, f(x_i))$ will lie over the graph. Hence every term in the midpoint sum will underestimate the area under the graph of f and every term in the trapezoidal sum will overestimate it. Thus

$$M_n \,<\, \int_a^b f(x)\,dx \,<\, T_n$$

no matter what the choice of n. If $f''(x) < 0$ on $[a,\, b]$, then Figs. 5.9.11 and 5.9.12 show that the inequalities will be reversed.

Chapter 5 Miscellaneous Problems

C05S0M.001: $\displaystyle\int (5x^{-3} - 2x^{-2} + x^2)\ dx = -\frac{5}{2}x^{-2} + 2x^{-1} + \frac{1}{3}x^3 + C.$

C05S0M.002: $\displaystyle\int (x^{1/2} + 3x + 3x^{3/2} + x^2)\ dx = \frac{2}{3}x^{3/2} + \frac{3}{2}x^2 + \frac{6}{5}x^{5/2} + \frac{1}{3}x^3 + C.$

C05S0M.003: $\displaystyle\int (1-3x)^9\ dx = -\frac{1}{30}(1-3x)^{10} + C.$

C05S0M.004: $\displaystyle\int 7(2x+3)^{-3}\ dx = -\frac{7}{4}(2x+3)^{-2} + C.$

C05S0M.005: $\displaystyle\int (9+4x)^{1/3}\ dx = \frac{3}{16}(9+4x)^{4/3} + C.$

C05S0M.006: $\displaystyle\int 24(6x+7)^{-1/2}\ dx = 8(6x+7)^{1/2} + C.$

C05S0M.007: $\displaystyle\int x^3(1+x^4)^5\ dx = \frac{1}{24}(1+x^4)^6 + C.$

C05S0M.008: $\displaystyle\int 3x^2(4+x^3)^{1/2}\ dx = \frac{2}{3}(4+x^3)^{3/2} + C.$

C05S0M.009: $\displaystyle\int x(1-x^2)^{1/3}\ dx = -\frac{3}{8}(1-x^2)^{4/3} + C.$

C05S0M.010: $\displaystyle\int 3x(1+3x^2)^{-1/2}\ dx = (1+3x^2)^{1/2} + C.$

C05S0M.011: $\displaystyle\int (7\cos 5x - 5\sin 7x)\ dx = \frac{1}{35}(25\cos 7x + 49\sin 5x) + C.$

C05S0M.012: $\displaystyle\int 5\sin^3 4x\ \cos 4x\ dx = \frac{5}{16}\sin^4 4x + C.$

C05S0M.013: If $u = x^4$, then $du = 4x^3\ dx$, so that $x^3\ dx = \frac{1}{4}\ du$. Hence

$$\int x^3(1+x^4)^{1/2}dx = \int \frac{1}{4}(1+u)^{1/2}\ du = \frac{1}{4}\cdot\frac{2}{3}(1+u)^{3/2} + C = \frac{1}{6}(1+x^4)^{3/2} + C.$$

C05S0M.014: If $u = \sin x$ then $du = \cos x\ dx$. Hence

$$\int \sin^2 x\ \cos x\ dx = \int u^2\ du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C.$$

C05S0M.015: If $u = 1 + x^{1/2}$ then $du = \frac{1}{2}x^{-1/2}\ dx$, so that $x^{-1/2}\ dx = 2\ du$. Therefore

$$\int \frac{x^{-1/2}}{\left(1+x^{1/2}\right)^2}\ dx = \int \frac{2}{u^2}\ du = \int 2u^{-2}\ du = -2u^{-1} + C = -\frac{2}{1+\sqrt{x}} + C.$$

C05S0M.016: If $u = x^{1/2}$ then $du = \dfrac{1}{2}x^{-1/2}\,dx$, so that $x^{-1/2}\,dx = 2\,du$. Hence

$$\int \frac{x^{-1/2}}{\left(1+x^{1/2}\right)^2}\,dx = \int \frac{2}{(1+u)^2}\,du = \int 2(1+u)^{-2}\,du = -2(1+u)^{-1} + C = -\frac{2}{1+\sqrt{x}} + C.$$

C05S0M.017: If $u = 4x^3$ then $du = 12x^2\,dx$, so that $x^2\,dx = \dfrac{1}{12}\,du$. Hence

$$\int x^2 \cos 4x^3\,dx = \int \frac{1}{12}\cos u\,du = \frac{1}{12}\sin u + C = \frac{1}{12}\sin 4x^3 + C.$$

C05S0M.018: If $u = x + 1$, then $x = u - 1$ and $dx = du$. Hence

$$\int x(x+1)^{14}\,dx = \int (u-1)u^{14}\,du = \int (u^{15} - u^{14})\,du = \frac{1}{16}u^{16} - \frac{1}{15}u^{15} + C = \frac{1}{16}(x+1)^{16} - \frac{1}{15}(x+1)^{15} + C.$$

C05S0M.019: If $u = x^2 + 1$, then $du = 2x\,dx$, so that $x\,dx = \dfrac{1}{2}\,du$. Thus

$$\int x(x^2+1)^{14}\,dx = \int \frac{1}{2}u^{14}\,du = \frac{1}{30}u^{15} + C = \frac{1}{30}(x^2+1)^{15} + C.$$

An unpleasant alternative is to expand $(x^2+1)^{14}$ using the binomial formula, multiply by x, then integrate the resulting polynomial to obtain

$$\begin{aligned} &\frac{1}{30}x^{30} + \frac{1}{2}x^{28} + \frac{7}{2}x^{26} + \frac{91}{6}x^{24} + \frac{91}{2}x^{22} + \frac{1001}{10}x^{20} + \frac{1001}{6}x^{18} \\ &\quad + \frac{429}{2}x^{14} + \frac{1001}{6}x^{12} + \frac{1001}{10}x^{10} + \frac{91}{2}x^8 + \frac{91}{6}x^6 + \frac{7}{2}x^4 + \frac{1}{2}x^2 + C. \end{aligned}$$

C05S0M.020: If $u = x^4$, then $du = 4x^3\,du$, so that $x^3\,dx = \dfrac{1}{4}\,du$. Then

$$\int x^3 \cos x^4\,dx = \int \frac{1}{4}\cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin x^4 + C.$$

C05S0M.021: If $u = 4 - x$ then $x = 4 - u$ and $dx = -\,du$. Hence

$$\begin{aligned} \int x(4-x)^{1/2}\,dx &= \int (u-4)u^{1/2}\,du = \int (u^{3/2} - 4u^{1/2})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{5}(4-x)^{5/2} - \frac{8}{3}(4-x)^{3/2} + C. \end{aligned}$$

C05S0M.022: If $u = x^4 + x^2$, then $du = (4x^3 + 2x)\,dx$, so that $(x + 2x^3)\,dx = \dfrac{1}{2}\,du$. Thus

$$\int (x+2x^3)(x^4+x^2)^{-3}\,dx = \int \frac{1}{2}u^{-3}\,du = -\frac{1}{4}u^{-2} + C = -\frac{1}{4(x^4+x^2)^2} + C.$$

C05S0M.023: If $u = x^4$, then $du = 4x^3\,dx$, so that $2x^3\,dx = \dfrac{1}{2}\,du$. Therefore

$$\int 2x^3(1+x^4)^{-1/2}\,dx = \int \frac{1}{2}(1+u)^{-1/2}\,du = (1+u)^{1/2} + C = \sqrt{1+x^4}\, + C.$$

C05S0M.024: If $u = x^2 + x$, then $du = (2x+1)\,dx$, so that

$$\int (x^2+x)^{-1/2}(2x+1)\,dx = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{x^2+x}\, + C.$$

C05S0M.025: $y(x) = \displaystyle\int_0^x (3t^2+2t)\,dt + 5 = \Big[t^3+t^2\Big]_0^x + 5 = x^3 + x^2 + 5.$

C05S0M.026: $y(x) = \displaystyle\int_4^x 3t^{1/2}\,dt + 20 = \Big[2t^{3/2}\Big]_4^x + 20 = 2x^{3/2} - 16 + 20 = 2x^{3/2} + 4.$

C05S0M.027: If $\dfrac{dy}{dx} = (2x+1)^5$, then $y(x) = \dfrac{1}{12}(2x+1)^6 + C$. Then

$$2 = y(0) = \frac{1}{12} + C \quad \text{implies that} \quad C = \frac{23}{12},$$

and therefore $y(x) = \dfrac{1}{12}(2x+1)^6 + \dfrac{23}{12}$.

C05S0M.028: $y(x) = \displaystyle\int_4^x 2(t+5)^{-1/2}\,dt + 3 = \Big[4(t+5)^{1/2}\Big]_4^x + 3 = 4(x+5)^{1/2} - 12 + 3 = 4\sqrt{x+5}\, - 9.$

C05S0M.029: $y(x) = \displaystyle\int_1^x t^{-1/3}\,dt + 1 = \Big[\frac{3}{2}t^{2/3}\Big]_1^x + 1 = \frac{3}{2}x^{2/3} - \frac{3}{2} + 1 = \frac{3x^{2/3}-1}{2}.$

C05S0M.030: $y(x) = \displaystyle\int_0^x (1-\cos t)\,dt = \Big[t - \sin t\Big]_0^x = x - \sin x.$

C05S0M.031: First convert 90 mi/h to 132 ft/s (just multiply by $\frac{22}{15}$). Let $x(t)$ denote the distance (in feet) the automobile travels after its brakes are first applied at time $t = 0$ (s). Then $x(t) = -11t^2 + 132t$, so the automobile first comes to a stop when $v(t) = x'(t) = -22t + 132 = 0$; that is, when $t = 6$. Therefore the total distance it travels while braking will be $x(6) = 396$ (ft).

C05S0M.032: If the stone is dropped at time $t = 0$ (s), then its altitude at time t will be

$$y(t) = -11250t^2 + 450 \quad \text{(ft)}.$$

The stone reaches the ground when $y(t) = 0$, so that $t = \frac{1}{5}$. Its impact velocity will be $x'(t) = v(t) = -22500t$ evaluated when $t = \frac{1}{5}$; that is, $v\left(\frac{1}{5}\right) = -4500$ (ft/s). Thus the stone remains aloft for 0.2 s and its impact speed will be 4500 ft/s.

C05S0M.033: Let v_0 denote the initial velocity of the automobile and let $x(t)$ denote the distance it has skidded since its brakes were applied at time $t = 0$ (units are in feet and seconds). Then $x(t) = -20t^2 + v_0t$. Let T denote the time at which the automobile first comes to a stop. Then

$$x(T) = 180 \qquad \text{and} \qquad x'(T) = 0.$$

That is,

$$-20T^2 + v_0 T = 180 \qquad \text{and} \qquad -40T + v_0 = 0.$$

The second of these equations implies that $v_0 = 40T$, and substitution of this datum in the first of these equations yields $-20T^2 + 40T^2 = 180$, so that $T^2 = 9$. Thus $T = 3$, and therefore $v_0 = 120$. Hence the initial velocity of the automobile was 120 feet per second, slightly less than 82 miles per hour (slightly less than 132 kilometers per hour).

C05S0M.034: Assume that the car begins its journey at time $t = 0$ and at position $x = 0$ (units are in feet and seconds). Then $x(t) = 4t^2$ because both initial position and initial velocity are zero. It will reach a speed of 60 miles per hour—that is, 88 feet per second—when $x'(t) = 88$; that is, when $y = 11$. At that point the car will have traveled a distance of $x(11) = 484$ feet.

C05S0M.035: Let q denote the acceleration of gravity on the planet Zorg. First consider the ball dropped from a height of 20 feet. Then its altitude at time t will be $y(t) = -\frac{1}{2}qt^2 + 20$ (because its initial velocity is $v_0 = 0$). Then the information that $x(2) = 0$ yields the information that $q = 10$. Now suppose that the ball is dropped from an initial height of 200 feet. Then its altitude at time t will be $y(t) = -5t^2 + 200$, so the ball will reach then ground when $5t^2 = 200$; that is, when $t = 2\sqrt{10}$. Its velocity during its descent will be $v(t) = y'(t) = -10t$, so its impact velocity will be $v\big(2\sqrt{10}\,\big) = -20\sqrt{10}$ feet per second. Thus its impact speed will be $|\,v\big(2\sqrt{10}\,\big)\,| = 20\sqrt{10} \approx 63.25$ feet per second.

C05S0M.036: First we need to find the initial velocity v_0 you can impart to the ball that you throw straight upward. Let $y(t)$ denote the altitude of the ball at time t (units are in feet and seconds), with the throw occurring at time $t = 0$. Then $y(t) = -16t^2 + v_0 t$ (we make the simplifying assumption that $y_0 = y(0) = 0$). The ball reaches its maximum height when its velocity $v(t) = y'(t) = -32t + v_0$ is zero; let T denote the time at which that event occurs. Then

$$x(T) = 144 \qquad \text{when} \qquad v(T) = 0;$$

that is, we must solve simultaneously the equations

$$-16T^2 + v_0 T = 144 \qquad \text{and} \qquad -32T + v_0 = 0.$$

The second of these equations yields $v_0 = 32T$, and substitution in the first yields $-16T^2 + 32T^2 = 144$, so that $T = 3$, and thus $v_0 = 96$.

We now assume that you can impart the same initial velocity to the ball on both Zorg and Mesklin (the latter assumption is surely invalid, but it's clearly the intent of the problem). On Zorg, the altitude function of the ball will be $y(t) = -5t^2 + 96t$, so its velocity will be $v(t) = -10t + 96$. The ball will reach its maximum altitude when $v(t) = 0$, thus when $t = 9.6$. Thus the maximum altitude reached by the ball will be $y(9.6) = 460.8$ (feet). On Mesklin, the altitude function of the ball will be $y(t) = -11250t^2 + 96t$, so its velocity at time t will be $v(t) = -22500t + 96$. The ball reaches its maximum altitude when $v(t) = 0$, so that $t = \frac{8}{1875} \approx 0.004267$ (seconds). Its maximum altitude will therefore be $y\big(\frac{8}{1875}\big) = \frac{128}{625} = 0.2048$ feet; that is, only 2.4576 inches, a little less than 6.25 centimeters.

C05S0M.037: First we need to find the deceleration constant a of the car. Let $x(t)$ denote the distance (in feet) the car has skidded at time t (in seconds) if its brakes are applied at time $t = 0$. Then $x(t) = -\frac{1}{2}at^2 + 44t$. (We converted 30 miles per hour to 44 feet per second.) Thus in the first skid, if the car first comes to a stop at time T, then both

$$x(T) = 44 \qquad \text{and} \qquad x'(T) = 0.$$

Because the velocity of the car is $v(t) = x'(t) = -at + 44$, we must solve simultaneously the equations

$$-\frac{1}{2}aT^2 + 44T = 44 \qquad \text{and} \qquad -aT + 44 = 0.$$

The second of these equations yields $T = \dfrac{44}{a}$, then substitution in the first equation yields

$$\begin{aligned}
&-\frac{1}{2}\cdot\frac{44^2}{a} + \frac{44^2}{a} = 44;\\
&-\frac{22}{a} + \frac{44}{a} = 1;\\
&\frac{22}{a} = 1; \qquad a = 22.
\end{aligned}$$

Now suppose that the initial velocity of the car is 60 miles per hour; that is, 88 feet per second. With the same meaning of $x(t)$ as before, we now have $x(t) = -11t^2 + 88t$. Thus $v(t) = x'(t) = -22t + 88$, so the car comes to a stop when $t = 4$. The distance it now skids will be $x(t) = 176$ feet. The point of the problem is that doubling the initial speed of the car *quadruples* the stopping distance.

C05S0M.038: When the fuel is exhausted, the acceleration of the rocket is that solely due to gravity, so the velocity stops increasing and begins to decrease. In Fig. 5.MP.1 this appears to occur about at time $t = 1.8$. When the parachute opens, the velocity stops decreasing because of the upward force due to the parachute; this appears to occur close to time $t = 3.2$. The rocket reached its maximum altitude when its velocity was zero, close to time $t = 2.8$. The rocket landed at time $t = 5$. Its maximum height was about 240 feet and the pole atop which it landed was about 110 feet high. One of the more interesting aspects of this problem is that during the free fall of the rocket, its velocity changed from about 200 feet per second at time $t = 1.8$ to about -80 feet per second at time $t = 3.2$, implying that the acceleration of gravity on the planet where this all took place is about 200 feet per second per second.

C05S0M.039: $\displaystyle\sum_{i=1}^{100} 17 = 17\cdot\sum_{i=1}^{100} 1 = 17\cdot 100 = 1700.$

C05S0M.040: $\displaystyle\sum_{k=1}^{100}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{99}-\frac{1}{100}\right) + \left(\frac{1}{100}-\frac{1}{101}\right)$

$\displaystyle\quad = 1 - \frac{1}{101} = \frac{100}{101}.$

C05S0M.041: $\displaystyle\sum_{n=1}^{10}(3n-2)^2 = 9\cdot\sum_{n=1}^{10} n^2 - 12\cdot\sum_{n=1}^{10} n + 4\cdot\sum_{n=1}^{10} 1 = 9\cdot\frac{10\cdot 11\cdot 21}{6} - 12\cdot\frac{10\cdot 11}{2} + 4\cdot 10 = 2845.$

C05S0M.042: $\displaystyle\sum_{n=1}^{16}\sin\frac{n\pi}{2} = 4\cdot(1+0-1+0) = 0.$

C05S0M.043: On $[1, 2]$, we have $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\Delta x}{\sqrt{x_i^\star}} = \int_1^2 \frac{1}{\sqrt{x}}\,dx = \Big[2\sqrt{x}\Big]_1^2 = 2\sqrt{2} - 2.$

C05S0M.044: On $[0, 3]$, we have $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left[\,(x_i^\star)^2 - 3x_i^\star\,\right]\Delta x = \int_0^3 (x^2 - 3x)\,dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^3 = -\frac{9}{2}.$

C05S0M.045: On [0, 1], we have $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^\star \sqrt{1+(x_i^\star)^2}\ \Delta x = \int_0^1 2\pi x\sqrt{1+x^2}\ dx$

$$= \left[\frac{2\pi}{3}(1+x^2)^{3/2}\right]_0^1 = \frac{2\pi}{3}\left(2\sqrt{2}-1\right).$$

C05S0M.046: First note that

$$\lim_{n\to\infty} \frac{1^{10}+2^{10}+3^{10}+4^{10}+\cdots+n^{10}}{n^{11}} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{10} \cdot \frac{1}{n} \tag{1}$$

is the limit of Riemann sums for $f(x) = x^{10}$ on [0, 1] with $x_i^\star = i/n$ and $\Delta x = 1/n$. Hence the value of the limit in Eq. (1) is

$$\int_0^1 x^{10}\ dx = \left[\frac{1}{11}x^{11}\right]_0^1 = \frac{1}{11}.$$

Alternatively, one can show that

$$\sum_{k=1}^{n} k^{10} = \frac{n(n+1)(2x+1)(n^2-n+1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)}{66}.$$

Then division by n^{11} yields

$$\frac{5}{66n^{10}} - \frac{1}{2n^8} + \frac{1}{n^6} - \frac{1}{n^4} + \frac{5}{6n^2} + \frac{1}{2n} + \frac{1}{11},$$

and the value of the limit as $n \to +\infty$ is now clear.

C05S0M.047: If $f(x) \equiv c$ (a constant), then for every partition of $[a, b]$ and every selection for each such partition, we have $f(x_i^\star) = c$. Therefore

$$\sum_{i=1}^{n} f(x_i^\star)\ \Delta x = \sum_{i=1}^{n} c \cdot \frac{b-a}{n} = c(b-a) \cdot \sum_{i=1}^{n} \frac{1}{n} = c(b-a) \cdot n \cdot \frac{1}{n} = c(b-a).$$

Then, because every Riemann sum is equal to $c(b-a)$, this is also the limit of those Riemann sums. Therefore, by definition,

$$\int_a^b f(x)\ dx = c(b-a).$$

C05S0M.048: Because f is continuous on $[a, b]$, the definite integral of f exists there; that is, the appropriate Riemann sums have a limit as $\Delta x \to 0$. For every partition of $[a, b]$ and every selection for every such partition, we have $f(x_i^\star) \geqq 0$, so every Riemann sum is a sum of nonnegative numbers. Therefore their limit is nonnegative (if the limit were negative, then at least one Riemann sum would have to be negative). Therefore

$$\int_a^b f(x)\ dx \ \geqq\ 0.$$

C05S0M.049: Given: f continuous on $[a, b]$ and $f(x) > 0$ there. Let m be the global minimum value of f on $[a, b]$; then $m > 0$. Hence, by the second comparison property (Section 5.5),

$$\int_a^b f(x)\,dx \geqq m(b-a) > 0.$$

C05S0M.050: $\displaystyle \int_0^1 (1-x^2)^3\,dx = \int_0^1 (1-3x^2+3x^4-x^6)\,dx = \left[x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{3}{5} - \frac{1}{7} = \frac{16}{35}.$

C05S0M.051: $\displaystyle \int \left[\,(2x)^{1/2} - (3x^3)^{-1/2}\,\right] dx = \int \left(x^{1/2}\sqrt{2} - \frac{\sqrt{3}}{3}x^{-3/2}\right) dx = \frac{2\sqrt{2}}{3}x^{3/2} + \frac{2\sqrt{3}}{3x^{1/2}} + C.$

C05S0M.052: $\displaystyle \int \frac{\left(1+x^{1/3}\right)^2}{x^{1/2}}\,dx = \int (x^{-1/2} + 2x^{-1/6} + x^{1/6})\,dx = 2x^{1/2} + \frac{12}{5}x^{5/6} + \frac{6}{7}x^{7/6} + C.$

C05S0M.053: $\displaystyle \int \frac{4-x^3}{2x^2}\,dx = \int \left(2x^{-2} - \frac{1}{2}x\right) dx = -2x^{-1} - \frac{1}{4}x^2 + C.$

C05S0M.054: $\displaystyle \int_0^1 \frac{dt}{(3-2t)^2} = \left[\frac{1}{2(3-2t)}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$

C05S0M.055: $\displaystyle \int x^{1/2}\cos x^{3/2}\,dx = \frac{2}{3}\sin x^{3/2} + C.$

C05S0M.056: $\displaystyle \int_0^2 x^2(9-x^3)^{1/2}\,dx = \left[-\frac{2}{9}(9-x^3)^{3/2}\right]_0^2 = -\frac{2}{9} - (-6) = \frac{52}{9}.$

C05S0M.057: $\displaystyle \int \frac{1}{t^2}\sin\frac{1}{t}\,dt = \cos\frac{1}{t} + C.$

C05S0M.058: $\displaystyle \int_1^2 \frac{2t+1}{(t^2+t)^{1/2}}\,dt = \left[2(t^2+t)^{1/2}\right]_1^2 = 2\sqrt{6} - 2\sqrt{2}\,.$

C05S0M.059: $\displaystyle \int \frac{u^{1/3}}{(1+u^{4/3})^3}\,du = -\frac{3}{8}(1+u^{4/3})^{-2} + C.$

C05S0M.060: $\displaystyle \int_0^{\pi/4} (\cos t)^{-1/2}\sin t\,dt = \Big[-2(\cos t)^{1/2}\Big]_0^{\pi/4} = -(2^{3/4}) - (-2) = 2 - 2^{3/4}.$

C05S0M.061: $\displaystyle \int_1^4 \frac{(1+t^{1/2})^2}{t^{1/2}}\,dt = \int_1^4 (t^{1/2} + 2 + t^{-1/2})\,dt = \left[\frac{2}{3}t^{3/2} + 2t + 2t^{1/2}\right]_1^4 = \frac{52}{3} - \frac{14}{3} = \frac{38}{3}.$

C05S0M.062: $\displaystyle \int \frac{1}{u^2}\left(1-\frac{1}{u}\right)^{1/3} du = \frac{3}{4}\left(1-\frac{1}{u}\right)^{4/3} + C.$

C05S0M.063: Let $u = \dfrac{1}{x}$, so that $x = \dfrac{1}{u}$ and $dx = -\dfrac{1}{u^2}\,du$. Then

$$I = \int \frac{(4x^2-1)^{1/2}}{x^4}\,dx = \int -\frac{u^4}{u^2}\left(\frac{4}{u^2} - 1\right)^{1/2} du = -\int u^2\left(\frac{4}{u^2} - 1\right)^{1/2} du.$$

Next move one copy of u from outside the square root to inside, where it becomes u^2, and we see

$$I = -\int u(4-u^2)^{1/2}\,du = \frac{1}{3}(4-u^2)^{3/2} + C = \frac{1}{3}\left(4 - \frac{1}{x^2}\right)^{3/2} + C.$$

To make this answer more appealing, multiply numerator and denominator by x^3; when the x^3 in the numerator is moved into the 3/2-power, it becomes x^2 and the final version of the answer is

$$I = \frac{(4x^2-1)^{3/2}}{3x^3} + C.$$

C05S0M.064: The area is $\displaystyle\int_{-1}^{1}(1-x^3)\,dx = \left[x - \frac{1}{4}x^4\right]_{-1}^{1} = \frac{3}{4} - \left(-\frac{5}{4}\right) = 2.$

C05S0M.065: The area is $\displaystyle\int_{0}^{1}(x^4-x^5)\,dx = \left[\frac{1}{5}x^5 - \frac{1}{6}x^6\right]_{0}^{1} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}.$

C05S0M.066: Solve $3y^2 - 6 = y^2$ to find that the two curves cross where $y = a = -\sqrt{3}$ and $y = b = \sqrt{3}$. Hence the area between them is

$$\int_a^b (6-2y^2)\,dy = \left[6y - \frac{2}{3}y^3\right]_a^b = 4\sqrt{3} - \left(-4\sqrt{3}\right) = 8\sqrt{3}.$$

C05S0M.067: Solve $x^4 = 2 - x^2$ to find that the two curves cross where $x = \pm 1$. Hence the area between them is

$$\int_{-1}^{1}(2-x^2-x^4)\,dx = \left[2x - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{1} = \frac{22}{15} - \left(-\frac{22}{15}\right) = \frac{44}{15}.$$

C05S0M.068: Solve $x^4 = 2x^2 - 1$ for $x = \pm 1$ to find where the curves cross. The area between them is then

$$\int_{-1}^{1}(x^4-2x^2+1)\,dx = \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^{1} = \frac{8}{15} - \left(-\frac{8}{15}\right) = \frac{16}{15}.$$

C05S0M.069: Solve $(x-2)^2 = 10 - 5x$ to find that the two curves cross where $x = -3$ and where $x = 2$. The area between them is

$$\int_{-3}^{2}(10-5x-(x-2)^2)\,dx = \left[6x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-3}^{2} = \frac{22}{3} - \left(-\frac{27}{2}\right) = \frac{125}{6}.$$

C05S0M.070: Solve $x^{2/3} = 2 - x^2$ to find that the two curves cross where $x = \pm 1$. Thus the area between them is

$$\int_{-1}^{1}(2-x^2-x^{2/3})\,dx = \left[2x - \frac{1}{3}x^3 - \frac{3}{5}x^{5/3}\right]_{-1}^{1} = \frac{16}{15} - \left(-\frac{16}{15}\right) = \frac{32}{15}.$$

C05S0M.071: If $y = \sqrt{2x - x^2}$, then $y \geqq 0$ and $x^2 - 2x + y^2 = 0$, so that $(x-1)^2 + y^2 = 1$. Therefore the graph of the integrand is the top half of a circle of radius 1 centered at (1, 0), and so the value of the integral is the area of that semicircle:

$$\int_0^2 \sqrt{2x - x^2}\ dx = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2}.$$

C05S0M.072: If $y = \sqrt{6x - 5 - x^2}$, then $y \geqq 0$ and $x^2 - 6x + 5 + y^2 = 0$, so that $(x-3)^2 + y^2 = 4$. Hence the graph of the integrand is the top half of a circle of radius 2 centered at (3, 0), and so the value of the integral is the area of that semicircle:

$$\int_1^5 \sqrt{6x - 5 - x^2}\ dx = \frac{1}{2} \cdot \pi \cdot 2^2 = 2\pi.$$

C05S0M.073: If

$$x^2 = 1 + \int_1^x \sqrt{1 + [f(t)]^2}\ dt,$$

then differentiation of both sides of this *identity* with respect to x (using Part 1 of the fundamental theorem of calculus, Section 5.6) yields

$$2x = \sqrt{1 + [f(x)]^2},$$

so that $4x^2 = 1 + [f(x)]^2$. Therefore if $x > 1$, one solution is $f(x) = \sqrt{4x^2 - 1}$.

C05S0M.074: Let

$$F(x) = \int_a^x \phi(t)\ dt.$$

Then $G(x) = F(h(x))$, so that $G'(x) = F'(h(x)) \cdot h'(x)$. But $F'(x) = \phi(x)$ by the fundamental theorem of calculus (Section 5.6). Therefore $G'(x) = \phi(h(x)) \cdot h'(x)$.

C05S0M.075: Because $f(x) = \sqrt{1 + x^2}$ is increasing on $[0, 1]$, the left-endpoint approximation will be an underestimate of the integral and the right-endpoint approximation will be an overestimate. With $n = 5$, $\Delta x = \frac{1}{5}$, and $x_i = i \cdot \Delta x$, we find the left-endpoint approximation to be

$$\sum_{i=1}^{5} f(x_{i-1})\ \Delta x \approx 1.10873$$

and the right-endpoint approximation is

$$\sum_{i=1}^{5} f(x_i)\ \Delta x \approx 1.19157.$$

The average of the two approximations is 1.15015 and half their difference is 0.04142, and therefore

$$\int_0^1 \sqrt{1 + x^2}\ dx = 1.15015 \pm 0.04143.$$

The true value of this integral is approximately 1.1477935747. When we used $n = 4$ subintervals the error in the approximation was larger than 0.05.

C05S0M.076: With $n = 6$, $a = 0$, $b = \pi$, $\Delta x = (b - a)/n$, and $x_i = i \cdot \Delta x$, the trapezoidal approximation is

$$T_6 = \frac{1}{12}\left[\sqrt{2} + 2\left(1 + \frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2} + \sqrt{1 - \frac{\sqrt{3}}{2}} + \sqrt{1 + \frac{\sqrt{3}}{2}}\right)\right]\pi \ \approx\ 2.8122538625.$$

Simpson's approximation is

$$S_6 = \frac{1}{18}\left(4 + 2\sqrt{2} + \sqrt{6} + 4\sqrt{1 - \frac{\sqrt{3}}{2}} + 4\sqrt{1 + \frac{\sqrt{3}}{2}}\right)\pi \approx 2.8285015468.$$

The exact value of the integral is

$$\int_0^\pi \sqrt{1 - \cos x}\ dx = \sqrt{2}\int_0^\pi \sqrt{\frac{1 - \cos x}{2}}\ dx = \sqrt{2}\int_0^\pi \sqrt{\sin^2 \frac{x}{2}}\ dx$$
$$= \sqrt{2}\int_0^\pi \left|\sin \frac{x}{2}\right|\ dx = \sqrt{2}\int_0^\pi \sin \frac{x}{2}\ dx = \left[-\left(2\sqrt{2}\right)\cos \frac{x}{2}\right]_0^\pi = 2\sqrt{2}.$$

C05S0M.077: $M_5 \approx 0.2866736772$ and $T_5 \approx 0.2897075147$. Because the graph of the integrand is concave upward on the interval $[1,\ 2]$,

$$M_5\ <\ \int_1^2 \frac{1}{x + x^2}\ dx\ <\ T_5$$

for the reasons given in the solution of Problem 33 of Section 5.9.

C05S0M.078: First,

$$(x_{i-1})^2 \leqq \frac{1}{3}\left((x_{i-1})^2 + x_{i-1}x_i + (x_i)^2\right) \leqq (x_i)^2.$$

Therefore $x_{i-1} \leqq x_i^\star \leqq x_i$. Then

$$(x_i^\star)^2\,(x_i - x_{i-1}) = \frac{1}{3}\left((x_{i-1})^2 + x_{i-1}x_i + (x_i)^2\right)(x_i - x_{i-1}) = \frac{1}{3}\left((x_i)^3 - (x_{i-1})^3\right),$$

and when such expressions are summed for $i = 1, 2, 3, \dots, n$, the result is $\dfrac{1}{3}\left(b^3 - a^3\right)$.

Because all Riemann sums for $f(x) = x^2$ on $[a,\ b]$ have the same limit, this must be the same limit as the limit of the particular Riemann sums of this problem. This shows that

$$\int_a^b x^2\ dx = \frac{1}{3}(b^3 - a^3).$$

C05S0M.079: Suppose that $0 < a < b$, that n is a positive integer, that $\mathcal{P} = \{x_0,\ x_1,\ x_2,\ \dots,\ x_n\}$ is a partition of $[a,\ b]$, and that $\Delta x_i = x_i - x_{i-1}$ for $1 \leqq i \leqq n$. Let $x_i^\star = \sqrt{x_{i-1}x_i}$ for $1 \leqq i \leqq n$. Then $S = \{x_1^\star,\ x_2^\star,\ \dots,\ x_n^\star\}$ is a selection for $\mathcal{P}$ because

$$x_{i-1} = \sqrt{(x_{i-1})^2} < \sqrt{x_{i-1}x_i} < \sqrt{(x_i)^2} = x_i$$

for $1 \leqq i \leqq n$. Next,

$$\sum_{i=1}^{n} \frac{1}{(x_i^\star)^2}\,\Delta x_i = \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{x_{i-1}x_i} = \sum_{i=1}^{n}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right)$$

$$= \left(\frac{1}{x_0} - \frac{1}{x_1}\right) + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \left(\frac{1}{x_2} - \frac{1}{x_3}\right) + \cdots + \left(\frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}\right) + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right)$$

$$= \frac{1}{x_0} - \frac{1}{x_n} = \frac{1}{a} - \frac{1}{b}.$$

Therefore

$$\sum_{i=1}^{n} \frac{1}{(x_i^\star)^2}\,\Delta x_i \to \frac{1}{a} - \frac{1}{b}$$

as $|\mathcal{P}| \to 0$. But

$$\sum_{i=1}^{n} \frac{1}{(x_i^\star)^2}\,\Delta x_i \quad \text{is a Riemann sum for} \quad \int_a^b \frac{1}{x^2}\,dx.$$

Because f is continuous on $[a,\,b]$, all such Riemann sums converge to the same limit, which must therefore be the same as the particular limit just computed. Therefore

$$\int_a^b \frac{1}{x^2}\,dx \;=\; \frac{1}{a} - \frac{1}{b}.$$

C05S0M.080: Suppose that $0 < a < b$, that n is a positive integer, that $\mathcal{P} = \{x_0,\, x_1,\, x_2,\, \dots,\, x_n\}$ is a partition of $[a,\,b]$, and that $\Delta x_i = x_i - x_{i-1}$ for $1 \leqq i \leqq n$. Let

$$\sqrt{x_i^\star} = \frac{\frac{2}{3}\left[\,(x_i)^{3/2} - (x_{i-1})^{3/2}\,\right]}{x_i - x_{i-1}}$$

for $1 \leqq i \leqq n$. Then $S = \{x_1^\star,\, x_2^\star,\, \dots,\, x_n^\star\}$ is a selection for $\mathcal{P}$ because

$$\sqrt{x_i^\star} = \frac{\frac{2}{3}\left[\left(\sqrt{x_i}\,\right)^3 - \left(\sqrt{x_{i-1}}\,\right)^3\right]}{\left(\sqrt{x_i}\,\right)^2 - \left(\sqrt{x_{i-1}}\,\right)^2}$$

$$= \frac{2}{3} \cdot \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\sqrt{x_i} - \sqrt{x_{i-1}}} \cdot \frac{\left(\sqrt{x_i}\,\right)^2 + \sqrt{x_i x_{i-1}} + \left(\sqrt{x_{i-1}}\,\right)^2}{\sqrt{x_i} + \sqrt{x_{i-1}}} = \frac{2}{3} \cdot \frac{x_i + \sqrt{x_i x_{i-1}} + x_{i-1}}{\sqrt{x_i} + \sqrt{x_{i-1}}}.$$

At this point we'd like to be able to claim that the last term in the last equation is less than

$$\frac{2}{3} \cdot \frac{x_i + x_i + x_i}{2\sqrt{x_i}} = \frac{x_i}{\sqrt{x_i}} = \sqrt{x_i}\,,$$

because this would establish that $x_i^\star < x_i$. But while we increase the numerator by replacing x_{i-1} with the larger x_i, we are also increasing the denominator. So we need a technical lemma whose proof you may prefer to ignore.

Lemma: If $0 < a < b$, then

$$\frac{b+\sqrt{ab}+a}{\sqrt{b}+\sqrt{a}} < \frac{3b}{2\sqrt{b}} = \frac{3\sqrt{b}}{2}.$$

Proof: Suppose that $0 < a < b$. Then

$$\begin{aligned}
&\sqrt{a} \;<\; \sqrt{b};\\
&a \;<\; \sqrt{ab};\\
&2a \;<\; a+\sqrt{ab} \;<\; b+\sqrt{ab};\\
&2b+2\sqrt{ab}+2a \;<\; 3b+3\sqrt{ab};\\
&2(b+\sqrt{ab}+a) \;<\; 3\sqrt{b}\,\left(\sqrt{b}+\sqrt{a}\right);\\
&\frac{b+\sqrt{ab}+a}{\sqrt{b}+\sqrt{a}} < \frac{3\sqrt{b}}{2}.
\end{aligned}$$

This concludes the proof of the lemma.

Therefore, if $0 < x_{i-1} < x_i$, then

$$\frac{x_i+\sqrt{x_i x_{i-1}}+x_{i-1}}{\sqrt{x_i}+\sqrt{x_{i-1}}} \;<\; \frac{3}{2}\sqrt{x_i}.$$

It now follows that $\sqrt{x_i^\star} \;<\; \sqrt{x_i}$, and thus that $x_i^\star \;<\; x_i$ for $1 \leqq i \leqq n$. In much the same way, one can establish that $x_{i-1} < x_i^\star$ for $1 \leqq i \leqq n$. Hence S is indeed a selection for $\mathcal{P}$.

Consequently,

$$\begin{aligned}
\sum_{i=1}^{n} \sqrt{x_i^\star}\;\Delta x_i &= \frac{2}{3}\cdot\sum_{i=1}^{n}\left[(x_i)^{3/2}-(x_{i-1})^{3/2}\right]\\
&= \frac{2}{3}\left[(x_1)^{3/2}-(x_0)^{3/2}+(x_2)^{3/2}-(x_1)^{3/2}+\cdots+(x_n)^{3/2}-(x_{n-1})^{3/2}\right]\\
&= \frac{2}{3}\cdot\left(b^{3/2}-a^{3/2}\right).
\end{aligned}$$

Finally, because $f(x)=\sqrt{x}$ is continuous on $[a,\,b]$, all the Riemann sums for f there converge to the same limit. Some of these sums have been shown to converge to $\frac{2}{3}\left(b^{3/2}-a^{3/2}\right)$. Therefore they all converge to that limit, and thus

$$\int_a^b \sqrt{x}\;dx \;=\; \frac{2}{3}\left(b^{3/2}\;-\;a^{3/2}\right).$$

Section 6.1

C06S01.001: With $a = 0$ and $b = 1$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^\star \,\Delta x = \int_0^1 2x\, dx = \Big[\, x^2 \,\Big]_0^1 = 1.$

C06S01.002: With $a = 1$ and $b = 2$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\Delta x}{(x_i^\star)^2} = \int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}.$

C06S01.003: With $a = 0$ and $b = 1$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} (\sin \pi x_i^\star)\,\Delta x = \int_0^1 \sin \pi x\, dx = \left[-\frac{1}{\pi}\cos \pi x\right]_0^1 = \frac{2}{\pi}.$

C06S01.004: With $a = -1$ and $b = 3$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \Big[3\,(x_i^\star)^2 - 1\Big]\,\Delta x = \int_{-1}^{3} (3x^2 - 1)\, dx = \Big[\, x^3 - x \,\Big]_{-1}^{3} = 24.$

C06S01.005: With $a = 0$ and $b = 4$,

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i^\star \sqrt{(x_i^\star)^2 + 9}\;\Delta x = \int_0^4 x\sqrt{x^2+9}\, dx = \left[\frac{1}{3}(x^2+9)^{3/2}\right]_0^4 = \frac{125}{3} - 9 = \frac{98}{3}.$$

C06S01.006: The limit is $\displaystyle\int_2^4 x^2\, dx = \left[\frac{1}{3}x^3\right]_2^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3} \approx 18.666666666667.$

C06S01.007: The limit is $\displaystyle\int_{-1}^{3} (2x-1)\, dx = \Big[\, x^2 - x \,\Big]_{-1}^{3} = 6 - 2 = 4.$

C06S01.008: The limit is $\displaystyle\int_0^4 \sqrt{2x+1}\, dx = \left[\frac{1}{3}(2x+1)^{3/2}\right]_0^4 = 9 - \frac{1}{3} = \frac{26}{3}.$

C06S01.009: The limit is $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{x^2+16}}\, dx = \Big[\sqrt{x^2+16}\,\Big]_{-3}^{0} = 4 - 5 = -1.$

C06S01.010: The limit is $\displaystyle\int_0^{\sqrt{\pi}} x\cos x^2\, dx = \left[\frac{1}{2}\sin x^2\right]_0^{\sqrt{\pi}} = 0 - 0 = 0.$

C06S01.011: With $a = 1$ and $b = 4$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^\star f(x_i^\star)\;\Delta x = \int_1^4 2\pi x f(x)\, dx$. Compare this with Eq. (2) in Section 6.3.

C06S01.012: With $a = -1$ and $b = 1$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^\star)\right]^2\;\Delta x = \int_{-1}^{1} \left[f(x)\right]^2\, dx.$

C06S01.013: With $a = 0$ and $b = 10$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f(x_i^\star)\right]^2}\;\Delta x = \int_0^{10} \sqrt{1 + \left[f(x)\right]^2}\, dx.$

C06S01.014: With $a = -2$ and $b = 3$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi m_i \sqrt{1 + \left[f(m_i)\right]^2}\;\Delta x = \int_{-2}^{3} 2\pi x\sqrt{1 + \left[f(x)\right]^2}\, dx.$$

C06S01.015: $M = \displaystyle\int_0^{100} \frac{1}{5}x\,dx = \left[\frac{1}{10}x^2\right]_0^{100} = 1000 - 0 = 1000$ (grams).

C06S01.016: $M = \displaystyle\int_0^{25} (60 - 2x)\,dx = \Big[60x - x^2\Big]_0^{25} = 875 - 0 = 875$ (grams).

C06S01.017: $M = \displaystyle\int_0^{10} x(10 - x)\,dx = \left[5x^2 - \frac{1}{3}x^3\right]_0^{10} = \frac{500}{3} - 0 = \frac{500}{3}$ (grams).

C06S01.018: $M = \displaystyle\int_0^{10} 10\sin\frac{\pi x}{10}\,dx = \left[-\frac{100}{\pi}\cos\frac{\pi x}{10}\right]_0^{10} = \frac{100}{\pi} - \left(-\frac{100}{\pi}\right) = \frac{200}{\pi}$ (grams).

C06S01.019: The net distance is

$$\int_0^{10} (-32)\,dt = \Big[-32t\Big]_0^{10} = -320$$

and the total distance is 320.

C06S01.020: The net distance is

$$\int_1^5 (2t + 10)\,dt = \Big[t^2 + 10t\Big]_1^5 = 75 - 11 = 64$$

and because $v(t) = 2t + 10 \geqq 0$ for $1 \leqq t \leqq 5$, this is the total distance as well.

C06S01.021: The net distance is

$$\int_0^{10} (4t - 25)\,dt = \Big[2t^2 - 25t\Big]_0^{10} = 200 - 250 = -50.$$

Because $v(t) = 4t - 25 \leqq 0$ for $0 \leqq t \leqq 6.25$ and $v(t) \geqq 0$ for $6.25 \leqq t \leqq 10$, the total distance is

$$-\int_0^{6.25} v(t)\,dt + \int_{6.25}^{10} v(t)\,dt = \frac{625}{8} + \frac{225}{8} = \frac{425}{4} = 106.25.$$

C06S01.022: Because $v(t) \geqq 0$ for $0 \leqq t \leqq 5$, the net and total distance are both equal to

$$\int_0^5 |2t - 5|\,dt = \int_0^{2.5} (5 - 2t)\,dt + \int_{2.5}^5 (2t - 5)\,dt = \frac{25}{4} + \frac{25}{4} = \frac{25}{2} = 12.5.$$

C06S01.023: The net distance is

$$\int_{-2}^3 4t^3\,dt = \Big[t^4\Big]_{-2}^3 = 81 - 16 = 65.$$

Because $v(t) \leqq 0$ for $-2 \leqq t \leqq 0$, the total distance is

$$-\int_{-2}^0 4t^3\,dt + \int_0^3 4t^3\,dt = 16 + 81 = 97.$$

C06S01.024: The net distance is

$$\int_{0.1}^{1}\left(t-\frac{1}{t^2}\right)\,dt=\left[\frac{1}{2}t^2+\frac{1}{t}\right]_{0.1}^{1}=\frac{3}{2}-\frac{2001}{200}=-\frac{1701}{200}=-8.505.$$

Because $v(t) \leqq 0$ for $0.1 \leqq t \leqq 1$, the total distance is 8.505.

C06S01.025: Because $v(t) = \sin 2t \geqq 0$ for $0 \leqq t \leqq \pi/2$, the net distance and the total distance are both equal to

$$\int_0^{\pi/2}\sin 2t\,dt=\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2}=\frac{1}{2}-\left(-\frac{1}{2}\right)=1.$$

C06S01.026: The net distance is

$$\int_0^{\pi/2}\cos 2t\,dt=\left[\frac{1}{2}\sin 2t\right]_0^{\pi/2}=0-0=0.$$

Because $v(t) = \cos 2t \leqq 0$ for $\pi/4 \leqq t \leqq \pi/2$, the total distance is

$$\int_0^{\pi/4}\cos 2t\,dt-\int_{\pi/4}^{\pi/2}\cos 2t\,dt=\frac{1}{2}+\frac{1}{2}=1.$$

C06S01.027: The net distance is

$$\int_{-1}^{1}\cos \pi t\,dt=\left[\frac{1}{\pi}\sin \pi t\right]_{-1}^{1}=0-0=0.$$

Because $v(t) = \cos \pi t \leqq 0$ for $-1 \leqq t \leqq -0.5$ and for $0.5 \leqq t \leqq 1$, the total distance is

$$-\int_{-1}^{-0.5}\cos \pi t\,dt+\int_{-0.5}^{0.5}\cos \pi t\,dt-\int_{0.5}^{1}\cos \pi t\,dt=\frac{1}{\pi}+\frac{2}{\pi}+\frac{1}{\pi}=\frac{4}{\pi}.$$

C06S01.028: The net distance is

$$\int_0^{\pi}(\sin t+\cos t)\,dt=\Big[\sin t-\cos t\Big]_0^{\pi}=1-(-1)=2.$$

But $v(t) = \sin t + \cos t \leqq 0$ for $3\pi/4 \leqq t \leqq \pi$, so the total distance is

$$\int_0^{3\pi/4}(\sin t+\cos t)\,dt-\int_{3\pi/4}^{\pi}(\sin t+\cos t)\,dt=1+\sqrt{2}+\sqrt{2}-1=2\sqrt{2}.$$

C06S01.029: The net distance is

$$\int_0^{10}(t^2-9t+14)\,dt=\left[\frac{1}{3}t^3-\frac{9}{2}t^2+14t\right]_0^{10}=\frac{70}{3}\approx 23.333333,$$

but because $v(t) = t^2 - 9t + 14 \leqq 0$ for $2 \leqq t \leqq 7$, the total distance is

$$\int_0^{2}v(t)\,dt-\int_2^{7}v(t)\,dt+\int_7^{10}v(t)\,dt=\frac{38}{3}+\frac{125}{6}+\frac{63}{2}=65.$$

C06S01.030: The net distance is

$$\int_0^6 (t^3 - 8t^2 + 15t)\ dt = \left[\frac{1}{4}t^4 - \frac{8}{3}t^3 + \frac{15}{2}t^2\right]_0^6 = 18 - 0 = 18.$$

Because $v(t) = t^3 - 8t^2 + 15t < 0$ for $3 \leqq t \leqq 5$, the total distance is

$$\int_0^3 (t^3 - 8t^2 + 15t)\ dt - \int_3^5 (t^3 - 8t^2 + 15t)\ dt + \int_5^6 (t^3 - 8t^2 + 15t)\ dt = \frac{63}{4} + \frac{16}{3} + \frac{91}{12} = \frac{86}{3} \approx 28.666667.$$

C06S01.031: If $v(t) = t^3 - 7t + 4$ for $0 \leqq t \leqq 3$, then $v(t) < 0$ for $\alpha = 0.602705 < t < \beta = 2.29240$ (numbers with decimal points are approximations), so the net distance is

$$\int_0^3 (t^3 - 7t + 4)\ dt = \left[\frac{1}{4}t^4 - \frac{7}{2}t^2 + 4t\right]_0^3 = \frac{3}{4}$$

but the total distance is approximately

$$\int_0^\alpha (t^3 - 7t + 4)\ dt - \int_\alpha^\beta (t^3 - 7t + 4)\ dt + \int_\beta^3 (t^3 - 7t + 4)\ dt \approx 1.17242 + 3.49165 + 3.06923 = 7.73330.$$

C06S01.032: Because $v(t) = t^3 - 5t^2 + 10 < 0$ if $\alpha = 1.175564 < t < \beta = 4.50790$ (numbers with decimal points are approximations), the net distance is

$$\int_0^5 (t^3 - 5t^2 + 10)\ dt = \left[\frac{1}{4}t^4 - \frac{5}{3}t^3 + 10t\right]_0^5 = -\frac{25}{12} \approx -2.083333,$$

but the total distance is

$$\int_0^\alpha (t^3 - 5t^2 + 10)\ dt - \int_\alpha^\beta (t^3 - 5t^2 + 10)\ dt + \int_\beta^5 (t^3 - 5t^2 + 10)\ dt \approx 10.9126 + 15.2724 + +2.27654 \approx 28.4615.$$

C06S01.033: Here, $v(t) = t \sin t - \cos t$ is negative for $0 \leqq t < \alpha = 0.860333589019$ (numbers with decimals are approximations), so the net distance is

$$\int_0^\pi (t \sin t - \cos t)\ dt = \Big[-t\ \cos t\Big]_0^\pi = \pi$$

but the total distance is

$$-\int_0^\alpha (t \sin t - \cos t)\ dt + \int_\alpha^\pi (t \sin t - \cos t)\ dt \approx 0.561096338191 + 3.702688991781 = 4.263785329972.$$

C06S01.034: The velocity $v(t) = \sin t + t^{1/2} \cos t$ is negative for $\alpha = 2.167455 < t < \beta = 5.128225$ (numbers with decimals are approximations), so the net distance is

$$\int_0^{2\pi} (\sin t + t^{1/2} \cos t)\ dt = -0.430408$$

and the total distance is

$$\int_0^{\alpha} (\sin t + t^{1/2} \cos t)\ dt - \int_{\alpha}^{\beta} (\sin t + t^{1/2} \cos t)\ dt + \int_{\beta}^{2\pi} (\sin t + t^{1/2} \cos t)\ dt$$

$$\approx 2.02380 + 4.05657 + 1.60236 = 7.68273.$$

We used *Mathematica's* `NIntegrate` command for all four definite integrals. The numbers α and β were found using Newton's method.

C06S01.035: If n is a large positive integer, $\Delta x = r/n$, $x_i = i \cdot \Delta x$, and $x_i^\star$ is chosen in the interval $[x_{i-1}, x_i]$ for $1 \leqq i \leqq n$, then the area of the annular ring between x_{i-1} and x_i is approximately $2\pi x_i^\star\ \Delta x$ and its average density is approximately $\rho\,(x_i^\star)$, so a good approximation to the total mass of the disk is

$$\sum_{i=1}^{n} 2\pi x_i^\star \rho\,(x_i^\star)\ \Delta x.$$

But this is a Riemann sum for

$$\int_0^r 2\pi x \rho(x)\ dx,$$

and therefore such sums approach this integral as $\Delta x \to 0$ and $n \to +\infty$ because $2\pi x \rho(x)$ is (we presume) continuous for $0 \leqq x \leqq r$. But such Riemann sums also approach the total mass M of the disk and this establishes the equation in Problem 35.

C06S01.036: The mass is $M = \displaystyle\int_0^{10} 2\pi x^2\ dx = \left[\frac{2}{3}\pi x^3\right]_0^{10} = \frac{2000\pi}{3} \approx 2094.395102.$

C06S01.037: The mass is $M = \displaystyle\int_0^{5} 2\pi x(25 - x^2)\ dx = \left[-\frac{1}{2}\pi(x^4 - 50x^2)\right]_0^{5} = \frac{625\pi}{2} \approx 981.747704.$

C06S01.038: The maximum height will be

$$\int_0^5 (160 - 32t)\ dt = \Big[160t - 16t^2\Big]_0^5 = 400 \quad \text{(feet)}.$$

C06S01.039: The amount of water that flows into the tank from time $t = 10$ to time $t = 20$ is

$$\int_{10}^{20} (100 - 3t)\ dt = \left[100t - \frac{3}{2}t^2\right]_{10}^{20} = 1400 - 850 = 550 \quad \text{(gallons)}.$$

C06S01.040: Answer: $\displaystyle\int_0^{20} 1000(16 + t)\ dt = \Big[16000t + 500t^2\Big]_0^{20} = 520000.$

C06S01.041: Answer:

$$375000 + \int_0^{20} \left[(1000(16 + t) - 1000\left(5 + \frac{1}{2}t\right)\right]\ dt = 375000 + \Big[11000t + 250t^2\Big]_0^{20} = 695000.$$

C06S01.042: Let n be a positive integer, let $\mathcal{P} = \{t_0, t_1, t_2, \ldots, t_n\}$ be a partition of the interval $[0, 365]$, let $\Delta t = 365/n$, and let $t_i^\star$ be a point in $[t_{i-1}, t_i]$ for $1 \leqq i \leqq n$. Then the approximate rainfall in Charleston in the time interval $t_{i-1} \leqq t \leqq t_i$ will be $r\,(t_i^\star)\ \Delta t$. Hence the total rainfall in a year will be approximately

$$\sum_{i=1}^{n} r\left(t_i^\star\right)\ \Delta t.$$

But this is a Riemann sum for

$$R = \int_0^{365} r(t)\ dt,$$

and if $r(t)$ is piecewise continuous (I've lived there; I can assure you that $r(t)$ is not continuous, but merely piecewise continuous) then these Riemann sums approach R as a limit (as $n \to +\infty$). Thus R gives the average total annual rainfall in Charleston.

C06S01.043: We solved $r(0) = 0.1$ and $r(182.5) = 0.3$ for $a = 0.2$ and $b = 0.1$. Then we found that

$$\int_0^{365} r(t)\ dt = \left[\frac{73}{4\pi} \cdot \left(\frac{4\pi t}{365} - \sin \frac{2\pi t}{365} \right) \right]_0^{365} = 73$$

inches per year, average annual rainfall.

C06S01.044: Let n be a positive integer, let $\Delta t = (b - a)/n$, let $\mathcal{P} = \{t_0,\, t_1,\, t_2,\, \dots,\, t_n\}$ be a regular partition of $[a,\, b]$, and let $t_i^\star$ be a number in $[t_{i-1},\, t_i]$ for $1 \leqq i \leqq n$. Then the amount of water that flows into the tank in the interval $[t_{i-1},\, t_i]$ will be approximately $r\left(t_i^\star\right)\ \Delta t$. Hence the total amount of water that flows into the tank between times $t = a$ and $t = b$ will be approximately

$$\sum_{i=1}^{n} r\left(t_i^\star\right)\ \Delta t.$$

The error in this approximation will approach zero as $t \to +\infty$, and the sum itself is a Riemann sum for $r(t)$ on $[a,\, b]$, and therefore—if r is piecewise continuous (physical considerations dictate that it must be)—the amount of water that flows into the tank during that interval must be

$$Q = \int_a^b r(t)\ dt.$$

C06S01.045: If $f(x) = x^{1/3}$ on $[0,\, 1]$, n is a positive integer, $\Delta x = 1/n$, and $x_i = x_i^\star = i \cdot \Delta x$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/3}}{n^{4/3}} = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^\star\right)\ \Delta x = \int_0^1 x^{1/3}\ dx = \left[\frac{3}{4} x^{4/3} \right]_0^1 = \frac{3}{4}.$$

C06S01.046: Let r denote the radius of the spherical ball and partition the interval $[0,\, r]$ into n subintervals all having the same length $\Delta x = r/n$. Let $x_i = i \cdot \Delta x$, so that $\mathcal{P} = \{x_0,\, x_1,\, x_2,\, \dots,\, x_n\}$ is a regular partition of $[0,\, r]$. Let $x_i^\star$ be the midpoint of $[x_{i-1},\, x_i]$, so that $\{x_i^\star\}$ is a selection for $\mathcal{P}$. Then the volume of the spherical shell with inner radius x_{i-1} and outer radius x_i will be approximately $4\pi\left(x_i^\star\right)^2\ \Delta x$, so that the total volume of the spherical ball will be closely approximated (if n is large) by

$$\sum_{i=1}^{n} 4\pi\left(x_i^\star\right)^2\ \Delta x.$$

But this sum is a Riemann sum for $f(x) = 4\pi x^2$ on the interval $[0,\, r]$. The error in this approximation to the volume V of the spherical ball will approach zero as $n \to +\infty$, but it also approaches the integral shown next, and therefore

$$V = \int_0^r 4\pi x^2\,dx = \left[\frac{4}{3}\pi x^3\right]_0^r = \frac{4}{3}\pi r^3.$$

C06S01.047: Let n be a large positive integer, let $\Delta x = 1/n$, and let $x_i = i \cdot \Delta x$. Then $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$ is a regular partition of $[0, 1]$. Let $x_i^\star$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then the weight of the spherical shell with inner radius x_{i-1} and outer radius x_i will be approximately

$$100\,(1 + x_i^\star) \cdot 4\pi\,(x_i^\star)^2\ \Delta x$$

(the approximate volume of the shell multiplied by its approximate average density). Therefore the total weight of the ball will be approximately

$$\sum_{i=1}^{n} 100\,(1 + x_i^\star) \cdot 4\pi\,(x_i^\star)^2\ \Delta x.$$

This is a Riemann sum for $f(x) = 100(1 + x) \cdot 4\pi x^2$ on $[0, 1]$, and such sums approach the total weight W of the ball as $n \to +\infty$. Therefore

$$W = \int_0^1 100(1 + x) \cdot 4\pi x^2\,dx = \left[\frac{400}{3}\pi x^3 + 100\pi x^4\right]_0^1 = \frac{700}{3}\pi \approx 733.038286\ \ \text{(pounds)}.$$

C06S01.048: Given $v(x) = k\cos\left(\dfrac{\pi x}{2r}\right)$, the flow rate is

$$\begin{aligned} F &= \int_0^r 2\pi kx\cos\left(\frac{\pi x}{2r}\right)\,dx = \frac{4}{\pi}\cdot\left[2kr^2\cos\left(\frac{\pi x}{2r}\right) + \pi krx\sin\left(\frac{\pi x}{2r}\right)\right]_0^r \\ &= 4kr^2 - \frac{8kr^2}{\pi} = \frac{4k(\pi - 2)r^2}{\pi}. \end{aligned}$$

C06S01.049: Because the pressure P is inversely proportional to the fourth power of the radius r, the values $r = 1.00,\ 0.95,\ 0.90,\ 0.85,\ 0.80,$ and 0.75 yield the values $r^{-4} = 1.00,\ 1.22774,\ 1.52416,\ 1.91569,\ 2.44141,$ and 3.16049. We subtract 1 from each of the latter and then multiply by 100 to obtain *percentage increase* in pressure (and multiply each value of r by 100 to convert to percentages). Result:

100	0.000
95	22.774
90	52.416
85	91.569
80	144.141
75	216.049

C06S01.050: The amount of dye injected is $A = 4$ (in milligrams) and the concentration function is

$$c(t) = \frac{250t}{(3 + t^2)^2}, \quad 0 \leqq t \leqq 10$$

(t in seconds). Then

$$\int_0^{10} c(t)\,dt = \left[-\frac{125}{3 + t^2}\right]_0^{10} = \frac{125}{3} - \frac{125}{103} = \frac{12500}{309}.$$

Thus the patient's cardiac output is approximately

$$F = 60A \cdot \frac{309}{12500} = \frac{3708}{625} = 5.9328$$

liters per minute (the factor of 60 in the second expression converts seconds to minutes). Our experience with experimental measurements of biological phenomena suggests that, for the sake of honesty, the answer should be rounded to 5.9 L/min.

C06S01.051: Given: $A = 4.5$ (mg). Simpson's rule applied to the concentration data

$$c = 0, \quad 2.32, \quad 9.80, \quad 10.80, \quad 7.61, \quad 4.38, \quad 2.21, \quad 1.06, \quad 0.47, \quad 0.18, \quad 0.0$$

measured at the times $t = 0,\ 1,\ 2,\ \ldots,\ 10$ (seconds) yields

$$\begin{aligned}\int_0^{10} c(t)\,dt \approx S_{10} &= \frac{1}{3}\big[1 \cdot 0 + 4 \cdot (2.32) + 2 \cdot (9.8) + 4 \cdot (10.8) + 2 \cdot (7.61) + 4 \cdot (4.38)\\ &\qquad + 2 \cdot (2.21) + 4 \cdot (1.06) + 2 \cdot (0.47) + 4 \cdot (0.18) + 1 \cdot 1\big] = \frac{1919}{50} = 38.38.\end{aligned}$$

Hence—multiplying by 60 to convert second to minutes—the cardiac output is approximately

$$F \approx \frac{60A}{38.38} \approx 7.0349$$

L/min (liters per minute).

The *Mathematica* code is straightforward, although you must remember that an array's initial subscript is 1 rather than zero (unless you decree it otherwise):

```
c = {0, 232/100, 98/10, 108/10, 761/100, 438/100, 221/100, 106/100, 47/100, 18/100, 0};

c[[1]] + c[[11]] + 4*Sum[c[[i]], {i, 2, 10, 2}] + 2*Sum[c[[i]], {i, 3, 9, 2}]
```

$$\frac{5757}{50}$$

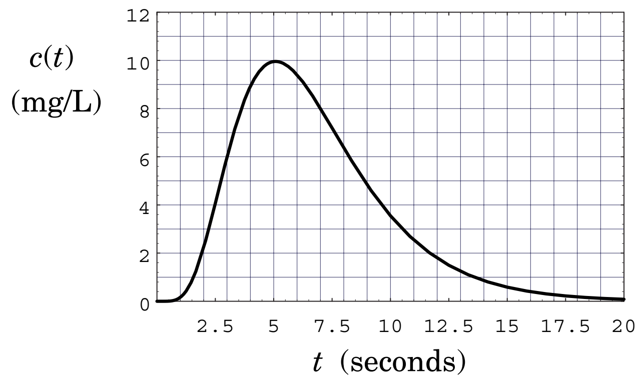
```
N[ 5757/(3*50), 12 ]
```

38.3800000000

```
N[ (60*45/10)/(3838/100), 12 ]
```

7.03491401772

C06S01.052: Here we have $A = 5.5$, $a = 200$, $b = 8$, $c = 7.1$, $d = 0.5$, and $f(t) = at^b \exp(-ct^d)$ yielding the graph of concentration (in mg/L) as a function of t (in seconds). Result:



Next,

$$I = \int_0^{20} f(t)\,dt \approx 67.4709,$$

so the cardiac output of the patient is

$$F \approx \frac{60A}{I} \approx 4.891$$

liters per minute.

Section 6.2

C06S02.001: The volume is $V = \displaystyle\int_0^1 \pi x^4\,dx = \left[\frac{1}{5}\pi x^5\right]_0^1 = \frac{\pi}{5}$.

C06S02.002: The volume is $V = \displaystyle\int_0^4 \pi x\,dx = \left[\frac{1}{2}\pi x^2\right]_0^4 = 8\pi - 0 = 8\pi$.

C06S02.003: The volume is $V = \displaystyle\int_0^4 \pi y\,dy = \left[\frac{1}{2}\pi y^2\right]_0^4 = 8\pi$.

C06S02.004: The volume is $V = \displaystyle\int_{0.1}^1 \frac{\pi}{x^2}\,dx = \left[-\frac{\pi}{x}\right]_{0.1}^1 = -\pi - (-10\pi) = 9\pi$.

C06S02.005: The volume is $V = \displaystyle\int_0^\pi \pi\sin^2 x\,dx = \pi\int_0^\pi \frac{1-\cos 2x}{2}\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi = \frac{1}{2}\pi^2$.

C06S02.006: The volume is $V = \displaystyle\int_{-3}^3 \pi(9-x^2)^2\,dx = \pi\left[\frac{1}{5}x^5 - 6x^3 + 81x\right]_{-3}^3 = \frac{1296\pi}{5} \approx 814.300816$.

C06S02.007: Rotation of the given figure around the x-axis produces annular rings of inner radius $y = x^2$ and outer radius $y = \sqrt{x}$ (because $x^2 \leqq \sqrt{x}$ if $0 \leqq x \leqq 1$). Hence the volume of the solid is

$$V = \int_0^1 \pi\left[\left(\sqrt{x}\right)^2 - x^4\right]\,dx = \pi\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{10}\pi.$$

C06S02.008: Rotation of the given region around the vertical line $x = 5$ produces annular rings with inner radius $5 - \sqrt{y}$ and outer radius $5 - \frac{1}{4}y$. Hence the volume of the solid is

$$V = \int_0^{16} \pi\left[\left(5 - \frac{1}{4}y\right)^2 - \left(5 - \sqrt{y}\right)^2\right]\,dy = \pi\int_0^{16}\left(10y^{1/2} - \frac{7}{2}y + \frac{1}{16}y^2\right)\,dy$$
$$= \frac{\pi}{48}\left[320y^{3/2} - 84y^2 + y^3\right]_0^{16} = 64\pi - 0 = 64\pi \approx 201.062.$$

C06S02.009: Rotation of the region between the given curves around the x-axis produces annular rings with outer radius $8 - x^2$ and inner radius x^2, so the volume of the solid is

$$V = \int_{-2}^2 \pi\left[(8-x^2)^2 - x^4\right]\,dx = \pi\int_{-2}^2 (64 - 16x^2)\,dx$$
$$= \pi\left[64x - \frac{16}{3}x^3\right]_{-2}^2 = \frac{512\pi}{3} \approx 536.1651462126580460309578.$$

C06S02.010: Rotation of the region between the given curves around the y-axis produces annular rings with outer radius $y+6$ and inner radius y^2; solution of the equation $y+6 = y^2$ yields the limits of integration $y = -2$ and $y = 3$. Hence the volume of the solid in question is

$$V = \int_{-2}^{3} \pi \left[(y+6)^2 - y^4 \right] dy = \pi \int_{-2}^{3} (36 + 12y + y^2 - y^4)\, dy$$

$$= \pi \left[36y + 6y^2 + \frac{1}{3}y^3 - \frac{1}{5}y^5 \right]_{-2}^{3} = \frac{612\pi}{5} - \left(-\frac{664\pi}{15} \right) = \frac{500\pi}{3} \approx 523.598776.$$

C06S02.011: Volume: $V = \displaystyle\int_{-1}^{1} \pi(1-x^2)^2\, dx = \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^{1} = \frac{16\pi}{15} \approx 3.351032.$

C06S02.012: Volume: $V = \displaystyle\int_{0}^{1} \pi(x-x^3)^2\, dx = \pi \left[\frac{1}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7 \right]_{0}^{1} = \frac{8\pi}{105} \approx 0.2393594403.$

C06S02.013: A horizontal cross section "at" y has radius $x = \sqrt{1-y}$, so the volume of the solid is

$$\int_0^1 \pi(1-y)\, dy = \pi \left[y - \frac{1}{2}y^2 \right]_0^1 = \frac{\pi}{2}.$$

C06S02.014: The curves meet where $x = \pm 2$ and rotation of the region bounded by them around the x-axis produces annular regions with inner radius 2 and outer radius $6 - x^2$. Hence the volume of the solid is

$$V = \int_{-2}^{2} \pi \left[(6-x^2)^2 - 4 \right] dx = \pi \int_{-2}^{2} (x^4 - 12x^2 + 32)\, dx$$

$$= \pi \left[\frac{1}{5}x^5 - 4x^3 + 32x \right]_{-2}^{2} = \frac{192\pi}{5} - \left(-\frac{192\pi}{5} \right) = \frac{384\pi}{5} \approx 241.274315795696120713931.$$

C06S02.015: The region between the two curves extends from $y = 2$ to $y = 6$ and the radius of a horizontal cross section "at" y is $x = \sqrt{6-y}$. Therefore the volume of the solid is

$$V = \int_2^6 \pi(6-y)\, dy = \pi \left[6y - \frac{1}{2}y^2 \right]_2^6 = 18\pi - 10\pi = 8\pi.$$

C06S02.016: The region bounded by the two curves extends from $y = 0$ to $y = 1$. When it is rotated around the vertical line $x = 2$, it generates annular regions with outer radius $2 + \sqrt{1-y}$ and inner radius $2 - \sqrt{1-y}$. Hence the volume of the solid is

$$V = \int_0^1 \pi \left[\left(2 + \sqrt{1-y}\right)^2 - \left(2 - \sqrt{1-y}\right)^2 \right] dy = \pi \int_0^1 8(1-y)^{1/2}\, dy$$

$$= \pi \left[-\frac{16}{3}(1-y)^{3/2} \right]_0^1 = 0 - \left(-\frac{16\pi}{3} \right) = \frac{16\pi}{3} \approx 16.755161.$$

C06S02.017: When the region bounded by the given curves is rotated around the horizontal line $y = -1$, it generates annular regions with outer radius $x - x^3 + 1$ and inner radius 1. Hence the volume of the solid thereby generated is

$$V = \int_0^1 \pi\left[(x - x^3 + 1)^2 - 1^2\right]\,dx = \pi\int_0^1 (2x + x^2 - 2x^3 - 2x^4 + x^6)\,dx$$

$$= \pi\left[x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 - \frac{2}{5}x^5 + \frac{1}{7}x^7\right]_0^1 = \frac{121\pi}{210} - 0 = \frac{121\pi}{210} \approx 1.8101557671.$$

C06S02.018: We assume that the region intended is the one in the first quadrant bounded by the given curves (the volume is the same if the region in the second quadrant is used, so it doesn't really matter). When this region is rotated around the x-axis, it generates annular regions with outer radius 4 and inner radius x^2. Hence the volume of the solid thereby generated is

$$V = \int_0^2 \pi(16 - x^4)\,dx = \pi\left[16x - \frac{1}{5}x^5\right]_0^2 = \frac{128\pi}{5} \approx 80.4247719318987069.$$

C06S02.019: When the region bounded by the given curves is rotated around the y-axis, it generates circular regions; the one "at" y has radius $x = \sqrt{y}$, so the volume generated is

$$V = \int_0^4 \pi y\,dy = \pi\left[\frac{1}{2}y^2\right]_0^4 = 8\pi.$$

C06S02.020: When the region bounded by the given curves is rotated around the x-axis, it generates circular regions; the one "at" x has radius $y = \sqrt{16 - x}$, so the volume generated is

$$V = \int_0^{16} \pi(16 - x)\,dx = \pi\left[16x - \frac{1}{2}x^2\right]_0^{16} = 128\pi.$$

C06S02.021: When the region bounded by the given curves is rotated around the horizontal line $y = -2$, it generates annular regions with outer radius $\sqrt{x} - (-2)$ and inner radius $x^2 - (-2)$. Hence the volume of the solid thereby generated is

$$V = \int_0^1 \pi\left[\left(\sqrt{x} + 2\right)^2 - (2 + x^2)^2\right]\,dx = \pi\int_0^1 (4x^{1/2} + x - 4x^2 - x^4)\,dx$$

$$= \pi\left[\frac{8}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{49\pi}{30} \approx 5.1312680009.$$

C06S02.022: When the region bounded by the given curves is rotated around the horizontal line $y = -1$, it generates annular regions with outer radius $8 - x^2 + 1$ and inner radius $x^2 + 1$, and therefore the volume of the solid thus generated is

$$V = \int_{-2}^2 \pi\left[\,(9 - x^2)^2 - (x^2 + 1)^2\,\right]\,dx = 2\pi\int_0^2 (80 - 20x^2)\,dx$$

$$= 2\pi\left[80x - \frac{20}{3}x^3\right]_0^2 = \frac{640\pi}{3} \approx 670.206433.$$

C06S02.023: When the region bounded by the given curves is rotated around the vertical line $x = 3$, it generates annular regions with outer radius $3 - y^2$ and inner radius $3 - \sqrt{y}$. Hence the volume of the solid thereby generated is

$$V = \int_0^1 \pi\left[(3-y^2)^2 - (3-\sqrt{y}\,)^2\right]\,dy = \pi\int_0^1 (6y^{1/2} - y - 6y^2 + y^4)\,dy$$

$$= \pi\left[4y^{3/2} - \frac{1}{2}y^2 - 2y^3 + \frac{1}{5}y^5\right]_0^1 = \frac{17\pi}{10} \approx 5.340708.$$

C06S02.024: When the region bounded by the given curves is rotated around the vertical line $x = 4$, it generates annular regions with locations varying from $y = 0$ to $y = 8$. If $0 \leqq y \leqq 4$ then the outer radius of such an annulus is $4+\sqrt{y}$ and its inner radius is $4-\sqrt{y}$. The top half of the solid generated (for $4 \leqq y \leqq 8$) is the mirror image of the bottom half, thus we find the volume of the solid thereby generated by doubling the volume of the bottom half. So the total volume of the solid is

$$V = 2\int_0^4 \pi\left[(4+\sqrt{y}\,)^2 - (4-\sqrt{y}\,)^2\right]\,dy = 2\pi\int_0^4 16y^{1/2}\,dy$$

$$= 2\pi\left[\frac{32}{3}y^{3/2}\right]_0^4 = \frac{512\pi}{3} \approx 536.1651462126580460309578.$$

As a check of the validity of this answer, and as a glance ahead to Section 6.3, the method of nested cylindrical shells proceeds as follows. A typical shell with centerline $x = 4$ and located "at" position x has radius $4-x$ and height $8-2x^2$. To fill the solid, such shells are located between $x = -2$ and $x = 2$, and thus the total volume of the solid is

$$V = \int_{-2}^2 2\pi(4-x)(8-2x^2)\,dx = \pi\int_{-2}^2 (4x^3 - 16x^2 - 16x + 64)\,dx$$

$$= \pi\left[x^4 - \frac{16}{3}x^3 - 8x^2 + 64x\right]_{-2}^2 = \frac{208\pi}{3} + \frac{304\pi}{3} = \frac{512\pi}{3}.$$

C06S02.025: The volume generated by rotation of R around the x-axis is

$$V = \int_0^\pi \pi\sin^2 x\,dx = \pi\int_0^\pi \frac{1-\cos 2x}{2}\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^\pi = \frac{\pi^2}{2} \approx 4.9348022005.$$

C06S02.026: The volume is

$$V = \int_{-1}^1 \pi\cos^2\frac{\pi x}{2}\,dx = \pi\int_{-1}^1 \frac{1+\cos \pi x}{2}\,dx = \pi\left[\frac{1}{2}x + \frac{1}{2}\sin \pi x\right]_{-1}^1 = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

C06S02.027: The curves $y = \cos x$ and $y = \sin x$ cross at $\pi/4$ and the former is above the latter for $0 \leqq x < \pi/4$. So the region between them, when rotated around the x-axis, generates annular regions with outer radius $\cos x$ and inner radius $\sin x$. Therefore the volume of the solid generated will be

$$V = \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x)\,dx = \pi\int_0^{\pi/4} \cos 2x\,dx = \pi\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{\pi}{2}.$$

C06S02.028: The curve $y = \cos x$ and the horizontal line $y = \frac{1}{2}$ cross where $x = -\frac{1}{3}\pi$ and where $x = \frac{1}{3}\pi$, and the curve is above the line between those two points. So when the region they bound is rotated around

the x-axis, it generates annular regions with outer radius $\cos x$ and inner radius $\frac{1}{2}$. So the volume of the solid thereby generated will be

$$V = \int_{-\pi/3}^{\pi/3} \pi \left(\cos^2 x \, - \, \frac{1}{4} \right) dx = \pi \int_{-\pi/3}^{\pi/3} \frac{1 + 2\cos 2x}{4} \, dx = \pi \left[\frac{1}{4} (x + \sin 2x) \right]_{-\pi/3}^{\pi/3}$$

$$= \frac{\pi}{24} \left(3\sqrt{3} + 2\pi \right) - \left[-\frac{\pi}{24} \left(3\sqrt{3} + 2\pi \right) \right] = \frac{\pi}{12} \left(3\sqrt{3} + 2\pi \right) \approx 3.0052835900.$$

C06S02.029: The volume is

$$V = \int_0^{\pi/4} \pi \tan^2 x \, dx = \pi \int_0^{\pi/4} \frac{\sin^2 x}{\cos^2 x} \, dx = \pi \int_0^{\pi/4} \frac{1 - \cos^2 x}{\cos^2 x} \, dx$$

$$= \pi \int_0^{\pi/4} (\sec^2 x - 1) \, dx = \pi \Big[(\tan x) - x \Big]_0^{\pi/4} = \frac{\pi(4 - \pi)}{4} \approx 0.6741915533.$$

C06S02.030: When the region bounded by the given curves is rotated around the x-axis, it produces annular regions with outer radius 1 and inner radius $\tan x$. Hence the volume of the solid generated is

$$V = \int_0^{\pi/4} \pi \left(1 - \tan^2 x \right) \, dx = \pi \int_0^{\pi/4} \left(1 - \sec^2 x + 1 \right) \, dx$$

$$= \pi \Big[2x - \tan x \Big]_0^{\pi/4} = \frac{\pi(\pi - 2)}{2} \approx 1.7932095470.$$

C06S02.031: The two curves cross near $a = -0.532089$, $b = 0.652704$, and $c = 2.87939$. When the region between the curves for $a \leqq x \leqq b$ is rotated around the x-axis, the solid it generates has approximate volume

$$V_1 = \int_a^b \pi \left[(x^3 + 1)^2 - (3x^2)^2 \right] \, dx \approx 1.68838 - (-1.39004) = 2.99832.$$

When the region between the curves for $b \leqq x \leqq c$ is rotated around the x-axis, the solid it generates has approximate volume

$$V_2 = \int_b^c \pi \left[(3x^2)^2 - (x^3 + 1)^2 \right] \, dx \approx 265.753 - (-1.68838) \approx 267.442.$$

The total volume generated is thus $V_1 + V_2 \approx 270.440$.

C06S02.032: The two curves cross near $a = -1.28378$ and $b = 1.53375$. When the region between the curves is rotated around the x-axis, the volume of the solid it generates is

$$V = \int_a^b \pi \left[(x + 4)^2 - (x^4)^2 \right] \, dx = \pi \left[16x + 4x^2 + \frac{1}{3}x^3 - \frac{1}{9}x^9 \right]_a^b \approx 94.0394 - (-42.7288) \approx 136.768.$$

C06S02.033: The two curves cross near $a = -0.8244962453$ and $b = 0.8244962453$. When the region between them is rotated around the x-axis, the volume of the solid it generates is

$$V \approx \int_a^b \pi \left[(\cos^2 x) - x^4 \right] \, dx = \frac{\pi}{20} \left[10x - 4x^5 + 5\sin 2x \right]_a^b \approx 1.83871 - (-1.83871) \approx 3.67743.$$

C06S02.034: The two curves cross near $a = 0.3862368706$ and $b = 1.9615690350$. When the region between them is rotated around the x-axis, the volume of the solid generated is approximately

$$\begin{aligned} V &= \int_a^b \pi\left[\sin^2 x - (x-1)^4\right]\,dx \\ &= \frac{\pi}{20}\Big[40x^2 - 10x - 40x^3 + 20x^4 - 4x^5 - 5\sin 2x\Big]_a^b \approx 2.48961 - (-0.51503) \approx 3.00464. \end{aligned}$$

C06S02.035: The two curves cross at the two points where $x = 6$, and the one on the right meets the x-axis at $(3, 0)$. When the region they bound is rotated around the x-axis, the volume of the solid it generates is

$$V = \int_0^6 \pi x\,dx - \int_3^6 2\pi(x-3)\,dx = \pi\left[\frac{1}{2}x^2\right]_0^6 - \pi\Big[x^2 - 6x\Big]_3^6 = (18\pi - 0) - (0 + 9\pi) = 9\pi.$$

C06S02.036: The top half of the ellipse is the graph of the function

$$f(x) = \frac{b}{a}\sqrt{a^2 - x^2}, \qquad -a \leqq x \leqq a.$$

Hence when the region bounded above by the graph of $y = f(x)$ and below by the x-axis is rotated around the x-axis, the volume of the ellipsoid thereby swept out will be

$$V = \int_{-a}^a \pi\,[\,f(x)\,]^2\,dx = \pi\left[\,\frac{b^2x(3a^2 - x^2)}{3a^2}\,\right]_{-a}^a = \frac{4}{3}\pi ab^2.$$

C06S02.037: The right half of the ellipse is the graph of the function

$$g(y) = \frac{a}{b}\sqrt{b^2 - y^2}, \qquad -b \leqq y \leqq b.$$

Hence when the region bounded on the right by the graph of $x = g(y)$ and on the left by the y-axis is rotated around the y-axis, the volume of the ellipsoid thereby swept out will be

$$V = \int_{-b}^b \pi\,[\,g(y)\,]^2\,dy = \pi\left[\,\frac{a^2y(3b^2 - y^2)}{3b^2}\,\right]_{-a}^a = \frac{4}{3}\pi a^2 b.$$

C06S02.038: The volume of the part of the solid from $x = 1$ to $x = b$ is

$$V(b) = \int_1^b \frac{\pi}{x^4}\,dx = \pi\left[-\frac{1}{3x^3}\right]_1^b = \frac{\pi}{3}\left(1 - \frac{1}{b^3}\right).$$

Hence the volume of the unbounded solid is $\displaystyle\lim_{b\to\infty} V(b) = \frac{\pi}{3}$.

C06S02.039: Locate the base of the observatory in the xy-plane with the center at the origin and the diameter AB on the x-axis. Then the boundary of the base has equation $x^2 + y^2 = a^2$. A typical vertical cross-section of the observatory has as *its* base a chord of that circle perpendicular to the x-axis at [say] x, so that the length of this chord is $2\sqrt{a^2 - x^2}$. The square of this length gives the area of that vertical cross section, and it follows that the volume of the observatory is

$$V = \int_{-a}^a 4(a^2 - x^2)\,dx = \left[\frac{4}{3}(3a^2x - x^3)\right]_{-a}^a = \frac{8}{3}a^3 - \left(-\frac{8}{3}a^2\right) = \frac{16}{3}a^3.$$

C06S02.040: Locate the base of the solid in the xy-plane with the center at the origin and the diameter AB on the x-axis. Then the boundary of the base has equation $x^2+y^2=a^2$. A typical vertical cross-section of the solid has as *its* base a chord of that circle perpendicular to the x-axis at [say] x, so that the length of this chord is $2\sqrt{a^2-x^2}$. This chord is the diameter of that semicircular cross section, which therefore has radius $\sqrt{a^2-x^2}$ and thus area $\frac{1}{2}\pi(a^2-x^2)$. Hence the volume of the solid is

$$\int_{-a}^{a}\frac{\pi}{2}(a^2-x^2)\,dx=\pi\left[\frac{x(3a^2-x^2)}{6}\right]_{-a}^{a}=\frac{2}{3}\pi a^3,$$

exactly as expected, for after all the solid is a hemisphere of radius a.

C06S02.041: Locate the base of the solid in the xy-plane with the center at the origin and the diameter AB on the x-axis. Then the boundary of the base has equation $x^2+y^2=a^2$. A typical vertical cross-section of the base has as *its* base a chord of that circle perpendicular to the x-axis at [say] x, so that the length of this chord is $2\sqrt{a^2-x^2}$. This chord is one of the three equal sides of the vertical cross section—which is an equilateral triangle—and it follows that this triangle has

$$\text{Base:}\qquad b=2\sqrt{a^2-x^2}\qquad\text{and height:}\qquad h=\frac{\sqrt{3}}{2}b=\sqrt{3a^2-3x^2}.$$

So the area of this triangle is $\frac{1}{2}bh=\left(\sqrt{3}\right)(a^2-x^2)$, and therefore the volume of the solid is

$$\int_{-a}^{a}\left(\sqrt{3}\right)(a^2-x^2)\,dx=\left[\frac{\sqrt{3}}{3}(3a^2x-x^3)\right]_{-a}^{a}=\frac{4\sqrt{3}}{3}a^3.$$

C06S02.042: A cross section of the solid perpendicular to the x-axis at x has base of length $\sqrt{x}-x^2$, so its area is $\left(\sqrt{x}-x^2\right)^2=x-2x^{5/2}+x^4$. So the volume of the solid is

$$\int_0^1\left(x-2x^{5/2}+x^4\right)\,dx=\left[\frac{1}{2}x^2-\frac{4}{7}x^{7/2}+\frac{1}{5}x^5\right]_0^1=\frac{9}{70}.$$

C06S02.043: The volume of the paraboloid is

$$V_p=\int_0^h 2\pi px\,dx=\left[\pi px^2\right]_0^h=\pi ph^2.$$

If r is the radius of the cylinder, then the equation $y^2=2px$ yields $r^2=2ph$, so the volume of the cylinder is $V_c=\pi r^2h=2\pi ph^2=2V_p$.

C06S02.044: Consider a cross section of the pyramid parallel to its base and at distance x from the vertex of the pyramid. Similar triangles show that the length y and width z of this cross section are proportional to x, so that its area is $f(x)=kx^2$. Moreover, $f(h)=A$, so that $kh^2=A$, and therefore $k=A/(h^2)$. Thus the total volume of the pyramid will be

$$V=\int_0^h f(x)\,dx=\int_0^h\frac{A}{h^2}\cdot x^2\,dx=\left[\frac{Ax^3}{3h^2}\right]_0^h=\frac{Ah^3}{3h^2}=\frac{1}{3}Ah.$$

C06S02.045: Consider a cross section of the pyramid parallel to its base and at distance x from the vertex of the pyramid. Similar triangles show that the lengths of the edges of this triangular cross section are proportional to x. If the edges have lengths p, q, and r, then Heron's formula tells us that the area of this triangular cross section is

$$\frac{1}{4}\sqrt{(p+q+r)(p+q-r)(p-q+r)(q+r-p)}\ ,$$

and thus the area $g(x)$ of this triangular cross section is proportional to x^2; that is, $g(x) = kx^2$ for some constant k. But $g(h) = A$, which tells us that $kh^2 = A$ and thus that $k = A/(h^2)$. So the total volume of the pyramid will be

$$V = \int_0^h g(x)\ dx = \int_0^h \frac{A}{h^2}\cdot x^2\ dx = \left[\frac{Ax^3}{3h^2}\right]_0^h = \frac{Ah^3}{3h^2} = \frac{1}{3}Ah.$$

C06S02.046: Set up a coordinate system in which the origin is at the center of the sphere and the sphere-with-hole is symmetric around the y-axis. Then a horizontal cross section of the solid "at" location y is an annular region with inner radius 3 and outer radius $x = \sqrt{25 - y^2}$. Therefore the volume of the sphere-with-hole is

$$V = \int_{-4}^5 \pi(25 - y^2 - 9)\ dy = 2\pi\left[\frac{1}{3}(48y - y^3)\right]_0^4 = \frac{256\pi}{3} \approx 268.082573.$$

As an independent check, the volume of the sphere-*without*-hole is about 523.598776.

C06S02.047: Set up a coordinate system in which one cylinder has axis the x-axis and the other has axis the y-axis. Introduce a z-axis perpendicular to the xy-plane and passing through (0, 0). We will find the volume of the eighth of the intersection that lies in the *first octant,* where x, y, and z are nonnegative, then multiply by 8 to find the answer.

A cross section of the eighth perpendicular to the z-axis (thus parallel to the xy-plane) is a square; if this cross section meets the z-axis at z, then one of its edges lies in the yz-plane and reaches from the z-axis (where $y = 0$) to the side of the cylinder symmetric around the x-axis. That cylinder has equation the same as the equation as the circle in which it meets the yz-plane: $y^2 + z^2 = a^2$. Hence the edge of the square under consideration has length $y = \sqrt{a^2 - z^2}$. So the area of the square cross section "at" z is $a^2 - z^2$. So the volume of the eighth of the solid in the first octant is

$$\int_0^a (a^2 - z^2)\ dz = \left[a^2 z - \frac{1}{3}z^3\right]_0^a = a^3 - \frac{1}{3}a^3 = \frac{2}{3}a^3.$$

Therefore the total volume of the intersection of the two cylinders is $\dfrac{16}{3}a^3$.

As an independent check, it's fairly easy to see that the sphere of radius a centered at the origin is enclosed in the intersection and occupies most of the volume of the intersection; the ratio of the volume of the intersection to the volume of that sphere is

$$\frac{\frac{16}{3}a^3}{\frac{4}{3}\pi a^3} = \frac{4}{\pi} \approx 1.273240,$$

a very plausible result.

C06S02.048: Set up a coordinate system in which the center of the sphere is located at the origin and the flat part of the spherical segment is perpendicular to the y-axis. Now consider a cross section of the spherical segment, lying between $y = r - h$ and $y = r$ on the y-axis, and "at" location y. This cross section is a circular disk of radius $x = \sqrt{r^2 - y^2}$, so its has cross-sectional area $\pi(r^2 - y^2)$. Therefore the volume of the spherical segment will be

$$V(h) = \int_{r-h}^{r} \pi(r^2 - y^2)\ dy = \pi \left[r^2 y - \frac{1}{3}y^3 \right]_{r-h}^{r} = \pi \left[\frac{2}{3}r^3 - \frac{1}{3}(r-h)(3r^2 - (r-h)^2) \right] = \frac{1}{3}\pi h^2 (3r - h).$$

Note that the answer is dimensionally correct (the product of three lengths; see page 156 of the text). It also meets the "test of extremes," known in the past as "the exception that proves the rule," which in modern translation would be "the exceptional case that *tests* the rule." Specifically, $V(0) = 0$, $V(r) = \frac{2}{3}\pi r^3$, and $V(2r) = \frac{4}{3}\pi r^3$, exactly as should be the case.

C06S02.049: The cross section of the torus perpendicular to the y-axis "at" y is an annular ring with outer radius $x = b + \sqrt{a^2 - y^2}$ and inner radius $x = b - \sqrt{a^2 - y^2}$, a consequence of the fact that the circular disk that generates the torus has equation $(x - b)^2 + y^2 = a^2$. So the cross section has area

$$\pi \left[\left(b + \sqrt{a^2 - y^2} \right)^2 - \left(b - \sqrt{a^2 - y^2} \right)^2 \right] = 4\pi b \sqrt{a^2 - y^2}.$$

Therefore the volume of the torus is

$$V = 4\pi b \int_{-a}^{a} \sqrt{a^2 - y^2}\ dy = 4\pi b \cdot \frac{1}{2}\pi a^2$$

because the integral is the area of a semicircle of radius a centered at the origin. Therefore $V = 2\pi^2 a^2 b$.

C06S02.050: Simpson's approximation is

$$S_4 = \frac{25\pi}{3} \cdot \left(60^2 + 4 \cdot 55^2 + 2 \cdot 50^2 + 4 \cdot 35^2 + 0^2 \right) = \frac{640000\pi}{3} \approx 670206.433.$$

C06S02.051: First, r is the value of $y = R - kx^2$ when $x = \frac{1}{2}h$, so $r = R - \frac{1}{4}kh^2 = R - \delta$ where $4\delta = kh^2$. Next, the volume of the barrel is

$$\begin{aligned}
V &= 2\int_0^{h/2} \pi(R - kx^2)^2\ dx = 2\pi \int_0^{h/2} \left(R^2 - 2Rkx^2 + k^2x^4 \right)\ dx \\
&= 2\pi \left[R^2 x - \frac{2}{3}Rkx^3 + \frac{1}{5}k^2x^5 \right]_0^{h/2} = 2\pi \left(\frac{1}{2}R^2 h - \frac{1}{12}Rkh^3 + \frac{1}{160}k^2h^5 \right) \\
&= \pi h \left(R^2 - \frac{1}{6}Rkh^2 + \frac{1}{80}k^2h^4 \right) = \pi h \left(R^2 - \frac{1}{6}R \cdot 4\delta + \frac{1}{80} \cdot 16\delta^2 \right) \\
&= \pi h \left(R^2 - \frac{2}{3}R\delta + \frac{1}{5}\delta^2 \right) = \frac{\pi h}{3} \cdot \left(3R^2 - 2R\delta + \frac{3}{5}\delta^2 \right) \\
&= \frac{\pi h}{3} \cdot \left(2R^2 + R^2 - 2R\delta + \delta^2 - \frac{2}{5}\delta^2 \right) = \frac{\pi h}{3} \cdot \left(2R^2 + (R - \delta)^2 - \frac{2}{5}\delta^2 \right) = \frac{\pi h}{3} \cdot \left(2R^2 + r^2 - \frac{2}{5}\delta^2 \right).
\end{aligned}$$

C06S02.052: Suppose that the depth of water in the clepsydra is h. A horizontal cross section of the water "at" y, $0 \leqq y \leqq h$, is a circular disk of radius $x = (y/k)^{1/4}$, so the total volume of water in the clepsydra will be

$$V(h) = \int_0^h \pi \cdot \frac{y^{1/2}}{k^{1/2}}\ dy = \frac{2\pi h^{3/2}}{3k^{1/2}}.$$

Thus if the depth of water is y, we find that its volume is

$$V(y) = \frac{2\pi y^{3/2}}{3k^{1/2}}.$$

But even without this computation the fundamental theorem of calculus tells us that

$$\frac{dV}{dy} = \frac{\pi}{k^{1/2}} \cdot y^{1/2}.$$

Therefore—using Torricelli's law—

$$-cy^{1/2} = \frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = \frac{\pi}{k^{1/2}} \cdot y^{1/2} \cdot \frac{dy}{dt},$$

and now it follows that

$$\frac{dy}{dt} = -\frac{c\sqrt{k}}{\pi},$$

which is a constant. That is, the water level falls at a constant rate. To use the clepsydra as a clock (as it was used in ancient Egypt, Greece, and Rome), put a ruler vertically in the clepsydra. The ancient Egyptians were so sophisticated with these clocks that they had different rulers for different water temperatures, as the less viscous warm water would run out at a slightly greater (but still constant) rate. According to the *American Heritage Dictionary of the English Language* (Boston: Houghton Mifflin, 1969, 1970), the name of the device is derived from the Greek word *klepsudra,* "water stealer" (from the "stealthy" flow of the water), from the words *kleptein,* "to steal" (an English relative is *kleptomania*) and *hudōr*, "water" (an English relative is *hydrosphere*).

C06S02.053: The factors 27 and 3.3 in the following computations convert cubic feet to cubic yards and cubic yards to dollars. The trapezoidal approximation gives

$$T_6 = \frac{(10)(3.3)}{(2)(27)} \cdot (1513 + 2 \cdot 882 + 2 \cdot 381 + 2 \cdot 265 + 2 \cdot 151 + 2 \cdot 50 + 0) \approx 3037.83$$

and Simpson's approximation gives

$$S_6 = \frac{(10)(3.3)}{(3)(27)} \cdot (1513 + 4 \cdot 882 + 2 \cdot 381 + 4 \cdot 265 + 2 \cdot 151 + 4 \cdot 50 + 0) \approx 3000.56.$$

To the nearest hundred dollars, each answer rounds to \$3000.

C06S02.054: Please don't give this solution away, particularly to a differential equations student. Note first that if $V(t)$ is the volume of water in the bowl at time t, then $dV/dt = -k \cdot A(t)$ where k is a positive constant. Let $y(t)$ be the depth of water in the bowl at time t. Consider the time interval $[t,\, t+\Delta t]$. Assume that the water level drops by the amount Δy. Then the change in the volume of water in the bowl is

$$\Delta V \;\approx\; A(t)\,\Delta y$$

where $A(t)$ is the area of the water surface at time t. Thus

$$\frac{\Delta V}{\Delta t} \;\approx\; A(t) \cdot \frac{\Delta y}{\Delta t},$$

and if we let $\Delta t \to 0$, the error in this approximation will approach zero; therefore

$$\begin{aligned} A(t)\cdot\frac{dy}{dt} &= \frac{dV}{dt};\\ A(t)\cdot\frac{dy}{dt} &= -kA(t);\\ \frac{dy}{dt} &= -k. \end{aligned}$$

That is, the water level in the bowl drops at a constant rate.

C06S02.055: First "finish" the frustum; that is, complete the cone of which it is a frustum. We measure all distances from the vertex of the completed cone and perpendicular to the bases of the frustum. Let H be the height of the cone and suppose that $H - h \leqq y \leqq H$, so that a cross section of the cone at distance y from its vertex is a circular cross section of the frustum. The area $A(y)$ of such a cross section is proportional to the square of its radius, which is proportional to y^2, so that $A(y) = ky^2$ where k is a positive proportionality constant. By similar triangles,

$$\frac{H-h}{r} = \frac{H}{R}, \qquad \text{and so} \qquad H = \frac{hR}{R-r}.$$

Also, $A(H) = kH^2 = \pi R^2$, and it follows that

$$k = \frac{\pi R^2}{H^2}, \qquad \text{so that} \qquad A(y) = \frac{\pi R^2 y^2}{H^2}.$$

Therefore the volume of the frustum is

$$V = \int_{H-h}^{H} \frac{\pi R^2 y^2}{H^2}\,dy = \frac{\pi R^2}{3H^2}\cdot\Big(H^3 - (H-h)^3\Big).$$

Next note that $A(H-h) = \pi r^2 = \dfrac{\pi R^2}{H^2}(H-h)^2$. Therefore

$$\begin{aligned} V &= \frac{\pi R^2}{3H^2}\big(3H^2h - 3Hh^2 + h^3\big) = \frac{\pi R^2}{3h^2R^2}(R-r)^2\left(\frac{3h^3R^2}{(R-r)^2} - \frac{3h^3R}{R-r} + h^3\right)\\ &= \frac{\pi}{3h^2}\Big(3h^3R^2 - 3h^3R(R-r) + h^3(R-r)^2\Big) = \frac{\pi h}{3}\Big(3R^2 - 3R(R-r) + (R-r)^2\Big)\\ &= \frac{\pi h}{3}(3R^2 - 3R^2 + 3rR + R^2 - 2Rr + r^2) = \frac{\pi h}{3}(R^2 + Rr + r^2). \end{aligned}$$

C06S02.056: The solid is formed by rotating around the x-axis the plane region above the x-axis and common to the two circular disks bounded by the two circles with equations

$$\left(x + \tfrac{1}{2}a\right)^2 + y^2 = a^2 \qquad \text{and} \qquad \left(x - \tfrac{1}{2}a\right)^2 + y^2 = a^2.$$

Let R denote the half of that region that lies in the first quadrant. Then the solid of Problem 56 has volume V double that obtained when R is revolved around the x-axis. The curve that forms the upper boundary of R has equation

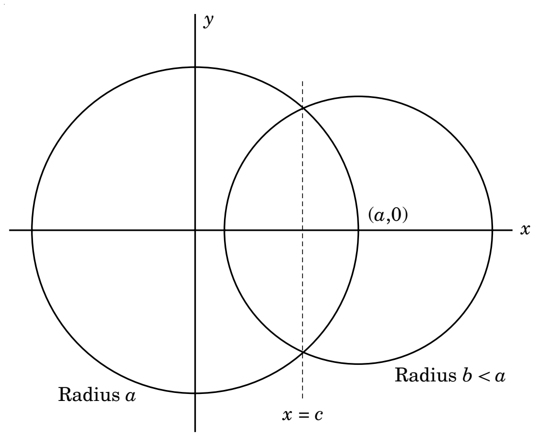
$$y = f(x) = \frac{1}{2}\sqrt{3a^2 - 4ax - 4x^2}\,,$$

and therefore

$$V = 2\int_0^{a/2} \pi\,[f(x)]^2\;dx = 2\pi\left[\frac{3}{4}a^2x - \frac{1}{2}ax^2 - \frac{1}{3}x^3\right]_0^{a/2} = \frac{5}{12}\pi a^3.$$

Note that the answer is dimensionally correct. Moreover, the solid occupies 31.25% of the volume of either sphere, and thus the answer also passes the test of plausibility (see page 155 of the text).

C06S02.057: The solid of intersection of the two spheres can be generated by rotating around the x-axis the region R common to the two circles shown in the following figure.



The circle on the left, of radius a and centered at the origin, has equation $x^2 + y^2 = a^2$. The circle on the right, of radius $b < a$ and centered at $(a, 0)$, has equation $(x-a)^2 + y^2 = b^2$. The x-coordinate of their two points of intersection can be found by solving

$$x^2 - a^2 \;=\; (x-a)^2 - b^2 \qquad \text{for} \qquad x = \frac{2a^2 - b^2}{2a} = c.$$

The part of R above the x-axis is comprised of two smaller regions, one to the left of the vertical line $x = c$ and one to its right. The width of the region R_1 on the left—measured along the x-axis—is

$$h_1 = c - (a-b) = \frac{2a^2 - b^2}{2a} - a + b = \frac{b(2a-b)}{2a}.$$

The width of the region R_2 on the right is

$$h_2 = a - c = \frac{b^2}{2a}.$$

One formula will tell us the volume of the solid generated by rotation of either R_1 or R_2 around the x-axis. Suppose that C is the circle of radius r centered at the origin and that $0 < h < r$. Let us find the volume generated by rotation of the region above the x-axis, within the circle, and to the right of the vertical line $x = r - h$ around the x-axis. It is

$$V(r,\,h) \;=\; \int_{r-h}^{r} \pi(r^2 - x^2)\;dx \;=\; \pi\left[r^2x - \frac{1}{3}x^3\right]_{r-h}^{r} \;=\cdots=\; \frac{\pi h^2}{3}(3r - h)$$

(we worked this problem by hand—it is not difficult, merely tedious—but checked our results with *Mathematica* 3.0). Hence the volume of intersection of the two original spheres is

$$V(b,\,h_1) + V(a,\,h_2) \;=\cdots=\; \frac{\pi b^3}{12a}(8a - 3b)$$

(also by hand, tedious but not difficult, and checked with *Mathematica*).

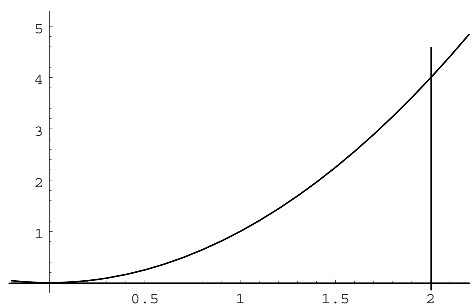
C06S02.058: As $n \to \infty$, $x_{i-1} \to x_i$. Continuity of f then implies that $f(x_{i-1}) \to f(x_i)$, so that

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{3} \left\{ [f(x_{i-1})]^2 + f(x_{i-1})f(x_i) + [f(x_i)]^2 \right\} \Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{3} \cdot 3\,[f(x_i)]^2 \Delta x \\ &= \int_a^b \pi\,[f(x)]^2\, dx. \end{aligned}$$

Section 6.3

C06S03.001: The region R, bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$, is shown next. If we rotate R around the y-axis, then the vertical strip in R "at" x will move around a circle of radius x and the height of the strip will be $y = x^2$, so the volume generated will be

$$\int_0^2 2\pi x^3\,dx = 2\pi\left[\frac{1}{4}x^4\right]_0^2 = 8\pi \approx 25.1327412287.$$



C06S03.002: The region R to be rotated around the y-axis is bounded above by the graph of $y = \sqrt{x}$ and below by the graph of $y = -\sqrt{x}$. Hence the vertical strip "at" x has height $2\sqrt{x}$ and is rotated around a circle of radius x, so the volume swept out by R is

$$V = \int_0^4 4\pi x^{3/2}\,dx = \left[\frac{8}{5}\pi x^{5/2}\right]_0^4 = \frac{256\pi}{5} \approx 160.8495438638.$$

C06S03.003: To obtain all the cylindrical shells, we need only let x range from 0 to 5, so the volume of the solid is

$$V = \int_0^5 2\pi x(25 - x^2)\,dx = \pi\left[25x^2 - \frac{1}{2}x^4\right]_0^5 = \frac{625\pi}{2} \approx 981.7477042468.$$

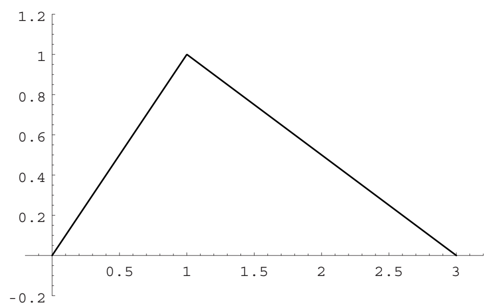
C06S03.004: The volume is $\displaystyle\int_0^2 2\pi x(8 - 2x^2)\,dx = 2\pi\left[4x^2 - \frac{1}{2}x^4\right]_0^2 = 16\pi \approx 50.2654824574.$

C06S03.005: The volume is $\displaystyle\int_0^2 2\pi x(8 - 2x^2)\,dx = 2\pi\left[4x^2 - \frac{1}{2}x^4\right]_0^2 = 16\pi \approx 50.2654824574.$

C06S03.006: The volume is $\displaystyle\int_0^3 2\pi y(9 - y^2)\,dy = \pi\left[9y^2 - \frac{1}{2}y^4\right]_0^3 = \frac{81\pi}{2} \approx 127.2345024704.$

C06S03.007: A horizontal strip of the region R (shown next) "at" y stretches from $x = y$ to $x = 3 - 2y$ and thus has length $3 - 3y$. Hence the volume swept out by rotation of R around the x-axis is

$$V = \int_0^1 2\pi y(3 - 3y)\,dy = \pi\left[3y^2 - 2y^3\right]_0^1 = \pi \approx 3.1415926536.$$



C06S03.008: The horizontal strip of the region R "at" y moves around a circle of radius $5 - y$ and has length $y^{1/2} - \frac{1}{2}y$, so the volume swept out by R is

$$V = \int_0^4 2\pi(5-y)\left(y^{1/2} - \frac{1}{2}y\right)\,dy = 2\pi\left[\frac{10}{3}y^{3/2} - \frac{5}{4}y^2 - \frac{2}{5}y^{5/2} + \frac{1}{6}y^3\right]_0^4$$

$$= 2\pi\left(\frac{80}{3} - 20 - \frac{64}{5} + \frac{32}{3}\right) = \frac{136\pi}{15} \approx 28.4837733925.$$

C06S03.009: A horizontal strip of the region R "at" y $(0 \leqq y \leqq 2)$ stretches from $y^2/4$ to $x = (y/2)^{1/2}$ and is rotated around a circle of radius y, so the volume swept out when R is rotated around the x-axis is

$$V = \int_0^4 2\pi y\left[\left(\frac{y}{2}\right)^{1/2} - \frac{y^2}{4}\right]\,dy = 2\pi\int_0^4\left(\frac{\sqrt{2}}{2}y^{3/2} - \frac{1}{4}y^3\right)\,dy$$

$$= 2\pi\left[\frac{\sqrt{2}}{5}y^{5/2} - \frac{1}{16}y^4\right]_0^2 = \frac{6\pi}{5} \approx 3.7699111843.$$

C06S03.010: The volume is $\displaystyle\int_0^3 2\pi x(3x - x^2)\,dx = 2\pi\left[x^3 - \frac{1}{4}x^4\right]_0^3 = \frac{27\pi}{2} \approx 42.4115008235.$

C06S03.011: The graph of $y = 4x - x^3$ crosses the x-axis at $x = -2$, $x = 0$, and $x = 2$. It is below the x-axis for $-2 < x < 0$ and above it for $0 < x < 2$. A vertical strip "at" x for $0 \leqq x \leqq 2$ has height $4x - x^3$ and moves around a circle of radius x, whereas a vertical strip "at" x for $-2 \leqq x \leqq 0$ has height $-(4x - x^3)$ and moves around a circle of radius $-x$. Therefore the total volume swept out when the given region is rotated around the y-axis is

$$V = \int_0^2 2\pi x(4x - x^3)\,dx + \int_{-2}^0(-2\pi x)(x^3 - 4x)\,dx = \int_{-2}^2 2\pi x(4x - x^3)\,dx$$

$$= 2\pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^2 = \frac{256\pi}{15} \approx 53.6165146213.$$

C06S03.012: A horizontal strip "at" y of the given region has length $x = y^3 - y^4$ and is rotated around a circle of radius $y - (-2) = y + 2$, so the volume swept out when this region is rotated around the horizontal line $y = -2$ is

$$V = \int_0^1 2\pi(y+2)(y^3 - y^4)\,dy = 2\pi\left[\frac{1}{2}y^4 - \frac{1}{5}y^5 - \frac{1}{6}y^6\right]_0^1 = \frac{4\pi}{15} \approx 0.8377580410.$$

C06S03.013: The volume is $\displaystyle\int_0^1 2\pi x(x - x^3)\,dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{4\pi}{15} \approx 0.8377580410.$

C06S03.014: The volume is $\displaystyle\int_0^4 2\pi y(16 - y^2)\,dy = 2\pi\left[8y^2 - \frac{1}{4}y^4\right]_0^4 = 128\pi \approx 402.1238596595.$

C06S03.015: A vertical strip "at" x of the given region has height $y = x - x^3$ and moves around a circle of radius $2 - x$, so the volume the region sweeps out is

$$V = \int_0^1 2\pi(2-x)(x - x^3)\,dx = 2\pi\left[x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{11\pi}{15} \approx 2.3038346126.$$

C06S03.016: The volume is $\displaystyle\int_0^2 2\pi x \cdot x^3\,dx = 2\pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{64\pi}{5} \approx 40.2123859659.$

C06S03.017: If $0 \leqq x \leqq 2$, then a vertical strip of the given region "at" x has height x^3 and moves around a circle of radius $3 - x$, so the volume swept out is

$$V = \int_0^2 2\pi(3-x)x^3\,dx = 2\pi\left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^2 = \frac{56\pi}{5} \approx 35.1858377202.$$

C06S03.018: If $0 \leqq y \leqq 8$, then a horizontal strip of the given region "at" y has length $2 - y^{1/3}$ and moves around a circle of radius y, so the volume generated by rotation of that region around the x-axis is

$$V = \int_0^8 2\pi y(2 - y^{1/3})\,dy = 2\pi\left[y^2 - \frac{3}{7}y^{7/2}\right]_0^8 = 2\pi\left(64 - \frac{384}{7}\right) = \frac{128\pi}{7} \approx 57.4462656656.$$

C06S03.019: If $-1 \leqq x \leqq 1$, then a vertical strip of the given region "at" x has height x^2 and moves around a circle of radius $2 - x$, so the volume generated by rotating the given region around the vertical line $x = 2$ is

$$V = \int_{-1}^1 2\pi(2-x)x^2\,dx = 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{-1}^1 = 2\pi\left(\frac{5}{12} + \frac{11}{12}\right) = \frac{8\pi}{3} \approx 8.3775804096.$$

C06S03.020 The volume is $\displaystyle\int_0^1 2\pi x(x - x^2)\,dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6} \approx 0.5235987756.$

C06S03.021: The volume is $\displaystyle\int_0^1 2\pi y(y^{1/2} - y)\,dy = 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = \frac{2\pi}{15} \approx 0.4188790205.$

C06S03.022: The volume is $\displaystyle\int_0^1 2\pi(2-y)(y^{1/2} - y)\,dy = 2\pi\int_0^1 (2y^{1/2} - 2y - y^{3/2} + y^2)\,dy$

$$= 2\pi\left[\frac{4}{3}y^{3/2} - y^2 - \frac{2}{5}y^{5/2} + \frac{1}{3}y^3\right]_0^1 = 2\pi\left(\frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3}\right) = \frac{8\pi}{15} \approx 1.6755160819.$$

C06S03.023: If $0 \leqq x \leqq 1$, then a vertical strip of the given region "at" x has height $x - x^2$ and moves around a circle of radius $x-(-1) = x+1$, so the volume swept out by rotating the region around the vertical line $x = -1$ is

$$V = \int_0^1 2\pi(x+1)(x-x^2)\,dx = 2\pi \int_0^1 (x-x^3)\,dx = 2\pi\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{2} \approx 1.5707963268.$$

C06S03.024: The given region R is symmetric around the x-axis, so the volume it generates in rotating around the x-axis can be found by rotating the top half of R around the x-axis. If $0 \leqq y \leqq 1$, then a horizontal strip of R "at" y has length $2-2y^2$ and moves around a circle of radius y, so the volume generated by R is

$$V = \int_0^1 2\pi y(2-2y^2)\,dy = 2\pi\left[y^2 - \frac{1}{2}y^4\right]_0^1 = \pi \approx 3.14159265358979323846.$$

You may wish to contrast this solution with the one using the methods of Section 6.2 (and thereby check this answer). Using symmetry of the given region around the vertical line $x = 1$, the result is

$$V = 2\int_1^2 \pi(2-x)\,dx = 2\pi\left[2x - \frac{1}{2}x^2\right]_1^2 = 4\pi - 3\pi = \pi.$$

C06S03.025: A horizontal strip of the given region "at" y (where $-1 \leqq y \leqq 1$) has length $2-2y^2$ and moves around a circle of radius $1-y$, so the volume generated is

$$\begin{aligned} V &= \int_{-1}^1 2\pi(1-y)(2-2y^2)\,dy = 2\pi\int_{-1}^1 (2-2y-2y^2+2y^3)\,dy \\ &= 2\pi\left[2y - y^2 - \frac{2}{3}y^3 + \frac{1}{2}y^4\right]_{-1}^1 = 2\pi\left(\frac{5}{6}+\frac{11}{6}\right) = \frac{16\pi}{3} \approx 16.7551608191455639384674 3. \end{aligned}$$

C06S03.026: The volume is

$$V = \int_0^4 2\pi x(4x-x^2)\,dx = 2\pi\left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3} \approx 134.0412865531645115.$$

The methods of Section 6.2 are somewhat unwieldy in this problem. They lead in a natural way to

$$\begin{aligned} V &= \pi\int_0^4 \left[\left(2+\sqrt{4-y}\right)^2 - \left(2-\sqrt{4-y}\right)^2\right]dy \\ &= \pi\int_0^4 8\sqrt{4-y}\,dy = \left[-\frac{16}{3}\pi(4-y)^{3/2}\right]_0^4 = 0 + \frac{128\pi}{3} = \frac{128\pi}{3}. \end{aligned}$$

C06S03.027: If $0 \leqq x \leqq 4$, then a vertical strip of the given region "at" x has height $4x - x^2$ and is rotated around a circle of radius $x+1$, so the volume generated is

$$V = \int_0^4 2\pi(x+1)(4x-x^2)\,dx = 2\pi\left[2x^2 + x^3 - \frac{1}{4}x^4\right]_0^4 = 64\pi \approx 201.0619298297467672616.$$

C06S03.028: The volume is

$$\int_0^1 2\pi(y+1)(y^{1/2}-y^2)\,dy \;=\; 2\pi\left[\frac{2}{3}y^{3/2}+\frac{2}{5}y^{5/2}-\frac{1}{3}y^3-\frac{1}{4}y^4\right]_0^1 \;=\; \frac{29\pi}{30} \;\approx 3.0368728984701334638 47.$$

The methods of Section 6.2 lead in a natural way to the volume integral

$$V \;=\; \pi\int_0^1 (2x^{1/2}+x-2x^2-x^4)\,dx \;=\; \frac{\pi}{30}\left[40x^{3/2}+15x^2-20x^3-6x^5\right]_0^1 \;=\; \frac{29\pi}{30}.$$

C06S03.029: The curves cross to the right of the y-axis at the points $x = a = 0.17248$ and $x = b = 1.89195$ (numbers with decimals are approximations). The quadratic is above the cubic if $a < x < b$, so the volume of the region generated by rotation of R around the y-axis is

$$V = \int_a^b 2\pi x(6x-x^2-x^3-1)\,dx = 2\pi\left[-\frac{1}{5}x^5-\frac{1}{4}x^4+2x^3-\frac{1}{2}x^2\right]_a^b \approx 23.2990983139.$$

C06S03.030: The curves cross at $x = a = 0.506586$ and $x = b = 1.95208$ (numbers with decimals are approximations). The linear function is above the quartic for $a < x < b$, so the volume generated by rotation of r around the y-axis is

$$V = \int_a^b 2\pi x(10x-5-x^4)\,dx = 2\pi\left[\frac{10}{3}x^3-\frac{5}{2}x^2-\frac{1}{6}x^6\right]_a^b \approx 39.3184699459.$$

C06S03.031: We used Newton's method to find that the two curves cross at $x = a = -0.8241323123$ and $x = b = -a$. But because the region R is symmetric around the y-axis, the interval of integration must be $[0,\,b]$. Also $\cos x \geqq x^2$ on this interval, so the volume generated by rotation of R around the y-axis is

$$\int_0^b 2\pi x(\cos x\,-x^2)\,dx = 2\pi\left[\cos x\,+x\sin x\,-\frac{1}{4}x^4\right]_0^b \approx 1.0602688478.$$

C06S03.032: The two curves clearly cross where $x = 0$. We used Newton's method to find that they also cross where $x = b = 1.4055636328$ (numbers with decimals are approximations). Because $\cos x \geqq (x-1)^2$ on $[0,\,b]$, the volume generated by rotation of R around the y-axis is

$$\int_0^b 2\pi x[\cos x\,-(x-1)^2]\,dx = 2\pi\left[\cos x+x\sin x\,-\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{1}{2}x^2\right]_0^b \approx 2.7556103644.$$

C06S03.033: We used Newton's method to find that the two curves cross where $x = a = 0.1870725959$ and $x = b = 1.5758806791$. Because $\cos x \geqq 3x^2-6x+2$ on $[a,\,b]$, the volume generated by rotation of R around the y-axis is

$$V = \int_a^b 2\pi x(\cos x\,-3x^2+6x-2)\,dx = 2\pi\left[\cos x\,+x\sin x\,-x^2+2x^3-\frac{3}{4}x^4\right]_a^b \approx 8.1334538068.$$

C06S03.034: The region R is symmetric around the y-axis and the two curves cross to the right of the y-axis where $x = b = 1.7878717268$ (numbers with decimal points are approximations). Because $3\cos x \geqq -\cos 4x$ for $0 < x < b$, the volume generated when R is rotated around the y-axis is

$$V = \int_0^b 2\pi x(3\cos x + \cos 4x)\,dx = 2\pi\left[3\cos x + \frac{1}{16}\cos 4x + 3x\sin x + \frac{1}{4}x\sin 4x\right]_0^b \approx 12.0048972158.$$

C06S03.035: The slant side of the cone is the graph of

$$f(x) = h - \frac{hx}{r} \quad \text{for} \quad 0 \leqq x \leqq r.$$

Therefore the volume of a cone of radius r and height h is

$$V = \int_0^r 2\pi x f(x)\,dx = 2\pi\left[\frac{1}{2}hx^2 - \frac{hx^3}{3r}\right]_0^r = \frac{1}{3}\pi r^2 h.$$

C06S03.036: The volume of the paraboloid is

$$V = \int_0^{\sqrt{2ph}} 2\pi y\left(h - \frac{y^2}{2p}\right)dy = 2\pi\left[\frac{1}{2}hy^2 - \frac{y^4}{8p}\right]_0^{\sqrt{2ph}} = 2\pi\left(ph^2 - \frac{1}{2}ph^2\right) = \pi ph^2.$$

C06S03.037: The top half of the ellipse is the graph of

$$y = f(x) = \frac{b}{a}(a^2 - x^2)^{1/2}, \quad -a \leqq x \leqq a.$$

The height of a vertical cross section of the ellipse "at" the number x is therefore $2f(x)$, so the volume of the ellipsoid will be

$$V = \int_0^a 4\pi x f(x)\,dx = \frac{4\pi b}{a}\int_0^a x(a^2 - x^2)^{1/2}\,dx = \frac{4\pi b}{a}\left[-\frac{1}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{4\pi b}{3a}\cdot a^3 = \frac{4}{3}\pi a^2 b.$$

Note the lower limit of integration: zero, not $-a$.

C06S03.038: Place the sphere of radius r with its center at the origin in the vertical xy-plane, so that the sphere intersects the xy-plane in the circle with equation $x^2 + y^2 = r^2$. We will find the volume of the spherical segment that occupies the region for which $r - h \leqq y \leqq r$ where $0 \leqq h \leqq 2r$. Note first that if the horizontal line $y = r - h$ meets the circle at the point where $x = a > 0$, then $a^2 + (r - h)^2 = r^2$, and it follows that $a = \sqrt{2rh - h^2}$. So if x is between 0 and a, then the vertical strip above x from $y = r - h$ to the top of the sphere (where $y = \sqrt{r^2 - x^2}$) has length

$$(r^2 - x^2)^{1/2} - (r - h)$$

and will be rotated around a circle of radius x. Hence the volume of the spherical segment is

$$\begin{aligned}
V &= 2\pi\int_0^a x\left[\sqrt{r^2 - x^2} - (r - h)\right]\,dx = \frac{\pi}{3}\left[(r^2 - x^2)\left(3(r - h) - 2\sqrt{r^2 - x^2}\right)\right]_0^a \\
&= \frac{\pi}{3}\left[(r^2 - 2rh + h^2)\left(3(r - h) - 2\sqrt{r^2 - 2rh + h^2}\right) - r^2\big(3(r - h) - 2r\big)\right] \\
&= \frac{\pi}{3}\left[(r - h)^2\big(3(r - h) - 2(r - h)\big) - 3r^2(r - h) + 2r^3\right] = \frac{\pi}{3}\left[(r - h)^3 - 3r^3 + 3r^2h + 2r^3\right] \\
&= \frac{\pi}{3}(r^3 - 3r^2h + 3rh^2 - h^3 - r^3 + 3r^2h) = \frac{\pi}{3}(3rh^2 - h^3) = \frac{\pi h^2}{3}(3r - h).
\end{aligned}$$

C06S03.039: The torus is generated by rotating the circular disk D in the xy-plane around the y-axis; the boundary of D has equation $(x-b)^2 + y^2 = a^2$ where $0 < a \leqq b$. Thus D has its center at $(b, 0)$ on the positive x-axis. If $b - a \leqq x \leqq b + a$, then a vertical slice through D at x has height $2\sqrt{a^2 - (x-b)^2}$, and so the volume of the torus is

$$V = \int_{b-a}^{b+a} 2\pi x \cdot 2\sqrt{a^2 - (x-b)^2}\ dx.$$

The substitution $u = x - b$, with $x = u + b$ and $dx = du$, transforms this integral into

$$V = \int_{-a}^{a} 2\pi(u+b) \cdot 2\sqrt{a^2 - u^2}\ du = 4\pi \int_{-a}^{a} \left[u(a^2-u^2)^{1/2} + b(a^2-u^2)^{1/2} \right] du$$

$$= 4\pi \left[-\frac{1}{3}(a^2-u^2)^{3/2} \right]_{-a}^{a} + 4\pi b \int_{-a}^{a} \sqrt{a^2-u^2}\ du = 0 + 4\pi b \cdot \frac{1}{2}\pi a^2 = 2\pi^2 a^2 b$$

because the last integral represents the area of a semicircle of radius a.

C06S03.040: The two curves meet at $(-1, 1)$ and $(2, 4)$ and the graph of the linear equation is above that of the quadratic for $-1 < x < 2$. If a vertical strip of the region R they bound, above the point x on the x-axis, is rotated around the vertical line $x = -2$, then it has height $x + 2 - x^2$ and moves around a circle of radius $x + 1$. Hence the volume of the solid generated in part (a) is

$$V = \int_{-1}^{2} 2\pi(x+2)(x+2-x^2)\ dx = 2\pi \int_{-1}^{2} (4 + 4x - x^2 - x^3)\ dx$$

$$= 2\pi \left[4x + 2x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_{-1}^{2} = 2\pi\left(\frac{28}{3} + \frac{23}{12} \right) = \frac{45\pi}{2} \approx 70.6858347058.$$

But if that vertical strip is rotated around the vertical line $x = 3$, then the radius of its circular path is now $3 - x$, and therefore the volume of the solid generated in part (b) is

$$V = \int_{-1}^{2} 2\pi(3-x)(x+2-x^2)\ dx = 2\pi \int_{-1}^{2} (6 + x - 4x^2 + x^3)\ dx$$

$$= 2\pi \left[6x + \frac{1}{2}x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \right]_{-1}^{2} = 2\pi\left(\frac{22}{3} + \frac{47}{12} \right) = \frac{45\pi}{2} \approx 70.6858347058.$$

The equality of the answers in parts (a) and (b) is merely a coincidence.

C06S03.041: If $-a \leqq x \leqq a$, then the vertical cross section through the disk "at" x has length $2\sqrt{a^2 - x^2}$ and moves through a circle of radius $a - x$, so the volume of the so-called *pinched torus* the disk generates is

$$V = \int_{-a}^{a} 2\pi(a-x) \cdot 2(a^2-x^2)^{1/2}\ dx$$

$$= 4\pi a \int_{-a}^{a} (a^2-x^2)^{1/2}\ dx - 4\pi \left[-\frac{1}{3}(a^2-x^2)^{3/2} \right]_{-a}^{a} = 4\pi a \cdot \frac{1}{2}\pi a^2 + 0 = 2\pi^2 a^3.$$

The value of the last integral is $\frac{1}{2}\pi a^2$ because it represents the area of a semicircle of radius a.

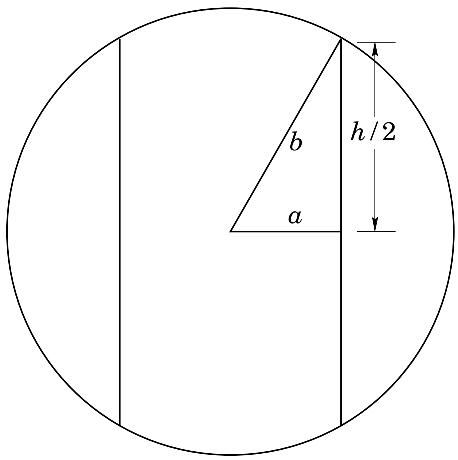
C06S03.042: First, $D_x(\sin x - x\cos x) = \cos x - \cos x + x \sin x = x \sin x$. Therefore

$$\int x \sin x \, dx = \sin x - x \cos x + C.$$

Next, the volume of the solid of part (b) is

$$V = \int_0^{\pi} 2\pi x \sin x \, dx = 2\pi \Big[\sin x - x \cos x \Big]_0^{\pi} = 2\pi \cdot \pi = 2\pi^2 \approx 19.7392088021787172376689 82.$$

C06S03.043: The next figure shows the central cross section of the sphere-with-hole; the radius of the hole is a, its height is h, and the radius of the sphere is b.



The figure shows that $\frac{1}{2}h = \sqrt{b^2 - a^2}$, and substitution in the volume formula $V = \frac{4}{3}\pi(b^2 - a^2)^{3/2}$ of Example 2 yields

$$V = \frac{4}{3}\pi \cdot \left(\frac{h}{2}\right)^3 = \frac{\pi}{6}h^3,$$

quite independent of the values of a or b.

C06S03.044: The method of cross sections yields volume

$$V = \int_0^{16} \pi(25 - y - 9)\, dy = \pi \left[16y - \frac{1}{2}y^2 \right]_0^{16} = 128\pi \approx 402.1238596595.$$

The method of cylindrical shells yields volume

$$V = \int_3^5 2\pi x(25 - x^2)\, dx = 2\pi \left[\frac{25}{2}x^2 - \frac{1}{4}x^4 \right]_3^5 = \frac{625\pi}{2} - \frac{369\pi}{2} = 128\pi.$$

C06S03.045: (a) The method of cross sections yields volume

$$V = \int_0^5 \pi x(5 - x)^2\, dx = \frac{\pi}{12}\left[3x^4 - 40x^3 + 150x^2 \right]_0^5 = \frac{625\pi}{12} \approx 163.6246173745.$$

(b) The method of cylindrical shells yields volume

$$V = \int_0^5 4\pi x^{3/2}(5-x)\,dx = 2\pi\left[4x^{5/2} - \frac{4}{7}x^{7/2}\right]_0^5 = \frac{400\pi\sqrt{5}}{7} \approx 401.4179846309.$$

(c) The method of cylindrical shells yields volume

$$V = \int_0^5 4\pi x^{1/2}(5-x)^2\,dx = 4\pi\left[\frac{50}{3}x^{3/2} - 4x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^5 = \frac{1600\pi\sqrt{5}}{21} \approx 535.2239795079.$$

C06S03.046: (a) The method of cross sections yields volume

$$V = \int_{-3}^0 \pi x^2(x+3)\,dx = \pi\left[x^3 + \frac{1}{4}x^4\right]_{-3}^0 = \frac{27\pi}{4} \approx 21.2057504117.$$

(b) In the method of parallel slabs, we must note that the radius of the vertical strip "at" x is $-x$ and that $\sqrt{x^2} = -x$ because $x \leqq 0$. Thus the volume is

$$V = \int_{-3}^0 (-2\pi x)(-2x)(x+3)^{1/2}\,dx = 4\pi\int_{-3}^0 x^2(x+3)^{1/2}\,dx.$$

The substitution $u = x+3$ converts the last integral into

$$\begin{aligned} V &= 4\pi\int_0^3 (u-3)^2u^{1/2}\,du = 4\pi\int_0^3 (9u^{1/2} - 6u^{3/2} + u^{5/2})\,du \\ &= 4\pi\left[6u^{3/2} - \frac{12}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^3 = 4\pi\cdot\frac{144\sqrt{3}}{35} = \frac{576\pi\sqrt{3}}{35} \approx 89.5498657542. \end{aligned}$$

(c) The method of cylindrical shells yields volume

$$V = \int_{-3}^0 2\pi(x+3)(-2x)(x+3)^{1/2}\,dx = -4\pi\int_{-3}^0 x(x+3)^{3/2}\,dx.$$

Then the substitution $u = x+3$ transforms the last integral into

$$\begin{aligned} V &= -4\pi\int_0^3 (u-3)u^{3/2}\,du = -4\pi\int_0^3 (u^{5/2} - 3u^{3/2})\,du \\ &= -4\pi\left[\frac{2}{7}u^{7/2} - \frac{6}{5}u^{5/2}\right]_0^3 = \frac{432\pi\sqrt{3}}{35} \approx 67.1623993156. \end{aligned}$$

C06S03.047: Given

$$f(x) = 1 + \frac{x^2}{5} - \frac{x^4}{500} \quad \text{and} \quad g(x) = \frac{x^4}{10000},$$

the curves cross where $f(x) = g(x)$. The *Mathematica* command

```
Solve[ f[x] == g[x], x ]
```

returns two complex conjugate solutions and two real solutions, $x = \pm 10$. Hence the volume of the solid obtained by rotating the region between the two curves around the y-axis can be computed in this way (we include extra steps for the reader's benefit):

```
Integrate[ 2*Pi*x*(f[x] - g[x]), x ]
```

$$\pi x^2 + \frac{\pi x^4}{10} - \frac{7\pi x^6}{10000}$$

```
(% /.  x → 10) − (% /.  x → 0)
```

$$400\pi$$

```
N[%, 20]
```

$$1256.6370614359172954$$

To find the volume of water the birdbath will hold when full, we need to find the highest points on the graph of $y = f(x)$.

```
Solve[ D[ f[x], x] == 0, x ]
```

$$\{\{ x \to 0 \}, \{ x \to -5\sqrt{2} \}, \{ x \to 5\sqrt{2} \}\}$$

```
f[ 5*Sqrt[2] ]
```

$$6$$

Thus the amount of water the birdbath will hold can be found as follows:

```
Integrate[ 2*Pi*x*(6 - f[x]), x ]
```

$$5\pi x^2 - \frac{\pi x^4}{10} + \frac{\pi x^6}{1500}$$

```
(% /.  x → 5*Sqrt[2]) - (% /.  x → 0)
```

$$\frac{250\pi}{3}$$

```
N[%, 20]
```

$$261.79938779914943654$$

Section 6.4

Note: In problems 1–20, we will also provide the exact answer (when the antiderivative is elementary) and an approximation to the exact answer (in every case). This information is not required of students working these problems, but it provides an opportunity for extra practice for them in techniques of integration (after they complete Chapter 7) and in numerical integration (which most of them have already completed). The approximations are correct (or correctly rounded) to the number of decimal places shown.

C06S04.001: The length is $\displaystyle\int_0^1 \sqrt{1+4x^2}\ dx = \frac{1}{4}\left(2\sqrt{5} + \sinh^{-1} 2\right) \approx 1.4789428575.$

C06S04.002: The length is $\displaystyle\int_1^3 \frac{1}{2}(4+25x^3)^{1/2}\ dx \approx 14.7554.$ (The antiderivative is nonelementary.)

C06S04.003: The length is $\displaystyle\int_0^2 \left[1+36x^2(x-1)^2\right]^{1/2}\ dx \approx 6.6617.$

C06S04.004: The length is $\displaystyle\int_{-1}^1 \frac{1}{3}\left[9+16x^{2/3}\right]^{1/2}\ dx \approx 2.85552.$

C06S04.005: The length is $\displaystyle\int_0^{100} (1+4x^2)^{1/2}\ dx = 50\sqrt{40001} + \frac{1}{4}\sinh^{-1} 200 \approx 10001.6228669180.$

C06S04.006: The length is $\displaystyle\int_0^1 \left[1+4(y-2)^2\right]^{1/2}\ dy$

$$= \sqrt{17} + \frac{1}{2}\left(\sinh^{-1} 4 - \sinh^{-1} 2 - 2\sqrt{5}\right) \approx 3.1678409049.$$

C06S04.007: The length is $\displaystyle\int_{-1}^2 (1+16y^6)^{1/2}\ dy \approx 18.2471.$

C06S04.008: The length of the graph of $y = x^2$ *in the first quadrant* for $1 \leqq y \leqq 4$ is its length for $1 \leqq x \leqq 2$, and is therefore

$$L_1 = \int_1^2 \sqrt{1+4x^2}\ dx = \sqrt{17} - \frac{\sqrt{5}}{2} - \frac{1}{4}\sinh^{-1} 2 + \frac{1}{4}\sinh^{-1} 4 \approx 3.167840904888.$$

If the part of the graph of $y = x^2$ in the second quadrant is to be included in the total length, simply double the previous answer. The straight-line distance from $(1, 1)$ to $(2, 4)$ is $\sqrt{10} \approx 3.16227766$, and hence the answer found here is plausible.

C06S04.009: The length is $\displaystyle\int_1^2 \frac{\sqrt{x^4+1}}{x^2}\ dx \approx 1.13209039330591770397.$

The antiderivative appears to be nonelementary. The *Mathematica* 3.0 command

```
Integrate[ (Sqrt[x∧4 + 1])/(x∧2), x ]
```

elicits a response involving the complete elliptic integral of the second kind. The command

```
Integrate[ (Sqrt[x∧4 + 1])/(x∧2), { x, 1, 2 } ]
```

produces the exact answer

$$\sqrt{2} - \frac{1}{2}\sqrt{17} - 2(-1)^{1/4}\text{EllipticE}\left[\frac{1}{2}\cos^{-1} i,\ 2\right] + 2(-1)^{1/4}\text{EllipticE}\left[\frac{1}{2}\cos^{-1} 4i,\ 2\right].$$

Because the straight-line distance from (1, 1) to (2, 0.5) is $\frac{1}{2}\sqrt{5} \approx 1.1180339987$, we may conclude that the answer in the first line is plausible.

C06S04.010: The arc in question occupies parts of both the first and fourth quadrants. It can be described as the graph of

$$x = g(y) = \sqrt{4 - y^2}, \quad -2 \leqq y \leqq 2.$$

The arc-length element is

$$\sqrt{1 + [g'(y)]^2}\ dy = \left(1 + \frac{y^2}{4 - y^2}\right)^{1/2} dy = \frac{2}{\sqrt{4 - y^2}}\ dy.$$

Therefore the length of the arc (and the answer to Problem 10) is

$$A = \int_{-2}^{2} \frac{2}{\sqrt{4 - y^2}}\ dy.$$

This integral is a so-called *improper integral*, which we discuss in some detail in Section 8.8. It is not a Riemann integral because the integrand is not defined (indeed, becomes unbounded) at the endpoints of the interval $-2 \leqq y \leqq 2$ of integration. In Section 8.8 such an integral is defined using limits, thereby becoming an extension of the Riemann integral:

$$A = \lim_{\alpha \to 2^-} \int_{-\alpha}^{\alpha} \frac{2}{\sqrt{4 - y^2}}\ dy = \lim_{\alpha \to 2^-} \left[2 \arcsin \frac{y}{2}\right]_{-\alpha}^{\alpha} = \lim_{\alpha \to 2^-} \left(2 \arcsin \frac{\alpha}{2} - 2 \arcsin \frac{-\alpha}{2}\right)$$

$$= 2 \arcsin 1 - 2 \arcsin(-1) = \pi - (-\pi) = 2\pi \approx 6.2831853071795864769252867665590.$$

When we take up parametric curves (Section 10.4) and integral computations with parametric curves (Section 10.5), it will become clear that the improper integral we obtained here is merely an artifact of the method used to find arc length in this problem; more powerful methods that lead to much easier integration problems (and avoid improper integrals) will then be available to the students.

C06S04.011: The surface area is $\displaystyle\int_0^4 2\pi x^2(1 + 4x^2)^{1/2}\ dx = \frac{\pi}{32}\left(1032\sqrt{65} - \sinh^{-1} 8\right) \approx 816.5660537285.$

C06S04.012: The surface area is $\displaystyle\int_0^4 2\pi x(1 + 4x^2)^{1/2}\ dx = \frac{\pi}{6}\left(65^{3/2} - 1\right) \approx 273.8666397863.$

C06S04.013: The surface area is $\displaystyle\int_0^1 2\pi(x - x^2)(4x^2 - 4x + 2)^{1/2}\ dx$

$$= \frac{\pi}{16}\left(\sqrt{2} + 5\sinh^{-1} 1\right) \approx 1.1429666793.$$

C06S04.014: The surface area is $\displaystyle\int_0^1 2\pi(4 - x^2)(1 + 4x^2)^{1/2}\ dx$

$$= \frac{5\pi}{32}\left(22\sqrt{5} + 13\sinh^{-1} 2\right) \approx 33.3601584259.$$

C06S04.015: The surface area is $\displaystyle\int_0^1 2\pi(2-x)(1+4x^2)^{1/2}\,dx$

$$= \frac{\pi}{6}\left(7\sqrt{5} + 1 + 6\sinh^{-1} 2\right) \approx 13.2545305651.$$

C06S04.016: The surface area is $\displaystyle\int_0^1 2\pi(x-x^3)(9x^4-6x^2+2)^{1/2}\,dx$

$$= \frac{\pi}{27}\left(5\sqrt{2} + \sqrt{5} + 3\sinh^{-1} 1 + 3\sinh^{-1} 2\right) \approx 1.8945156885.$$

C06S04.017: The surface area is $\displaystyle\int_1^4 \pi(4x+1)^{1/2}\,dx = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right) \approx 30.846489697142435.$

C06S04.018: The surface area is $\displaystyle\int_1^4 2\pi x\left(1+\frac{1}{4x}\right)^{1/2}\,dx = \pi\int_1^4 (4x^2+x)^{1/2}\,dx$

$$= \frac{\pi}{32}\left(132\sqrt{17} - 18\sqrt{5} + \sinh^{-1} 2 - \sinh^{-1} 4\right) \approx 49.4162355383.$$

C06S04.019: Let $f(x) = x^{3/2}$. The surface area element is

$$ds = \sqrt{1+[f'(x)]^2}\,dx = \frac{1}{2}\sqrt{9x+4}\,dx.$$

Hence the surface area is

$$A = \int_1^4 2\pi(x+1)\cdot\frac{1}{2}\sqrt{9x+4}\,dx = \pi\int_1^4 (x+1)\sqrt{9x+4}\,dx.$$

Techniques of Chapter 8 can be used to evaluate the antiderivative, and thus to find the exact value of the definite integral. One method is to begin with the substitution $u = 9x+4$. Space prohibits our providing complete details, but the result (condensed) is

$$A = \pi\left[\left(\frac{2}{5}x^2 + \frac{98}{135}x + \frac{296}{1215}\right)\sqrt{9x+4}\right]_1^4 = \frac{32\pi}{1215}\left(725\sqrt{10} - 52\sqrt{13}\right) \approx 174.184387843870197332.$$

C06S04.020: The surface area is $\displaystyle\int_1^4 \pi(2+x^{5/2})(4+25x^3)^{1/2}\,dx \approx 3615.28.$

C06S04.021: The length is $\displaystyle\int_0^2 (1+2x^2)\,dx = \left[x + \frac{2}{3}x^3\right]_0^2 = \frac{22}{3}.$

C06S04.022: The length is $\displaystyle\int_1^5 y^{1/2}\,dy = \left[\frac{2}{3}y^{3/2}\right]_1^5 = \frac{10\sqrt{5}-2}{3} \approx 6.7868932583.$

C06S04.023: First let $f(x) = \dfrac{1}{6}x^3 + \dfrac{1}{2x}$. Then

$$1+[f'(x)]^2 = 1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2 = 1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4} = \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4} = \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2.$$

Therefore the length is

$$\frac{1}{2}\int_1^3 (x^2 + x^{-2})\,dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{x}\right]_1^3 = \frac{13}{3} - \left(-\frac{1}{3}\right) = \frac{14}{3}.$$

C06S04.024: First let $g(y) = \dfrac{1}{8}y^4 + \dfrac{1}{4y^2}$. Then

$$1 + [g'(y)]^2 = 1 + \left(\frac{1}{2}y^3 - \frac{1}{2}y^{-3}\right)^2 = 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6} = \left(\frac{1}{2}y^3 + \frac{1}{2}y^{-3}\right)^2.$$

Therefore the length is

$$\frac{1}{2}\int_1^2 (y^3 + y^{-3})\,dy = \frac{1}{2}\left[\frac{1}{4}y^4 - \frac{1}{2}y^{-2}\right]_1^2 = \frac{31}{16} - \left(-\frac{1}{8}\right) = \frac{33}{16} = 2.0625.$$

C06S04.025: First solve for $y = f(x) = \dfrac{2x^6 + 1}{8x^2} = \dfrac{1}{4}x^4 + \dfrac{1}{8}x^{-2}$. Then

$$1 + [f'(x)]^2 = 1 + \left(x^3 - \frac{1}{4}x^{-3}\right)^2 = 1 + x^6 - \frac{1}{2} + \frac{1}{16}x^{-6} = x^6 + \frac{1}{2} + \frac{1}{16}x^{-6} = \left(x^3 + \frac{1}{4}x^{-3}\right)^2.$$

Therefore the length is

$$\int_1^2 \left(x^3 + \frac{1}{4}x^{-3}\right)dx = \left[\frac{1}{4}x^4 - \frac{1}{8}x^{-2}\right]_1^2 = \frac{127}{32} - \frac{1}{8} = \frac{123}{32} = 3.84375.$$

C06S04.026: First solve for $x = g(y) = \dfrac{3 + 4y^4}{12y} = \dfrac{1}{4}y^{-1} + \dfrac{1}{3}y^3$. Then

$$1 + [g'(y)]^2 = 1 + \left(y^2 - \frac{1}{4}y^{-2}\right)^2 = 1 + y^4 - \frac{1}{2} + \frac{1}{16}y^{-4} = y^4 + \frac{1}{2} + \frac{1}{16}y^{-4} = \left(y^2 + \frac{1}{4}y^{-2}\right)^2.$$

Therefore the length is

$$\int_1^2 \left(y^2 + \frac{1}{4}y^{-2}\right)dy = \left[\frac{1}{3}y^3 - \frac{1}{4}y^{-1}\right]_1^2 = \frac{61}{24} - \frac{1}{12} = \frac{59}{24} \approx 2.4583333333.$$

C06S04.027: First solve for $y = f(x) = 2x^{2/3}$. Then

$$1 + [f'(x)]^2 = 1 + \frac{16}{9x^{2/3}} = \frac{9x^{2/3} + 16}{9x^{2/3}},$$

so that the length is

$$\int_1^8 \frac{1}{3}x^{-1/3}\left(9x^{2/3} + 16\right)^{1/2}\,dx = \left[\frac{1}{27}(9x^{2/3} + 16)^{3/2}\right]_1^8 = \frac{104\sqrt{13} - 125}{27} \approx 9.2584197277131439438663.$$

C06S04.028: The given curve has two branches, but the endpoints given in the problem determine that when we solve for $y = f(x)$, we find that $f(x) = 3 + 2(x + 2)^{3/2}$. (It really doesn't matter, because the two branches are symmetric around the x-axis.) Then

$$1 + [f'(x)]^2 = (19 + 9x)^{1/2},$$

and therefore the length of the given arc is

$$L = \int_{-1}^{5} (19 + 9x)^{1/2}\,dx = \left[\frac{2}{27}(19 + 9x)^{3/2}\right]_{-1}^{2} = \frac{74\sqrt{37} - 20\sqrt{20}}{27} \approx 14.328847186618543346.$$

C06S04.029: Because

$$1 + [f'(x)]^2 \;=\; 1 + \left(\frac{1}{2}x^{-1/2}\right)^2 \;=\; \frac{4x+1}{4x},$$

the surface area is

$$\int_0^1 \frac{1}{2}\pi x^{1/2}\left(\frac{4x+1}{x}\right)^{1/2}\,dx = \int_0^1 \pi(4x+1)^{1/2}\,dx = \left[\frac{1}{6}\pi(4x+1)^{3/2}\right]_0^1 = \frac{5\sqrt{5} - 1}{6}\pi \approx 5.3304135003.$$

C06S04.030: Because $1 + [f'(x)]^2 = 1 + 9x^4$, the surface area is

$$\int_1^2 2\pi x^3(1 + 9x^4)^{1/2}\,dx = \left[\frac{1}{27}\pi(1 + 9x^4)^{3/2}\right]_1^2 = \frac{145\sqrt{145} - 10\sqrt{10}}{27}\pi \approx 199.4804797017.$$

C06S04.031: First,

$$1 + [f'(x)]^2 = 1 + \left(x^4 - \frac{1}{4x^4}\right)^2 = x^8 + \frac{1}{2} + \frac{1}{16x^8} = \left(x^4 + \frac{1}{4x^4}\right)^2.$$

Therefore the surface area of revolution is

$$\begin{aligned}\int_1^2 2\pi x\left(x^4 + \frac{1}{4x^4}\right)dx &= \pi\int_1^2 \left(2x^5 + \frac{1}{2}x^{-3}\right)dx\\ &= \pi\left[\frac{1}{3}x^6 - \frac{1}{4}x^{-2}\right]_1^2 = \frac{339}{16}\pi \approx 66.5624943479.\end{aligned}$$

C06S04.032: Let $g(y) = \dfrac{1}{8}y^4 + \dfrac{1}{4}y^{-2}$. Then

$$1 + [g'(y)]^2 = \frac{1}{4}\left(y^6 + 2 + y^{-6}\right) = \frac{1}{4}\left(y^3 + y^{-3}\right)^2.$$

Therefore the surface area is

$$\int_1^2 \pi\left(y^4 + y^{-2}\right)dy = \pi\left[\frac{1}{5}y^5 - y^{-1}\right]_1^2 = \frac{67\pi}{10} \approx 21.0486707791.$$

C06S04.033: Let $f(x) = (3x)^{1/3}$. Then

$$\sqrt{1 + [f'(x)]^2} = \left(1 + \frac{1}{(3x)^{4/3}}\right)^{1/2}.$$

Therefore the surface area of revolution is

$$\int_0^9 2\pi x\left(1+\frac{1}{(3x)^{4/3}}\right)^{1/2}\,dx = \left[\frac{\pi}{27x}\left(9x^2+(3x)^{2/3}\right)^{3/2}\right]_0^9 = \frac{82\sqrt{82}-1}{9}\pi \approx 258.8468426921.$$

C06S04.034: Let $f(x) = \dfrac{2}{3}x^{3/2}$. Then $\sqrt{1+[f'(x)]^2} = \sqrt{1+x}$. Therefore the surface area is

$$A = \int_1^2 2\pi x(1+x)^{1/2}\,dx.$$

The substitution $u = x+1,\ x = u-1,\ du = dx$ then yields

$$\begin{aligned}
A &= \int_2^3 2\pi(u-1)u^{1/2}\,du = 2\pi\int_2^3\left(u^{3/2}-u^{1/2}\right)\,du = 2\pi\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_2^3\\
&= 2\pi\left[\frac{2}{5}\left(9\sqrt{3}-4\sqrt{2}\right)-\frac{2}{3}\left(3\sqrt{3}-2\sqrt{2}\right)\right] = \frac{2\pi}{15}\left(54\sqrt{3}-24\sqrt{2}-30\sqrt{3}+20\sqrt{2}\right)\\
&= \frac{2\pi}{15}\left(24\sqrt{3}-4\sqrt{2}\right) = \frac{8\pi}{15}\left(6\sqrt{3}-\sqrt{2}\right) \approx 15.0429363296306960339 6.
\end{aligned}$$

C06S04.035: Let $f(x) = (2x-x^2)^{1/2}$. Then

$$f'(x) = \frac{1}{2}(2x-x^2)^{1/2}(2-2x) = \frac{1-x}{(2x-x^2)^{1/2}}.$$

Therefore

$$1+[f'(x)]^2 = 1+\frac{(1-x)^2}{2x-x^2} = \frac{2x-x^2+1-2x+x^2}{2x-x^2} = \frac{1}{2x-x^2}.$$

Hence the surface area of revolution is

$$\begin{aligned}
A &= \int_0^2 2\pi f(x)\sqrt{1+[f'(x)]^2}\,\,dx = \int_0^2 2\pi(2x-x^2)^{1/2}\cdot\frac{1}{(2x-x^2)^{1/2}}\,dx\\
&= \int_0^2 2\pi\,dx = \Big[2\pi x\Big]_0^2 = 4\pi \approx 12.5663706143591729538 5.
\end{aligned}$$

Note that the integrand is not defined at either endpoint of the interval of integration, but the discontinuities there are removable—we used this in the transition to the last line of computations—and hence there is no real problem. (Is it clear that changing the value of an integrable function at only one or two points in its domain cannot affect the integrability of that function or the values of its definite integrals there?)

C06S04.036: The length of one arch of the sine curve is

$$L_1 = \int_0^\pi\left(1+\cos^2 x\right)^{1/2}\,dx.$$

To find the arc length L_2 of half the ellipse, we take $y = \left(2-2x^2\right)^{1/2},\ -1 \leqq x \leqq 1$. Then

$$\frac{dy}{dx} = -\frac{2x}{\left(2-2x^2\right)^{1/2}}, \quad\text{so}\quad 1+\left(\frac{dy}{dx}\right)^2 = \frac{1+x^2}{1-x^2}.$$

Thus

$$L_2 = \int_{-1}^{1} \left(\frac{1+x^2}{1-x^2} \right)^{1/2} dx.$$

Let $x = \cos u$. Then $dx = -\sin u\, du$, and

$$L_2 = \int_{\pi}^{0} \left(\frac{1+\cos^2 u}{\sin^2 u} \right)^{1/2} (-\sin u)\, du = \int_0^{\pi} \left(1+\cos^2 u\right)^{1/2} du = L_1.$$

This concludes the solution, but an additional comment is appropriate. The *Mathematica* command

```
Integrate[ Sqrt[ 1 + (Cos[u])∧2 ], {u, 0, Pi} ]
```

elicits the response

$$\left(2\sqrt{2}\right) \text{EllipticE}\left(\frac{1}{2}\right).$$

Because *elliptic integrals* are involved, this strongly suggests that the antiderivative of $\sqrt{1+\cos^2 u}$ is nonelementary. In this case none of the techniques of Chpater 7 will produce an antiderivative; y ou should therefore, if necessary, approximate the value of L_1 by using Simpson's approximation. Your result should be close to *Mathematica's* approximation 3.820197789027712017904762.

C06S04.037: We take $f(x) = \sqrt{1+\cos^2 x}$, $\Delta x = \pi/6$, $x_i = i \cdot \Delta x$, and compute

$$\begin{aligned} S_6 &= \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6) \right] \\ &= \frac{\pi}{18} \left(4 + 2\sqrt{2} + 2\sqrt{5} + 4\sqrt{7} \right) \approx 3.8194031934. \end{aligned}$$

C06S04.038: We take $f(x) = \sqrt{1+4x^2}$, $\Delta x = 1/10$, $x_i = i \cdot \Delta x$, and compute

$$\begin{aligned} S_{10} &= \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_8) + 4f(x_9) + f(x_{10}) \right] \\ &= \frac{1}{30} \left[1 + \sqrt{5} + \frac{2}{5}\left(\sqrt{29} + \sqrt{41} + \sqrt{61} + \sqrt{89} \right) + \frac{4}{5}\left(5\sqrt{2} + \sqrt{26} + \sqrt{34} + \sqrt{74} + \sqrt{106} \right) \right] \\ &\approx 1.4789423874. \end{aligned}$$

The exact value of the arc length is

$$\int_0^1 \sqrt{1+4x^2}\, dx = \frac{2\sqrt{5} + \sinh^{-1} 2}{4} \approx 1.4789428575.$$

C06S04.039: The line segment from $(r_1, 0)$ to (r_2, h) is part of the line with equation

$$y = f(x) = \frac{h(x-r_1)}{r_2 - r_1},$$

and

$$\sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{h^2}{(r_2-r_1)^2}}\ .$$

Therefore the area of the conical frustum is

$$\int_{r_1}^{r_2} 2\pi x\sqrt{1+[f'(x)]^2}\ dx = \pi\left[x^2\left(\frac{(r_2-r_1)^2+h^2}{(r_2-r_1)^2}\right)^{1/2}\right]_{r_1}^{r_2} = \pi(r_1+r_2)\sqrt{h^2+(r_2-r_1)^2}\ .$$

With $\overline{r} = (r_1+r_2)/2$ the "average radius" of the frustum and $L = \sqrt{h^2+(r_2-r_1)^2}$ its slant height, this yields the result in Eq. (6) of the text.

C06S04.040: The spherical surface S of radius r centered at the origin may be generated by rotating the graph of $f(x) = \sqrt{r^2-x^2}$ around the x-axis for $-r \leqq x \leqq r$. Because

$$\sqrt{1+[f'(x)]^2} = \frac{r}{\sqrt{r^2-x^2}},$$

the surface area of S is

$$\int_{-r}^{r} 2\pi(r^2-x^2)^{1/2}\cdot\frac{r}{(r^2-x^2)^{1/2}}\ dx = \int_{-r}^{r} 2\pi r\ dx = \Big[2\pi rx\Big]_{-r}^{r} = 2\pi r^2 - (-2\pi r^2) = 4\pi r^2.$$

A perceptive student may notice that we have glossed over a problem with an improper integral here (Section 8.8). Perhaps such a student could be informed that the first simplification is valid for $-r < x < r$, and that changing the value of an integrable function at two points cannot change the fact of its integrability nor can it change the value of the definite integral.

C06S04.041: Let $f(x) = (1-x^{2/3})^{3/2}$ for $0 \leqq x \leqq 1$. Then the graph of f is the part of the astroid that lies in the first quadrant in Fig. 6.4.16. Next,

$$\sqrt{1+[f'(x)]^2} = \frac{1}{x^{1/3}},$$

so it would appear that the total length of the astroid is

$$4\int_0^1 \frac{1}{x^{1/3}}\ dx.$$

But this integral is not a Riemann integral—the integrand approaches $+\infty$ as $x \to 0^+$. (It is an *improper integral*, the topic of Section 9.8.) But we can avoid the difficulty at $x = 0$ by integrating from the *midpoint* of the graph of f to $x = 1$. That midpoint occurs where $y = x$, so that $2x^{2/3} = 1$, and thus $x = a = \frac{1}{4}\sqrt{2}$. Therefore the total length of the astroid is

$$L = 8\int_a^1 \frac{1}{x^{1/3}}\ dx = 8\left[\frac{3}{2}x^{2/3}\right]_a^1 = 8\cdot\left(\frac{3}{2}-\frac{3}{4}\right) = 6.$$

C06S04.042: The surface area is

$$2\int_0^1 \frac{2\pi x}{x^{1/3}}\ dx = 2\int_0^1 2\pi x^{2/3}\ dx = \left[\frac{12}{5}\pi x^{5/3}\right]_0^1 = \frac{12\pi}{5} \approx 7.5398223686.$$

The integral is improper, but see the remarks attached to the solution of Problem 40.

C06S04.043: We will solve this problem by first rotating Fig. 6.4.18 through an angle of 90°. Let $f(x) = (r^2 - x^2)^{1/2}$. Think of the sphere as generated by rotation of the graph of f around the x-axis for $-r \leqq x \leqq r$. Suppose that the spherical zone Z of height h is the part of the sphere between $x = a$ and $x = a + h$, where

$$-r \leqq a \leqq a + h \leqq r.$$

Also,

$$2\pi f(x)\sqrt{1 + [f'(x)]^2} = 2\pi(r^2 - x^2)^{1/2} \cdot \frac{r}{(r^2 - x^2)^{1/2}} = 2\pi r.$$

Therefore the area of the spherical zone Z is

$$A = \int_a^{a+h} 2\pi r\, dx = \Big[2\pi r x\Big]_a^{a+h} = 2\pi r(a + h) - 2\pi r a = 2\pi r h.$$

As noted in the statement of the problem, the area of Z depends only on the radius r of the sphere and the height (width) h of the zone and not on the location of the two planes (at a and $a + h$) that determine the zone. You can also "test" the answer by substituting the value $2r$ for h.

C06S04.044: The top half of the loop is the graph of

$$f(x) = \frac{x\sqrt{4 - x^2}}{4\sqrt{2}}$$

for $0 \leqq x \leqq 2$. Next,

$$\sqrt{1 + [f'(x)]^2} = \sqrt{\frac{(6 - x^2)^2}{8(4 - x^2)}} = \frac{\sqrt{2}}{4} \cdot \frac{6 - x^2}{(4 - x^2)^{1/2}}.$$

Thus

$$2\pi f(x)\sqrt{1 + [f'(x)]^2} = \frac{\pi x(6 - x^2)}{8},$$

and therefore the surface area generated by rotating the loop of Fig. 6.4.17 around the x-axis is

$$A = \int_0^2 \frac{\pi x(6 - x^2)}{8}\, dx = \left[\frac{\pi}{32}x^2(12 - x^2)\right]_0^2 = \pi - 0 = \pi.$$

C06S04.045: The right-hand endpoint of the cable is located at the point (S, H), so that $H = kS^2$. It follows that

$$y(x) = \frac{H}{S^2}x^2, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{2H}{S^2}x.$$

Therefore the total length of the cable is

$$L = \int_{-S}^{S} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_0^S \sqrt{1 + \frac{4H^2}{S^4}x^2}\, dx.$$

C06S04.046: The following *Mathematica* computations yield Simpson's approximation to the integral in question. We used s in place of S and h in place of H, and first let

```
s = 8000; h = 380;
```

and then we set up the integrand

$$f(x) = 2\sqrt{1 + \frac{4h^2x^2}{s^4}}\,.$$

```
f[x_] := 2*Sqrt[1 + 4*h*h*x*x/(s*s*s*s)]

n = 20; delta = s/n; x[i_] := i*delta

(delta/3)*(N[f[x[0]],20] + N[f[x[n]],20] +
   4*Sum[N[f[x[i]],20], { i, 1, n - 1, 2 } ] +
   2*Sum[N[f[x[i]],20], { i, 2, n - 2, 2 } ])
```

16024.034190838711974

Next, $n = 40$ yielded the (probably) better approximation 16024.034190963156976. We repeated with $n = 80$, $n = 160$, $n = 320$, $n = 640$, and $n = 1280$. The last two agreed in the first eleven digits to the right of the decimal; with $n = 1280$ we obtained 16024.034190971453043.

If you prefer exact results,

$$\int f(x)\,dx = x\sqrt{\frac{s^4 + 4h^2x^2}{s^4}} + \frac{s^2}{2h}\sinh^{-1}\left(\frac{2hx}{s^2}\right) + C,$$

and (with the given values of s and h)

$$\begin{aligned}\int_0^s f(x)\,dx &= \sqrt{1 + \frac{4h^2}{s^2}} + \frac{s^2}{2h}\sinh^{-1}\left(\frac{2h}{s}\right)\\ &= 40\sqrt{40361} + \frac{1600000}{19}\sinh^{-1}\left(\frac{19}{200}\right) \approx 16024.034190971453050453137866697.\end{aligned}$$

Section 6.5

C06S05.001: The work is $W = \int_a^b F(x)\,dx = \int_{-2}^{1} 10\,dx = \Big[10x\Big]_{-2}^{1} = 30.$

C06S05.002: The work is $W = \int_a^b F(x)\,dx = \int_1^5 (3x-1)\,dx = \left[\frac{3}{2}x^2 - x\right]_1^5 = \frac{65}{2} - \frac{1}{2} = 32.$

C06S05.003: The work is $W = \int_a^b F(x)\,dx = \int_1^{10} 10x^{-2}\,dx = \left[-\frac{10}{x}\right]_1^{10} = -1-(-10) = 9.$

C06S05.004: The work is $W = \int_a^b F(x)\,dx = \int_0^4 -3x^{1/2}\,dx = \Big[-2x^{3/2}\Big]_0^4 = -16-0 = -16.$

C06S05.005: The work is $W = \int_a^b F(x)\,dx = \int_{-1}^{1} \sin \pi x\,dx = \left[-\frac{1}{\pi}\cos \pi x\right]_{-1}^{1} = \frac{1}{\pi} - \frac{1}{\pi} = 0.$

C06S05.006: The spring constant is 10 N/m, so the force function for this spring is $F(x) = 10x$. Hence the work done is

$$W = \int_0^{-2/5} 10x\,dx = \Big[5x^2\Big]_0^{-2/5} = \frac{4}{5} = 0.8 \quad (\text{N}\cdot\text{m}).$$

C06S05.007: The spring constant is 30 lb/ft, so the force function for this spring is $F(x) = 30x$. Hence the work done is

$$W = \int_0^1 30x\,dx = \Big[15x^2\Big]_0^1 = 15 \quad (\text{ft}\cdot\text{lb}).$$

C06S05.008: The force function is $F(x) \equiv 100$, so the work done is

$$W = \int_0^{10} 100\,dx = \Big[100x\Big]_0^{10} = 1000 \quad (\text{ft}\cdot\text{lb}).$$

C06S05.009: With $k = 16 \times 10^9$, we find that the work done is

$$W = \int_{5000}^{6000} \frac{k}{x^2}\,dx = \left[-\frac{k}{x}\right]_{5000}^{6000} = \frac{k}{30000} = \frac{16}{3} \times 10^5 \quad (\text{mi}\cdot\text{lb}).$$

We multiply by 5280 to convert the answer to 2.816×10^9 ft·lb.

C06S05.010: With water density $\rho = 62.4$ (lb/ft^3) and cross-sectional area function $A(y) \equiv 25\pi$, we find that the work done to fill the tank is

$$W = \int_0^{10} \rho y A(y)\,dy = \Big[780\pi y^2\Big]_0^{10} = 78000\pi \approx 245044.226980 \quad (\text{ft}\cdot\text{lb}).$$

C06S05.011: By similar triangles, if the height of water in the tank is y and the radius of the circular water surface is r, then $r = \frac{1}{2}(10-y)$. Hence the area of the water surface is $A(y) = \frac{1}{4}\pi(10-y)^2$. With water density $\rho = 62.4$ (lb/ft^3), the work done to fill the tank is

$$W = \int_0^{10} \rho y A(y)\,dy = \left[\frac{13}{10}\pi(3y^4 - 80y^3 + 600y^2)\right]_0^{10} = 13000\pi \approx 40840.704497 \quad (\text{ft}\cdot\text{lb}).$$

C06S05.012: By similar triangles, if the height of water in the tank is y and the radius of the circular water surface is r, then $r = \frac{1}{2}y$. Hence the area of the water surface is $A(y) = \frac{1}{4}\pi y^2$. With water density $\rho = 62.4$ (lb/ft^3), the work done to fill the tank is

$$W = \int_0^{10} \rho y A(y)\,dy = \left[\frac{39}{10}\pi y^4\right]_0^{10} = 39000\pi \approx 122522.113490 \quad (\text{ft}\cdot\text{lb}).$$

C06S05.013: The work is $\displaystyle\int_0^5 50(y+10)\cdot 5\pi y\,dy = \pi\left[\frac{250}{3}y^3 + 1250y^2\right]_0^5 = \frac{125000\pi}{3} \approx 130899.694 \quad (\text{ft}\cdot\text{lb}).$

C06S05.014: Suppose that the bottom of the tank is located where $y = a$ and the top where $y = b$. Let n be a positive integer and let $\mathcal{P} = \{y_0, y_1, y_2, \ldots, y_n\}$ be a partition of the interval $[a, b]$. Let $S = \{y_1^\star, y_2^\star, \ldots, y_n^\star\}$ be a selection for $\mathcal{P}$. Consider the liquid in the tank between the levels y_{i-1} and y_i where $1 \leqq i \leqq n$. Let $A(y_i^\star)$ be the cross-sectional area of the liquid at level $y_i^\star$ Then the weight of the liquid between y_{i-1} and y_i will be approximately $\rho A(y_i^\star)\,\Delta y_i$ (where $\Delta y_i = y_i - y_{i-1}$) and this liquid must be lifted the approximate distance $h - y_i^\star$ in pumping all the liquid in the tank to the level $y = h$. Hence the total work done will be approximately

$$\sum_{i=1}^{n} \rho(h - y_i^\star)A(y_i^\star)\,\Delta y_i.$$

But the error in these approximations will approach zero as $n \to +\infty$ and the maximum of the Δy_i approaches zero. Because this sum is a Riemann sum, it has as its limit the following integral; therefore the work to pump all the liquid in the tank to the level $y = h$ will be

$$W = \int_a^b \rho(h - y)A(y)\,dy.$$

C06S05.015: $\displaystyle W = \int_0^{10} \rho(15 - y)\cdot 25\pi\,dy = \pi\Big[23400y - 780y^2\Big]_0^{10} = 156000\pi \approx 490088.454 \quad (\text{ft}\cdot\text{lb}).$

C06S05.016: Set up the following coordinate system: The x-axis and y-axis cross at the center of one end of the tank, so that the equation of the circular (vertical) cross section of the tank is $x^2 + y^2 = 9$. Then the gasoline must be lifted to the level $y = 10$. A horizontal cross section of the tank at level y is a rectangle of length 10 and width $2x$ where x and y satisfy the equation $x^2 + y^2 = 9$ of the end of the tank, and thus $2x = 2(9 - y^2)^{1/2}$. Thus the amount of work required to pump all the gasoline in the tank into automobiles will be

$$\begin{aligned}
W &= \int_{-2}^{3} \left(2\sqrt{9 - y^2}\right)\cdot 10\cdot(10 - y)\cdot 45\,dy \\
&= 900\int_{-3}^{3}\left(10\sqrt{9 - y^2} - y\sqrt{9 - y^2}\right)dy \\
&= 9000\cdot\frac{1}{2}\cdot\pi\cdot 9 + 900\left[\frac{1}{3}(9 - y^2)^{3/2}\right]_{-3}^{3} \\
&= 40500\pi \approx 127234.5 \quad (\text{ft}\cdot\text{lb}).
\end{aligned}$$

Now assume that the tank uses a 1.341 hp motor. If it were to operate at 100% efficiency, it would pump all the gasoline in

$$\frac{40500\pi}{33000 \cdot 1.341} \approx 2.875161$$

minutes, using 1 kW for 2.875161 minutes. This would amount to 0.047919356 kWh, costing about 0.345 cents. Assuming that the pump is only 30% efficient, the actual cost would be about 1.15 cents.

C06S05.017: With water density $\rho = 62.4$ lb/ft^3, the work will be

$$\int_{-10}^{10} \rho(50+y)\cdot\pi(100-y^2)\,dy = \frac{26}{5}\pi\left[60000y + 600y^2 - 200y^3 - 3y^4\right]_{-10}^{10} = 4160000\pi \approx 13069025 \quad \text{(ft·lb)}.$$

C06S05.018: Place the origin at the center of the base of the hemisphere, at indicated in Fig. 6.6.16. Then the cross-sectional area at y is $A(y) = \pi(100 - y^2)$ for $0 \leqq y \leqq 10$ and the oil that ends up at level y is lifted a total distance $60 + y$, so the work is

$$\int_0^{10} 50(y+60)A(y)\,dy = \pi\left[300000y + 2500y^2 - 1000y^3 - \frac{25}{2}y^4\right]_0^{10} = 2125000\pi \approx 6675884 \quad \text{(ft·lb)}.$$

C06S05.019: Let $y = 0$ at the surface of the water in the well, so the top of the well is at $y = 100$. The weight of water in the bucket is $100 - \frac{1}{4}y$ when the bucket is at level y, so the total work done in lifting the water to the top of the well is

$$W = \int_0^{100}\left(100 - \frac{y}{4}\right)dy = \left[100y - \frac{1}{8}y^2\right]_0^{100} = 8750 \quad \text{(ft·lb)}.$$

C06S05.020: Set up a coordinate system in which the bottom of the rope is initially located at $y = 0$ and the top of the building at $y = 100$. When the bottom of the rope has been lifted to position y $(0 \leqq y \leqq 100)$, then the weight of the part of the rope still dangling from the top is $(100-y)/4$ (pounds). The work required to lift the rope a short distance Δy is approximately $\frac{1}{4}(100-y)\,\Delta y$, and therefore the total work to lift the rope to the top of the building will be

$$W = \int_0^{100} \frac{1}{4}(100-y)\,dy = \left[25y - \frac{1}{8}y^2\right]_0^{100} = 1250 \quad \text{(ft·lb)}. \tag{1}$$

This solution in effect *partitions the process* of lifting the rope to the top of the building. You may prefer the alternative of *partitioning the rope.* Imagine a short section of the rope near position y and of length Δy. This section of rope weighs $\frac{1}{4}\,\Delta y$ pounds and will be lifted the distance $100 - y$ to the top of the building, so the work to lift this short section of rope will be $\frac{1}{4}(100-y)\,\Delta y$. So the total work to lift all such short sections of the rope will be exactly the same as that shown in Eq. (1).

Finally, the answer may be checked in the following way: Stiffen the rope and turn it 90° around its center (where $y = 50$). This requires no net work as the rope will balance at its center. Then lift the 25-lb rope 50 feet to the top of the building, requiring $25 \cdot 50 = 1250$ ft·lb of work.

C06S05.021: The weight of the rope and the water will be $w(y) = 100 + (100-y)/4$ when the bucket is y feet above the water surface. So the work to lift the rope and the water to the top of the well will be

$$W = \int w(y)\,dy = \left[125y - \frac{1}{8}y^2\right]_0^{100} = 11250 \quad \text{(ft·lb)}.$$

C06S05.022: The work is $W = \int_{x_1}^{x_2} A \cdot p(Ax)\,dx$. Let $V = Ax$; then $dV = A\,dx$. Therefore

$$W = \int_{V_1}^{V_2} pV\,dV.$$

C06S05.023: Given: $pV^{1.4} = c$. When $p_1 = 200,\ V_1 = 50$. Hence

$$c = 200 \cdot 50^{7/5} = 200 \cdot 50^{7/5}, \quad \text{so} \quad p = \frac{c}{V^{7/5}} = 200 \cdot \left(\frac{50}{V}\right)^{7/5}.$$

Therefore the work done by the engine in each cycle is

$$W = \int_{50}^{500} 200 \cdot \left(\frac{50}{V}\right)^{7/5} dV = \left[200 \cdot 50^{7/5} \cdot \left(-\frac{5}{2}V^{-2/5}\right)\right]_{50}^{500} = 2500(10 - 10^{3/5}) \approx 15047.320736.$$

The answer is in "inch-pounds" (in.·lb); divide by 12 to convert the answer into $W \approx 1253.94339468$ foot-pounds.

C06S05.024: Set up a coordinate system in which the center of the hemisphere is at the origin, with a diameter lying on the x-axis and the y-axis perpendicular to the base, so that the highest point of the hemisphere has coordinates (0, 60). Now imagine a horizontal thin circular slice of its contents at position y and having radius x, so that $x^2 + y^2 = 3600$. If the thickness of this slice is dy, then its volume is $dV = \pi\left(3600 - y^2\right)\,dy$, so its weight is $40\pi\left(3600 - y^2\right)\,dy$. This is the force acting on the slice, which is to be lifted a distance $60 - y$, so the work used in lifting this slice is $40\pi\left(3600 - y^2\right)(60 - y)\,dy$. Therefore the total work to pump all the liquid to the level of the top of the tank is

$$W = \int_0^{60} 40\pi\left(3600 - y^2\right)(60 - y)\,dy = 216000000\pi \approx 6.78584 \times 10^8 \quad (\text{ft·lb}).$$

C06S05.025: $W = \int_0^1 60\pi(1 - y)\sqrt{y}\,dy = 16\pi \approx 50.265482$ (ft·lb).

C06S05.026: Set up a coordinate system in which the center of the tank is at the origin and the ground surface coincides with the horizontal line $y = -3$. Imagine a horizontal cross section of the tank at position y, $-3 \leqq y \leq 3$. The equation of the circle $x^2 + y^2 = 9$ gives us the width $2x$ of this rectangular cross section: $2x = 2\left(9 - y^2\right)^{1/2}$. The length of the cross section is 20, so if we denote its thickness by dy then its volume is $40\left(9 - y^2\right)^{1/2}\,dy$. To fill this slab with gasoline weighing 40 pounds per cubic foot, which is to be lifted the distance $y + 3$ feet, requires $dW = 1600(y + 3)\left(9 - y^2\right)^{1/2}\,dy$ ft·lb of work. So the work required to fill the tank is

$$\begin{aligned} W &= \int_{-3}^{3} 1600(y+3)\left(9 - y^2\right)^{1/2}\,dy \\ &= 1600\int_{-3}^{3} y\left(9 - y^2\right)^{1/2}\,dy + 4800\int_{-3}^{3}\left(9 - y^2\right)^{1/2}\,dy. \end{aligned}$$

The first integral is zero because it involves the evaluation of $\left(9 - y^2\right)^{3/2}$ at $y = 3$ and at $y = -3$. The second is the product of 4800 and the area of a semicircle of radius 3, so the answer is that the total work is $21600\pi \approx 67858.401318$ ft·lb.

C06S05.027: It is convenient to set up a coordinate system in which the center of the tank is at the origin, the x-axis horizontal, and the y-axis vertical. A horizontal cross section at y is circular with radius x satisfying $x^2 + y^2 = 144$, so the work to fill the tank is

$$W = \int_{-12}^{12} (50\pi)(y+12)\left(144 - y^2\right)\,dy = \pi\left[86400y + 3600y^2 - 200y^3 - \frac{25}{2}y^4\right]_{-12}^{12}$$

$$= 950400\pi - (-432000\pi) = 1382400\pi \approx 4.342938 \times 10^6 \quad (\text{ft·lb}).$$

C06S05.028: Set up a coordinate system in which the y-axis is vertical, $y = 0$ corresponds to the bottom of the cage, and $y = 40$ to its top. The work done in lifting the monkey from $y = 0$ to $y = 40$ is $20 \cdot 40 = 800$ ft·lb. Now we compute the work done in lifting the chain.

Suppose that the free end of the chain is lifted from $y = 0$ to $y = 10$ and is at position y. Then the length of chain lifted is y, so its weight is $\frac{1}{2}y$. Then the free end of the chain is lifted from $y = 10$ to $y = 40$. When the free end is at position y, part of the chain is doubled; let z denote the length of each part that is doubled. The length of the part of the chain not doubled is $40 - y$. So $40 - y + 2z = 50$, and it follows that $z = 5 + \frac{1}{2}y$. So the weight of the chain lifted is $\frac{1}{2}z = \frac{5}{2} + \frac{1}{4}y$. Hence the work done to lift the chain is

$$\int_0^{10} \frac{1}{2}y\,dy + \int_{10}^{40}\left(\frac{5}{2} + \frac{1}{4}y\right)dy = 25 + 262.5 = 287.5 \quad (\text{ft·lb}).$$

Therefore the total amount of work done in lifting monkey and chain is $800 + 287.5 = 1087.5$ ft·lb.

C06S05.029: Let the string begin its journey stretched out straight along the x-axis from $x = 0$ to $x = 500\sqrt{2}$. Imagine it reaching its final position by simply pivoting at the origin up to a $45°$ angle while remaining straight. A small segment of the string initially at location x and of length dx is lifted from $y = 0$ to the final height $y = x/\sqrt{2}$, so the total work done in lifting the string is

$$W = \int_0^{500\sqrt{2}} \frac{x}{16\sqrt{2}}\,dx = \left[\frac{1}{32\sqrt{2}}x^2\right]_0^{500\sqrt{2}} = \frac{15625\sqrt{2}}{2} \qquad (\text{ft·oz}).$$

Divide by 16 to convert the answer into ft·lb. Answer: Approximately 690.533966 ft·lb. To check the answer without using calculus, note that the string is lifted an average distance of 250 feet. Multiply this by the weight of the string in pounds to obtain the answer in ft·lb.

C06S05.030: Set up a coordinate system in which the center of the sphere is located at the origin. The cross section of the liquid in the tank "at" position y $(-R \leqq y \leqq R)$ is circular with radius $x = \sqrt{R^2 - y^2}$. Hence the work to fill the tank is

$$W = \int_{-R}^{R} (y + H)\,\rho\pi\left(R^2 - y^2\right)\,dy$$

$$= \rho\pi\left[R^2Hy + \frac{1}{2}R^2y^2 - \frac{1}{3}Hy^3 - \frac{1}{4}y^4\right]_{-R}^{R}$$

$$= \rho\pi\left(2R^3H - \frac{2}{3}R^3H\right) = \frac{4}{3}\pi\rho R^3H.$$

This is the product of the volume $\frac{4}{3}\pi R^3$ of the tank, the weight density ρ of the liquid, and the distance H from the ground to the center of the tank, and the result in Problem 30 now follows.

C06S05.031: Set up a coordinate system with the y-axis vertical and the x-axis coinciding with the bottom of one end of the trough. A horizontal section of the trough at y is $2-y$ feet below the water surface, so the total force on the end of the trough is given by

$$F = \int_0^2 (2)(2-y)\rho\,dy = \rho\Big[4y - y^2\Big]_0^2 = 4\rho = 249.6 \quad \text{(lb)}.$$

C06S05.032: Set up a coordinate system in which the origin is at the lowest point of the triangular end of the trough and the y-axis is vertical. A narrow horizontal strip at height y has width $2x = \frac{2}{3}y\sqrt{3}$. Therefore the total force on the end of the trough is given by

$$F = \int_0^{\frac{3}{2}\sqrt{3}} \rho\left(\frac{3}{2}\sqrt{3} - y\right)\left(\frac{2}{3}y\sqrt{3}\right)\,dy = \rho\int_0^{\frac{3}{2}\sqrt{3}}\left(3y - \frac{2}{3}y^2\sqrt{3}\right)\,dy$$

$$= \rho\left[\frac{3}{2}y^2 - \frac{2\sqrt{3}}{9}y^3\right]_0^{\frac{3}{2}\sqrt{3}} = \frac{27}{8}\rho = 210.6 \quad \text{(lb)}.$$

C06S05.033: Set up a coordinate system in which one end of the trough lies in the [vertical] xy-plane with its base on the x-axis and bisected by the y-axis. Thus the trapezoidal end of the trough has vertices at the points $(1, 0)$, $(-1, 0)$, $(2, 3)$, and $(-2, 3)$. Because the width of a horizontal section at height y is $2x = \frac{2}{3}(y+3)$, the total force on the end of the trough is

$$F = \int_0^3 \rho\frac{2}{3}(y+3)(3-y)\,dy = \rho\left[6y - \frac{2}{9}y^3\right]_0^3 = 12\rho = 748.8 \quad \text{(lb)}.$$

C06S05.034: Describe the end of the tank by the inequality $x^2 + y^2 \leqq 16$, so that a horizontal section at level y has width $2x = 2\left(16-y^2\right)^{1/2}$. Note that $\rho = 50$ in this problem. Then the total force on the end of the tank is

$$F = \int_{-4}^4 \rho(2)(4-y)\left(16-y^2\right)^{1/2}\,dy = 2\rho\int_{-4}^4 4\left(16-y^2\right)^{1/2}\,dy - 2\rho\int_{-4}^4 y\left(16-y^2\right)^{1/2}\,dy.$$

The last integral involves the evaluation of $(16-y^2)^{3/2}$ at $y = 4$ and at $y = -4$, so its value is zero. The next-to-last is the product of 8ρ and the area of a semicircle of radius 4, so its value is

$$(8\rho)(8\pi) = 64\rho\pi = 3200\pi \approx 10053.1 \text{ (lb)}.$$

C06S05.035: Let $\rho = 62.4$ lb/ft^3 and set up a coordinate system in which the y-axis is vertical and $y = 0$ is the location of the bottom of the square gate. Then the pressure at a horizontal cross section of the plate at location y will be $(15-y)\rho$ and the area of the cross section will be $5\,dy$, so the total force on the gate will be

$$F = \int_0^5 5\rho(15-y)\,dy = \rho\left[75y - \frac{5}{2}y^2\right]_0^5 = 19500 \quad \text{(lb)}.$$

C06S05.036: Let $\rho = 62.4$, as usual. Put the origin at the center of the circle. Then the force on the gate is

$$F = \int_{-3}^{3} 2\rho(13-y)\left(9-y^2\right)^{1/2}\,dy = 26\rho\int_{-3}^{3}\left(9-y^2\right)^{1/2}\,dy - 2\rho\int_{-3}^{3} y\left(9-y^2\right)^{1/2}\,dy.$$

The last integral involves the evaluation of $(9-y^2)^{1/2}$ at $y=3$ and at $y=-3$, so its value is zero. The next-to-last is the product of 26ρ and the area of a semicircle of radius 3, so its value is the product of 26ρ and $\frac{9}{2}\pi$: $117\pi\rho \approx 22936.139645$ (lb).

C06S05.037: Let ρ denote the density of water, as usual. Place the origin at the low vertex of the triangular gate with the y-axis vertical. Then a horizontal cross section of the gate at y has width $\frac{8}{5}y$ and depth $15-y$, so the total force on the gate will be

$$F = \int_0^5 \frac{8}{5}\rho y(15-y)\,dy = \rho\left[12y^2 - \frac{8}{15}y^3\right]_0^5 = 14560 \quad \text{(lb)}.$$

C06S05.038: Place the origin at the center of the diameter of the gate with the y-axis vertical; let $\rho = 62.4$ lb/ft^3 denote the density of water. The equation of the semicircle is $x^2+y^2=16$, $y \leqq 0$. So the force on the gate is

$$F = \int_{-4}^{0} 2\rho(10-y)\left(16-y^2\right)^{1/2}\,dy = 20\rho\int_{-4}^{0}\left(16-y^2\right)^{1/2}\,dy - 2\rho\int_{-4}^{0} y\left(16-y^2\right)^{1/2}\,dy.$$

The next-to-last integral is the product of 20ρ and the area of a quarter-circle of radius 4, so its value is $80\pi\rho$. Therefore

$$F = 80\pi\rho - 2\rho\left[-\frac{1}{3}(16-y^2)^{3/2}\right]_{-4}^{0} = 80\pi\rho + \frac{128}{3}\rho \approx 18345.230527 \quad \text{(lb)}.$$

C06S05.039: Set up a coordinate system in which the bottom of the *vertical* face of the dam lies on the x-axis and the y-axis is vertical, with the origin at the center of the bottom of the vertical face of the dam. The vertical face occupies the interval $0 \leqq y \leqq 100$; form a regular partition of this interval and suppose that $[y_{i-1},\, y_i]$ is one of the subintervals in this partition. Horizontal lines perpendicular to the vertical face through the endpoints of this interval determine a strip on the slanted face of length 200 (units are in feet) and (by similar triangles) width

$$\frac{\sqrt{100^2+30^2}}{100}(y_i - y_{i-1}) = \frac{\sqrt{100^2+30^2}}{100}\,\Delta y.$$

If $y_i^\star$ lies in this interval, then the strip on the slanted face opposite $[y_{i-1},\, y_i]$ has approximate depth $100-y_i^\star$. So the total force of water on this strip—acting normal to the slanted face—is

$$200\rho(100-y_i^\star)\cdot\frac{\sqrt{100^2+30^2}}{100}\,\Delta y$$

where $\rho = 62.4$ lb/ft^3 is the density of the water. Therefore the total force the water exerts on the slanted face of the dam—normal to that face—is

$$\begin{aligned} F &= \int_0^{100} 200\rho(100-y)\cdot\frac{\sqrt{100^2+30^2}}{100}\,dy \\ &= \rho(109)^{1/2}\left[2000y - 10y^2\right]_0^{100} = 100000\rho\sqrt{109} \approx 6.514721\times 10^7 \quad \text{(lb)}. \end{aligned}$$

The horizontal component of this force can be found by multiplying the total force by $10/\sqrt{109}$, and is thus 6.24×10^7 (lb).

C06S05.040: Given

$$f(x) = 1 + \frac{1}{5}x^2 - \frac{1}{500}x^4,$$

water filling the birdbath occupies the space region S generated by rotation around the y-axis of the plane region R bounded on the left by the y-axis, above by the line $y = 6$, and below and on the right by the graph of $f(x)$ for $0 \leqq x \leqq \alpha = 5\sqrt{2}$. With measurements in inches, fresh water has weight density

$$\rho = \frac{62.4}{1728}$$

pounds per cubic inch; we will convert to ft·lb at the end of the solution.

Solution (a): By the method of nested cylindrical shells. Such a shell, with centerline the y-axis and meeting the positive x-axis at x, has radius x, height $6 - f(x)$ and the water comprising it has been lifted an average distance of

$$40 + f(x) + \frac{6 - f(x)}{2}$$

inches. Hence the work to lift all the water from ground level to fill the space region S is

$$\begin{aligned} W &= \int_0^\alpha 2\pi\rho x\,[6 - f(x)] \cdot \left[40 + f(x) + \frac{6 - f(x)}{2}\right] dx \\ &= \pi \cdot \left[\frac{377x^2}{48} - \frac{533x^4}{3600} + \frac{403x^6}{540000} + \frac{13x^8}{3600000} - \frac{13x^{10}}{900000000}\right]_0^\alpha = \frac{28925\pi}{216}. \end{aligned}$$

We divide by 12 to convert the answer to

$$\frac{28925\pi}{2592} \approx 35.058089315233 \quad \text{(ft·lb)}.$$

Solution (b): By the method of parallel slabs. A horizontal slab "at" location y, $1 \leqq y \leqq 6$, has radius

$$x = \sqrt{50 - 10\sqrt{30 - 5y}}\,.$$

Hence the work to pump water originally at ground level to fill the region S will be

$$\begin{aligned} W &= \int_1^6 \pi\rho(40 + y) \cdot \left(50 - 10\sqrt{30 - 5y}\right) dy \\ &= \pi \cdot \left[\frac{650y}{9} + \frac{65y^2}{72} + \left(\frac{2756}{45} - \frac{1261y}{135} - \frac{13y^2}{90}\right)\sqrt{30 - 5y}\right]_1^6 = \frac{28925\pi}{216}, \end{aligned}$$

and this solution concludes in the same way, and with the same result, as the previous solution.

Section 6.6

C06S06.001: By symmetry, the centroid is located at $(2, 3)$.

C06S06.002: By symmetry, the centroid is at $(2, 3)$.

C06S06.003: By symmetry, the centroid is at $(1, 1)$.

C06S06.004: The area of the triangle is $\dfrac{9}{2}$ and

$$M_y = \int_0^3 x(3-x)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \frac{27}{2} - \frac{18}{2} = \frac{9}{2}.$$

Hence $\overline{x} = 1$, and by symmetry $\overline{y} = 1$ as well.

C06S06.005: The area of the triangular region is $A = 4$, and

$$M_y = \int_0^4 x\left(2 - \frac{1}{2}x\right)dx = \left[x^2 - \frac{1}{6}x^3\right]_0^4 = 16 - \frac{32}{3} = \frac{16}{3}.$$

Hence $\overline{x} = \dfrac{4}{3}$. Next,

$$M_x = \int_0^4 \frac{1}{2}\cdot\left(\frac{4-x}{2}\right)^2 dx = \frac{1}{8}\int_0^4 (16 - 8x + x^2)\,dx = \frac{1}{8}\left[16x - 4x^2 + \frac{1}{3}x^3\right]_0^4 = \frac{8}{3},$$

and therefore $\overline{y} = \dfrac{2}{3}$.

C06S06.006: The area of the triangular region is $A = 1$. Moreover, $\overline{x} = 1$ by symmetry. Using additivity of moments,

$$M_x = 2\int_0^1 \frac{1}{2}x^2\,dx = \frac{1}{3}.$$

Therefore $\overline{y} = \dfrac{1}{3}$.

C06S06.007: The area of the region is

$$A = \int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}.$$

Next,

$$M_y = \int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4 \quad \text{and}$$

$$M_x = \frac{1}{2}\int_0^2 x^4\,dx = \left[\frac{1}{10}x^5\right]_0^2 = \frac{16}{5}.$$

Therefore $(\overline{x}, \overline{y}) = \left(\dfrac{3}{2}, \dfrac{6}{5}\right)$.

C06S06.008: The area of the region is

$$A = 2\int_0^3 (9 - x^2)\,dx = 2\left[9x - \frac{1}{3}x^3\right]_0^3 = 2\cdot(27-9) = 36.$$

Also $\overline{x} = 0$ by symmetry. Finally,

$$M_x = \frac{1}{2}\int_{-3}^3 (9^2 - x^4)\,dx = \left[81x - \frac{1}{5}x^5\right]_0^3 = 243 - \frac{243}{5} = \frac{972}{5}.$$

Therefore $\overline{y} = \dfrac{972}{5\cdot 36} = \dfrac{27}{5}$.

C06S06.009: The area of the region is

$$A = 2\int_0^2 (4 - x^2)\,dx = 2\left[4x - \frac{1}{3}x^3\right]_0^2 = 2\cdot\left(8 - \frac{8}{3}\right) = \frac{32}{3}.$$

Now $\overline{x} = 0$ by symmetry, but

$$\begin{aligned} M_x &= -\frac{1}{2}\int_{-2}^2 (4-x^2)^2\,dx = -\int_0^2 (x^2-4)^2\,dx = -\int_0^2 (x^4 - 8x^2 + 16)\,dx \\ &= -\left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x\right]_0^2 = -\left(\frac{32}{5} - \frac{64}{3} + 32\right) = -\frac{256}{15}. \end{aligned}$$

Therefore $\overline{y} = -\dfrac{256}{15}\cdot\dfrac{3}{32} = -\dfrac{8}{5}$.

C06S06.010: By symmetry, $\overline{x} = 0$. Next,

$$M_x = \frac{1}{2}\int_{-2}^2 (x^2+1)^2\,dx = \frac{206}{15}.$$

The area of the region is $\dfrac{28}{3}$, and hence $\overline{y} = \dfrac{103}{70}$.

C06S06.011: The area of the region is

$$A = 2\int_0^2 (4 - x^2)\,dx = \frac{32}{3}, \quad \text{and}$$

$$M_x = \int_{-2}^2 \frac{1}{2}(4-x^2)^2\,dx = \frac{256}{15}.$$

Therefore $\overline{y} = \dfrac{256\cdot 3}{15\cdot 32} = \dfrac{8}{5}$; $\overline{x} = 0$ by symmetry.

C06S06.012: The area of the region is

$$A = \int_0^3 (18 - 2x^2)\,dx = 36 \quad \text{and}$$

$$M_x = \int_0^3 \frac{1}{2}\cdot 18\cdot(18 - 2x^2)\,dx = 324.$$

Therefore $(\overline{x}, \overline{y}) = (0, 9)$. Alternatively, you can find the centroid by symmetry.

C06S06.013: The area of the region is $A = 1$. Moreover,

$$M_y = \int_0^1 3x^3\, dx = \frac{3}{4} \quad \text{and} \quad M_x = \int_0^1 \frac{9}{2}x^4\, dx = \frac{9}{10}.$$

Therefore $(\overline{x}, \overline{y}) = \left(\dfrac{3}{4}, \dfrac{9}{10}\right)$.

C06S06.014: The area of the region is

$$\begin{aligned} A &= \int_0^4 \sqrt{x}\, dx = \frac{16}{3}, \\ M_y &= \int_0^4 x^{3/2}\, dx = \frac{64}{5}, \quad \text{and} \\ M_x &= \int_0^2 y(4 - y^2)\, dy = 4. \end{aligned}$$

Therefore $(\overline{x}, \overline{y}) = \left(\dfrac{12}{5}, \dfrac{3}{4}\right)$.

C06S06.015: The parabola and the line meet at the two points $P(-3, -3)$ and $Q(2, 2)$. Hence

$$\begin{aligned} A &= \int_{-3}^2 (6 - x - x^2)\, dx = \frac{125}{6}, \\ M_y &= \int_{-3}^2 (6x - x^2 - x^3)\, dx = -\frac{125}{12}, \quad \text{and} \\ M_x &= \int_{-3}^2 \left[\frac{1}{2}(6 - x^2)^2 - \frac{1}{2}x^2\right] dx = \frac{125}{3}. \end{aligned}$$

Therefore the centroid is located at the point $\left(-\dfrac{1}{2}, 2\right)$.

C06S06.016: First note that $\overline{x} = \overline{y}$ by symmetry. Next,

$$\begin{aligned} A &= \int_0^1 \left(\sqrt{x} - x^2\right)\, dx = \frac{1}{3} \quad \text{and} \\ M_y &= \int_0^1 \left(x^{3/2} - x^3\right)\, dx = \frac{3}{20}. \end{aligned}$$

Therefore the centroid is located at the point $\left(\dfrac{9}{20}, \dfrac{9}{20}\right)$.

C06S06.017: The region has area $A = \dfrac{1}{12}$. Next,

$$\begin{aligned} M_y &= \int_0^1 (x^3 - x^4)\, dx = \frac{1}{20} \quad \text{and} \\ M_x &= \frac{1}{2}\int_0^1 (x^4 - x^6)\, dx = \frac{1}{35}. \end{aligned}$$

Thus $(\overline{x}, \overline{y}) = \left(\frac{3}{5}, \frac{12}{35}\right)$.

C06S06.018: By symmetry, $\overline{x} = \frac{\pi}{2}$. Next,

$$A = \int_0^\pi \sin x \, dx = 2 \quad \text{and}$$

$$M_x = \int_0^\pi \frac{1}{2} \sin^2 x \, dx = \frac{\pi}{4}.$$

Therefore $\overline{y} = \frac{\pi}{8}$.

C06S06.019: First note that $\overline{y} = \overline{x}$ by symmetry. Next,

$$M_y = \int_0^r x(r^2 - x^2)^{1/2} \, dx = \left[-\frac{1}{3}(r^2 - x^2)^{3/2}\right]_0^r = \frac{1}{3}r^3$$

and the area of the quarter circle is $\frac{1}{4}\pi r^2$. Therefore the centroid is at

$$(\overline{x}, \overline{y}) = \left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right).$$

C06S06.020: By Pappus's first theorem,

$$\left(\frac{1}{4}\pi r^2\right) \cdot (2\pi\overline{x}) = \frac{2}{3}\pi r^3.$$

Therefore $\overline{x} = \dfrac{\frac{1}{3}r^3}{\frac{1}{4}\pi r^2} = \dfrac{4r}{3\pi}$. By symmetry, $\overline{y} = \overline{x}$.

C06S06.021: By symmetry, $\overline{y} = \overline{x}$. Because $y = (r^2 - x^2)^{1/2}$,

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{r^2 - x^2}.$$

So

$$M_y = \int_0^r \frac{rx}{(r^2 - x^2)^{1/2}} \, dx = \Big[-r(r^2 - x^2)^{1/2}\Big]_0^r = r^2.$$

The length of the quarter circle is $\frac{1}{2}\pi r$, so $\overline{x} = \frac{2r}{\pi}$.

C06S06.022: By Pappus's second theorem,

$$\left(\frac{1}{2}\pi r\right) \cdot (2\pi\overline{x}) = 2\pi r^2.$$

Therefore—with the aid of symmetry—$\overline{x} = \dfrac{2r}{\pi} = \overline{y}$.

C06S06.023: First,

$$M_y = \int_0^r x\left(h - \frac{hx}{r}\right)\,dx = \frac{1}{6}hr^2 \quad \text{and} \quad A = \frac{1}{2}rh.$$

Therefore $\overline{x} = r/3$. By interchanging the roles of x and y, we find that $\overline{y} = \overline{x}$.

Next, the midpoint of the hypotenuse is $(r/2,\ h/2)$ and its slope is $-h/r$. The line L from $(0,\ 0)$ to the midpoint has equation

$$y = \frac{h}{r}x.$$

If $x = r/3$, then $y = h/3$, so $(r/3,\ h/3)$ lies on the line L. The distance from $(0,\ 0)$ to $(r/3,\ h/3)$ is

$$D_1 = \frac{1}{3}(r^2 + h^2)^{1/2};$$

the distance from $(0,\ 0)$ to $(r/2,\ h/2)$ is

$$D_2 = \frac{1}{2}(r^2 + h^2)^{1/2}.$$

Therefore

$$\frac{D_1}{D_2} = \frac{2}{3},$$

and this concludes the proof.

C06S06.024: $V = \left(2\pi\frac{r}{3}\right)\cdot\left(\frac{1}{2}rh\right) = \frac{1}{3}\pi r^2 h.$

C06S06.025: $A = \left(2\cdot\frac{\pi r}{2}\right)\sqrt{r^2 + h^2} = \pi r(r^2 + h^2)^{1/2} = \pi rL.$

C06S06.026: (a) Part (1): the rectangle. Its area is $A = r_2 h$, and thus

$$M_y = \int_0^{r_2} xh\,dx = \frac{1}{2}hr_2^2;$$

$$M_x = \int_0^{r_2} \frac{1}{2}h^2\,dx = \frac{1}{2}h^2 r_2.$$

Part (2): the triangle. Its area is $A = \dfrac{1}{2}h(r_2 - r_1)$. By the result in Problem 23,

$$\overline{x} = r_2 + \frac{1}{3}(r_1 - r_2) = \frac{1}{3}(r_1 + 2r_2),$$

and $\overline{y} = h/3$. Therefore

$$M_y = A\overline{x} = \frac{h}{6}(r_1 + 2r_2)(r_1 - r_2) \quad \text{and}$$

$$M_x = A\overline{y} = \frac{h^2}{6}(r_1 - r_2).$$

Part (3): the trapezoid. By additivity of moments,

$$M_y = \frac{1}{2}hr_2^2 + \frac{h}{6}(r_1 + 2r_2)(r_1 - r_2) = \frac{h}{6}(r_1^2 + r_1 r_2 + r_2^2);$$

$$M_x = \frac{1}{2}h^2 r_2 + \frac{h^2}{6}(r_1 - r_2) = \frac{h^2}{6}(r_1 + 2r_2).$$

Answer to (a):

$$\overline{x} = \frac{1}{A}M_y = \frac{r_1^2 + r_1 r_2 + r_2^2}{3(r_1 + r_2)},$$

$$\overline{y} = \frac{1}{A}M_x = \frac{h(r_1 + 2r_2)}{3(r_1 + r_2)}.$$

Answer to (b): By Pappus's first theorem,

$$V = 2\pi\overline{x}A = 2\pi M_y = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2).$$

C06S06.027: The radius of revolution is $\frac{1}{2}(r_1 + r_2)$, so the lateral area is

$$A = 2\pi \cdot \left(\frac{r_1 + r_2}{2}\right)\left[(r_2 - r_1)^2 + h^2\right]^{1/2} = \pi(r_1 + r_2)L.$$

C06S06.028: The lateral surface is generated by rotating around the axis of the cylinder a vertical line of length h. Its midpoint is at $(r, h/2)$ and the radius of the circle around which the midpoint moves is r, so the lateral surface area is $A = (2\pi r)h$. Alternatively, from Problem 27,

$$A = \pi(r_1 + r_2)L = 2\pi rh.$$

C06S06.029: The semicircular region has centroid (x, y) where $x = 0$ (by symmetry) and $y = b + \frac{4a}{3\pi}$ (by earlier work). So, for the semicircular region,

$$M_x = \left(b + \frac{4a}{3\pi}\right) \cdot \left(\frac{1}{2}\pi a^2\right).$$

For the rectangle, we have

$$M_x = \frac{b}{2} \cdot 2ab = ab^2.$$

The sum of these two moments is the moment of the entire region:

$$M_x = \frac{2}{3}a^3 + \frac{\pi}{2}a^2 b + ab^2.$$

When we divide this moment by the area $2ab + \frac{1}{2}\pi a^2$ of the entire region, we find that

$$\overline{y} = \frac{4a^2 + 3\pi ab + 6b^2}{12b + 3\pi a}.$$

Of course $\overline{x} = 0$ by symmetry.

For Part (b), we use the fact that the radius of the circle of rotation is $\overline{y}$, so the volume generated by rotation around the x-axis is

$$V = 2\pi\overline{y}A = 2\pi\overline{y}\left(2ab + \frac{1}{2}\pi a^2\right)$$

$$= \pi\overline{y}(4ab + \pi a^2) \;=\; \pi a\overline{y}(4b + \pi a) = \frac{1}{3}\pi a\overline{y}(12b + 3\pi a);$$

that is,

$$V \;=\; \frac{1}{3}\pi a(4a^2 + 3\pi ab + 6b^2).$$

C06S06.030: (a) First note that

$$A \;=\; \int_0^h \sqrt{2py}\; dy \;=\; \frac{2}{3}h^{3/2}\sqrt{2p}\,.$$

Now $r^2 = 2ph$, so

$$A \;=\; \frac{2}{3}h\sqrt{2ph} \;=\; \frac{2}{3}rh. \quad \text{But}$$

$$M_y \;=\; \int_0^h \frac{1}{2}(2py)\; dy \;=\; \frac{1}{2}ph^2,$$

so $\overline{x} = \dfrac{ph^2/2}{2rh/3} = \dfrac{3ph}{4r}$. But $ph = \dfrac{1}{2}r^2$, so $\overline{x} \;=\; \frac{3}{8}r$.

Part (b): $V = 2\pi\overline{x}A = 2\pi M_y = \pi ph^2$. But $ph = \dfrac{1}{2}r^2$, so

$$V \;=\; \frac{1}{2}\pi r^2 h.$$

C06S06.031: Let $f(x) = x$ and $g(x) = x^2$. The area of the region bounded by the graphs of f and g is

$$A \;=\; \int_0^1 (x - x^2)\; dx \;=\; \frac{1}{2} - \frac{1}{3} \;=\; \frac{1}{6}.$$

The moments with respect to the coordinate axes are

$$M_y \;=\; \int_0^1 (x^2 - x^3)\; dx \;=\; \frac{1}{3} - \frac{1}{4} \;=\; \frac{1}{12} \quad \text{and}$$

$$M_x \;=\; \int_0^1 \frac{1}{2}(x^2 - x^4)\; dx \;=\; \frac{1}{2}\cdot\left(\frac{1}{3} - \frac{1}{5}\right) \;=\; \frac{1}{15}.$$

Therefore the centroid is located at the point

$$(\overline{x},\, \overline{y}) \;=\; C\left(\frac{1}{2},\, \frac{2}{5}\right).$$

The axis L of rotation is the line $y = x$; the line through the centroid perpendicular to L has equation

$$y \;=\; \frac{9}{10} - x$$

and this perpendicular meets L at the point $P\left(\dfrac{9}{20},\, \dfrac{9}{20}\right)$. The distance from P to C is

$$d \;=\; \sqrt{\left(\frac{1}{2} - \frac{9}{20}\right)^2 + \left(\frac{2}{5} - \frac{9}{20}\right)^2} \;=\; \frac{\sqrt{2}}{20}.$$

Because d is the radius of the circle through which C is rotated, the volume generated is (by the first theorem of Pappus)

$$V = 2\pi d A = 2\pi \cdot \frac{\sqrt{2}}{20} \cdot \frac{1}{6} = \frac{\pi\sqrt{2}}{60} \approx 0.07404804897.$$

C06S06.032: We let $f(x) = x^m$ and $g(x) = x^n$. Then we used *Mathematica* 3.0:

```
A = Integrate[ f[x] − g[x], { x, 0, 1 } ]
```

$$\frac{1}{1+m} - \frac{1}{1+n}$$

Then we compute the moments:

```
My = Integrate[ x*(f[x] − g[x]), { x, 0, 1 } ];
Mx = Integrate[ (1/2)*( (f[x])∧2 − (g[x])∧2 ), { x, 0, 1 } ];
```

Thus the centroid has coordinates

```
{ xc, yx } = { My/A, Mx/A } // Simplify
```

$$\left\{ \frac{(1+m)(1+n)}{(2+m)(2+n)}, \frac{1+m+n+mn}{1+2m+2n+4mn} \right\}$$

For selected values of m and with $n = m + 1$ we check to see if it's `True` that the centroid lies within the region:

```
m = 1; n = m + 1;
yc < xc∧m
yc > xc∧n
```

True

True

```
m = 2; n = m + 1;
yc < xc∧m
yc > xc∧n
```

True

True

```
m = 3; n = m + 1;
yc < xc∧m
yc > xc∧n
```

False

True

Therefore if $m = 3$ and $n = 4$, then the centroid does *not* lie within the region.

Chapter 6 Miscellaneous Problems

C06S0M.001: The net distance is

$$\int_0^3 v(t)\,dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t\right]_0^3 = -\frac{3}{2} - 0 = -\frac{3}{2}.$$

Because $v(t) < 0$ for $0 < t < 2$, the total distance is

$$-\int_0^2 v(t)\,dt + \int_2^3 v(t)\,dt = \left(\frac{10}{3} - 0\right) + \left(-\frac{3}{2} + \frac{10}{3}\right) = \frac{31}{6} \approx 5.166667.$$

C06S0M.002: Because $t^2 - 4 < 0$ for $1 < t < 2$ but $t^2 - 4 > 0$ for $2 < t < 4$, whereas $v(t) \geqq 0$ for $1 \leqq t \leqq 4$, the net and total distance are both

$$-\int_1^2 (t^2-4)\,dt + \int_2^4 (t^2-4)\,dt = -\left[\frac{1}{3}t^3 - 4t\right]_1^2 + \left[\frac{1}{3}t^3 - 4t\right]_2^4 = \frac{5}{3} + \frac{32}{3} = \frac{37}{3} \approx 12.333333.$$

C06S0M.003: Because $v(t) < 0$ for $0 < t < \frac{1}{2}$ but $v(t) > 0$ for $\frac{1}{2} < t < \frac{3}{2}$, the net distance is

$$\int_0^{3/2} v(t)\,dt = \left[-\cos\left(\frac{1}{2}\pi(2t-1)\right)\right]_0^{3/2} = 1 - 0 = 1$$

and the total distance is

$$-\int_0^{1/2} v(t)\,dt + \int_{1/2}^{3/2} v(t)\,dt = (1-0) + (1-(-1)) = 3.$$

C06S0M.004: The volume is $\displaystyle\int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$

C06S0M.005: The volume is $\displaystyle\int_1^4 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3} \approx 4.666667.$

C06S0M.006: The volume is $\displaystyle\int_1^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_1^2 = 4 - \frac{1}{4} = \frac{15}{4} = 3.75.$

C06S0M.007: The volume is $\displaystyle\int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{15} \approx 0.4188790205.$

C06S0M.008: The volume is $\displaystyle\int_{-1}^1 x^{100}\,dx = \left[\frac{1}{101}x^{101}\right]_{-1}^1 = \frac{2}{101} \approx 0.0198019802.$

C06S0M.009: Between time $t = 0$ and time $t = 12$, the rainfall in inches is

$$\int_0^{12} \frac{1}{12}(t+6)\,dt = \left[\frac{1}{2}t + \frac{1}{24}t^2\right]_0^{12} = 12 - 0 = 12.$$

C06S0M.010: The curves meet at $(0,\ 0)$ and at $(1,\ 1)$, and the quadratic is higher than the cubic between those points. A cross section of the solid perpendicular to the x-axis is a square of base length $2x - x^2 - x^3$ and thus the cross section has area $A(x) = (2x - x^2 - x^3)^2$. Hence the volume of the solid is

$$V = \int_0^1 A(x)\,dx = \left[\frac{4}{3}x^3 - x^4 - \frac{3}{5}x^5 + \frac{1}{3}x^6 + \frac{1}{7}x^7\right]_0^1 = \frac{22}{105} \approx 0.2095238095.$$

C06S0M.011: The region R of Problem 10 is bounded above by the graph of $f(x) = 2x - x^2$ and below by the graph of $g(x) = x^3$ for $0 \leqq x \leqq 1$. A cross section of the solid S of Problem 11 perpendicular to the x-axis at x is an annular region with outer radius $f(x)$ and inner radius $g(x)$, thus of cross-sectional area $A(x) = \pi\,[\,f(x)\,]^2 - \pi\,[\,g(x)\,]^2$. Therefore the volume of S is

$$V = \int_0^1 A(x)\,dx = \pi\left[\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{41}{105}\pi - 0 \approx 1.2267171314.$$

C06S0M.012: The region R of this problem is bounded above by the graph of $f(x) = x^2 + 1$ and below by the graph of $g(x) = 2x^4$ for $-1 \leqq x \leqq 1$ and (important) is symmetric around the y-axis. If R is rotated around the x-axis, the solid S that it generates has as cross sections perpendicular to the x-axis at x annular regions with outer radius $f(x)$ and inner radius $g(x)$, thus of cross-sectional area $A(x) = \pi\,[\,f(x)\,]^2 - \pi\,[\,g(x)\,]^2$. Therefore the volume of S is

$$V_1 = \int_{-1}^1 A(x)\,dx = \pi\left[x + \frac{2}{3}x^3 + \frac{1}{5}x^5 - \frac{4}{9}x^9\right]_{-1}^1 = \frac{64\pi}{45} - \left(-\frac{64\pi}{45}\right) = \frac{128\pi}{45} \approx 8.9360857702.$$

If R is rotated around the y-axis to form the solid T, then (using the symmetry of R around the y-axis) the method of cylindrical shells yields the volume of T as

$$V_2 = \int_0^1 2\pi x\,[\,f(x) - g(x)\,]\,dx = \pi\left[x^2 + \frac{1}{2}x^4 - \frac{2}{3}x^6\right]_0^1 = \frac{5\pi}{6} \approx 2.6179938780.$$

C06S0M.013: Each cross section perpendicular to the x-axis has area $A(x) = \frac{1}{16}\pi$, so the total mass of the helix is

$$m = \int_0^{20} (8.5)\cdot A(x)\,dx = \left[\frac{17\pi}{32}\,x\right]_0^{20} = \frac{85\pi}{8} \approx 33.3794219444 \quad \text{(grams)}.$$

C06S0M.014: Most of the natural ways to solve this problem involve algebraic difficulties. For example, the side of the frustum should *not* be part of the graph of $y = mx$, even though this would seem to yield the simplest choice. In each case, both the method of cross sections and the method of cylindrical shells lead to difficulties. Here's the simplest solution we've found. Write r for r_1 and s for r_2. Sketch a trapezoid in the first quadrant with vertices at (0, 0), (h, 0), (h, s), and (0, r). Then an equation of the top edge of the trapezoid is

$$y = r + \frac{s - r}{h}x.$$

The frustum is produced by rotating the trapezoidal region around the x-axis, and its volume is

$$V = \int_0^h \pi\left(r + \frac{s-r}{h}x\right)^2 dx = \pi\int_0^h \left[r^2 + \frac{2r(s-r)}{h}x + \left(\frac{s-r}{h}\right)^2 x^2\right] dx$$

$$= \pi\left[r^2x + \frac{r(s-r)}{h}x^2 + \frac{1}{3}\left(\frac{s-r}{h}\right)^2 x^3\right]_0^h = \pi\left(r^2h + r(s-r)h + \frac{1}{3}(s-r)^2h\right)$$

$$= \frac{1}{3}\pi h(3r^2 + 3rs - 3r^2 + s^2 - 2rs + r^2) = \frac{\pi h}{3}\left(r^2 + rs + s^2\right) = \frac{\pi h}{3}\left(r_1^2 + r_1r_2 + r_2^2\right).$$

C06S0M.015: Let z denote the distance from P to the origin. A horizontal cross-section of the elliptical cone "at" z (thus at distance z from P) is an ellipse with major axis and minor axis each proportional to z. So the area $A(z)$ of this cross section is proportional to z^2: $A(z) = kz^2$ where k is a positive constant. But

$$A(h) = kh^2 = \pi ab$$

by the result of Problem 47 of Section 5.8, and hence $k = \pi ab/h^2$. Therefore $A(z) = \pi abz^2/h^2$. So the volume of the elliptical cone is

$$V = \int_0^h \frac{\pi ab}{h^2} z^2 \, dz = \left[\frac{\pi ab}{3h^2} z^3 \right]_0^h = \frac{1}{3}\pi abh,$$

one-third the product of the area of the base and the height of the elliptical cone.

C06S0M.016: Because $(a - h, r)$ lies on the ellipse, $\left(\dfrac{a-h}{a}\right)^2 + \left(\dfrac{r}{b}\right)^2 = 1$. Therefore $r^2 = \dfrac{2ah - h^2}{a^2}\, b^2$. And so

$$V = \int_{a-h}^a \pi y^2 \, dx = \int_{a-h}^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi \frac{b^2h^2}{3a^2} (3a - h)\,.$$

But $r^2 = \dfrac{b^2}{a^2}\, h(2a - h)$, so $\dfrac{b^2}{a^2}\, h = \dfrac{r^2}{2a - h}$. Therefore

$$V = \tfrac{1}{3}\pi r^2 h \,\frac{3a - h}{2a - h}.$$

C06S0M.017: Because $(a + h,\, r)$ lies on the hyperbola,

$$\frac{(a + h)^2}{a^2} - \frac{r^2}{b^2} = 1.$$

It follows that

$$r^2 = \frac{b^2(2ah + h^2)}{a^2}. \tag{1}$$

Moreover, the equation of the hyperbola may be written in the form

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

Therefore the "segment of the hyperboloid" has volume

$$\begin{aligned} V &= \int_a^{a+h} \pi y^2 \, dx = \frac{\pi b^2}{a^2} \int_a^{a+h} (x^2 - a^2)\, dx = \frac{\pi b^2}{a^2} \left[\frac{1}{3}x^3 - a^2 x \right]_a^{a+h} \\ &= \frac{\pi b^2}{3a^2} \left[x^3 - 3a^2 x \right]_a^{a+h} = \frac{\pi b^2}{3a^2} \left(a^3 + 3a^2h + 3ah^2 + h^3 - 3a^3 - 3a^2h - a^3 + 3a^3 \right) \\ &= \frac{1}{3}\pi \frac{b^2}{a^2} h^2 (3a + h). \end{aligned}$$

But by Eq. (1), $b^2 = \dfrac{a^2r^2}{2ah + h^2}$. So $V = \dfrac{1}{3}\pi \dfrac{h^2}{a^2}(3a + h)\, \dfrac{a^2r^2}{h(2a + h)} = \dfrac{1}{3}\pi r^2 h\, \dfrac{3a + h}{2a + h}$.

C06S0M.018: $V(t) = \displaystyle\int_1^t \pi\,(f(x))^2\,dx = \pi\left(1 - \frac{1}{t}\right)$, so $V'(t) = \pi\,(f(t))^2 = \dfrac{\pi}{t^2}$. Therefore $f(x) = \dfrac{1}{x}$.

C06S0M.019: $V = \displaystyle\int_1^t \pi\,(f(x))^2\,dx = \frac{\pi}{6}\left[(1+3t)^2 - 16\right]$. Thus

$$\pi\,(f(x))^2 = \frac{\pi}{6}\,[(2)(1+3x)(3)] = \pi(1+3x).$$

Therefore $f(x) = \sqrt{1+3x}\,$.

C06S0M.020: $V(t) = \displaystyle\int_1^t 2\pi x f(x)\,dx = \tfrac{2}{9}\pi\left(\left(1+3t^2\right)^{3/2} - 8\right)$, so

$$V'(t) = 2\pi t f(t) = \tfrac{2}{9}\pi\left(\tfrac{3}{2}\sqrt{1+3t^3}\,(6t)\right) = 2\pi t\,\sqrt{1+3t^2}\,.$$

Therefore $f(x) = \sqrt{1+3x^2}\,$.

C06S0M.021: The graphs of $f(x) = \sin\left(\frac{1}{2}\pi x\right)$ and $g(x) = x$ cross at $(0,\,0)$ and $(1,1)$, and $g(x) < f(x)$ if $0 < x < 1$. When the region they bound is rotated around the y-axis, the method of cylindrical shells yields the volume of the solid thus generated to be

$$V = \int_0^1 2\pi x\,[\,f(x) - g(x)\,]\,dx = 2\pi\int_0^1 \left[\,\left(x\sin\frac{\pi x}{2}\right) - x^2\,\right]\,dx.$$

Now let $u = \dfrac{\pi x}{2}$, so that $x = \dfrac{2u}{\pi}$. This substitution yields

$$\begin{aligned} V &= 2\pi\int_0^{\pi/2}\left(\frac{2}{\pi}u\sin u\; - \;\frac{4}{\pi^2}u^2\right)\cdot\frac{2}{\pi}\,du = \int_0^{\pi/2}\left(\frac{8}{\pi}\,u\sin u - \frac{16}{\pi^2}u^2\right)\,du \\ &= \left[\frac{8}{\pi}(\sin u\; - \;u\cos u)\; - \;\frac{16}{3\pi^2}\,u^3\right]_0^{\pi/2} = \frac{8}{\pi} - \frac{16}{3\pi^2}\cdot\frac{\pi^3}{8} = \frac{8}{\pi} - \frac{2\pi}{3} \approx 0.4520839871. \end{aligned}$$

C06S0M.022: If $-1 \leqq x \leqq 2$, then a thin vertical strip of the region above x is rotated in a circle of radius $x+2$. Therefore the volume generated is

$$\begin{aligned} V &= \int_{-1}^2 2\pi(x+2)(x+2-x^2)\,dx = \pi\left[8x + 4x^2 - \frac{2}{3}x^3 - \frac{1}{2}x^4\right]_{-1}^2 \\ &= \frac{56\pi}{3} - \left(-\frac{23\pi}{6}\right) = \frac{45\pi}{2} \approx 70.6858347058. \end{aligned}$$

C06S0M.023: $\dfrac{dy}{dx} = \tfrac{1}{2}x^{1/2} - \tfrac{1}{2}x^{-1/2}$, so

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\tfrac{1}{2}x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)^2. \tag{1}$$

So the length of the curve is

$$L = \int_1^4 \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)dx = \left[x^{1/2} + \frac{1}{3}x^{3/2}\right]_1^4 = \frac{14}{3} - \frac{4}{3} = \frac{10}{3}.$$

C06S0M.024: We use the result in Eq. (1) in the solution of Problem 23: $ds = \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)\,dx$. The graph of $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ lies below the x-axis for $1 \leqq x \leqq 3$ and above it for $3 \leqq x \leqq 4$. Hence the radius of the cirle of rotation is $-f(x)$ in the former case and $f(x)$ in the latter case. So the area of the left-hand part of the surface is

$$\begin{aligned} A_L &= -\int_1^3 2\pi f(x)\,ds = 2\pi \int_1^3 \left(\frac{1}{2} + \frac{1}{3}x - \frac{1}{6}x^2\right)\,dx \\ &= 2\pi \left[\frac{1}{2}x + \frac{1}{6}x^2 - \frac{1}{18}x^3\right]_1^3 = 2\pi\left(\frac{3}{2} + \frac{3}{2} - \frac{3}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{18}\right) = \frac{16\pi}{9}. \end{aligned}$$

The area of the right-hand part of the surface is

$$\begin{aligned} A_R &= \int_1^3 2\pi f(x)\,ds = 2\pi \left[\frac{1}{18}x^3 - \frac{1}{6}x^2 - \frac{1}{2}x\right]_3^4 \\ &= 2\pi\left(\frac{32}{9} - \frac{8}{3} - 2 - \frac{3}{2} + \frac{3}{2} + \frac{3}{2}\right) = \frac{7\pi}{9}. \end{aligned}$$

Therefore the total area of the surface of revolution around the x-axis is

$$A_L + A_R = \frac{16\pi}{9} + \frac{7\pi}{9} = \frac{23\pi}{9} \approx 8.0285145592.$$

There is no such difficulty in part (b), in which the graph of f is rotated around the y-axis. The area of the surface thereby generated is

$$\begin{aligned} A &= \int_1^4 2\pi x\,ds = 2\pi \int_1^4 \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{3/2}\right)\,dx \\ &= 2\pi\left[\frac{1}{3}x^{3/2} + \frac{1}{5}x^{5/2}\right]_1^4 = 2\pi\left(\frac{8}{3} + \frac{32}{5} - \frac{1}{3} - \frac{1}{5}\right) = \frac{256\pi}{15} \approx 53.6165146213. \end{aligned}$$

C06S0M.025: Let $x = f(y) = \frac{3}{8}(y^{4/3} - 2y^{2/3})$. Then

$$1 + [\,f'(y)\,]^2 = 1 + \frac{9}{64}\left(\frac{4}{3}y^{1/3} - \frac{4}{3}y^{-1/3}\right)^2 = \frac{1}{4}y^{2/3} + \frac{1}{2} + \frac{1}{4}y^{-2/3} = \frac{(1 + y^{2/3})^2}{4y^{2/3}},$$

and therefore $ds = \frac{1}{2}(y^{1/3} + y^{-1/3})\,dy$. Hence the length of the graph of g from $y = 1$ to $y = 8$ is

$$L = \int_1^8 1\,ds = \int_1^8 \frac{1}{2}(y^{1/3} + y^{-1/3})\,dy = \left[\frac{3}{8}y^{4/3} + \frac{3}{4}y^{2/3}\right]_1^8 = 9 - \frac{9}{8} = \frac{63}{8} = 7.875.$$

C06S0M.026: Let $x = g(y) = \frac{3}{8}(y^{4/3} - 2y^{2/3})$, $1 \leqq y \leqq 8$. As in the solution of Problem 25, we find that $ds = \frac{1}{2}(y^{1/3} + y^{-1/3})\,dy$. So the surface area generated by revolving the graph of g around the x-axis will be

$$A = \int_1^8 2\pi y\,ds = \pi \int_1^8 (y^{4/3} + y^{2/3})\,dy = \pi\left[\frac{3}{7}y^{7/3} + \frac{3}{5}y^{5/3}\right]_1^8 = \frac{2592\pi}{35} - \frac{36\pi}{35} = \frac{2556\pi}{35} \approx 229.4260235022.$$

But the graph of $x = g(y)$ crosses the y-axis where $y = a = 2\sqrt{2}$, so two integrals are required to find the surface area generated by rotating the graph around the y-axis. They are

$$A_1 = -\int_1^a 2\pi g(y)\,ds = -\frac{3\pi}{8}\int_1^a (y^{5/3} - y - 2y^{1/3})\,dy = -\pi\left[\frac{9}{64}y^{8/3} - \frac{3}{16}y^2 - \frac{9}{16}y^{4/3}\right]_1^a = \frac{57\pi}{64}$$

and

$$A_2 = \int_a^8 2\pi g(y)\,ds = \frac{3\pi}{8}\int_a^8 (y^{5/3} - y - 2y^{1/3})\,dy = \pi\left[\frac{9}{64}y^{8/3} - \frac{3}{16}y^2 - \frac{9}{16}y^{4/3}\right]_a^8 = \frac{33\pi}{2}.$$

Therefore the answer in part (b) is $A_1 + A_2 = \dfrac{1113\pi}{64} \approx 54.63425974.$

C06S0M.027: Let $f(x) = \frac{1}{3}x^{3/2} - x^{1/2},\ 1 \leqq x \leqq 4$. Then

$$1 + [f'(x)]^2 = 1 + \left(\frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1/2}\right)^2 = \frac{1}{4}x + \frac{1}{2} + \frac{1}{4}x^{-1} = \left(\frac{1}{2}x^{1/2} + \frac{1}{2}x^{-1/2}\right)^2.$$

Therefore $ds = \frac{1}{2}\left(x^{1/2} + x^{-1/2}\right)\,dx$. Therefore the area of the surface generated when the graph of f is rotated around the vertical line $x = 1$ is

$$\begin{aligned} A &= \int_1^4 2\pi(x-1)\,ds = \pi\int_1^4 (x^{3/2} - x^{-1/2})\,dx = \pi\left[\frac{2}{5}x^{5/2} - 2x^{1/2}\right]_1^4 \\ &= \frac{44\pi}{5} - \left(-\frac{8\pi}{5}\right) = \frac{52\pi}{5} \approx 32.6725635973. \end{aligned}$$

C06S0M.028: $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{r^2 - x^2}}$, so $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{r^2}{r^2 - x^2}$. Therefore

$$A = \int_a^b 2\pi\sqrt{r^2 - x^2}\,\frac{r}{\sqrt{r^2 - x^2}}\,dx = \int_a^b 2\pi r\,dx = \Big[2\pi r x\Big]_a^b = 2\pi r(b - a) = 2\pi r h.$$

C06S0M.029: This is merely a matter of substituting $2r$ for h in the area formula $A = 2\pi r h$ derived in Problem 28. Thus the area of a sphere of radius r is $A = 2\pi r \cdot 2r = 4\pi r^2$.

C06S0M.030: Let $f(x) = 2x^3$ and $g(x) = 2\sqrt{x}$. The region R bounded by the graphs of f and g lies in the first quadrant and the two curves cross at the origin and at (1, 2). The graph of f is also the graph of $x = h(y) = (y/2)^{1/3}$ and the graph of g is also the graph of $x = j(y) = (y/2)^2$. The graph of g is above the graph of f on the interval $0 \leqq x \leqq 1$ and the graph of j is to the left of the graph of h on the interval $0 \leqq y \leqq 2$.

Part (a): R is rotated around the x-axis, generating a solid of volume V_1. To find V_1 by the method of cross sections, we first simplify

$$[g(x)]^2 - [f(x)]^2 = 4x - 4x^6,$$

and therefore

$$V_1 = \pi\int_0^1 (4x - 4x^6)\,dx = \pi\left[2x^2 - \frac{4}{7}x^7\right]_0^1 = \frac{10\pi}{7} \approx 4.4879895051.$$

To find V_1 by the method of cylindrical shells, we evaluate

$$\int_0^2 2\pi y\,[h(y)-j(y)]\;dy = \pi\int_0^2 \left(2^{2/3}y^{4/3}-\frac{1}{2}y^3\right)dy = \pi\left[\frac{3\cdot 2^{2/3}}{7}y^{7/3}-\frac{1}{8}y^4\right]_0^2 = \frac{10\pi}{7}.$$

Part (b): R is rotated around the y-axis, generating a solid of volume V_2. To find V_2 by the method of cylindrical shells, we evaluate

$$V_2 = \int_0^2 2\pi x\,[g(x)-f(x)]\;dx = \pi\int_0^1 (4x^{3/2}-4x^4)\,dx = \pi\left[\frac{8}{5}x^{5/2}-\frac{4}{5}x^5\right]_0^1 = \frac{4\pi}{5} \approx 2.5132741229.$$

To find V_2 by the method of cross sections, we first simplify

$$[h(y)]^2-[j(y)]^2 = \frac{y^{2/3}}{2^{2/3}}-\frac{y^4}{16} = \frac{1}{16}\left(8\cdot 2^{1/3}y^{2/3}-y^4\right).$$

Then

$$V_2 = \int_0^2 \pi\cdot\frac{1}{16}\left(8\cdot 2^{1/3}y^{2/3}-y^4\right)\,dy = \frac{\pi}{80}\left[24\cdot 2^{1/3}y^{5/3}-y^5\right]_0^2 = \frac{4\pi}{5}.$$

Part (c): R is rotated around the horizontal line $y=-1$, generating a solid of volume V_3. To find V_3 by the method of cross sections, we simplify

$$[g(x)+1]^2-[f(x)+1]^2 = 4x^{1/2}+4x-4x^3-4x^6.$$

Then

$$V_3 = \int_0^1 \pi(4x^{1/2}+4x-4x^3-4x^6)\,dx = \frac{\pi}{21}\left[56x^{3/2}+42x^2-21x^4-12x^7\right]_0^1 = \frac{65\pi}{21} \approx 9.7239772611.$$

To find V_3 by the method of cylindrical shells, we first simplify the integrand:

$$2\pi(y+1)\,[h(y)-j(y)] = \frac{1}{2}\pi(y+1)(2\cdot 2^{2/3}y^{1/3}-y^2) = \pi\left(2^{2/3}y^{1/3}+2^{2/3}y^{4/3}-\frac{1}{2}y^2-\frac{1}{2}y^3\right).$$

Then

$$V_3 = \pi\int_0^2 \left(2^{2/3}y^{1/3}+2^{2/3}y^{4/3}-\frac{1}{2}y^2-\frac{1}{2}y^3\right)\,dy$$

$$= \pi\left[\frac{3\cdot 2^{2/3}}{4}y^{4/3}+\frac{3\cdot 2^{2/3}}{7}y^{7/3}-\frac{1}{6}y^3-\frac{1}{8}y^4\right]_0^2 = \frac{65\pi}{21}.$$

Part (d): Finally, R is rotated around the vertical line $x=2$, thereby generating a solid of volume V_4. To evaluate V_4 by the method of cross sections, we first simplify

$$[2-j(y)]^2-[2-h(y)]^2 = 2^{5/3}y^{1/3}-\frac{y^{2/3}}{2^{2/3}}-y^2+\frac{y^4}{16}.$$

Then

$$V_4 = \pi \int_0^2 \left(2^{5/3} y^{1/3} - \frac{y^{2/3}}{2^{2/3}} - y^2 + \frac{y^4}{16} \right) dy$$

$$= \pi \left[\frac{3 \cdot 2^{2/3}}{2} y^{4/3} - \frac{3 \cdot 2^{1/3}}{10} y^{5/3} - \frac{1}{3} y^3 + \frac{1}{80} y^5 \right]_0^2 = \frac{38\pi}{15} \approx 7.9587013891.$$

To evaluate V_4 by the method of cylindrical shells, we first simplify

$$(2 - x)\,[g(x) - f(x)] = 4x^{1/2} - 2x^{3/2} - 4x^3 + 2x^4.$$

Then

$$V_4 = 2\pi \int_0^1 \left(4x^{1/2} - 2x^{3/2} - 4x^3 + 2x^4 \right) dx = 2\pi \left[\frac{8}{3} x^{3/2} - \frac{4}{5} x^{5/2} - x^4 + \frac{2}{5} x^5 \right]_0^1 = \frac{38\pi}{15}.$$

C06S0M.031: Denote the spring constant by K. The information given in the problem yields

$$\int_2^5 K(x - L)\,dx = 5 \int_2^3 K(x - L)\,dx;$$

$$\int_2^5 (x - L)\,dx = 5 \int_2^3 (x - L)\,dx;$$

$$\left[\frac{1}{2}(x - L)^2 \right]_2^5 = 5 \left[\frac{1}{2}(x - L)^2 \right]_2^3;$$

$$(5 - L)^2 - (2 - L)^2 = 5(3 - L)^2 - 5(2 - L)^2;$$

$$25 - 10L + L^2 - 4 + 4L - L^2 = 45 - 30L + 5L^2 - 20 + 20L - 5L^2;$$

$$4L = 4.$$

Therefore the natural length of the spring is $L = 1$ (ft).

C06S0M.032: Set up a coordinate system in which $y = 50$ is the position of the windlass and the lowest point P of the cable is initially at $y = 0$. When P is at location y $(0 \leqq y \leqq 50)$, the length of the cable is $50 - y$, so the total weight on the windlass is $1000 + 5 \cdot (50 - y)$ (lb). Therefore the work to wind in 25 feet of the cable is

$$W = \int_0^{25} (1000 + 250 - 5y)\,dy = \left[1250y - \frac{5}{2} y^2 \right]_0^{25} = 29687.5 \quad \text{(ft·lb)}.$$

C06S0M.033: Set up a coordinate system in which the center of the tank is at the origin and the y-axis is vertical. A horizontal cross section of the oil at positive y $(-R \leqq y \leqq R)$ is circular with radius $x = \sqrt{R^2 - y^2}$, so its area is $\pi(R^2 - y^2)$. Hence the work to pump the oil to its final position $y = 3R$ is

$$W = \int_{-R}^{R} (3R - y)\pi\rho(R^2 - y^2)\,dy = \pi\rho \int_{-R}^{R} (y^3 - 3Ry^2 - R^2 y + 3R^3)\,dy$$

$$= \pi\rho \left[\frac{1}{4} y^4 - Ry^3 - \frac{1}{2} R^2 y^2 + 3R^3 y \right]_{-R}^{R} = \pi\rho \left(\frac{7}{4} R^4 + \frac{9}{4} R^4 \right) = 4\pi\rho R^4.$$

C06S0M.034: Set up a coordinate system with the axis of the cone lying on the y-axis and with a diameter of the base of the cone lying on the x-axis. Now a horizontal slice of the cone at height y has radius given by $x = \frac{1}{2}(1-y)$; the units here are in feet. Therefore the work done in building the anthill is

$$W = \int_0^1 \frac{1}{4}(150y)\pi(1-y)^2\, dy = \frac{\pi}{4}\left[\frac{75}{2}y^4 - 100y^3 + 75y^2\right]_0^1 = \frac{25}{8}\pi \approx 9.82 \quad \text{(ft·lb)}.$$

C06S0M.035: Set up a coordinate system in which the center of the earth is at the origin and the hole extends upward along the vertical y-axis, with its top where $y = R$, the radius of the earth in feet. A 1-pound weight at position y ($0 \leqq y \leqq R$) weighs y/R pounds, so the total work to lift the weight from $y = 0$ to $y = R$ is

$$W = \int_0^R \frac{y}{R}\, dy = \left[\frac{y^2}{2R}\right]_0^R = \frac{R}{2} = \frac{3960 \cdot 5280}{2} = 10454400 \quad \text{(ft·lb)}.$$

The assumption of constant density of the earth is required to draw the conclusion that the gravitational force is proportional to the distance from the center of the earth.

C06S0M.036: Set up a coordinate system in which the center of the earth is at the origin and the hole extends along the nonnegative y-axis from $y = 0$ to $y = R = 3960 \cdot 5280$, the radius of the earth in feet. Imagine a thin cylindrical horizontal slab of dirt (or basalt, or whatever) in the hole at distance y from the center of the earth. As it moves from its initial position y to its final position R, its weight varies: If it is at position u, $y \leqq u \leqq R$, then its weight will be

$$(350\pi) \cdot \left(\frac{u}{R}\right)\, du$$

where du denotes its thickness. The total work required to lift this slab from its initial position ($u = y$) to the surface ($u = R$) is then

$$\int_y^R 350\pi\frac{u}{R}\, du = 350\pi\left[\frac{u^2}{2R}\right]_y^R = \frac{350\pi}{2R}\left(R^2 - y^2\right).$$

Therefore the total work required to lift all the dirt (or basalt, or whatever) from the hole to the surface of the earth is

$$W = \int_0^R \frac{350\pi}{2R}\left(R^2 - y^2\right)\, dy = \frac{350\pi}{2R}\left[R^2y - \frac{1}{3}y^3\right]_0^R = \frac{350\pi R^2}{3} \approx 1.6023407560 \times 10^{17} \quad \text{(ft·lb)}.$$

It is intriguing to note that the answer may be written in the form

$$W = \int_0^R \left(\int_y^R 350\pi\frac{u}{R}\, du\right) dy.$$

C06S0M.037: If the coordinate system is chosen with the origin at the midpoint of the bottom of the dam and with the x-axis horizontal, then the equation of the slanted edge of the dam is $y = 2x - 200$ (with units in feet). Therefore the width of the dam at level y is $2x = y + 200$. Let $\rho = 62.4$ be the density of water in pounds per cubic foot. Then the total force on the dam is

$$F = \int_0^{100} \rho(100-y)(y+200)\, dy = \rho\left[20000y - 50y^2 - \frac{1}{3}y^3\right]_0^{100} = \frac{3500000\rho}{3} = 72800000 \quad \text{(lb)}.$$

C06S0M.038: The answer may be obtained from the answer to Problem 37 by multiplying the latter by $\sec 30°$: The force is $2/\sqrt{3}$ times as great, or approximately 8.4062199194×10^7 pounds. The analytical approach here is to introduce the additional factor $\sec(\pi/6)$ into the integral in the solution of Problem 37, but because this factor is a constant, one may as well simply multiply the answer by the same factor.

C06S0M.039: The volume of the solid is

$$V = \int_0^c 2\pi\left(y + \frac{1}{c}\right)\frac{2}{c}\sqrt{y}\ dy = \frac{4\pi}{c}\cdot\left[\frac{2}{5}y^{5/2} + \frac{2}{3c}y^{3/2}\right]_0^c = 8\pi\left(\frac{1}{5}c^{3/2} + \frac{1}{3}c^{-1/2}\right).$$

It is clear that there is no maximum volume, because $V \to +\infty$ as $c \to 0^+$. But $V \to +\infty$ as $c \to +\infty$ as well, so there is a minimum volume; $V'(c) = 0$ when $c = \frac{1}{3}\sqrt{5}$, so this value of c minimizes V.

C06S0M.040: Here we have

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(x^4 + \frac{1}{4x^4}\right)^2.$$

Therefore

$$\begin{aligned}
L &= \int_1^2 \left(x^4 + \frac{1}{4x^4}\right) dx = \frac{3011}{480}, \\
M_y &= \int_1^2 \left(x^5 + \frac{1}{4x^3}\right) dx = \frac{339}{32}, \qquad \text{and} \\
M_x &= \int_1^2 \left(\frac{1}{5}x^5 + \frac{1}{12x^3}\right)\cdot\left(x^4 + \frac{1}{4x^4}\right) dx = \int_1^2 \left(\frac{x^9}{5} + \frac{x}{20} + \frac{x}{12} + \frac{1}{48x^7}\right) dx \\
&= \left[\frac{x^{10}}{50} + \frac{x^2}{40} + \frac{x^2}{24} - \frac{1}{288x^6}\right]_1^2 = \frac{1057967}{51200} = 20.66341796875.
\end{aligned}$$

Therefore

$$\overline{x} = \frac{M_y}{L} = \frac{5085}{3011} \approx 1.68880770508 \quad \text{and} \quad \overline{y} = \frac{M_x}{L} = \frac{3173901}{963520} \approx 3.294 - 6862388.$$

C06S0M.041: Here,

$$\begin{aligned}
L &= \int_1^2 \frac{1}{2}(y^3 + y^{-3})\ dy = \frac{33}{16}, \\
M_y &= \int_1^2 \frac{1}{2}\cdot\left(\frac{1}{8}y^4 + \frac{1}{4}y^{-2}\right)\cdot(y^3 + y^{-3})\ dy = \frac{1179}{512}, \qquad \text{and} \\
M_x &= \int_1^2 \frac{1}{2}(y^4 + y^{-2})\ dy = \frac{67}{20}.
\end{aligned}$$

Therefore

$$\overline{x} = \frac{393}{352} \approx 1.116477 \quad \text{and} \quad \overline{y} = \frac{268}{165} \approx 1.624242.$$

C06S0M.042: First,

$$1+\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}\right)^2.$$

Therefore

$$L = \int_1^4 \frac{1}{2}(x^{1/2}+x^{-1/2})\,dx = \frac{10}{3},$$

$$M_y = \int_1^4 \frac{1}{2}(x^{3/2}+x^{1/2})\,dx = \frac{128}{15}, \quad \text{and}$$

$$M_x = \int_1^4 \frac{1}{2}\cdot\left(\frac{1}{3}x^{3/2}-x^{1/2}\right)\cdot(x^{1/2}+x^{-1/2})\,dx = -\frac{1}{2}.$$

Therefore

$$\overline{x} = \frac{128}{15}\cdot\frac{3}{10} = \frac{64}{25} \quad \text{and} \quad \overline{y} = -\frac{1}{2}\cdot\frac{3}{10} = -\frac{3}{20}.$$

C06S0M.043: To begin with,

$$1+\left(\frac{dx}{dy}\right)^2 = \left(\frac{1}{2}y^{1/3}+\frac{1}{2}y^{-1/3}\right)^2;$$

it follows that the length of the curve is $L = \dfrac{63}{8}$. Next,

$$M_y = \int_1^8 \frac{3}{8}\cdot\frac{1}{2}\cdot(y^{4/3}-2y^{2/3})\cdot(y^{1/3}+y^{-1/3})\,dy = \frac{999}{128} = 7.8046875 \quad \text{and}$$

$$M_x = \int_1^8 \frac{1}{2}(y^{4/3}+y^{2/3})\,dy = \frac{1278}{35} \approx 36.51428571.$$

Therefore

$$\overline{x} = \frac{111}{112} \approx 0.991071427 \quad \text{and} \quad \overline{y} = \frac{1136}{245} \approx 4.636734694.$$

C06S0M.044: The two curves meet at (0, 0) and at (1, 1). So

$$A = \int_0^1 (2x-x^2-x^3)\,dx = \frac{5}{12},$$

$$M_y = \int_0^1 (2x^2-x^3-x^4)\,dx = \frac{13}{60}, \quad \text{and}$$

$$M_x = \int_0^1 \frac{1}{2}(2x-x^2-x^3)^2\,dx = \frac{41}{210}.$$

Therefore

$$\overline{x} = \frac{13}{25} = 0.52 \quad \text{and} \quad \overline{y} = \frac{82}{175} \approx 0.4685714286.$$

C06S0M.045: The curves meet at $(2, 1)$, at $(0, 0)$, and at $(2, -1)$. It follows that $\overline{y} = 0$ by symmetry and that we may compute $\overline{x}$ by using only the upper half of the figure. In that case we have

$$A = \int_0^1 (y^2 + 1 - 2y^4)\,dy = \frac{14}{15} \quad \text{and}$$

$$M_y = \int_0^1 \frac{1}{2}\left[(y^2+1)^2 - (2y^4)^2\right]\,dy = \frac{32}{45}.$$

Therefore $\overline{x} = \dfrac{16}{21} \approx 0.7619047619$.

C06S0M.046: Given a triangle in the plane, set up a coordinate system in such a way that the lowest vertex of the triangle is at the origin, there is a vertex in the first quadrant at (a, b), and a vertex in the second quadrant at $(-a, c)$. Thus the y-axis passes through the midpoint of the side opposite the vertex at the origin, and hence a median of the triangle lies on the y-axis. We will show that the y-coordinate of the centroid also lies on the y-axis. Then, by rotating the triangle to plane the other two vertices at the origin in a similar way, we may conclude that $\overline{y}$ lies on all three medians. Then interchange the roles of x and y to conclude that $\overline{x}$ lies on the intersection of the medians as well.

The left side of the triangle has equation $y = h(x) = -cx/a$, the right side has equation $y = g(x) = bx/a$, and the top side has equation

$$y = f(x) = b + \frac{b-c}{2a}(x-a).$$

Hence the moment of the triangle with respect to the y-axis is $M_x = M_L + M_R$ where M_L denotes its moment to the left of the y-axis and M_R its moment to the right. Now

$$\begin{aligned} M_R &= \int_0^a \left(bx + \frac{b-c}{2a}x^2 - \frac{b-c}{2}x - \frac{b}{a}x^2\right)\,dx \\ &= \left[\frac{1}{2}bx^2 + \frac{b-c}{6a}x^3 - \frac{b-c}{4}x^2 - \frac{b}{3a}x^3\right]_0^a = \frac{1}{2}ba^2 + \frac{b-c}{6}a^2 - \frac{b-c}{4}a^2 - \frac{b}{3}a^2 \\ &= \frac{6b + 2b - 2c - 3b + 3c - 4b}{12}a^2 = \frac{b+c}{12}a^2. \end{aligned}$$

Moreover,

$$M_L = -\int_{-a}^0 \left(bx + \frac{b-c}{2a}x^2 - \frac{b-c}{2}x + \frac{c}{a}x^2\right)\,dx = \cdots = \frac{b+c}{12}a^2$$

by extremely similar computations. Thus the triangle balances on the y-axis, and therefore $\overline{y} = 0$. In light of the opening remarks, this completes the proof.

C06S0M.047: $2\pi\overline{y} \cdot \dfrac{\pi ab}{2} = \dfrac{4}{3}\pi ab^2$, and it follows that $\overline{y} = \dfrac{4b}{3\pi}$.

C06S0M.048: Note that $\overline{x} = \overline{y}$. The area of the quarter ring is

$$A = \frac{1}{4}(\pi b^2 - \pi a^2) = \frac{\pi}{4}(b^2 - a^2),$$

and the volume generated by rotating it around the x-axis is

$$V = \frac{2}{3}\pi b^3 - \frac{2}{3}\pi a^3.$$

Therefore $V = \dfrac{2}{3}\pi(b^3 - a^3) = (2\pi\overline{y}) \cdot \dfrac{\pi}{4} \cdot (b^2 - a^2)$.
Part (a): Consequently

$$\overline{y} = \frac{\frac{2}{3}\pi(b^3 - a^3)}{\frac{1}{2}\pi^2(b^2 - a^2)} = \frac{4(b^2 + ab + a^2)}{3\pi(b + a)} = \overline{x}.$$

Part (b): $\displaystyle\lim_{b \to a} \overline{x} = \frac{12a^2}{(3\pi)(2a)} = \frac{2a}{\pi} = \lim_{b \to a} \overline{y}$.

C06S0M.049: (a) The area A of the triangle T can be computed in several ways; we chose the most direct which, elementary, is easy to do by hand. Let O denote the vertex of the triangle at (0, 0), $C = C(c,\,0)$, $A = A(a,\,0)$, $B = B(a,\,b)$, and $D = D(c,\,d)$. Then A is the area of triangle OCD *plus* the area of trapezoid $CABD$ *minus* the area of triangle OAB:

$$\begin{aligned} A &= \frac{cd}{2} + \frac{(a-c)(b+d)}{2} - \frac{ab}{2} \\ &= \frac{cd}{2} + \frac{ab}{2} + \frac{ad}{2} - \frac{bc}{2} - \frac{cd}{2} - \frac{ab}{2} = \frac{ad - bc}{2}. \end{aligned}$$

(b) In Problem 46 we saw that the centroid of a triangle lies on the intersection of its medians. From plane geometry we also know that the point of intersection is two-thirds of the way from any vertex to the midpoint of the opposite side. The midpoint of L has y-coordinate $(b + d)/2$, and hence

$$\overline{y} = \frac{2}{3} \cdot \frac{b+d}{2} = \frac{b+d}{3}.$$

(c) $V = 2\pi\overline{y}A = 2\pi \cdot \dfrac{b+d}{3} \cdot \dfrac{ad-bc}{2} = \dfrac{1}{3}\pi(b+d)(ad-bc)$.

(d) $\dfrac{1}{2}pw = A = \dfrac{ad-bc}{2}$, so $p = \dfrac{ad-bc}{w}$.

(e) $S = 2\pi \cdot \dfrac{b+d}{2} \cdot w = \pi w(b+d)$.

(f) $V = 2\pi\overline{y}A = 2\pi \cdot \dfrac{b+d}{3} \cdot \dfrac{1}{2}pw = \pi pw \cdot \dfrac{b+d}{3} = \dfrac{1}{3}pS$.

C06S0M.050: Let $n = 2k$. Inscribe the $2k$-gon with opposite vertices on the x-axis. Let T be one of the triangles formed by a side of the polygon and two radii of the circle. The perpendicular from the origin to the midpoint of the side of the polygon has length (in the notation of Problem 49)

$$p = r\cos\left(\frac{\pi}{k}\right).$$

By part (f) of Problem 49,

$$V = \frac{1}{3}\left(r\cos\frac{\pi}{k}\right)S.$$

Now let $k \to +\infty$ and replace S with $4\pi r^2$ to obtain Archimedes' result

$$V = \frac{4}{3}\pi r^3.$$

C06S0M.051: A *Mathematica* solution: First let $f(x) = x^m$ and $g(x) = x^n$ where m and n are positive integers and $n > m$.

```
a = Integrate[ f[x] - g[x], { x, 0, 1 } ]
```

$$\frac{1}{m+1} - \frac{1}{n+1}$$

```
area = a /.  m → 1
```

$$\frac{1}{2} - \frac{1}{n+1}$$

Formulas (11) and (12) in the text give the moments

```
my = Integrate[ x*(f[x] - g[x]), { x, 0, 1 } ];
mx = Integrate[ (1/2)*((f[x])^2 - (g[x])^2), { x, 0, 1 } ];
My = my /.  m → 1
```

$$\frac{n-1}{3(n+2)}$$

```
Mx = mx /.  m → 1
```

$$\frac{\frac{1}{3} - \frac{1}{2n+1}}{2}$$

Hence the centroid has coordinates

```
{ xc, yc } = { My/area, Mx/area } // Simplify
```

$$\left\{ \frac{(n+1)}{3(n+2)}, \frac{2(n+1)}{3(2n+1)} \right\}$$

```
Limit[ { xc, yc }, n → Infinity ]
```

$$\left\{ \frac{2}{3}, \frac{1}{3} \right\}$$

Obviously this is the centroid of the triangle with vertices (0, 0), (1, 0), and (1, 1)—which the area of the region bounded by the graphs of f and g "exhausts" as $n \to +\infty$.

C06S0M.052: By Example 1 in Section 6.6, the semicircular disk has area and centroid

```
a1 = 9*Pi/2;
c1 = { 0, 4/Pi };
```

Hence its moment with respect to the x-axis is

```
mx1 = a1*c1[[2]]
```

$$18$$

(Recall that the *Mathematica* command `list[[n]]` extracts the nth entry from the k-dimensional array `list = { a1, a2, a3, ..., ak }`.)

The square are area and centroid

```
a2 = 4;
```

```
c2 = { 0, -1 };
```

so its moment with respect to the x-axis is

```
mx2 = a2*c2[[2]]
```

$$-4$$

Therefore the region R has area and x-moment

```
a = a1 + a2
```

$$4 + \frac{9\pi}{2}$$

```
mx = mx1 + mx2
```

$$14$$

So the y-coordinate of its centroid is

```
yc - mx/a // Simplify
```

$$\frac{28}{8 + 9\pi}$$

A numerical approximation to this result is

```
N[yc]
```

$$0.771896$$

The radius of revolution around the line $y = -4$ is

```
r = 4 + yc // Together
```

$$\frac{12(5 + 3\pi)}{8 + 9\pi}$$

So the volume of revolution is

```
v = 2*Pi*r*a // Simplify
```

$$12\pi(3\pi + 5)$$

Section 7.1

C07S01.001: If $f(x) = e^{2x}$, then $f'(x) = e^{2x} \cdot D_x(2x) = 2e^{2x}$.

C07S01.002: If $f(x) = e^{3x-1}$, then $f'(x) = e^{3x-1} \cdot D_x(3x-1) = 3e^{3x-1}$.

C07S01.003: If $f(x) = \exp(x^2)$, then $f'(x) = \left[\exp(x^2)\right] \cdot D_x(x^2) = 2x\exp(x^2)$.

C07S01.004: If $f(x) = e^{4-x^3}$, then $f'(x) = e^{4-x^3} \cdot D_x(4-x^3) = -3x^2e^{4-x^3}$.

C07S01.005: If $f(x) = e^{1/x^2}$, then $f'(x) = e^{1/x^2} \cdot D_x(1/x^2) = -\dfrac{2}{x^3}\,e^{1/x^2}$.

C07S01.006: If $f(x) = x^2\exp(x^3)$, then $f'(x) = 2x\exp(x^3) + x^2 \cdot 3x^2\exp(x^3) = (2x+3x^4)\exp(x^3)$.

C07S01.007: If $g(t) = t\exp(t^{1/2})$, then $g'(t) = \exp(t^{1/2}) + t \cdot \dfrac{1}{2}t^{-1/2}\exp(t^{1/2}) = \dfrac{2+\sqrt{t}}{2}\exp(t^{1/2})$.

C07S01.008: If $g(t) = (e^{2t}+e^{3t})^7$, then $g'(t) = 7(e^{2t}+e^{3t})^6(2e^{2t}+3e^{3t})$.

C07S01.009: If $g(t) = (t^2-1)e^{-t}$, then $g'(t) = 2te^{-t} - (t^2-1)e^{-t} = (1+2t-t^2)e^{-t}$.

C07S01.010: If $g(t) = (e^t - e^{-t})^{1/2}$, then $g'(t) = \dfrac{1}{2}(e^t-e^{-t})^{-1/2}(e^t+e^{-t})$.

C07S01.011: If $g(t) = e^{\cos t} = \exp(\cos t)$, then $g'(t) = (-\sin t)\exp(\cos t)$.

C07S01.012: If $f(x) = xe^{\sin x} = x\exp(\sin x)$, then

$$f'(x) = \exp(\sin x) + (x\cos x)\exp(\sin x) = e^{\sin x}(1 + x\cos x).$$

C07S01.013: If $g(t) = \dfrac{1-e^{-t}}{t}$, then $g'(t) = \dfrac{te^{-t} - (1-e^{-t})}{t^2} = \dfrac{te^{-t}+e^{-t}-1}{t^2}$.

C07S01.014: If $f(x) = e^{-1/x}$, then $f'(x) = \dfrac{1}{x^2}e^{-1/x}$.

C07S01.015: If $f(x) = \dfrac{1-x}{e^x}$, then

$$f'(x) = \frac{(-1)e^x - (1-x)e^x}{(e^x)^2} = \frac{-1-1+x}{e^x} = \frac{x-2}{e^x}.$$

C07S01.016: If $f(x) = \exp\left(\sqrt{x}\,\right) + \exp\left(-\sqrt{x}\,\right)$, then

$$f'(x) = \frac{1}{2}x^{-1/2}\exp\left(\sqrt{x}\,\right) - \frac{1}{2}x^{-1/2}\exp\left(-\sqrt{x}\,\right) = \frac{\exp\left(\sqrt{x}\,\right) - \exp\left(-\sqrt{x}\,\right)}{2\sqrt{x}}.$$

C07S01.017: If $f(x) = \exp(e^x)$, then $f'(x) = e^x\exp(e^x)$.

C07S01.018: If $f(x) = \left(e^{2x}+e^{-2x}\right)^{1/2}$, then

$$f'(x) = \frac{1}{2}\left(e^{2x}+e^{-2x}\right)^{-1/2}\left(2e^{2x}-2e^{-2x}\right) = \frac{e^{2x}-e^{-2x}}{\sqrt{e^{2x}+e^{-2x}}}.$$

C07S01.019: If $f(x) = \sin(2e^x)$, then $f'(x) = 2e^x \cos(2e^x)$.

C07S01.020: If $f(x) = \cos(e^x + e^{-x})$, then $f'(x) = (e^{-x} - e^x)\sin(e^x + e^{-x})$.

C07S01.021: If $f(x) = \ln(3x-1)$, then $f'(x) = \dfrac{1}{3x-1} \cdot D_x(3x-1) = \dfrac{3}{3x-1}$.

C07S01.022: If $f(x) = \ln(4-x^2)$, then $f'(x) = \dfrac{2x}{x^2-4}$.

C07S01.023: If $f(x) = \ln\big[(1+2x)^{1/2}\big]$, then $f'(x) = \dfrac{\frac{1}{2} \cdot 2(1+2x)^{-1/2}}{(1+2x)^{1/2}} = \dfrac{1}{1+2x}$.

C07S01.024: If $f(x) = \ln\big[(1+x)^2\big]$, then $f'(x) = \dfrac{2(1+x)}{(1+x)^2} = \dfrac{2}{1+x}$.

C07S01.025: If $f(x) = \ln\big[(x^3-x)^{1/3}\big] = \dfrac{1}{3}\ln(x^3-x)$, then $f'(x) = \dfrac{3x^2-1}{3(x^3-x)}$.

C07S01.026: If $f(x) = \ln\big[(\sin x)^2\big] = 2\ln(\sin x)$, then $f'(x) = \dfrac{2\cos x}{\sin x} = 2\cot x$.

C07S01.027: If $f(x) = \cos(\ln x)$, then $f'(x) = -\dfrac{\sin(\ln x)}{x}$.

C07S01.028: If $f(x) = (\ln x)^3$, then $f'(x) = \dfrac{3(\ln x)^2}{x}$.

C07S01.029: If $f(x) = \dfrac{1}{\ln x}$, then (by the reciprocal rule) $f'(x) = -\dfrac{1}{x(\ln x)^2}$.

C07S01.030: If $f(x) = \ln(\ln x)$, then $f'(x) = \dfrac{1}{x \ln x}$.

C07S01.031: If $f(x) = \ln\big[x(x^2+1)^{1/2}\big]$, then

$$f'(x) = \frac{(x^2+1)^{1/2} + x^2(x^2+1)^{-1/2}}{x(x^2+1)^{1/2}} = \frac{2x^2+1}{x(x^2+1)}.$$

C07S01.032: If $g(t) = t^{3/2}\ln(t+1)$, then

$$g'(t) = \frac{3}{2}t^{1/2}\ln(t+1) + \frac{t^{3/2}}{t+1} = \frac{t^{1/2}\,[\,2t + 3\ln(t+1) + 3t\ln(t+1)\,]}{2(t+1)}.$$

C07S01.033: If $f(x) = \ln\cos x$, then $f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x$.

C07S01.034: If $f(x) = \ln(2\sin x) = (\ln 2) + \ln(\sin x)$, then $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$.

C07S01.035: If $f(t) = t^2\ln(\cos t)$, then $f'(t) = 2t\ln(\cos t) - \dfrac{t^2 \sin t}{\cos t} = t\,[\,2\ln(\cos t) - t\tan t\,]$.

C07S01.036: If $f(x) = \sin(\ln 2x)$, then $f'(x) = [\cos(\ln 2x)\,] \cdot \dfrac{2}{2x} = \dfrac{\cos(\ln 2x)}{x}$.

C07S01.037: If $g(t) = t(\ln t)^2$, then

$$g'(t) = (\ln t)^2 + t \cdot \frac{2 \ln t}{t} = (2 + \ln t) \ln t.$$

C07S01.038: If $g(t) = t^{1/2} \left[\cos(\ln t) \right]^2$, then

$$g'(t) = \frac{1}{2} t^{-1/2} \left[\cos(\ln t) \right]^2 + 2t^{1/2} \left[\cos(\ln t) \right] \cdot \frac{-\sin(\ln t)}{t} = \frac{\left[\cos(\ln t) \right] \left[\cos(\ln t) - 4 \sin(\ln t) \right]}{2t^{1/2}}.$$

C07S01.039: Because $f(x) = 3 \ln(2x+1) + 4 \ln(x^2 - 4)$, we have

$$f'(x) = \frac{6}{2x+1} + \frac{8x}{x^2-4} = \frac{22x^2 + 8x - 24}{(2x+1)(x^2-4)}.$$

C07S01.040: If

$$f(x) = \ln \left(\frac{1-x}{1+x} \right)^{1/2} = \frac{1}{2} \ln(1-x) - \frac{1}{2} \ln(1+x),$$

then

$$f'(x) = -\frac{1}{2(1-x)} - \frac{1}{2(1+x)} = \frac{1}{(x+1)(x-1)}.$$

C07S01.041: If

$$f(x) = \ln \left(\frac{4-x^2}{9+x^2} \right)^{1/2} = \frac{1}{2} \ln(4-x^2) - \frac{1}{2} \ln(9+x^2),$$

then

$$f'(x) = -\frac{x}{4-x^2} - \frac{x}{9+x^2} = \frac{13x}{(x^2-4)(x^2+9)}.$$

C07S01.042: If

$$f(x) = \ln \frac{\sqrt{4x-7}}{(3x-2)^3} = \frac{1}{2} \ln(4x-7) - 3 \ln(3x-2),$$

then

$$f'(x) = \frac{2}{4x-7} - \frac{9}{3x-2} = \frac{59-30x}{(3x-2)(4x-7)}.$$

C07S01.043: If

$$f(x) = \ln \frac{x+1}{x-1} = \ln(x+1) - \ln(x-1), \quad \text{then} \quad f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = -\frac{2}{(x-1)(x+1)}.$$

C07S01.044: If

$$f(x) = x^2 \ln \frac{1}{2x+1} = -x^2 \ln(2x+1), \quad \text{then} \quad f'(x) = -\frac{2x^2}{2x+1} - 2x\ln(2x+1).$$

C07S01.045: If

$$g(t) = \ln \frac{t^2}{t^2+1} = 2\ln t - \ln(t^2+1), \quad \text{then} \quad g'(t) = \frac{2}{t} - \frac{2t}{t^2+1} = \frac{2}{t(t^2+1)}.$$

C07S01.046: If

$$f(x) = \ln \frac{\sqrt{x+1}}{(x-1)^3} = \frac{1}{2}\ln(x+1) - 3\ln(x-1), \quad \text{then} \quad f'(x) = \frac{1}{2(x+1)} - \frac{3}{x-1} = -\frac{5x+7}{2(x-1)(x+1)}.$$

C07S01.047: Given: $y = 2^x$. Then

$$\ln y = \ln(2^x) = x\ln 2;$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln 2;$$
$$\frac{dy}{dx} = y(x)\cdot \ln 2 = 2^x \ln 2.$$

C07S01.048: Given: $y = x^x$. Then

$$\ln y = \ln(x^x) = x\ln x;$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \ln x;$$
$$\frac{dy}{dx} = y(x)\cdot(1+\ln x) = x^x(1+\ln x).$$

C07S01.049: Given: $y = x^{\ln x}$. Then

$$\ln y = \ln\left(x^{\ln x}\right) = (\ln x)\cdot(\ln x) = (\ln x)^2;$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2\ln x}{x};$$
$$\frac{dy}{dx} = y(x)\cdot\frac{2\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}.$$

C07S01.050: Given: $y = (1+x)^{1/x}$. Then

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x);$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2} = \frac{x - \ln(1+x) - x\ln(1+x)}{x^2(1+x)};$$
$$\frac{dy}{dx} = y(x)\cdot\frac{x-\ln(1+x)-x\ln(1+x)}{x^2(1+x)} = \frac{[\,x-\ln(1+x)-x\ln(1+x)\,]\cdot(1+x)^{1/x}}{x^2(1+x)}.$$

C07S01.051: Given: $y = (\ln x)^{\sqrt{x}}$. Then

$$\ln y = \ln (\ln x)^{\sqrt{x}} = x^{1/2} \ln (\ln x)\,;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} x^{-1/2} \ln (\ln x) + \frac{x^{1/2}}{x \ln x};$$

$$\frac{dy}{dx} = y(x) \cdot \left[\frac{\ln (\ln x)}{2x^{1/2}} + \frac{1}{x^{1/2} \ln x} \right];$$

$$\frac{dy}{dx} = \frac{2 + (\ln x) \ln (\ln x)}{2x^{1/2} \ln x} \cdot (\ln x)^{\sqrt{x}}\,.$$

C07S01.052: Given: $y = (3 + 2^x)^x$. Then

$$\ln y = \ln (3 + 2^x)^x = x \ln (3 + 2^x)\,;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x \cdot 2^x \cdot \ln 2}{3 + 2^x} + \ln (3 + 2^x)\,;$$

$$\frac{dy}{dx} = y(x) \cdot \frac{x \cdot 2^x \cdot (\ln 2) + 3 \ln (3 + 2^x) + 2^x \ln (3 + 2^x)}{3 + 2^x};$$

$$\frac{dy}{dx} = \frac{x \cdot 2^x \cdot (\ln 2) + 3 \ln (3 + 2^x) + 2^x \ln (3 + 2^x)}{3 + 2^x} \cdot (3 + 2^x)^x\,.$$

C07S01.053: If $y = (1 + x^2)^{3/2}(1 + x^3)^{-4/3}$, then

$$\ln y = \frac{3}{2} \ln(1 + x^2) - \frac{4}{3} \ln(1 + x^3);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3x}{1 + x^2} - \frac{4x^2}{1 + x^3} = \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)};$$

$$\frac{dy}{dx} = y(x) \cdot \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)} = \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)} \cdot \frac{(1 + x^2)^{3/2}}{(1 + x^3)^{4/3}};$$

$$\frac{dy}{dx} = \frac{(3x - 4x^2 - x^4)(1 + x^2)^{1/2}}{(1 + x^3)^{7/3}}.$$

C07S01.054: If $y = (x + 1)^x$, then

$$\ln y = \ln(x + 1)^x = x \ln(x + 1);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{x + 1} + \ln(x + 1);$$

$$\frac{dy}{dx} = \left[\frac{x}{x + 1} + \ln(x + 1) \right] \cdot (x + 1)^x.$$

C07S01.055: If $y = (x^2 + 1)^{x^2}$, then

$$\ln y = \ln(x^2+1)^{x^2} = x^2 \ln(x^2+1);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2x^3}{x^2+1} + 2x\ln(x^2+1);$$

$$\frac{dy}{dx} = y(x) \cdot \left[\frac{2x^3}{x^2+1} + 2x\ln(x^2+1) \right] = \left[\frac{2x^3}{x^2+1} + 2x\ln(x^2+1) \right] \cdot (x^2+1)^{x^2}.$$

C07S01.056: If $y = \left(1 + \dfrac{1}{x}\right)^x$, then

$$\ln y = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) = x\ln(x+1) - x\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{x+1} + \ln(x+1) - 1 - \ln x;$$

$$\frac{dy}{dx} = \left[\frac{x}{x+1} + \ln(x+1) - 1 - \ln x \right] \cdot \left(1+\frac{1}{x}\right)^x.$$

C07S01.057: Given: $y = (\sqrt{x}\,)^{\sqrt{x}}$. Then

$$\ln y = \ln\left(\sqrt{x}\,\right)^{\sqrt{x}} = x^{1/2}\ln\left(x^{1/2}\right) = \frac{1}{2}x^{1/2}\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2x^{1/2}} + \frac{\ln x}{4x^{1/2}} = \frac{2+\ln x}{4\sqrt{x}};$$

$$\frac{dy}{dx} = \frac{(2+\ln x)\left(\sqrt{x}\,\right)^{\sqrt{x}}}{4\sqrt{x}}.$$

C07S01.058: If $y = x^{\sin x}$, then

$$\ln y = (\sin x)\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\sin x}{x} + (\cos x)\ln x;$$

$$\frac{dy}{dx} = \frac{\sin x + x(\cos x)\ln x}{x} \cdot \left(x^{\sin x}\right).$$

C07S01.059: If $f(x) = xe^{2x}$, then $f'(x) = e^{2x} + 2xe^{2x}$, so the slope of the graph of $y = f(x)$ at $(1,\ e^2)$ is $f'(1) = 3e^2$. Hence an equation of the line tangent to the graph at that point is $y - e^2 = 3e^2(x-1)$; that is, $y = 3e^2x - 2e^2$.

C07S01.060: If $f(x) = e^{2x}\cos x$, then $f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$, so the slope of the graph of $y = f(x)$ at the point $(0,\ 1)$ is $f'(0) = 2$. So an equation of the line tangent to the graph at that point is $y - 1 = 2(x-0)$; that is, $y = 2x + 1$.

C07S01.061: If $f(x) = x^3\ln x$, then $f'(x) = x^2 + 3x^2\ln x$, so the slope of the graph of $y = f(x)$ at the point $(1,\ 0)$ is $f'(1) = 1$. Hence an equation of the line tangent to the graph at that point is $y - 0 = 1\cdot(x-1)$; that is, $y = x - 1$.

C07S01.062: If

$$f(x) = \frac{\ln x}{x^2}, \qquad \text{then} \qquad f'(x) = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}.$$

Hence the slope of the graph of $y = f(x)$ at the point $(e,\, e^{-2})$ is $f'(e) = -1/e^3$. Therefore an equation of the line tangent to the graph at that point is

$$y - \frac{1}{e^2} = -\frac{1}{e^3}(x - e); \qquad \text{that is,} \qquad y = \frac{2e - x}{e^3}.$$

C07S01.063: If $f(x) = e^{2x}$, then

$$f'(x) = 2e^{2x}, \quad f''(x) = 4e^{2x}, \quad f'''(x) = 8e^{2x}, \quad f^{(4)}(x) = 16e^{2x}, \quad \text{and} \quad f^{(5)}(x) = 32e^{2x}.$$

It appears that $f^{(n)}(x) = 2^n e^{2x}$.

C07S01.064: If $f(x) = xe^x$, then

$$f'(x) = (x+1)e^x, \quad f''(x) = (x+2)e^x, \quad f'''(x) = (x+3)e^x, \quad f^{(4)}(x) = (x+4)e^x, \quad \text{and} \quad f^{(5)}(x) = (x+5)e^x.$$

It appears that $f^{(n)}(x) = (x+n)e^x$.

C07S01.065: If $f(x) = e^{-x/6}\sin x$, then

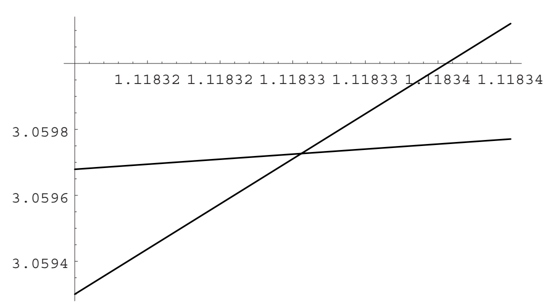
$$f'(x) = -\frac{1}{6}e^{-x/6}\sin x + e^{-x/6}\cos x = \frac{6\cos x - \sin x}{6e^{x/6}}.$$

Hence the first local maximum point for $x > 0$ occurs when $x = \arctan 6$ and the first local minimum point occurs when $x = \pi + \arctan 6$. The corresponding y-coordinates are, respectively,

$$\frac{6}{e^{(\arctan 6)/6}\sqrt{37}} \qquad \text{and} \qquad -\frac{6}{e^{(\pi + \arctan 6)/6}\sqrt{37}}.$$

C07S01.066: Given $f(x) = e^{-x/6}\sin x$, let $g(x) = e^{-x/6}$ and $h(x) = -e^{-x/6}$. We solve the equation $f(x) = g(x)$ by hand; the x-coordinate of the first point of tangency is $\pi/2$. Similarly, the x-coordinate of the second point of tangency is $3\pi/2$. These are *not* the same as $\arctan 6$ and $\pi + \arctan 6$.

C07S01.067: The viewing window $1.11831 \leqq x \leqq 1.11834$ shows the intersection of the two graphs near 1.11833 (see the figure that follows this solution). Thus, to three decimal places, the indicated solution of $e^x = x^{10}$ is 1.118.



C07S01.068: The viewing rectangle with $35.771515 \leqq x \leqq 35.771525$ reveals a solution of $e^x = x^{10}$ near $x = 35.75152$. Therefore this solution is approximately 3.58×10^1. Newton's method applied to $f(x) = e^x - x^{10}$ reveals the more accurate approximation 35.7715206396.

C07S01.069: We first let

$$f(k) = \left(1 + \frac{1}{10^k}\right)^{10^k}.$$

Then *Mathematica* 3.0 yields the following approximations:

k	$f(k)$ (rounded)
1	2.593742460100
2	2.704813829422
3	2.716923932236
4	2.718145926825
5	2.718268237174
6	2.718280469319
7	2.718281692545
8	2.718281814868
9	2.718281827100
10	2.718281828323
11	2.718281828445
12	2.718281828458
13	2.718281828459
14	2.718281828459
15	2.718281828459
16	2.718281828459045099
17	2.719291929459045222
18	2.718281828459045234
19	2.718281828459045235
20	2.718281828459045235
21	2.718281828459045235

C07S01.070: If $y = u^v$ where all are differentiable functions of x, then $\ln y = v \ln u$. With $u'(x)$ denoted simply by u', etc., we now have

$$\frac{1}{y}y' = v' \ln u + \frac{vu'}{u}.$$

Thus $y' = u^v\, v' \ln u + \dfrac{u^v vu'}{u} = vu^{v-1}u' + u^v(\ln u)v'$.

(a) If u is constant, this implies that $\dfrac{dy}{dx} = u^{v(x)}(\ln u)v'(x)$.

(b) If v is constant, this implies that $\dfrac{dy}{dx} = v\,(u(x))^{v-1}\,u'(x)$.

C07S01.071: Solution:

$$\ln y = \ln u + \ln v + \ln w - \ln p - \ln q - \ln r;$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} - \frac{1}{p}\cdot\frac{dp}{dx} - \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx};$$

$$\frac{dy}{dx} = y\cdot\left(\frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} - \frac{1}{p}\cdot\frac{dp}{dx} - \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}\right).$$

The solution makes the generalization obvious.

C07S01.072: Suppose by way of contradiction that $\log_2 3$ is a rational number. Then $\log_2 3 = p/q$ where p and q are positive integers (both positive because $\log_2 3 > 0$). Thus $2^{p/q} = 3$, so that $2^p = 3^q$. But if p and q are *positive* integers, then 2^p is even and 3^q is odd, so they cannot be equal. Therefore the assumption that $\log_2 3$ is rational leads to a contradiction, and thus $\log_2 3$ is irrational.

C07S01.073: (a): If $f(x) = \log_{10} x$, then the definition of the derivative yields

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{1}{h}\log_{10}(1+h) = \lim_{h\to 0}\log_{10}(1+h)^{1/h}.$$

(b): When $h = 0.1$ the value of $\log_{10}(1+h)^{1/h}$ is approximately 0.4139. With $h = 0.01$ we get 0.4321, with $h = 0.001$ we get 0.4341, and with $h = \pm 0.0001$ we get 0.4343.

C07S01.074: Because $\exp(\ln x) = x$, we see first that

$$10^x = \exp(\ln 10^x) = \exp(x\ln 10) = e^{x\ln 10}.$$

Hence

$$D_x\,10^x = D_x\left(e^{x\ln 10}\right) = e^{x\ln 10}\ln 10 = 10^x\ln 10.$$

Thus, by the chain rule, if u is a differentiable function of x, then

$$D_x 10^u = (10^u\ln 10)\,\frac{du}{dx}.$$

Finally, if $u(x) = \log_{10} x$, so that $10^u \equiv x$, then differentiation of this last identity yields

$$(10^u\ln 10)\frac{du}{dx} \equiv 1, \qquad \text{so that} \qquad \frac{du}{dx} = D_x\log_{10} x = \frac{1}{x\ln 10} \approx \frac{0.4343}{x}.$$

Section 7.2

C07S02.001: You don't need l'Hôpital's rule to evaluate this limit, but you may use it:

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{2x} = \frac{1}{2}.$$

C04S08.002: You don't need l'Hôpital's rule to evaluate this limit, but you may use it:

$$\lim_{x\to \infty} \frac{3x-4}{2x-5} = \lim_{x\to \infty} \frac{3}{2} = \frac{3}{2}.$$

C07S02.003: You don't need l'Hôpital's rule to evaluate this limit, but you may use it (twice):

$$\lim_{x\to \infty} \frac{2x^2-1}{5x^2+3x} = \lim_{x\to \infty} \frac{4x}{10x+3} = \lim_{x\to \infty} \frac{4}{10} = \frac{2}{5}.$$

C07S02.004: You don't need l'Hôpital's rule to evaluate this limit (apply the definition of the derivative to the evaluation of $f'(0)$ where $f(x) = e^{3x}$), but you may use it:

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = \lim_{x\to 0} \frac{3e^{3x}}{1} = 3.$$

C07S02.005: Without l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} x \cdot \frac{\sin x^2}{x^2} = 0 \cdot 1 = 0.$$

(We used Theorem 1 of Section 2.3, $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$, and the product law for limits.)

With l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} \frac{2x\cos x^2}{1} = 2 \cdot 0 \cdot 1 = 0.$$

C07S02.006: You don't need l'Hôpital's rule to evaluate this limit (see the solution to Problem 3 of Section 2.3), but you may use it:

$$\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{2}x^{-1/2}\sin x^{1/2}}{1} = \lim_{x\to 0^+} \frac{\sin x^{1/2}}{2x^{1/2}} = \frac{1}{2}$$

by Theorem 1 of Section 2.3. If you prefer a "pure" l'Hôpital's rule solution, you should substitute $x = u^2$ to obtain

$$\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{x} = \lim_{u\to 0} \frac{1-\cos u}{u^2} = \lim_{u\to 0} \frac{\sin u}{2u} = \lim_{u\to 0} \frac{\cos u}{2} = \frac{1}{2}.$$

Note that it was necessary to apply l'Hôpital's rule twice in the second solution.

C07S02.007: You may *not* use l'Hôpital's rule! The numerator is approaching zero but the denominator is not. Hence use the quotient law for limits (Section 2.2):

$$\lim_{x\to 1} \frac{x-1}{\sin x} = \frac{\displaystyle\lim_{x\to 1}(x-1)}{\displaystyle\lim_{x\to 1}\sin x} = \frac{0}{\sin 1} = 0.$$

Note that illegal use of l'Hôpital's rule in this problem will result in the incorrect value $\sec 1 \approx 1.8508157177$ for the limit.

C07S02.008: The numerator and denominator of the fraction are both approaching zero, so you may try using l'Hôpital's rule (twice):

$$\lim_{x \to 0} \frac{1 - \cos x}{x^3} = \lim_{x \to 0} \frac{\sin x}{3x^2} = \lim_{x \to 0} \frac{\cos x}{6x}.$$

But the latter limit does not exist (the left-hand limit is $-\infty$ and the right-hand limit is $+\infty$). So the hypotheses of Theorem 1 of Section 7.2 are not satisfied; this is a case in which l'Hôpital's rule (as stated in Theorem 1) has failed. Other measures are needed. By l'Hôpital's rule (or by Problem 3 of Section 2.3), we have

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Therefore

$$\frac{1 - \cos x}{x^3} \approx \frac{1}{2x}$$

if x is close to zero. Consequently $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x^3}$ does not exist.

C04S08.009: Without l'Hôpital's rule we might need to resort to the Taylor series methods of Section 11.9 to evaluate this limit. But l'Hôpital's rule may be applied (twice):

$$\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}.$$

C07S02.010: You may not apply l'Hôpital's rule: The numerator is approaching zero but the denominator is not. But this means that the quotient law of limits (Section 2.2) may be applied instead:

$$\lim_{z \to \pi/2} \frac{1 + \cos 2z}{1 - \sin 2z} = \frac{\displaystyle\lim_{z \to \pi/2} (1 + \cos 2z)}{\displaystyle\lim_{z \to \pi/2} (1 - \sin 2z)} = \frac{1 - 1}{1 - 0} = 0.$$

This problem would be more interesting if the denominator were $1 - \sin z$.

C07S02.011: The numerator and denominator are both approaching zero, so l'Hôpital's rule may be applied:

$$\begin{aligned} \lim_{u \to 0} \frac{u \tan u}{1 - \cos u} &= \lim_{u \to 0} \frac{\tan u + u \sec^2 u}{\sin u} \\ &= \lim_{u \to 0} \left(\sec u + \frac{u}{\sin u} \cdot \sec^2 u \right) = 1 + 1 \cdot 1^2 = 2. \end{aligned}$$

Note that we used the sum law for limits (Section 2.2) and the fact that

$$\lim_{u \to 0} \frac{u}{\sin u} = 1,$$

a consequence of Theorem 1 of Section 2.3 and the quotient law for limits.

C07S02.012: Numerator and denominator are both approaching zero, so l'Hôpital's rule may be applied (twice):

$$\lim_{x\to 0}\frac{x-\tan x}{x^3} = \lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}$$
$$= \lim_{x\to 0}\left(-\frac{2\sec^2 x\tan x}{6x}\right) = -\frac{1}{3}\left(\lim_{x\to 0}\frac{\sec^2 x}{\cos x}\cdot\frac{\sin x}{x}\right) = -\frac{1}{3}\cdot\frac{1^2}{1}\cdot 1 = -\frac{1}{3}.$$

C07S02.013: $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^{1/10}} = \lim_{x\to\infty}\frac{1}{x\cdot\frac{1}{10}x^{-9/10}} = \lim_{x\to\infty}\frac{10}{x^{1/10}} = 0.$

C04S08.014: Several applications of l'Hôpital's rule yield

$$\lim_{r\to\infty}\frac{e^r}{(r+1)^4} = \lim_{r\to\infty}\frac{e^r}{4(r+1)^3} = \lim_{r\to\infty}\frac{e^r}{12(r+1)^2} = \lim_{r\to\infty}\frac{e^r}{24(r+1)} = \frac{e^r}{24} = +\infty.$$

Even though the limit does not exist, the hypotheses of Theorem 1 of Section 7.2 are all satisfied, so the answer is correct.

C07S02.015: $\displaystyle\lim_{x\to 10}\frac{\ln(x-9)}{x-10} = \lim_{x\to 10}\frac{1}{1\cdot(x-9)} = 1.$

C07S02.016: $\displaystyle\lim_{t\to\infty}\frac{t^2+1}{t\ln t} = \lim_{t\to\infty}\frac{2t}{1+\ln t} = \lim_{t\to\infty}\frac{2}{t^{-1}} = \lim_{t\to\infty}(2t) = +\infty.$

C07S02.017: Always verify that the hypotheses of l'Hôpital's rule are satisfied.

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x\sin x} = \lim_{x\to 0}\frac{e^x-e^{-x}}{x\cos x+\sin x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{2\cos x - x\sin x} = \frac{1+1}{2-0} = 1.$$

C07S02.018: As $x\to(\pi/2)^-$, $\tan x\to+\infty$ and $\cos x\to 0^+$, so that $\ln(\cos x)\to-\infty$. Hence l'Hôpital's rule may be tried:

$$\lim_{x\to(\pi/2)^-}\frac{\tan x}{\ln(\cos x)} = \lim_{x\to(\pi/2)^-}\frac{\sec^2 x\,\cos x}{-\sin x} = \lim_{x\to(\pi/2)^-}\frac{-\sec x}{\sin x} = -\infty$$

because $\sin x\to 1$ and $\sec x\to+\infty$ as $x\to(\pi/2)^-$.

C07S02.019: Methods of Section 2.3 may be used, or l'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 3x}{\tan 5x} = \lim_{x\to 0}\frac{3\cos 3x}{5\sec^2 5x} = \frac{3\cdot 1}{5\cdot 1} = \frac{3}{5}.$$

C07S02.020: $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{x} = \lim_{x\to 0}\frac{e^x+e^{-x}}{1} = 2.$

C07S02.021: The factoring techniques of Section 2.2 work well here, or l'Hôpital's rule yields

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1} = \lim_{x\to 1}\frac{3x^2}{2x} = \frac{3}{2}.$$

C07S02.022: $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x^4-16} = \lim_{x\to 2}\frac{3x^2}{4x^3} = \frac{12}{32} = \frac{3}{8}.$

C07S02.023: Both numerator and denominator approach $+\infty$ as x does, so we may attempt to find the limit with l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x+\sin x}{3x+\cos x} = \lim_{x\to\infty} \frac{1+\cos x}{3-\sin x},$$

but the latter limit does not exist (and so the equals mark in the previous equation is invalid). The reason: As $x \to +\infty$, x runs infinitely many times through numbers of the form $n\pi$ where n is a positive even integer. At such real numbers the value of

$$f(x) = \frac{1+\cos x}{3-\sin x}$$

is $\frac{2}{3}$. But x also runs infinitely often through numbers of the form $n\pi$ where n is a positive odd integer. At these real numbers the value of $f(x)$ is 0. Because $f(x)$ takes on these two distinct values infinitely often as $x \to +\infty$, $f(x)$ has no limit as $x \to +\infty$.

This does not imply that the limit given in Problem 22 does not exist. (Read Theorem 1 carefully.) In fact, the limit does exist, and we have here the rare phenomenon of failure of l'Hôpital's rule. Other techniques must be used to solve this problem. Perhaps the simplest is this:

$$\lim_{x\to\infty} \frac{x+\sin x}{3x+\cos x} = \lim_{x\to\infty} \frac{1+\dfrac{\sin x}{x}}{3-\dfrac{\cos x}{x}} = \frac{1+0}{3-0} = \frac{1}{3}.$$

C07S02.024: First we try l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to\infty} \frac{(x^2+4)^{1/2}}{x} &= \lim_{x\to\infty} \frac{\frac{1}{2}(x^2+4)^{-1/2}\cdot 2x}{1} = \lim_{x\to\infty} \frac{x}{(x^2+4)^{1/2}}\\ &= \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+4)^{-1/2}\cdot 2x} = \lim_{x\to\infty} \frac{(x^2+4)^{1/2}}{x}.\end{aligned}$$

Another failure of l'Hôpital's rule! Here are two ways to find the limit.

$$\left(\lim_{x\to\infty} \frac{(x^2+4)^{1/2}}{x}\right)^2 = \lim_{x\to\infty} \left(\frac{(x^2+4)^{1/2}}{x}\right)^2 = \lim_{x\to\infty} \frac{x^2+4}{x^2} = \lim_{x\to\infty} \frac{2x}{2x} = \lim_{x\to\infty} \frac{1}{1} = 1.$$

Therefore the original limit is also 1. Second method:

$$\lim_{x\to\infty} \frac{(x^2+4)^{1/2}}{x} = \lim_{x\to\infty} \left(\frac{x^2+4}{x^2}\right)^{1/2} = \lim_{x\to\infty} \left(1+\frac{4}{x^2}\right)^{1/2} = \sqrt{1} = 1.$$

Methods of Section 11.9 may also be used, but—while quite general—are a bit unwieldy in this case.

C07S02.025: $\displaystyle \lim_{x\to 0} \frac{2^x-1}{3^x-1} = \lim_{x\to 0} \frac{2^x \ln 2}{3^x \ln 3} = \frac{\ln 2}{\ln 3} \approx 0.6309297536.$

C07S02.026: You may apply l'Hôpital's rule, but watch what happens:

$$\lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} \frac{2^x \ln 2}{3^x \ln 3} = \lim_{x\to\infty} \frac{2^x(\ln 2)^2}{3^x(\ln 3)^2} = \lim_{x\to\infty} \frac{2^x(\ln 2)^3}{3^x(\ln 3)^3} = \cdots.$$

If we knew that the original limit existed and was finite, we could conclude that it must be zero, but we don't even know that it exists. But

$$\left(\frac{3}{2}\right)^{3x} = \left(\frac{27}{8}\right)^x > 3^x > e^x,$$

so

$$\lim_{x\to\infty} \left(\frac{3}{2}\right)^x = +\infty, \qquad \text{and therefore} \qquad \lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} \left(\frac{2}{3}\right)^x = 0.$$

C07S02.027: You can solve this problem without l'Hôpital's rule, but if you intend to use it you should probably proceed as follows:

$$\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{\sqrt{4x^2-x}} = \lim_{x\to\infty} \left(\frac{x^2-1}{4x^2-x}\right)^{1/2} = \left(\lim_{x\to\infty} \frac{x^2-1}{4x^2-x}\right)^{1/2}$$

$$= \left(\lim_{x\to\infty} \frac{2x}{8x-1}\right)^{1/2} = \left(\lim_{x\to\infty} \frac{2}{8}\right)^{1/2} = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{2}.$$

C07S02.028: As in the previous solution,

$$\lim_{x\to\infty} \frac{\sqrt{x^3+x}}{\sqrt{2x^3-4}} = \left(\lim_{x\to\infty} \frac{x^3+x}{2x^3-4}\right)^{1/2} = \left(\lim_{x\to\infty} \frac{3x^2+1}{6x^2}\right)^{1/2}$$

$$= \left(\lim_{x\to\infty} \frac{6x}{12x}\right)^{1/2} = \left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}.$$

C07S02.029: $\displaystyle \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1\cdot(1+x)} = 1.$

C07S02.030: It would be easier to establish first that

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = 0,$$

and it would follow immediately that the given limit is zero as well. But let's see how well a direct approach succeeds.

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x\ln x} = \lim_{x\to\infty} \frac{\dfrac{1}{x\ln x}}{1+\ln x} = \lim_{x\to\infty} \frac{1}{(1+\ln x)(x\ln x)} = 0$$

because

$$\frac{1}{(1+\ln x)(x\ln x)} < \frac{1}{x} \qquad \text{if} \qquad x > 1.8554$$

and $1/x \to 0$ as $x \to \infty$.

C07S02.031: Three applications of l'Hôpital's rule yield

$$\lim_{x\to 0} \frac{2e^x - x^2 - 2x - 2}{x^3} = \lim_{x\to 0} \frac{2e^x - 2x - 2}{3x^2} = \lim_{x\to 0} \frac{2e^x - 2}{6x} = \lim_{x\to 0} \frac{2e^x}{6} = \frac{1}{3}.$$

C07S02.032: Three applications of l'Hôpital's rule yield

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{3x^2}=\lim_{x\to 0}\frac{-\sin x-2\sec^2 x\,\tan x}{6x}$$
$$=\lim_{x\to 0}\frac{-\cos x-4\sec^2 x\,\tan^2 x-2\sec^4 x}{6}=\frac{-1-0-2}{6}=-\frac{1}{2}.$$

C07S02.033: $\displaystyle\lim_{x\to 0}\frac{2-e^x-e^{-x}}{2x^2}=\lim_{x\to 0}\frac{e^{-x}-e^x}{4x}=\lim_{x\to 0}\frac{-e^{-x}-e^x}{4}=-\frac{1}{2}.$

C07S02.034: $\displaystyle\lim_{x\to 0}\frac{e^{3x}-e^{-3x}}{2x}=\lim_{x\to 0}\frac{3e^{3x}+3e^{-3x}}{2}=3.$

C07S02.035: $\displaystyle\lim_{x\to\pi/2}\frac{2x-\pi}{\tan 2x}=\lim_{x\to\pi/2}\frac{2}{2\sec^2 2x}=\frac{1}{\sec^2\pi}=1.$

C07S02.036: The "direct approach" yields

$$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\sec x\,\tan x}{\sec^2 x}=\lim_{x\to\pi/2}\frac{\tan x}{\sec x}=\lim_{x\to\pi/2}\frac{\sec^2 x}{\sec x\,\tan x}=\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\cdots\,.$$

Proceed instead as follows (*without* l'Hôpital's rule):

$$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{1}{\cos x}\cdot\frac{\cos x}{\sin x}=\lim_{x\to\pi/2}\frac{1}{\sin x}=\frac{1}{1}=1.$$

C07S02.037: $\displaystyle\lim_{x\to 2}\frac{x-2\cos\pi x}{x^2-4}=\lim_{x\to 2}\frac{1+2\pi\sin\pi x}{2x}=\frac{1+0}{4}=\frac{1}{4}.$

C07S02.038: $\displaystyle\lim_{x\to 1/2}\frac{2x-\sin\pi x}{4x^2-1}=\lim_{x\to 1/2}\frac{2-\pi\cos\pi x}{8x}=\frac{2-\pi\cdot 0}{4}=\frac{1}{2}.$

C07S02.039: We first simplify (using laws of logarithms), *then* apply l'Hôpital's rule:

$$\lim_{x\to 0^+}\frac{\ln(2x)^{1/2}}{\ln(3x)^{1/3}}=\lim_{x\to 0^+}\frac{\frac{1}{2}(\ln 2+\ln x)}{\frac{1}{3}(\ln 3+\ln x)}=\frac{3}{2}\left(\lim_{x\to 0^+}\frac{\dfrac{1}{x}}{\dfrac{1}{x}}\right)=\frac{3}{2}.$$

C07S02.040: One application of l'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\ln(1+x)}{\ln(1-x^2)}\overset{?}{=}\lim_{x\to 0}\frac{x-1}{2x}.$$

The limit on the right-hand side does not exist, so we use left-hand and right-hand limits:

$$\lim_{x\to 0^+}\frac{\ln(1+x)}{\ln(1-x^2)}=\lim_{x\to 0^+}\frac{x-1}{2x}=-\infty\qquad\text{and}\qquad\lim_{x\to 0^-}\frac{\ln(1+x)}{\ln(1-x^2)}=\lim_{x\to 0^-}\frac{x-1}{2x}=+\infty.$$

Therefore the original limit does not exist.

C07S02.041: Two applications of l'Hôpital's rule yield

$$\lim_{x\to 0}\frac{\exp(x^3)-1}{x-\sin x}=\lim_{x\to 0}\frac{3x^2\exp(x^3)}{1-\cos x}=\lim_{x\to 0}\frac{(6x+9x^4)\exp(x^3)}{\sin x}$$
$$=\lim_{x\to 0}\frac{(6+9x^3)\exp(x^3)}{\dfrac{\sin x}{x}}=\frac{(6+0)\cdot 1}{1}=6.$$

Alternatively, three applications of l'Hôpital's rule yield

$$\lim_{x\to 0}\frac{\exp(x^3)-1}{x-\sin x}=\lim_{x\to 0}\frac{3x^2\exp(x^3)}{1-\cos x}=\lim_{x\to 0}\frac{(6x+9x^4)\exp(x^3)}{\sin x}$$
$$=\lim_{x\to 0}\frac{(6+54x^3+27x^6)\exp(x^3)}{\cos x}=\frac{(6+54\cdot 0+27\cdot 0)\cdot 1}{1}=6.$$

In this problem the Taylor series methods of Section 11.9 are considerably simpler.

C07S02.042: The technique of multiplying numerator and denominator by the conjugate of the numerator succeeds just as it did in Sections 2.2 and 2.3. Use of l'Hôpital's rule yields

$$\lim_{x\to 0}\frac{(1+3x)^{1/2}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2}(1+3x)^{-1/2}\cdot 3}{1}=\lim_{x\to 0}\frac{3}{2(1+3x)^{1/2}}=\frac{3}{2}.$$

C07S02.043: $\displaystyle\lim_{x\to 0}\frac{(1+4x)^{1/3}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{3}(1+4x)^{-2/3}\cdot 4}{1}=\lim_{x\to 0}\frac{4}{3(1+4x)^{2/3}}=\frac{4}{3}.$

C07S02.044: Multiplication of numerator and denominator by the conjugate of the numerator is one way to find this limit; l'Hôpital's rule yields

$$\lim_{x\to 0}\frac{(3+2x)^{1/2}-(3+x)^{1/2}}{x}=\lim_{x\to 0}\frac{\frac{1}{2}(3+2x)^{-1/2}\cdot 2-\frac{1}{2}(3+x)^{-1/2}}{1}$$
$$=\lim_{x\to 0}\left[\frac{1}{(3+2x)^{1/2}}-\frac{1}{2(3+x)^{1/2}}\right]=\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{3}}=\frac{1}{2\sqrt{3}}.$$

C07S02.045: If you want to use the conjugate technique to find this limit, you need to know that the conjugate of $a^{1/3}-b^{1/3}$ is $a^{2/3}+a^{1/3}b^{1/3}+b^{2/3}$, and the algebra becomes rather long. Here l'Hôpital's rule is probably the easy way:

$$\lim_{x\to 0}\frac{(1+x)^{1/3}-(1-x)^{1/3}}{x}=\lim_{x\to 0}\frac{\frac{1}{3}(1+x)^{-2/3}+\frac{1}{3}(1-x)^{-2/3}}{1}$$
$$=\lim_{x\to 0}\left[\frac{1}{3(1+x)^{2/3}}+\frac{1}{3(1-x)^{2/3}}\right]=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}.$$

C07S02.046: $\displaystyle\lim_{x\to\pi/4}\frac{1-\tan x}{4x-\pi}=\lim_{x\to\pi/4}\frac{-\sec^2 x}{4}=\frac{-\left(\sqrt{2}\,\right)^2}{4}=-\frac{1}{2}.$

C07S02.047: $\displaystyle\lim_{x\to 0}\frac{\ln(1+x^2)}{e^x-\cos x}=\lim_{x\to 0}\frac{2x}{(1+x^2)(e^x+\sin x)}=\frac{0}{(1+0)(1+0)}=0.$

C07S02.048: Because the numerator and denominator are both approaching zero as $x \to 2$, it should be possible to factor $x - 2$ out of each, cancel, and proceed without l'Hôpital's rule. But if we use l'Hôpital's rule, the result is

$$\lim_{x\to 2} \frac{x^5 - 5x^2 - 12}{x^{10} - 500x - 24} = \lim_{x\to 2} \frac{5x^4 - 10x}{10x^9 - 500} = \frac{80 - 20}{5120 - 500} = \frac{60}{4620} = \frac{1}{77}.$$

The factor-and-cancel technique yields

$$\lim_{x\to 2} \frac{x^5 - 5x^2 - 12}{x^{10} - 500x - 24} = \lim_{x\to 2} \frac{(x-2)(x^4 + 2x^3 + 4x^2 + 3x + 6)}{(x-2)(x^9 + 2x^8 + 4x^7 + 8x^6 + 16x^5 + 32x^4 + 64x^3 + 128x^2 + 256x + 12)}$$

$$= \lim_{x\to 2} \frac{x^4 + 2x^3 + 4x^2 + 3x + 6}{x^9 + 2x^8 + 4x^7 + 8x^6 + 16x^5 + 32x^4 + 64x^3 + 128x^2 + 256x + 12}$$

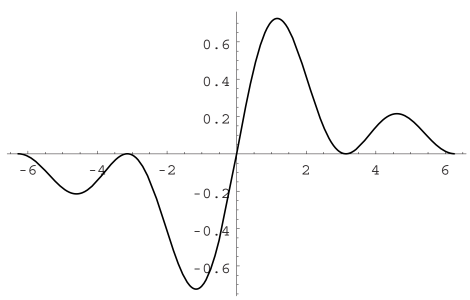
$$= \frac{16 + 16 + 16 + 6 + 6}{512 + 512 + 512 + 512 + 512 + 512 + 512 + 512 + 512 + 12} = \frac{60}{4620} = \frac{1}{77}.$$

In this problem l'Hôpital's rule seems the better choice.

C07S02.049: If $f(x) = \dfrac{\sin^2 x}{x}$, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2\sin x\, \cos x}{1} = 2 \cdot 0 \cdot 1 = 0.$$

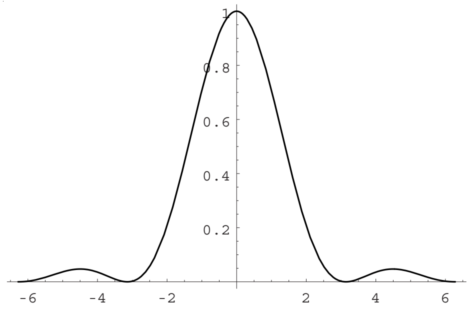
The graph of $y = f(x)$ is next.



C07S02.050: We don't need l'Hôpital's rule—we could use Theorem 1 in Section 2.3—but we'll use the rule anyway:

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \frac{2\sin x\, \cos x}{2x} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \cos x\right) = \left(\lim_{x\to 0} \frac{\cos x}{1}\right) \cdot 1 = 1 \cdot 1 = 1.$$

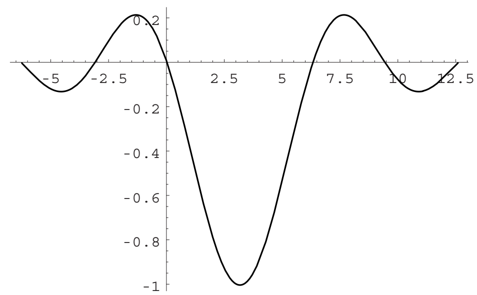
The graph is next.



C07S02.051: Here we have

$$\lim_{x \to \pi} \frac{\sin x}{x - \pi} = \lim_{x \to \pi} \frac{\cos x}{1} = \cos \pi = -1.$$

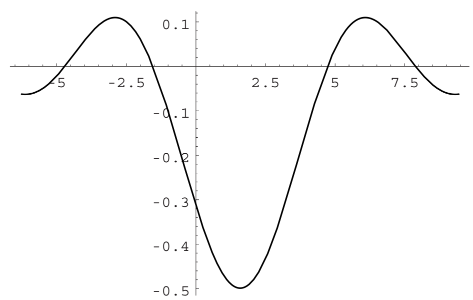
The graph is next.



C07S02.052: One application of l'Hôpital's rule yields

$$\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi} = \lim_{x \to \pi/2} \frac{-\sin x}{2} = -\frac{1}{2}.$$

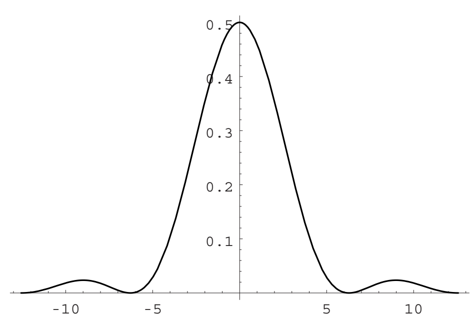
The graph is next.



C07S02.053: Two applications of l'Hôpital's rule yield

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

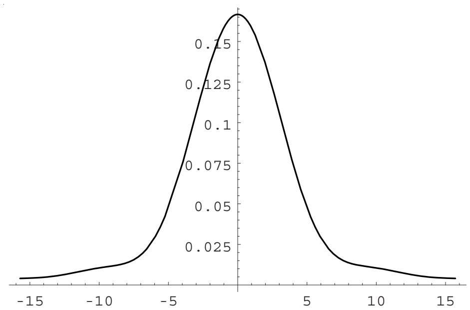
The graph is next.



C07S02.054: Three applications of l'Hôpital's rule yield

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

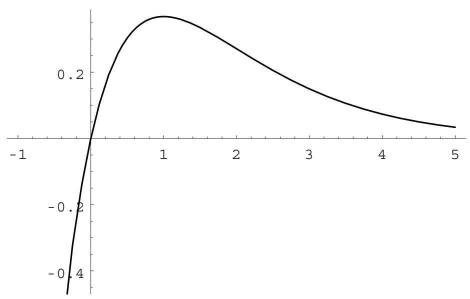
The graph is next.



C07S02.055: As $x \to -\infty$, $f(x) = xe^{-x} \to -\infty$, but

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Also $f'(x) = (1-x)e^{-x}$ and $f''(x) = (x-2)e^{-x}$. It follows that the graph of f is increasing for $x < 1$, decreasing for $x > 1$, concave downward for $x < 2$, and concave upward for $x > 2$. The positive x-axis is a horizontal asymptote and the only intercept is $(0, 0)$. The graph of $y = f(x)$ is shown next.



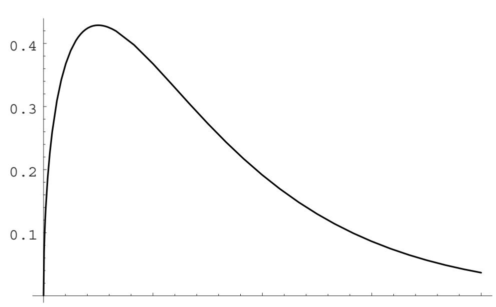
C07S02.056: Given $f(x) = x^{1/2}e^{-x}$, we first use l'Hôpital's rule:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^{1/2}}{e^x} = \lim_{x\to\infty} \frac{1}{2x^{1/2}e^x} = 0.$$

Next,

$$f'(x) = \frac{1-2x}{2x^{1/2}e^x} \qquad \text{and} \qquad f''(x) = \frac{4x^2-4x-1}{4x^{3/2}e^x}.$$

So the graph of f is increasing for $0 < x < \frac{1}{2}$ and decreasing for $x > \frac{1}{2}$. There is an inflection point where $x = \frac{1}{2}\left(1+\sqrt{2}\right)$; the y-coordinate is approximately 0.3285738758. The positive x-axis is a horizontal asymptote and the only intercept is $(0, 0)$. The graph of $y = f(x)$ is shown next.



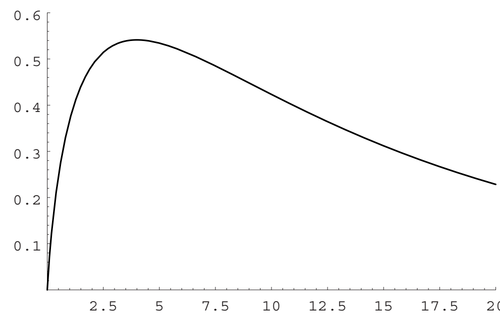
C07S02.057: If $f(x) = x \exp\left(-x^{1/2}\right)$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\exp\left(x^{1/2}\right)} = \lim_{x\to\infty} \frac{2x^{1/2}}{\exp\left(x^{1/2}\right)} = \lim_{x\to\infty} \frac{2x^{1/2}}{x^{1/2}\exp\left(x^{1/2}\right)} = \lim_{x\to\infty} \frac{2}{\exp\left(x^{1/2}\right)} = 0.$$

Thus the positive x-axis is a horizontal asymptote. Next,

$$f'(x) = \frac{2 - x^{1/2}}{2\exp\left(x^{1/2}\right)} \qquad \text{and} \qquad f''(x) = \frac{x^{1/2} - 3}{4x^{1/2}\exp\left(x^{1/2}\right)}.$$

Hence the graph of f is increasing for $0 < x < 4$ and decreasing for $x > 4$; it is concave downward if $0 < x < 9$ and concave upward if $x > 9$. The only intercept is $(0, 0)$. The graph of $y = f(x)$ is shown next.



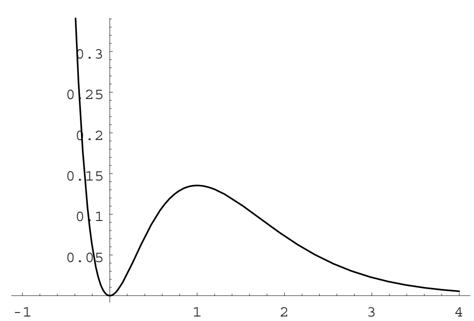
C07S02.058: Given: $f(x) = x^2e^{-2x}$. By l'Hôpital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0.$$

So the positive x-axis is a horizontal asymptote. Also note that $f(x) \to +\infty$ as $x \to -\infty$. Next,

$$f'(x) = 2x(1-x)e^{-2x} \qquad \text{and} \qquad f''(x) = 2(2x^2 - 4x + 1)e^{-2x},$$

and it follows that the graph of f is decreasing if $x < 0$ and if $x > 1$, increasing if $0 < x < 1$; it is concave upward for $x < a = \frac{1}{2}\left(2 - \sqrt{2}\right)$ and for $x > b = \frac{1}{2}\left(2 + \sqrt{2}\right)$. It is concave downward for $a < x < b$, so there are inflection points where $x = a$ and where $x = b$. The graph of $y = f(x)$ is next.



C07S02.059: Given: $f(x) = \dfrac{\ln x}{x}$. So

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$$

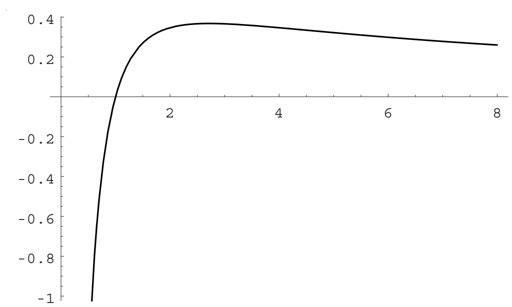
because the numerator is approaching $-\infty$ and the denominator is approaching 0 through positive values. Next, using l'Hôpital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{1\cdot x} = 0,$$

so the positive y-axis is a horizontal asymptote and the negative y-axis is a vertical asymptote. Moreover,

$$f'(x) = \frac{1-\ln x}{x^2} \qquad \text{and} \qquad f''(x) = \frac{-3+2\ln x}{x^3},$$

and thus the graph of f is increasing if $0 < x < e$, decreasing if $x > e$, concave downward for $0 < x < e^{3/2}$, and concave upward if $x > e^{3/2}$. The inflection point where $x = e^{3/2}$ is not visible because the curvature of the graph is very small for $x > 3$. The graph of $y = f(x)$ is next.



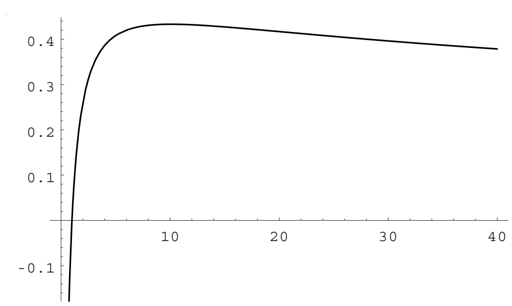
C07S02.060: Given:

$$f(x) = \frac{\ln x}{x^{1/2}+x^{1/3}}.$$

We used *Mathematica* to find $f'(x)$ and solve $f'(x) = 0$, and thus discovered that there is a global maximum near (10.094566, 0.433088). Similarly, we found an inflection point near (20.379823, 0.416035). Clearly $f(x) \to -\infty$ as $x \to 0^+$, so the negative y-axis is a vertical asymptote. Also

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}+x^{1/3}} = \lim_{x\to\infty} \frac{1}{x\left(\frac{1}{2}x^{-1/2}+\frac{1}{3}x^{-2/3}\right)} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}x^{1/2}+\frac{1}{3}x^{1/3}} = 0,$$

so the positive x-axis is a horizontal asymptote. The graph of $y = f(x)$ is next.



C07S02.061: The computation in the solution of Problem 55 establishes that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \tag{1}$$

in the case $n = 1$. Suppose that Eq. (1) holds for $n = k$, a positive integer. Then (by l'Hôpital's rule)

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x} = (k+1)\left(\lim_{x\to\infty} \frac{x^k}{e^x}\right) = (k+1)\cdot 0 = 0.$$

Therefore, by induction, Eq. (1) holds for every positive integer n. Now suppose that $k > 0$ (a positive number not necessarily an integer). Let n be an integer larger than k. Then, for x large positive, we have

$$0 < \frac{x^k}{e^x} < \frac{x^n}{e^x}.$$

Therefore, by the squeeze law for limits, $\displaystyle\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ for every positive number k.

C07s02.062: Suppose that k is a positive real number. Then (by l'Hôpital's rule)

$$\lim_{x\to\infty} \frac{\ln x}{x^k} = \lim_{x\to\infty} \frac{1}{kx^{k-1}\cdot x} = \lim_{x\to\infty} \frac{1}{kx^k} = 0.$$

C07S02.063: Given: $f(x) = x^n e^{-x}$ where n is a positive integer larger than 1. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

by the result in Problem 61. So the positive x-axis is a horizontal asymptote. Next,

$$f'(x) = \frac{(n-x)x^{n-1}}{e^x} \qquad \text{and} \qquad f''(x) = \frac{(x^2 - 2nx + n^2 - n)x^{n-2}}{e^x}.$$

Therefore $f'(x) = 0$ at the two points $(0, 0)$ and $(n, n^n e^{-n})$. We consider only the part of the graph for which $x > 0$, and the graph of f is increasing for $0 < x < n$ and decreasing if $x > n$, so there is a local maximum at $x = n$. Next, $f''(x) = 0$ when $x = a = n - \sqrt{n}$ and when $x = b = n + \sqrt{n}$. It is easy to establish that $f''(x) > 0$ if $0 < x < a$ and if $x > b$, but that $f''(x) < 0$ if $a < x < b$. (Use the fact that the graph of $g(x) = x^2 - 2nx + n^2 - n$ is a parabola opening upward.) Therefore the graph of f has two inflection points for $x > 0$.

C07S02.064: Given: $f(x) = x^{-k}\ln x$ where k is a positive constant. Then

$$f'(x) = \frac{1 - k\ln x}{x^{k+1}},$$

and the sign of $f'(x)$ is the same as the sign of $1 - k\ln x$, which is positive if $0 < x < e^{1/k}$ but negative if $x > e^{1/k}$. Hence the graph of f will have a single local maximum where $x = e^{1/k}$. Next,

$$f''(x) = \frac{k^2\ln x + k\ln x - 2k - 1}{x^{k+2}},$$

so $f''(x) = 0$ when

$$x = \exp\left(\frac{2k+1}{k^2+k}\right),$$

so the graph of f has at most one inflection point. Moreover, if x is near zero then $f''(x) < 0$, whereas $f''(x) > 0$ if x is large positive. Therefore the graph of f has exactly one inflection point. Finally, the result in Problem 62 shows that the positive x-axis is a horizontal asymptote.

C07S02.065: The substitution $y = \dfrac{1}{x}$ yields

$$\lim_{x\to 0^+} x^k \ln x = \lim_{y\to\infty} \frac{-\ln y}{y^k} = -\left(\lim_{y\to\infty} \frac{1}{ky^k}\right) = 0.$$

C07S02.066: First suppose that $n = -k < 0$ where k is a positive integer. Then

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{1}{x(\ln x)^k} = 0.$$

Next suppose that $n = 0$. Then

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Moreover, if $n = 1$, then by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Assume that

$$\lim_{x\to\infty} \frac{(\ln x)^k}{x} = 0$$

for some positive integer k. Then, by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{(\ln x)^{k+1}}{x} = \lim_{x\to\infty} \frac{(k+1)(\ln x)^k}{x} = (k+1) \lim_{x\to\infty} \frac{(\ln x)^k}{x} = (k+1)\cdot 0 = 0.$$

Therefore, by induction for positive n, $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^n}{x} = 0$ for every integer n.

C07S02.067: In the following computations we take derivatives with respect to h in the first step. By l'Hôpital's rule,

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0} \frac{f'(x+h) + f'(x-h)}{2} = \frac{2f'(x)}{2} = f'(x).$$

The continuity of $f'(x)$ is needed for two reasons: It implies that f is also continuous, so the first numerator approaches zero as $h \to 0$; moreover, continuity of $f'(x)$ is needed to ensure that $f'(x+h)$ and $f'(x-h)$ both approach $f'(x)$ as $h \to 0$.

C07S02.068: In the following computations we take derivatives with respect to h in the first two steps. By l'Hôpital's rule,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2} = \frac{2f''(x)}{2} = f''(x).\end{aligned}$$

The continuity of $f''(x)$ is needed for the following reasons: We needed to know that $f''(x+h)$ and $f''(x-h)$ both approach $f''(x)$ as $h \to 0$. We also needed to know that f' was continuous so that the second numerator approaches zero as $h \to 0$. There is a third reason, which you will see when you discover the reason for the presence of the term $-2f(x)$ in the first numerator.

C07S02.069: If

$$f(x) = \frac{(2x - x^4)^{1/2} - x^{1/3}}{1 - x^{4/3}},$$

then both the numerator $n(x) = (2x - x^4)^{1/2} - x^{1/3}$ and the denominator $d(x) = 1 - x^{4/3}$ approach zero as $x \to 1$, and both are differentiable, so l'Hôpital's rule may be applied. After simplifications we find that

$$n'(x) = \frac{3x^{2/3} - 6x^{11/3} - (2x - x^4)^{1/2}}{3x^{2/3}(2x - x^4)^{1/2}} \qquad \text{and} \qquad d'(x) = -\frac{4x^{1/3}}{3}.$$

Therefore

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{n'(x)}{d'(x)} = \lim_{x\to 1} \frac{-3x^{2/3} + 6x^{11/3} + (2x - x^4)^{1/2}}{4x(2x - x^4)^{1'2}} = \frac{-3 + 6 + 1}{4 \cdot 1} = \frac{4}{4} = 1.$$

C07S02.070: We are to show that if $f(x)$ and $g(x)$ both approach zero as $x \to +\infty$, both $f'(x)$ and $g'(x)$ exist for arbitrarily large values of x, and the second limit in the next line exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Following the *Suggestion*, we let $F(t) = f(1/t)$ and $G(t) = g(1/t)$. Then, with $t = 1/x$, we have

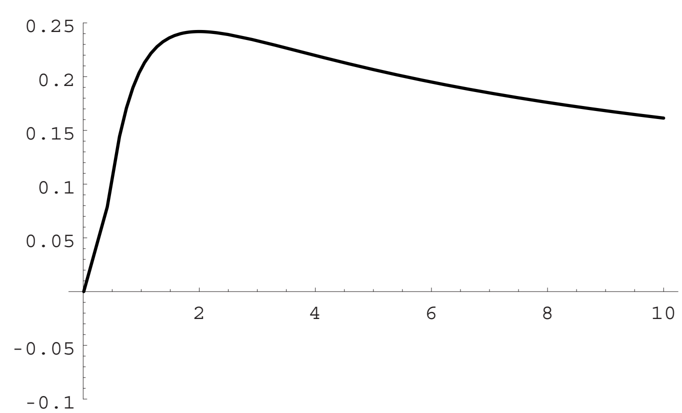
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{F(t)}{G(t)} = \lim_{t\to 0^+} \frac{F'(t)}{G'(t)}$$

provided that the last limit exists. Note that F and G are differentiable if $t > 0$ and t is close to zero. Hence

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{F'(t)}{G'(t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

C07S02.071: $\displaystyle \lim_{x\to\infty} \left(\frac{x}{e}\right)^x \geqq \lim_{x\to\infty} \left(\frac{e^2}{e}\right)^x = \lim_{x\to\infty} e^x = +\infty.$

C07S02.072: The graph of $C(t)$ for the case $A = 1$, $k = 1$, and $x = 2$ is next.



Next,

$$\frac{dC}{dt} = \frac{A}{\sqrt{k\pi t}} \left(\frac{x^2}{4kt^2} - \frac{1}{2t}\right) \exp\left(-\frac{x^2}{4kt}\right).$$

Now $dC/dt = 0$ when $2x^2t = 4kt^2$, so $t = 0$ or $t = x^2/(2k)$. The general shape of the graph shown here makes it clear that the former yields the minimum of $C(t)$ (define $C(0) = 0$ and C will be continuous on $[0, +\infty]$) and the latter yields the maximum, and the maximum pollutant concentration is the corresponding value of $C(t)$; that is,

$$C_{\max} = \frac{A}{x}\sqrt{\frac{2}{\pi e}}.$$

C07S02.073: If $f(x) = x^n e^{-x}$ (with n a fixed positive integer), then $f'(x) = (n-x)x^{n-1}e^{-x}$. Because $f(x) \geqq 0$ for $x \geqq 0$, $f(0) = 0$, and $f(x) \to 0$ as $x \to +\infty$, $f(x)$ must have a maximum value, and the critical point where $x = n$ is the sole candidate. Hence the global maximum value of $f(x)$ is $f(n) = n^n e^{-n}$.

Next, $f(n-1) = (n-1)^n e^{-(n-1)} < n^n e^{-n}$, so

$$\left(\frac{n-1}{n}\right)^n < \frac{e^{n-1}}{e^n} = \frac{1}{e}.$$

Therefore

$$e < \left(\frac{n}{n-1}\right)^n = \left(\frac{n-1}{n}\right)^{-n} = \left(1 - \frac{1}{n}\right)^{-n}.$$

Also $f(n+1) = (n+1)^n e^{-(n+1)} < \dfrac{n^n}{e^n}$. Therefore, by similar computations,

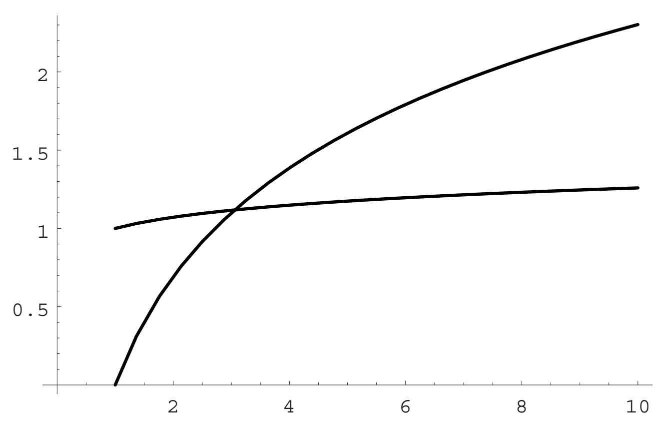
$$\left(1 + \frac{1}{n}\right)^n < e.$$

When we substitute $n = 10^6$ (using a computer algebra program, of course) we find that

$$2.7182804690 < e < 2.7182831877$$

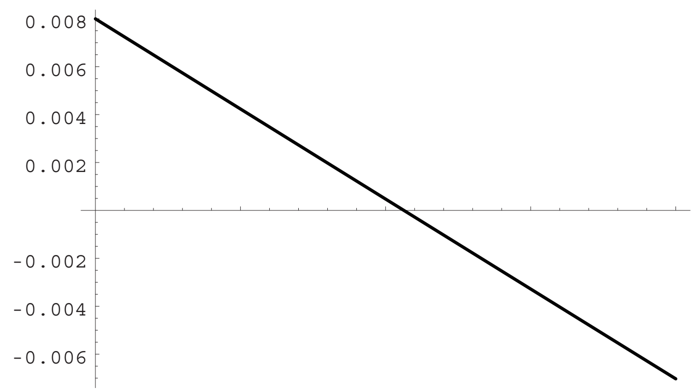
(round *down* on the left, *up* on the right). Thus, to five places, $e = 2.71828$.

C07S02.074: We used *Mathematica* 3.0 to plot the graphs of $y = \ln x$ and $y = x^{1/10}$ on the interval $[1, 10]$. The result is shown next.



The graph makes it clear that a solution of $\ln x = x^{1/10}$ is close to $x_0 = 3.1$. With this initial estimate, a few iterations of Newton's method yields the approximate solution $x_1 \approx 3.05972667962080885461$.

Next we let $f(x) = (\ln x) - x^{1/10}$ and followed the suggestion in Problem 74. The graph of f finally crossed the x-axis when viewed on the interval $[10^{15}, 10^{16}]$. A few magnifications yielded the graph shown next.



The scale on the x-axis ranges from 3.42×10^{15} to 3.44×10^{15}. Thus we have $x_2 \approx 3.43\times10^{15}$. A few iterations of Newton's method soon yielded the more accurate approximation $x_2 \approx 3.4306311214078012027 8 \times 10^{15}$.

C07S02.075: Let $f(x) = x$ and $g(x) = x^2$. The area of the region bounded by the graphs of f and g is

$$A = \int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The moments with respect to the coordinate axes are

$$M_y = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad \text{and}$$

$$M_x = \int_0^1 \frac{1}{2}(x^2 - x^4)\,dx = \frac{1}{2}\cdot\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{1}{15}.$$

Therefore the centroid is located at the point

$$(\overline{x}, \overline{y}) = C\left(\frac{1}{2}, \frac{2}{5}\right).$$

The axis L of rotation is the line $y = x$; the line through the centroid perpendicular to L has equation

$$y = \frac{9}{10} - x$$

and this perpendicular meets L at the point $P\left(\frac{9}{20}, \frac{9}{20}\right)$. The distance from P to C is

$$d = \sqrt{\left(\frac{1}{2} - \frac{9}{20}\right)^2 + \left(\frac{2}{5} - \frac{9}{20}\right)^2} = \frac{\sqrt{2}}{20}.$$

Because d is the radius of the circle through which C is rotated, the volume generated is (by the first theorem of Pappus)

$$V = 2\pi dA = 2\pi\cdot\frac{\sqrt{2}}{20}\cdot\frac{1}{6} = \frac{\pi\sqrt{2}}{60} \approx 0.07404804897.$$

C07S02.076: We let $f(x) = x^m$ and $g(x) = x^n$. Then we used *Mathematica* 3.0:

```
A = Integrate[ f[x] − g[x], { x, 0, 1 } ]
```

$$\frac{1}{1+m} - \frac{1}{1+n}$$

Then we compute the moments:

```
My = Integrate[ x∗(f[x] − g[x]), { x, 0, 1 } ];
Mx = Integrate[ (1/2)∗( (f[x])∧2 − (g[x])∧2 ), { x, 0, 1 } ];
```

Thus the centroid has coordinates

```
{ xc, yx } = { My/A, Mx/A } // Simplify
```

$$\left\{ \frac{(1+m)(1+n)}{(2+m)(2+n)}, \frac{1+m+n+mn}{1+2m+2n+4mn} \right\}$$

For selected values of m and with $n = m + 1$ we check to see if it's `True` that the centroid lies within the region:

```
m = 1; n = m + 1;
yc < xc∧m
yc > xc∧n
```

True

True

```
m = 2; n = m + 1;
yc < xc∧m
yc > xc∧n
```

True

True

```
m = 3; n = m + 1;
yc < xc∧m
yc > xc∧n
```

False

True

Therefore if $m = 3$ and $n = 4$, then the centroid does *not* lie within the region.

Section 7.3

C07S03.001: We use Theorem 1 of Section 2.3 and the quotient and product laws for limits in Section 2.2:

$$\lim_{x\to 0} x\cot x = \lim_{x\to 0} \frac{x}{\sin x}\cdot \cos x = 1\cdot 1 = 1.$$

C07S03.002: $\displaystyle \lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x\cos x}{x\sin x} = \lim_{x\to 0}\frac{x\sin x}{x\cos x+\sin x} = \lim_{x\to 0}\frac{x\cos x+\sin x}{2\cos x - x\sin x} = 0.$

C07S03.003: Both the "numerator" $\ln\dfrac{7x+8}{4x+8}$ and the denominator x approach zero as $x\to 0$, so l'Hôpital's rule may be applied. The trick is to use a law of logarithms to make the "numerator" easier to differentiate.

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x}\cdot\ln\frac{7x+8}{4x+8} &= \lim_{x\to 0}\frac{1}{x}\cdot[\ln(7x+8)-\ln(4x+8)]\\ &= \lim_{x\to 0}\frac{1}{1}\left(\frac{7}{7x+8}-\frac{4}{4x+8}\right) = \lim_{x\to 0}\frac{24}{(7x+8)(4x+8)} = \frac{24}{64} = \frac{3}{8}.\end{aligned}$$

C07S03.004: $\displaystyle \lim_{x\to 0^+}(\sin x)(\ln\sin x) = \lim_{x\to 0^+}\frac{\ln\sin x}{\csc x} = \lim_{x\to 0^+}\frac{\cos x}{-\csc x\,\cot x\,\sin x} = \lim_{x\to 0^+}(-\sin x) = 0.$

C07S03.005: $\displaystyle \lim_{x\to 0} x^2\csc^2 x = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = 1^2 = 1.$

C07S03.006: $\displaystyle \lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty}\frac{\ln x}{e^x} = \lim_{x\to\infty}\frac{1}{xe^x} = 0.$

C07S03.007: $\displaystyle \lim_{x\to\infty} x\left(e^{1/x}-1\right) = \lim_{x\to\infty}\frac{e^{1/x}-1}{x^{-1}} = \lim_{x\to\infty}\frac{-(x^{-2}e^{1/x})}{-(x^{-2})} = \lim_{x\to\infty} e^{1/x} = 1.$

C07S03.008: Combine into a single fraction, then apply l'Hôpital's rule twice:

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{\ln(x-1)}\right) &= \lim_{x\to 2}\frac{\ln(x-1)-(x-2)}{(x-2)\ln(x-1)}\\ &= \lim_{x\to 2}\frac{-\dfrac{x-2}{x-1}}{\dfrac{x-2}{x-1}+\ln(x-1)} = \lim_{x\to 2}\frac{-\dfrac{1}{(x-1)^2}}{\dfrac{x}{(x-1)^2}} = -\frac{1}{2}.\end{aligned}$$

C07S03.009: $\displaystyle \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1}} = \lim_{x\to 0^+}\frac{1}{-x\cdot x^{-2}} = \lim_{x\to 0^+}(-x) = 0.$

C07S03.010: $\displaystyle \lim_{x\to \pi/2}(\tan x)(\cos 3x) = \lim_{x\to \pi/2}\frac{\cos 3x}{\cot x} = \lim_{x\to \pi/2}\frac{-3\sin 3x}{-\csc^2 x} = \frac{3}{-1} = -3.$

C07S03.011: $\displaystyle \lim_{x\to \pi}(x-\pi)\csc x = \lim_{x\to \pi}\frac{x-\pi}{\sin x} = \lim_{x\to \pi}\frac{1}{\cos x} = -1.$

C07S03.012: $\displaystyle \lim_{x\to\infty}(x-\sin x)\exp(-x^2) = \lim_{x\to\infty}\frac{x-\sin x}{\exp(x^2)} = \lim_{x\to\infty}\frac{1-\cos x}{2x\exp(x^2)} = 0.$

The last equality results from the observation that $0 \leqq 1 - \cos x \leqq 2$ for all x, whereas $2x\exp(x^2) \to +\infty$ as $x \to +\infty$.

C07S03.013: First combine terms to form a single fraction, apply l'Hôpital's rule once, then multiply each term in numerator and denominator by $2x^{1/2}$. Result:

$$\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{(\cos x) - \frac{1}{2}x^{-1/2}}{x^{1/2}\cos x + \frac{1}{2}x^{-1/2}\sin x}$$

$$= \lim_{x\to 0^+} \frac{(2x^{1/2}\cos x) - 1}{2x\cos x + \sin x} = -\infty.$$

The last limit follows because the numerator is approaching -1 as $x \to 0^+$, while the denominator is approaching zero through positive values.

C07S03.014: First combine terms to form a single fraction, then apply l'Hôpital's rule twice:

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \lim_{x\to 0} \frac{e^x}{2e^x + xe^x} = \frac{1}{2}.$$

C07S03.015: First combine terms to form a single fraction, then (if necessary) apply l'Hôpital's rule:

$$\lim_{x\to 1^+}\left(\frac{x}{x^2 + x - 2} - \frac{1}{x - 1}\right) = \lim_{x\to 1^+}\left(\frac{x}{(x-1)(x+2)} - \frac{x+2}{(x-1)(x+2)}\right) = \lim_{x\to 1^+} \frac{2}{(1-x)(2+x)} = -\infty.$$

Note that l'Hôpital's rule is neither used nor required.

C07S03.016: First multiply numerator and denominator (the denominator is 1) by the conjugate of the numerator; l'Hôpital's rule is not required.

$$\lim_{x\to\infty}\left(\sqrt{x+1} - \sqrt{x}\,\right) = \lim_{x\to\infty} \frac{\sqrt{x+1} - \sqrt{x}}{1} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{x + 1 - x}{\sqrt{x+1} + \sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0.$$

C07S03.017: In this solution we first combine the two terms into a single fraction, apply l'Hôpital's rule a first time, make algebraic simplifications, then apply the rule a second time.

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right) = \lim_{x\to 0} \frac{\ln(1+x) - x}{x\ln(1+x)}$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{1+x} - 1}{\dfrac{x}{1+x} + \ln(1+x)} = \lim_{x\to 0} \frac{1 - (1+x)}{x + (1+x)\ln(1+x)}$$

$$= \lim_{x\to 0} \frac{-x}{x + (1+x)\ln(1+x)} = \lim_{x\to 0} \frac{-1}{1 + 1 + \ln(1+x)} = -\frac{1}{2}$$

C07S03.018: First multiply the "numerator" and the denominator (which is 1) by the conjugate of the numerator.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=\lim_{x\to\infty}\frac{\sqrt{x^2+x}-\sqrt{x^2-x}}{1}\cdot\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{1+\dfrac{1}{x}}+\sqrt{1-\dfrac{1}{x}}}=\frac{2}{1+1}=1.$$

The transition from the second line to the third did not involve l'Hôpital's rule. Instead we divided each term in numerator and denominator by x, which becomes x^2 when moved under the radical because $x>0$.

C07S03.019: The conjugate of $a^{1/3}-b^{1/3}$ is $a^{2/3}+a^{1/3}b^{1/3}+b^{2/3}$ because

$$(a^{1/3}-b^{1/3})(a^{2/3}+a^{1/3}b^{1/3}+b^{2/3})=a-b.$$

Therefore we multiply "numerator" and denominator (which is 1) by the conjugate of the numerator. The result:

$$\lim_{x\to\infty}\left[(x^3+2x+5)^{1/3}-x\right]$$

$$=\lim_{x\to\infty}\frac{\left[(x^3+2x+5)^{1/3}-x\right]\cdot\left[(x^3+2x+5)^{2/3}+x(x^3+2x+5)^{1/3}+x^2\right]}{(x^3+2x+5)^{2/3}+x(x^3+2x+5)^{1/3}+x^2}$$

$$=\lim_{x\to\infty}\frac{x^3+2x+5-x^3}{(x^3+2x+5)^{2/3}+x(x^3+2x+5)^{1/3}+x^2}$$

$$=\lim_{x\to\infty}\frac{2x+5}{(x^3+2x+5)^{2/3}+x(x^3+2x+5)^{1/3}+x^2}$$

$$=\lim_{x\to\infty}\frac{\dfrac{2}{x}+\dfrac{5}{x^2}}{\left(1+\dfrac{2}{x}+\dfrac{5}{x^2}\right)^{2/3}+\left(1+\dfrac{2}{x}+\dfrac{5}{x^2}\right)^{1/3}+1}=\frac{0}{1+1+1}=0.$$

There was no need—certainly, no temptation—to use l'Hôpital's rule.

C07S03.020: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to0^+}x^x\right)=\lim_{x\to0^+}\ln(x^x)=\lim_{x\to0^+}x\ln x$$

$$=\lim_{x\to0^+}\frac{\ln x}{\dfrac{1}{x}}=\lim_{x\to0^+}\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=\lim_{x\to0^+}(-x)=0.$$

Therefore $\displaystyle\lim_{x\to0^+}x^x=e^0=1.$

C07S03.021: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 0^+} x^{\sin x}\right) = \lim_{x\to 0^+} \ln\left(x^{\sin x}\right) = \lim_{x\to 0^+} (\sin x)(\ln x)$$
$$= \lim_{z\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1}{-x \csc x\, \cot x} = \lim_{x\to 0^+} \frac{\tan x}{-\dfrac{x}{\sin x}} = \frac{0}{-1} = 0.$$

Therefore $\displaystyle\lim_{x\to 0^+} x^{\sin x} = e^0 = 1$.

C07S03.022: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to\infty}\left[\frac{2x-1}{2x+1}\right]^x\right) = \lim_{x\to\infty}\left[x\ln\left(\frac{2x-1}{2x+1}\right)\right] = \lim_{x\to\infty}\frac{\ln(2x-1)-\ln(2x+1)}{\dfrac{1}{x}}$$
$$= \lim_{x\to\infty}\left(\frac{-2x^2}{2x-1}+\frac{2x^2}{2x+1}\right) = \lim_{x\to\infty}\frac{-4x^3-2x^2+4x^3-2x^2}{4x^2-1} = \lim_{x\to\infty}\frac{-4x^2}{4x^2-1} = -1.$$

Therefore $\displaystyle\lim_{x\to\infty}\left[\frac{2x-1}{2x+1}\right]^x = e^{-1} = \frac{1}{e}$.

C07S03.023: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to\infty}(\ln x)^{1/x}\right) = \lim_{x\to\infty}\ln(\ln x)^{1/x} = \lim_{x\to\infty}\frac{\ln(\ln x)}{x} = \lim_{x\to\infty}\frac{1}{x\ln x} = 0.$$

Therefore $\displaystyle\lim_{x\to\infty}(\ln x)^{1/x} = e^0 = 1$.

C07S03.024: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x\right) = \lim_{x\to\infty}\ln\left(1-\frac{1}{x^2}\right)^x = \lim_{x\to\infty} x\ln\left(1-\frac{1}{x^2}\right)$$
$$= \lim_{x\to\infty}\frac{\ln\left(\dfrac{x^2-1}{x^2}\right)}{\dfrac{1}{x}} = \lim_{x\to\infty}\frac{\ln(x^2-1)-2\ln x}{\dfrac{1}{x}} = \lim_{x\to\infty}\frac{\dfrac{2x}{x^2-1}-\dfrac{2}{x}}{-\dfrac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{2x}{1-x^2} = \lim_{x\to\infty}\frac{2}{-2x} = 0.$$

Therefore $\displaystyle\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)^x = e^0 = 1$.

C07S03.025: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}\right) = \lim_{x\to 0}\ln\left(\frac{\sin x}{x}\right)^{1/x^2} = \lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x}$$
$$= \lim_{x\to 0}\frac{\ln(\sin x)-\ln x}{x^2} = \lim_{x\to 0}\frac{\dfrac{\cos x}{\sin x}-\dfrac{1}{x}}{2x} = \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}$$
$$= \lim_{x\to 0}\frac{-x\sin x}{2x^2\cos x+4x\sin x} = \lim_{x\to 0}\frac{-\dfrac{\sin x}{x}}{2\cos x+\dfrac{4\sin x}{x}} = -\frac{1}{2\cdot 1+4\cdot 1} = -\frac{1}{6}.$$

Therefore $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6} \approx 0.846481724890614.$

This limit is particularly resistant to numerical approximation by a conventional (hand-held) calculator. To simulate the behavior of such a calculator, we used *Mathematica* 3.0 and defined

```
g[x_] := (N[ (Sin[x])/x, 14 ])∧(N[ 1/(x*x), 14 ])        (1)
```

thus asking the computer to carry 14 digits (and no more) in its computations of values of

$$f(x) = \left(\frac{\sin x}{x}\right)^{1/x^2} . \tag{2}$$

Here are the results.

x	$g(x)$
0.1	0.846435
0.01	0.846481
0.001	0.846482
0.0001	0.846482
0.00001	0.846482
0.000001	0.846501
0.0000001	0.837247
0.00000001	1.0
0.000000001	1.0
0.0000000001	1.0
0.00000000001	$1.848091019 \times 10^{-482164}$
0.000000000001	1.0
0.0000000000001	1.0
0.00000000000001	1.0
0.000000000000001	1.0
0.0000000000000001	1.0

Of course, *Mathematica* is capable of essentially arbitrarily great accuracy (if you have the time and the memory). When we replaced the parameter 14 in Eq. (1) with 200 and asked for results to 20 places, here's what we found.

x	$g(x)$
0.1	0.8464346695553817290
0.01	0.8464812546201339178
0.001	0.8464817201879375391
0.0001	0.8464817248435873115
0.00001	0.8464817248901438064

0.000001	0.8464817248906093714
0.0000001	0.8464817248906140270
0.00000001	0.8464817248906140736
0.000000001	0.8464817248906140740

and all such values of x through 10^{-20} gave the same answer as that in the last row, which agrees with $e^{-1/6}$ to the number of digits shown. We also tried direct evaluation of $f(1/10^n)$, but exhausted time and memory when $n = 4$. Results with *Derive* 2.56 with precision set to `Exact: 60 Digits` were similar to those in the second table here. The typical hand-held calculator we tried (TRS-80 Pocket Computer PC-2, Hewlett-Packard HP-29C, Casio fx-300v, and Texas Instruments TI-35 plus) generally gave results similar to those in the first table, except jumping to the value 1 somewhere between $x = 10^{-4}$ and $x = 10^{-6}$. Although I'm no expert at computer algebra programs, I seemed to get the best results with *Maple* V Version 5.1: With $f(x)$ defined as in Eq. (1), the command

```
evalf(f(1.0*10∧(−25)),100);
```

agreed with the exact value of the limit with 50 digits correct to the right of the decimal point, and this program's approximations began to break down—and only slightly—at $x = 10^{-35}$. If this discussion provokes any student's interest in numerical mathematics and its computer implementation, it has been worth the space.

C07S03.026: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 0^+}(1+2x)^{1/(3x)}\right) = \lim_{x\to 0^+}\ln(1+2x)^{1/(3x)} = \lim_{x\to 0^+}\frac{\ln(1+2x)}{3x} = \lim_{x\to 0^+}\frac{\dfrac{2}{1+2x}}{3} = \frac{2}{3}.$$

Therefore $\displaystyle\lim_{x\to 0^+}(1+2x)^{1/(3x)} = e^{2/3} \approx 1.9477340411.$

C07S03.027: Apply the natural logarithm function:

$$\begin{aligned}\ln\left(\lim_{x\to\infty}\left(\cos\frac{1}{x^2}\right)^{(x^4)}\right) &= \lim_{x\to\infty}\ln\left(\cos\frac{1}{x^2}\right)^{(x^4)} = \lim_{x\to\infty}x^4\ln\left(\cos\frac{1}{x^2}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(\cos\dfrac{1}{x^2}\right)}{x^{-4}} = \lim_{x\to\infty}\frac{2\tan\dfrac{1}{x^2}}{-4x^{-5}\cdot x^3} = \lim_{x\to\infty}\frac{\tan\dfrac{1}{x^2}}{-2x^{-2}}\\ &= \lim_{x\to\infty}\frac{-\dfrac{2}{x^3}\sec^2\dfrac{1}{x^2}}{4x^{-3}} = \lim_{x\to\infty}\frac{-\sec^2\dfrac{1}{x^2}}{2} = -\frac{1}{2}.\end{aligned}$$

Therefore $\displaystyle\lim_{x\to\infty}\left(\cos\frac{1}{x^2}\right)^{(x^4)} = e^{-1/2} \approx 0.6065306597.$

C07S03.028: As $x \to 0^+$, $\sin x \to 0$ through positive values and $\sec x \to 1$. So this is not an indeterminate form, and $\displaystyle\lim_{x\to 0^+}(\sin x)^{\sec x} = 0.$

C07S03.029: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 0^+}(x+\sin x)^x\right)=\lim_{x\to 0^+}\ln(x+\sin x)^x=\lim_{x\to 0^+}x\ln(x+\sin x)$$

$$=\lim_{x\to 0^+}\frac{\ln(x+\sin x)}{x^{-1}}=\lim_{x\to 0^+}\frac{\dfrac{1+\cos x}{x+\sin x}}{-(x^{-2})}=\lim_{x\to 0^+}\frac{-x^2(1+\cos x)}{x+\sin x}$$

$$=\lim_{x\to 0^+}\frac{x^2\sin x-2x(1+\cos x)}{1+\cos x}=\frac{0\cdot 0-2\cdot 0\cdot 2}{1+1}=0.$$

Therefore $\displaystyle\lim_{x\to 0^+}(x+\sin x)^x=e^0=1.$

C07S03.030: $\displaystyle\lim_{x\to\pi/2}(\tan x-\sec x)=\lim_{x\to\pi/2}\frac{(\sin x)-1}{\cos x}=\lim_{x\to\pi/2}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0.$

C07S03.031: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 1}x^{1/(1-x)}\right)=\lim_{x\to 1}\ln x^{1/(1-x)}=\lim_{x\to 1}\frac{\ln x}{1-x}=\lim_{x\to 1}\frac{1}{-x}=-1.$$

Therefore $\displaystyle\lim_{x\to 1}x^{1/(1-x)}=e^{-1}\approx 0.3678795512.$

C07S03.032: Apply the natural logarithm function:

$$\ln\left(\lim_{x\to 1^+}(x-1)^{\ln x}\right)=\lim_{x\to 1^+}\ln(x-1)^{\ln x}=\lim_{x\to 1^+}(\ln x)\ln(x-1)$$

$$=\lim_{x\to 1^+}\frac{\ln(x-1)}{(\ln x)^{-1}}=\lim_{x\to 1^+}\frac{\dfrac{1}{x-1}}{-\dfrac{1}{x}(\ln x)^{-2}}=\lim_{x\to 1^+}\frac{x(\ln x)^2}{1-x}$$

$$=\lim_{x\to 1^+}\frac{(\ln x)^2+2\ln x}{-1}=\frac{0^2+2\cdot 0}{-1}=0.$$

Therefore $\displaystyle\lim_{x\to 1^+}(x-1)^{\ln x}=e^0=1.$

C07S03.033: First combine the two terms to form a single fraction, then apply l'Hôpital's rule, and finally simplify:

$$\lim_{x\to 2^+}\left(\frac{1}{(x^2-4)^{1/2}}-\frac{1}{x-2}\right)=\lim_{x\to 2^+}\frac{x-2-(x^2-4)^{1/2}}{(x^2-4)^{1/2}(x-2)}=\lim_{x\to 2^+}\frac{1-x(x^2-4)^{-1/2}}{x(x^2-4)^{-1/2}(x-2)+(x^2-4)^{1/2}}$$

$$=\lim_{x\to 2^+}\frac{(x^2-4)^{1/2}-x}{x(x-2)+(x^2-4)}=-\infty$$

because, in the last limit, the numerator is approaching -2 while the denominator is approaching zero through *positive* values.

C07S03.034: Let $Q=x^5-3x^4+17$. We plan to multiply by the conjugate of $Q^{1/5}-x$.

$$\lim_{x\to\infty} \left[(x^5 - 4x^4 + 17)^{1/5} - x \right] = \lim_{x\to\infty} (Q^{1/5} - x)$$

$$= \lim_{x\to\infty} \frac{Q - x^5}{Q^{4/5} + Q^{3/5}x + Q^{2/5}x^2 + Q^{1/5}x^3 + x^4}$$

$$= \lim_{x\to\infty} \frac{17 - 3x^4}{Q^{4/5} + Q^{3/5}x + Q^{2/5}x^2 + Q^{1/5}x^3 + x^4}.$$

Now *carefully* divide each term in numerator and denominator by x^4. The numerator becomes

$$\frac{17}{x^4} - 3,$$

which approaches -3 as $x \to +\infty$. The first term in the denominator becomes

$$\frac{(x^5 - 3x^4 + 17)^{4/5}}{x^4} = \left(\frac{x^5 - 3x^4 + 17}{x^5} \right)^{4/5} = \left(1 - \frac{3}{x} + \frac{17}{x^5} \right)^{4/5},$$

which approaches 1 as $x \to +\infty$. The second term in the denominator becomes

$$\frac{x \cdot (x^5 - 3x^4 + 17)^{3/5}}{x^4} = \left(\frac{x^5 - 3x^4 + 17}{x^5} \right)^{3/5} = \left(1 - \frac{3}{x} + \frac{17}{x^5} \right)^{3/5},$$

which also approaches 1 as $x \to +\infty$, as do the third, fourth, and fifth terms in the demoninator. Therefore

$$\lim_{x\to\infty} \left[(x^5 - 4x^4 + 17)^{1/5} - x \right] = -\frac{3}{5}.$$

C07S03.035: Given: $f(x) = x^{1/x}$ for $x > 0$. We plotted the graph of $y = f(x)$ on the interval $10^{-6} \leqq x \leqq 1$ and obtained strong evidence that $f(x)$ approaches zero as $x \to 0^+$. We also plotted $y = f(x)$ on the interval $100 \leqq x \leqq 1000$ and obtained some evidence that $f(x) \to 1$ as $x \to +\infty$. Then we verified these limits with l'Hôpital's rule as follows:

$$\ln \left(\lim_{x\to\infty} x^{1/x} \right) = \lim_{x\to\infty} \frac{1}{x} \ln x = \lim_{x\to\infty} \frac{1}{x} = 0,$$

so that $\lim_{x\to\infty} x^{1/x} = e^0 = 1$.

But $\lim_{x\to 0^+} x^{1/x}$ is not indeterminate, because the exponent is approaching $+\infty$; this limit is clearly zero.

The graph that follows this solution indicates that the global maximum value of $f(x)$ occurs close to 2.71828 (surely no coincidence). We found that

$$f'(x) = \frac{x^{1/x}(1 - \ln x)}{x^2},$$

and it follows that the maximum value of $f(x)$ is $f(e) = e^{1/e} \approx 1.4446678610$.

C07S03.036: Given: $f(x) = x^{1/(x^2)}$ for $x > 0$. We plotted $f(x)$ on the interval $0.01 \leqq x \leqq 1$ and obtained strong evidence that $f(x) \to 0$ as $x \to 0^+$. We plotted $f(x)$ for $10 \leqq x \leqq 100$ and obtained some evidence that $f(x) \to 1$ as $x \to +\infty$. The first conjecture is correct because, as $x \to 0^+$, the exponent in $f(x)$ is approaching $+\infty$ and therefore this limit is not an indeterminate form; it is clearly zero. Then

$$\ln\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1}{2x \cdot x} = 0,$$

and therefore $\lim_{x\to\infty} f(x) = e^0 = 1$.

Next we plotted $y = f(x)$ for $1 \leqq x \leqq 4$ and saw a clear maximum near where $x = 1.65$. We found that

$$f'(x) = \frac{(1 - 2\ln x)x^{1/(x^2)}}{x^3},$$

and it follows that the maximum is $f\left(\sqrt{e}\,\right) = e^{1/(2e)} \approx 1.2019433685$.

C07S03.037: Given: $f(x) = (x^2)^{1/x}$ for $x > 0$. We plotted $y = f(x)$ for $0.01 \leqq x \leqq 1$ and obtained strong evidence that $f(x) \to 0$ as $x \to 0^+$. We plotted $f(x)$ for $100 \leqq x \leqq 1000$ and obtained weak evidence that $f(x) \to 1$ as $x \to +\infty$. (These two graphs follow this solution.) The first limit is clear because x^2 is approaching zero while the exponent $1/x$ is approaching $+\infty$. For the second limit, we found

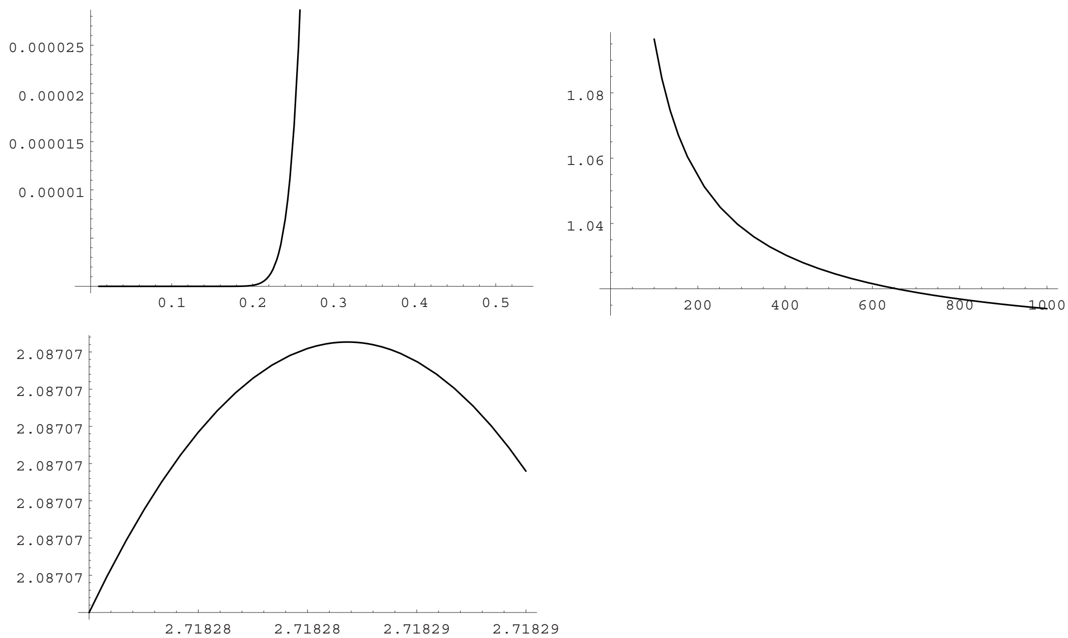
$$\ln\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0,$$

so that $\lim_{x\to\infty} f(x) = e^0 = 1$.

Next we plotted $f(x)$ for $2.71827 \leqq x \leqq 2.71829$ (by the "method of successive zooms") and saw a clear maximum near where $x = 2.71828$. (The graph follows this solution.) We found that

$$f'(x) = \frac{(2 - 2\ln x)(x^2)^{1/x}}{x^2},$$

and it follows that the maximum value of $f(x)$ is $f(e) = e^{2/e} \approx 2.0870652286$.



C07S03.038: Given: $f(x) = x^{-x}$, $x > 0$. We plotted $y = f(x)$ for $0.001 \leqq x \leqq 1$; the graph is strong evidence that $f(x) \to 1$ as $x \to 0^+$. Then we plotted $y = f(x)$ for $10 \leqq x \leqq 20$; the graph is very strong evidence that $f(x) \to 0$ as $x \to +\infty$. To be sure, we computed

$$\ln\left(\lim_{x\to 0^+} f(x)\right) = \lim_{x\to 0^+}(-x\ln x) = \lim_{x\to 0^+} \frac{-\ln x}{x^{-1}} = \lim_{x\to 0^+} x = 0.$$

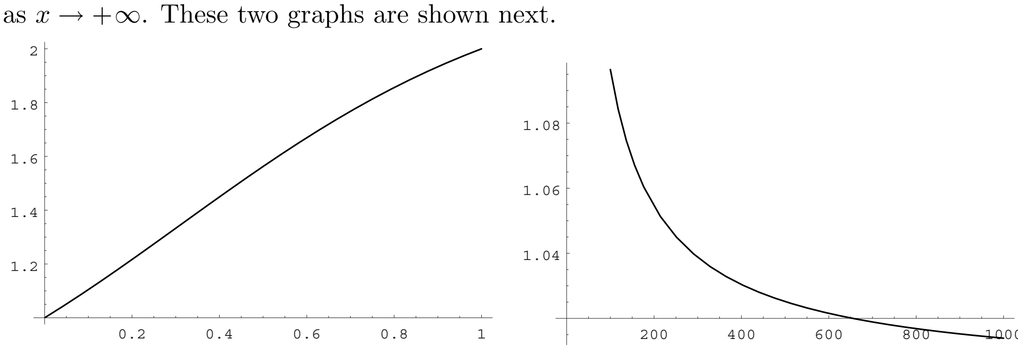
It is clear that $\displaystyle\lim_{x\to\infty} \frac{1}{x^x} = 0$; we are not dealing with an indeterminate form here.

The first graph showed a global maximum near the point where $x = 0.4$. We found that

$$f'(x) = -\frac{1+\ln x}{x^x},$$

and therefore the global maximum value of $f(x)$ is $f(e^{-1}) = e^{1/e} \approx 1.44466678610$.

C07S03.039: We graphed $f(x) = (1 + x^2)^{1/x}$ for $0.001 \leqq x \leqq 1$; the graph is extremely strong evidence that $f(x) \to 1$ as $x \to 0^+$. Then we graph $y = f(x)$ for $100 \leqq x \leqq 1000$; the graph is weak evidence that $f(x) \to 1$ as $x \to +\infty$. These two graphs are shown next.



To be sure about these limits, we computed

$$\ln\left(\lim_{x\to 0^+} f(x)\right) = \lim_{x\to 0^+} \frac{\ln(1+x^2)}{x} = \lim_{x\to 0^+} \frac{2x}{1+x^2} = 0,$$

and therefore $f(x) \to e^0 = 1$ as $x \to 0^+$. Next,

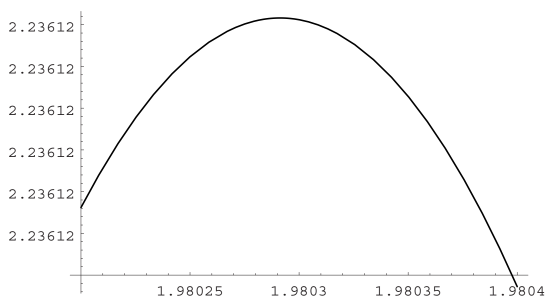
$$\ln\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty} \frac{\ln(1+x^2)}{x} = \lim_{x\to\infty} \frac{2x}{1+x^2} = 0,$$

and therefore $f(x) \to e^0 = 1$ as $x \to +\infty$.

Then we used the "method of successive zooms" and thereby found that the graph of $y = f(x)$ for $1.9802 \leqq x \leqq 1.9804$ shows a maximum near where $x = 1.9803$. (The graph follows this solution.) Then we found that

$$f'(x) = \frac{(1+x^2)^{1/x}\left[2x^2 - (1+x^2)\ln(1+x^2)\right]}{x^2(1+x^2)}$$

but could not solve the transcendental equation $2x^2 = (1+x^2)\ln(1+x^2)$ exactly. So we used Newton's method to solve $f'(x) = 0$, and our conclusion is that the global maximum value of $f(x)$ is approximately $2.2361202715 \approx f(1.9802913004)$.



C07S03.040: Given:

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \qquad \text{for} \qquad x > 0.$$

We plotted $y = f(x)$ for $0.001 \leqq x \leqq 0.2$; the graph is very strong evidence that $f(x) \to 1$ as $x \to 0^+$. Then we plotted $y = f(x)$ for $10 \leqq x \leqq 1000$; the graph is good evidence that $f(x) \to 1$ as $x \to +\infty$. Both these limits are indeterminate, so we used l'Hôpital's rule:

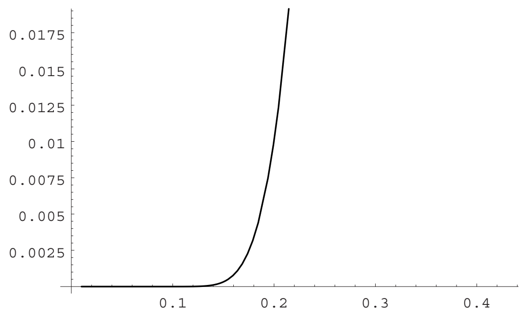
$$\begin{aligned}\ln\left(\lim_{x\to 0^+}\left(1+\frac{1}{x^2}\right)^x\right) &= \lim_{x\to 0^+} x\ln\left(\frac{x^2+1}{x^2}\right) = \lim_{x\to 0^+} \frac{\ln(x^2+1) - 2\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{\dfrac{2x}{x^2+1} - \dfrac{2}{x}}{-(x^{-2})} \\ &= \lim_{x\to 0^+} x^2\cdot\left(\frac{2}{x} - \frac{2x}{x^2+1}\right) = \lim_{x\to 0^+} \frac{x^2(2x^2+2-2x^2)}{x(x^2+1)} = \lim_{x\to 0^+} \frac{2x}{x^2+1} = 0.\end{aligned}$$

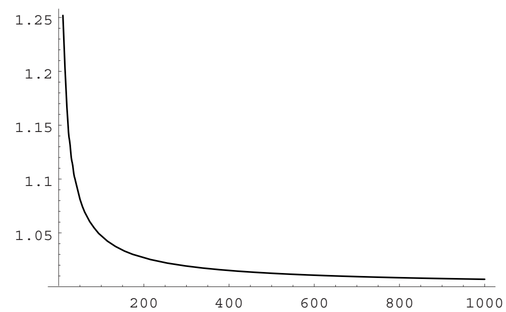
Repeat these computations with $x \to 0^+$ replaced with $x \to +\infty$ to discover the same limit, zero. Thus $f(x) \to e^0 = 1$ as $x \to 0^+$ and as $x \to +\infty$.

Next we used the method of successive zooms to find that $f(x)$ has a global maximum just a little to the right of the point where $x = 0.5$. Solving $f'(x) = 0$ exactly seemed hopeless; we used Newton's method to solve $f'(x) = 0$ to find that the global maximum value of $f(x)$ is approximately

$$f(0.5049762122) \approx 2.2361202715.$$

C07S03.041: Given: $f(x) = (x + \sin x)^{1/x}$. The graph of $y = f(x)$ for $0.01 \leqq x \leqq 1$ provides strong evidence that $f(x) \to 0$ as $x \to 0^+$. The graph of $y = f(x)$ for $10 \leqq x \leqq 1000$ provides fairly good evidence that $f(x) \to 1$ as $x \to +\infty$. These graphs are shown next.





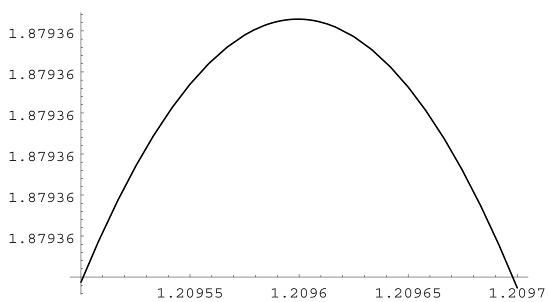
Next, we verified these limits as follows:

$$\ln\left(\lim_{x\to\infty} (x + \sin x)^{1/x}\right) = \lim_{x\to\infty} \frac{\ln(x + \sin x)}{x} = \lim_{x\to\infty} \frac{1 + \cos x}{x + \sin x} = 0,$$

and therefore $\lim\limits_{x\to\infty} (x + \sin x)^{1/x} = e^0 = 1$.

But $(x + \sin x)^{1/x}$ is not indeterminate as $x \to 0^+$ because if x is very small and positive, then $x + \sin x$ is positive and near zero while $1/x$ is very large positive. Therefore $\lim\limits_{x\to 0^+} (x + \sin x)^{1/x} = 0$.

Then a plot of $y = f(x)$ for $0.5 \leqq x \leqq 2$ revealed a global maximum near where $x = 1.2$. A plot of f for $1.2095 \leqq x \leqq 1.2097$ (by the "method of repeated zooms") showed the maximum near the midpoint of that interval. That graph is shown next. The equation $f'(x) = 0$ appeared to be impossible to solve exactly, so we used Newton's method to find that the maximum of $f(x)$ is very close to $(1.2095994645, 1.8793598343)$.



C07S03.042: Given: $f(x) = \left[\exp(1/x^2)\right]^{(\cos x - 1)}$. A plot of $y = f(x)$ for $0.01 \leqq x \leqq 1$ indicated that as $x \to 0^+$, $f(x)$ approaches a number between 0.60 and 0.61. A plot of $y = f(x)$ for $10 \leqq x \leqq 1000$ indicated oscillations between 0.9999 and 1.0, but dying out very rapidly while maintaining an upper bound of 1.0. Thus there is evidence that $f(x) \to 1$ as $x \to +\infty$. Analytically, we compute these limits as follows:

$$\begin{aligned}\ln\left(\lim_{x\to 0^+} \left[\exp(1/x^2)\right]^{(\cos x - 1)}\right) &= \lim_{x\to 0^+} (\cos x - 1)\ln\left(\exp(1/x^2)\right) \\ &= \lim_{x\to 0^+} \frac{\cos x - 1}{x^2} = \lim_{x\to 0^+} \frac{-\sin x}{2x} = -\frac{1}{2}.\end{aligned}$$

Therefore $f(x) \to e^{-1/2} \approx 0.6065306597$ as $x \to 0^+$. As $x \to +\infty$, $\cos x - 1$ varies between 0 and -2 while $\exp(1/x^2) \to 1$ from above (and quite rapidly). Hence $f(x) \to 1$ as $x \to +\infty$; note that $f(x) \leqq 1$ for all $x > 0$.

Moreover, when $\cos x - 1 = 0$ (which happens at each integral multiple of 2π), $f(x) = 1$, and this is the maximum value of $f(x)$. Examination of

$$f'(x) = -\frac{2f(x)(\cos x - 1)}{x^3} - \frac{f(x)\sin x}{x^2}$$

makes it clear that $f'(2n\pi) = 0$ for every positive integer n, although this does not establish that there are no other locations of maxima. What is clear is that 1 is the maximum value of $f(x)$ for $x > 0$.

C07S03.043: Note that in using l'Hôpital's rule we are computing derivatives with respect to h.

$$\ln\left(\lim_{h\to 0}(1+hx)^{1/h}\right) = \lim_{h\to 0}\frac{\ln(1+hx)}{h} = \lim_{h\to 0}\frac{x}{1+hx} = x,$$

and therefore $\lim_{h\to 0}(1+hx)^{1/h} = e^x$.

C07S03.044: The implication (through the notation) is that $n \to +\infty$ while assuming only positive *integral* values. We need to differentiate with respect to n, thus we let n run through positive *real* values. Then if n is later restricted to positive integral values, the limit will be the same.

$$\begin{aligned}
\ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) &= \lim_{n\to\infty}\frac{\ln\left(1+\dfrac{x}{n}\right)}{\dfrac{1}{n}} = \lim_{n\to\infty}\frac{\ln\dfrac{n+x}{n}}{\dfrac{1}{n}}\\
&= \lim_{n\to\infty}\frac{\ln(n+x)-\ln n}{\dfrac{1}{n}} = \lim_{n\to\infty}\frac{\dfrac{1}{n+x}-\dfrac{1}{n}}{-\dfrac{1}{n^2}}\\
&= \lim_{n\to\infty}\frac{nx}{n+x} = \lim_{n\to\infty}\frac{x}{1+\dfrac{x}{n}} = x.
\end{aligned}$$

Therefore $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = e^x$.

C07S03.045: We let $f(x) = x^{\tan x}$ and applied Newton's method to the equation $f'(x) = 0$ with initial guess $x_0 = 0.45$. Results: $x_1 = 0.4088273642$, $x_2 = 0.4099763617$, $x_3 = x_4 = 0.4099776300$; $f(x_4) = 0.6787405265$.

C07S03.046: Suppose that $n \geqq 2$. Let $Q = [p(x)]^{1/n}$. Then

$$\begin{aligned}
[p(x)]^{1/n} - x &= Q - x\\
&= \frac{Q^n - x^n}{Q^{n-1} + Q^{n-2}x + Q^{n-3}x^2 + \cdots + Qx^{n-2} + x^{n-1}}\\
&= \frac{a_1 + \dfrac{a_2}{x} + \dfrac{a_3}{x^2}\cdots + \dfrac{a_n}{x^{n-1}}}{\dfrac{Q^{n-1}}{x^{n-1}} + \dfrac{Q^{n-2}}{x^{n-2}} + \cdots + 1}.
\end{aligned}$$

Note that there are n terms in the last denominator and, apart from the last, each has the form

$$\frac{Q^{n-k}}{x^{n-k}} = \frac{[p(x)]^{(n-k)/n}}{x^{n-k}} = \left[\frac{p(x)}{x^n}\right]^{(n-k)/n}$$

where k is an integer and $1 \leqq k \leqq n-1$. It now follows that, for each such k,

$$\frac{Q^{n-k}}{x^{n-k}} = \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right)^{(n-k)/n} \to 1$$

as $x \to +\infty$. Therefore

$$\lim_{x\to\infty} \left([p(x)]^{1/n} - x\right) = \lim_{x\to\infty} \frac{a_1 + \dfrac{a_2}{x} + \dfrac{a_3}{x^2} + \cdots + \dfrac{a_n}{x^{n-1}}}{n} = \frac{a_1}{n}.$$

If $n = 1$, then

$$\lim_{x\to\infty} \left([p(x)]^{1/n} - x\right) = \lim_{x\to\infty} (x + a_1 - x) = a_1 = \frac{a_1}{1} = \frac{a_1}{n}.$$

This concludes the proof.

C07S03.047: Replace b with x to remind us that it's the only variable in this problem; note also that $0 < x < a$. The surface area of the ellipsoid is then

$$\begin{aligned} A(x) &= 2\pi a x \left[\frac{x}{a} + \frac{a}{(a^2 - x^2)^{1/2}} \arcsin \frac{(a^2 - a^2)^{1/2}}{a}\right] \\ &= 2\pi x^2 + 2\pi a^2 \cdot \frac{x \arcsin \dfrac{(a^2 - x^2)^{1/2}}{a}}{(a^2 - x^2)^{1/2}}. \end{aligned}$$

Therefore

$$\lim_{x\to a^-} A(x) = 2\pi a^2 + 2\pi a^2 \left(\lim_{x\to a^-} \frac{x}{a} \cdot \frac{\arcsin \dfrac{(a^2 - x^2)^{1/2}}{a}}{\dfrac{(a^2 - x^2)^{1/2}}{a}}\right).$$

Let $u = \dfrac{(a^2 - x^2)^{1/2}}{a}$. Then $u \to 0^+$ as $x \to a^-$. Hence

$$\begin{aligned} \lim_{x\to a^-} A(x) &= 2\pi a^2 + 2\pi a^2 \left(\lim_{x\to a^-} \frac{x}{a}\right) \cdot \left(\lim_{u\to 0^+} \frac{\arcsin u}{u}\right) \\ &= 2\pi a^2 + 2\pi a^2 \left(\lim_{u\to 0^+} \frac{1}{\sqrt{1-u^2}}\right) = 4\pi a^2. \end{aligned}$$

C07S03.048: Part (a): We plan to show that $dA/dn > 0$, so it will follow that A is an increasing function of n. We assume throughout that $A_0 > 0$, that $0 < r \leqq 1$, and that $n \geqq 1$. For the purpose of computing dA/dn, we let n take on all real values in its range, not merely positive integral values. Now

$$\ln A = \ln A_0 + nt \ln\left(1 + \frac{r}{n}\right),$$

so

$$\frac{1}{A} \cdot \frac{dA}{dn} = t\ln\left(\frac{n+r}{n}\right) + nt\left(\frac{1}{n+r} - \frac{1}{n}\right) = t\ln\left(\frac{n+r}{n}\right) - \frac{rt}{n+r}.$$

Because A and t are positive, it suffices to show that the function defined by

$$f(n) = \frac{1}{n+r}\left[-r + (n+r)\ln\left(\frac{n+r}{n}\right)\right] \tag{1}$$

is positive-valued for $r > 0$ and $n \geqq 1$. We substitute $n = rx$ to simplify f; the right-hand side in Eq. (1) takes the form

$$g(x) = \frac{1}{r+rx}\left[-r + (r+rx)\ln\left(\frac{r+rx}{rx}\right)\right] = -\frac{1}{1+x} + \ln\left(1+\frac{1}{x}\right).$$

It remains to show that $g(x)$ is positive-valued. If n is a positive integer and $r \leqq 1$, then $x > 1$, so $g(x)$ "starts" with the positive value $g(1) \approx 0.193147$. Moreover, $g(x)$ thenceforth decreases because its derivative,

$$g'(x) = -\frac{1}{x(1+x)^2},$$

is negative for all $x > 0$. Finally, it is obvious from the definition of g that $g(x) \to 0$ as $x \to +\infty$. Therefore $g(x)$ remains positive, because if it once took on the value zero, it would thereafter attain a negative value z, and then (continuing to decrease) would remain forever less than z—in which case it could not approach zero as $x \to +\infty$.

Part (b): L'Hôpital's rule yields

$$\begin{aligned}
\ln\left(\lim_{n\to\infty}\left[1+\frac{r}{n}\right]^{nt}\right) &= \lim_{n\to\infty}\left(nt\ln\frac{n+r}{n}\right) = \lim_{n\to\infty} t\cdot\frac{\ln(n+r)-\ln n}{\frac{1}{n}} \\
&= \lim_{n\to\infty}\left(\frac{-tn^2}{n+r} + \frac{tn^2}{n}\right) = \lim_{n\to\infty}\frac{-tn^3+tn^3+tn^2r}{n(n+r)} \\
&= \lim_{n\to\infty}\frac{tnr}{n+r} = \lim_{n\to\infty}\frac{rt}{1+\frac{r}{n}} = rt.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt},$$

and the result in Part (b) follows immediately.

C07S03.049: Given: $f(x) = |\ln x|^{1/x}$ for $x > 0$. The graph of $y = f(x)$ for $0.2 \leqq x \leqq 0.3$ shows $f(x)$ taking on values in excess of 10^{28}, so it seems quite likely that $f(x) \to +\infty$ as $x \to 0^+$. Indeed, this is the case, because as $x \to 0^+$, we see that $|\ln x| \to +\infty$ and also the exponent $1/x$ is increasing without bound.

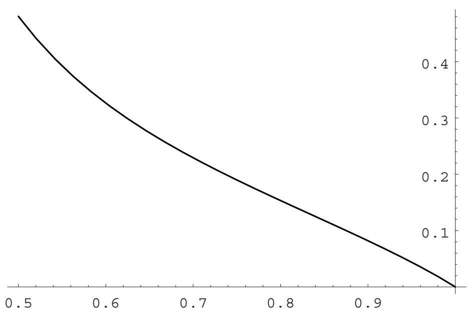
Next we show the graph of $y = f(x)$ for $0.5 \leqq x \leqq 1$ (on the left) and for $0.9 \leqq x \leqq 1.1$ (on the right). The first indicates an inflection point near where $x = 0.8$ and the second shows a clear global minimum at $(1, 0)$. To find the inflection point, we redefined $f(x) = (-\ln x)^{1/x}$ and computed

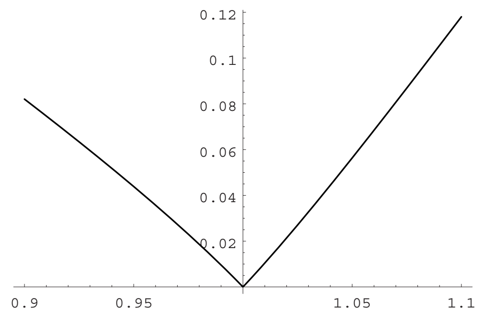
$$f''(x) = \frac{f(x)}{x^4(\ln x)^2}\left(1 - x - 3x\ln x - 2(\ln x)\ln(-\ln x) + 2x(\ln x)^2\ln(-\ln x) + (\ln x)^2(\ln(-\ln x))^2\right)$$

(assisted by *Mathematica,* of course). We let

$$g(x) = 1 - x - 3x\ln x - 2(\ln x)\ln(-\ln x) + 2x(\ln x)^2\ln(-\ln x) + (\ln x)^2(\ln(-\ln x))^2$$

and applied Newton's method to solve the equation $g(x) = 0$. With initial guess $x_0 = 0.8$, six iterations yielded over 20 digits of accuracy, and the inflection point shown in the figure is located close to $(0.8358706352, 0.1279267691)$.





It is clear that $f(x) > 0$ if $0 < x < 1$ and if $x > 1$, so the graph of f has a global minimum at $(1, 0)$. Still using $f(x) = (-\ln x)^{1/x}$, we computed

$$f'(x) = \frac{f(x)(1 - (\ln x)(\ln(-\ln x)))}{x^2\ln x},$$

and thereby found that

$$\lim_{x\to 1^-} f'(x) = -1.$$

Then we redefined $f(x) = (\ln x)^{1/x}$ and computed

$$f'(x) = \frac{f(x)}{x^2\ln x}\left(1 - (\ln x)\ln(\ln x)\right).$$

Then we found that

$$\lim_{x\to 1^+} f'(x) = 1.$$

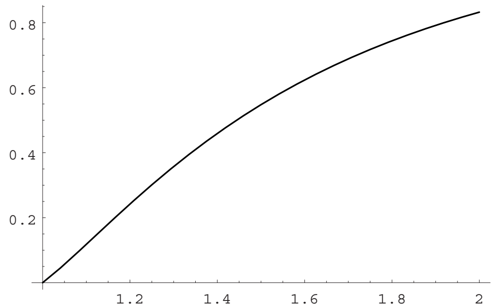
Thus the shape of the graph of f near $(1, 0)$ is quite similar to the shape of $y = |x|$ near $(0, 0)$.

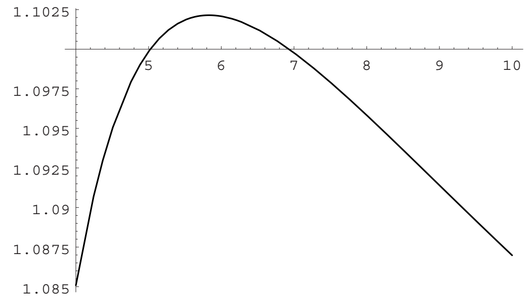
Next we plotted the graph of $y = f(x)$ for $1 \leqq x \leqq 2$ (shown next, on the left) and for $4 \leqq x \leqq 10$ (next, on the right). The first of these indicates an inflection point near where $x = 1.2$ and the second shows a clear local maximum near where $x = 5.8$. To locate the inflection point more accurately, we computed

$$f''(x) = \frac{f(x)}{x^4(\ln x)^2}\left(1 - x - 3x\ln x - 2(\ln x)\ln(\ln x) + 2x(\ln x)^2\ln(\ln x) + (\ln x)^2(\ln(\ln x))^2\right)$$

and applied Newton's method to the solution of $g(x) = 0$, where

$$g(x) = 1 - x - 3x\ln x - 2(\ln x)\ln(\ln x) + 2x(\ln x)^2\ln(\ln x) + (\ln x)^2(\ln(\ln x))^2.$$





Beginning with the initial guess $x_0 = 1.2$, seven iterations yielded more than 20 digits of accuracy; the inflection point is located close to (1.1163905964, 0.1385765415).

To find the local maximum, we applied Newton's method to the equation $g(x) = 0$, where

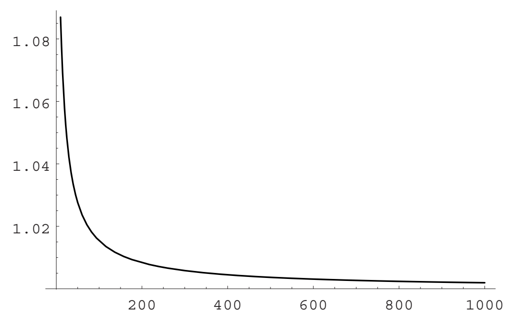
$$g(x) = 1 - (\ln x)\ln(\ln x).$$

Beginning with the initial guess $x_0 = 5.8$, six iterations yielded more than 20 digits of accuracy; the local maximum is very close to (5.8312001357, 1.1021470392).

Finally, we plotted $y = f(x)$ for $10 \leqq x \leqq 1000$ (shown after this solution). The change in concavity indicates that there must be yet another inflection point near where $x = 9$. Newton's method again yielded its approximate coordinates as (8.9280076968, 1.0917274397). The last graph also suggests that $f(x) \to 1$ as $x \to +\infty$. This is indeed the case;

$$\ln\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{1}{x \ln x} = 0,$$

and therefore $f(x) \to e^0 = 1$ as $x \to +\infty$.



C07S03.050: We applied the techniques of the previous solution to $f(x) = |\ln x|^{1/|\ln x|}$, $x > 0$. The graph of $y = f(x)$ for $0.0001 \leqq x \leqq 0.4$ shows a maximum near where $x = 0.06$. We redefined $f(x) = (-\ln x)^{-1/\ln x}$ and found that

$$f'(x) = \frac{f(x)(-1 + \ln(-\ln x))}{x(\ln x)^2},$$

and it is easy to solve the equation $f'(x) = 0$ for $x = e^{-e}$. Hence there is a local maximum at $(e^{-e}, e^{1/e}) \approx$ (0.0659880358, 1.4446678610). The limit of $f(x)$ as $x \to 0^+$ is not clear from the graph, but

$$\ln\left(\lim_{x\to 0^+} f(x)\right) = \lim_{x\to 0^+} \frac{-x}{x \ln x} = 0,$$

and therefore $f(x) \to e^0 = 1$ as $x \to 0^+$. The graph of $y = f(x)$ also indicates an inflection point near where $x = 0.5$, and application of Newton's method to the equation $f''(x) = 0$ with initial guess $x_0 = 0.5$ reveals that its coordinates are very close to (0.5070215891, 0.5657817947).

The graph of $y = f(x)$ for $0.5 \leqq x \leqq 1.5$ suggests a horizontal tangent and global minimum at (1, 0), but $f(1)$ is not defined. Nevertheless, as $x \to 1$, the base $|\ln x|$ approaches zero through positive values while the exponent $1/|\ln x|$ approaches $+\infty$, so $f(x) \to 0$ as $x \to 1$. We collected graphical and numerical evidence that $f'(x) \to 0$ as $x \to 1$ but have no formal proof.

The graph of $y = f(x)$ for $1.2 \leqq x \leqq 2$ indicates an inflection point near where $x = 1.7$. We rewrote f in the form $f(x) = (\ln x)^{1/\ln x}$ and applied Newton's method to the equation $f''(x) = 0$ to find that the inflection point is close to (1.6903045007, 0.2929095074). The graph also indicates a local maximum near where $x = 15$. Because

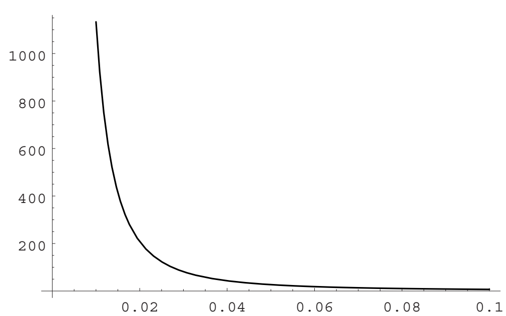
$$f'(x) = \frac{f(x)(1 - \ln(\ln x))}{x(\ln x)^2},$$

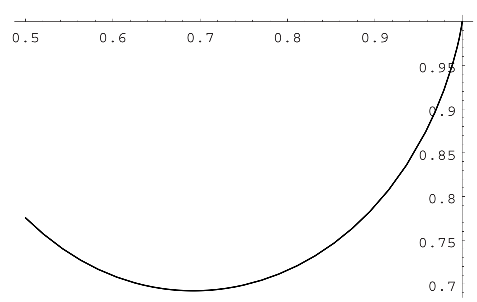
it is easy to solve $f'(x) = 0$ for $x = e^e$, so that extremum is located at $(e^e,\ e^{1/e}) \approx (15.154262,\ 1.444668)$. Hence both local maxima are global maxima. The graph also shows an inflection point near where $x = 26$, and Newton's method reveals its approximate coordinates to be (26.5384454497, 1.4364458579). The graph of $y = f(x)$ for $400 \leqq x \leqq 4000$ did not indicate any particular limit as $x \to +\infty$, but

$$\ln\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty} \frac{-x}{x \ln x} = 0,$$

and therefore $f(x) \to 1$ as $x \to +\infty$.

C07S03.051: Given: $f(x) = |\ln x|^{|\ln x|}$ for $x > 0$. The graph of $y = f(x)$ for $0.01 \leqq x \leqq 0.1$ indicates that $f(x) \to +\infty$ as $x \to 0^+$, and this is clear. (That graph is next, on the left.) The graph also indicates a local minimum near where $x = 0.7$ (that graph is next, on the right.)





We rewrote $f(x)$ in the form $f(x) = (-\ln x)^{-\ln x}$ for $0 < x < 1$ and found that

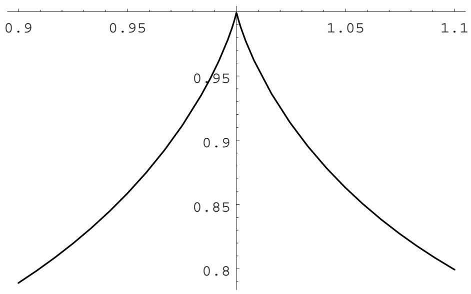
$$f'(x) = -\frac{(-\ln x)^{-\ln x}(1 + \ln(-\ln x))}{x}. \tag{1}$$

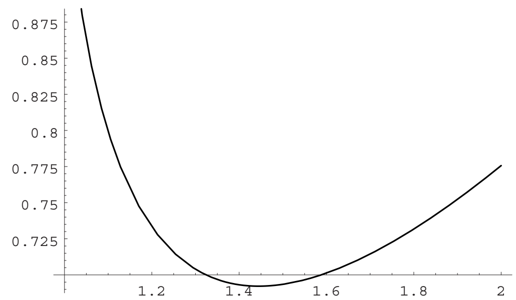
Then it is easy to solve $f'(x) = 0$ for $x = e^{-1/e}$. So the graph of $y = f(x)$ has a local minimum at $(e^{-1/e},\ e^{-1/e})$. Both the abscissa and the ordinate are approximately 0.6922006276. Next, the graph shows a cusp at the point (1, 1); there is a local maximum at that point, and $|f'(x)| \to +\infty$ as $x \to 1$. To see why, rewrite $f(x) = (\ln x)^{\ln x}$ for $x > 1$. Then

$$f'(x) = \frac{(\ln x)^{\ln x}(1 + \ln(\ln x))}{x}, \tag{2}$$

and it is clear that $f'(x) \to -\infty$ as $x \to 1^+$. You can also use Eq. (1) to show that $f'(x) \to +\infty$ as $x \to 1^-$. This is not apparent from the graph of $y = f(x)$ for $0.9 \leqq x \leqq 1.1$, shown next (on the left).

Next we plotted $y = f(x)$ for $1 \leqq x \leqq 2$ and found another local minimum near where $x = 1.4$. (The graph is next, on the right.) It is easy to solve $f'(x) = 0$ (use Eq. (2)), and you'll find that the coordinates of this point are $(e^{1/e},\ e^{-1/e})$, so the two local minima are actually global minima. Finally, it's clear that $f(x) \to +\infty$ as $x \to +\infty$.





C07S03.052: If α is a fixed real number, then

$$\ln\left(\lim_{x\to 0}\left[\exp\left(-\frac{1}{x^2}\right)\right]^{\alpha x^2}\right) = \lim_{x\to 0}\alpha x^2 \ln\left(\exp\left(-\frac{1}{x^2}\right)\right) = \lim_{x\to 0}\frac{-\alpha x^2}{x^2} = -\alpha.$$

Therefore

$$\lim_{x\to 0}\left[\exp\left(-\frac{1}{x^2}\right)\right]^{\alpha x^2} = e^{-\alpha}.$$

This means that the indeterminate form 0^0 may take on any positive real number value because that is the range of values of $e^{-\alpha}$.

The only way that

$$\lim_{x\to a}\,[f(x)]^{g(x)}$$

could be negative would be if $f(x)$ were negative for all x near a, but then $g(x)$ would take on irrational values and therefore the expression $[f(x)]^{g(x)}$ would be undefined. Hence the limit of a 0^0 indeterminate form cannot be negative.

The 0^0 form

$$\lim_{x\to 0^+} x^{1/|\ln x|^{1/2}} \tag{1}$$

has the value zero because

$$\ln\left(\lim_{x\to 0^+} x^{1/|\ln x|^{1/2}}\right) = \lim_{x\to 0^+}\frac{\ln x}{|\ln x|^{1/2}} = \lim_{x\to 0^+} -\frac{-\ln x}{(-\ln x)^{1/2}} = \lim_{x\to 0^+} -(-\ln x)^{1/2} = -\infty.$$

The 0^0 form

$$\lim_{x\to 0^+} x^{-1/(-\ln x)^{1/3}} \tag{2}$$

has the value $+\infty$ because

$$\ln\left(\lim_{x\to 0^+} x^{-1/(-\ln x)^{1/3}}\right) = \lim_{x\to 0^+} \frac{-\ln x}{(-\ln x)^{1/3}} = \lim_{x\to 0^+} (-\ln x)^{2/3} = +\infty.$$

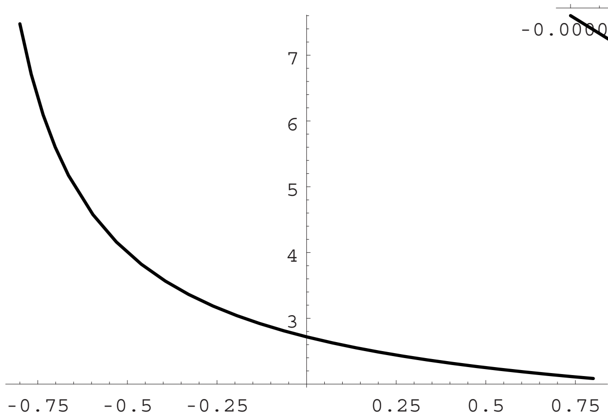
Our thanks to Ted Shifrin for the examples in (1) and (2).

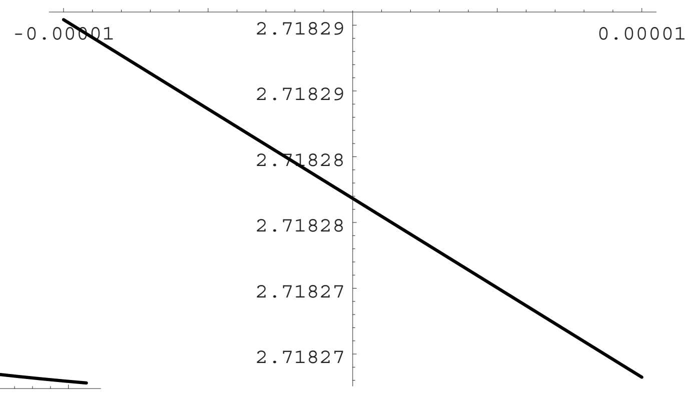
C07S03.053: The figure on the left shows the graph of $y = f(x)$ on the interval $[-1, 1]$. The figure on the right shows the graph of $y = f(x)$ on the interval $[-0.00001, 0.00001]$. It is clear from the second figure that $e \approx 2.71828$ to five places. When the removable discontinuity at $x = 0$ is removed in such a way to make f continuous there, the y-intercept will be e because

$$\ln\left[\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to 0} x\ln\left(\frac{x+1}{x}\right) = \lim_{x\to 0} \frac{\ln(x+1)-\ln x}{\dfrac{1}{x}} = \lim_{x\to 0}\left(\frac{-x^2}{x+1}+\frac{x^2}{x}\right)$$

$$= \lim_{x\to 0} \frac{-x^3+x^2+x^2}{x(x+1)} = \lim_{x\to 0} \frac{x^2}{x(x+1)} = \lim_{x\to 0} \frac{x}{x+1} = 1,$$

and therefore $\displaystyle\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = e^1 = e.$





C07S03.054: Part (c): If $v(t) = \dfrac{mg}{k}\left(1-e^{-kt/m}\right)$, then

$$\lim_{m\to\infty} v(t) = \lim_{m\to\infty} \frac{g}{k}\cdot\frac{1-e^{-kt/m}}{\dfrac{1}{m}} = \lim_{m\to\infty} \frac{g}{k}\cdot\frac{-\dfrac{kt}{m^2}e^{-kt/m}}{-\dfrac{1}{m^2}}$$

$$= \lim_{m\to\infty} \frac{g}{k}\cdot kte^{-kt/m} = \lim_{m\to\infty} gte^{-kt/m} = gt\cdot 1 = gt$$

because $k > 0$ and $t > 0$.

Section 7.4

C07S04.001: If $f(x) = 10^x$, then $f'(x) = 10^x \ln 10$ by Eq. (28).

C07S04.002: If $f(x) = 2^{1/x^2}$, then $f'(x) = -\dfrac{2}{x^3} \cdot \left(2^{1/x^2} \ln 2\right)$ by Eq. (29).

C07S04.003: If $f(x) = \dfrac{3^x}{4^x} = \left(\dfrac{3}{4}\right)^x$, then $f'(x) = \left(\dfrac{3}{4}\right)^x \ln \dfrac{3}{4}$ by Eq. (28).

C07S04.004: If $f(x) = \log_{10} \cos x$, then $f'(x) = -\dfrac{\sin x}{(\ln 10) \cos x}$ by Eq. (40).

C07S04.005: If $f(x) = 7^{\cos x}$, then $f'(x) = -\left(7^{\cos x} \ln 7\right) \cdot \sin x$ by Eq. (29).

C07S04.006: If $f(x) = 2^x \cdot 3^{x^2}$, then

$$f'(x) = (2^x \ln 2) \cdot 3^{x^2} + \left(3^{x^2} \ln 3\right) \cdot 2x \cdot 2^x$$

by the product rule and Eq. (29).

C07S04.007: If $f(x) = 2^{x\sqrt{x}} = 2^{x^{3/2}}$, then $f'(x) = \dfrac{3}{2} x^{1/2} \left(2^{x^{3/2}} \ln 2\right)$.

C07S04.008: If $f(x) = \log_{100} 10^x = x \log_{100} 10 = \dfrac{1}{2} x$, then $f'(x) \equiv \dfrac{1}{2}$.

C07S04.009: If $f(x) = 2^{\ln x}$, then $f'(x) = \dfrac{1}{x} \left(2^{\ln x} \ln 2\right)$.

C07S04.010: If $f(x) = 7^{8^x} = 7^{(8^x)}$, then $f'(x) = 7^{8^x} \cdot 8^x \cdot (\ln 7) \cdot (\ln 8)$.

C07S04.011: If $f(x) = 17^x$, then $f'(x) = 17^x \ln 17$.

C07S04.012: If $f(x) = 2^{\sqrt{x}}$, then $f'(x) = \dfrac{1}{2} x^{-1/2} \left(2^{\sqrt{x}} \ln 2\right)$.

C07S04.013: If $f(x) = 10^{1/x}$, then $f'(x) = -\dfrac{1}{x^2} \left(10^{1/x} \ln 10\right)$.

C07S04.014: If $f(x) = 3^{\sqrt{1-x^2}}$, then $f'(x) = -x(1-x^2)^{-1/2} \left(3^{\sqrt{1-x^2}} \ln 3\right)$.

C07S04.015: If $f(x) = 2^{2^x} = 2^{(2^x)}$, then $f'(x) = 2^{2^x} \cdot 2^x \cdot (\ln 2)^2$.

C07S04.016: If $f(x) = \log_2 x$, then $f'(x) = \dfrac{1}{x \ln 2}$ by Eq. (39).

C07S04.017: If $f(x) = \log_3 \sqrt{x^2+4} = \dfrac{1}{2} \log_3(x^2+4)$, then by Eq. (40) we find that

$$f'(x) = \frac{1}{2} \cdot \frac{1}{(x^2+4) \ln 3} \cdot 2x = \frac{x}{(x^2+4) \ln 3}.$$

C07S04.018: If $f(x) = \log_{10}(e^x) = x \log_{10} e$, then $f'(x) = \log_{10} e = \dfrac{1}{\ln 10}$ (by Eq. (40)).

C07S04.019: If $f(x) = \log_3(2^x) = x\log_3 2$, then $f'(x) = \log_3 2 = \dfrac{\ln 2}{\ln 3}$ (by Eq. (40)).

C07S04.020: If $f(x) = \log_{10}(\log_{10} x)$, then by Eq. (40)

$$f(x) = (\log_{10} e)\ln(\log_{10} x) = (\log_{10} e)\ln[(\log_{10} e)\ln x] = (\log_{10} e)[\ln(\log_{10} e) + \ln(\ln x)].$$

Therefore

$$f'(x) = (\log_{10} e)\left(\frac{1}{x\ln x}\right) = \frac{1}{(x\ln x)\ln 10}.$$

C07S04.021: If $f(x) = \log_2(\log_3 x)$, then by Eq. (35)

$$f(x) = (\log_2 e)\ln(\log_3 x) = (\log_2 e)\ln[(\log_3 e)\ln x] = (\log_2 e)[\ln(\log_3 e) + \ln(\ln x)].$$

Therefore

$$f'(x) = (\log_2 e)\left(\frac{1}{x\ln x}\right) = \frac{1}{(x\ln x)\ln 2}.$$

C07S04.022: If $f(x) = \pi^x + x^\pi + \pi^\pi$, then $f'(x) = \pi^x\ln\pi + \pi x^{\pi-1}$.

C07S04.023: If $f(x) = \exp(\log_{10} x)$, then $f'(x) = \dfrac{\exp(\log_{10} x)}{x\ln 10}$. *Mathematica* 3.0 reports that

$$f'(x) = \frac{x^{-1+(1/\ln 10)}}{\ln 10},$$

but a moment's work shows that the two answers are the same.

C07S04.024: If $f(x) = \pi^{x^3} = \pi^{\left(x^3\right)}$, then $f'(x) = (3x^2)\pi^{x^3}\ln\pi$ by Eq. (29).

C07S04.025: $\displaystyle\int 3^{2x}\,dx = \frac{3^{2x}}{2\ln 3} + C$ by Eq. (30).

C07S04.026: $\displaystyle\int x\cdot 10^{-x^2}\,dx = -\frac{10^{-x^2}}{2\ln 10} + C.$

C07S04.027: $\displaystyle\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \frac{2\cdot 2^{\sqrt{x}}}{\ln 2} + C.$

Comment: The easiest way to find an antiderivative such as this is to make an "educated guess" as to the form of the answer, differentiate it, and then modify the guess so it becomes correct. Here, for example, we guess $2^{\sqrt{x}}$ for the antiderivative. Then

$$D_x\left(2^{\sqrt{x}}\right) = \left(2^{\sqrt{x}}\ln 2\right)\cdot D_x\left(x^{1/2}\right) = \left(2^{\sqrt{x}}\ln 2\right)\cdot\frac{1}{2\sqrt{x}} = \frac{\ln 2}{2}\cdot\frac{2^{\sqrt{x}}}{\sqrt{x}}.$$

Thus we should multiply $2^{\sqrt{x}}$ by $\dfrac{2}{\ln 2}$ to correct it.

Of course this technique will succeed only if the correction consists of multiplication by a *constant.* If something else is needed, make a better guess or try integration by substitution.

C07S04.028: $\displaystyle\int \frac{10^{1/x}}{x^2}\,dx = -\frac{10^{1/x}}{\ln 10} + C.$

C07S04.029: Given: $\displaystyle\int x^2 \cdot 7^{x^3+1}\,dx.$ Let $u = x^3 + 1$. Then $du = 3x^2\,dx,$ so that $x^2\,dx = \dfrac{1}{3}\,du$. Thus

$$\int x^2 \cdot 7^{x^3+1}\,dx = \int \frac{1}{3}7^u\,du = \frac{7^u}{3\ln 7} + C = \frac{7^{x^3+1}}{3\ln 7} + C.$$

C07S04.030: First, $x\log_{10} x = x\,(\log_{10} e)\ln x = \dfrac{x\ln x}{\ln 10}$. Thus

$$\int \frac{1}{x\log_{10} x}\,dx = \int \frac{\ln 10}{x\ln x}\,dx = (\ln 10)\int \frac{1}{x\ln x}\,dx = (\ln 10)\cdot\ln(\ln x) + C.$$

C07S04.031: $\displaystyle\int \frac{\log_2 x}{x}\,dx = \int \frac{(\log_2 e)\ln x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx = \frac{1}{2\ln 2}(\ln x)^2 + C.$

C07S04.032: If necessary, use the substitution $u = 2^x$. In any case,

$$\int (2^x)\,3^{(2^x)}\,dx = \frac{3^{(2^x)}}{(\ln 3)(\ln 2)} + C.$$

C07S04.033: Taking logarithms transforms the equation $R = kW^m$ into $\ln R = \ln k + m\ln W$, an equation linear in the two unknown coefficients $\ln k$ and m. We put the data given in Fig. 7.4.13 into the array

```
datapoints = { {25, 131}, {67, 103}, {127, 88}, {175, 81}, {240, 75}, {975, 53} },
```

then entered the *Mathematica* 3.0 command

```
logdatapoints = N[Log[datapoints], 10]
```

to obtain the logarithms of the values of W and R to ten significant figures. We then set up the graph

```
pts = ListPlot[ logdatapoints ];
```

to see if the data points lay on a straight line. They very nearly did. Next we used *Mathematica's* `Fit` command to find the coefficients of the equation of the straight line that best fit the data points (by minimizing the sum of the squares of the deviations of the data points from the straight line—see Miscellaneous Problem 51 of Chapter 13). The command

```
Fit[ logdatapoints, {1, x}, x ]
```

finds the best-fitting linear combination $a \cdot 1 + b \cdot x$ to the given data. The result was

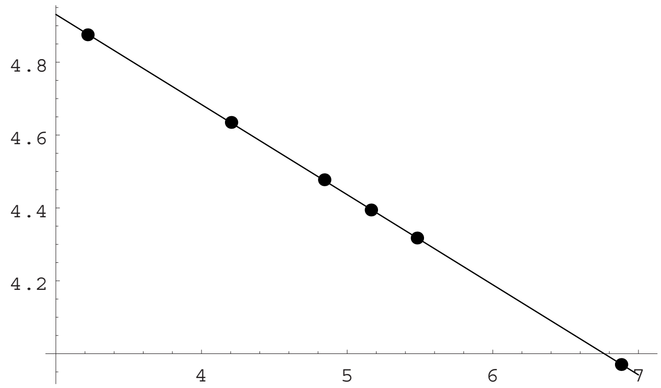
```
5.67299196 − 0.24730011 x
```

which told us that $\ln k \approx 5.67299$, so that $k \approx 290.903$, and that $m \approx -0.2473$. Thus we obtained the formula $R = (290.903)\cdot W^{-0.2473}$. This formula predicts the following values for R:

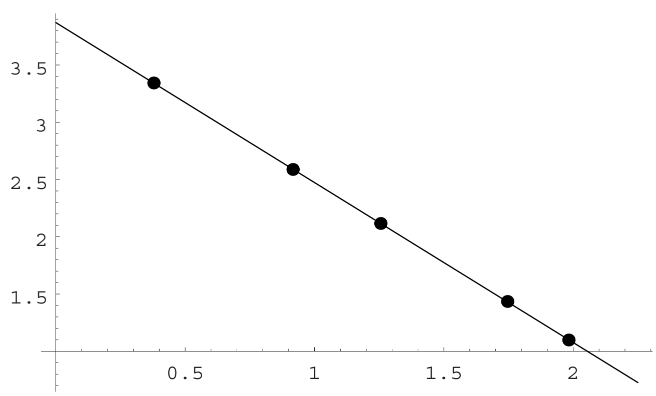
W	predicted R	experimental R
25	131.231	131
67	102.839	103
127	87.7966	88

175	81.1045	81
240	75.0105	75
975	53.0356	53

The agreement between the predicted and experimental results is quite good. The graph of the logarithms of the data points and the line $y = 5.67299 - (0.2473)x$ is shown next.



C07S04.034: We solved this problem in the same way as Problem 33. The straight line that best fits the logarithms of the data points has equation $y = 3.87045 - (1.39711)x$, so that $k = \exp(3.87045) \approx 47.9640$ and $m \approx -1.39711$. Thus we find the formula $p = (47.9640)V^{-1.39711}$. The agreement with the experimental data is even better than in Problem 33. The best-fitting straight line and the logarithms of the data points are shown next.



C07S04.035: If $f(x) = x \cdot 2^{-x}$, then $f'(x) = (1 - x \ln 2) \cdot 2^{-x}$, so $f'(x) = 0$ when $x = a = 1/\ln 2$. Because $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$, we have found the highest point on the graph of f; it is

$$\left(\frac{1}{\ln 2}, \ \frac{1}{2^{1/(\ln 2)} \ln 2} \right) \approx (1.4426950409, \ 0.5307378454) .$$

C07S04.036: Clearly the graphs of $f(x) = 2^{-x}$ and $g(x) = (x-1)^2$ cross at the point $(0, 1)$. When the graphs of f and g are plotted with the same coordinate axes, it becomes evident that the other point of intersection has x-coordinate near $x_0 = 1.6$. We applied Newton's method to $h(x) = f(x) - g(x)$ and found the sequence $x_1 = 1.5789150927$, $x_2 = 1.5786206934$, and $x_3 = x_4 = a = 1.5786206361$ of improving approximations. Because $f(x) > g(x)$ if $0 < x < a$, the area bounded by the graphs of f and g is

$$A = \int_0^a [\,f(x) - g(x)\,]\ dx = \left[x^2 - x - \frac{1}{3}x^3 - \frac{2^{-x}}{\ln 2}\right]_0^a \approx 0.5617703346.$$

C07S04.037: Let $f(x) = 2^{-x}$ and $g(x) = (x-1)^2$. We saw in the solution of Problem 36 that the graphs meet where $x = 0$ and where $x = a \approx 1.5786206361$ (obtained by applying Newton's method to the equation $f(x) - g(x) = 0$ with initial estimate $x_0 = 1.6$). There we saw also that $f(x) > g(x)$ if $0 < x < a$. So the method of parallel cross sections gives the volume of revolution around the x-axis as

$$V = \int_0^a \pi\left[2^{-2x} - (x-1)^4\right]\ dx = \pi\left[2x^2 - x - 2x^3 + x^4 - \frac{1}{5}x^5 - \frac{2^{-2x}}{2\ln 2}\right]_0^a \approx 1.343088216395.$$

C07S04.038: Let $f(x) = 3^{2-x}$ and $g(x) = (3x-4)^2$. We plotted the graphs of f and g for $0.5 \leqq x \leqq 2$ and saw that the graphs cross near where $x = 0.6$ and near where $x = 1.7$. We applied Newton's method to the equation $f(x) - g(x) = 0$ using these values as initial estimates and obtained the following results:

n	First x_n	Second x_n
1	0.622814450454	1.722194918580
2	0.623229033865	1.721721015948
3	0.623229171888	1.721720799346
4	0.623229171888	1.721720799346

Let a be the first x_4 and b the second. Note that $f(x) > g(x)$ for $a < x < b$. So the area of the region bounded by the graphs of f and g is approximately

$$A = \int_a^b [f(x) - g(x)]\,dx = \left[12x^2 - 16x - 3x^3 - \frac{3^{2-x}}{\ln 3}\right]_a^b \approx 1.645167893979.$$

C07S04.039: By definition of z, x, and y, respectively, we have

$$a^z = c, \quad a^x = b, \quad \text{and} \quad b^y = c.$$

Therefore $a^{xy} = b^y = c = a^z$. Because $a > 0$ and $a \neq 1$, it now follows that $z = xy$.

C07S04.040: $\displaystyle\lim_{x\to 0^+} \frac{1}{1+2^{1/x}} = \lim_{k\to\infty} \frac{1}{1+2^k} = 0; \quad \lim_{x\to 0^-} \frac{1}{1+2^{1/x}} = \lim_{k\to -\infty} \frac{1}{1+2^k} = 1.$

C07S04.041: Beginning with the equation $x^y = 2$, we first write $y \cdot \ln x = \ln 2$, then differentiate implicitly with respect to x to obtain

$$\frac{y}{x} + \frac{dy}{dx}\ln x = 0.$$

Thus

$$\frac{dy}{dx} = -\frac{y}{x\ln x} = -\frac{\ln 2}{x(\ln x)^2}.$$

Section 7.5

C07S05.001: $\arcsin\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$ because $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$ and $-\dfrac{\pi}{2} \leqq \dfrac{\pi}{6} \leqq \dfrac{\pi}{2}$. Similarly,

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \qquad \arcsin\left(\frac{1}{2}\sqrt{2}\right) = \frac{\pi}{4}, \qquad \text{and} \qquad \arcsin\left(-\frac{1}{2}\sqrt{3}\right) = -\frac{\pi}{3}.$$

C07S05.002: $\arccos\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$ because $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$ and $0 \leqq \dfrac{\pi}{3} \leqq \pi$. Similarly,

$$\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \arccos\left(\frac{1}{2}\sqrt{2}\right) = \frac{\pi}{4}, \qquad \text{and} \qquad \arccos\left(-\frac{1}{2}\sqrt{3}\right) = \frac{5\pi}{6}.$$

C07S05.003: $\arctan(0) = 0$ because $\tan(0) = 0$ and $-\dfrac{\pi}{2} < 0 < \dfrac{\pi}{2}$. Similarly,

$$\arctan(1) = \frac{\pi}{4}, \qquad \arctan(-1) = -\frac{\pi}{4}, \qquad \text{and} \qquad \arctan\left(\sqrt{3}\right) = \frac{\pi}{3}.$$

C07S05.004: $\operatorname{arcsec}(1) = 0$ because $\sec(0) = 1$ and $0 \leqq 0 \leqq \pi$. Similarly,

$$\operatorname{arcsec}(-1) = \pi, \qquad \operatorname{arcsec}(2) = \frac{\pi}{3}, \qquad \text{and} \qquad \operatorname{arcsec}\left(-\sqrt{2}\right) = \frac{3\pi}{4}.$$

C07S05.005: If $f(x) = \sin^{-1}\left(x^{100}\right)$, then $f'(x) = \dfrac{100x^{99}}{\sqrt{1 - x^{200}}}$.

C07S05.006: If $f(x) = \arctan(e^x)$, then $f'(x) = \dfrac{e^x}{1 + e^{2x}}$.

C07S05.007: If $f(x) = \sec^{-1}(\ln x)$, then $f'(x) = \dfrac{1}{x\,|\ln x|\sqrt{(\ln x)^2 - 1}}$.

C07S05.008: If $f(x) = \ln\left(\tan^{-1} x\right)$, then $f'(x) = \dfrac{1}{(1 + x^2)\arctan x}$.

C07S05.009: If $f(x) = \arcsin(\tan x)$, then $f'(x) = \dfrac{\sec^2 x}{\sqrt{1 - \tan^2 x}}$.

C07S05.010: If $f(x) = x\arctan x$, then $f'(x) = \dfrac{x}{1 + x^2} + \arctan x$.

C07S05.011: If $f(x) = \sin^{-1} e^x$, then $f'(x) = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$.

C07S05.012: If $f(x) = \arctan\sqrt{x}$, then $f'(x) = \dfrac{1}{2(1 + x)\sqrt{x}}$.

C07S05.013: If $f(x) = \cos^{-1} x + \sec^{-1}\left(\dfrac{1}{x}\right)$ and $0 < x < 1$, then

$$f'(x) = -\frac{1}{\sqrt{1-x^2}} - \frac{1}{x^2 \cdot \dfrac{1}{x} \cdot \sqrt{\left(\dfrac{1}{x}\right)^2 - 1}}$$

$$= -\frac{1}{\sqrt{1-x^2}} - \frac{1}{x\sqrt{x^{-2}-1}} = -\frac{2}{\sqrt{1-x^2}}.$$

But if $-1 < x < 0$, then

$$f'(x) = -\frac{1}{\sqrt{1-x^2}} - \frac{1}{x^2 \cdot \left|\dfrac{1}{x}\right| \cdot \sqrt{x^{-2}-1}} = -\frac{1}{\sqrt{1-x^2}} + \frac{1}{x^2 \cdot \dfrac{1}{x} \cdot \sqrt{x^{-2}-1}}$$

$$= -\frac{1}{\sqrt{1-x^2}} - \frac{1}{(-x)\sqrt{x^{-2}-1}} = -\frac{2}{\sqrt{1-x^2}}.$$

In the last line in the second derivation, we needed to replace $x < 0$ with $-x > 0$ in order to move it under the radical.

C07S05.014: If $f(x) = \operatorname{arccot}\bigl(x^{-2}\bigr)$, then

$$f'(x) = -\frac{1}{1+x^{-4}} \cdot \bigl(-2x^{-3}\bigr) = \frac{1}{1+x^{-4}} \cdot \frac{2x}{x^4} = \frac{2x}{x^4+1}.$$

C07S05.015: If $f(x) = \csc^{-1} x^2$, then

$$f'(x) = -\frac{2x}{|x^2|\sqrt{x^4-1}} = -\frac{2x}{x^2\sqrt{x^4-1}} = -\frac{2}{x\sqrt{x^4-1}}.$$

C07S05.016: If $f(x) = \arccos\left(x^{-1/2}\right)$, then

$$f'(x) = -\frac{1}{\sqrt{1-x^{-1}}} \cdot \left(-\frac{1}{2}x^{-3/2}\right) = \frac{1}{2x\sqrt{x}\,\sqrt{1-x^{-1}}} = \frac{1}{2x\sqrt{x-1}}.$$

Antidifferentiation of the last expression is quite easy if you make the substitution $x = u^2+1$, $dx = 2u\,du$. You can verify that one antiderivative is $g(x) = \arctan\sqrt{x-1}$. Thus the difference between $f(x)$ and $g(x)$ is a constant. What is that constant, and on what interval is the resulting equality valid? (Answers: Zero; $x \geqq 1$.)

C07S05.017: If $f(x) = \dfrac{1}{\arctan x} = (\arctan x)^{-1}$, then $f'(x) = -\dfrac{1}{(1+x^2)(\arctan x)^2}$.

C07S05.018: If $f(x) = (\arcsin x)^2$, then $f'(x) = \dfrac{2\arcsin x}{\sqrt{1-x^2}}$.

C07S05.019: If $f(x) = \tan^{-1}(\ln x)$, then

$$f'(x) = \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x} = \frac{1}{x\left[1+(\ln x)^2\right]}.$$

C07S05.020: If $f(x) = \operatorname{arcsec}\sqrt{x^2+1}$, then

$$f'(x) = \frac{1}{\left|\sqrt{x^2+1}\right|\sqrt{(x^2+1)-1}} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{(x^2+1)\sqrt{x^2}} = \frac{x}{(x^2+1)|x|}.$$

C07S05.021: If $f(x) = \tan^{-1} e^x + \cot^{-1} e^{-x}$, then

$$f'(x) = \frac{e^x}{1+e^{2x}} - \frac{-e^{-x}}{1+e^{-2x}} = \frac{2e^x}{1+e^{2x}} = \frac{2}{e^x + e^{-x}}.$$

The last expression is $\operatorname{sech} x$, the *hyperbolic secant* of x (Section 7.6). Thus we have accidentally found an antiderivative of the hyperbolic secant function.

C07S05.022: If $f(x) = \exp(\arcsin x)$, then $f'(x) = \dfrac{\exp(\arcsin x)}{\sqrt{1-x^2}}$.

C07S05.023: If $f(x) = \sin(\arctan x)$, then

$$f'(x) = \frac{\cos(\arctan x)}{1+x^2}.$$

But this problem has a twist. A reference right triangle with acute angle $\theta = \arctan x$, adjacent side 1, opposite side x, and hypotenuse $\sqrt{1+x^2}$ shows that

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

Therefore, by the quotient rule,

$$f'(x) = \frac{(1+x^2)^{1/2} - x^2(1+x^2)^{-1/2}}{1+x^2} = \frac{1+x^2-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}.$$

The second version of the derivative is potentially more useful than the first version.

C07S05.024: If $f(x) = \sec(\sec^{-1} e^x)$, then $f(x) = e^x$ wherever it is defined (for $x \geqq 0$), so $f'(x) = e^x$ (if $x > 0$).

C07S05.025: If $f(x) = \dfrac{\arctan x}{(1+x^2)^2}$, then

$$f'(x) = \frac{(1+x^2)^2 \cdot \dfrac{1}{1+x^2} - 4x(1+x^2)\arctan x}{(1+x^2)^4} = \frac{1-4x\arctan x}{(1+x^2)^3}.$$

C07S05.026: If $f(x) = (\sin^{-1} 2x^2)^{-2}$, then

$$f'(x) = (-2)(\sin^{-1} 2x^2)^{-3} \cdot \frac{1}{\sqrt{1-4x^4}} \cdot 4x = -\frac{8x}{(\sin^{-1} 2x^2)^3\sqrt{1-4x^4}}.$$

C07S05.027: Given: $\tan^{-1} x + \tan^{-1} y = \dfrac{\pi}{2}$:

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} \cdot \frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{1+y^2}{1+x^2}.$$

So the slope of the line tangent to the graph at $P(1, 1)$ is -1, and therefore an equation of that line is $y - 1 = -(x-1)$; that is, $y = 2 - x$.

C07S05.028: Given: $\sin^{-1} x + \sin^{-1} y = \dfrac{\pi}{2}$:

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

So the slope of the line tangent to the graph at $P\left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$ is $-\frac{1}{3}\sqrt{3}$, and therefore an equation of that line is

$$y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right); \qquad \text{that is,} \qquad y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}.$$

C07S05.029: Given: $(\sin^{-1} x)(\sin^{-1} y) = \dfrac{\pi^2}{16}$:

$$\frac{\sin^{-1} y}{\sqrt{1-x^2}} + \frac{\sin^{-1} x}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{(1-y^2)^{1/2}\sin^{-1} y}{(1-x^2)^{1/2}\sin^{-1} x}.$$

So the slope of the line tangent to the graph at $P\left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$ is -1, and therefore an equation of that line is

$$y - \frac{1}{2}\sqrt{2} = -\left(x - \frac{1}{2}\sqrt{2}\right); \qquad \text{that is,} \qquad y = -x + \sqrt{2}\,.$$

C07S05.030: Given: $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 = \dfrac{5\pi^2}{36}$:

$$\frac{2\sin^{-1} x}{\sqrt{1-x^2}} + \frac{2\sin^{-1} y}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = -\frac{(1-y^2)^{1/2}\sin^{-1} x}{(1-x^2)^{1/2}\sin^{-1} y}.$$

So the slope of the line tangent to the graph at $P\left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$ is $-\frac{1}{6}\sqrt{3}$. Therefore an equation of that line is

$$y - \frac{1}{2}\sqrt{3} = -\frac{1}{6}\sqrt{3}\left(x - \frac{1}{2}\right); \qquad \text{that is,} \qquad y = \frac{\sqrt{3}}{12}(7-2x).$$

C07S05.031: $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

C07S05.032: $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\arcsin x\Big]_0^{1/2} = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$

C07S05.033: $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec} x\Big]_{\sqrt{2}}^2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \approx 0.261799387799.$

In comparison, *Mathematica* 3.0 yields the result

$$\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \left[-\arctan\left(\frac{1}{\sqrt{x^2-1}}\right)\right]_{\sqrt{2}}^2 = -\frac{\pi}{6} + \frac{\pi}{4} = \frac{\pi}{12}.$$

C07S05.034: $\displaystyle\int_{-2}^{-2/\sqrt{3}} \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec}|x|\Big]_{-2}^{-2/\sqrt{3}} = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}.$

The answer is negative because the integrand is negative for $-2 \leqq x \leqq -\dfrac{2}{\sqrt{3}}$.

C07S05.035: Let $x = 3u$. Then $dx = 3\,du$, and as x ranges from 0 to 3, u ranges from 0 to 1. Therefore

$$\int_0^3 \frac{1}{9+x^2}\,dx = \int_0^1 \frac{3}{9+9u^2}\,du = \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{3}\Big[\arctan u\Big]_0^1 = \frac{\pi}{12} - 0 = \frac{\pi}{12} \approx 0.261799387799.$$

Alternatively, the antiderivative can be expressed as a function of x before evaluation:

$$\int_0^3 \frac{1}{9+x^2}\,dx = \int_0^1 \frac{3}{9+9u^2}\,du = \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{3}\Big[\arctan u\Big]_0^1$$

$$= \frac{1}{3}\left[\arctan\left(\frac{x}{3}\right)\right]_0^3 = \frac{\pi}{12} - 0 = \frac{\pi}{12} \approx 0.261799387799.$$

Note that in the latter case the original x-limits of integration must be restored before evaluation of the antiderivative.

C07S05.036: Let $x = 4u$. Then $dx = 4\,du$, and as x ranges from 0 to $\sqrt{12} = 2\sqrt{3}$, u ranges from 0 to $\frac{1}{2}\sqrt{3}$. Thus

$$\int_0^{\sqrt{12}} \frac{1}{\sqrt{16-x^2}}\,dx = \int_0^{\sqrt{3}/2} \frac{4}{\sqrt{16-16u^2}}\,du = \int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du = \Big[\arcsin u\Big]_0^{\sqrt{3}/2} = \frac{\pi}{3} - 0 = \frac{\pi}{3}.$$

C07S05.037: Let $u = 2x$, so that $4x^2 = u^2$ and $dx = \frac{1}{2}\,du$. Then

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin 2x + C.$$

C07S05.038: Let $u = \frac{3}{2}x$. Then $x = \frac{2}{3}u$, $dx = \frac{2}{3}\,du$, and $9x^2 + 4 = 4u^2 + 4$. Thus

$$\int \frac{1}{9x^2+4}\,dx = \frac{2}{3}\int \frac{1}{4u^2+4}\,du = \frac{1}{6}\arctan u + C = \frac{1}{6}\arctan\frac{3x}{2} + C.$$

C07S05.039: Let $x = 5u$. Then $dx = 5\,du$ and $x^2 - 25 = 25u^2 - 25 = 25(u^2-1)$. Thus

$$\int \frac{1}{x\sqrt{x^2-25}}\,dx = \int \frac{5}{5\cdot 5u\sqrt{u^2-1}}\,du = \frac{1}{5}\operatorname{arcsec}|u| + C = \frac{1}{5}\operatorname{arcsec}\frac{|x|}{5} + C.$$

Mathematica 3.0 reports that

$$\int \frac{1}{x\sqrt{x^2-25}}\,dx = C - \frac{1}{5}\arctan\left(\frac{5}{\sqrt{x^2-25}}\right).$$

C07S05.040: Let $u = \frac{2}{3}x$, so that $2x = 3u$ and $dx = \frac{3}{2}\,du$. Then

$$\int \frac{1}{x(4x^2-9)^{1/2}}\,dx = \frac{3}{2}\int \frac{1}{\frac{3}{2}u(9u^2-9)^{1/2}}\,du$$

$$= \int \frac{1}{3u(u^2-1)^{1/2}}\,du = \frac{1}{3}\operatorname{arcsec}|u| + C = \frac{1}{3}\operatorname{arcsec}\frac{2|x|}{3} + C.$$

C07S05.041: $\displaystyle \int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \arctan(e^x) + C.$

If you prefer integration by substitution, use $u = e^x$, so that $du = e^x\,dx$. Then

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan(e^x) + C.$$

C07S05.042: Let $u = \frac{1}{5}x^3$. Then $x^3 = 5u$, $3x^2\,dx = 5\,du$, and $x^6 + 25 = 25(u^2+1)$. Thus

$$\int \frac{x^2}{x^6+25}\,dx = \frac{5}{3}\int \frac{1}{25(u^2+1)}\,du = \frac{1}{15}\arctan u + C = \frac{1}{15}\arctan\left(\frac{1}{5}x^3\right) + C.$$

C07S05.043: Let $u = \frac{1}{5}x^3$. Then $5u = x^3$, $3x^2\,dx = 5\,du$, and $x^6 - 25 = 25(u^2-1)$. So

$$\int \frac{1}{x\sqrt{x^6-25}}\,dx = \int \frac{3x^2}{3x^3(x^6-25)^{1/2}}\,dx = \int \frac{5}{3\cdot 5u\,[25(u^2-1)]^{1/2}}\,du$$

$$= \frac{1}{15}\int \frac{1}{u(u^2-1)^{1/2}}\,du = \frac{1}{15}\operatorname{arcsec}|u| + C = \frac{1}{15}\operatorname{arcsec}\frac{|x^3|}{5} + C.$$

Mathematica 3.0 gives the answer in the form $C + \dfrac{1}{15}\arctan\left(\dfrac{1}{5}\sqrt{x^6-25}\right)$.

C07S05.044: Let $u = x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\,dx$, so $x^{1/2}\,dx = \frac{2}{3}\,du$ and

$$\int \frac{x^{1/2}}{1+x^3}\,dx = \frac{2}{3}\int \frac{1}{1+u^2}\,du = \frac{2}{3}\arctan u + C = \frac{2}{3}\arctan\left(x^{3/2}\right) + C.$$

C07S05.045: The radicand is

$$x(1-x) = x - x^2 = -(x^2-x) = -\frac{1}{4}(4x^2-4x)$$

$$= -\frac{1}{4}(4x^2-4x+1) + \frac{1}{4} = \frac{1}{4}\left[1-(2x-1)^2\right] = \frac{1}{4}(1-u^2)$$

if we let $u = 2x-1$. If so, $du = 2\,dx$, and then

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{1-u^2}}\,du = \arcsin u + C = \arcsin(2x-1) + C.$$

The more "obvious" substitution $x = u^2$, so that $u = x^{1/2}$ and $dx = 2u\,du$, leads to

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = \int \frac{2u}{\sqrt{u^2(1-u^2)}}\,du = 2\int \frac{u}{|u|\sqrt{1-u^2}}\,du$$

$$= 2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin u + C = 2\arcsin\sqrt{x} + C.$$

Replacement of $|u|$ with u here is permitted because $u = \sqrt{x} > 0$. Test your skill at trigonometry by showing that $f(x) = \arcsin(2x-1)$ and $g(x) = 2\arcsin\sqrt{x}$ differ by a constant (if $0 \leqq x \leqq 1$).

C07S05.046: Let $u = \sec x$, so that $du = \sec x\,\tan x\,dx$. Then

$$\int \frac{\sec x\,\tan x}{1+\sec^2 x}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u\, + C = \arctan(\sec x)\, + C.$$

Mathematica 3.0 returns the amazing answer $C - \dfrac{[\arctan(\cos x)]\,(3+\cos 2x)\sec^2 x}{2(1+\sec^2 x)}$.

C07S05.047: Let $u = x^{50}$, so that $du = 50x^{49}\,dx$. Then

$$\int \frac{x^{49}}{1+x^{100}}\,dx = \frac{1}{50}\int \frac{1}{1+u^2}\,du = \frac{1}{50}\arctan u\, + C = \frac{1}{50}\arctan\left(x^{50}\right)\, + C.$$

C07S05.048: Let $u = x^5$, so that $du = 5x^4\,dx$. Then

$$\int \frac{x^4}{\sqrt{1-x^{10}}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{5}\arcsin u\, + C = \frac{1}{5}\arcsin\left(x^5\right)\, + C.$$

C07S05.049: $\displaystyle\int \frac{1}{x\,[1+(\ln x)^2]}\,dx = \arctan(\ln x) + C.$ (Use the substitution $u = \ln x$ if necessary.)

C07S05.050: $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx = \frac{1}{2}(\arctan x)^2 + C.$ (Use the substitution $u = \arctan x$ if necessary.)

C07S05.051: $\displaystyle\int_0^1 \frac{1}{1+(2x-1)^2}\,dx = \left[\frac{1}{2}\arctan(2x-1)\right]_0^1 = \frac{\pi}{8} - \left(-\frac{\pi}{8}\right) = \frac{\pi}{4} \approx 0.7853981634.$

If you prefer integration by substitution, let $u = 2x-1$, $du = 2\,dx$, and do not forget to change the limits of integration to $u = -1$ and $u = 1$.

C07S05.052: $\displaystyle\int_0^1 \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int_0^1 \frac{4x^3}{1+x^4}\,dx = \frac{1}{4}\Big[\ln(1+x^4)\Big]_0^1 = \frac{1}{4}(\ln 2 - \ln 1) = \frac{\ln 2}{4} \approx 0.1732867951.$

If you prefer integration by substitution, let $u = 1+x^4$, $du = 4x^3\,dx$, and do not forget to change the limits of integration to $u = 1$ and $u = 2$.

C07S05.053: Let $u = \ln x$ (if necessary) to find that

$$\int_1^e \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \Big[\arcsin(\ln x)\Big]_1^e = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

C07S05.054: $\displaystyle\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \Big[\operatorname{arcsec} x\Big]_1^2 = \frac{\pi}{3} - 0 = \frac{\pi}{3}.$

C07S05.055: If $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2}\,dx$. . Moreover, $u = 1$ when $x = 1$ and $u = \sqrt{3}$ when $x = 3$. Therefore

$$\begin{aligned}\int_1^3 \frac{1}{2x^{1/2}(1+x)}\,dx &= \int_1^3 \frac{\frac{1}{2}x^{-1/2}}{1+(x^{1/2})^2}\,dx = \int_1^{\sqrt{3}} \frac{1}{1+u^2}\,du\\ &= \Big[\arctan u\Big]_1^{\sqrt{3}} = \Big[\arctan\left(\sqrt{x}\,\right)\Big]_1^3 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \approx 0.2617993878.\end{aligned}$$

C07S05.056: First note that $\cos^{-1} x = C - \sin^{-1} x$ for some constant C for all x in the interval $(0, 1)$. In particular,

$$C = \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

Therefore $\sin^{-1} x + \cos^{-1} x = \pi/2$ if $0 < x < 1$. This formula also holds if $x = 0$ and if $x = 1$, and therefore it holds for all x in the closed interval $0 \leqq x \leqq 1$.

C07S05.057: Suppose that $u < -1$ and let $x = -u$. Then $x > 0$, so

$$y = \operatorname{arcsec} |u| = \operatorname{arcsec} x,$$

and then the chain rule yields

$$\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du} = \frac{1}{|x|\sqrt{x^2 - 1}} \cdot (-1) = \frac{-1}{x\sqrt{(-x)^2 - 1}} = \frac{1}{(-x)\sqrt{(-x)^2 - 1}} = \frac{1}{u\sqrt{u^2 - 1}}.$$

C07S05.058: If $a > 0$ and $u = ax$, then

$$du = a\ dx, \quad a^2 - u^2 = a^2 - a^2x^2 = a^2(1 - x^2), \quad \text{and} \quad \sqrt{a^2 - u^2} = a\sqrt{1 - x^2}.$$

Therefore

$$\int \frac{1}{\sqrt{a^2 - u^2}}\ du = \int \frac{a}{a\sqrt{1 - x^2}}\ dx = \arcsin x + C = \arcsin\left(\frac{u}{a}\right) + C.$$

C07S05.059: If $a > 0$ and $u = ax$, then $du = a\ dx$ and $a^2 + u^2 = a^2 + a^2x^2 = a^2(1 + x^2)$. So

$$\int \frac{1}{a^2 + u^2}\ du = \int \frac{a}{a^2(1 + x^2)}\ dx = \frac{1}{a}\arctan x + C = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C.$$

C07S05.060: If $a > 0$ and $u = ax$, then

$$du = a\ dx, \quad u^2 - a^2 = a^2x^2 - a^2 = a^2(x^2 - 1), \quad \text{and} \quad \sqrt{u^2 - a^2} = a\sqrt{x^2 - 1}\,.$$

Therefore

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\ du = \int \frac{a}{a^2x\sqrt{x^2 - 1}}\ dx = \frac{1}{a}\operatorname{arcsec}|x| + C = \frac{1}{a}\operatorname{arcsec}\left|\frac{u}{a}\right| + C.$$

C07S05.061: If $x > 1$, then

$$f'(x) = \frac{1}{x^2\sqrt{1 - \dfrac{1}{x^2}}} = \frac{1}{x\sqrt{x^2 - \dfrac{x^2}{x^2}}} = \frac{1}{|x|\sqrt{x^2 - 1}}\,.$$

If $x < -1$, then

$$f'(x) = \frac{1}{x^2\sqrt{1 - \dfrac{1}{x^2}}} = \frac{1}{(-x)^2\sqrt{1 - \dfrac{1}{x^2}}} = \frac{1}{(-x)\sqrt{(-x)^2 - \dfrac{(-x)^2}{x^2}}} = \frac{1}{|x|\sqrt{x^2 - 1}}\,.$$

C07S05.062: If $x > 1$ or if $x < -1$, then

$$D_x \cos^{-1}\left(\frac{1}{x}\right) = -\frac{1}{\sqrt{1 - \dfrac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{1 - \dfrac{1}{x^2}}}\,.$$

Now apply the result in Problem 61. But at this point you can conclude only that

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) + C_i$$

for some constant C_1 if $x > 1$ and for some possibly different constant C_2 if $x < -1$. Substitute $x = 2$ and then $x = -2$ in the last equation to verify that $C_i = 0$ in each case.

C07S05.063: Let g denote the "alternative secant function," so that $y = g(x)$ if and only if $\sec y = x$ and either $0 \leqq y < \pi/2$ or $\pi \leqq y < 3\pi/2$. We differentiate implicitly the *identity* $\sec y = x$ on both the intervals $0 < y < \pi/2$ and $\pi < y < 3\pi/2$ and find that

$$(\sec y\, \tan y)\frac{dy}{dx} = 1, \qquad \text{so that} \qquad g'(x) = \frac{1}{\sec y\, \tan y} = \pm\frac{1}{x\sqrt{\sec^2 y - 1}} = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Then Fig. 7.5.13 shows that $g'(x) < 0$ if $x < -1$ and that $g'(x) > 0$ if $x > 1$. Therefore the choice of the plus sign in the previous equation is correct in both cases:

$$g'(x) = \frac{1}{x\sqrt{x^2-1}}\,.$$

C07S05.064: We begin with the identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\, \tan B}\,. \tag{1}$$

Let $x = \tan A$, $y = \tan B$, and suppose that $xy < 1$. We will treat only the case in which x and y are both positive; the other three cases are similar. In this case, Eq. (1) implies that $0 < A + B < \pi/2$, so we may apply the inverse tangent function to both sides of the identity in (1) to obtain

$$A + B = \arctan\frac{x+y}{1-xy}\,,$$

and therefore

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}.$$

Now we turn to part (b). We have

$$\arctan\frac{\dfrac{1}{2} + \dfrac{1}{3}}{1 - \dfrac{1}{6}} = \arctan\frac{\dfrac{5}{6}}{\dfrac{5}{6}} = \arctan 1 = \frac{\pi}{4}.$$

$$\arctan\frac{\dfrac{1}{3} + \dfrac{1}{3}}{1 - \dfrac{1}{9}} + \arctan\frac{1}{7} = \arctan\frac{\dfrac{2}{3}}{\dfrac{8}{9}} + \arctan\frac{1}{7} = \arctan\frac{3}{4} + \arctan\frac{1}{7}$$

$$= \arctan \frac{\dfrac{3}{4} + \dfrac{1}{7}}{1 - \dfrac{3}{28}} = \arctan \frac{25}{25} = \frac{\pi}{4}.$$

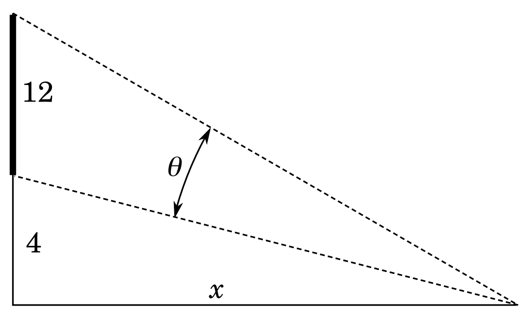
$$\arctan \frac{\dfrac{120}{119} - \dfrac{1}{239}}{1 + \dfrac{120}{119} \cdot \dfrac{1}{239}} = \arctan \frac{\dfrac{28561}{28441}}{\dfrac{28561}{28441}} = \frac{\pi}{4}.$$

$$2\arctan \frac{1}{5} = \arctan \frac{\dfrac{2}{5}}{1 - \dfrac{1}{25}} = \arctan \frac{10}{24} = \arctan \frac{5}{12};$$

$$4\arctan \frac{1}{5} = \arctan \frac{\dfrac{10}{12}}{1 - \dfrac{25}{144}} = \arctan \frac{120}{119};$$

$$4\arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4} \quad \text{(by (iii))}.$$

C07S05.065: See the following figure for the meanings of the variables.



We are required to maximize the angle θ, and from the figure and the data given in the problem we may express θ as a function of the distance x of the billboard from the motorist:

$$\theta = \theta(x) = \arctan \frac{16}{x} - \arctan \frac{4}{x}, \quad 0 < x < +\infty.$$

After simplifications we find that

$$\frac{d\theta}{dx} = \frac{4}{x^2 + 16} - \frac{16}{x^2 + 256}.$$

The solution of $\theta'(x) = 0$ certainly maximizes θ because θ is near zero if x is near zero and if x is large positive. We solve $\theta'(x) = 0$:

$$4(x^2 + 16) = x^2 + 256; \qquad 3x^2 = 192;$$

$$x^2 = 64; \qquad x = 8.$$

Answer: The billboard should be placed so that it will be 8 meters (horizontal distance) from the eyes of passing motorists. Many alert students have pointed out that such a billboard wouldn't be visible long

enough to be effective. This illustrates that once you have used mathematics to solve a problem, you must *interpret* the results!

C07S05.066: As in the figure that follows this solution, assume that the observer's eyes are at the height L above the floor and at horizontal distance W from the painting. Let h be the height of the painting and let y be the distance from the floor to the bottom of the painting. We are to maximize the angle $\theta = \theta_1 + \theta_2$ where

$$\theta_1 = \arctan \frac{y+h-L}{W} \qquad \text{and} \qquad \theta_2 = \arctan \frac{L-y}{W}.$$

With the aid of the arctangent addition formula (part (a) of Problem 64) we find—after simplifications—that

$$\theta(y) = \arctan \frac{Wh}{W^2 + (y-L)(y-L+h)}. \tag{1}$$

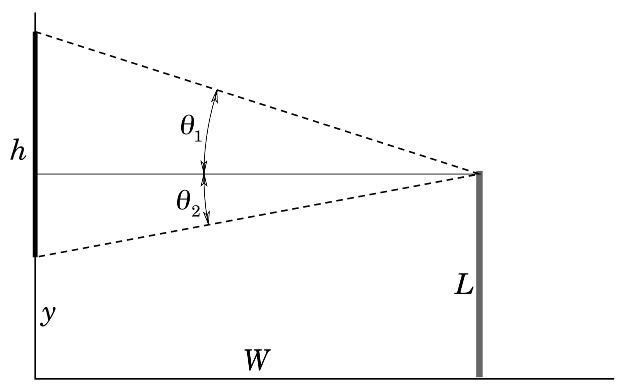
Now θ is maximized when $\tan\theta$ is maximized (because the tangent function is increasing on $(0, \pi/2)$), and this occurs when the denominator of the fraction in Eq. (1) is minimal. So we let $f(y) = W^2+(y-L)(y-L+h)$ and apply calculus:

$$f'(y) = y - L + y - L + h = 2y - 2L + h;$$
$$f'(y) = 0 \qquad \text{when} \qquad y = L - \frac{h}{2}.$$

This value of y clearly minimizes $f(y)$, and the center of the painting is then at height

$$y + \frac{h}{2} = L - \frac{h}{2} + \frac{h}{2} = L$$

above the floor—exactly at the height of the observer's eyes.



C07S05.067: If $f(x) = (a^2 - x^2)^{1/2}$, then

$$1 + [f'(x)]^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2},$$

so the circumference of a circle of radius a is

$$C = 8\int_0^{a/\sqrt{2}} \frac{a}{\sqrt{a^2 - x^2}}\,dx = 8 \cdot \left[a \arcsin\left(\frac{x}{a}\right)\right]_0^{a/\sqrt{2}} = 8\left(\frac{\pi a}{4} - 0\right) = 2\pi a.$$

The integration was carried out by using the result in Problem 58.

C07S05.068: By the method of nested cylindrical shells, the volume is

$$V = \int_0^1 \frac{2\pi x}{1+x^4}\,dx = \Big[\pi \arctan(x^2)\Big]_0^1 = \frac{\pi^2}{4} - 0 = \frac{\pi^2}{4} \approx 2.4674011003.$$

C07S05.069: Let A_a denote the area under the graph of $y(x)$ for $0 \leqq x \leqq a$. Therefore

$$A_a = \int_0^a \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_0^a = \arctan a.$$

Then $\displaystyle\lim_{a\to\infty} A_a = \frac{\pi}{2}$ by Eq. (2) and Fig. 7.5.4.

C07S05.070: Let y be the height of the elevator (measured upward from ground level) and let θ be the angle that your line of sight to the elevator makes with the horizontal ($\theta > 0$ if you are looking up, $\theta < 0$ if down). You're to maximize $\dfrac{d\theta}{dt}$ given $\dfrac{dy}{dt} = -25$.

$$\tan\theta = \frac{y-100}{50}, \quad \text{so} \quad \theta = \tan^{-1}\left(\frac{y-100}{50}\right).$$

Therefore

$$\frac{d\theta}{dt} = \frac{d\theta}{dy}\cdot\frac{dy}{dt} = -25\cdot\frac{1/50}{1+[(y-100)/50]^2} = \frac{-25\cdot 50}{2500+(y-100)^2}. \tag{1}$$

To find the value of y that maximizes $f(y) = d\theta/dt$, we need only minimize the last denominator in Eq. (1): $y = 100$. Answer: The elevator has maximum apparent speed when it's at eye level.

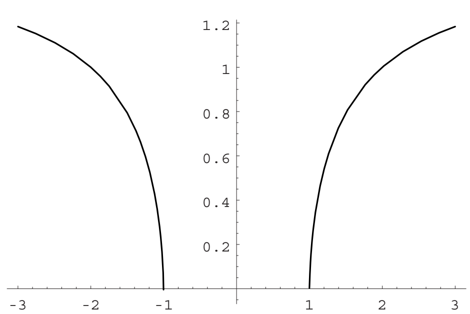
C07S05.071: For $x > 1$: $f(x) = \text{arcsec}\, x + A$:

$$1 = f(2) = \frac{\pi}{3} + 1, \quad \text{so} \quad A = 1 - \frac{\pi}{3}.$$

For $x < -1$: $f(x) = -\text{arcsec}\, x + B$;

$$1 = f(-2) = -\frac{2\pi}{3} + B, \quad \text{so} \quad B = 1 + \frac{2\pi}{3}.$$

Therefore $f(x) = \text{arcsec}\, x + 1 - \dfrac{\pi}{3}$ if $x > 1$, $f(x) = -\text{arcsec}\, x + 1 + \dfrac{2\pi}{3}$ if $x < -1$. The graph of $y = f(x)$ is next.



C07S05.072: If $|x| < 1$, then

$$D_x \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{1+\dfrac{x^2}{1-x^2}} \cdot \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}.$$

Therefore

$$\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = C + \arcsin x$$

for some constant C if $-1 < x < 1$. Now substitute $x = 0$ to show that $C = 0$.

C07S05.073: If

$$f(x) = \arctan(x^2-1)^{1/2} \quad \text{for} \quad x > 1,$$

then

$$f'(x) = \frac{1}{1+(x^2-1)} \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x = \frac{x}{x^2(x^2-1)^{1/2}} = \frac{1}{x\sqrt{x^2-1}}.$$

Therefore $\operatorname{arcsec} x = C + \arctan(x^2-1)^{1/2}$ for some constant C for all $x > 1$. Now substitute $x = \sqrt{2}$ to show that $C = 0$.

If

$$g(x) = \pi - \arctan(x^2-1)^{1/2} \quad \text{for} \quad x < -1,$$

then (using the earlier result)

$$g'(x) = -\frac{1}{x\sqrt{x^2-1}}.$$

Therefore

$$\operatorname{arcsec} x = C + \pi - \arctan\left(\sqrt{x^2-1}\right)$$

for some constant C for all $x < -1$. Now substitute $x = -\sqrt{2}$ to show that $C = 0$.

C07S05.074: The graph of f on $[0.001, 5]$ indicates a global maximum near $x = 1.4$. We used Newton's method to show that its location is close to (1.3917452003, 0.8033644570).

C07S05.075: The graph of f on $[0, 8]$ (following this solution) indicates a global maximum near $x = 2.7$. We used Newton's method to show that its location is close to (2.6892200292, 0.9283427321).

C07S05.076: The graph of f on $[1, 30]$ indicates a global maximum near $x = 8.3$. We used Newton's method to show that its location is close to (8.3332645728, 1.3345303526).

Section 7.6

C07S06.001: If $f(x) = \cosh(3x - 2)$, then $f'(x) = 3\sinh(3x - 2)$.

C07S06.002: If $f(x) = \sinh\sqrt{x}$, then $f'(x) = \left(\cosh x^{1/2}\right) \cdot D_x\left(x^{1/2}\right) = \dfrac{\cosh\sqrt{x}}{2\sqrt{x}}$.

C07S06.003: If $f(x) = x^2 \tanh\left(\dfrac{1}{x}\right)$, then

$$f'(x) = 2x\tanh\left(\frac{1}{x}\right) + \left[x^2 \operatorname{sech}^2\left(\frac{1}{x}\right)\right] \cdot \left[-\frac{1}{x^2}\right] = 2x\tanh\left(\frac{1}{x}\right) - \operatorname{sech}^2\left(\frac{1}{x}\right).$$

C07S06.004: If $f(x) = \operatorname{sech} e^{2x}$, then $f'(x) = -2e^{2x}\operatorname{sech} e^{2x}\ \tanh e^{2x}$.

C07S06.005: If $f(x) = \coth^3 4x = (\coth 4x)^3$, then

$$f'(x) = 4 \cdot (3\coth 4x)^2\left[-(\operatorname{csch} 4x)^2\right] = -12\coth^2 4x\ \operatorname{csch}^2 4x.$$

C07S06.006: If $f(x) = \ln\sinh 3x$, then

$$f'(x) = \frac{1}{\sinh 3x}(\cosh 3x) \cdot 3 = \frac{3\cosh 3x}{\sinh 3x} = 3\coth 3x.$$

C07S06.007: If $f(x) = e^{\operatorname{csch} x}$, then $f'(x) = e^{\operatorname{csch} x} \cdot D_x \operatorname{csch} x = -e^{\operatorname{csch} x}\operatorname{csch} x \coth x$.

C07S06.008: If $f(x) = \cosh\ln x = \cosh(\ln x)$, then

$$f'(x) = (\sinh\ln x) \cdot D_x(\ln x) = \frac{\sinh(\ln x)}{x} = \frac{x - x^{-1}}{2x} = \frac{x^2 - 1}{2x^2}.$$

C07S06.009: If $f(x) = \sin(\sinh x)$, then

$$f'(x) = [\cos(\sinh x)] \cdot D_x(\sinh x) = (\cosh x)\ \cos(\sinh x).$$

C07S06.010: If $f(x) = \tan^{-1}(\tanh x)$, then

$$f'(x) = \frac{1}{1 + \tanh^2 x} \cdot D_x(\tanh x) = \frac{\operatorname{sech}^2 x}{1 + \tanh^2 x} = \frac{1 - \tanh^2 x}{1 + \tanh^2 x} = \frac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}.$$

C07S06.011: If $f(x) = \sinh x^4 = \sinh(x^4)$, then $f'(x) = 4x^3\cosh x^4$.

C07S06.012: If $f(x) = \sinh^4 x = (\sinh x)^4$, then $f'(x) = 4\sinh^3 x\ \cosh x$.

C07S06.013: If $f(x) = \dfrac{1}{x + \tanh x}$, then (by the reciprocal rule)

$$f'(x) = -\frac{1 + \operatorname{sech}^2 x}{(x + \tanh x)^2} = \frac{(\tanh^2 x) - 2}{(x + \tanh x)^2}.$$

C07S06.014: If $f(x) = \cosh^2 x - \sinh^2 x$, then $f(x) \equiv 1$ by Eq. (4), and therefore $f'(x) \equiv 0$. Alternatively, $f'(x) = 2\cosh x\, \sinh x - 2\sinh x\, \cosh x \equiv 0$.

C07S06.015: If necessary, use the substitution $u = x^2$, $du = 2x\, dx$ to show that

$$\int x \sinh x^2\, dx = \frac{1}{2} \cosh x^2 + C.$$

C07S06.016: By Eq. (11), $\cosh^2 3u = \frac{1}{2}(\cosh 6u + 1)$. Therefore

$$\int \cosh^2 3u\, dx = \frac{1}{12} \sinh 6u + \frac{1}{2} u + C = \frac{1}{6} \sinh 3u\, \cosh 3u + \frac{1}{2} u + C$$

(the last equality by Eq. (9)).

C07S06.017: By Eq. (5) we have

$$\int \tanh^2 3x\, dx = \int \left(1 - \operatorname{sech}^2 3x\right) dx = x - \frac{1}{3} \tanh 3x + C.$$

C07S06.018: Let $u = \sqrt{x} = x^{1/2}$. Then $x = u^2$, so $dx = 2u\, du$. Therefore (with the aid of Eq. (17))

$$\int \frac{\operatorname{sech} \sqrt{x}\, \tanh \sqrt{x}}{\sqrt{x}}\, dx = \int \frac{\operatorname{sech} u\, \tanh u}{u} \cdot 2u\, du = -2 \operatorname{sech} u + C = -2 \operatorname{sech} \sqrt{x} + C.$$

C07S06.019: Let $u = \sinh 2x$. Then $du = 2\cosh 2x\, dx$, so

$$\int \sinh^2 2x\, \cosh 2x\, dx = \int \frac{1}{2} u^2\, du = \frac{1}{6} u^3 + C = \frac{1}{6} \sinh^3 2x + C.$$

C07S06.020: $\displaystyle\int \tanh 3x\, dx = \int \frac{\sinh 3x}{\cosh 3x}\, dx = \frac{1}{3} \ln(\cosh 3x) + C.$

C07S06.021: $\displaystyle\int (\cosh x)^{-3} \sinh x\, dx = \frac{(\cosh x)^{-2}}{-2} + C = -\frac{1}{2} \operatorname{sech}^2 x + C.$

C07S06.022: By Eqs. (11) and (12),

$$\begin{aligned}
\sinh^4 x = \left(\sinh^2 x\right)^2 &= \left[\frac{1}{2}(\cosh 2x - 1)\right]^2 = \frac{1}{4}\left(\cosh^2 2x - 2\cosh 2x + 1\right) \\
&= \frac{1}{4}\left[\frac{1}{2}(\cosh 4x + 1) - 2\cosh 2x + 1\right] = \frac{3}{8} - \frac{1}{2}\cosh 2x + \frac{1}{8}\cosh 4x.
\end{aligned}$$

Therefore

$$\int \sinh^4 x\, dx = \frac{1}{32}(\sinh 4x - 8\sinh 2x + 12x) + C.$$

C07S06.023: Let $u = \coth x$. Then $du = -\operatorname{csch}^2 x\, dx$. So

$$\int \coth x\, \operatorname{csch}^2 x\, dx = -\int u\, du = -\frac{1}{2} u^2 + C = -\frac{1}{2} \coth^2 x + C = -\frac{1}{2} \operatorname{csch}^2 x + C_1.$$

C07S06.024: This is not easy. Here's one solution:

$$\int \operatorname{sech} x\, dx = \int \frac{2}{e^x + e^{-x}}\, dx = \int \frac{2e^x}{1+(e^x)^2}\, dx = 2\arctan\left(e^x\right) + C.$$

If multiplication of numerator and denominator by e^x seems artificial, try a substitution: Let $u = e^x$, so that $du = e^x\, dx$, and thus $dx = e^{-x}\, du = u^{-1}\, du$. Then

$$\int \operatorname{sech} x\, dx = \int \frac{2}{e^x + e^{-x}}\, dx = \int \frac{2u^{-1}}{u + u^{-1}}\, du = \int \frac{2}{u^2+1}\, du = 2\arctan u + C = 2\arctan\left(e^x\right) + C.$$

Mathematica 3.0 apparently uses a rationalizing substitution like that found in the discussion following Miscellaneous Problem 134 of Chapter 7 (but adapted to hyperbolic rather than trigonometric integrals); it obtains

$$\int \operatorname{sech} x\, dx = 2\arctan\left(\tanh\frac{x}{2}\right) + C.$$

See also Problem 21 of Section 7.5 and its solution for yet another antiderivative.

C07S06.025: If necessary, let $u = 1 + \cosh x$ or let $u = \cosh x$. But simply "by inspection,"

$$\int \frac{\sinh x}{1+\cosh x}\, dx = \ln\left(1+\cosh x\right) + C.$$

C07S06.026: Let $u = \ln x$. Then $du = \dfrac{1}{x}\, dx$. So

$$\int \frac{\sinh(\ln x)}{x}\, dx = \int \sinh u\, du = \cosh u\, + C = \cosh(\ln x) + C.$$

Alternatively, first simplify the integrand:

$$\frac{\sinh(\ln x)}{x} = \frac{\exp(\ln x) - \exp(-\ln x)}{2x} = \frac{x - x^{-1}}{2x} = \frac{x^2-1}{2x^2} = \frac{1}{2}\left(1 - \frac{1}{x^2}\right).$$

Then

$$\int \frac{\sinh(\ln x)}{x}\, dx = \frac{1}{2}\int \left(1 - \frac{1}{x^2}\right) dx = \frac{1}{2}\left(x + \frac{1}{x}\right) + C = \frac{x^2+1}{2x} + C.$$

C07S06.027: One solution:

$$\int \frac{1}{(e^x+e^{-x})^2}\, dx = \frac{1}{4}\int \left(\frac{2}{e^x+e^{-x}}\right)^2 dx = \frac{1}{4}\int \operatorname{sech}^2 x\, dx = \frac{1}{4}\tanh x\, + C.$$

Another solution: Let $u = e^x$, so that $du = e^x\, dx$; thus $dx = e^{-x}\, du = \dfrac{1}{u}\, du$. Then

$$\begin{aligned}\int \frac{1}{(e^x+e^{-x})^2}\, dx &= \int \frac{u^{-1}}{(u+u^{-1})^2}\, du = \int \frac{u^{-1}}{u^2+2+u^{-2}}\, dx \\ &= \int \frac{u}{u^4+2u^2+1}\, dx = \int u(u^2+1)^{-2}\, du = -\frac{1}{2}(u^2+1)^{-1} + C \\ &= -\frac{1}{2}\cdot\frac{1}{e^{2x}+1} + C = -\frac{1}{2}\cdot\frac{e^{-x}}{e^x+e^{-x}} + C = -\frac{1}{4e^x}\cdot\frac{2}{e^x+e^{-x}} + C = -\frac{\operatorname{sech} x}{4e^x} + C.\end{aligned}$$

The two answers appear quite different, but they are both correct (although they differ by the constant $\frac{1}{4}$).

C07S06.028: Let $u = e^x - e^{-x}$ if you wish, but by inspection we find

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left|e^x - e^{-x}\right| + C.$$

You may also continue the previous calculations as follows:

$$\cdots = \ln\left|e^x - e^{-x}\right| - \ln 2 + C_1 = \ln\left|\frac{e^x - e^{-x}}{2}\right| + C_1 = \ln|\sinh x| + C_1.$$

Mathematica 3.0 obtains

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = -x + \ln|-1 + e^{2x}| + C.$$

C07S06.029: $f'(x) = \dfrac{1}{\sqrt{1+4x^2}} \cdot D_x\,(2x) = \dfrac{2}{\sqrt{1+4x^2}}.$

C07S06.030: $f'(x) = \dfrac{1}{\sqrt{(x+1)^2-1}} \cdot D_x\left(x^2 - 1\right) = \dfrac{2x}{\sqrt{x^4+2x^2}} = \dfrac{2x}{|x|\sqrt{x^2+2}}.$

C07S06.031: $f'(x) = \dfrac{1}{1-(\sqrt{x}\,)^2} \cdot D_x\left(x^{1/2}\right) = \dfrac{1}{2(1-x)\sqrt{x}}.$

C07S06.032: If $f(x) = \coth^{-1}\left(x^2+1\right)^{1/2}$, then

$$\begin{aligned} f'(x) &= \frac{1}{1-\left[(x^2+1)^{1/2}\right]^2} \cdot D_x\left(x^2+1\right)^{1/2} \\ &= \frac{1}{1-x^2-1} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = -\frac{1}{x^2} \cdot \frac{x}{\sqrt{x^2+1}} = -\frac{1}{x\sqrt{x^2+1}}. \end{aligned}$$

(Compare this result with Eq. (33).)

C07S06.033: If $f(x) = \operatorname{sech}^{-1}\left(\dfrac{1}{x}\right)$, then

$$\begin{aligned} f'(x) &= -\frac{1}{\dfrac{1}{x}\sqrt{1-\dfrac{1}{x^2}}} \cdot D_x\left(\frac{1}{x}\right) \\ &= -\frac{x}{\sqrt{1-\dfrac{1}{x^2}}} \cdot \left(-\frac{1}{x^2}\right) = \frac{x}{x^2\sqrt{1-\dfrac{1}{x^2}}} = \frac{x}{|x|\,\sqrt{x^2-1}}. \end{aligned}$$

(You need to write x^2 in the form $|x|^2$ in order to move one copy of $|x|$ underneath the radical, where it becomes x^2. Compare this result with Eq. (29).)

C07S06.034: If $f(x) = \operatorname{csch}^{-1} e^x$, then

$$f'(x) = -\frac{1}{|e^x|\,\sqrt{1+e^{2x}}} \cdot e^x = -\frac{1}{\sqrt{1+e^{2x}}}.$$

C07S06.035: If $f(x) = \left(\sinh^{-1} x\right)^{3/2}$, then

$$f'(x) = \frac{3}{2}\left(\sinh^{-1} x\right)^{1/2} \cdot D_x\left(\sinh^{-1} x\right) = \frac{3\left(\sinh^{-1} x\right)^{1/2}}{2(1+x^2)^{1/2}}.$$

C07S06.036: If $f(x) = \sinh^{-1}(\ln x)$, then

$$f'(x) = \frac{1}{\sqrt{1+(\ln x)^2}} \cdot D_x(\ln x) = \frac{1}{x\sqrt{1+(\ln x)^2}}.$$

C07S06.037: If $f(x) = \ln\left(\tanh^{-1} x\right)$, then

$$f'(x) = \frac{1}{\tanh^{-1} x} \cdot D_x\left(\tanh^{-1} x\right) = \frac{1}{(1-x^2)\tanh^{-1} x}.$$

C07S06.038: If $f(x) = \dfrac{1}{\tanh^{-1} 3x}$, then

$$f'(x) = -\frac{1}{\left(\tanh^{-1} 3x\right)^2} \cdot D_x\left(\tanh^{-1} x\right) = -\frac{3}{(1-9x^2)\left(\tanh^{-1} 3x\right)^2}.$$

C07S06.039: Let $x = 3u$. Then $dx = 3\,du$, so

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{3}{\sqrt{9u^2+9}}\,du = \int \frac{1}{\sqrt{u^2+1}}\,du = \text{arcsinh}\,u + C = \text{arcsinh}\,\frac{x}{3} + C.$$

C07S06.040: Let $y = \dfrac{3}{2}u$. Then $dy = \dfrac{3}{2}\,du$, so

$$\int \frac{1}{\sqrt{4y^2-9}}\,dy = \frac{3}{2}\int \frac{1}{\sqrt{9u^2-9}}\,du = \frac{1}{2}\int \frac{1}{\sqrt{u^2-1}}\,du = \frac{1}{2}\text{arccosh}\,u + C = \frac{1}{2}\text{arccosh}\left(\frac{2}{3}y\right) + C.$$

C07S06.041: Let $x = 2u$. Then $dx = 2\,du$, so

$$I = \int_{1/2}^{1} \frac{1}{4-x^2}\,dx = \int_{x=1/2}^{1} \frac{2}{4-4u^2}\,du = \frac{1}{2}\int_{x=1/2}^{1} \frac{1}{1-u^2}\,du$$

$$= \frac{1}{2}\Big[\tanh^{-1} u\Big]_{x=1/2}^{1} = \frac{1}{2}\left[\tanh^{-1}\left(\frac{x}{2}\right)\right]_{1/2}^{1} = \frac{1}{2}\left[\tanh^{-1}\left(\frac{1}{2}\right) - \tanh^{-1}\left(\frac{1}{4}\right)\right].$$

Now use Eq. (36) to transform the answer into a more familiar form:

$$I = \frac{1}{4}\left(\ln\frac{1+\frac{1}{2}}{1-\frac{1}{2}} - \ln\frac{1+\frac{1}{4}}{1-\frac{1}{4}}\right) = \frac{1}{4}\left(\ln 3 - \ln\frac{5}{3}\right) = \frac{1}{4}\ln\frac{9}{5} \approx 0.1469466662.$$

C07S06.042: Use the substitution in the solution of Problem 41, but now we must use Eq. (42b) rather than Eq. (42a):

$$I = \int_5^{10} \frac{1}{4 - x^2}\,dx = \frac{1}{2}\left[\coth^{-1}\left(\frac{x}{2}\right)\right]_5^{10} = \frac{1}{2}\left[\coth^{-1}(5) - \coth^{-1}\left(\frac{5}{2}\right)\right]$$

$$= \frac{1}{4}\left(\ln\frac{5+1}{5-1} - \ln\frac{\frac{5}{2}+1}{\frac{5}{2}-1}\right) = \frac{1}{4}\left(\ln\frac{3}{2} - \ln\frac{7}{3}\right) = \frac{1}{4}\ln\frac{9}{14} \approx -0.1104581881.$$

C07S06.043: Let $x = \dfrac{2}{3}u$. Then $\sqrt{4 - 9x^2} = \sqrt{4 - 4u^2}$ and $dx = \dfrac{2}{3}\,du$. Therefore

$$\int \frac{1}{x\sqrt{4 - 9x^2}}\,dx = \frac{2}{3}\int \frac{1}{\frac{4}{3}u\sqrt{1 - u^2}}\,du$$

$$= \frac{1}{2}\int \frac{1}{u\sqrt{1 - u^2}}\,du = -\frac{1}{2}\operatorname{sech}^{-1}|u| + C = -\frac{1}{2}\operatorname{sech}^{-1}\left|\frac{3}{2}x\right| + C.$$

C07S06.044: Let $x = 5u$: $dx = 5\,du$, $\sqrt{x^2 + 25} = 5\sqrt{u^2 + 1}$. So

$$\int \frac{1}{x\sqrt{x^2 + 25}}\,dx = \int \frac{5}{25u\sqrt{u^2 + 1}}\,du$$

$$= \frac{1}{5}\int \frac{1}{u\sqrt{u^2 + 1}}\,du = -\frac{1}{5}\operatorname{csch}^{-1}|u| + C = -\frac{1}{5}\operatorname{csch}^{-1}\left|\frac{x}{5}\right| + C.$$

C07S06.045: Let $u = e^x$: $du = e^x\,dx$. Hence

$$\int \frac{e^x}{\sqrt{e^{2x} + 1}}\,dx = \int \frac{1}{\sqrt{u^2 + 1}}\,du = \sinh^{-1} u + C = \sinh^{-1}(e^x) + C.$$

C07S06.046: Let $u = x^2$: $du = 2x\,dx$ and $\sqrt{x^4 - 1} = \sqrt{u^2 - 1}$. So

$$\int \frac{x}{\sqrt{x^4 - 1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u^2 - 1}}\,du = \frac{1}{2}\cosh^{-1} u + C = \frac{1}{2}\cosh^{-1}\left(x^2\right) + C.$$

C07S06.047: Let $u = e^x$: $x = \ln u$ and $dx = \dfrac{1}{u}\,du$. Thus

$$\int \frac{1}{\sqrt{1 - e^{2x}}}\,dx = \int \frac{1}{u\sqrt{1 - u^2}}\,du = -\operatorname{sech}^{-1}|u| + C = -\operatorname{sech}^{-1}(e^x) + C.$$

C07S06.048: Let $u = \sin x$: $du = \cos x\,dx$. Therefore

$$\int \frac{\cos x}{\sqrt{1 + \sin^2 x}}\,dx = \int \frac{1}{\sqrt{1 + u^2}}\,du = \sinh^{-1} u + C = \sinh^{-1}(\sin x) + C.$$

C07S06.049: $\sinh x\,\cosh y + \cosh x\,\sinh y - \sinh(x + y)$

$$= \frac{e^x - e^{-x}}{2}\cdot\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\cdot\frac{e^y - e^{-y}}{2} - \frac{e^{x+y} - e^{-x-y}}{2}$$

$$= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y} - 2e^{x+y} + 2e^{-x-y}\right) = 0.$$

Therefore $\sinh(x + y) = \sinh x\,\cosh y + \cosh x\,\sinh y$ for all real numbers x and y.

C07S06.050: Given $\cosh^2 x - \sinh^2 x = 1$ (for all x), we divide both sides by $\cosh^2 x$ (which is never zero) to find that

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}; \qquad \text{that is,} \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

(for all x). If we instead divide by sides by $\sinh^2 x$ (which is zero only when $x = 0$), we find that

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x}; \qquad \text{that is,} \qquad \coth^2 x - 1 = \operatorname{csch}^2 x$$

if $x \neq 0$.

C07S06.051: Substitute $y = x$ in the identity

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

to prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$. Then, with the aid of Eq. (4), we find that

$$\cosh 2x = 2\cosh^2 x - 1, \qquad \text{so that} \qquad \cosh^2 x = \frac{1}{2}(1 + \cosh 2x).$$

C07S06.052: First, $x'(t) = kA \sinh kt + kB \cosh kt$, and therefore

$$x''(t) = k^2 A \cosh kt + k^2 B \sinh kt = k^2 x(t).$$

C07S06.053: The length is

$$L = \int_0^a \sqrt{1 + \sinh^2 x}\ dx = \int_0^a \cosh x\ dx = \Big[\sinh x\Big]_0^a = \sinh a.$$

C07S06.054: The volume is

$$\begin{aligned} V &= \int_0^\pi \pi \sinh^2 x\ dx = \frac{\pi}{4}\int_0^\pi \left(e^{2x} - 2 + e^{-2x}\right) dx = \frac{\pi}{4}\left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_0^\pi \\ &= \frac{\pi}{4}\left(\frac{1}{2}e^{2\pi} - 2\pi - \frac{1}{2}e^{-2\pi} - \frac{1}{2} + \frac{1}{2}\right) = \frac{\pi}{4}(-2\pi + \sinh 2\pi) \approx 205.3515458383. \end{aligned}$$

C07S06.055: Beginning with the equation

$$A(\theta) = \frac{1}{2}\cosh\theta \sinh\theta - \int_1^{\cosh\theta} (x^2 - 1)^{1/2}\ dx,$$

we take the derivative of each side with respect to θ (using the fundamental theorem of calculus on the right-hand side). The result is

$$\begin{aligned} A'(\theta) &= \frac{1}{2}\cosh^2\theta + \frac{1}{2}\sinh^2\theta - \Big[(\cosh^2\theta - 1)^{1/2}\sinh\theta\Big] \\ &= \frac{1}{2}\cosh^2\theta + \frac{1}{2}\sinh^2\theta - \sinh^2\theta = \frac{1}{2}\left(\cosh^2\theta - \sinh^2\theta\right) = \frac{1}{2}. \end{aligned}$$

Therefore $A(\theta) = \frac{1}{2}\theta + C$ for some constant C. Evaluation of both sides of this equation when $\theta = 0$ yields the information that $C = 0$, and therefore

$$A(\theta) = \frac{1}{2}\theta.$$

C07S06.056: By l'Hôpital's rule,

$$\lim_{x\to 0} \frac{\sinh x}{x} = \lim_{x\to 0} \frac{\cosh x}{1} = \cosh 0 = 1.$$

We do not require l'Hôpital's rule for the other two limits:

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1;$$

$$\lim_{x\to\infty} \frac{\cosh x}{e^x} = \lim_{x\to\infty} \frac{e^x + e^{-x}}{2e^x} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}.$$

C07S06.057: Let $y = \sinh^{-1} 1$. Then

$$1 = \sinh y = \frac{e^y - e^{-y}}{2}; \qquad e^y - e^{-y} = 2;$$

$$e^{2y} - 2e^y - 1 = 0; \qquad u^2 - 2u - 1 = 0 \quad \text{where} \quad u = e^y;$$

$$u = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

But $u = e^y > 0$, so $u = 1 + \sqrt{2}$. Hence

$$\sinh^{-1} 1 = y = \ln u = \ln\left(1 + \sqrt{2}\right) \approx 0.8813735870.$$

C07S06.058: If $x \neq 0$, then by Eq. (34),

$$\sinh^{-1} \frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right) = \operatorname{csch}^{-1} x$$

by Eq. (39).

C07S06.059: Let $y = \sinh^{-1} x$ and remember that $\cosh y > 0$ for all y. Hence

$$\sinh y = x; \qquad (\cosh y)\,\frac{dy}{dx} = 1;$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\cosh^2 y}} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

C07S06.060: Let $y = \operatorname{sech}^{-1} x,\ 0 < x \leqq 1$. Recall that $\operatorname{sech} y > 0$ for all y. Hence:

$$\operatorname{sech} y = x; \qquad (-\operatorname{sech} y\,\tanh y)\,\frac{dy}{dx} = 1;$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \, \tanh y} = \pm\frac{1}{x\sqrt{1-\operatorname{sech}^2 y}} = \pm\frac{1}{x\sqrt{1-x^2}}.$$

Now $x > 0$ but $\dfrac{dy}{dx} < 0$ (because $y = \operatorname{sech}^{-1} x$ is decreasing on $(0,\,1)$). Therefore

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}, \qquad 0 < x < 1.$$

C07S06.061: Let

$$f(x) = \tanh^{-1} x \qquad \text{and} \qquad g(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

for $-1 < x < 1$. Equation (36) states that $f(x) = g(x)$. To prove this, note that

$$f'(x) = \frac{1}{1-x^2}$$

and $g(x) = \dfrac{1}{2}\ln(1+x) - \dfrac{1}{2}\ln(1-x)$, so that

$$g'(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1-x+1+x}{2(1-x^2)} = \frac{1}{1-x^2},$$

and therefore $f'(x) = g'(x)$. Hence $f(x) = g(x) + C$ for some constant C and for all x in $(-1,\,1)$. To evaluate C, note that

$$0 = f(x) = \tanh^{-1} 0 = g(0) + C = \frac{1}{2}\ln 1 + C = 0 + C = C.$$

Therefore $f(x) = g(x)$ if $-1 < x < 1$.

C07S06.062: Given: $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$:

$$e^y - e^{-y} = 2x; \qquad\qquad e^{2y} - 2xe^y - 1 = 0;$$

$$u^2 - 2xu - 1 = 0 \quad \text{where} \quad u = e^y; \qquad u = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1};$$

$$u = x + \sqrt{x^2+1} \quad \text{because } e^y > 0.$$

Therefore $y = \ln u = \ln\left(x + \sqrt{x^2+1}\right)$ for all real x.

C07S06.063: Let $u = e^y$. Then

$$x = \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{e^{2y}+1}{e^{2y}-1} = \frac{u^2+1}{u^2-1}.$$

Consequently

$$u^2 + 1 = xu^2 - x; \qquad (x-1)u^2 = x+1; \qquad u^2 = \frac{x+1}{x-1}.$$

Therefore $y = \ln u = \dfrac{1}{2}\ln u^2 = \dfrac{1}{2}\ln\dfrac{x+1}{x-1}$ for all x such that $|x| > 1$.

C07S06.064: Let $f(x) = \coth^{-1} x$ and $g(x) = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$ for $|x| > 1$. Then

$$f'(x) = \frac{1}{1-x^2}$$

and $g(x) = \dfrac{1}{2}\ln(x+1) - \dfrac{1}{2}\ln(x-1)$, so

$$g'(x) = \frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) = \frac{(x-1)-(x+1)}{2(x^2-1)} = -\frac{1}{x^2-1} = \frac{1}{1-x^2}.$$

Therefore there are constants C_1 and C_2 such that

$$f(x) = g(x) + C_1 \quad \text{if} \quad x > 1 \qquad \text{and} \qquad f(x) = g(x) + C_2 \quad \text{if} \quad x < -1.$$

Let us now express $y = \coth^{-1} 2$ in a more manageable form.

$$\begin{aligned}
&\coth y = 2; && \frac{e^y + e^{-y}}{e^y - e^{-y}} = 2;\\
&e^y + e^{-y} = 2e^y - 2e^{-y}; && 3e^{-y} = e^y;\\
&e^{2y} = 3; && 2y = \ln 3;\\
&y = \frac{1}{2}\ln 3; && \coth^{-1} 2 = \frac{1}{2}\ln 3.
\end{aligned}$$

Thus

$$C_1 = f(2) - g(2) = \frac{1}{2}\ln 3 - \frac{1}{2}\ln\frac{2+1}{2-1} = 0.$$

Therefore Eq. (37) holds for all $x > 1$. Repeat this argument, unchanged except for a few minus signs, with $y = \coth^{-1}(-2)$ to show that $C_2 = 0$ as well. This establishes Eq. (37).

C07S06.065: Let

$$f(x) = \operatorname{csch}^{-1} x \qquad \text{and} \qquad g(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

for $x \neq 0$. Then

$$f'(x) = -\frac{1}{|x|\sqrt{1+x^2}}.$$

If $x > 0$, then $g(x) = \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$. Thus

$$\begin{aligned}
g'(x) &= \frac{x}{1+(1+x^2)^{1/2}} \cdot \frac{x \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x - 1 - (1+x^2)^{1/2}}{x^2}\\
&= \frac{1}{1+(1+x^2)^{1/2}} \cdot \frac{x^2(1+x^2)^{-1/2} - 1 - (1+x^2)^{1/2}}{x}\\
&= \frac{1}{1+(1+x^2)^{1/2}} \cdot \frac{x^2 - (1+x^2)^{1/2} - 1 - x^2}{x(1+x^2)^{1/2}}\\
&= -\frac{1}{x(1+x^2)^{1/2}} = -\frac{1}{|x|\sqrt{1+x^2}}.
\end{aligned}$$

But if $x < 0$, then $g(x) = \ln\left(\dfrac{1 - \sqrt{1+x^2}}{x}\right)$, and hence

$$\begin{aligned}
g'(x) &= \frac{x}{1-(1+x^2)^{1/2}} \cdot \frac{-x \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x - 1 + (1+x^2)^{1/2}}{x^2} \\
&= \frac{1}{1-(1+x^2)^{1/2}} \cdot \frac{-x^2(1+x^2)^{-1/2} - 1 + (1+x^2)^{1/2}}{x} \\
&= \frac{1}{1-(1+x^2)^{1/2}} \cdot \frac{-x^2 - (1+x^2)^{1/2} + 1 + x^2}{x(1+x^2)^{1/2}} \\
&= \frac{1}{x(1+x^2)^{1/2}} = -\frac{1}{|x|\sqrt{1+x^2}}.
\end{aligned}$$

Therefore there exist constants C_1 and C_2 such that

$$f(x) = g(x) + C_1 \quad \text{if} \quad x > 0 \qquad \text{and} \qquad f(x) = g(x) + C_2 \quad \text{if} \quad x < 0.$$

Let us now express $u = \operatorname{csch}^{-1} 1$ in a more useful way.

$$\begin{aligned}
&\operatorname{csch} u = 1; &&\frac{2}{e^u - e^{-u}} = 1; \\
&e^u - e^{-u} = 2; &&e^{2u} - 1 = 2e^u; \\
&e^{2u} - 2e^u - 1 = 0; &&e^u = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}\,.
\end{aligned}$$

Therefore $e^u = 1 + \sqrt{2}$, and so $f(1) = \operatorname{csch}^{-1} 1 = u = \ln\left(1+\sqrt{2}\right)$. But

$$g(1) = \ln\left(\frac{1}{1} + \frac{\sqrt{2}}{1}\right) = \ln\left(1+\sqrt{2}\right).$$

Therefore $C_1 = 0$, and so

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

if $x > 0$. This argument may be repeated for $u = \operatorname{csch}^{-1}(-1)$ with few changes other than minus signs here and there, and it follows that $C_2 = 0$ as well. This establishes Eq. (39).

C07S06.066: A plot of $f(x) = x + 2$ and $g(x) = \cosh x$ for $-1 \leqq x \leqq 2.5$ reveals intersections near $x = -0.7$ and $x = 2.1$. We applied Newton's method to the equation $f(x) - g(x) = 0$ and found that the curves cross very close to the two points

$$(-0.7252637249,\ 1.2747362751) \quad \text{and} \quad (2.0851860142,\ 4.0851860142).$$

With a the abscissa of the first of these points and b the abscissa of the second, the area between the two curves is

$$A = \int_a^b [f(x) - g(x)]\ dx = \left[\frac{1}{2}(4x + x^2 - 2\sinh x)\right]_a^b \approx 2.7804546672.$$

C07S06.067: Given: $f(x) = e^{-2x} \tanh x$. First,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^{2x}(e^x + e^{-x})} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{e^{2x} + 1} = 0.$$

Next, a plot of $y = f(x)$ for $0 \leqq x \leqq 1$ reveals a local maximum near where $x = 0.45$. We applied Newton's method to solve $f'(x) = 0$ numerically and after seven iterations found that the maximum is close to the point $(0.4406867935, 0.1715728753)$. Indeed,

$$f'(x) = -\frac{e^{4x} - 2e^{2x} - 1}{2e^{2x}(e^{2x} + 1)^2},$$

so $f'(x) = 0$ when $e^{4x} - 2e^{2x} - 1 = 0$; that is, when

$$e^{2x} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}, \qquad \text{so that} \qquad x = \frac{1}{2}\ln\left(1 + \sqrt{2}\right) \approx 0.4406867935.$$

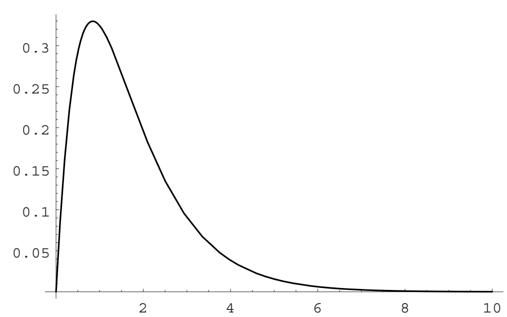
C07S06.068: If $f(x) = e^{-x} \sinh^{-1} x$, then by l'Hôpital's rule,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sinh^{-1} x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x\sqrt{1+x^2}} = 0.$$

Next, a plot of $y = f(x)$ for $0 \leqq x \leqq 2$ reveals a local maximum near where $x = 0.85$. The equation $f'(x) = 0$ is transcendental and we were unable to solve it exactly, but Newton's method revealed that the high point on the graph of f is close to $(0.8418432341, 0.3296546569)$. Note: To solve $f'(x) = 0$ you must solve

$$\frac{1}{2}\ln(x^2 + 1) = x + \frac{1}{\sqrt{x^2+1}}.$$

The graph of $y = f(x)$ for $0 \leqq x \leqq 10$ is next. It provides convincing evidence that the extremum we found is a global maximum.



C07S06.069: Given: $y(x) = y_0 + \dfrac{1}{k}(-1 + \cosh kx)$. Then

$$\frac{dy}{dx} = \sinh kx \qquad \text{and} \qquad \frac{d^2y}{dx^2} = k\cosh kx.$$

So

$$k\sqrt{1 + [y'(x)]^2} = k\sqrt{1 + \sinh^2 kx} = k\sqrt{\cosh^2 kx} = k\cosh kx.$$

Therefore $\dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Moreover,

$$y(0) = y_0 + \frac{1}{k}(-1 + \cosh 0) = y_0 + \frac{0}{k} = y_0$$

and

$$y'(0) = \sinh(k \cdot 0) = \sinh(0) = 0.$$

C07S06.070: Using the coordinate system in Fig. 7.6.4 with units in feet, the cable has the shape of the graph of

$$y(x) = 30 + \frac{1}{k}(-1 + \cosh kx).$$

We also know that

$$50 = y(100) = 30 + \frac{1}{k}(-1 + \cosh 100k),$$

and it follows that $g(k) = 0$ where $g(k) = \cosh(100k) - 20k - 1$. A plot of $y = g(x)$ reveals a solution near $x = 0.004$, and Newton's method reveals the more accurate approximation $k \approx 0.003948435453$. We then found that the approximate length of the high-voltage line is

$$L = 2\int_0^{100} \sqrt{1 + [y'(x)]^2}\ dx = 2\int_0^{100} \cosh kx\ dx = 2 \cdot \left[\frac{1}{k}\sinh kx\right]_0^{100} \approx 205.2373736258 \quad \text{(ft)}.$$

Chapter 7 Miscellaneous Problems

C07S0M.001: If $f(x) = \cos(1 - e^{-x})$, then $f'(x) = -e^{-x}\sin(1 - e^{-x})$.

C07S0M.002: If $f(x) = \sin^2(e^{-x}) = [\sin(e^{-x})]^2$, then $f'(x) = -2e^{-x}\sin(e^{-x})\,\cos(e^{-x})$.

C07S0M.003: If $f(x) = \ln(x + e^{-x})$, then $f'(x) = \dfrac{1 - e^{-x}}{x + e^{-x}}$.

C07S0M.004: If $f(x) = e^x \cos 2x$, then $f'(x) = e^x \cos 2x - 2e^x \sin 2x$.

C07S0M.005: If $f(x) = e^{-2x}\sin 3x$, then $f'(x) = 3e^{-2x}\cos 3x - 2e^{-2x}\sin 3x$.

C07S0M.006: If $g(t) = \ln(te^{t^2}) = (\ln t) + t^2 \ln e = (\ln t) + t^2$, then

$$g'(t) = \frac{1}{t} + 2t = \frac{1 + 2t^2}{t}.$$

C07S0M.007: If $g(t) = 3(e^t - \ln t)^5$, then $g'(t) = 15(e^t - \ln t)^4\left(e^t - \dfrac{1}{t}\right)$.

C07S0M.008: If $g(t) = \sin(e^t)\,\cos(e^{-t})$, then

$$g'(t) = e^t\cos(e^t)\,\cos(e^{-t}) + e^{-t}\sin(e^t)\,\sin(e^{-t}).$$

C07S0M.009: If $f(x) = \dfrac{2 + 3x}{e^{4x}}$, then

$$f'(x) = \frac{3e^{4x} - 4(2 + 3x)e^{4x}}{(e^{4x})^2} = \frac{3 - 8 - 12x}{e^{4x}} = -\frac{12x + 5}{e^{4x}}.$$

C07S0M.010: If $g(t) = \dfrac{1 + e^t}{1 - e^t}$, then $g'(t) = \dfrac{(1 - e^t)e^t + (1 + e^t)e^t}{(1 - e^t)^2} = \dfrac{2e^t}{(1 - e^t)^2}$.

C07S0M.011: Given $xe^y = y$, we apply D_x to both sides and find that

$$e^y + xe^y\frac{dy}{dx} = \frac{dy}{dx}; \qquad (1 - xe^y)\frac{dy}{dx} = e^y;$$
$$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}; \qquad \frac{dy}{dx} = \frac{e^y}{1 - y}.$$

In the last step we used the fact that $xe^y = y$ to simplify the denominator.

C07S0M.012: Given $\sin(e^{xy}) = x$, we apply D_x to both sides and find that

$$[\cos(e^{xy})]\cdot e^{xy}\cdot\left(y + x\frac{dy}{dx}\right) = 1; \qquad xe^{xy}[\cos(e^{xy})]\cdot\frac{dy}{dx} = 1 - ye^{xy}\cos(e^{xy});$$
$$\frac{dy}{dx} = \frac{1 - ye^{xy}\cos(e^{xy})}{xe^{xy}\cos(e^{xy})}.$$

C07S0M.013: Given $e^x + e^y = e^{xy}$, we apply D_x to both sides and find that

$$e^x + e^y \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right); \qquad (e^y - xe^{xy})\frac{dy}{dx} = ye^{xy} - e^x;$$

$$\frac{dy}{dx} = \frac{ye^{xy} - e^x}{e^y - xe^{xy}}.$$

C07S0M.014: Given $x = ye^y$, we apply D_x to both sides and obtain

$$1 = e^y \frac{dy}{dx} + ye^y \frac{dy}{dx}; \qquad \frac{dy}{dx} = \frac{1}{e^y + ye^y} = \frac{y}{ye^y + y^2e^y} = \frac{y}{x + xy}.$$

We used the fact that $ye^y = x$ in the simplification in the last step.

Here is an alternative approach to finding dy/dx. Beginning with $x = ye^y$, we differentiate with respect to y and find that

$$\frac{dx}{dy} = e^y + ye^y, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{e^y + ye^y}.$$

C07S0M.015: Given $e^{x-y} = xy$, we apply D_x to both sides and find that

$$e^{x-y}\left(1 - \frac{dy}{dx}\right) = y + x\frac{dy}{dx}; \qquad \left(x + e^{x-y}\right)\frac{dy}{dx} = e^{x-y} - y;$$

$$\frac{dy}{dx} = \frac{e^{x-y} - y}{e^{x-y} + x}; \qquad \frac{dy}{dx} = \frac{xy - y}{xy + x} = \frac{(x-1)y}{(y+1)x}.$$

We used the fact that $e^{x-y} = xy$ to make the simplification in the last step.

C07S0M.016: Given $x \ln y = x + y$, we apply D_x to both sides and find that

$$\begin{aligned}
\ln y + \frac{x}{y}\cdot\frac{dy}{dx} &= 1 + \frac{dy}{dx};\\
\left(\frac{x}{y} - 1\right)\cdot\frac{dy}{dx} &= 1 - \ln y;\\
\frac{x-y}{y}\cdot\frac{dy}{dx} &= 1 - \ln y;\\
\frac{dy}{dx} &= \frac{y(1 - \ln y)}{x - y}.
\end{aligned}$$

C07S0M.017: Given: $y = \sqrt{(x^2-4)\sqrt{2x+1}}$. Thus

$$\ln y = \ln\left[(x^2-4)(2x+1)^{1/2}\right]^{1/2} = \frac{1}{2}\ln\left[(x^2-4)(2x+1)^{1/2}\right] = \frac{1}{2}\left[\ln(x^2-4) + \frac{1}{2}\ln(2x+1)\right].$$

Therefore

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{x^2-4} + \frac{1}{2(2x+1)} = \frac{5x^2+2x-4}{2(x^2-4)(2x+1)},$$

and so

$$\frac{dy}{dx} = y(x)\cdot\frac{5x^2+2x-4}{2(x^2-4)(2x+1)} = \frac{(5x^2+2x-4)\sqrt{(x^2-4)\sqrt{2x+1}}}{2(x^2-4)(2x+1)}.$$

C07S0M.018: Given: $y = (3-x^2)^{1/2}(x^4+1)^{-1/4}$. Thus

$$\ln y = \frac{1}{2}\ln(3-x^2) - \frac{1}{4}\ln(x^4+1),$$

and therefore

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\frac{x}{3-x^2} - \frac{x^3}{x^4+1} = \frac{x(3x^2+1)}{(x^2-3)(x^4+1)}.$$

Thus

$$\frac{dy}{dx} = y(x)\cdot\frac{x(3x^2+1)}{(x^2-3)(x^4+1)} = -\frac{x(3x^2+1)(3-x^2)^{1/2}}{(3-x^2)(x^4+1)(x^4+1)^{1/4}} = -\frac{x(3x^2+1)}{(3-x^2)^{1/2}(x^4+1)^{5/4}}.$$

C07S0M.019: Given: $y = \left[\dfrac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right]^{1/3}$. Then

$$\ln y = \frac{1}{3}\left[\ln(x+1)+\ln(x+2)-\ln(x^2+1)-\ln(x^2+2)\right];$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{x+1}+\frac{1}{x+2}-\frac{2x}{x^2+1}-\frac{2x}{x^2+2}\right);$$

$$\frac{dy}{dx} = y(x)\cdot\frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)(x+2)(x^2+1)(x^2+2)};$$

$$\frac{dy}{dx} = \frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)(x+2)(x^2+1)(x^2+2)}\cdot\left[\frac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right]^{1/3};$$

$$\frac{dy}{dx} = \frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)^{2/3}(x+2)^{2/3}(x^2+1)^{4/3}(x^2+2)^{4/3}}.$$

C07S0M.020: If $y = (x+1)^{1/2}(x+2)^{1/3}(x+3)^{1/4}$, then

$$\ln y = \frac{1}{2}\ln(x+1)+\frac{1}{3}\ln(x+2)+\frac{1}{4}\ln(x+3);$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2(x+1)}+\frac{1}{3(x+2)}+\frac{1}{4(x+3)};$$

$$\frac{dy}{dx} = y(x)\cdot\frac{13x^2+55x+54}{12(x+1)(x+2)(x+3)} = \frac{13x^2+55x+54}{12(x+1)^{1/2}(x+2)^{2/3}(x+3)^{3/4}}.$$

C07S0M.021: If $y = x^{(e^x)}$, then

$$\ln y = e^x \ln x; \qquad \frac{1}{y} \cdot \frac{dy}{dx} = \frac{e^x}{x} + e^x \ln x;$$

$$\frac{dy}{dx} = y(x) \cdot \frac{(1 + x \ln x)e^x}{x}; \qquad \frac{dy}{dx} = \frac{(1 + x \ln x)e^x}{x} \cdot \left(x^{(e^x)}\right).$$

C07S0M.022: Given: $y = (\ln x)^{\ln x}$, $x > 1$. Then

$$\ln y = (\ln x) \ln (\ln x);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \ln (\ln x) + \frac{\ln x}{x \ln x} = \frac{1 + \ln (\ln x)}{x};$$

$$\frac{dy}{dx} = \frac{1 + \ln (\ln x)}{x} \cdot (\ln x)^{\ln x}.$$

C07S0M.023: By l'Hôpital's rule,

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}.$$

Without l'Hôpital's rule,

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x + 2)(x - 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}.$$

C07S0M.024: By l'Hôpital's rule, $\displaystyle\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2 \cos 2x}{1} = \frac{2 \cdot 1}{1} = 2.$

C07S0M.025: By l'Hôpital's rule, $\displaystyle\lim_{x \to \pi} \frac{1 + \cos x}{(x - \pi)^2} = \lim_{x \to \pi} \frac{-\sin x}{2(x - \pi)} = \lim_{x \to \pi} \frac{-\cos x}{2} = \frac{1}{2}.$

C07S0M.026: By l'Hôpital's rule (applied three times),

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

C07S0M.027: By l'Hôpital's rule (applied three times),

$$\begin{aligned} \lim_{t \to 0} \frac{\tan t - \sin t}{t^3} &= \lim_{t \to 0} \frac{\sec^2 t \, - \, \cos t}{3t^2} \\ &= \lim_{t \to 0} \frac{2 \sec^2 t \tan t + \sin t}{6t} = \lim_{t \to 0} \frac{4 \sec^2 t \tan^2 t + 2 \sec^4 t + \cos t}{6} = \frac{1}{2}. \end{aligned}$$

C07S0M.028: By l'Hôpital's rule, $\displaystyle\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x \to \infty} \frac{1}{\ln x} = 0.$

C07S0M.029: By l'Hôpital's rule,

$$\lim_{x \to 0} (\cot x) \ln(1 + x) = \lim_{x \to 0} \frac{\ln(1 + x)}{\tan x} = \lim_{x \to 0} \frac{1}{(1 + x) \sec^2 x} = \frac{1}{1 \cdot 1} = 1.$$

C07S0M.030: By l'Hôpital's rule and the product rule for limits,

$$\lim_{x\to 0^+}(e^{1/x}-1)\tan x = \lim_{x\to 0^+}\frac{e^{1/x}-1}{\cot x} = \lim_{x\to 0^+}\frac{e^{1/x}}{x^2\csc^2 x} = \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^2 e^{1/x} = 1\cdot\left(\lim_{x\to 0^+} e^{1/x}\right) = +\infty.$$

Recall that l'Hôpital's rule is valid even if the resulting limit is $+\infty$ or $-\infty$, although we are stretching the hypotheses of the product rule a little here. We need a lemma.

Lemma: If

$$\lim_{x\to a^+} f(x) = p > 0 \qquad \text{and} \qquad \lim_{x\to a^+} g(x) = +\infty,$$

then $\lim_{x\to a^+} f(x)\cdot g(x) = +\infty$.

Proof: Given $M > 0$, choose $\delta > 0$ such that, if $x - a < \delta$ then $f(x) > p/2$ and $g(x) > 2M/p$. Then, for such x, $f(x)\cdot g(x) > (p/2)\cdot 2M/p = M$. Because M may be arbitrarily large positive, this implies that $f(x)\cdot g(x)$ takes on arbitrarily large values if $x > a$ and x is near a. That is,

$$\lim_{x\to a^+} f(x)\cdot g(x) = +\infty.$$

C07S0M.031: After combining the two fractions, we apply l'Hôpital's rule once, then use a little algebra:

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{1-\cos x}\right) = \lim_{x\to 0}\frac{1-\cos x - x^2}{x^2(1-\cos x)} = \lim_{x\to 0}\frac{(\sin x)-2x}{2x(1-\cos x)+x^2\sin x} = \lim_{x\to 0}\frac{\dfrac{\sin x}{x}-2}{2(1-\cos x)+x\sin x}.$$

Now if x is close to (but not equal to) zero, $(\sin x)/x \approx 1$, so the numerator in the last limit is near -1. Moreover, for such x, $\cos x < 1$ and x and $\sin x$ have the same sign, so the denominator in the last limit is close to zero *and positive.* Therefore the limit is $-\infty$.

C07S0M.032: We don't need l'Hôpital's rule here, although it may be applied. Without it we obtain

$$\lim_{x\to\infty}\left(\frac{x^2}{x+2}-\frac{x^3}{x^2+3}\right) = \lim_{x\to\infty}\frac{3x^2-2x^3}{x^3+2x^2+3x+6} = \lim_{x\to\infty}\frac{\dfrac{3}{x}-2}{1+\dfrac{2}{x}+\dfrac{3}{x^2}+\dfrac{6}{x^3}} = \frac{0-2}{1+0+0+0} = -2.$$

Using l'Hôpital's rule (three times) we find that

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{x^2}{x+2}-\frac{x^3}{x^2+3}\right) &= \lim_{x\to\infty}\frac{3x^2-2x^3}{x^3+2x^2+3x+6}\\ &= \lim_{x\to\infty}\frac{6x-6x^2}{3x^2+4x+3} = \lim_{x\to\infty}\frac{6-12x}{6x+4} = \lim_{x\to\infty}\frac{-12}{6} = -2.\end{aligned}$$

C07S0M.033: Here it is easier not to use l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2-x-1}-\sqrt{x}\right) &= \lim_{x\to\infty}\frac{x^2-x-1-x}{\sqrt{x^2-x-1}+\sqrt{x}}\\ &= \lim_{x\to\infty}\frac{x^2-2x-1}{\sqrt{x^2-x-1}+\sqrt{x}} = \lim_{x\to\infty}\frac{x-2-\dfrac{1}{x}}{\sqrt{1-\dfrac{1}{x}-\dfrac{1}{x^2}}-\sqrt{\dfrac{1}{x}}} = +\infty.\end{aligned}$$

C07S0M.034: $\ln\left(\lim_{x\to\infty} x^{1/x}\right) = \lim_{x\to\infty} \ln\left(x^{1/x}\right) = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1}{x} = 0.$ Therefore $\lim_{x\to\infty} x^{1/x} = 1.$

C07S0M.035: First we need an auxiliary result:

$$\lim_{x\to\infty} 2xe^{-2x} = \lim_{x\to\infty} \frac{2x}{e^{2x}} = \lim_{x\to\infty} \frac{2}{2e^{2x}} = 0.$$

Then

$$\begin{aligned}\ln\left(\lim_{x\to\infty} (e^{2x}-2x)^{1/x}\right) &= \lim_{x\to\infty} \ln(e^{2x}-2x)^{1/x} = \lim_{x\to\infty} \frac{\ln(e^{2x}-2x)}{x} \\ &= \lim_{x\to\infty} \frac{2e^{2x}-2}{e^{2x}-2x} = \lim_{x\to\infty} \frac{2-2e^{-2x}}{1-2xe^{-2x}} = \frac{2-0}{1-0} = 2.\end{aligned}$$

Therefore $\lim_{x\to\infty} (e^{2x}-2x)^{1/x} = e^2.$

C07S0M.036: Given: $\lim_{x\to\infty} \left[1-\exp(-x^2)\right]^{1/x^2}.$

$$\begin{aligned}\ln\left(\lim_{x\to\infty} \left[1-\exp(-x^2)\right]^{1/x^2}\right) &= \lim_{x\to\infty} \ln\left[1-\exp(-x^2)\right]^{1/x^2} = \lim_{x\to\infty} \frac{\ln\left(1-\exp(-x^2)\right)}{x^2} \\ &= \lim_{x\to\infty} \frac{2x\exp(-x^2)}{2x\left[1-\exp(-x^2)\right]} = \lim_{x\to\infty} \frac{\exp(-x^2)}{1-\exp(-x^2)} = \frac{0}{1-0} = 0.\end{aligned}$$

Therefore $\lim_{x\to\infty} \left[1-\exp(-x^2)\right]^{1/x^2} = e^0 = 1.$

C07S0M.037: This is one of the most challenging problems in the book. We deeply regret publication of this solution. First let $u = 1/x$. Then

$$L = \lim_{x\to\infty} x\cdot\left[\left(1+\frac{1}{x}\right)^x - e\right] = \lim_{u\to 0^+} \frac{(1+u)^{1/u}-e}{u}.$$

Apply l'Hôpital's rule once:

$$L = \lim_{u\to 0^+} (1+u)^{1/u}\left(\frac{u-(1+u)\ln(1+u)}{u^2(1+u)}\right).$$

Now apply the product rule for limits!

$$L = e\cdot\left(\lim_{u\to 0^+} \frac{u-(1+u)\ln(1+u)}{u^2(1+u)}\right).$$

Finally apply l'Hôpital's rule twice:

$$L = e\cdot\left(\lim_{u\to 0^+} \frac{1-1-\ln(1+u)}{2u+3u^2}\right) = e\cdot\left(\lim_{u\to 0^+} \frac{-1}{(1+u)(2+6u)}\right) = e\cdot\frac{-1}{1\cdot 2} = -\frac{e}{2}.$$

Most computer algebra programs cannot evaluate the original limit.

C07S0M.038: First replace b with x to remind us what the variable in this problem is. Thus

$$A(x) = 2\pi ax\left[\frac{x}{a} + \frac{a}{\sqrt{x^2-a^2}}\ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\right].$$

Then

$$\begin{aligned}\lim_{x\to a} A(x) &= \lim_{x\to a^+}\left[2\pi x^2 + \frac{2\pi a^2 x}{(x^2-a^2)^{1/2}}\ln\left(\frac{x+(x^2-a^2)^{1/2}}{a}\right)\right]\\ &= 2\pi a^2 + 2\pi a^2\left[\lim_{x\to a^+}\frac{x}{(x^2-a^2)^{1/2}}\ln\left(\frac{x+(x^2-a^2)^{1/2}}{a}\right)\right].\end{aligned}$$

Then apply l'Hôpital's rule to the limit that remains:

$$\begin{aligned}&\lim_{x\to a}\frac{x\ln(x+(x^2-a^2)^{1/2})-x\ln a}{(x^2-a^2)^{1/2}}\\ &= \lim_{x\to a}\frac{\ln\big(x+(x^2-a^2)^{1/2}\big)+\dfrac{x}{x+(x^2-a^2)^{1/2}}\cdot\left(1+\dfrac{1}{2}(x^2-a^2)^{-1/2}\cdot 2x\right)-\ln a}{\dfrac{1}{2}(x^2-a^2)^{-1/2}\cdot 2x}\\ &= \lim_{x\to a^+}\frac{(x^2-a^2)^{1/2}\ln\big(x+(x^2-a^2)^{1/2}\big)+\dfrac{x}{x+(x^2-a^2)^{1/2}}\cdot\Big((x^2-a^2)^{1/2}+x\Big)-(x^2-a^2)^{1/2}\ln a}{x}\\ &= \lim_{x\to a^+}\frac{(x^2-a^2)^{1/2}\ln\big(x+(x^2-a^2)^{1/2}\big)+x-(x^2-a^2)^{1/2}\ln a}{x} = 1.\end{aligned}$$

Therefore $A(b)\to 4\pi a^2$ as $b\to a^+$.

C07S0M.030: Let $u = 1-2x$. Then $dx = -\frac{1}{2}\,du$, and so

$$\int\frac{dx}{1-2x} = -\frac{1}{2}\int\frac{1}{u}\,du = -\frac{1}{2}\ln|u|+C = -\frac{1}{2}\ln|1-2x|+C.$$

C07S0M.040: Let $u = 1+x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\,dx$, so that $x^{1/2}\,dx = \frac{2}{3}\,du$. Hence

$$\int\frac{x^{1/2}}{1+x^{3/2}}\,dx = \frac{2}{3}\int\frac{1}{u}\,du = \frac{2}{3}\ln u + C = \frac{2}{3}\ln\left(1+x^{3/2}\right)+C.$$

C07S0M.041: Let $u = 1+6x-x^2$. Then $du = (6-2x)\,dx$, so that $(3-x)\,dx = \frac{1}{2}\,du$. Thus

$$\int\frac{3-x}{1+6x-x^2}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|1+6x-x^2|+C.$$

C07S0M.042: Let $u = e^x+e^{-x}$. Then $du = (e^x-e^{-x})\,dx$, so that

$$\int\frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx = \int\frac{1}{u}\,du = (\ln u)+C = \ln\left(e^x+e^{-x}\right)+C.$$

C07S0M.043: Let $u = 2+\cos x$. Then $du = -\sin x\,dx$, and thus

$$\int\frac{\sin x}{2+\cos x}\,dx = -\int\frac{1}{u}\,du = -(\ln u)+C = -\ln(2+\cos x)+C.$$

C07S0M.044: $\displaystyle\int\frac{e^{-1/x^2}}{x^3}\,dx = \frac{1}{2}e^{-1/x^2}+C.$ (Optional substitution: $u = -1/x^2$.)

C07S0M.045: Let $u = 10^{\sqrt{x}}$. Then

$$du = (10^{\sqrt{x}} \ln 10) \cdot \frac{1}{2} x^{-1/2}\, dx = \frac{10^{\sqrt{x}}}{\sqrt{x}} \cdot \frac{\ln 10}{2}\, dx.$$

Therefore

$$\int \frac{10^{\sqrt{x}}}{\sqrt{x}}\, dx = \int \frac{2}{\ln 10}\, du = \frac{2u}{\ln 10} + C = \frac{2 \cdot 10^{\sqrt{x}}}{\ln 10} + C.$$

C07S0M.046: Let $u = \ln x$. Then $du = \dfrac{1}{x}\, dx$, so that

$$\int \frac{1}{x(\ln x)^2}\, dx = \int \frac{1}{u^2}\, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

C07S0M.047: Let $u = 1 + e^x$. Then $du = e^x\, dx$, and thus

$$\int e^x (1 + e^x)^{1/2}\, dx = \int u^{1/2}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C.$$

C07S0M.048: Let $u = 1 + \ln x$. Then $du = \dfrac{1}{x}\, dx$, and therefore

$$\int \frac{1}{x} (1 + \ln x)^{1/2}\, dx = \int u^{1/2}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + \ln x)^{3/2} + C.$$

C07S0M.049: $\displaystyle \int 2^x \cdot 3^x\, dx = \int 6^x\, dx = \frac{6^x}{\ln 6} + C.$

C07S0M.050: Let $u = 1 + x^{2/3}$. Then $du = \dfrac{2}{3} x^{-1/3}\, dx$. Hence

$$\int \frac{dx}{x^{1/3}(1 + x^{2/3})} = \frac{3}{2} \int \frac{1}{u}\, du = \frac{3}{2} (\ln u) + C = \frac{3}{2} \ln(1 + x^{2/3}) + C.$$

C07S0M.051: The revenue realized upon selling after t months will be

$$f(t) = B \cdot \left(2 + \frac{t}{12}\right) \cdot 2^{-t/12}, \qquad \text{for which} \qquad f'(t) = B \cdot \frac{12 - 24 \ln 2 - t \ln 2}{144 \cdot 2^{t/12}}.$$

Thus $f'(t) = 0$ when

$$t = \frac{12 - 24 \ln 2}{\ln 2} \approx -6.68765951.$$

But this value of t is negative, and in addition $f'(t) < 0$ for all larger values of t. Thus the revenue is a decreasing function of t for all $t \geqq 0$. Therefore the grain should be sold immediately.

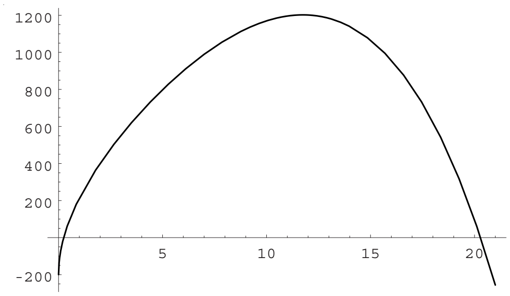
C07S0M.052: The profit will be

$$f(t) = 800 \exp\left(\tfrac{1}{2}\sqrt{t}\right) - 1000 \exp\left(\tfrac{1}{10} t\right).$$

Now

$$f'(t) = \frac{100\left[2\exp\left(\frac{1}{2}\sqrt{t}\right) - t^{1/2}\exp\left(\frac{1}{10}t\right)\right]}{t^{1/2}},$$

We used Newton's method to solve $f'(t) = 0$ and found the solution to be approximately 11.7519277504. To make sure that this value of t maximizes the profit, we graphed $f(t)$ for $0 \leqq t \leqq 21$ (the graph is shown following this solution). The profit upon cutting and selling after about 11.75 years will be approximately \$1202.37.



C07S0M.053: If lots composed of x pooled samples are tested, there will be $1000/x$ lots, so there will be $1000/x$ tests. In addition, if a lot tests positive, there will be x additional tests. The probability of a lot testing positive is $1-(0.99)^x$, so the expected number of lots that require additional tests will be the product of the number of lots and the probability $1-(0.99)^x$ that a lot tests positive. Hence the total number of tests to be expected will be

$$f(x) = \frac{1000}{x} + \frac{1000}{x}\left[1-(0.99)^x\right]\cdot x = \frac{1000}{x} + 1000 - 1000(0.99)^x$$

if $x \geqq 2$. Next,

$$f'(x) = -\frac{1000}{x^2} - 1000(0.99)^x\ln(0.99);$$

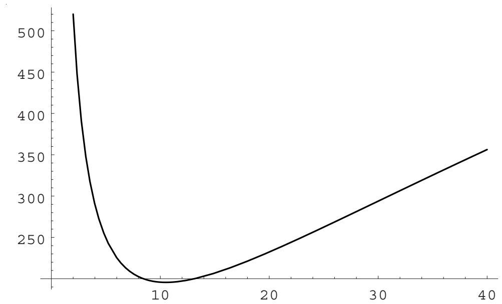
$$f'(x) = 0 \quad \text{when} \quad \frac{1}{x^2} = (0.99)^x\ln(100/99);$$

$$x^2 = \frac{(0.99)^{-x}}{\ln(100/99)};$$

$$x = \frac{(0.99)^{-x/2}}{\sqrt{\ln(100/99)}}.$$

The form of the last equation is exactly what we need to implement the method of repeated substitution (see Problems 23 through 25 of Section 3.8). We substitute our first "guess" $x_0 = 10$ into the right-hand side of the last equation, thus obtaining a "better" (we hope) value x_1 and continue this process until the digits in these successive approximations stabilize. Results: $x_1 = 10.488992$, $x_2 = 10.514798$, $x_3 = 10.516161$, $x_4 = 10.516233$, and $x_5 = 10.516237 = x_6$. The method is not as fast as Newton's method but the formula is simpler. (The graph of $y = f(x)$ follows this solution to convince you that we have actually found the minimum value of f.) We must use an integral number of samples, so we find that $f(10) \approx 195.618$ and $f(11) = 195.571$, so there should be 90 lots of 11 samples each and one lot of 10 for the most economical results. Alternatively, it might be simpler to use 10 samples in every lot; the extra cost would be only about

24 cents. The total cost of the batch method will be about \$978, significantly less than the \$5000 cost of testing each sample individually.



C07S0M.054: If $f(x) = \frac{1}{2}x^2 - \frac{1}{4}\ln x$, then

$$1 + [f'(x)]^2 = 1 + \left(x - \frac{1}{4x}\right)^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2.$$

Therefore the arc length is

$$\int_1^e \left(x + \frac{1}{4x}\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{4}\ln x\right]_1^e = \frac{1}{4} + \frac{1}{2}e^2 - \frac{1}{2} = \frac{2e^2 - 1}{4} \approx 3.4445280495.$$

C07S0M.055: If $f(x) = \sin^{-1} 3x$, then $f'(x) = \dfrac{1}{\sqrt{1 - (3x)^2}} \cdot D_x\,(3x) = \dfrac{3}{\sqrt{1 - 9x^2}}$.

C07S0M.056: If $f(x) = \tan^{-1} 7x$, then $f'(x) = \dfrac{7}{1 + 49x^2}$.

C07S0M.057: If $g(t) = \sec^{-1} t^2$, then

$$g'(t) = \frac{1}{|t^2|\sqrt{(t^2)^2 - 1}} \cdot D_t\left(t^2\right) = \frac{2t}{t^2\sqrt{t^4 - 1}} = \frac{2}{t\sqrt{t^4 - 1}}.$$

C07S0M.058: If $g(t) = \tan^{-1} e^t$, then $g'(t) = \dfrac{e^t}{1 + e^{2t}}$.

C07S0M.059: If $f(x) = \sin^{-1}(\cos x)$, then

$$f'(x) = \frac{1}{\sqrt{1 - (\cos x)^2}} \cdot D_x\,(\cos x) = -\frac{\sin x}{\sqrt{1 - \cos^2 x}} = -\frac{\sin x}{|\sin x|}.$$

C07S0M.060: If $f(x) = \sinh^{-1} 2x$, then $f'(x) = \dfrac{2}{\sqrt{1 + 4x^2}}$.

C07S0M.061: If $g(t) = \cosh^{-1} 10t$, then $g'(t) = \dfrac{1}{\sqrt{(10t)^2 - 1}} \cdot D_t\,(10t) = \dfrac{10}{\sqrt{100t^2 - 1}}$, $t > \frac{1}{10}$.

C07S0M.062: If $h(u) = \tanh^{-1}\left(\dfrac{1}{u}\right)$, then

$$h'(u) = \frac{1}{1 - \left(\dfrac{1}{u}\right)^2} \cdot D_u\left(\frac{1}{u}\right) = -\frac{u^2}{u^2-1} \cdot \frac{1}{u^2} = -\frac{1}{u^2-1} = \frac{1}{1-u^2}$$

provided that $|u| > 1$.

C07S0M.063: If $f(x) = \sin^{-1}\left(\dfrac{1}{x^2}\right)$, then

$$f'(x) = \frac{1}{\sqrt{1 - \left(\dfrac{1}{x^2}\right)^2}} \cdot D_x\left(\frac{1}{x^2}\right) = \frac{1}{\sqrt{1 - \dfrac{1}{x^4}}} \cdot \frac{-2}{x^3} = -\frac{2}{x\sqrt{x^4-1}}.$$

C07S0M.064: If $f(x) = \tan^{-1}\left(\dfrac{1}{x}\right)$, then

$$f'(x) = \frac{1}{1 + \dfrac{1}{x^2}} \cdot D_x\left(\frac{1}{x}\right) = \frac{x^2}{x^2+1} \cdot \frac{-1}{x^2} = -\frac{1}{x^2+1}$$

provided that $x \neq 0$.

C07S0M.065: If $f(x) = \arcsin\sqrt{x}$, then

$$f'(x) = \frac{1}{\sqrt{1 - (\sqrt{x}\,)^2}} \cdot D_x\left(x^{1/2}\right) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{1-x}\,\sqrt{x}}.$$

C07S0M.066: If $f(x) = x\sec^{-1}x^2$, then

$$f'(x) = \frac{x}{|x^2|\,\sqrt{(x^2)^2-1}} \cdot D_x\left(x^2\right) + \sec^{-1}x^2 = \frac{2}{\sqrt{x^4-1}} + \sec^{-1}x^2$$

C07S0M.067: If $f(x) = \tan^{-1}(x^2+1)$, then

$$f'(x) = \frac{1}{1+(x^2+1)^2} \cdot D_x(x^2+1) = \frac{2x}{x^4+2x^2+2}.$$

C07S0M.068: If $f(x) = \sin^{-1}\sqrt{1-x^2}$, then

$$f'(x) = \frac{1}{\sqrt{1-(1-x^2)}} \cdot D_x(1-x^2)^{1/2} = \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = -\frac{x}{|x|\sqrt{1-x^2}}.$$

C07S0M.069: If $f(x) = e^x \sinh e^x$, then $f'(x) = e^{2x}\cosh e^x + e^x \sinh e^x$.

C07S0M.070: If $f(x) = \ln\cosh x$, then $f'(x) = \dfrac{\sinh x}{\cosh x} = \tanh x$.

C07S0M.071: If $f(x) = \tanh^2 3x + \operatorname{sech}^2 3x$, then $f(x) \equiv 1$ by Eq. (5) in Section 7.6. Therefore $f'(x) \equiv 0$. Alternatively, $f'(x) = 6\tanh 3x\ \operatorname{sech}^2 3x - 6\operatorname{sech} 3x\ \operatorname{sech} 3x\ \tanh 3x \equiv 0$.

C07S0M.072: If $f(x) = \sinh^{-1}\sqrt{x^2 - 1}$, then

$$f'(x) = \frac{1}{\sqrt{1 + \left(\sqrt{x^2 - 1}\right)^2}} \cdot D_x(x^2 - 1)^{1/2} = \frac{1}{\sqrt{1 + x^2 - 1}} \cdot \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x = \frac{x}{|x|\sqrt{x^2 - 1}}.$$

C07S0M.073: If $f(x) = \cosh^{-1}\sqrt{x^2 + 1}$, then

$$f'(x) = \frac{1}{\sqrt{\left(\sqrt{x^2 + 1}\right)^2 - 1}} \cdot D_x(x^2 + 1)^{1/2} = \frac{1}{\sqrt{x^2}} \cdot \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x = \frac{x}{|x|\sqrt{x^2 + 1}}.$$

C07S0M.074: If $f(x) = \tanh^{-1}(1 - x^2)$, then

$$f'(x) = \frac{1}{1 - (1 - x^2)^2} \cdot (-2x) = \frac{-2x}{2x^2 - x^4} = \frac{2}{x^3 - 2x}.$$

C07S0M.075: Let $u = 2x$. Then $du = 2\,dx$, so

$$\int \frac{1}{\sqrt{1 - 4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1 - u^2}}\,du = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} 2x + C.$$

C07S0M.076: Let $u = 2x$. Then $du = 2\,dx$, so

$$\int \frac{1}{1 + 4x^2}\,dx = \frac{1}{2}\int \frac{1}{1 + u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan 2x + C.$$

C07S0M.077: Let $x = 2u$. Then $dx = 2\,du$ and $\sqrt{4 - x^2} = \sqrt{4 - 4u^2} = 2\sqrt{1 - u^2}$. Therefore

$$\int \frac{1}{\sqrt{4 - x^2}}\,dx = \int \frac{2}{2\sqrt{1 - u^2}}\,du = \arcsin u + C = \arcsin\left(\frac{x}{2}\right) + C.$$

C07S0M.078: Let $x = 2u$. Then $dx = 2\,du$ and $4 + x^2 = 4(1 + u^2)$. Thus

$$\int \frac{1}{4 + x^2}\,dx = \int \frac{2}{4(1 + u^2)}\,du = \frac{1}{2}\int \frac{1}{1 + u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C.$$

C07S0M.079: Let $u = e^x$. Then $du = e^x\,dx$ and $\sqrt{1 - e^{2x}} = \sqrt{1 - u^2}$. Hence

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx = \int \frac{1}{\sqrt{1 - u^2}}\,du = \arcsin u + C = \arcsin e^x + C.$$

C07S0M.080: Let $u = x^2$. Then $du = 2x\,dx$ and $1 + x^4 = 1 + u^2$. Therefore

$$\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{1}{1 + u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan x^2 + C.$$

Mathematica 3.0 returns the answer

$$-\frac{1}{2}\arctan\left(\frac{1}{x^2}\right) + C,$$

demonstrating that your intelligence, guided by experience, is still required to interpret what the computer tells you.

C07S0M.081: Let $u = \frac{2}{3}x$, so that $x = \frac{3}{2}u$. Then $dx = \frac{3}{2}\,du$ and $\sqrt{9-4x^2} = \sqrt{9-9u^2} = 3\sqrt{1-u^2}$. So

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \frac{3}{2}\cdot\frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin u\, + C = \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C.$$

C07S0M.082: Let $u = \frac{2}{3}x$, so that $x = \frac{3}{2}u$. Then $dx = \frac{3}{2}\,du$ and $9 + 4x^2 = 9 + 9u^2 = 9(1+u^2)$. Thus

$$\int \frac{1}{9+4x^2}\,dx = \frac{3}{2}\cdot\frac{1}{9}\int \frac{1}{1+u^2}\,du = \frac{1}{6}\arctan u\, + C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C.$$

C07S0M.083: The integrand resembles the derivative of the inverse tangent of *something*, so we let $u = x^3$. Then $du = 3x^2\,dx$, so that $x^2\,dx = \frac{1}{3}du$. Then

$$\int \frac{x^2}{1+x^6}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u\, + C = \frac{1}{3}\arctan x^3\, + C.$$

C07S0M.084: If necessary, use the substitution $u = \sin x$, but by inspection,

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \arctan(\sin x) + C.$$

Note that while expressions such as $\sin(\arctan x)$ can be simplified to algebraic functions, expressions such as $\arctan(\sin x)$ normally cannot be further simplified.

C07S0M.085: Let $u = 2x$. Then $du = 2\,dx$ and $\sqrt{4x^2-1} = \sqrt{u^2-1}$. Thus

$$\int \frac{1}{x\sqrt{4x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\frac{1}{2}u\sqrt{u^2-1}}\,du = \int \frac{1}{u\sqrt{u^2-1}}\,du = \operatorname{arcsec}|u| + C = \operatorname{arcsec}|2x| + C.$$

Mathematica 3.0 returns the equivalent alternative answer

$$\int \frac{1}{x\sqrt{4x^2-1}}\,dx = -\arctan\left(\frac{1}{\sqrt{4x^2-1}}\right) + C,$$

which in turn is equal to $\arctan\left(\sqrt{4x^2-1}\right) + C$.

C07S0M.086: The integrand resembles the derivative of the inverse secant function, so we try the substitution $u = x^2$. Then $du = 2x\,dx$ and $\sqrt{x^4-1} = \sqrt{u^2-1}$, and thus

$$\int \frac{1}{x\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{2x}{x^2\sqrt{x^4-1}}\,dx = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{1}{2}\operatorname{arcsec}|u|\, + C = \frac{1}{2}\operatorname{arcsec} x^2\, + C.$$

Mathematica 3.0 returns the equivalent answer

$$\int \frac{1}{x\sqrt{x^4-1}}\,dx = -\frac{1}{2}\arctan\left(\frac{1}{\sqrt{x^4-1}}\right) + C,$$

which itself is equal to $\frac{1}{2}\arctan\left(\sqrt{x^4-1}\right) + C$.

C07S0M.087: The integrand is slightly evocative of the derivative of the inverse secant function, so we try the substitution $u = e^x$, so that $\sqrt{e^{2x} - 1} = \sqrt{u^2 - 1}$. Then $du = e^x\,dx$, so

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du = \operatorname{arcsec}|u| + C = \operatorname{arcsec}(e^x) + C.$$

Mathematica 3.0 returns the equivalent answer $\arctan\left(\sqrt{e^{2x}-1}\right) + C$.

C07S0M.088: Use the substitution $u = x^3$ if necessary, but by inspection

$$\int x^2 \cosh x^3\,dx = \frac{1}{3}\sinh x^3 + C.$$

C07S0M.089: Let $u = \sqrt{x}$ if necessary, but by inspection an antiderivative is $f(x) = k\cosh x^{1/2}$ for some constant k. Because

$$f'(x) = \left(k\sinh x^{1/2}\right)\cdot D_x\left(x^{1/2}\right) = \frac{k\sinh x^{1/2}}{2x^{1/2}},$$

it follows that $k = 2$ and therefore that

$$\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx = 2\cosh\sqrt{x} + C.$$

C07S0M.090: Let $u = 3x - 2$ if necessary, but by inspection an antiderivative is $f(x) = k\tanh(3x-2)$ for some constant k. Because $f'(x) = 3k\operatorname{sech}^2(3x-2)$, it follows that $k = \frac{1}{3}$, and thus that

$$\int \operatorname{sech}^2(3x-2)\,dx = \frac{1}{3}\tanh(3x-2) + C.$$

Mathematica 3.0 returns the equivalent answer

$$\int \operatorname{sech}^2(3x-2)\,dx = -\frac{1}{3}\tanh(2-3x) + C.$$

Figure 7.6.3 shows why the two answers are really the same.

C07S0M.091: Let $u = \arctan x$ if necessary, but evidently

$$\int \frac{\arctan x}{1+x^2}\,dx = \frac{1}{2}(\arctan x)^2 + C.$$

C07S0M.092: Let $u = 2x$, so that $\sqrt{4x^2-1} = \sqrt{u^2-1}$ and $du = 2\,dx$. Then

$$\int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u^2-1}}\,du = \frac{1}{2}\cosh^{-1}u + C = \frac{1}{2}\cosh^{-1}2x + C.$$

C07S0M.093: Let $u = \frac{2}{3}x$, so that $x = \frac{3}{2}u$. Then $dx = \frac{3}{2}\,du$ and $\sqrt{4x^2+9} = \sqrt{9u^2+9} = 3\sqrt{u^2+1}$. Therefore

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx = \frac{3}{2}\cdot\frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,du = \frac{1}{2}\sinh^{-1}u + C = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right) + C.$$

C07S0M.094: We expect to see an integrand containing $\sqrt{u^2+1}$, so let $u = x^2$. Then $du = 2x\ dx$, and thus

$$\int \frac{x}{\sqrt{x^4+1}}\ dx = \frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\ du = \frac{1}{2}\sinh^{-1} u\ + C = \frac{1}{2}\sinh^{-1} x^2\ + C.$$

C07S0M.095: The volume is

$$V = \int_0^{1/\sqrt{2}} \frac{2\pi x}{\sqrt{1-x^4}}\ dx.$$

Let $u = x^2$. Then $du = 2x\ dx$, and hence

$$\int \frac{2\pi x}{\sqrt{1-x^4}}\ dx = \pi\int \frac{1}{\sqrt{1-u^2}}\ du = \pi \arcsin u\ + C = \pi \arcsin x^2\ + C.$$

Therefore

$$V\ =\ \Big[\pi \arcsin x^2\Big]_0^{1/\sqrt{2}}\ =\ \frac{\pi^2}{6}\ =\ \zeta(2)\ \approx\ 1.6449340668482264364724 15.$$

See the Index in the textbook for references to further information on the Riemann zeta function $\zeta(z)$.

C07S0M.096: The volume is

$$V = \int_0^1 \frac{2\pi x}{\sqrt{x^4+1}}\ dx.$$

Let $u = x^2$. Then $du = 2x\ dx$, and therefore

$$\int \frac{2\pi x}{\sqrt{x^4+1}}\ dx = \int \frac{\pi}{\sqrt{u^2+1}}\ du = \pi \sinh^{-1} u\ + C = \pi \sinh^{-1} x^2\ + C.$$

Thus $V = \Big[\pi \sinh^{-1} x^2\Big]_0^1 = \pi \sinh^{-1} 1 = \pi \ln\left(1+\sqrt{2}\right) \approx 2.7689167860.$

C07S0M.097: We use some results in Section 7.6. By Eqs. (36) and (37),

$$\tanh^{-1}\left(\frac{1}{x}\right) = \frac{1}{2}\ln\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}} = \frac{1}{2}\ln\frac{x+1}{x-1} = \coth^{-1} x$$

(provided that $|x| > 1$). By Eq. (35), if $0 < x \leqq 1$ then

$$\cosh^{-1}\left(\frac{1}{x}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right) = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{\sqrt{x^2}}\right) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \operatorname{sech}^{-1} x$$

by Eq. (38). Note that $\sqrt{x^2} = x$ because $x > 0$.

C07S0M.098: If $k \neq 0$ and $x(t) = A\cosh kt + B\sinh kt$, then

$$x'(t) = kA\sinh kt + kB\cosh kt \qquad \text{and} \qquad x''(t) = k^2 A\cosh kt + k^2 B\sinh kt = k^2 x(t).$$

If $x(0) = 1$ and $x'(0) = 0$, then $A = 1$ and $kB = 0$, so $A = 1$ and $B = 0$. If $x(0) = 0$ and $x'(0) = 1$, then $A = 0$ and $kB = 1$, so $A = 0$ and $B = 1/k$.

C07S0M.099: The graphs of $y = \cos x$ and $y = \operatorname{sech} x$ are shown following this solution, graphed for $0 \leqq x \leqq 6$. One wonders if there is a solution of $\cos x = \operatorname{sech} x$ in the interval $(0, \pi/2)$. We graphed $y = f(x) = \operatorname{sech} x - \cos x$ for $0 \leqq x \leqq 1$ and it was clear that $f(x) > 0$ if $x > 0.25$. We graphed $y = f(x)$ for $0 \leqq x \leqq 0.25$ and it was clear that $f(x) > 0$ if $x > 0.06$. We repeated this process several times and could see that $f(x) > 0$ if $x > 0.0002$. Instability in the hardware or software made further progress along these lines impossible. Methods of Section 11.8 can be used to show the desired inequality, but Ted Shifrin provided the following elegant argument.

First, $\operatorname{sech} x \leqq 1 \leqq \sec x$ if x is in $I = [0, \pi/2)$. Moreover, $\operatorname{sech} x = \sec x$ only for $x = 0$ in that interval; otherwise, $\operatorname{sech} x < \sec x$ for x in $J = (0, \pi/2)$. So $\operatorname{sech}^2 x < \sec^2 x$ if x is in J.

But $\tanh x = \tan x$ if $x = 0$. Therefore $\tanh x < \tan x$ for x in J because $D_x \tanh x < D_x \tan x$ for such x. That is,

$$\frac{\sinh x}{\cosh x} < \frac{\sin x}{\cos x}$$

if x is in J. All the expressions involved here are positive, so

$$\sinh x \, \cos x < \cosh x \, \sin x$$

if x is in J. That is,

$$-\cosh x \, \sin x + \sinh x \, \cos x < 0$$

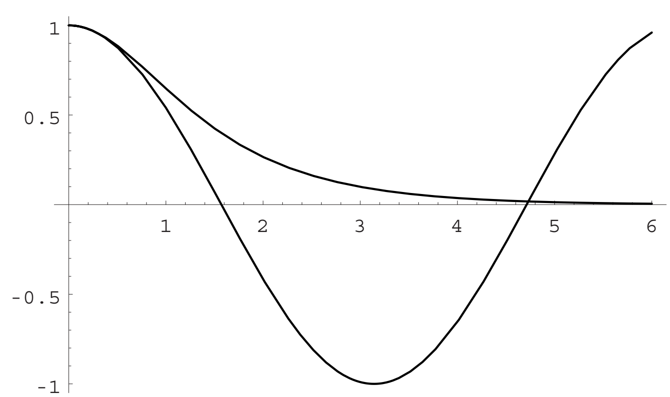
for x in J. But $\cosh x \, \cos x = 1$ if $x = 0$, and we have now shown that $D_x(\cosh x \, \cos x) < 0$ if x is in J. Therefore

$$\cosh x \, \cos x < 1$$

for x in J. In other words, $\operatorname{sech} x \geqq \cos x$ for x in I and $\operatorname{sech} x > \cos x$ for x in J.

Because $\cos x \leqq 0 < \operatorname{sech} x$ for $\pi/2 < x < 3\pi/2$ and because $\cos x$ is increasing, while $\operatorname{sech} x$ is decreasing, for $3\pi/2 \leqq x < 2\pi$, it follows that the least positive solution of $f(x) = 0$ is slightly larger than $3\pi/2$, about 4.7, exactly as the figure suggests.

We then applied Newton's method to the solution of $f(x) = 0$ with $x_0 = 4.7$, with the following results: $x_1 \approx 4.7300338216$, $x_2 \approx 4.7300407449$, $x_3 \approx 4.7300407449$. Thus x_3 is a good approximation to the least positive solution of $\cos x \cosh x = 1$.



C07S0M.100: If $f(x) = \sinh^{-1}(\tan x)$, then

$$f'(x) = \frac{\sec^2 x}{\sqrt{1+\tan^2 x}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = |\sec x|.$$

Therefore

$$\int \sec x \, dx = \sinh^{-1}(\tan x) + C$$

if $\sec x > 0$; that is, if k is an odd integer and $k\pi/2 < x < (k+2)\pi/2$. On the intervals where $\sec x < 0$, we find that

$$\int \sec x \, dx = -\sinh^{-1}(\tan x) + C.$$

Next, if $g(x) = \tan^{-1}(\sinh x)$, then

$$g'(x) = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x,$$

and therefore

$$\int \operatorname{sech} x \, dx = \tan^{-1}(\sinh x) + C$$

for all x.

C07S0M.101: Given $f(x) = x^{1/2}$, let $F(x) = f(x) - \ln x$. Then

$$F'(x) = \frac{1}{2x^{1/2}} - \frac{1}{x} = \frac{x^{1/2}-2}{2x},$$

so $F'(x) = 0$ when $x = 4$. Clearly F is decreasing on $(0, 4)$ and increasing on $(0, +\infty)$, so the global minimum value of $F(x)$ is

$$F(4) = 2 - \ln 4 = 2 - 2\ln 2 > 2 - 2 \cdot 1$$

because $\ln 2 < 1$. Therefore $f(x) > \ln x$ for all $x > 0$.

For part (b), we need to solve $x^{1/3} - \ln x = 0$. The iteration of Newton's method takes the form

$$x \longleftarrow x - \frac{x^{1/3} - \ln x}{\dfrac{1}{3}x^{-2/3} - \dfrac{1}{x}}.$$

Beginning with $x_0 = 100$, we get $x_5 \approx 93.354461$.

For part (c), suppose that $j(x) = x^{1/p}$ is tangent to the graph of $g(x) = \ln x$ at the point $(q, \ln q)$. Then $q^{1/p} = \ln q$ and $j'(q) = g'(q)$. Hence

$$\frac{1}{p}q^{(1/p)-1} = \frac{1}{q}; \qquad q^{1/p} = p;$$

$$p = \ln q = \ln p^p = p \ln p; \qquad \ln p = 1, \quad \text{so} \quad p = e.$$

Section 8.2

C08S02.001: Let $u = 2 - 3x$. Then $du = -3\,dx$, and so

$$\int (2-3x)^4\,dx = -\frac{1}{3}\int (2-3x)^4(-3)\,dx = -\frac{1}{3}\int u^4\,du = -\frac{1}{3}\cdot\frac{1}{5}u^5 + C = -\frac{1}{15}(2-3x)^5 + C.$$

C08S02.002: Let $u = 1 + 2x$. Then $x = \frac{1}{2}(u-1)$ and $dx = \frac{1}{2}\,du$, so that

$$\int \frac{1}{(1+2x)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2u} + C = -\frac{1}{2(1+2x)} + C.$$

C08S02.003: Let $u = 2x^3 - 4$. Then $du = 6x^2\,dx$, so that

$$\int x^2(2x^3-4)^{1/2}\,dx = \frac{1}{6}\int (2x^3-4)^{1/2}\cdot 6x^2\,dx = \frac{1}{6}\int u^{1/2}\,du = \frac{1}{6}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{9}(2x^3-4)^{3/2} + C.$$

C08S02.004: Let $u = 5 + 2t^2$. Then $du = 4t\,dt$, and so

$$\int \frac{5t}{5+2t^2}\,dt = \frac{5}{4}\int \frac{4t}{5+2t^2}\,dt = \frac{5}{4}\int \frac{1}{u}\,du = \frac{5}{4}\ln|u| + C = \frac{5}{4}\ln(5+2t^2) + C.$$

C08S02.005: Let $u = 2x^2 + 3$. Then $du = 4x\,dx$, and so

$$\int 2x(2x^2+3)^{-1/3}\,dx = \frac{1}{2}\int (2x^2+3)^{-1/3}\cdot 4x\,dx = \frac{1}{2}\int u^{-1/3}\,du = \frac{1}{2}\cdot\frac{3}{2}u^{2/3} + C = \frac{3}{4}(2x^2+3)^{2/3} + C.$$

C08S02.006: Let $u = x^2$. Then $du = 2x\,dx$, and therefore

$$\int x\sec^2 x^2\,dx = \frac{1}{2}\int \left(\sec x^2\right)^2\cdot 2x\,dx = \frac{1}{2}\int (\sec u)^2\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan(x^2) + C = \frac{1}{2}\tan x^2 + C.$$

C08S02.007: Let $u = y^{1/2}$, so that $du = \frac{1}{2}y^{-1/2}\,dy$. Then

$$\int y^{-1/2}\left(\cot y^{1/2}\right)\left(\csc y^{1/2}\right)\,dy = 2\int \cot u\,\csc u\,du = -2\csc u + C = -2\csc\sqrt{y} + C.$$

C08S02.008: Let $u = \pi(2x+1)$. Then $du = 2\pi\,dx$, and hence

$$\int \sin\pi(2x+1)\,dx = \frac{1}{2\pi}\int \sin u\,du = -\frac{1}{2\pi}\cos u + C = -\frac{1}{2\pi}\cos\pi(2x+1) + C.$$

C08S02.009: Let $u = 1 + \sin\theta$. Then $du = \cos\theta\,d\theta$, and therefore

$$\int (1+\sin\theta)^5\cos\theta\,d\theta = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}(1+\sin\theta)^6 + C.$$

C08S02.010: Let $u = 4 + \cos 2x$. Then $du = -2\sin 2x\,dx$, and thus

$$\int \frac{\sin 2x}{4+\cos 2x}\,dx = -\frac{1}{2}\int \frac{-2\sin 2x}{4+\cos 2x}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln(4+\cos 2x) + C.$$

C08S02.011: Let $u = -\cot x$. Then $du = \csc^2 x\, dx$. So

$$\int e^{-\cot x} \csc^2 x\, dx = \int e^u\, du = e^u + C = e^{-\cot x} + C = \exp(-\cot x) + C.$$

C08S02.012: Let $u = (x+4)^{1/2}$. Then $du = \frac{1}{2}(x+4)^{-1/2}\, dx$. Thus

$$\int \frac{\exp\left((x+4)^{1/2}\right)}{(x+4)^{1/2}}\, dx = 2\int e^u\, du = 2e^u + C = 2\exp\left((x+4)^{1/2}\right) + C.$$

C08S02.013: Let $u = \ln t$. Then $du = \dfrac{1}{t}\, dt$, so

$$\int \frac{(\ln t)^{10}}{t}\, dt = \int u^{10}\, du = \frac{1}{11}u^{11} + C = \frac{1}{11}(\ln t)^{11} + C.$$

C08S02.014: Let $u = 1 - 9t^2$. Then $du = -18t\, dt$. Hence

$$\int \frac{t}{\sqrt{1-9t^2}}\, dt = -\frac{1}{18}\int (1-9t^2)^{-1/2}\cdot(-18t)\, dt = -\frac{1}{18}\int u^{-1/2}\, du = -\frac{1}{18}\cdot 2u^{1/2} + C = -\frac{1}{9}\sqrt{1-9t^2} + C.$$

C08S02.015: Let $u = 3t$, so that $du = 3\, dt$. Then

$$\int \frac{1}{\sqrt{1-9t^2}}\, dt = \frac{1}{3}\int \frac{1}{\sqrt{1-9t^2}}\cdot 3\, dt = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\, du = \frac{1}{3}\arcsin u + C = \frac{1}{3}\arcsin(3t) + C.$$

C08S02.016: Let $u = 1 + e^{2x}$. Then $du = 2e^{2x}\, dx$ and thus

$$\int \frac{e^{2x}}{1+e^{2x}}\, dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+e^{2x}\right) + C.$$

C08S02.017: Let $u = e^{2x}$. Then $du = 2e^{2x}\, dx$. Therefore

$$\int \frac{e^{2x}}{1+e^{4x}}\, dx = \frac{1}{2}\int \frac{2e^{2x}}{1+(e^{2x})^2}\, dx = \frac{1}{2}\int \frac{1}{1+u^2}\, du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\left(e^{2x}\right) + C.$$

C08S02.018: Let $u = \arctan x$. Then $du = \dfrac{1}{1+x^2}\, dx$, so that

$$\int \frac{\exp(\arctan x)}{1+x^2}\, dx = \int e^u\, du = e^u + C = \exp(\arctan x) + C.$$

C08S02.019: Let $u = x^2$, so that $du = 2x\, dx$, and so

$$\int \frac{3x}{\sqrt{1-x^4}}\, dx = \frac{3}{2}\int \frac{2x}{\sqrt{1-x^4}}\, dx = \frac{3}{2}\int \frac{1}{\sqrt{1-u^2}}\, du = \frac{3}{2}\arcsin u + C = \frac{3}{2}\arcsin\left(x^2\right) + C.$$

C08S02.020: Let $u = \sin 2x$. Then $du = 2\cos 2x\, dx$, and thus

$$\int \sin^3 2x\, \cos 2x\, dx = \frac{1}{2}\int u^3\, du = \frac{1}{8}u^4 + C = \frac{1}{8}\sin^4 2x + C.$$

The *Mathematica* 3.0 command

```
Integrate[ ( (Sin[2*x]∧3 )*Cos[2*x], x ] + C
```

produces exactly the same response, as does the analogous command

```
int( sin(2*x)∧3*cos(2*x), x) + C;
```

in *Maple* V Ver. 5.1 and a similar command in *Derive* 2.56.

C08S02.021: Let $u = \tan 3x$. Then $du = 3\sec^2 3x\ dx$. Hence

$$\int (\tan 3x)^4 \sec^2 3x\ dx = \frac{1}{3}\int (\tan 3x)^4 (3\sec^2 3x)\ dx = \frac{1}{3}\int u^4\ du = \frac{1}{15}u^5 + C = \frac{1}{15}\tan^5 3x\ + C.$$

C08S02.022: Let $u = 2t$. Then $du = 2\ dt$, so that

$$\int \frac{1}{1+4t^2}\ dt = \frac{1}{2}\int \frac{1}{1+4t^2}\cdot 2\ dt = \frac{1}{2}\int \frac{1}{1+u^2}\ du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(2t) + C.$$

C08S02.023: Let $u = \sin\theta$. Then $du = \cos\theta\ d\theta$. Thus

$$\int \frac{\cos\theta}{1+\sin^2\theta}\ d\theta = \int \frac{1}{1+u^2}\ du = \arctan u + C = \arctan(\sin\theta) + C.$$

C08S02.024: Let $u = 1 + \tan\theta$. Then $du = \sec^2\theta\ d\theta$, so that

$$\int \frac{\sec^2\theta}{1+\tan\theta}\ d\theta = \int \frac{1}{u}\ du = \ln|u| + C = \ln|1+\tan\theta| + C.$$

A *Maple* V ver. 5.1 command similar to the one in the solution of Problem 20 elicits the response

$$-\ln\left(\tan\left(\frac{t}{2}\right) - 1\right) - \ln\left(\tan\left(\frac{t}{2}\right) + 1\right) + \ln\left(\tan^2\left(\frac{t}{2}\right) - 2\tan\left(\frac{t}{2}\right) - 1\right) + C.$$

Derive 2.56 yields the antiderivative

$$\ln\frac{\sin t\ +\ \cos t}{\cos t}\ + C$$

and *Mathematica* 3.0 returns the antiderivative in the form

$$-\frac{[\ln\cos t]\,(\cos t + \sin t)\sec t}{1+\tan t} + \frac{[\ln(\cos t + \sin t)]\,(\cos t + \sin t)\sec t}{1+\tan t} + C.$$

C08S02.025: Let $u = 1 + x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}\ dx$, and so

$$\int (1+x^{1/2})^4\cdot x^{-1/2}\ dx = 2\int (1+x^{1/2})^4\cdot \frac{1}{2}x^{-1/2}\ dx = 2\int u^4\ du = \frac{2}{5}u^5 + C = \frac{2}{5}\left(1+\sqrt{x}\right)^5 + C.$$

C08S02.026: Let $u = t^{2/3} - 1$: $du = \frac{2}{3}t^{-1/3}\ dt$ and

$$\int t^{-1/3}(t^{2/3}-1)^{1/2}\,dt = \frac{3}{2}\int (t^{2/3}-1)^{1/2}\cdot\frac{2}{3}t^{-1/3}\,dt = \frac{3}{2}\int u^{1/2}\,du = \frac{3}{2}\cdot\frac{2}{3}u^{3/2}+C = (t^{2/3}-1)^{3/2}+C.$$

C08S02.027: Let $u = \arctan t$. Then $du = \dfrac{1}{1+t^2}\,dt$, and thus

$$\int \frac{1}{(1+t^2)\arctan t}\,dt = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\arctan t| + C.$$

C08S02.028: Let $u = 1+\sec 2x$. Then $du = 2\sec 2x\,\tan 2x\,dx$. Therefore

$$\int \frac{\sec 2x\,\tan 2x}{(1+\sec 2x)^{3/2}}\,dx = \frac{1}{2}\int \frac{2\sec 2x\,\tan 2x}{(1+\sec 2x)^{3/2}}\,dx = \frac{1}{2}\int u^{-3/2}\,du = \frac{1}{2}\cdot(-2)u^{-1/2}+C = -\frac{1}{\sqrt{1+\sec 2x}}+C.$$

C08S02.029: Let $u = e^x$. Then $du = e^x\,dx$. The first equality that follows is motivated by the similarity of the integrand to part of the derivative of the inverse secant function:

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{e^x}{|e^x|\sqrt{(e^x)^2-1}}\,dx = \int \frac{1}{|u|\sqrt{u^2-1}}\,du$$

$$= \operatorname{arcsec} u + C = \operatorname{arcsec}(e^x) + C = \arctan\left(\sqrt{e^{2x}-1}\right) + C.$$

To obtain the last equality, and thus the answer in the form given by *Mathematica* 3.0, draw a right triangle and label an acute angle θ. Let the hypotenuse have length e^x and the side adjacent θ have length 1. Then $\sec\theta = e^x$, so that $\theta = \operatorname{arcsec}(e^x)$. By the Pythagorean theorem, the side opposite θ has length $\sqrt{e^{2x}-1}$, and it follows that $\theta = \arctan\left(\sqrt{e^{2x}-1}\right)$.

C08S02.030: Let $u = \exp\left(x^2\right) = e^{(x^2)}$. Then $du = 2x\exp\left(x^2\right)\,dx = 2xe^{(x^2)}\,dx$, and so

$$\int \frac{x}{\sqrt{e^{(2x^2)}-1}}\,dx = \frac{1}{2}\int \frac{2xe^{(x^2)}}{\left|e^{(x^2)}\right|\sqrt{\left(e^{(x^2)}\right)^2-1}}\,dx = \frac{1}{2}\int \frac{1}{|u|\sqrt{u^2-1}}\,du$$

$$= \frac{1}{2}\operatorname{arcsec} u + C = \frac{1}{2}\operatorname{arcsec}\left(\exp(x^2)\right) + C.$$

C08S02.031: Let $u = x-2$; then $x = u+2$ and $dx = du$. Thus

$$\int x^2(x-2)^{1/2}\,dx = \int (u+2)^2u^{1/2}\,du = \int \left(u^{5/2}+4u^{3/2}+4u^{1/2}\right)du$$

$$= \frac{2}{7}u^{7/2}+\frac{8}{5}u^{5/2}+\frac{8}{3}u^{3/2}+C = \frac{2}{7}(x-2)^{7/2}+\frac{8}{5}(x-2)^{5/2}+\frac{8}{3}(x-2)^{3/2}+C.$$

The answer is quite acceptable in this form, but simplifications are possible; for example, before replacing u with $x-2$ in the last line, we could proceed as follows:

$$\cdots = \frac{2u^{3/2}}{105}\left(15u^2+84u+140\right)+C = \frac{2}{105}(x-2)^{3/2}\left[15(x^2-4x+4)+84(x-2)+140\right]+C$$

$$= \frac{2}{105}(x-2)^{3/2}\left(15x^2-60x+60+84x-168+140\right)+C = \frac{2}{105}(x-2)^{3/2}(15x^2+24x+32)+C.$$

C08S02.032: Let $u = x + 3$: $x = u - 3$, $dx = du$, and

$$\int x^2(x+3)^{-1/2}\,dx = \int (u-3)^2 u^{-1/2}\,du = \int \left(u^{3/2} - 6u^{1/2} + 9u^{-1/2}\right)\,du$$

$$= \frac{2}{5}u^{5/2} - 4u^{3/2} + 18u^{1/2} + C = \frac{2}{5}(x+3)^{5/2} - 4(x+3)^{3/2} + 18(x+3)^{1/2} + C.$$

The answer is quite acceptable in this form, but simplifications are possible; for example, before replacing u with $x+3$ in the last line, we could proceed as follows:

$$\cdots = \frac{2}{5}u^{1/2}(u^2 - 10u + 45) + C = \frac{2}{5}(x+3)^{1/2}\left[(x+3)^2 - 10(x+3) + 45\right] + C$$

$$= \frac{2}{5}(x+3)^{1/2}\left(x^2 + 6x + 9 - 10x - 30 + 45\right) + C = \frac{2}{5}(x+3)^{1/2}(x^2 - 4x + 24) + C.$$

Maple V ver. 5.1 and *Derive* 2.56 return the antiderivative in forms similar to the first shown here; *Mathematica* 3.0 gives an answer quite similar to the second version here.

C08S02.033: Let $u = 2x + 3$, so that $x = \frac{1}{2}(u-3)$ and $ds = \frac{1}{2}\,du$. Then

$$\int x(2x+3)^{-1/2}\,dx = \frac{1}{2}\int x(2x+3)^{-1/2}\cdot 2\,dx = \frac{1}{2}\int \frac{1}{2}(u-3)u^{-1/2}\,du$$

$$= \frac{1}{4}\int \left(u^{1/2} - 3u^{-1/2}\right)\,du = \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 6u^{1/2}\right) + C = \frac{1}{6}(2x+3)^{3/2} - \frac{3}{2}(2x+3)^{1/2} + C.$$

If you need further simplifications, proceed as follows before replacing u with $2x+3$:

$$\cdots = \frac{1}{6}u^{1/2}(u-9) + C = \frac{1}{6}(2x+3)^{1/2}(2x-6) + C = \frac{1}{3}(x-3)\sqrt{2x+3} + C.$$

C08S02.034: Let $u = x - 1$. Then $x = u + 1$ and $dx = du$. Thus

$$\int x(x-1)^{1/3}\,dx = \int (u+1)u^{1/3}\,du = \int \left(u^{4/3} + u^{1/3}\right)\,du$$

$$= \frac{3}{7}u^{7/3} + \frac{3}{4}u^{4/3} + C = \frac{3}{7}(x-1)^{7/3} + \frac{3}{4}(x-1)^{4/3} + C.$$

If you desire additional simplifications, proceed as follows before replacing u with $x-1$ in the last line:

$$\cdots = \frac{3}{28}u^{4/3}(4u+7) + C = \frac{3}{28}(x-1)^{4/3}(4x+3) + C.$$

C08S02.035: Let $u = x + 1$, so that $x = u - 1$ and $dx = du$. Then

$$\int x(x+1)^{-1/3}\,dx = \int (u-1)u^{-1/3}\,du = \int \left(u^{2/3} - u^{-1/3}\right)\,du$$

$$= \frac{3}{5}u^{5/3} - \frac{3}{2}u^{2/3} + C = \frac{3}{5}(x+1)^{5/3} - \frac{3}{2}(x+1)^{2/3} + C.$$

If additional simplifications are required, proceed as follows before replacing u with $x+1$ in the last line:

$$\cdots = 3u^{2/3}\left(\frac{1}{5}u - \frac{1}{2}\right) + C = \frac{3}{10}u^{2/3}(2u-5) + C = \frac{3}{10}(x+1)^{2/3}(2x-3) + C.$$

C08S02.036: Let $u = 3x$. Then $du = 3\,dx$, and thus

$$I = \int \frac{1}{100 + 9x^2}\,dx = \frac{1}{3}\int \frac{1}{10^2 + u^2}\,du = \frac{1}{3}\left(\frac{1}{10}\arctan\frac{u}{10}\right) + C = \frac{1}{30}\arctan\left(\frac{3x}{10}\right) + C.$$

Derive 2.56 returns an identical answer, as do *Mathematica* 3.0 and *Maple* V ver. 5.1.

C08S02.037: Let $u = 3x$. Then $du = 3\,dx$, and thus

$$I = \int \frac{1}{100 - 9x^2}\,dx = \frac{1}{3}\int \frac{1}{10^2 - u^2}\,du = \frac{1}{3}\left(\frac{1}{20}\ln\left|\frac{u+10}{u-10}\right|\right) + C = \frac{1}{60}\ln\left|\frac{3x+10}{3x-10}\right| + C.$$

Mathematica 3.0 and *Maple* V ver. 5.1 return the answer

$$I = C - \frac{1}{60}\ln(-10 + 3x) + \frac{1}{60}\ln(10 + 3x).$$

The only difference is that both omit the absolute value symbols. The antiderivative produced by *Derive* 2.56 is almost the same:

$$I \;=\; -\frac{1}{60}\ln\frac{3x - 10}{3x + 10} + C.$$

C08S02.038: Let $u = 2x$. Then $du = 2\,dx$, and thus

$$\begin{aligned}\int (9 - 4x^2)^{1/2} &= \frac{1}{2}\int (3^2 - u^2)^{1/2}\,du \\ &= \frac{1}{2}\left[\frac{u}{2}(3^2 - u^2)^{1/2} + \frac{3^2}{2}\arcsin\frac{u}{3}\right] + C = \frac{1}{2}x\sqrt{9 - 4x^2} + \frac{9}{4}\arcsin\left(\frac{2x}{3}\right) + C.\end{aligned}$$

Maple V ver. 5.1, *Derive* 2.56, and *Mathematica* 3.0 return almost exactly the same answer.

C08S02.039: Let $u = 3x$, so that $du = 3\,dx$. Then

$$\begin{aligned}J &= \int (4 + 9x^2)^{1/2}\,dx = \frac{1}{3}\int (4 + u^2)^{1/2}\,du \\ &= \frac{1}{3}\left[\frac{1}{2}u(4 + u^2)^{1/2} + 2\ln\left|u + (4 + u^2)^{1/2}\right|\right] = \frac{1}{2}x(4 + 9x^2)^{1/2} + \frac{2}{3}\ln\left(3x + (4 + 9x^2)^{1/2}\right) + C.\end{aligned}$$

Mathematica 3.0 and *Maple* V ver. 5.1 return instead

$$J = C + \frac{1}{2}x\sqrt{4 + 9x^2} + \frac{2}{3}\text{arcsinh}\left(\frac{3x}{2}\right).$$

But Eq. (34) in Section 7.6 makes it clear that the two answers agree. Here's how: Eq. (34) tells us that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

for all x. Thus

$$\sinh^{-1}\left(\frac{3x}{2}\right) = \ln\left(\frac{3x}{2} + \sqrt{\left(\frac{3x}{2}\right)^2 + 1}\,\right) = \ln\left(\frac{3x}{2} + \sqrt{\frac{9x^2+4}{4}}\,\right)$$

$$= \ln\left(\frac{3x + \sqrt{9x^2+4}}{2}\right) = \ln\left(3x + \sqrt{4+9x^2}\,\right) - \ln 2.$$

Therefore the two answers differ by a constant, as expected. *Derive* 2.56 yields the antiderivative in the first form given here.

C08S02.040: Let $u = 4x$. Then $du = 4\,dx$, and thus

$$K = \int \frac{1}{(16x^2+9)^{1/2}}\,dx = \frac{1}{4}\int \frac{1}{(u^2+9)^{1/2}}\,du = \frac{1}{4}\ln\left|u + \sqrt{u^2+9}\,\right| + C = \frac{1}{4}\ln\left(4x + \sqrt{16x^2+9}\,\right) + C.$$

Mathematica 3.0 returns instead the equivalent answer

$$K = C + \frac{1}{4}\operatorname{arcsinh}\left(\frac{4x}{3}\right);$$

see the solution of Problem 39 for an explanation of why the two are equivalent.

C08S02.041: Let $u = 4x$; then $du = 4\,dx$ and we obtain

$$I = \int \frac{x^2}{\sqrt{16x^2+9}}\,dx = \frac{1}{4}\int \frac{\frac{1}{16}u^2}{\sqrt{u^2+9}}\,du = \frac{1}{64}\int \frac{u^2}{\sqrt{u^2+9}}\,du$$

$$= \frac{1}{64}\left(\frac{1}{2}u\sqrt{u^2+9} - \frac{9}{2}\ln\left|u + \sqrt{u^2+9}\,\right|\right) + C$$

$$= \frac{1}{32}x\sqrt{16x^2+9} - \frac{9}{128}\ln\left(4x + \sqrt{16x^2+9}\,\right) + C.$$

Mathematica 3.0 returns the equivalent answer

$$I = C + \frac{1}{32}x\sqrt{9+16x^2} - \frac{9}{128}\operatorname{arcsinh}\left(\frac{4x}{3}\right).$$

See the solution of Problem 39 for an explanation of why the two antiderivatives are equivalent (that is, they differ by a constant).

C08S02.042: Let $u = 4x$. Then $du = 4\,dx$ and thus

$$J = \int \frac{x^2}{\sqrt{25+16x^2}}\,dx = \frac{1}{4}\int \frac{\frac{1}{16}u^2}{\sqrt{25+u^2}}\,du = \frac{1}{64}\int \frac{u^2}{\sqrt{25+u^2}}\,du$$

$$= \frac{1}{64}\left(\frac{1}{2}u\sqrt{25-u^2} - \frac{25}{2}\ln\left|u + \sqrt{25+u^2}\,\right|\right) + C$$

$$= \frac{1}{32}x\sqrt{25+16x^2} - \frac{25}{128}\ln\left(4x + \sqrt{25+16x^2}\,\right) + C.$$

Mathematica 3.0 returns the equivalent answer

$$J = C + \frac{1}{32}x\sqrt{25+16x^2} - \frac{25}{128}\text{arcsinh}\left(\frac{4x}{5}\right).$$

See the solution of Problem 39 for an explanation of why the two answers differ only by a constant.

C08S02.043: Let $u = 4x$, so that $du = 4\,dx$. Thus

$$\begin{aligned}\int x^2\sqrt{25-16x^2}\,dx &= \frac{1}{4}\int \frac{1}{16}u^2\sqrt{25-u^2}\,du \\ &= \frac{1}{64}\left[\frac{1}{8}u(2u^2-25)\sqrt{25-u^2} + \frac{625}{8}\arcsin\frac{u}{5}\right] + C \\ &= \frac{1}{128}x(32x^2-25)\sqrt{25-16x^2} + \frac{625}{512}\arcsin\left(\frac{4x}{5}\right) + C.\end{aligned}$$

Mathematica 3.0 and *Maple* V ver. 5.1 return an almost identical answer.

C08S02.044: Let $u = x^2$. Then $du = 2x\,dx$, and therefore

$$\begin{aligned}\int x(4-x^4)^{1/2}\,dx = \frac{1}{2}\int(4-u^2)^{1/2}\,du &= \frac{1}{2}\left(\frac{1}{2}u\sqrt{4-u^2} + 2\arcsin\frac{u}{2}\right) + C \\ &= \frac{1}{4}x^2\sqrt{4-x^4} + \arcsin\left(\frac{x^2}{2}\right) + C.\end{aligned}$$

Mathematica 3.0 returns an almost identical answer.

C08S02.045: Let $u = e^x$. Then $du = e^x\,dx$, and thus

$$\begin{aligned}K = \int e^x\sqrt{9+e^{2x}}\,dx = \int\sqrt{9+u^2}\,du &= \frac{1}{2}u\sqrt{9+u^2} + \frac{9}{2}\ln\left|u+\sqrt{9+u^2}\right| + C \\ &= \frac{1}{2}e^x\sqrt{9+e^{2x}} + \frac{9}{2}\ln\left(e^x+\sqrt{9+e^{2x}}\right) + C.\end{aligned}$$

Mathematica 3.0 returns the equivalent answer

$$K = C + \frac{1}{2}e^x\sqrt{9+e^{2x}} + \frac{9}{2}\text{arcsinh}\left(\frac{e^x}{3}\right).$$

See the solution of Problem 39 for the reason that the two answers differ only by a constant.

C08S02.046: Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$I = \int\frac{\cos x}{(\sin^2 x)\sqrt{1+\sin^2 x}}\,dx = \int\frac{1}{u^2\sqrt{1+u^2}}\,du = -\frac{\sqrt{1+u^2}}{u} + C = -(1+\sin^2 x)^{1/2}\csc x + C.$$

Mathematica 3.0 returns instead

$$I = C - \frac{\sqrt{3-\cos 2x}}{\sqrt{2}}\csc x.$$

But the two answers are exactly the same because

$$(1+\sin^2 x)^{1/2} - \frac{(3-\cos 2x)^{1/2}}{\sqrt{2}} = (1+\sin^2 x)^{1/2} - \left(\frac{3-\cos 2x}{2}\right)^{1/2}$$

$$= (1+\sin^2 x)^{1/2} - \left(1 + \frac{1-\cos 2x}{2}\right)^{1/2} = (1+\sin^2 x)^{1/2} - (1+\sin^2 x)^{1/2} = 0.$$

Derive 2.56 and *Maple* V ver. 5.1 express the antiderivative in almost the same form as the first answer here.

C08S02.047: Let $u = x^2$, so that $du = 2x\,dx$. Then

$$J = \int \frac{(x^4-1)^{1/2}}{x}\,dx = \frac{1}{2}\int \frac{(x^4-1)^{1/2}}{x^2}\cdot 2x\,dx = \frac{1}{2}\int \frac{(u^2-1)^{1/2}}{u}\,du$$

$$= \frac{1}{2}\left(\sqrt{u^2-1} - \operatorname{arcsec} u\right) + C = \frac{1}{2}\sqrt{x^4-1} - \frac{1}{2}\operatorname{arcsec}\left(x^2\right) + C.$$

Mathematica 3.0 and *Maple* V ver. 5.1 return the antiderivative in the form

$$J = C + \frac{1}{2}\sqrt{-1+x^4} + \frac{1}{2}\arctan\left(\frac{1}{\sqrt{-1+x^4}}\right).$$

Derive 2.56 returns the antiderivative in the form

$$J = \frac{1}{2}\sqrt{x^4-1} - \frac{1}{2}\arctan\left(\sqrt{x^4-1}\right)$$

(like most computer algebra programs, *Derive* omits the constant of integration). Because

$$\arctan\left(\frac{1}{x}\right) = \pi - \arctan x$$

if $x \neq 0$, the last two answers differ by a constant. See the solution to Problem 29 for an explanation of why our answer is the same as the one given by *Derive*.

C08S02.048: Let $u = 4e^x$. Then $du = 4e^x\,dx$, and therefore

$$K = \int \frac{e^{3x}}{\sqrt{25+16e^{2x}}}\,dx = \frac{1}{4}\int \frac{e^{2x}}{\sqrt{25+16e^{2x}}}\cdot 4e^x\,dx = \frac{1}{64}\int \frac{u^2}{\sqrt{25+u^2}}\,du$$

$$= \frac{1}{64}\left(\frac{1}{2}u\sqrt{25+u^2} - \frac{25}{2}\ln\left|u+\sqrt{25+u^2}\,\right|\right) + C$$

$$= \frac{1}{32}e^x\sqrt{25+16e^{2x}} - \frac{25}{128}\ln\left(4e^x + \sqrt{25+16e^{2x}}\,\right) + C.$$

Maple V ver. 5.1 and *Mathematica* 3.0 both report that

$$K = C + \frac{1}{32}e^x\sqrt{25+16e^{2x}} - \frac{25}{128}\operatorname{arcsinh}\left(\frac{4e^x}{5}\right).$$

See the solution to Problem 39 for the reason why this answer differs from the first only by a constant. *Derive* 2.56 yields an antiderivative essentially the same as ours.

C08S02.049: Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, and hence

$$I = \int \frac{(\ln x)^2}{x} \sqrt{1 + (\ln x)^2}\ dx = \int u^2 \sqrt{1+u^2}\ du = \frac{1}{8}u(2u^2+1)\sqrt{1+u^2} - \frac{1}{8}\ln\left|u + \sqrt{1+u^2}\,\right| + C$$

$$= \frac{1}{8}(\ln x)\left[2(\ln x)^2 + 1\right]\sqrt{1+(\ln x)^2} - \frac{1}{8}\ln\left(\ln x + \sqrt{1+(\ln x)^2}\,\right) + C.$$

Maple V ver. 5.1, *Mathematica* 3.0, and *Derive* 2.56 all yield

$$I = C - \frac{1}{8}\text{arcsinh}(\ln x) + \sqrt{1+(\ln x)^2}\left[\frac{1}{8}\ln x + \frac{1}{4}(\ln x)^3\right],$$

which differs from our first answer only by a constant (see the explanation in the solution of Problem 39).

C08S02.050: Let $u = 2x^3$, so that $du = 6x^2\ dx$. Then

$$J = \int x^8(4x^6-1)^{1/2}\ dx = \frac{1}{6}\int x^6(4x^6-1)^{1/2}\cdot 6x^2\ dx = \frac{1}{6}\int \frac{1}{4}u^2(u^2-1)^{1/2}\ du = \frac{1}{24}\int u^2(u^2-1)^{1/2}\ du$$

$$= \frac{1}{24}\left(\frac{1}{8}u(2u^2-1)\sqrt{u^2-1} - \frac{1}{8}\ln\left|u + \sqrt{u^2-1}\,\right|\right) + C$$

$$= \frac{1}{96}x^3(8x^6-1)\sqrt{4x^6-1} - \frac{1}{192}\ln\left(2x^3 + \sqrt{4x^6-1}\,\right) + C.$$

Derive 2.56 returns the same answer in slightly expanded form. *Mathematica* 3.0 yields

$$J = C + \sqrt{-1+4x^6}\left(-\frac{1}{96}x^3 + \frac{1}{12}x^9\right) - \frac{1}{192}\ln\left(4x^3 + 2\sqrt{-1+4x^6}\,\right),$$

which differs from the others only by the constant $\ln 2$, because

$$\ln\left(4x^3 + 2\sqrt{-1+4x^6}\,\right) = \ln\left(2\left[2x^3 + \sqrt{4x^6-1}\,\right]\right) = \ln 2 + \ln\left(2x^3 + \sqrt{4x^6-1}\,\right).$$

C08S02.051: The substitution is illegal: $x = \sqrt{u} \geqq 0$, but $x < 0$ for many x in $[-1, 1]$. To use this substitution correctly, let

$$x = u^{1/2}, \quad dx = \frac{1}{2u^{1/2}}\ du, \quad 0 \leqq x \leqq 1.$$

Then

$$\int_0^1 x^2\ dx = \int_0^1 \frac{1}{2}u^{1/2}\ du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{1}{3}.$$

Then let

$$x = -u^{1/2}, \quad dx = -\frac{1}{2u^{1/2}}\ du, \quad -1 \leqq x \leqq 0.$$

Then

$$\int_{-1}^0 x^2\ dx = \int_1^0 -\frac{1}{2}u^{1/2}\ du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{1}{3}.$$

And, finally,

$$\int_{-1}^{1} x^2\,dx = \int_{-1}^{0} x^2\,dx + \int_{0}^{1} x^2\,dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

C08S02.052: The substitution $u = x + 2$, $x = u - 2$, $dx = du$ yields

$$\int \frac{1}{x^2 + 4x + 5}\,dx = \int \frac{1}{1 + (x+2)^2}\,dx = \int \frac{1}{1 + u^2}\,du = \arctan u + C = \arctan(x+2) + C.$$

C08S02.053: The substitution $u = x - 1$, $x = u + 1$, $dx = du$ yields

$$\int \frac{1}{\sqrt{2x - x^2}}\,dx = \int \frac{1}{\sqrt{1 - (x-1)^2}}\,dx = \int \frac{1}{\sqrt{1 - u^2}}\,du = \arcsin u + C = \arcsin(x-1) + C.$$

C08S02.054: The binomial theorem yields

$$\begin{aligned}\frac{1}{6}(1 + \ln x)^6 &= \frac{1}{6}\left[1 + 6\ln x + 15(\ln x)^2 + 20(\ln x)^3 + 15(\ln x)^4 + 6(\ln x)^5 + (\ln x)^6\right]\\ &= \frac{1}{6} + \ln x + \frac{5}{2}(\ln x)^2 + \frac{10}{3}(\ln x)^3 + \frac{5}{2}(\ln x)^4 + (\ln x)^5 + \frac{1}{6}(\ln x)^6,\end{aligned}$$

which differs from the machine's answer by exactly $\dfrac{1}{6}$, a constant.

C08S02.055: First note that

$$D_x\left(\frac{1}{2}\tan^{-1} x^2\right) = \frac{x}{1 + x^4}$$

for all x and that

$$D_x\left(-\frac{1}{2}\tan^{-1} x^{-2}\right) = -\frac{1}{2}\cdot\frac{1}{1 + x^{-4}}\cdot(-2x^{-3}) = \frac{x^4 \cdot x^{-3}}{x^4 + 1} = \frac{x}{1 + x^4}$$

provided that $x \neq 0$. Therefore

$$\frac{1}{2}\tan^{-1} x^2 + \frac{1}{2}\tan^{-1} x^{-2} = C_1,$$

a constant, for all $x > 0$ and

$$\frac{1}{2}\tan^{-1} x^2 + \frac{1}{2}\tan^{-1} x^{-2} = C_2,$$

a constant, for all $x < 0$. Moreover, substitution of $x = 1$ yields

$$C_1 = \frac{1}{2}\tan^{-1} 1 + \frac{1}{2}\tan^{-1} 1 = \frac{\pi}{2}$$

and substitution of $x = -1$ yields

$$C_2 = \frac{1}{2}\tan^{-1}(-1)^2 + \frac{1}{2}\tan^{-1}(-1)^2 = \frac{\pi}{2}$$

.

Therefore

$$\frac{1}{2}\tan^{-1} x^2 = \frac{\pi}{2} - \frac{1}{2}\tan^{-1} x^{-2}$$

for all $x \neq 0$.

C08S02.056: Equation (34) in Section 6.9 tells us that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\,\right)$ for all x.

C08S02.057: Here is a *Mathematica* solution:

```
G = (x/2)*Sqrt[x∧2 + 1] + (1/2)*Log[x + Sqrt[x∧2 + 1]];
```

```
D[G,x]
```

$$\frac{x^2}{2\sqrt{1+x^2}} + \frac{\sqrt{1+x^2}}{2} + \frac{1 + \dfrac{x}{\sqrt{1+x^2}}}{2\left(x + \sqrt{1+x^2}\,\right)}$$

```
% // Together
```

$$\sqrt{1+x^2}$$

```
H = (1/8)*((x + Sqrt[x∧2 + 1])∧2 + 4*Log[x + Sqrt[x∧2 + 1]]
− (x + Sqrt[x∧2 + 1])∧(−2))
```

$$\frac{1}{8}\left[-\left(x+\sqrt{1+x^2}\,\right)^{-2} + \left(x+\sqrt{1+x^2}\,\right)^2 + 4\ln\left(x+\sqrt{1+x^2}\,\right)\right]$$

```
DH = D[H,x]
```

$$\frac{1}{8}\left[\frac{2\left(1+\dfrac{x}{\sqrt{1+x^2}}\right)}{\left(x+\sqrt{1+x^2}\,\right)^3} + \frac{4\left(1+\dfrac{x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}} + 2\left(1+\frac{x}{\sqrt{1+x^2}}\right)\left(x+\sqrt{1+x^2}\,\right)\right]$$

```
DH // Together
```

$$\frac{1+3x^2+2x^4+2x\sqrt{1+x^2}\,+2x^3\sqrt{1+x^2}}{\left(x+\sqrt{1+x^2}\,\right)^2\sqrt{1+x^2}}$$

```
DH // FullSimplify
```

$$\sqrt{1+x^2}$$

The only thing we might add is that because $G(0) = H(0)$, it now follows that $G(x) = H(x)$ for all x.

Section 8.3

C08S03.001: Let $u = x$ and $dv = e^{2x}\,dx$: $du = dx$ and choose $v = \frac{1}{2}e^{2x}$. Then

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$$

C08S03.002: Let $u = x^2$ and $dv = e^{2x}\,dx$: $du = 2x\,dx$ and choose $v = \frac{1}{2}e^{2x}$. Then

$$\int x^2e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$

The last equality follows from the result in Problem 1.

C08S03.003: Let $u = t$ and $dv = \sin t\,dt$: $du = dt$ and choose $v = -\cos t$. Then

$$\int t\sin t\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C.$$

C08S03.004: Let $u = t^2$ and $dv = \sin t\,dt$: $du = 2t\,dt$ and choose $v = -\cos t$. Then

$$\int t^2\sin t\,dt = -t^2\cos t + 2\int t\cos t\,dt.$$

Next, let $u = t$ and $dv = \cos t\,dt$: $du = dt$ and choose $v = \sin t$. Then

$$\int t^2\sin t\,dt = -t^2\cos t + 2\left(t\sin t - \int \sin t\,dt\right) = -t^2\cos t + 2t\sin t + 2\cos t + C.$$

C08S03.005: Let $u = x$ and $dv = \cos 3x\,dx$: $du = dx$ and choose $v = \dfrac{1}{3}\sin 3x$. Then

$$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C.$$

C08S03.006: Let $u = \ln x$ and $dv = x\,dx$: $du = \dfrac{1}{x}\,dx$ and choose $v = \dfrac{1}{2}x^2$. Then

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

C08S03.007: Let $u = \ln x$ and $dv = x^3\,dx$: $du = \dfrac{1}{x}\,dx$ and choose $v = \dfrac{1}{4}x^4$. Then

$$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C.$$

C08S03.008: Let $u = \cos 3z$ and $dv = e^{3z}\,dz$: $du = -3\sin 3z\,dz$ and choose $v = \dfrac{1}{3}e^{3z}$. Then

$$\int e^{3z}\cos 3z\,dz = \frac{1}{3}e^{3z}\cos 3z + \int e^{3z}\sin 3z\,dz.$$

Next let $u = \sin 3z$ and $dv = e^{3z}\,dz$: $du = 3\cos 3z\,dz$ and choose $v = \dfrac{1}{3}e^{3z}$. Then

$$\int e^{3z}\cos 3z\,dz = \frac{1}{3}e^{3z}\cos 3z + \frac{1}{3}e^{3z}\sin 3z - \int e^{3z}\cos 3z\,dz;$$

$$2\int e^{3z}\cos 3z\,dz = \frac{1}{3}e^{3z}\cos 3z + \frac{1}{3}e^{3z}\sin 3z + 2C;$$

$$\int e^{3z}\cos 3z\,dz = \frac{1}{6}e^{3z}(\cos 3z + \sin 3z) + C.$$

C08S03.009: Let $u = \arctan x$ and $dv = dx$: $du = \dfrac{1}{1+x^2}\,dx$ and choose $v = x$. Then

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

C08S03.010: Let $u = \ln x$ and $dv = \dfrac{1}{x^2}\,dx$: $du = \dfrac{1}{x}\,dx$ and choose $v = -\dfrac{1}{x}$. Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

C08S03.011: Let $u = \ln y$ and $dv = y^{1/2}\,dy$: $du = \dfrac{1}{y}\,dy$ and choose $v = \dfrac{2}{3}y^{3/2}$. Then

$$\int y^{1/2}\ln y\,dy = \frac{2}{3}y^{3/2}\ln y - \frac{2}{3}\int y^{1/2}\,dy = \frac{2}{3}y^{3/2}\ln y - \frac{4}{9}y^{3/2} + C.$$

C08S03.012: Let $u = x$ and $dv = \sec^2 x\,dx$: $du = dx$ and choose $v = \tan x$. Then

$$\int x\sec^2 x\,dx = x\tan x - \int \frac{\sin x}{\cos x}\,dx = x\tan x + \ln|\cos x| + C.$$

C08S03.013: Let $u = (\ln t)^2$ and $dv = dt$: $du = \dfrac{2\ln t}{t}\,dt$ and choose $v = t$. Then

$$\int (\ln t)^2\,dt = t(\ln t)^2 - 2\int \ln t\,dt.$$

Next let $u = \ln t$ and $dv = dt$: $du = \dfrac{1}{t}\,dt$ and choose $v = t$. Thus

$$\int (\ln t)^2\,dt = t(\ln t)^2 - 2\left(t\ln t - \int 1\,dt\right) = t(\ln t)^2 - 2t\ln t + 2t + C.$$

C08S03.014: Let $u = (\ln t)^2$ and $dv = t\,dt$. Then $du = \dfrac{2\ln t}{t}\,dt$; choose $v = \dfrac{1}{2}t^2$. Thus

$$\int t(\ln t)^2\,dt = \frac{1}{2}(t\ln t)^2 - \int t\ln t\,dt = \frac{1}{2}(t\ln t)^2 - \frac{1}{2}t^2\ln t + \frac{1}{4}t^2 + C.$$

(The last equality follows from the result in Problem 6.)

C08S03.015: Let $u = x$ and $dv = (x+3)^{1/2}\,dx$: $du = dx$ and choose $v = \dfrac{2}{3}(x+3)^{3/2}$. Then

$$\int x(x+3)^{1/2}\,dx = \frac{2}{3}x(x+3)^{3/2} - \frac{2}{3}\int (x+3)^{3/2}\,dx = \frac{2}{3}x(x+3)^{3/2} - \frac{4}{15}(x+3)^{5/3} + C$$
$$= (x+3)^{3/2}\left(\frac{2}{3}x - \frac{4}{15}x - \frac{4}{5}\right) + C = (x+3)^{3/2}\left(\frac{6x-12}{15}\right) + C$$
$$= \frac{2}{5}(x-2)(x+3)^{3/2} + C = \frac{2}{5}(x^2+x-6)\sqrt{x+3}\, + C.$$

C08S03.016: Let $u = x^2$ and $dv = x(1-x^2)^{1/2}$: $du = 2x\,dx$; choose $v = -\frac{1}{3}(1-x^2)^{3/2}$. Then

$$\int x^3(1-x^2)^{1/2}\,dx = -\frac{1}{3}x^2(1-x^2)^{3/2} + \frac{2}{3}\int x(1-x^2)^{3/2}\,dx$$
$$= -\frac{1}{3}x^2(1-x^2)^{3/2} - \frac{2}{15}(1-x^2)^{5/2} + C = -(1-x^2)^{3/2}\left(\frac{1}{3}x^2 + \frac{2}{15}(1-x^2)\right) + C$$
$$= -(1-x^2)^{3/2}\left(\frac{3x^2+2}{15}\right) + C = \frac{1}{15}(3x^4 - x^2 - x)\sqrt{1-x^2}\, + C.$$

C08S03.017: Let $u = x^3$ and $dv = x^2(x^3+1)^{1/2}\,dx$: $du = 3x^2\,dx$ and choose $v = \frac{2}{9}(x^3+1)^{3/2}$. Then

$$\int x^5(x^3+1)^{1/2}\,dx = \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{2}{3}\int x^2(x^3+1)^{3/2}\,dx = \frac{2}{9}x^3(x^3+1)^{3/2} - \frac{4}{45}(x^3+1)^{5/2} + C$$
$$= \frac{1}{45}(x^3+1)^{3/2}\left[10x^3 - 4(x^3+1)\right] + C = \frac{1}{45}(x^3+1)^{3/2}(6x^3-4) + C$$
$$= \frac{2}{45}(x^3+1)^{3/2}(3x^3-2) + C = \frac{2}{45}(x^3+1)^{1/2}(3x^6+x^3-2) + C.$$

C08S03.018: Let $u = \sin\theta$ and $dv = \sin\theta\,d\theta$: $du = \cos\theta\,d\theta$ and choose $v = -\cos\theta$. Then

$$\int \sin^2\theta\,d\theta = -\sin\theta\,\cos\theta + \int \cos^2\theta\,d\theta$$
$$= -\sin\theta\,\cos\theta + \int (1-\cos^2\theta)\,d\theta = -\sin\theta\,\cos\theta + \theta - \int \sin^2\theta\,d\theta;$$
$$2\int \sin^2\theta\,d\theta = \theta - \sin\theta\,\cos\theta + 2C;$$
$$\int \sin^2\theta\,d\theta = \frac{1}{2}(\theta - \sin\theta\,\cos\theta) + C.$$

C08S03.019: Let $u = \csc\theta$ and $dv = \csc^2\theta\,d\theta$: $du = -\csc\theta\,\cot\theta$ and choose $v = -\cot\theta$. Then

$$\int \csc^3\theta\,d\theta = -\csc\theta\,\cot\theta - \int \csc\theta\,\cot^2\theta\,d\theta$$
$$= -\csc\theta\,\cot\theta - \int (\csc\theta)(\csc^2\theta - 1)\,d\theta = -\csc\theta\,\cot\theta - \int \csc^3\theta\,d\theta + \int \csc\theta\,d\theta;$$
$$2\int \csc^3\theta\,d\theta = -\csc\theta\,\cot\theta + \ln|\csc\theta - \cot\theta| + 2C;$$
$$\int \csc^3\theta\,d\theta = -\frac{1}{2}\csc\theta\,\cot\theta + \frac{1}{2}\ln|\csc\theta - \cot\theta| + C.$$

Mathematica 3.0 returns the antiderivative in the form

$$C - \frac{1}{2}\cot\theta\,\csc\theta - \frac{1}{2}\ln\left(\cos\frac{\theta}{2}\right) + \frac{1}{2}\ln\left(\sin\frac{\theta}{2}\right),$$

whereas *Maple* V ver. 5.1 yields an answer that is essentially the same as the one we obtained "by hand."

C08S03.020: Let $u = \sin(\ln t)$ and $dv = dt$: $du = \dfrac{1}{t}\cos(\ln t)$ and choose $v = t$. Then

$$\int \sin(\ln t)\,dt = t\sin(\ln t) - \int \cos(\ln t)\,dt.$$

Now let $u = \cos(\ln t)$ and $dv = dt$: $du = -\dfrac{1}{t}\sin(\ln t)\,dt$, and choose $v = t$. Thus

$$\begin{aligned}\int \sin(\ln t)\,dt &= t\sin(\ln t) - t\cos(\ln t) - \int \sin(\ln t)\,dt;\\ \int \sin(\ln t)\,dt &= \frac{1}{2}t\sin(\ln t) - \frac{1}{2}t\cos(\ln t) + C.\end{aligned}$$

C08S03.021: Let $u = \arctan x$ and $dv = x^2\,dx$: $du = \dfrac{1}{1+x^2}\,dx$ and choose $v = \dfrac{1}{3}x^3$. Then

$$\begin{aligned}\int x^2\arctan x\,dx &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx\\ &= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right) + C = \frac{1}{3}x^2\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C.\end{aligned}$$

C08S03.022: Let $u = \ln(1+x^2)$ and $dv = dx$: $du = \dfrac{2x}{1+x^2}\,dx$ and choose $v = x$. Then

$$\begin{aligned}\int \ln(1+x^2)\,dx &= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx\\ &= x\ln(1+x^2) - \int\left(2 - \frac{2}{1+x^2}\right)dx = x\ln(1+x^2) - 2x + 2\arctan x\, + C.\end{aligned}$$

C08S03.023: Let $u = \operatorname{arcsec}(x^{1/2})$ and $dv = dx$: $du = \dfrac{1}{2x(x-1)^{1/2}}\,dx$ and choose $v = x$. Then

$$\int \operatorname{arcsec}(x^{1/2})\,dx = x\operatorname{arcsec}(x^{1/2}) - \frac{1}{2}\int (x-1)^{-1/2}\,dx = x\operatorname{arcsec}(x^{1/2}) - (x-1)^{1/2} + C.$$

C08S03.024: Let $u = \arctan(x^{1/2})$ and $dv = x\,dx$: then $du = \dfrac{1}{2(1+x)x^{1/2}}\,dx$; choose $v = \dfrac{1}{2}x^2 - \dfrac{1}{2}$ (for a sly reason). Then

$$\begin{aligned}\int x\arctan(x^{1/2})\,dx &= \frac{x^2-1}{2}\arctan(x^{1/2}) - \frac{1}{4}\int \frac{x^2-1}{(x+1)x^{1/2}}\,dx\\ &= \frac{x^2-1}{2}\arctan(x^{1/2}) - \frac{1}{4}\int \frac{x-1}{x^{1/2}}\,dx = \frac{x^2-1}{2}\arctan(x^{1/2}) - \frac{1}{4}\int (x^{1/2} - x^{-1/2})\,dx\\ &= \frac{x^2-1}{2}\arctan(x^{1/2}) - \frac{1}{6}x^{3/2} + \frac{1}{2}x^{1/2} + C.\end{aligned}$$

C08S03.025: Let $u = \arctan(x^{1/2})$ and $dv = dx$: $du = \dfrac{1}{2(x+1)x^{1/2}}\,dx$ and cleverly choose $v = x + 1$. Then

$$\int \arctan(x^{1/2})\,dx = (x+1)\arctan(x^{1/2}) - \frac{1}{2}\int x^{-1/2}\,dx = (x+1)\arctan(x^{1/2}) - x^{1/2} + C.$$

C08S03.026: Let $u = x^2$ and $dv = \cos 4x\,dx$: $du = 2x\,dx$ and choose $v = \dfrac{1}{4}\sin 4x$. Then

$$\int x^2\cos 4x\,dx = \frac{1}{4}x^2\sin 4x - \frac{1}{2}\int x\sin 4x\,dx.$$

Next let $u = x$ and $dv = \sin 4x\,dx$. Then $du = dx$; choose $v = -\dfrac{1}{4}\cos 4x$. Hence

$$\begin{aligned}\int x^2\cos 4x\,dx &= \frac{1}{4}x^2\sin 4x - \frac{1}{2}\left(-\frac{1}{4}x\cos 4x + \frac{1}{4}\int \cos 4x\,dx\right)\\ &= \frac{1}{4}x^2\sin 4x + \frac{1}{8}x\cos 4x - \frac{1}{32}\sin 4x + C.\end{aligned}$$

C08S03.027: Let $u = x$ and $dv = \csc^2 x\,dx$: $du = dx$; choose $v = -\cot x$. Then

$$\int x\csc^2 x\,dx = -x\cot x + \int \frac{\cos x}{\sin x}\,dx = -x\cot x + \ln|\sin x| + C.$$

C08S03.028: Let $u = \arctan x$ and $dv = x\,dx$: $du = \dfrac{1}{1+x^2}\,dx$; choose $v = \dfrac{1}{2}x^2 + \dfrac{1}{2}$. Then

$$\int x\arctan x\,dx = \frac{x^2+1}{2}\arctan x - \int \frac{1}{2}\,dx = \frac{x^2+1}{2}\arctan x - \frac{1}{2}x + C.$$

C08S03.029: Let $u = x^2$ and $dv = x\cos x^2\,dx$: $du = 2x\,dx$ and choose $v = \dfrac{1}{2}\sin x^2$. Then

$$\int x^3\cos x^2\,dx = \frac{1}{2}x^2\sin x^2 - \int x\sin x^2\,dx = \frac{1}{2}x^2\sin x^2 + \frac{1}{2}\cos x^2 + C.$$

C08S03.030: Suppose that a and b are nonzero real constants. Choose $u = e^{ax}$ and $dv = \sin bx\,dx$. Then $du = ae^{ax}\,dx$; choose $v = -\dfrac{1}{b}\cos bx$. Then

$$I = \int e^{ax}\sin bs\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx.$$

Now let $u = e^{ax}$ and $dv = \cos bx\,dx$. Then $du = ae^{ax}\,dx$; choose $v = \dfrac{1}{b}\sin bx$. Thus

$$\begin{aligned}I &= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I;\\ \frac{a^2+b^2}{b^2}I &= \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{b^2} + C_1;\\ I &= \int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C.\end{aligned}$$

$$\text{So}\quad \int e^{-3x}\sin 4x\,dx = -\frac{3\sin 4x + 4\cos 4x}{25}e^{-3x} + C.$$

C08S03.031: Let $u = \ln x$ and $dv = x^{-3/2}\,dx$: $du = \dfrac{1}{x}\,dx$ and choose $v = -2x^{-1/2}$. Then

$$\int \frac{\ln x}{x^{3/2}}\,dx = -\frac{2\ln x}{x^{1/2}} + 2\int x^{-3/2}\,dx = -\frac{2\ln x}{x^{1/2}} - \frac{4}{x^{1/2}} + C.$$

C08S03.032: Let $u = x^4$ and $dv = \dfrac{x^3}{(1+x^4)^{3/2}}\,dx$: $du = 4x^3\,dx$ and choose $v = -\dfrac{1}{2}(1+x^4)^{-1/2}$. Then

$$\begin{aligned}\int \frac{x^7}{(1+x^4)^{3/2}}\,dx &= -\frac{1}{2}x^4(1+x^4)^{-1/2} + 2\int \frac{x^3}{(1+x^4)^{1/2}}\,dx\\ &= -\frac{x^4}{2\sqrt{1+x^4}} + \sqrt{1+x^4} + C = \frac{2(1+x^4)-x^4}{2\sqrt{1+x^4}} + C = \frac{x^4+2}{2\sqrt{x^4+1}} + C.\end{aligned}$$

C08S03.033: Let $u = x$ and $dv = \cosh x\,dx$: $du = dx$ and choose $v = \sinh x$. Then

$$\int x\cosh x\,dx = x\sinh x - \int \cosh x\,dx = x\sinh x - \cosh x + C.$$

C08S03.034: First method:

$$\begin{aligned}\int e^x\cosh x\,dx &= \tfrac{1}{2}\int \left(e^{2x}+1\right)\,dx = \tfrac{1}{4}e^{2x} + \tfrac{1}{2}x + C_1\\ &= \tfrac{1}{4}\left(e^{2x}+1\right) - \tfrac{1}{4} + \tfrac{1}{2}x + C_1 = \tfrac{1}{4}e^x\left(e^x + e^{-x}\right) + \tfrac{1}{2}x + C\\ &= \tfrac{1}{2}e^x\cosh x + \tfrac{1}{2}x + C.\end{aligned}$$

Second method: Presented because no integration by parts is used in the first method, although what follows is somewhat artificial.

$$\begin{bmatrix} u = e^x & dv = \cosh x\,dx\\ du = e^x\,dx & v = \sinh x\end{bmatrix} \qquad J = \int e^x\cosh x\,dx = e^x\sinh x - \int e^x\sinh x\,dx.$$

Now $e^x\sinh x = \frac{1}{2}\left(e^{2x}-1\right) = \frac{1}{2}\left(e^{2x}+1\right) - 1 = e^x\cosh x - 1$. Therefore

$$J = e^x\sinh x - J + \int 1\,dx; \quad \text{it follows that} \quad \int e^x\cosh x\,dx = \frac{1}{2}e^x\sinh x + \frac{1}{2}x + C.$$

Third method: It appears that *Mathematica* 3.0 simply writes $\cosh x$ in exponential form and then returns the antiderivative as

$$C + \frac{1}{4}e^{2x} + \frac{1}{2}x.$$

Fourth method: *Maple* V version 5.1 returns an answer that, because of its similarity to the second answer here, suggests that integration by parts is used:

$$\frac{1}{2}\cosh^2 x + \frac{1}{2}\cosh x\,\sinh x + \frac{1}{2}x + C.$$

C08S03.035: Let $t = x^2$. Then $dt = 2x\,dx$, so $\frac{1}{2}t\;dt = x^3\,dx$. This substitution transforms the given integral into

$$I = \frac{1}{2}\int t\sin t\, dt.$$

Then integrate by parts: Let $u = t$, $dv = \sin t\ dt$. Thus $du = dt$ and $v = -\cos t$, and hence

$$2I = -t\cos t + \int \cos t\ dt = -t\cos t + \sin t + C.$$

Therefore

$$\int x^3 \sin x^2\ dx = \frac{1}{2}\left(-x^2\cos x^2 + \sin x^2\right) + C.$$

C08S03.036: Let $t = x^4$. Then $dt = 4x^3\ dx$, so $x^7 = \frac{1}{4}t\ dt$. Thus the given integral becomes

$$I = \frac{1}{4}\int t\cos t\ dt.$$

Now let $u = t$ and $dv = \cos t\ dt$, so that $du = dt$ and $v = \sin t$. Hence

$$4I = t\sin t - \int \sin t\ dt = t\sin t + \cos t + C.$$

Therefore $I = \frac{1}{4}\left(x^4\sin x^4 + \cos x^4\right) + C$.

C08S03.037: Let $t = \sqrt{x}$, so that $x = t^2$ and $dx = 2t\ dt$. Thus

$$I = \int \exp\left(-\sqrt{x}\right)\ dx = \int 2t\ \exp(-t)\ dt.$$

Now let $u = 2t$ and $dv = \exp(-t)\,dt$. Then $du = 2\ dt$ and $v = -\exp(-t)$. Hence

$$I = -2t\ \exp(-t) + \int 2\exp(-t)\ dt = -2t\ \exp(-t) - 2\exp(-t) + C.$$

Therefore

$$I = -2\sqrt{x}\ \exp\left(-\sqrt{x}\right) - 2\exp\left(-\sqrt{x}\right) + C.$$

C08S03.038: Let $t = x^{3/2}$. Then $dt = \frac{3}{2}x^{1/2}\ dx$, so $t\ dt = \frac{3}{2}x^2\ dx$. Therefore

$$I = \int x^2\sin x^{3/2}\ dx = \frac{2}{3}\int t\sin t\ dt = \frac{2}{3}\left(-t\cos t + \sin t\right) + C.$$

(The integration by parts is the same as in the solution of Problem 35.) Therefore

$$I = \frac{2}{3}\left(-x^{3/2}\cos x^{3/2} + \sin x^{3/2}\right) + C.$$

C08S03.039: The volume is

$$V = \int_0^{\pi/2} 2\pi x\ \cos x\ dx = 2\pi\int_0^{\pi/2} x\ \cos x\ dx.$$

Let $u = x$ and $dv = \cos x\ dx$: $du = dx$ and choose $v = \sin x$. Then

$$\int x\,\cos x\;dx = x\,\sin x - \int \sin x\;dx = x\,\sin x + \cos x + C.$$

Therefore

$$V = 2\pi\Big[x\,\sin x + \cos x\Big]_0^{\pi/2} = 2\pi\left(\frac{\pi}{2} - 1\right) = \pi^2 - 2\pi \approx 3.5864190939.$$

C08S03.040: The volume is

$$V = \int_0^{\pi} 2\pi x \sin x\;dx = 2\pi \int_0^{\pi} x \sin x\;dx.$$

Let $u = x$ and $dv = \sin x\;dx$: $du = dx$ and choose $v = -\cos x$. Then

$$\int x \sin x\;dx = -x\cos x + \int \cos x\;dx = -x\cos x + \sin x + C.$$

Therefore

$$V = 2\pi\Big[-x\cos x + \sin x\Big]_0^{\pi} = 2\pi\,(\pi + 0 - 0 - 0) = 2\pi^2 \approx 19.7392088022.$$

C08S03.041: The volume is

$$V = \int_1^{e} 2\pi x \ln x\;dx = 2\pi \int_1^{e} x \ln x\;dx.$$

Let $u = \ln x$ and $dv = x\;dx$: $du = \dfrac{1}{x}\;dx$ and choose $v = \dfrac{1}{2}x^2$. Then

$$\int x \ln x\;dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\;dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

Therefore

$$V = 2\pi\left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^{e} = 2\pi\left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}\right) = \frac{\pi}{2}(e^2 + 1) \approx 13.1774985055.$$

C08S03.042: The volume is

$$V = \int_0^{1} 2\pi x e^{-x}\;dx = 2\pi \int_0^{1} x e^{-x}\;dx.$$

Let $u = x$ and $dv = e^{-x}\;dx$: $du = dx$ and choose $v = -e^{-x}$. Then

$$\int x e^{-x}\;dx = -xe^{-x} + \int e^{-x}\;dx = -xe^{-x} - e^{-x} + C.$$

Therefore

$$V = 2\pi\Big[-(x+1)e^{-x}\Big]_0^{1} = 2\pi\left(1 - \frac{2}{e}\right) = \frac{2\pi(e-2)}{e} \approx 1.6602759080.$$

C08S03.043: The curves intersect at the point $(a,\, b)$ in the first quadrant for which $a \approx 0.824132312$. The volume is

$$V = \int_0^a 2\pi x \left[(\cos x) - x^2\right]\,dx = 2\pi \int_0^a (x\cos x - x^3)\,dx.$$

To find the antiderivative of $x\cos x$, let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$; choose $v = \sin x$. Thus

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Therefore

$$V = 2\pi\left[x\sin x + \cos x - \frac{1}{4}x^4\right]_0^a \approx 1.06027.$$

C08S03.044: The curves intersect where $x = 0$ and where $x = a \approx 3.110367680$. The volume is

$$V = \int_0^a 2\pi x(10x - x^2 - e^x + 1)\,dx = 2\pi \int_0^a (10x^2 - x^3 - xe^x + x)\,dx.$$

Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$; choose $v = e^x$. Then

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = (x-1)e^x + C.$$

Therefore

$$V = 2\pi\left[\frac{10}{3}x^3 - \frac{1}{4}x^4 - (x-1)e^x + \frac{1}{2}x^2\right]_0^a \approx 209.907.$$

C08S03.045: The curves intersect where $x = 0$ and where $x = a \approx 2.501048238$. The volume is

$$V = \int_0^a 2\pi x\left[2x - x^2 + \ln(x+1)\right]\,dx = 2\pi \int_0^a \left[2x^2 - x^3 + x\ln(x+1)\right]\,dx.$$

Let $u = \ln(x+1)$ and $dv = x\,dx$. Then $du = \dfrac{1}{x+1}\,dx$; choose $v = \dfrac{1}{2}x^2 - \dfrac{1}{2}$. Then

$$\int x\ln(x+1)\,dx = \frac{x^2-1}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2-1}{x+1}\,dx = \frac{x^2-1}{2}\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x + C.$$

Therefore

$$V = 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{2}(x^2-1)\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x\right]_0^a \approx 22.7894.$$

C08S03.046: Let $u = \arctan x$ and $dv = 2x\,dx$: $du = \dfrac{1}{1+x^2}\,dx$ and choose $v = x^2 + 1$. Then

$$\int 2x\arctan x\,dx = (x^2+1)\arctan x - x + C.$$

Such a choice of v is permitted for the following reason. Suppose that K is a constant. Then if we use $v(x) + K$ rather than $v(x)$, the result is

$$u \cdot (v + K) - \int (v + K)\, du = uv + Ku - \int v\, du - \int K\, du$$

$$= uv + Ku - \int v\, du - Ku = uv - \int v\, du = \int u\, dv.$$

C08S03.047: First choose $u = xe^x$ and $dv = \cos x\, dx$. This yields

$$I = \int xe^x \cos x\, dx = xe^x \sin x - \int (x+1)\, e^x \sin x\, dx.$$

Now choose $u = (x + 1)\, e^x$ and $dv = \sin x\, dx$;

$$I = xe^x \sin x + (x+1)\, e^x \cos x - \int (x+2)\, e^x \cos x\; dx$$

$$= xe^x \sin x + (x+1)\, e^x \cos x - 2\int e^x \cos x\; dx - I.$$

Thus

$$2I = xe^x \sin x + (x+1)\, e^x \cos x - 2\int e^x \cos x\; dx. \tag{1}$$

Compute the right-hand integral by parts separately: Let $u = e^x$ and $dv = \cos x\; dx$. Then $du = e^x\; dx$; choose $v = \sin x$. Thus

$$J = \int e^x \cos x\; dx = e^x \sin x - \int e^x \sin x\; dx.$$

Now let $u = e^x$ and $dv = \sin x\; dx$. So $du = e^x\; dx$; choose $v \;\; -\cos x$. Thus

$$J = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\; dx\right)$$

$$= e^x \sin x + e^x \cos x - J.$$

Thus $J = \frac{1}{2}(\sin x + \cos x)e^x + C$. Substitute this result in Eq. (1), then solve for I:

$$I = \tfrac{1}{2}xe^x \cos x + \tfrac{1}{2}(x-1)\, e^x \sin x + C.$$

C08S03.048: Given: Constants A and B, neither zero, $A \neq B$, and $J = \displaystyle\int \sin Ax \cos Bx\;\, dx$. Let $u = \sin Ax$ and $dv = \cos Bx\, dx$. Result:

$$J = \frac{1}{B}\sin Ax\, \sin Bx + \frac{A}{B}\int \cos Ax\, \sin Bx\; dx.$$

In the second integral, let $u = \cos Ax$ and $dv = \sin Bx\; dx$ (the other choice doesn't work). You will find that

$$J = \frac{1}{B}\sin Ax\, \sin Bx + \frac{A}{B^2}\cos Ax\, \cos Bx + \frac{A^2}{B^2}J.$$

Now solve for J to obtain

$$J = \frac{B}{B^2 - A^2} \sin Ax \, \sin Bx + \frac{A}{B^2 - A^2} \cos Ax \, \cos Bx + C.$$

In particular, we get the integral in Problem 48 by choosing $A = 3$ and $B = 1$, thus obtaining

$$\int \sin 3x \, \cos x \, dx = -\frac{1}{8} \sin 3x \, \sin x - \frac{3}{8} \cos 3x \, \cos x + C.$$

See Problems 49–52 of Section 8.4 for a "better" way, which yields the antiderivative in the alternative form $-\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$, as do both *Mathematica* 3.0 and *Maple* V version 5.1.

C08S03.049: Let $u = x^n$ and $dv = e^x \, dx$: $du = nx^{n-1} \, dx$ and choose $v = e^x$. Then

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx, \qquad n \geqq 1.$$

C08S03.050: Let $u = x^{n-1}$ and $dv = xe^{-x^2} \, dx$: $du = (n-1)x^{n-2} \, dx$; choose $v = -\frac{1}{2}e^{-x^2}$. Then

$$\int x^n e^{-x^2} \, dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} \, dx, \qquad n \geqq 2.$$

C08S03.051: Let $u = (\ln x)^n$ and $dv = dx$: $du = \dfrac{n(\ln x)^{n-1}}{x} \, dx$; choose $v = x$. Then

$$\int (\ln x)^n \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx, \qquad n \geqq 1.$$

C08S03.052: Let $u = x^n$ and $dv = \cos x \, dx$: $du = nx^{n-1} \, dx$; choose $v = \sin x$. Then

$$\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx, \qquad n \geqq 1.$$

C08S03.053: Let $u = (\sin x)^{n-1}$ and $dv = \sin x \, dx$. Then $du = (n-1)(\sin x)^{n-2} \cos x \, dx$; choose $v = -\cos x$. Then

$$\begin{aligned}
I_n &= \int (\sin x)^n \, dx = -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} \cos^2 x \, dx \\
&= -(\sin x)^{n-1} \cos x + (n-1) \int (\sin x)^{n-2} \, dx - (n-1) \int (\sin x)^n \, dx; \\
nI_n &= -(\sin x)^{n-1} \cos x + (n-1) I_{n-2}; \\
I_n &= -\frac{1}{n} (\sin x)^{n-1} \cos x + \frac{n-1}{n} I_{n-2}, \qquad n \geqq 2.
\end{aligned}$$

C08S03.054: Let

$$J_n = \int (\cos x)^n \, dx, \qquad n \geqq 2.$$

Then let $u = (\cos x)^{n-1}$ and $dv = \cos x \, dx$: $du = -(n-1)(\cos x)^{n-2} \sin x \, dx$; choose $v = \sin x$. Hence

$$\begin{aligned} J_n &= (\cos x)^{n-1}\sin x + (n-1)\int (\cos x)^{n-2}\sin^2 x\, dx \\ &= (\cos x)^{n-1}\sin x + (n-1)\int (\cos x)^{n-2}\, dx - (n-1)\int (\cos x)^n\, dx; \\ nJ_n &= (\cos x)^{n-1}\sin x + (n-1)J_{n-2}; \\ J_n &= \frac{1}{n}(\cos x)^{n-1}\sin x + \frac{n-1}{n}J_{n-2}. \end{aligned}$$

C08S03.055: The formula in Problem 49 yields

$$\begin{aligned} \int_0^1 x^3 e^x\, dx &= \Big[x^3e^x\Big]_0^1 - 3\int_0^1 x^2e^x\, dx = e - 3\left(\Big[x^2e^x\Big]_0^1 - 2\int_0^1 xe^x\, dx\right) \\ &= e - 3e + 6\left(\Big[xe^x\Big]_0^1 - \int_0^1 e^x\, dx\right) = -2e + 6e - 6\Big[e^x\Big]_0^1 \\ &= 4e - 6e + 6 = 6 - 2e \approx 0.5634363431. \end{aligned}$$

C08S03.056: Let $J_n = \displaystyle\int_0^1 x^n e^{-x^2}\, dx$. Then from the solution of Problem 50 we conclude that

$$J_n = -\frac{1}{2e} + \frac{n-1}{2}J_{n-2}.$$

Therefore

$$\begin{aligned} J_5 &= -\frac{1}{2e} + 2J_3 = -\frac{1}{2e} - \frac{2}{2e} + 2\int_0^1 xe^{-x^2}\, dx \\ &= -\frac{1}{2e} - \frac{1}{e} + 2\left[-\frac{1}{2}e^{-x^2}\right]_0^1 \\ &= -\frac{3}{2e} + 1 - \frac{1}{e} = \frac{2e-5}{2e} \approx 0.0803013971. \end{aligned}$$

C08S03.057: $\displaystyle\int (\ln x)^3\, dx = x(\ln x)^3 - 3\int x(\ln x)^2\, dx - 2\int x\ln x\, dx - \int 1\, dx.$ Therefore

$$\int_1^e (\ln x)^3\, dx = \Big[x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x\Big]_1^e = e - 3e + 6e - 6e + 6 = 6 - 2e \approx 0.5634363431.$$

C08S03.058: Let $I_n = \displaystyle\int_0^{\pi/2} (\sin x)^n\, dx$ for $n \geqq 0$. By the result in Problem 53,

$$\begin{aligned} I_{2n} &= \left[-\frac{(\sin x)^{2n-1}\cos x}{2n}\right]_0^{\pi/2} + \frac{2n-1}{2n}I_{2n-2} \\ &= \frac{2n-1}{2n}I_{2n-2} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot I_{2n-4} \\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 1\, dx \\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}. \end{aligned}$$

Again using the result in Problem 53,

$$I_{2n+1} = \left[-\frac{(\sin x)^{2n}\cos x}{2n}\right]_0^{\pi/2} + \frac{2n}{2n+1}I_{2n-1} = \frac{2n}{2n+1}I_{2n-1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}I_{2n-3}$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot\int_0^{\pi/2}\sin x\,dx$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot\Big[-\cos x\Big]_0^{\pi/2}$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{4}{5}\cdot\frac{2}{3}.$$

C08S03.059: Part (a): Let $u = x + 10$: $x = u - 10$, $dx = du$. Thus

$$\int \ln(x+10)\,dx = \int \ln u\,du = u\ln u - u + C_1$$

$$= (x+10)\ln(x+10) - (x+10) + C_1 = (x+10)\ln(x+10) - x + C.$$

Part (b): Let $u = \ln(x+10)$ and $dv = dx$: $du = \dfrac{1}{x+10}\,dx$; choose $v = x$. Then

$$\int \ln(x+10)\,dx = x\ln(x+10) - \int\frac{x}{x+10}\,dx = x\ln(x+10) - \int\left(1 - \frac{10}{x+10}\right)dx$$

$$= x\ln(x+10) - x + 10\ln(x+10) + C = (x+10)\ln(x+10) - x + C.$$

Part (c): Let $u = \ln(x+10)$ and $dv = dx$: $du = \dfrac{1}{x+10}\,dx$; choose $v = x + 10$. Then

$$\int \ln(x+10)\,dx = (x+10)\ln(x+10) - \int 1\,dx = (x+10)\ln(x+10) - x + C.$$

C08S03.060: Let $u = \arctan x$ and $dv = x^3\,dx$: $du = \dfrac{1}{1+x^2}\,dx$; choose $v = \dfrac{x^4-1}{4}$. Then

$$\int x^3\arctan x\,dx = \frac{1}{4}(x^4-1)\arctan x - \frac{1}{4}\int\frac{x^4-1}{x^2+1}\,dx$$

$$= \frac{1}{4}(x^4-1)\arctan x - \frac{1}{4}\int(x^2-1)\,dx = \frac{1}{4}(x^4-1)\arctan x - \frac{1}{12}x^3 + \frac{1}{4}x + C.$$

C08S03.061: Part (a):

$$J_0 = \int_0^1 e^{-x}\,dx = \Big[-e^{-x}\Big]_0^1 = 1 - \frac{1}{e}.$$

If $n \geqq 1$, then let $u = x^n$ and $dv = e^{-x}\,dx$. Then $du = nx^{n-1}\,dx$; choose $v = -e^{-x}$. Thus

$$J_n = \Big[-x^ne^{-x}\Big]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx = nJ_{n-1} - \frac{1}{e}.$$

Part (b): If $n = 1$, then

$$n! - \frac{n!}{e}\sum_{k=0}^{n}\frac{1}{k!} = 1 - \frac{1}{e}\left(\frac{1}{0!}+\frac{1}{1!}\right) = 1 - \frac{2}{e}$$

and

$$J_1 = 1 \cdot J_0 - \frac{1}{e} = 1 - \frac{2}{e}.$$

Therefore the formula in part (b) holds if $n = 1$. Assume that

$$J_m = m! - \frac{m!}{e}\sum_{k=0}^{m}\frac{1}{k!}$$

for some integer $m \geqq 1$. Then

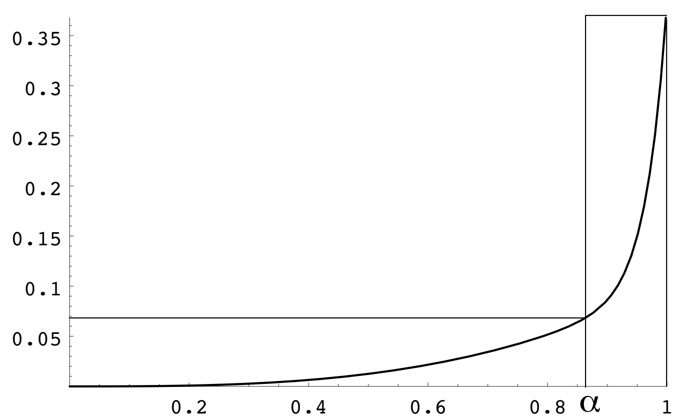
$$\begin{aligned} J_{m+1} &= (m+1)J_m - \frac{1}{e} = m!(m+1) - \frac{m!(m+1)}{e}\sum_{k=0}^{m}\frac{1}{k!} - \frac{1}{e} \\ &= (m+1)! - \left[\frac{(m+1)!}{e}\sum_{k=0}^{m}\frac{1}{k!} + \frac{(m+1)!}{(m+1)!e}\right] \\ &= (m+1)! - \frac{(m+1)!}{e}\left[\frac{1}{(m+1)!} + \sum_{k=0}^{m}\frac{1}{k!}\right] = (m+1)! - \frac{(m+1)!}{e}\sum_{k=0}^{m+1}\frac{1}{k!}. \end{aligned}$$

Therefore, by induction,

$$J_n = n! - \frac{n!}{e}\sum_{k=0}^{n}\frac{1}{k!}$$

for every integer $n \geqq 1$.

Part (c): The next figure will aid in understanding the following proof.



The curve represents the graph of $y = x^n e^{-x}$ on $[0, 1]$. (It really isn't; it's the graph of $y = \frac{1}{10}x^3 + \frac{10}{13}x^{30}e^{-x}$.) Given the positive integer k, choose the real number α, $0 < \alpha < 1$, so close to 1 that

$$\frac{1-\alpha}{e} < \frac{1}{2k}.$$

Because $\alpha^n \to 0$ as $n \to \infty$, choose the positive integer N so large that

$$\alpha^{N+1} < \frac{1}{2k}.$$

Then

$$x^N e^{-x} \leqq \alpha^N \quad \text{if} \quad 0 \leqq x \leqq \alpha \qquad \text{and} \qquad x^N e^{-x} \leqq \frac{1}{e} \quad \text{if} \quad \alpha \leqq x \leqq 1. \tag{1}$$

The area of the short wide rectangle in the figure is

$$\alpha \cdot \alpha^N e^{-\alpha} < \alpha \cdot \alpha^N = \alpha^{N+1}$$

and the area of the tall narrow rectangle there is $(1-\alpha)/e$. The inequalities in (1) show that the graph of $y = x^N e^{-x}$ is enclosed in the two rectangles, and hence

$$\int_0^1 x^N e^{-x}\,dx \leqq \alpha^{N+1} + \frac{1-\alpha}{e} < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

Moreover, if $n \geqq N$ and $0 \leqq x \leqq 1$, then $x^n e^{-x} \leqq x^N e^{-x}$. Therefore, for every positive integer k, there exists a positive integer N such that

$$0 \leqq \int_0^1 x^n e^{-x}\,dx < \frac{1}{k}$$

if $n \geqq N$. Let $k \to \infty$. By the squeeze law for limits,

$$\lim_{n\to\infty} \int_0^1 x^n e^{-x}\,dx = 0.$$

Therefore $J_n \to 0$ as $n \to +\infty$. ◀

Part (d): By part (c),

$$\lim_{n\to\infty} \frac{eJ_n}{n!} = 0 = \lim_{n\to\infty} \left(e - \sum_{k=0}^{n} \frac{1}{k!} \right).$$

Therefore $e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$. (See Eq. (20) in Section 7.4.)

C08S03.062: Let $u = (\ln x)^n$ and $dv = x^m\,dx$. Then $du = \dfrac{n(\ln x)^{n-1}}{x}\,dx$; we choose $v = \dfrac{x^{m+1}}{m+1}$. Thus

$$\int x^m (\ln x)^n\,dx = \frac{x^{n+1}}{m+1}(\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1}\,dx.$$

C08S03.063: The expansion of $(k\ln x - 2x^3 + 3x^2 + b)^4$ is a sum of 35 terms, including terms as formidable to antidifferentiate as

$$-32kx^9 \ln x, \qquad 54k^2x^4(\ln x)^2, \qquad \text{and} \qquad k^4(\ln x)^4,$$

as well as several polynomial terms. The reduction formula of Problem 62 handles the three shown here as follows:

$$\int x^9 \ln x \, dx = \frac{x^{10}}{10} \ln x - \frac{1}{10} \int x^9 \, dx = \frac{x^{10}}{10} \ln x - \frac{1}{100} x^{10} + C,$$

$$\begin{aligned}\int x^4 (\ln x)^2 \, dx &= \frac{x^5}{5} (\ln x)^2 - \frac{2}{5} \int x^4 \ln x \, dx \\ &= \frac{x^5}{5} (\ln x)^2 - \frac{2}{5} \left[\frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 \, dx \right] = \frac{x^5}{5} (\ln x)^2 - \frac{2}{5} \left[\frac{x^5}{5} \ln x - \frac{x^5}{25} \right] + C \\ &= \frac{x^5}{5} (\ln x)^2 - \frac{2x^5}{25} \ln x + \frac{2x^5}{125} + C,\end{aligned}$$

and

$$\begin{aligned}\int (\ln x)^4 \, dx &= x(\ln x)^4 - 4 \int (\ln x)^3 \, dx \\ &= x(\ln x)^4 - 4 \left[x(\ln x)^3 - 3 \int (\ln x)^2 \, dx \right] \\ &= x(\ln x)^4 - -4x(\ln x)^3 + 12 \left[x(\ln x)^2 - 2 \int \ln x \, dx \right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24 \left[x \ln x - \int 1 \, dx \right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x \ln x + 24x + C.\end{aligned}$$

A very patient person can in this way discover that the engineer's antiderivative is

$$\begin{aligned}&(b^4 - 4b^3k + 12b^2k^2 - 24bk^3 + 24k^4)x + \frac{4}{9}(9b^3 - 9b^2k + 6bk^2 - 2k^3)x^3 + \frac{1}{16}(-32b^3 + 24b^2k - 12bk^2 \\ &+ 3k^3)x^4 + \frac{54}{125}(25b^2 - 10bk + 2k^2)x^5 - \frac{2}{3}(18b^2 - 6bk + k^2)x^6 + \frac{12}{343}(441b + 98b^2 - 63k - 28bk + 4k^2)x^7 \\ &- \frac{27}{8}(8b - k)x^8 + \frac{1}{9}(81 + 144b - 16k)x^9 - \frac{4}{25}(135 + 20b - 2k)x^{10} + \frac{216}{11}x^{11} - 8x^{12} + \frac{16}{13}x^{13} \\ &- \frac{1}{14700}kx(-58800b^3 + 176400b^2k - 352800bk^2 + 352800k^3 - 176400b^2x^2 + 117600bkx^2 - 39200k^2x^2 \\ &+ 88200b^2x^3 - 44100bkx^3 + 11025k^2x^3 - 317520bx^4 + 63504kx^4 + 352800bx^5 - 58800kx^5 - 226800x^6 \\ &- 100800bx^6 + 14400kx^6 + 396900x^7 - 235200x^8 + 47040x^9) \ln x + \frac{1}{70}k^2x(420b^2 - 840bk + 840k^2 + 840bx^2 \\ &- 280kx^2 - 420bx^3 + 105kx^3 + 756x^4 - 840x^5 + 240x^6)(\ln x)^2 - 2k^3x(-2b + 2k - 2x^2 + x^3)(\ln x)^3 \\ &+ k^4x(\ln x)^4 + C.\end{aligned}$$

C08S03.064: Area:

$$A = \int_0^\pi \tfrac{1}{2}x^2 \sin x \, dx = \left[\tfrac{1}{2} \left(-x^2 \cos x + 2 \int x \cos x \, dx \right) \right]_0^\pi$$

$$
\begin{aligned}
&= \left[-\tfrac{1}{2}x^2 \cos x + x \sin x - \int \sin x \, dx \right]_0^\pi \\
&= \Big[-\tfrac{1}{2}x^2 \cos x + x \sin x + \cos x \Big]_0^\pi \\
&= \tfrac{1}{2}\pi^2 - 1 - 1 = \frac{\pi^2 - 4}{2}.
\end{aligned}
$$

Volume:

$$
\begin{aligned}
V &= \int_0^\pi 2\pi x \cdot \tfrac{1}{2}x^2 \sin x \, dx = \pi \int_0^\pi x^3 \sin x \, dx \\
&= \pi \Big[-x^3 \cos x + 3 \int x^2 \cos x \, dx \Big]_0^\pi \\
&= \pi \left[-x^3 \cos x + 3 \left(x^2 \sin x - 2 \int x \sin x \, dx \right) \right]_0^\pi \\
&= \pi \Big[-x^3 \cos x + 3x^2 \sin x - 6 \left(-x \cos x + \sin x \right) \Big]_0^\pi \\
&= \pi \Big[-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x \Big]_0^\pi \\
&= \pi \left(\pi^3 - 6\pi \right) = \pi^4 - 6\pi^2 = \pi^2 \left(\pi^2 - 6 \right).
\end{aligned}
$$

C08S03.065: Volume: $V = \displaystyle\int_0^\pi \pi \left(\tfrac{1}{2}x^2 \sin x \right)^2 \, dx = \frac{\pi}{4} \int_0^\pi x^4 \sin^2 x \, dx = \frac{\pi}{8} \int_0^\pi x^4 (1 - \cos 2x) \, dx.$

Let $u = 2x$: $x = \frac{1}{2}u$, $dx = \frac{1}{2}\,du$.

$$
\begin{aligned}
V &= \frac{\pi}{8} \int_0^{2\pi} \frac{u^4}{16} (1 - \cos u) \cdot \tfrac{1}{2} \, du \\
&= \frac{\pi}{256} \int_0^{2\pi} \left(u^4 - u^4 \cos u \right) \, du \\
&= \frac{\pi}{256} \left(\Big[\tfrac{1}{5}u^5 \Big]_0^{2\pi} - \int_0^{2\pi} u^4 \cos u \, du \right) \\
&= \frac{\pi}{256} \left(\tfrac{32}{5}\pi^5 - \left[u^4 \sin u - 4 \int u^3 \sin u \, du \right]_0^{2\pi} \right) \\
&= \frac{\pi^6}{40} - \frac{\pi}{256} \left[u^4 \sin u - 4 \Big(-u^3 \cos u + 3 \Big\{ u^2 \sin u - 2 \left[-u \cos u + \sin u \right] \Big\} \Big) \right]_0^{2\pi} \\
&= \frac{\pi^6}{40} - \frac{\pi}{256} \Big[4(2\pi)^3 - 24(2\pi) \Big] \\
&= \frac{\pi^6}{40} - \frac{\pi^4}{8} + \frac{3\pi^2}{16} = \frac{\pi^2}{80} \Big(2\pi^4 - 10\pi^2 + 15 \Big).
\end{aligned}
$$

C08S03.066: A *Mathematica* solution. Part (a):

```
a = 100*E∧(−t);
v = 100 + Integrate[a,t] // Together
```

$$\frac{100(-1 + e^t)}{e^t}$$

```
x = Integrate[v,t] − 100 // Together
```

$$\frac{100(1-e^t+te^t)}{e^t}$$

```
Limit[x, t → Infinity]
```

$$+\infty$$

That is, the particle moves arbitrarily far to the right along the x-axis. Part (b):

```
a = 100*(1 − t)*E∧(−t);
v = Integrate[a,t]
```

$$\frac{100t}{e^t}$$

```
x = 100 + Integrate[v,t] // Together
```

$$\frac{10(-1+e^t-t)}{e^t}$$

```
Limit[x, t → Infinity]
```

$$100$$

Thus, because x is an increasing function of t, $x(t)$ always remains less than 100; the particle moves only a finite distance to the right before effectively coming to a stop (because $v \to 0$ as $t \to +\infty$).

C08S03.067: A *Mathematica* solution:

```
f = x∧2;   g = 2∧x;
R = Plot[ { f, g }, { x, 1.5, 4.5 },
PlotStyle → { RGBColor[0,0,1], RGBColor[1,0,0] } ];
```

The different colors enable us to more easily distinguish the graphs. Area:

```
A = Integrate[ f − g, { x, 2, 4 } ]
```

$$-4\cdot\frac{-9\ln 2+6(\ln 2)^2+12\ln 8-16(\ln 2)(\ln 8)}{3(\ln 2)(\ln 8)}$$

```
A = A /.  { Log[8] → 3*Log[2] }
```

$$\frac{-4\left[27\ln 2-42(\ln 2)^2\right]}{9(\ln 2)^2}$$

(*Mathematica* writes `Log[x]` where we write $\ln x$.)

```
A = A // Simplify
```

$$\frac{-36+56\ln 2}{\ln 8}$$

```
A = A /.  { Log[8] → 3*Log[2] }
```

$$\frac{-36+56\ln 2}{3\ln 2}$$

Next we find the x-coordinate of the centroid.

```
xc = (1/A)*Integrate[ x*(f − g), { x, 2, 4 } ]
```

$$\frac{12\left[3+2\ln 2+15(\ln 2)^2-4\ln 16\right]}{(-36+56\ln 2)(\ln 2)}$$

```
xc = xc /.  { Log[16] → 4*Log[2] }
```

$$\frac{12\left[3-14\ln 2+15(\ln 2)^2\right]}{(-36+56\ln 2)(\ln 2)}$$

Finally, we find the y-coordinate of the centroid.

```
yc = (1/A)*(1/2)*Integrate[ f∧2 − g∧2, { x, 2, 4 } ]
```

$$24\cdot\frac{25\ln 2-10(\ln 2)(\ln 4)-40\ln(1024)+64(\ln 2)(\ln 1024)}{5(-36+56\ln 2)(\ln 1024)}$$

```
yc = yc /.  { Log[4] → 2*Log[2], Log[1024] → 10*Log[2] }
```

$$\frac{12\left[-375\ln 2+620(\ln 2)^2\right]}{(-36+56\ln 2)(25\ln 2)}$$

```
yc = yc // Simplify
```

$$\frac{225-372\ln 2}{45-70\ln 2}$$

Thus the centroid of the region has approximate coordinates (3.0904707864762604, 9.3317974433586819).

C08S03.068: Part (a): If m is a positive integer, then (because $0 < \sin x < 1$ if $0 < x < \pi/2$), we have $(\sin x)^m > (\sin x)^{m+1}$ if $0 < x < \pi/2$. Therefore by the comparison property for definite integrals,

$$I_{n2} \geqq I_{2n+1} \geqq I_{2n+2}$$

for every positive integer n. Part (b): By the result in Problem 58,

$$\frac{I_{2n+2}}{I_{2n}} = \left(\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2}\right)\cdot\left(\frac{\pi}{2}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{2n}{2n-1}\right) = \frac{2n+1}{2n+2}.$$

Therefore

$$\lim_{n\to\infty}\frac{I_{2n+2}}{I_{2n}} = \lim_{n\to\infty}\frac{2n+1}{2n+2} = 1.$$

Part (c): Because $I_{2n+2} \geqq I_{2n+1} \geqq I_{2n}$ for each positive integer n, we have

$$\frac{I_{2n+2}}{I_{2n}} \geqq \frac{I_{2n+1}}{I_{2n}} \geqq \frac{I_{2n}}{I_{2n}}$$

for each positive integer n. Therefore, by the squeeze law for limits,

$$\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1.$$

Part (d): But then, by the result in Problem 58,

$$\begin{aligned}
\frac{I_{2n+1}}{I_{2n}} &= \left(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n}{2n+1}\right)\cdot\left(\frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n}{2n-1}\right) \\
&= \frac{2}{\pi}\cdot\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}.
\end{aligned}$$

Hence, by Part (c),

$$\lim_{n\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}.$$

Section 8.4

C08S04.001: $\displaystyle\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C = \frac{1}{2}(x - \sin x\, \cos x) + C.$

C08S04.002: $\displaystyle\int \cos^2 5x\, dx = \int \frac{1+\cos 10x}{2}\, dx = \frac{1}{2} + \frac{1}{20}\sin 10x + C = \frac{1}{2}x + \frac{1}{10}\sin 5x\, \cos 5x + C.$

C08S04.003: $\displaystyle\int \sec^2 \frac{x}{2}\, dx = 2\tan \frac{x}{2} + C.$

C08S04.004: $\displaystyle\int \tan^2 \frac{x}{2}\, dx = \int \left(\sec^2 \frac{x}{2} - 1\right) dx = 2\tan \frac{x}{2} - x + C.$

C08S04.005: $\displaystyle\int \tan 3x\, dx = \int \frac{\sin 3x}{\cos 3x}\, dx = -\frac{1}{3}\ln|\cos 3x| + C = \frac{1}{3}\ln|\sec x| + C.$

C08S04.006: $\displaystyle\int \cot 4x\, dx = \int \frac{\cos 4x}{\sin 4x}\, dx = \frac{1}{4}\ln|\sin 4x| + C.$

C08S04.007: $\displaystyle\int \sec 3x\, dx = \frac{1}{3}\ln|\sec 3x + \tan 3x| + C.$

C08S04.008: $\displaystyle\int \csc 2x\, dx = \frac{1}{2}\ln|\csc 2x - \cot 2x| + C.$

C08S04.009: $\displaystyle\int \frac{1}{\csc^2 x}\, dx = \int \sin^2 x\, dx = \frac{1}{2}(x - \sin x\, \cos x) + C$ (by Problem 1).

C08S04.010: $\displaystyle\int \sin^2 x\, \cot^2 x\, dx = \int \frac{\sin^2 x\, \cos^2 x}{\sin^2 x}\, dx = \int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx$

$\displaystyle = \frac{1}{2}x + \frac{1}{4}\sin 2x + C = \frac{1}{2}(x + \sin x\, \cos x) + C.$

C08S04.011: $\displaystyle\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx = \int (\sin x - \cos^2 x\, \sin x)\, dx = \frac{1}{3}\cos^3 x - \cos x + C.$

C08S04.012: We use the reduction formula in Problem 53 of Section 7.3:

$$\begin{aligned}\int \sin^4 x\, dx &= -\frac{\sin^3 x\, \cos x}{4} + \frac{3}{4}\int \sin^2 x\, dx \\ &= -\frac{\sin^3 x\, \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x\, \cos x}{2} + \frac{1}{2}\int 1\, dx\right) = -\frac{1}{4}\sin^3 x\, \cos x - \frac{3}{8}\sin x\, \cos x + \frac{3}{8}x + C.\end{aligned}$$

C08S04.013: $\displaystyle\int \sin^2\theta\, \cos^3\theta\, d\theta = \int (\sin^2\theta)(1-\sin^2\theta)\cos\theta\, d\theta = \int (\sin^2\theta\, \cos\theta - \sin^4\theta\, \cos\theta)\, d\theta$

$\displaystyle = \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C.$

C08S04.014: $\displaystyle\int \sin^3 t\, \cos^3 t\, dt = \int (\sin t)(1-\cos^2 t)\cos^3 t\, dt = \int (\cos^3 t\, \sin t - \cos^5 t\, \sin t)\, dt$

$\displaystyle = \frac{1}{6}\cos^6 t - \frac{1}{4}\cos^4 t + C.$ Alternatively, $\displaystyle\int \sin^3 t\, \cos^3 t\, dt = \int (\sin^3 t)(1-\sin^2 t)\cos t\, dt$

$$= \int (\sin^3 t \, \cos t - \sin^5 t \, \cos t) \, dt = \frac{1}{4} \sin^4 t - \frac{1}{6} \sin^6 t + C.$$

C08S04.015: $\displaystyle \int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx = \int (\sin^4 x \, \cos x - 2 \sin^2 x \, \cos x + \cos x) \, dx$

$$= \frac{1}{5} \sin^5 x - \frac{2}{3} \sin^3 x + \sin x + C.$$

C08S04.016: $\displaystyle \int (\cos t)^{-3} \sin t \, dt = \frac{1}{2} (\cos t)^{-2} + C = \frac{1}{2} \sec^2 t + C.$ Alternatively, $\displaystyle \int \frac{\sin t}{\cos^3 t} \, dt$

$$= \int \sec^2 t \, \tan t \, dt = \int (\sec t)(\sec t \, \tan t) \, dt = \frac{1}{2} \sec^2 t + C.$$

C08S04.017: $\displaystyle \int (\sin^3 x)(\cos x)^{-1/2} \, dx = \int (1 - \cos^2 x)(\cos x)^{-1/2} \sin x \, dx$

$$= \int \Big[(\cos x)^{-1/2} \sin x - (\cos x)^{3/2} \sin x \Big] \, dx = \frac{2}{5} (\cos x)^{5/2} - 2(\cos x)^{1/2} + C.$$

C08S04.018: $\displaystyle \int \sin^3 \phi \, \cos^4 \phi \, d\phi = \int (\cos^4 \phi)(1 - \cos^2 \phi) \sin \phi \, d\phi = \int (\cos^4 \phi \, \sin \phi - \cos^6 \phi \, \sin \phi) \, d\phi$

$$= \frac{1}{7} \cos^7 \phi - \frac{1}{5} \cos^5 \phi + C.$$

C08S04.019: $\displaystyle \int \sin^5 2z \, \cos^2 2z \, dz = \int (1 - \cos^2 2z)^2 \cos^2 2z \, \sin 2z \, dz$

$$= \int (\cos^6 2z \, \sin 2z - 2 \cos^4 2z \, \sin 2z + \cos^2 2z \, \sin 2z) \, dz = -\frac{1}{14} \cos^7 2z + \frac{1}{5} \cos^5 2z - \frac{1}{6} \cos^3 2z + C.$$

The computer algebra program *Derive* 2.56 returns the answer

$$-\frac{1}{14} \sin^4 2z \, \cos^3 2z - \frac{2}{35} \sin^2 2z \, \cos^3 2z - \frac{4}{105} \cos^3 2z.$$

C08S04.020: $\displaystyle \int (\sin x)^{3/2} \cos^3 x \, dx = \int (\sin x)^{3/2} (1 - \sin^2 x) \cos x \, dx$

$$= \int \Big[(\sin x)^{3/2} \cos x - (\sin x)^{7/2} \cos x \Big] \, dx = \frac{2}{5} (\sin x)^{5/2} - \frac{2}{9} (\sin x)^{9/2} + C.$$

C08S04.021: $\displaystyle \int \frac{\sin^3 4x}{\cos^2 4x} \, dx = \int \frac{(1 - \cos^2 4x) \sin 4x}{\cos^2 4x} \, dx$

$$= \int \big[(\cos 4x)^{-2} \sin 4x - \sin 4x \big] \, dx = \frac{1}{4} (\cos 4x)^{-1} + \frac{1}{4} \cos 4x + C = \frac{1}{4} (\sec 4x + \cos 4x) + C.$$

C08S04.022: $\displaystyle \int \cos^6 4\theta \, d\theta = \int \left(\frac{1 + \cos 8\theta}{2} \right)^3 d\theta = \frac{1}{8} \int (1 + 3 \cos 8\theta + 3 \cos^2 8\theta + \cos^3 8\theta) \, d\theta$

$$= \frac{1}{8} \int \left[1 + 3 \cos 8\theta + \frac{3}{2} (1 + \cos 16\theta) + (1 - \sin^2 8\theta) \cos 8\theta \right] d\theta$$

$$= \frac{1}{8} \int \left(\frac{5}{2} + 4 \cos 8\theta + \frac{3}{2} \cos 16\theta - \sin^2 8\theta \, \cos 8\theta \right) d\theta$$

$$= \frac{1}{8} \left(\frac{5}{2} \theta + \frac{1}{2} \sin 8\theta + \frac{3}{32} \sin 16\theta - \frac{1}{24} \sin^3 8\theta \right) + C = \frac{5}{16} \theta + \frac{1}{16} \sin 8\theta + \frac{3}{256} \sin 16\theta - \frac{1}{192} \sin^3 8\theta + C.$$

C08S04.023: $\displaystyle\int \sec^4 t\,dt = \int (\sec^2 t)(1+\tan^2 t)\,dt = \int (\sec^2 t + \sec^2 t\,\tan^2 t)\,dt = \tan t + \frac{1}{3}\tan^3 t + C.$

C08S04.024: $\displaystyle\int \tan^3 x\,dx = \int (\sec^2 x - 1)\tan x\,dx = \int \left[(\sec x)(\sec x\,\tan x) - \frac{\sin x}{\cos x}\right]dx$

$$= \frac{1}{2}\sec^2 x + \ln|\cos x| + C.$$

C08S04.025: $\displaystyle\int \cot^3 2x\,dx = \int (\csc^2 2x - 1)\cot 2x\,dx = \int \left[(\csc 2x)(\csc 2x\,\cot 2x) - \frac{\cos 2x}{\sin 2x}\right]dx$

$$= -\frac{1}{4}\csc^2 2x - \frac{1}{2}\ln|\sin 2x| + C.$$

C08S04.026: $\displaystyle\int \tan\theta\,\sec^4\theta\,d\theta = \int (\sec^3\theta)(\sec\theta\,\tan\theta)\,d\theta = \frac{1}{4}\sec^4\theta + C.$ Alternatively,

$$\int \tan\theta\,\sec^4\theta\,d\theta = \int (\tan\theta)(1+\tan^2\theta)\sec^2\theta\,d\theta$$

$$= \int (\tan\theta\,\sec^2\theta + \tan^3\theta\,\sec^2\theta)\,d\theta = \frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta + C.$$

C08S04.027: $\displaystyle\int \tan^5 2x\,\sec^2 2x\,dx = \frac{1}{12}\tan^6 2x + C.$ Alternatively,

$$\int \tan^5 2x\,\sec^2 2x\,dx = \int (\sec^2 2x - 1)^2\sec^2 2x\,\tan 2x\,dx = \int (\sec^6 2x - 2\sec^4 2x + \sec^2 2x)\tan 2x\,dx$$

$$= \int (\sec^5 2x - 2\sec^3 2x + \sec 2x)(\sec 2x\,\tan 2x)\,dx = \frac{1}{12}\sec^6 2x - \frac{1}{4}\sec^4 2x + \frac{1}{4}\sec^2 2x + C.$$

C08S04.028: $\displaystyle\int \cot^3 x\,\csc^2 x\,dx = -\frac{1}{4}\cot^4 x + C.$

C08S04.029: $\displaystyle\int \csc^6 2t\,dt = \int (1+\cot^2 2t)^2\csc^2 2t\,dt$

$$= \int (\cot^4 2t\,\csc^2 2t + 2\cot^2 2t\,\csc^2 2t + \csc^2 2t)\,dt = -\frac{1}{10}\cot^5 2t - \frac{1}{3}\cot^3 2t - \frac{1}{2}\cot 2t + C.$$

C08S04.030: $\displaystyle\int \frac{\sec^4 t}{\tan^2 t}\,dt = \int \frac{(1+\tan^2 t)\sec^2 t}{\tan^2 t}\,dt = \int \left[(\tan t)^{-2}\sec^2 t + \sec^2 t\right]dt$

$$= -(\tan t)^{-1} + \tan t + C = \tan t - \cot t + C.$$

C08S04.031: $\displaystyle\int \frac{\tan^3\theta}{\sec^4\theta}\,d\theta = \int \frac{\sin^3\theta\,\cos^4\theta}{\cos^3\theta}\,d\theta = \int \sin^3\theta\,\cos\theta\,d\theta = \frac{1}{4}\sin^4\theta + C.$ Alternatively,

$$\int \frac{\tan^3\theta}{\sec^4\theta}\,d\theta = \int \frac{(\sec^2\theta - 1)\tan\theta}{\sec^4\theta}\,d\theta = \int \frac{\sec^2\theta - 1}{\sec^5\theta}\sec\theta\,\tan\theta\,d\theta$$

$$= \int \left[(\sec\theta)^{-3}\sec\theta\,\tan\theta - (\sec\theta)^{-5}\sec\theta\,\tan\theta\right]d\theta$$

$$= -\frac{1}{2}(\sec\theta)^{-2} + \frac{1}{4}(\sec\theta)^{-4} + C = \frac{1}{4}\cos^4\theta - \frac{1}{2}\cos^2\theta + C.$$

C08S04.032: $\displaystyle\int \frac{\cot^3 x}{\csc^2 x}\,dx = \int \frac{\cos^3 x\,\sin^2 x}{\sin^3 x}\,dx = \int \frac{1-\sin^2 x}{\sin x}\cos x\,dx$

$$= \int \left[(\sin x)^{-1}\cos x - \sin x\,\cos x\right]dx = \ln|\sin x| - \frac{1}{2}\sin^2 x + C.$$

C08S04.033: $\displaystyle\int \frac{\tan^3 t}{\sqrt{\sec t}}\,dt = \int (\sec t)^{-1/2}(\sec^2 t - 1)\tan t\,dt = \int \Big[(\sec t)^{3/2}\tan t - (\sec t)^{-1/2}\tan t\Big]\,dt$

$$= \int \Big[(\sec t)^{1/2}(\sec t\,\tan t) - (\sec t)^{-3/2}(\sec t\,\tan t)\Big]\,dt = \frac{2}{3}(\sec t)^{3/2} + 2(\sec t)^{-1/2} + C$$

$$= \frac{2}{3}(\sec t)^{3/2} + 2(\cos t)^{1/2} + C.$$

C08S04.034: $\displaystyle\int \frac{1}{\cos^4 2x}\,dx = \int \sec^4 2x\,dx = \int (1+\tan^2 2x)\sec^2 2x\,dx = \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C.$

C08S04.035: $\displaystyle\int \frac{\cot\theta}{\csc^3\theta}\,d\theta = \int \frac{\cos\theta\,\sin^3\theta}{\sin\theta}\,d\theta = \int \sin^2\theta\,\cos\theta\,d\theta = \frac{1}{3}\sin^3\theta + C.$

C08S04.036: $\displaystyle\int \sin^2 3\alpha\,\cos^2 3\alpha\,d\alpha = \frac{1}{4}\int (2\sin 3\alpha\,\cos 3\alpha)^2\,d\alpha = \frac{1}{4}\int \sin^2 6\alpha\,d\alpha = \frac{1}{4}\int \frac{1-\cos 12\alpha}{2}\,d\alpha$

$$= \frac{1}{8}\left(\alpha - \frac{1}{12}\sin 12\alpha\right) + C = \frac{1}{8}\alpha - \frac{1}{96}\sin 12\alpha + C.$$

C08S04.037: $\displaystyle\int \cos^3 5t\,dt = \int (1-\sin^2 5t)\cos 5t\,dt = \frac{1}{5}\sin 5t - \frac{1}{15}\sin^3 5t + C.$

C08S04.038: $\displaystyle\int \tan^4 x\,dx = \int (\sec^2 x - 1)\tan^2 x\,dx = \int \left[\sec^2 x\,\tan^2 x - (\sec^2 x - 1)\right]dx$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C.$$ Also see Problem 67.

C08S04.039: $\displaystyle\int \cot^4 3t\,dt = \int (\csc^2 3t - 1)\cot^2 3t\,dt = \int \left[\cot^2 3t\,\csc^2 3t - (\csc^2 3t - 1)\right]dt$

$$= -\frac{1}{9}\cot^3 3t + \frac{1}{3}\cot 3t + t + C.$$

C08S04.040: $\displaystyle\int \tan^2 2t\,\sec^4 2t\,dt = \int (\tan^2 2t)(1+\tan^2 2t)\sec^2 2t\,dt = \frac{1}{6}\tan^3 2t + \frac{1}{10}\tan^5 2t + C.$

C08S04.041: $\displaystyle\int \sin^5 2t\,(\cos 2t)^{3/2}\,dt = \int (1-\cos^2 2t)^2(\cos 2t)^{3/2}\sin 2t\,dt$

$$= \int \Big[(\cos 2t)^{3/2} - 2(\cos 2t)^{7/2} + (\cos 2t)^{11/2}\Big]\sin 2t\,dt$$

$$= -\frac{1}{5}(\cos 2t)^{5/2} + \frac{2}{9}(\cos 2t)^{9/2} - \frac{1}{13}(\cos 2t)^{13/2} + C.$$

C08S04.042: $\displaystyle\int (\cot^3\xi)(\csc\xi)^{3/2}\,d\xi = \int (\csc^2\xi - 1)(\csc\xi)^{3/2}\cot\xi\,d\xi$

$$= \int \left[(\csc\,\xi)^{5/2} - (\csc\,\xi)^{1/2}\right] \csc\,\xi\,\cot\,\xi\,d\xi = -\frac{2}{7}(\csc\,\xi)^{7/2} + \frac{2}{3}(\csc\,\xi)^{3/2} + C.$$

C08S04.043: $\displaystyle\int \frac{\tan x + \sin x}{\sec x}\,dx = \int (\sin x + \sin x\,\cos x)\,dx = \frac{1}{2}\sin^2 x - \cos x + C.$

C08S04.044: $\displaystyle\int \frac{\cot x + \csc x}{\sin x}\,dx = \int (\cot x\,\csc x + \csc^2 x)\,dx = -\csc x - \cot x + C.$

C08S04.045: The area is

$$A = \int_0^{\pi} \sin^3 x\,dx = \int_0^{\pi} (1 - \cos^2 x)\sin x\,dx = \left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^{\pi} = 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}.$$

C08S04.046: The area is $\displaystyle A = 2\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\,dx = 2\int_0^{\pi/4} \cos 2x\,dx = \Big[\sin 2x\Big]_0^{\pi/4} = 1.$

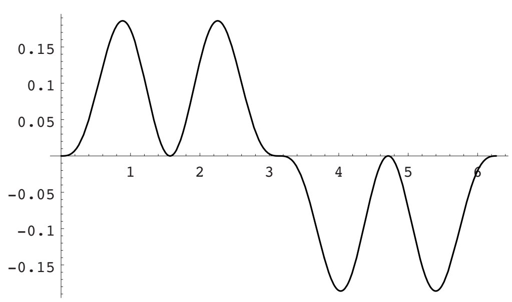
C08S04.047: The area is

$$A = \int_{\pi/4}^{\pi} (\sin^2 x - \sin x\,\cos x)\,dx = \int_{\pi/4}^{\pi} \left(\frac{1 - \cos 2x}{2} - \sin x\,\cos x\right)dx$$
$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{2}\sin^2 x\right]_{\pi/4}^{\pi} = \frac{\pi}{2} - \frac{\pi}{8} + \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3\pi + 4}{8}.$$

C08S04.048: The area is

$$A = \int_{\pi/4}^{5\pi/4} (\sin^3 x - \cos^3 x)\,dx = \int_{\pi/4}^{5\pi/4} \left[(1 - \cos^2 x)\sin x - (1 - \sin^2 x)\cos x\right]dx$$
$$= \left[-\cos x + \frac{1}{3}\cos^3 x - \sin x + \frac{1}{3}\sin^3 x\right]_{\pi/4}^{5\pi/4} = 4\left(\frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{24}\right) = \frac{5\sqrt{2}}{3}.$$

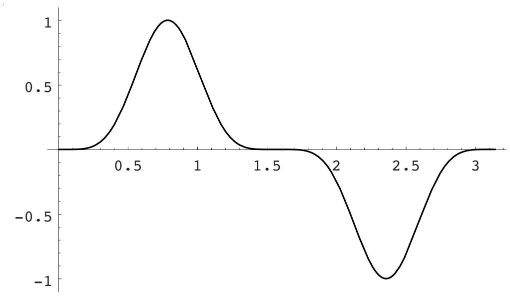
C08S04.049: The following graph makes it appear that the value of the integral is zero.



Sure enough,

$$\int_0^{2\pi} \sin^3 x\,\cos^2 x\,dx = \int_0^{2\pi} (1 - \cos^2 x)\cos^2 x\,\sin x\,dx = \left[-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right]_0^{2\pi} = 0.$$

C08S04.050: The following graph makes it appear that the value of the integral is zero.



Sure enough,

$$\int_0^{\pi} \sin^5 2x\,dx = \int_0^{\pi} (1-\cos^2 2x)^2 \sin 2x\,dx = \int_0^{\pi} (1-2\cos^2 2x + \cos^4 2x)\sin 2x\,dx$$

$$= \left[-\frac{1}{2}\cos 2x + \frac{1}{3}\cos^3 2x - \frac{1}{10}\cos^5 2x\right]_0^{\pi} = -\frac{1}{2} + \frac{1}{3} - \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = 0.$$

C08S04.051: The volume is

$$V = \int_0^{\pi} \pi \sin^4 x\,dx = 2\pi \int_0^{\pi/2} \sin^4 x\,dx = 2\pi \cdot \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} = \frac{3\pi^2}{8}$$

(we used the result in Problem 58 of Section 8.3 to evaluate the definite integral).

C08S04.052: The volume is

$$V = 2\int_0^{\pi/4} \pi\left[(\cos^2 x)^2 - (\sin^2 x)^2\right]\,dx = 2\pi \int_0^{\pi/4} (\cos^4 x - \sin^4 x)\,dx$$

$$= 2\pi \int_0^{\pi/4} (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)\,dx = 2\pi \int_0^{\pi/4} \cos 2x\,dx = \pi\Big[\sin 2x\Big]_0^{\pi/4} = \pi.$$

C08S04.053: The volume is

$$V = 2\pi \int_0^{\pi/3} (4 - \sec^2 x)\,dx = 2\pi\Big[4x - \tan x\Big]_0^{\pi/3} = \frac{2\pi}{3}\left(4\pi - 3\sqrt{3}\right) \approx 15.4361488842.$$

C08S04.054: The volume is

$$V = 2\pi \int_0^{\pi/3} (16\cos^2 x - \sec^2 x)\,dx = 2\pi \int_0^{\pi/3} (8 + 8\cos 2x - \sec^2 x)\,dx$$

$$= 2\pi\Big[8x + 4\sin 2x - \tan x\Big]_0^{\pi/3} = 2\pi\left(\frac{8\pi}{3} + 2\sqrt{3} - \sqrt{3}\right) = \frac{2\pi}{3}\left(8\pi + 3\sqrt{3}\right) \approx 63.5206863245.$$

C08S04.055: Part (a): The area is

$$A = \int_0^{\pi/4} (\sec^2 x - \tan^2 x)\,dx = \int_0^{\pi/4} 1\,dx = \frac{\pi}{4}.$$

Part (b): The volume is

$$\begin{aligned}
V &= \pi \int_0^{\pi/4} (\sec^4 x - \tan^4 x)\ dx = \pi \int_0^{\pi/4} (\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x)\ dx \\
&= \pi \int_0^{\pi/4} (\sec^2 x + \tan^2 x)\ dx = \pi \int_0^{\pi/4} (2\sec^2 x - 1)\ dx \\
&= \pi \Big[2\tan x - x \Big]_0^{\pi/4} = \frac{\pi}{4}(8 - \pi) \approx 3.8157842069.
\end{aligned}$$

C08S04.056: If $y = \ln(\cos x)$, then

$$\frac{dy}{dx} = -\tan x, \qquad \text{so} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = \sec^2 x.$$

Therefore the length of the graph is

$$L = \int_0^{\pi/4} \sec x\ dx = \Big[\ln(\sec x + \tan x) \Big]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right) \approx 0.8813735870.$$

C08S04.057: First way:

$$\int \tan x\ \sec^4 x\ dx = \int (\sec^3 x)(\sec x\ \tan x)\ dx = \frac{1}{4}\sec^4 x + C_1.$$

Second way:

$$\begin{aligned}
\int \tan x\ \sec^4 x\ dx &= \int (\tan x)(1 + \tan^2 x)\sec^2 x\ dx = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C_2 \\
&= \frac{1}{2}(\sec^2 x - 1) + \frac{1}{4}(\sec^2 x - 1)^2 + C_2 = \frac{1}{2}\sec^2 x - \frac{1}{2} + \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + \frac{1}{4} + C_2 \\
&= \frac{1}{4}\sec^4 x + C_1 \qquad \text{where} \qquad C_1 = C_2 - \frac{1}{4}.
\end{aligned}$$

C08S04.058: First way:

$$\int \cot^3 x\ dx = \int (\csc^2 x - 1)\cot x\ dx = \int \left(\cot x\ \csc^2 x - \frac{\cos x}{\sin x}\right) dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C_1.$$

Second way:

$$\begin{aligned}
\int \cot^3 x\ dx &= \int (\csc^2 x - 1)\cot x\ dx = \int \left[(\csc x)(\csc x\ \cot x) - \frac{\cos x}{\sin x}\right] dx \\
&= -\frac{1}{2}\csc^2 x - \ln|\sin x| + C_2 = -\frac{1}{2}(1 + \cot^2 x) - \ln|\sin x| + C_2 \\
&= -\frac{1}{2}\cot^2 x - \ln|\sin x| + C_1 \qquad \text{where} \qquad C_1 = C_2 - \frac{1}{2}.
\end{aligned}$$

C08S04.059: First, $\sin 3x\ \cos 5x = \dfrac{1}{2}\left[\sin(3x - 5x) + \sin(3x + 5x)\right] = \dfrac{1}{2}\sin 8x - \dfrac{1}{2}\sin 2x.$ Thus

$$\int \sin 3x \, \cos 5x \, dx = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C.$$

Because *Mathematica* 3.0 gives identical answers in this and the next two problems, it appears that it uses the same formulas for integrating such products. *Maple* V version 5.1 also gives the same answer in this problem; we did not check the following problems of the same type.

C08S04.060: First, $\sin 2x \, \sin 4x = \frac{1}{2}\left[\cos(2x - 4x) - \cos(2x + 4x)\right] = \frac{1}{2}\cos 2x - \frac{1}{2}\cos 6x$. Thus

$$\int \sin 2x \, \sin 4x \, dx = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C.$$

C08S04.061: First, $\cos x \, \cos 4x = \frac{1}{2}\left[\cos(x - 4x) + \cos(x + 4x)\right] = \frac{1}{2}\cos 3x + \frac{1}{2}\cos 5x$. Thus

$$\int \cos x \, \cos 4x \, dx = \frac{1}{6}\sin 3x + \frac{1}{10}\sin 5x + C.$$

C08S04.062: Part (a): $\sin mx \, \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$. Therefore

$$\int_0^{2\pi} \sin mx \, \sin nx \, dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{2\pi} = 0$$

because the sine of an integral multiple of π is zero.

Part (b): $\cos mx \, \sin nx = \frac{1}{2}\left[\sin(n-m)x + \sin(n+m)x\right]$. Therefore

$$\int_0^{2\pi} \cos mx \, \sin nx \, dx = \frac{1}{2}\left[\frac{\cos(n-m)x}{m-n} - \frac{\cos(n+m)x}{n+m}\right]_0^{2\pi} = 0$$

because the cosine of an integral multiple of 2π is 1.

Part (c): $\cos mx \, \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$. Thus

$$\int_0^{2\pi} \cos mx \, \cos nx \, dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_0^{2\pi} = 0$$

because the sine of an integral multiple of π is zero.

C08S04.063: $\displaystyle\int \sec x \, \csc x \, dx = \int \frac{\sec^2 x}{\tan x} \, dx = \ln|\tan x| + C.$

C08S04.064: First note that

$$\csc x = \frac{1}{\sin x} = \frac{1}{\sin(2 \cdot x/2)} = \frac{1}{2\sin(x/2)\,\cos(x/2)} = \frac{1}{2}\csc(x/2)\,\sec(x/2).$$

Therefore

$$\int \csc x \, dx = \frac{1}{2}\int \csc(x/2)\,\sec(x/2)\,dx.$$

Let $u = x/2$: $du = \frac{1}{2}\,dx$. Thus

$$\int \csc x\, dx = \int \csc u \sec u\, du = \ln|\tan u| + C = \ln\left|\tan\frac{x}{2}\right| + C.$$

C08S04.065: We use the substitution $x = \frac{1}{2}\pi - u,\ dx = -du$. Because (from Problem 64)

$$\int \csc x\, dx = \ln\left|\tan\frac{x}{2}\right| + C,$$

we have

$$-\int \csc\left(\frac{1}{2}\pi - u\right)du = \ln\left|\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)\right| + C.$$

But

$$\sin\left(\frac{1}{2}\pi - u\right) = \cos u \qquad \text{and so} \qquad \csc\left(\frac{1}{2}\pi - u\right) = \sec u.$$

Therefore

$$-\int \sec u\, du = \ln\left|\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)\right| C;$$

$$\int \sec u\, du = \ln\left|\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)\right|^{-1} + C;$$

$$\int \sec x\, dx = \ln\left|\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C.$$

C08S04.066: The following computation is sufficient:

$$\cot\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)} = \frac{\cos\frac{\pi}{4}\cos\frac{x}{2} + \sin\frac{\pi}{4}\sin\frac{x}{2}}{\sin\frac{\pi}{4}\cos\frac{x}{2} - \cos\frac{\pi}{4}\sin\frac{x}{2}}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}} = \frac{\cos^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$$

C08S04.067: The reduction formula in Eq. (12) tells us that if n is an integer and $n \geqq 2$, then

$$\int \tan^n x\, dx = \frac{(\tan x)^{n-1}}{n-1} - \int (\tan x)^{n-2} dx.$$

Hence

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \int \tan^2 x$$

$$= \frac{1}{3}\tan^3 x - \tan x + \int 1\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

Mathematica 3.0 gives the antiderivative in the form

$$\frac{1}{12}(\sec^3 x)(9x\cos x + 3x\cos 3x - 4\sin 3x).$$

Now

$$\begin{aligned}\cos 3x &= \cos 2x\,\cos x - \sin 2x\,\sin x = \cos^3 x - \sin^2 x\,\cos x - 2\sin^2 x\,\cos x\\ &= \cos^3 x - 3(1-\cos^2 x)\cos x = \cos^3 x + 3\cos^3 x - 3\cos x = 4\cos^3 x - 3\cos x\end{aligned}$$

and

$$\begin{aligned}\sin 3x &= \sin 2x\,\cos x + \cos 2x\,\sin x = 2\sin x\,\cos^2 x + \cos^2 x\,\sin x - \sin^3 x\\ &= 3\sin x\,\cos^2 x - (1-\cos^2 x)\sin x = 4\sin x\,\cos^2 x - \sin x.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{12}(\sec^3 x)(9x\cos x + 3x\cos 3x - 4\sin 3x)\\ &\quad= \frac{1}{12}(\sec^3 x)(9x\cos x + 12x\cos^3 x - 9x\cos x - 16\sin x\,\cos^2 x + 4\sin x)\\ &\quad= \frac{1}{12}(12x - 16\tan x + 4\sec^2 x\,\tan x) = \frac{1}{12}(12x - 16\tan x + 4(1+\tan^2 x)\tan x)\\ &\quad= \frac{1}{12}(12x - 16\tan x + 4\tan x + 4\tan^3 x) = x - \tan x + \frac{1}{3}\tan^3 x.\end{aligned}$$

The two antiderivatives are exactly the same. By contrast, **Maple** V version 5.1 gives essentially the same antiderivative as the one we derived "by hand."

C08S04.068: Example 9 shows that

$$\int \tan^6 x\,dx = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.$$

Mathematica 3.0 gives the antiderivative in the form

$$\frac{1}{240}(\sec^5 x)(-150x\cos x - 75x\cos 3x - 15x\cos 5x + 50\sin x + 25\sin 3x + 23\sin 5x).$$

To reconcile the two answers, we use two formulas from the solution of Problem 67:

$$\cos 3x = 4\cos^3 x - 3\cos x \qquad \text{and} \qquad \sin 3x = 4\sin x\,\cos^2 x - \sin x.$$

We will also need two more:

$$\begin{aligned}\cos 5x &= \cos 3x\,\cos 2x - \sin 3x\,\sin 2x\\ &= (4\cos^3 x - 3\cos x)(2\cos^2 x - 1) - (4\sin x\,\cos^2 x - \sin x)(2\sin x\,\cos x)\\ &= 8\cos^5 x - 6\cos^3 x + 4\cos^3 x + 3\cos x - 8\sin^2 x\,\cos^3 x + 2\sin^2 x\,\cos x\\ &= 8\cos^5 x - 10\cos^3 x + 3\cos x - 8(1-\cos^2 x)\cos^3 x + 2(1-\cos^2 x)\cos x\\ &= 8\cos^5 x - 10\cos^3 x + 3\cos x - 8\cos^3 x + 8\cos^5 x + 2\cos x - 2\cos^3 x\\ &= 16\cos^5 x - 20\cos^3 x + 5\cos x\end{aligned}$$

and

$$
\begin{aligned}
\sin 5x &= \sin 2x\,\cos 3x + \cos 2x\,\sin 3x \\
&= (2\sin x\,\cos x)(4\cos^3 x - 3\cos x) + (2\cos^2 x - 1)(4\sin x\,\cos^2 x - \sin x) \\
&= 8\sin x\,\cos^4 x - 6\sin x\,\cos^2 x + 8\sin x\,\cos^4 x - 2\sin x\,\cos^2 x - 4\sin x\,\cos^2 x + \sin x \\
&= 16\sin x\,\cos^4 x - 12\sin x\,\cos^2 x + \sin x.
\end{aligned}
$$

Therefore the third factor in *Mathematica's* answer becomes

$$
\begin{aligned}
&-150x\cos x - 75x(4\cos^3 x - 3\cos x) - 15x(16\cos^5 x - 20\cos^3 x + 5\cos x) \\
&\qquad + 50\sin x + 25(4\sin x\,\cos^2 x - \sin x) + 23(15\sin x\,\cos^4 x - 12\sin x\,\cos^2 x + \sin x) \\
&\quad = -150x\cos x - 300x\cos^3 x + 225x\cos x - 240x\cos^5 x + 300x\cos^3 x - 75x\cos x \\
&\qquad + 50\sin x + 100\sin x\,\cos^2 x - 25\sin x + 368\sin x\,\cos^4 x - 276\sin x\,\cos^2 x + 23\sin x \\
&\quad = -240x\cos^5 x + 48\sin x - 176\sin x\,\cos^2 x + 368\sin x\,\cos^4 x.
\end{aligned}
$$

Now multiply by $\sec^5 x$. The result is

$$
\begin{aligned}
&-240x + 48\sec^4 x\,\tan x - 176\sec^2 x\,\tan x + 368\tan x \\
&\quad = -240x + 48(1+\tan^2 x)^2\tan x - 176(1+\tan^2 x)\tan x + 368\tan x \\
&\quad = -240x + 48\tan x + 96\tan^3 x + 48\tan^5 x - 176\tan x - 176\tan^3 x + 368\tan x \\
&\quad = -240x + 240\tan x - 80\tan^3 x + 48\tan^5 x.
\end{aligned}
$$

Finally, divide by 240. Thus *Mathematica's* antiderivative turns out to equal

$$-x + \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x,$$

exactly the same as the result from Example 9. And, in conclusion, *Derive* 2.56 gives the antiderivative in exactly the same form as in Example 9; *Maple* V version 5.1 returns

$$\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - \arctan(\tan x).$$

Section 8.5

C08S05.001: $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$, so $\displaystyle\int \frac{x^2}{x+1}\,dx = \frac{1}{2}x^2 - x + \ln|x+1| + C.$

C08S05.002: $\dfrac{x^3}{2x-1} = \dfrac{1}{2}x^2 + \dfrac{1}{4}x + \dfrac{1}{8} + \dfrac{1}{8(2x-1)}$, so

$$\int \frac{x^3}{2x-1}\,dx = \frac{1}{6}x^3 + \frac{1}{8}x^2 + \frac{1}{8}x + \frac{1}{16}\ln|2x-1| + C.$$

C08S05.003: Given:

$$\frac{1}{x^2-3x} = \frac{1}{x(x-3)} = \frac{A}{x} + \frac{B}{x-3},$$

so $Ax - 3A + Bx = 1$, and thus $A + B = 0$ and $-3A = 1$. So

$$\frac{1}{x^2-3x} = \frac{-\frac{1}{3}}{x} + \frac{\frac{1}{3}}{x-3},$$

and therefore

$$\int \frac{1}{x^2-3x}\,dx = \frac{1}{3}\left(\ln|x-3| - \ln|x|\right) + C = \frac{1}{3}\ln\left|\frac{x-3}{x}\right| + C.$$

C08S05.004: $\dfrac{x}{x^2+4x} = \dfrac{1}{x+4}$, so $\displaystyle\int \frac{x}{x^2+4x}\,dx = \ln|x+4| + C.$

C08S05.005: $x^2 + x - 6 = (x-2)(x+3)$, so

$$\frac{1}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3}.$$

Therefore $Ax + 3A + Bx - 2B = 1$, so that $A + B = 0$ and $3A - 2B = 1$. Thus

$$\int \frac{1}{x^2+x-6}\,dx = \int\left(\frac{\frac{1}{5}}{x-2} - \frac{\frac{1}{5}}{x+3}\right)dx = \frac{1}{5}\left(\ln|x-2| - \ln|x+3|\right) + C.$$

C08S05.006: Division of numerator by denominator yields

$$\frac{x^3}{x^2+x-6} = x - 1 + \frac{7x-6}{x^2+x-6}.$$

Next,

$$\frac{7x-6}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3}$$

leads to $Ax + 3A + Bx - 2B = 7x - 6$, so that $A + B = 7$ and $3A - 2B = -6$. Thus

$$\int \frac{x^3}{x^2+x-6}\,dx = \int\left(x - 1 + \frac{\frac{8}{5}}{x-2} + \frac{\frac{27}{5}}{x+3}\right)dx = \frac{1}{2}x^2 - x + \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.$$

C08S05.007: $\dfrac{1}{x^3+4x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$ leads to $Ax^2 + 4A + Bx^2 + Cx = 1$. Thus

$$A+B=0, \quad C+0, \quad \text{and} \quad 4A=1.$$

It follows that $A=\frac{1}{4}$, $B=-\frac{1}{4}$, and $C=0$. Hence

$$\int \frac{1}{x^3+4x}\,dx = \int \left(\frac{\frac{1}{4}}{x} - \frac{\frac{1}{4}x}{x^2+4}\right) dx = \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4)+C.$$

C08S05.008: $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ leads to $Ax^2+A+Bx^2+Bx+Cx+C=1$, and hence

$$A+B=0, \quad B+C=0, \quad \text{and} \quad A+C=1.$$

Therefore

$$\int \frac{1}{(x+1)(x^2+1)}\,dx = \int \left(\frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\right) dx = \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + C.$$

C08S05.009: Division of denominator into numerator leads to

$$\frac{x^4}{x^2+4} = x^2 - 4 + \frac{16}{x^2+4},$$

and therefore

$$\int \frac{x^4}{x^2+4}\,dx = \frac{1}{3}x^3 - 4x + 8\arctan\left(\frac{x}{2}\right) + C.$$

C08S05.010: $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4}$ leads to

$$Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D=1,$$

so that

$$A+C=0, \quad B+D=0, \quad 4A+C=0, \quad \text{and} \quad 4B+D=1.$$

Thus

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx = \int \left(\frac{\frac{1}{3}}{x^2+1} - \frac{\frac{1}{3}}{x^2+4}\right) dx = \frac{1}{3}\arctan x - \frac{1}{6}\arctan\left(\frac{x}{2}\right) + C.$$

C08S05.011: Division of denominator into numerator yields

$$\int \frac{x-1}{x+1}\,dx = \int \left(1 - \frac{2}{x+1}\right) dx = x - 2\ln|x+1| + C.$$

C08S05.012: Division of denominator into numerator yields

$$\int \frac{2x^3-1}{x^2+1}\,dx = \int \left(2x - \frac{2x+1}{x^2+1}\right) dx = x^2 - \ln(x^2+1) - \arctan x + C.$$

C08S05.013: Division of denominator into numerator yields

$$\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \left(1 - \frac{1}{(x+1)^2}\right) dx = x + \frac{1}{x+1} + C.$$

C08S05.014: $\dfrac{2x-4}{x^2-x} = \dfrac{A}{x} + \dfrac{B}{x-1}$ leads to $Ax - A + Bx = 2x - 4$, so that

$$A + B = 2 \quad \text{and} \quad A = 4. \qquad \text{Thus} \quad B = -2.$$

Therefore

$$\int \frac{2x-4}{x^2-x}\,dx = \int \left(\frac{4}{x} - \frac{2}{x-1}\right) dx = 4\ln|x| - 2\ln|x-1| + C.$$

C08S05.015: $\dfrac{1}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$, so that $Ax + 2A + Bx - 2B = 1$. So

$$\int \frac{1}{x^2-4}\,dx = \int \left(\frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\right) dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C.$$

C08S05.016: Division of denominator into numerator yields

$$\frac{x^4}{x^2+4x+4} = x^2 - 4x + 12 - \frac{32x+48}{(x+2)^2}.$$

Next,

$$\frac{32x+48}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2},$$

so that $Ax + 2A + B = 32x + 48$. It now follows that

$$A = 32 \quad \text{and} \quad 2A + B = 48. \qquad \text{So} \quad B = -16.$$

Therefore

$$\begin{aligned}\int \frac{x^4}{x^2+4x+4}\,dx &= \int \left(x^2 - 4x + 12 - \frac{32}{x+2} + \frac{16}{(x+2)^2}\right) dx \\ &= \frac{1}{3}x^3 - 2x^2 + 12x - \frac{16}{x+2} - 32\ln|x+2| + C.\end{aligned}$$

C08S05.017: $\dfrac{x+10}{2x^2+5x-3} = \dfrac{A}{x+3} + \dfrac{B}{2x-1}$ yields $2Ax - A + Bx + 3B = x + 10$. Thus

$$\int \frac{x+10}{2x^2+5x-3}\,dx = \int \left(\frac{3}{2x-1} - \frac{1}{x+3}\right) dx = \frac{3}{2}\ln|2x-1| - \ln|x+3| + C.$$

C08S05.018: $\dfrac{x+1}{x^3-x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$ yields $Ax^2 - Ax + Bx - B + Cx^2 = x + 1$. Hence

$$A + C = 0, \quad -A + B = 1, \quad \text{and} \quad -B = 1.$$

Therefore $B = -1$, $A = -2$, and $C = 2$. Thus

$$\int \frac{x+1}{x^3 - x^2}\,dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2} \right) dx = 2\ln|x-1| - 2\ln|x| + \frac{1}{x} + C.$$

C08S05.019: $\dfrac{x^2+1}{x^3+2x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$ yields $A(x+1)^2 + Bx(x+1) + Cx = x^2 + 1$. So

$$A + B = 1, \quad 2A + B + C = 0, \quad \text{and} \quad A = 1.$$

Hence $B = 0$ and $C = -2$. Therefore

$$\int \frac{x^2+1}{x^3+2x^2+x}\,dx = \int \left(\frac{1}{x} - \frac{2}{(x+1)^2} \right) dx = \frac{2}{x+1} + \ln|x| + C.$$

C08S05.020: $\dfrac{x^2+x}{x^3-x^2-2x} = \dfrac{x(x+1)}{(x-2)x(x+1)} = \dfrac{1}{x-2}$. Therefore

$$\int \frac{x^2+x}{x^3-x^2-2x}\,dx = \int \frac{1}{x-2}\,dx = \ln|x-2| + C.$$

C08S05.021: $\dfrac{4x^3-7x}{x^4-5x^2+4} = \dfrac{A}{x-2} + \dfrac{B}{x-1} + \dfrac{C}{x+1} + \dfrac{D}{x+2}$ yields

$$A(x^3+2x^2-x-2) + B(x^3+x^2-4x-4) + C(x^3-x^2-4x+4) + D(x^3-2x^2-x+2) = 4x^3 - 7x.$$

Thus

$$\begin{aligned} A + B + C + D &= 4, \\ 2A + B - C - 2D &= 0, \\ -A - 4B - 4C - D &= -7, \\ -2A - 4B + 4C + 2D &= 0. \end{aligned}$$

This system of equations has the solution $A = \frac{3}{2}$, $B = \frac{1}{2}$, $C = \frac{1}{2}$, $D = \frac{3}{2}$. Therefore

$$\begin{aligned} \int \frac{4x^3-7x}{x^4-5x^2+4} &= \frac{1}{2} \int \left(\frac{3}{x-2} + \frac{1}{x-1} + \frac{1}{x+1} + \frac{3}{x+2} \right) dx \\ &= \frac{1}{2} \left(3\ln|x-2| + \ln|x-1| + \ln|x+1| + 3\ln|x+2| \right) + C. \end{aligned}$$

C08S05.022: $\dfrac{2x^2+3}{x^4-2x^2+1} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$ yields

$$A(x^3+x^2-x-1) + B(x^2+2x+1) + C(x^3-x^2-x+1) + D(x^2-2x+1) = 2x^3 + 3.$$

Thus

$$\begin{aligned}
A + C &= 2, \\
A + B - C + D &= 0, \\
-A + 2B - C - 2D &= 0, \\
-A + B + C + D &= 3.
\end{aligned}$$

This system has solution $A = -\frac{1}{4},\ B = \frac{5}{4},\ C = \frac{1}{4},\ D = \frac{5}{4}$. Therefore

$$\begin{aligned}
\int \frac{2x^2+3}{x^4-2x^2+1}\,dx &= \frac{1}{4}\int \left[\frac{5}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x+1} + \frac{5}{(x+1)^2}\right] dx \\
&= \frac{1}{4}\left[\ln|x+1| - \ln|x-1| - \frac{5}{x+1} - \frac{5}{x-1}\right] + C.
\end{aligned}$$

C08S05.023: $\dfrac{x^2}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$ yields

$$A(x^2+4x+4) + B(x+2) + C = x^2,$$

so that $A = 1,\ 4A + B = 0,$ and $4A + 2B + C = 0$. It follows that $B = -4$ and $C = 4$. Hence

$$\int \frac{x^2}{(x+2)^3}\,dx = \int \left(\frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3}\right) dx = \frac{4}{x+2} - \frac{2}{(x+2)^2} + \ln|x+2| + C.$$

C08S05.024: $\dfrac{x^2+x}{(x^2-4)(x+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{x+4}$ yields

$$A(x^2+6x+8) + B(x^2+2x-8) + C(x^2-4) = x^2+x,$$

so that $A + B + C = 1,\ 6A + 2B = 1,$ and $8A - 8B - 4C = 0$. It follows that $A = \frac{1}{4},\ B = -\frac{1}{4},$ and $C = 1$. Therefore

$$\int \frac{x^2+x}{(x^2-4)(x+4)}\,dx = \frac{1}{4}\int \left(\frac{1}{x-2} - \frac{1}{x+2} + \frac{4}{x+4}\right) dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2| + 4\ln|x+4|\right) + C.$$

C08S05.025: $\dfrac{1}{x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$, so $Ax^2 + A + Bx^2 + Cx = 1$. Thus $A + B = 0,\ C = 0,$ and $A = 1$. Therefore

$$\int \frac{1}{x^3+x}\,dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C = \frac{1}{2}\ln\left(\frac{x^2}{x^2+1}\right) + C.$$

C08S05.026: $\dfrac{6x^3-18x}{(x^2-1)(x^2-4)} = \dfrac{A}{x-2} + \dfrac{B}{x-1} + \dfrac{C}{x+1} + \dfrac{D}{x+2}$ leads to

$$A(x^3+2x^2-x-2) + B(x^3+x^2-4x-4) + C(x^3-x^2-4x+4) + D(x^3-2x^2-x+2) = 6x^3-18x.$$

Thus

$$
\begin{aligned}
A+B+C+D&=6,\\
2A+B-C-2D&=0,\\
-A-4B-4C-D&=-18,\\
-2A-4B+4C+2D&=0.
\end{aligned}
$$

It follows that $A=1,\ B=2,\ C=2,$ and $D=1.$ Therefore

$$
\begin{aligned}
\int \frac{6x^3-18x}{(x^2-1)(x^2-4)}\,dx &= \int \left(\frac{1}{x-2}+\frac{2}{x-1}+\frac{2}{x+1}+\frac{1}{x+2}\right)dx\\
&= \ln|x-2|+2\ln|x-1|+2\ln|x+1|+\ln|x+2|+C.
\end{aligned}
$$

C08S05.027: $\dfrac{x+4}{x^3+4x}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$ leads to $Ax^2+4A+Bx^2+Cx=x+4$. So

$$A+B=0,\quad C=1,\quad \text{and}\quad 4A=4.\qquad \text{So}\quad A=1,\quad B=-1.$$

Thus

$$\int \frac{x+4}{x^3+4x}\,dx=\int\left(\frac{1}{x}-\frac{x}{x^2+4}+\frac{1}{x^2+4}\right)dx=\ln|x|-\frac{1}{2}\ln(x^2+4)+\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C.$$

C08S05.028: $\dfrac{4x^4+x+1}{x^5+x^4}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x^4}+\dfrac{E}{x+1}$ implies that

$$A(x^4+x^3)+B(x^3+x^2)+C(x^2+x)+D(x+1)+Ex^4=4x^4+x+1.$$

Thus

$$
\begin{aligned}
A+E&=4,\\
A+B&=0,\\
B+C&=0,\\
C+D&=1,\\
D&=1.
\end{aligned}
$$

These equations are easily solve from the bottom up: $D=1,\ C=0,\ B=0,\ A=0,$ and $E=4.$ Therefore

$$\int \frac{4x^4+x+1}{x^5+x^4}\,dx=\int\left(\frac{1}{x^4}+\frac{4}{x+1}\right)dx=-\frac{1}{3x^3}+4\ln|x+1|+C.$$

C08S05.029: $\dfrac{x}{(x+1)(x^2+1)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$ yields $Ax^2+A+Bx^2+Bx+Cx+C=x$. Thus

$$A+B=0,\quad B+C=1,\quad \text{and}\quad A+C=0.$$

It follows that $A=-\frac{1}{2},\ B=\frac{1}{2},$ and $C=\frac{1}{2}.$ Therefore

$$\int \frac{x}{(x+1)(x^2+1)}\,dx=\frac{1}{2}\int\left(-\frac{1}{x+1}+\frac{x+1}{x^2+1}\right)dx=-\frac{1}{2}\ln|x+1|+\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan x+C.$$

C08S05.030: Rather than searching for unknown coefficients, we note that

$$\left(\frac{x+2}{x^2+4}\right)^2 = \frac{x^2+4x+4}{(x^2+4)^2} = \frac{x^2+4}{(x^2+4)^2} + \frac{4x}{(x^2+4)^2} = \frac{1}{x^2+4} + \frac{4x}{(x^2+4)^2}.$$

Therefore

$$\int \left(\frac{x+2}{x^2+4}\right)^2 dx = \int \frac{1}{x^2+4}\,dx + \int \frac{4x}{(x^2+4)^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{2}{x^2+1} + C.$$

C08S05.031: $\dfrac{x^2-10}{2x^4+9x^2+4} = \dfrac{A}{x^2+4} + \dfrac{B}{2x^2+1}$ implies that $2Ax^2 + A + Bx^2 + 4B = x^2 - 10$, and thus

$$2A + B = 1 \quad \text{and} \quad A + 4B = -10, \qquad \text{so that} \qquad A = 2 \quad \text{and} \quad B = -3.$$

Therefore

$$\int \frac{x^2-10}{2x^4+9x^2+4}\,dx = \int \left(\frac{2}{x^2+4} - \frac{3}{2x^2+1}\right) dx = \arctan\left(\frac{x}{2}\right) - \frac{3\sqrt{2}}{2}\arctan\left(x\sqrt{2}\right) + C.$$

A substitution to integrate the second fraction is $u = x\sqrt{2}$.

C08S05.032: $\dfrac{x^2}{x^4-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$, so that

$$A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + C(x^3-x) + D(x^2-1) = x^2.$$

Therefore

$$\begin{aligned} A + B + C &= 0, \\ A - B + D &= 1, \\ A + B - C &= 0, \\ A - B - D &= 0. \end{aligned}$$

It follows that $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$, and $D = \frac{1}{2}$. Thus

$$\int \frac{x^2}{x^4-1}\,dx = \frac{1}{4}\int \left(\frac{1}{x-1} - \frac{1}{x+1} + \frac{2}{x^2+1}\right) dx = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\arctan x + C.$$

C08S05.033: $\dfrac{x^3+x^2+2x+3}{x^4+5x^2+6} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{x^2+3}$, and so

$$Ax^3 + 3Ax + Bx^2 + 3B + Cx^3 + 2Cx + Dx^2 + 2D = x^3 + x^2 + 2x + 3.$$

Therefore

$$\begin{aligned} A + C &= 1, & B + D &= 1, \\ 3A + 2C &= 2, & 3B + 2D &= 3. \end{aligned}$$

It follows that $A = 0$, $B = 1$, $C = 1$, and $D = 0$. Hence

$$\int \frac{x^3+x^2+2x+3}{x^4+5x^2+6}\,dx = \int \left(\frac{1}{x^2+2} + \frac{x}{x^2+3} \right) dx = \frac{\sqrt{2}}{2} \arctan\left(\frac{x\sqrt{2}}{2} \right) + \frac{1}{2}\ln(x^2+3) + C.$$

C08S05.034: $\dfrac{x^2+4}{(x^2+1)^2(x^2+2)} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$ yields

$A(x^5+2x^3+x)+B(x^4+2x^2+1)+C(x^5+3x^3+2x)+D(x^4+3x^2+2)+E(x^3+2x)+F(x^2+2) = x^2+4.$

Therefore

$$\begin{array}{lll} A+C=0, & B+D=0, & 2A+3C+E=0, \\ 2B+3D+F=1, & A+2C+2E=0, & B+2D+2F=4. \end{array}$$

Note how the equations involving A, C, and E "separate" from those involving B, D, and F. This makes it easy to solve them for $A=0$, $B=2$, $C=0$, $D=-2$, $E=0$, and $F=3$. Therefore

$$\begin{aligned} \int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx &= \int \left(\frac{2}{x^2+2} - \frac{2}{x^2+1} + \frac{3}{(x^2+1)^2} \right) dx \\ &= \sqrt{2}\,\arctan\left(\frac{x\sqrt{2}}{2} \right) - \frac{1}{2}\arctan x + \frac{3x}{2(x^2+1)} + C. \end{aligned}$$

(Part of the solution of Problem 30 was used to integrate the third fraction.)

C08S05.035: Expand the denominator to $x^4-2x^3+2x^2-2x+1$ and then divide it into the numerator to find that

$$\frac{x^4+3x^2-4x+5}{(x^2+1)(x-1)^2} = 1 + \frac{2x^3+x^2-2x+4}{(x^2+1)(x-1)^2}.$$

Then

$$\frac{2x^3+x^2-2x+4}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

leads to

$$A(x^3-x^2+x-1)+B(x^2+1)+C(x^3-2x^2+x)+D(x^2-2x+1) = 2x^3+x^2-2x+4.$$

Therefore

$$\begin{array}{ll} A+C=2, & -A+B-2C+D=1, \\ A+C-2D=-2, & -A+B+D=4; \end{array}$$

the solution is $A=\frac{1}{2}$, $B=\frac{5}{2}$, $C=\frac{3}{2}$, $D=2$. Therefore

$$\begin{aligned} \int \frac{x^4+3x^2-4x+5}{(x^2+1)(x-1)^2}\,dx &= \int \left(1 + \frac{\frac{1}{2}}{x-1} + \frac{\frac{5}{2}}{(x-1)^2} + \frac{\frac{3}{2}x}{x^2+1} + \frac{2}{x^2+1} \right) dx \\ &= x + \frac{1}{2}\ln|x-1| - \frac{5}{2(x-1)} + \frac{3}{4}\ln(x^2+1) + 2\arctan x + C. \end{aligned}$$

C08S05.036: $\dfrac{2x^3+5x^2-x+3}{(x^2+x-2)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$ yields

$$A(x-1)(x^2+4x+4) + B(x^2+4x+4) + C(x+2)(x^2-2x+1) + D(x^2-2x+1) = 2x^3+5x^2-x+3;$$

$$A(x^3+3x^2-4) + B(x^2+4x+4) + C(x^3-3x+2) + D(x^2-2x+1) = 2x^3+5x^2-x+3.$$

Thus

$$A+C=2, \qquad 3A+B+D=5,$$

$$4B-3C-2D=-1, \qquad -4A+4B+2C+D=3.$$

It follows that $A=B=C=D=1$, and therefore

$$\begin{aligned}\int \frac{2x^3+5x^2-x+3}{(x^2+x-2)^2}\,dx &= \int \left(\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{x}{x+2} + \frac{1}{(x+2)^2}\right)dx \\ &= \ln|x-1| - \frac{1}{x-1} + \ln|x+2| - \frac{1}{x+2} + C.\end{aligned}$$

C08S05.037: Let $x = e^{2t}$; then $dx = 2e^{2t}\,dt$. Thus

$$\int \frac{e^{4t}}{(e^{2t}-1)^3}\,dt = \frac{1}{2}\int \frac{x}{(x-1)^3}\,dx.$$

Then $\dfrac{x}{(x-1)^3} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$ leads to $A(x^2-2x+1)+B(x-1)+C = x$, and so

$$A=0, \quad -2A+B=1, \quad \text{and} \quad A-B+C=0.$$

Therefore $B=C=1$. Hence

$$\int \frac{x}{(x-1)^3}\,dx = \int \left(\frac{1}{(x-1)^2} + \frac{1}{(x-1)^3}\right)dx = -\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C.$$

Finally,

$$\int \frac{e^{4t}}{(e^{2t}-1)^3}\,dt = -\frac{1}{2(x-1)} - \frac{1}{4(x-1)^2} + C = -\frac{1}{2(e^{2t}-1)} - \frac{1}{4(e^{2t}-1)^2} + C.$$

C08S05.038: Let $u = \sin\theta$; then $du = \cos\theta\,d\theta$. So

$$\begin{aligned}I &= \int \frac{\cos\theta}{\sin^2\theta - \sin\theta - 6}\,d\theta = \int \frac{1}{u^2-u-6}\,du = \frac{1}{5}\int\left(\frac{1}{u-3} - \frac{1}{u+2}\right)du \\ &= \frac{1}{5}\left(\ln|u-3| - \ln|u+2|\,\right) + C = \frac{1}{5}\left(\ln|-3+\sin\theta| - \ln|2+\sin\theta|\,\right) + C.\end{aligned}$$

C08S05.039: Let $u = \ln t$; then $du = \dfrac{1}{t}\,dt$. Therefore

$$
\begin{aligned}
J &= \int \frac{1+\ln t}{t(3+2\ln t)^2}\,dt = \int \frac{u+1}{(2u+3)^2}\,du = \frac{1}{2}\int\left(\frac{1}{2u+3} - \frac{1}{(2u+3)^2}\right)du \\
&= \frac{1}{4}\ln|2u+3| + \frac{1}{4}\cdot\frac{1}{2u+3} + C = \frac{1}{4}\ln|3+2\ln t| + \frac{1}{4(3+2\ln t)} + C.
\end{aligned}
$$

C08S05.040: Let $u = \tan t$. Then $du = \sec^2 t\,dt$. Hence

$$
\begin{aligned}
K &= \int \frac{\sec^2 t}{\tan^3 t + \tan^2 t}\,dt = \int \frac{1}{u^3+u^2}\,du = \int\left(-\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\right)du \\
&= -\ln|u| - \frac{1}{u} + \ln|1+u| + C = \ln\left|\frac{1+\tan t}{\tan t}\right| - \cot t + C \\
&= \ln\left|\frac{\cos t + \sin t}{\sin t}\right| - \cot t + C = \ln|\sin t + \cos t| - \ln|\sin t| - \cot t + C.
\end{aligned}
$$

C08S05.041: $\dfrac{x-9}{x^2-3x} = \dfrac{3}{x} - \dfrac{2}{x-3}$. So

$$
\int_1^2 \frac{x-9}{x^2-3x}\,dx = \Big[3\ln|x| - 2\ln|x-3|\Big]_1^2 = 3\ln 2 - (-2\ln 2) = 5\ln 2 \approx 3.4657359028.
$$

C08S05.042: $\dfrac{x+5}{3+2x-x^2} = \dfrac{1}{x+1} - \dfrac{2}{x-3}$. Hence

$$
\int_0^2 \frac{x+5}{3+2x-x^2}\,dx = \Big[\ln|x+1| - 2\ln|x-3|\Big]_0^2 = \ln 3 - (-2\ln 3) = 3\ln 3 \approx 3.2958368660.
$$

C08S05.043: $\dfrac{3x-15-2x^2}{x^3-9x} = \dfrac{1}{3}\left(\dfrac{5}{x} - \dfrac{7}{x+3} - \dfrac{4}{x-3}\right)$. Hence

$$
\begin{aligned}
\int_0^2 \frac{3x-15-2x^2}{x^3-9x}\,dx &= \frac{1}{3}\Big[5\ln|x| - 7\ln|x+3| - 4\ln|x-3|\Big]_0^2 \\
&= \frac{1}{3}(5\ln 2 - 7\ln 5 + 4\ln 2 + 7\ln 4) = \frac{1}{3}(23\ln 2 - 7\ln 5) \approx 1.5587732553.
\end{aligned}
$$

C08S05.044: $\dfrac{x^2+10x+16}{x^3+8x^2+16x} = \dfrac{1}{x} + \dfrac{2}{(x+4)^2}$. Thus

$$
\int_2^5 \frac{x^2+10x+16}{x^3+8x^2+16x}\,dx = \left[\ln|x| - \frac{2}{x+4}\right]_2^5 = -\frac{2}{9} + \ln 5 + \frac{1}{3} - \ln 2 = \frac{1}{9} - \ln 2 + \ln 5 \approx 1.0274018430.
$$

C08S05.045: $x\cdot\dfrac{x-9}{x^2-3x} = 1 - \dfrac{6}{x-3}$. Hence the volume is

$$
V = 2\pi\int_1^2\left(1 - \frac{6}{x-3}\right)dx = 2\pi\Big[x - 6\ln|x-3|\Big]_1^2 = 2\pi(1+6\ln 2) \approx 32.4142183908.
$$

C08S05.046: $x \cdot \dfrac{x+5}{3+2x-x^2} = -1 - \dfrac{6}{x-3} - \dfrac{1}{x+1}$. So the volume is

$$V = 2\pi \int_0^2 \left(-1 - \frac{6}{x-3} - \frac{1}{x+1}\right) dx = 2\pi \Big[-x - 6\ln|x-3| - \ln|x+1| \Big]_0^2$$

$$= 2\pi(-2 + 5\ln 3) \approx 21.9475523379.$$

C08S05.047: $x \cdot \dfrac{3x-15-2x^2}{x^3-9x} = -2 - \dfrac{4}{x-3} + \dfrac{7}{x+3}$. So the volume is

$$V = 2\pi \int_1^2 \left(-2 - \frac{4}{x-3} + \frac{7}{x+3}\right) dx$$

$$= 2\pi \Big[-2x - 4\ln|x-3| + 7\ln|x+3| \Big]_1^2 = 2\pi(7\ln 5 - 2 - 10\ln 2) \approx 14.66868411.$$

C08S05.048: $x \cdot \dfrac{x^2+10x+16}{x^3+8x^2+16x} = 1 + \dfrac{2}{x+4} - \dfrac{8}{(x+4)^2}$. So the volume is

$$V = 2\pi \int_2^5 \left(1 + \frac{2}{x+4} - \frac{8}{(x+4)^2}\right) dx$$

$$= 2\pi \left[x + 2\ln|x+4| + \frac{8}{x+4} \right]_2^5 = \frac{2\pi}{9}(23 + 18\ln 9 - 18\ln 6) \approx 21.1522539380.$$

C08S05.049: $\left(\dfrac{x-9}{x^2-3x}\right)^2 = \dfrac{4}{x} + \dfrac{9}{x^2} - \dfrac{4}{x-3} + \dfrac{4}{(x-3)^2}$. So the volume is

$$V = \pi \int_1^2 \left(\frac{x-9}{x^2-3x}\right)^2 dx = \pi \left[4\ln|x| - \frac{9}{x} - 4\ln|x-3| - \frac{4}{x-3} \right]_1^2 = \frac{\pi}{2}(13 + 16\ln 2) \approx 37.8410409708.$$

C08S05.050: $\left(\dfrac{x+5}{3+2x-x^2}\right)^2 = \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} - \dfrac{1}{x-3} + \dfrac{4}{(x-3)^2}$. So the volume is

$$V = \pi \int_0^2 \left(\frac{x+5}{3+2x-x^2}\right)^2 dx$$

$$= \pi \left[\ln|x+1| - \frac{1}{x+1} - \ln|x-3| - \frac{4}{x-3} \right]_0^2 = \frac{2\pi}{3}(5 + \ln 3) \approx 17.3747601024.$$

C08S05.051: The volume is $V = \displaystyle\int_0^1 \pi y^2\, dx$. Now $y^2 = \dfrac{1-x}{1+x}x^2 = -x^2 + 2x - 2 + \dfrac{2}{x+1}$, and so

$$V = \pi \int \left(-x^2 + 2x - 2 + \frac{2}{x+1}\right) dx$$

$$= \pi \left[-\frac{1}{3}x^3 + x^2 - 2x + 2\ln|x+1| \right]_0^1 = \frac{\pi}{3}(-4 + 6\ln 2) \approx 0.1663819758.$$

C08S05.052: Let $f(x) = \dfrac{x^2(1-x)}{1+x}$ for $0 \leqq x \leqq 1$.

Part (a): $[f(x)]^2 = x^4 - 4x^3 + 8x^2 - 12x + 16 - \dfrac{20}{x+1} + \dfrac{4}{(x+1)^2}$. So the volume is

$$\begin{aligned} V &= \pi \int_0^1 [f(x)]^2 \ dx \\ &= \pi \left[\frac{1}{5}x^5 - x^4 + \frac{8}{3}x^3 - 6x^2 + 16x - 20 \ln|x+1| - \frac{4}{x+1} \right]_0^1 = \frac{4\pi}{15}(52 - 75 \ln 2) \approx 0.0116963237. \end{aligned}$$

Part (b): $x \cdot f(x) = 2 - 2x + 2x^2 - x^3 - \dfrac{2}{x+1}$, so the volume is

$$\begin{aligned} V &= 2\pi \int_0^1 2x \cdot f(x) \ dx \\ &= 4\pi \left[2x - x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 - 2 \ln|x+1| \right]_0^1 = \frac{\pi}{3}(17 - 24 \ln 2) \approx 0.381669647913. \end{aligned}$$

C08S05.053: $f(x) = \dfrac{A}{x-7} + \dfrac{B}{x-5} + \dfrac{C}{x} + \dfrac{D}{x^2} = \dfrac{93}{x-7} + \dfrac{49}{x-5} - \dfrac{44}{x} + \dfrac{280}{x^2}$. Thus

$$\int f(x) \ dx = 93 \ln|x-7| + 49 \ln|x-5| - 44 \ln|x| - \frac{280}{x} + C,$$

both by *Mathematica* 3.0, by *Maple* V version 5.1, and by hand (except that the computer algebra programs omit the absolute value symbols).

C08S05.054: $f(x) = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x+7} + \dfrac{D}{(x+7)^2} = \dfrac{323}{x+3} - \dfrac{384}{(x+3)^2} - \dfrac{291}{x+7} - \dfrac{1324}{(x+7)^2}$. Thus

$$\int f(x) \ dx = 323 \ln|x+3| + \frac{384}{x+3} - 291 \ln|x+7| + \frac{1324}{x+7} + C,$$

both by *Mathematica* 3.0 and by hand (except that *Mathematica* omits the absolute value symbols).

C08S05.055: $f(x) = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{C}{x-3} + \dfrac{D}{x+5} + \dfrac{E}{(x+5)^2}$

$$= -\frac{\frac{104}{3}}{x-4} + \frac{48}{(x-4)^2} + \frac{\frac{567}{16}}{x-3} - \frac{\frac{37}{48}}{x+5} - \frac{\frac{39}{2}}{(x+5)^2}. \text{ Therefore}$$

$$\int f(x) \ dx = -\frac{104}{3} \ln|x-4| - \frac{48}{x-4} + \frac{567}{16} \ln|x-3| - \frac{37}{48} \ln|x+5| + \frac{39}{2(x+5)} + C,$$

both by hand and by *Mathematica* 3.0 (except that *Mathematica* omits the absolute value symbols).

C08S05.056: $f(x) = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{(x-2)^3} + \dfrac{D}{(x-2)^4} + \dfrac{E}{x+2} + \dfrac{F}{(x+2)^2}$

$$= \frac{\frac{375}{4}}{x-2} + \frac{\frac{26375}{16}}{(x-2)^2} + \frac{\frac{2625}{2}}{(x-2)^3} - \frac{375}{(x-2)^4} - \frac{\frac{375}{4}}{x+2} - \frac{\frac{375}{16}}{(x+2)^2}. \text{ Thus}$$

$$\int f(x) \ dx = \frac{375}{4} \ln|x-2| - \frac{26375}{16(x-2)} - \frac{2625}{4(x-2)^2} + \frac{125}{(x-2)^3} - \frac{375}{4} \ln|x+2| + \frac{375}{16(x+2)} + C,$$

both by hand and by *Mathematica* 3.0 (except that *Mathematica* omits the absolute value symbols).

C08S05.057: *Mathematica* yields the partial fraction decomposition

$$\frac{4}{2x-1}+\frac{6}{3x-1}+\frac{3}{(3x-1)^2}+\frac{2x}{x^2+25}+\frac{5}{x^2+25}$$

and the antiderivative

$$-\frac{1}{3x-1}-\arctan\frac{5}{x}+2\ln(1-3x)+2\ln(1-2x)+\ln(25+x^2).$$

By hand, we get

$$-\frac{1}{3x-1}+\arctan\frac{x}{5}+2\ln|3x-1|+2\ln|2x-1|+\ln(x^2+25)+C.$$

Mathematica normally omits the absolute value symbols in logarithmic integrals as well as the constant of integration; of course,

$$-\arctan\frac{5}{x} = \arctan\frac{x}{5}+C,$$

so the two answers are basically the same.

C08S05.058: The partial fraction decomposition of the integrand (using *Mathematica*) is

$$\frac{1}{x-2}+\frac{1}{x+2}+\frac{5}{25x^2+1}+\frac{100x}{25x^2+1}-\frac{100x}{(25x^2+1)^2},$$

and according to the computer algebra system, the antiderivative is

$$\frac{2}{25x^2+1}+\arctan(5x)+\ln(x^2-4)+2\ln(25x^2+1);$$

by hand, we obtained

$$\frac{2}{25x^2+1}+\arctan(5x)+\ln|x-2|+\ln|x+2|+2\ln(25x^2+1)+C.$$

C08S05.059: *Maple* V version 5.1 and *Mathematica* 3.0 both yield

$$\int\frac{ax^2+bx+c}{x^2(x-1)}\,dx = \frac{c}{x}+(a+b+c)\ln(x-1)-(b+c)\ln x+C.$$

The term including $\ln x$ drops out if we let $c=-b$, and then the term including $\ln(x-1)$ drops out if $a=0$. Thus to obtain a rational antiderivative, let $a=0$, $b\neq 0$ (but otherwise arbitrary), and $c=-b$.

C08S05.060: According to *Mathematica*,

$$\int\frac{ax^2+bx+c}{x^3(x-1)^2} = (a+2b+3c)\ln x-(a+2b+3c)\ln(x-1)-\frac{a+b+c}{x-1}-\frac{c}{2x^2}-\frac{b+2c}{x}+C.$$

The logarithmic terms drop out if $a+2b+3c=0$. Thus choose a and b not both zero and let $c=-(a+2b)/3$. For example, if $a=1$, $b=1$, and $c=-1$, then the antiderivative is

$$-\frac{1}{x-1}+\frac{1}{2x^2}+\frac{1}{x}+C.$$

C08S05.061: According to *Mathematica*,

$$\begin{aligned}\int \frac{ax^2+bx+c}{x^3(x-4)^4}\,dx \\ &= -\frac{16a+4b+c}{192(x-4)^3}+\frac{16a+8b+3c}{512(x-4)^2}-\frac{8a+6b+3c}{512(x-4)}-\frac{c}{512x^2}-\frac{b+c}{256x} \\ &\quad -\frac{(8a+8b+5c)\ln(x-4)}{2048}+\frac{(8a+8b+5c)\ln x}{2048}+C.\end{aligned}$$

The logarithmic terms drop out if $8a+8b+5c=0$. Hence choose a and b not both zero (but otherwise arbitrary) and let $c=-(8a+8b)/5$. For example, if $a=b=5$ and $c=-16$, then the antiderivative is

$$-\frac{7}{16(x-4)^3}+\frac{9}{64(x-4)^2}-\frac{11}{256(x-4)}+\frac{1}{32x^2}+\frac{11}{256x}+C.$$

Section 8.6

C08S06.001: Let $x = 4\sin\theta$. Then

$$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\cos\theta \qquad \text{and} \qquad dx = 4\cos\theta\, d\theta.$$

Therefore

$$\int \frac{1}{\sqrt{16 - x^2}}\, dx = \int \frac{1}{4\cos\theta} \cdot 4\cos\theta\, d\theta = \int 1\, d\theta = \theta + C = \arcsin\left(\frac{x}{4}\right) + C.$$

C08S06.002: Let $x = \dfrac{2}{3}\sin u$. Then

$$4 - 9x^2 = 4 - 4\sin^2 u = 4\cos^2 u \qquad \text{and} \qquad dx = \frac{2}{3}\cos u\, du.$$

Thus

$$\int \frac{1}{\sqrt{4 - 9x^2}}\, dx = \frac{2}{3}\int \frac{\cos u}{2}\cos u\, du = \frac{1}{3}u + C = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C.$$

C08S06.003: Let $x = 2\sin u$. Then $dx = 2\cos u\, du$ and

$$x^2\sqrt{4 - x^2} = (4\sin^2 u)\sqrt{4 - 4\sin^2 u} = 8\sin^2 u\, \cos u.$$

Thus

$$I = \int \frac{1}{x^2\sqrt{4 - x^2}}\, dx = \frac{2\cos u}{8\sin^2 u\, \cos u}\, du = \frac{1}{4}\csc^2 u\, du = -\frac{1}{4}\cot u + C.$$

The reference triangle with acute angle u, opposite side x, and hypotenuse 2 has adjacent side of length $\sqrt{4 - x^2}$, and thus

$$I = -\frac{1}{4} \cdot \frac{\sqrt{4 - x^2}}{x} + C = -\frac{\sqrt{4 - x^2}}{4x} + C.$$

C08S06.004: Let $x = 5\sec u$. Then $dx = 5\sec u\, \tan u\, du$ and $x^2\sqrt{x^2 - 25}\,, = 125\sec^2 u\, \tan u$. So

$$J = \int \frac{1}{x^2\sqrt{x^2 - 25}}\, dx = \int \frac{5\sec u\, \tan u}{125\sec^2 u\, \tan u}\, du = \frac{1}{25}\int \cos u\, du = \frac{1}{25}\sin u + C.$$

A reference triangle with acute angle u, adjacent side 5, and hypotenuse x has opposite side $\sqrt{x^2 - 25}$, and hence

$$J = \frac{\sqrt{x^2 - 25}}{25x} + C.$$

C08S06.005: Let $x = 4\sin u$. Then $dx = 4\cos u\, du$ and $\dfrac{x^2}{\sqrt{16 - x^2}} = \dfrac{16\sin^2 u}{4\cos u}$. Therefore

$$K = \int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int 16\sin^2 u\,du = 8\int (1-\cos 2u)\,du$$
$$= 8\left(u - \frac{1}{2}\sin 2u\right) + C = 8(u - \sin u\,cosu) + C.$$

A reference triangle with acute angle u, opposite side x, and hypotenuse 4 has adjacent side $\sqrt{16-x^2}$, and therefore

$$K = 8\left[\arcsin\left(\frac{x}{4}\right) - \frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4}\right] + C = 8\arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C.$$

C08S06.006: Let $x = \dfrac{3}{2}\sin u$. Then $dx = \dfrac{3}{2}\cos u\,du$ and

$$\frac{x^2}{\sqrt{9-4x^2}} = \frac{\frac{9}{4}\sin^2 u}{\sqrt{9-9\sin^2 u}} = \frac{3\sin^2 u}{4\cos u};$$
$$I = \int \frac{x^2}{\sqrt{9-4x^2}}\,dx = \frac{9}{8}\int \sin^2 u\,du = \frac{9}{16}\int (1-\cos 2u)\,u = \frac{9}{16}(u - \sin u\,\cos u) + C.$$

The reference triangle with acute angle u, opposite side $2x$, and hypotenuse 3 has adjacent side $\sqrt{9-4x^2}$ and thus

$$I = \frac{9}{16}\left[\arcsin\left(\frac{2x}{3}\right) - \frac{2x}{3}\cdot\frac{\sqrt{9-4x^2}}{3}\right] + C = \frac{9}{16}\arcsin\left(\frac{2x}{3}\right) - \frac{x\sqrt{9-4x^2}}{8} + C.$$

C08S06.007: Let $x = \dfrac{3}{4}\sin u$: $dx = \dfrac{3}{4}\cos u\,du$, $(9-16x^2)^{3/2} = (9-9\sin^2 u)^{3/2} = 27\cos^3 u$. Hence

$$J = \int \frac{1}{(9-16x^2)^{3/2}}\,dx = \frac{3}{4}\int \frac{\cos u}{27\cos^3 u}\,du = \frac{1}{36}\int \sec^2 u\,du = \frac{1}{36}\tan u + C.$$

The reference triangle with acute angle u, opposite side $4x$, and hypotenuse 3 has adjacent side $\sqrt{9-16x^2}$, and hence

$$J = \frac{1}{36}\cdot\frac{4x}{\sqrt{9-16x^2}} + C = \frac{x}{9\sqrt{9-16x^2}} + C.$$

C08S06.008: Let $x = \dfrac{5}{4}\tan u$: $25+16x^2 = 25\sec^2 u$ and $dx = \dfrac{5}{4}\sec^2 u\,du$. Thus

$$K = \int \frac{1}{(25+16x^2)^{3/2}}\,dx = \int \frac{1}{125\sec^3 u}\cdot\frac{5}{4}\sec^2 u\,du = \frac{1}{100}\int \cos u\,du = \frac{1}{100}\sin u + C.$$

The reference triangle with acute angle u has opposite side $4x$ and adjacent side 5, and thus hypotenuse $\sqrt{25+16x^2}$. Therefore

$$K = \frac{1}{100}\cdot\frac{4x}{\sqrt{25+16x^2}} + C = \frac{x}{25\sqrt{25+16x^2}} + C.$$

C08S06.009: Let $x = \sec\theta$: $\sqrt{x^2-1} = \tan\theta$, $dx = \sec\theta\,\tan\theta\,d\theta$. Thus

$$I = \int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\tan\theta}{\sec^2\theta}\,\sec\theta\,\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C.$$

The reference triangle with acute angle θ, adjacent side x, and hypotenuse x has opposite side $\sqrt{x^2-1}$. Therefore

$$I = \ln\left|x + \sqrt{x^2-1}\,\right| - \frac{\sqrt{x^2-1}}{x} + C.$$

C08S06.010: Let $x = 2\sin u$: $\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\cos u$, $dx = 2\cos u\,du$. Thus

$$J = \int x^3\sqrt{4-x^2}\,dx = \int (8\sin^3 u)(2\cos u)(2\cos u)\,du$$

$$= 32\int (1-\cos^2 u)\cos^2 u\,\sin u\,du = 32\left(\frac{1}{5}\cos^5 u - \frac{1}{3}\cos^3 u\right) + C.$$

The reference triangle with acute angle u, opposite side x, and hypotenuse 2 has adjacent side $\sqrt{4-x^2}$. Therefore

$$J = 32\left[\frac{1}{5}\cdot\frac{(4-x^2)^{5/2}}{32} - \frac{1}{3}\cdot\frac{(4-x^2)^{3/2}}{8}\right] + C = \frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + C$$

$$= (4-x^2)^{1/2}\left[\frac{1}{5}(4-x^2)^2 - \frac{4}{3}(4-x^2)\right] + C = \frac{1}{15}(4-x^2)^{1/2}\left[3(16-8x^2+x^4) - 20(4-x^2)\right] + C$$

$$= \frac{1}{15}(3x^4 - 4x^2 - 32)\sqrt{4-x^2} + C.$$

C08S06.011: Let $x = \dfrac{3}{2}\tan u$: $dx = \dfrac{3}{2}\sec^2 u\,du$, $9 + 4x^2 + 9 + 9\tan^2 u = 9\sec^2 u$. Thus

$$K = \int x^3\sqrt{9+4x^2}\,dx = \int \left(\frac{27}{8}\tan^3 u\right)(3\sec u)\left(\frac{3}{2}\sec^2 u\right)du$$

$$= \frac{243}{16}\int \sec^3 u\,\tan^3 u\,du = \frac{243}{16}\int (\sec^3 u)(\sec^2 u - 1)\tan u\,du$$

$$= \frac{243}{16}\int (\sec^4 u - \sec^2 u)\sec u\,\tan u\,du = \frac{243}{16}\left(\frac{1}{5}\sec^5 u - \frac{1}{3}\sec^3 u\right) + C.$$

The reference triangle with acute angle u, opposite side $2x$, and adjacent side 3 has hypotenuse $\sqrt{9+4x^2}$. Hence

$$K = \frac{243}{16}\left[\frac{1}{5}\cdot\frac{(9+4x^2)^{5/2}}{243} - \frac{1}{3}\cdot\frac{(9+4x^2)^{3/2}}{27}\right] + C = \frac{1}{80}(9+4x^2)^{5/2} - \frac{3}{16}(9+4x^2)^{3/2} + C$$

$$= \frac{1}{80}\left[(9+4x^2)^{5/2} - 15(9+4x^2)^{3/2}\right] + C = \frac{\sqrt{9+4x^2}}{80}\left[(9+4x^2)^2 - 15(9+4x^2)\right] + C$$

$$= \frac{\sqrt{9+4x^2}}{80}\left(16x^4 + 72x^2 + 81 - 60x^2 - 135\right) + C = \frac{\sqrt{9+4x^2}}{80}(16x^4 + 12x^2 - 54) + C$$

$$= \frac{1}{40}(8x^4 + 6x^2 - 27)\sqrt{9+4x^2} + C.$$

Such extensive algebraic simplifications are not normally necessary.

C08S06.012: Let $x = 5\tan u$: $dx = 5\sec^2 u\, du$, $x^2 + 25 + 25\sec^2 u$. Therefore

$$I = \int \frac{x^3}{\sqrt{x^2+25}}\, dx = \int \frac{125\tan^3 u}{5\sec u}\cdot 5\sec^2 u\, du$$

$$= \int 125\tan^3 u\, \sec u\, du = 125\int (\sec^2 u - 1)\sec u\, \tan u\, du = 125\left(\frac{1}{3}\sec^3 u - \sec u\right) + C.$$

The reference triangle with acute angle u, opposite side x, and adjacent side 5 has hypotenuse $\sqrt{x^2+25}$. Thus

$$I = 125\left[\frac{1}{3}\cdot\frac{(x^2+25)^{3/2}}{125} - \frac{(x^2+25)^{1/2}}{5}\right] + C = \frac{1}{3}(x^2+25)^{3/2} - 25(x^2+25)^{1/2} + C$$

$$= (x^2+25)^{1/2}\left[\frac{1}{3}(x^2+25) - 25\right] + C = \frac{1}{3}(x^2+25)^{1/2}(x^2-50) + C.$$

C08S06.013: Let $x = \frac{1}{2}\sin\theta$: $dx = \frac{1}{2}\cos\theta\, d\theta$, $1 - 4x^2 = 1 - \sin^2\theta = \cos^2\theta$. So

$$I = \int \frac{\sqrt{1-4x^2}}{x}\, dx = \int \frac{2\cos\theta}{\sin\theta}\cdot\frac{1}{2}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$$

$$= \int (\csc\theta - \sin\theta)\, d\theta = \ln|\csc\theta - \cot\theta| + \cos\theta + C.$$

The reference triangle with acute angle θ, opposite side $2x$, and hypotenuse 1 has adjacent side $\sqrt{1-4x^2}$. Therefore

$$I = \ln\left|\frac{1-\sqrt{1-4x^2}}{2x}\right| + \sqrt{1-4x^2} + C = \ln\left|\frac{1-1+4x^2}{2x\left(1+\sqrt{1-4x^2}\,\right)}\right| + \sqrt{1-4x^2} + C$$

$$= \ln(4x^2) - \ln|2x| - \ln\left(1+\sqrt{1-4x^2}\,\right) + \sqrt{1-4x^2} + C$$

$$= \ln 4 + 2\ln|x| - \ln 2 - \ln|x| - \ln\left(1+\sqrt{1-4x^2}\,\right) + \sqrt{1-4x^2} + C$$

$$= \ln|x| + \ln 2 - \ln\left(1+\sqrt{1-4x^2}\,\right) + \sqrt{1-4x^2} + C$$

$$= \ln|x| - \ln\left(1+\sqrt{1-4x^2}\,\right) + \sqrt{1-4x^2} + C_1$$

where $C_1 = C - \ln 2$.

C08S06.014: Let $x = \tan\theta$: $dx = \sec^2\theta\ d\theta,\ 1 + x^2 = \sec\theta$. Thus

$$\int \frac{1}{\sqrt{1+x^2}}\ dx = \int \frac{1}{\sec\theta}\sec^2\theta\ d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left(x + \sqrt{1+x^2}\right) + C.$$

C08S06.015: Let $x = \dfrac{3}{2}\tan u$: $9 + 4x^2 = 9 + 9\tan^2 u = 9\sec^2 u,\ ds = \dfrac{3}{2}\sec^2 u\ du$. Thus

$$J = \int \frac{1}{\sqrt{9+4x^2}}\ dx = \int \frac{1}{3\sec u}\cdot\frac{3}{2}\sec^2 u\ du = \frac{1}{2}\int \sec u\ du = \frac{1}{2}\ln|\sec u + \tan u| + C.$$

The reference triangle with acute angle u, opposite side $3x$, and adjacent side 3 has hypotenuse $\sqrt{9+4x^2}$. Therefore

$$J = \frac{1}{2}\ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{2x}{3}\right| + C = \frac{1}{2}\ln\left(2x + \sqrt{9+4x^2}\right) + C_1$$

where $C_1 = C - \dfrac{1}{2}\ln 3$.

C08S06.016: Let $x = \dfrac{1}{2}\tan u$: $dx = \dfrac{1}{2}\sec^2 u\ du,\ \sqrt{1+4x^2} = \sqrt{1+\tan^2 u} = \sec u$. So

$$K = \int \sqrt{1+4x^2}\ dx = \frac{1}{2}\int \sec^3 u\ du = \frac{1}{4}\sec u\,\tan u + \frac{1}{4}\ln|\sec u + \tan u| + C$$

by Formula 28 inside the back cover (or use the result in Example 6 of Section 8.3). The antiderivative of $\sec^3 x$ is easy to remember: It is the *average* of the derivative and antiderivative of $\sec x$ (merely a coincidence). Next, the reference triangle with acute angle u, opposite side $2x$, and adjacent side 1 has hypotenuse $\sqrt{1+4x^2}$, and therefore

$$K = \frac{1}{2}x\sqrt{1+4x^2} + \frac{1}{4}\ln\left(2x + \sqrt{1+4x^2}\right) + C.$$

C08S06.017: Let $x = 5\sin\theta$: $dx = 5\cos\theta\ d\theta,\ 25 - x^2 = 25\cos^2\theta$. Thus

$$I = \int \frac{x^2}{\sqrt{25-x^2}}\ dx = \int \frac{25\sin^2\theta}{5\cos\theta}\cdot 5\cos\theta\ d\theta = \frac{25}{2}\int (1 - \cos 2\theta)\ d\theta = \frac{25}{2}\left(\theta - \sin\theta\,\cos\theta\right) + C.$$

The reference triangle with acute angle θ, opposite side x, and hypotenuse 5 has adjacent side $\sqrt{25-x^2}$. Therefore

$$I = \frac{25}{2}\left[\arcsin\left(\frac{x}{5}\right) - \frac{x\sqrt{25-x^2}}{25}\right] + C = \frac{25}{2}\arcsin\left(\frac{x}{5}\right) - \frac{x\sqrt{25-x^2}}{2} + C.$$

C08S06.018: Let $x = 5\sin u$: $25 - x^2 - 25\cos^2 u,\ dx = 5\cos u\ du$. So

$$\begin{aligned} J &= \int \frac{x^3}{\sqrt{25-x^2}}\ dx = \int \frac{125\sin^3 u}{5\cos u}\cdot 5\cos u\ du \\ &= 125\int (1 - \cos^2 u)\sin u\ du = 125\left(\frac{1}{3}\cos^3 u - \cos u\right) + C. \end{aligned}$$

A reference triangle with acute angle u has opposite side x and hypotenuse 5, and thus adjacent side $\sqrt{25-x^2}$. Therefore

$$J = 125\left[\frac{1}{3}\cdot\frac{(25-x^2)^{3/2}}{125} - \frac{(25-x^2)^{1/2}}{5}\right] + C = \frac{1}{3}(25-x^2)^{3/2} - 25(25-x^2)^{1/2} + C$$

$$= \frac{1}{3}(25-x^2)^{1/2}(25-x^2-75) = -\frac{1}{3}(x^2+50)\sqrt{25-x^2} + C.$$

C08S06.019: Let $x = \tan\theta$: $1 + x^2 = \sec^2\theta$, $ds = \sec^2\theta\, d\theta$. Thus

$$K = \int \frac{x^2}{\sqrt{1+x^2}}\,dx = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \sec\theta\,\tan^2\theta\,d\theta = \int(\sec^3\theta - \sec\theta)\,d\theta.$$

For the antiderivatives, refer to Formulas 14 and 28 of the endpapers of the text or use the reduction formula in Example 6 of Section 8.3. Thus we obtain

$$K = \frac{1}{2}\sec\theta\,\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

A reference triangle with acute angle θ, opposite side x, and adjacent side 1 has hypotenuse $\sqrt{1+x^2}$. Therefore

$$K = \frac{1}{2}\left[x\sqrt{1+x^2} - \ln\left(x + \sqrt{1+x^2}\right)\right] + C.$$

C08S06.020: Let $x = \tan\theta$: $1+x^2 = \sec^2\theta$, $ds = \sec^2\theta\,d\theta$. Almost exactly as in the solution of Problem 19, we get

$$I = \int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \sec\theta\,\tan^3\theta\,d\theta = \int(\sec^2\theta - 1)\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\sec^3\theta - \sec\theta + C.$$

A reference triangle with acute angle θ, opposite side x, and adjacent side 1 has hypotenuse $\sqrt{1+x^2}$. Therefore

$$I = \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C = \frac{1}{3}\left[(1+x^2)^{3/2} - 3(1+x^2)^{1/2}\right] + C$$

$$= \frac{\sqrt{1+x^2}}{3}(1+x^2-3) + C = \frac{1}{3}(x^2-2)\sqrt{1+x^2} + C.$$

C08S06.021: Let $x = \frac{2}{3}\tan u$: $dx = \frac{2}{3}\sec^2 u\,du$, $4 + 9x^2 = 4 + 4\tan^2 u = 4\sec^2 u$. Therefore

$$J = \int \frac{x^2}{\sqrt{4+9x^2}}\,dx = \int \frac{\frac{4}{9}\tan^2 u}{2\sec u}\cdot\frac{2}{3}\sec^2 u\,du = \frac{4}{27}\int \sec u\,\tan^2 u\,du = \frac{4}{27}\int(\sec^3 u - \sec u)\,du$$

$$= \frac{4}{27}\left(\frac{1}{2}\sec u\,\tan u - \frac{1}{2}\ln|\sec u + \tan u|\right) + C = \frac{2}{27}(\sec u\,\tan u - \ln|\sec u + \tan u|) + C.$$

A reference triangle with acute angle u, opposite side $3x$, and adjacent side 2 yields hypotenuse of length $\sqrt{4+9x^2}$. Therefore

$$J = \frac{2}{27}\left[\frac{3x\sqrt{4+9x^2}}{4} - \ln\left(\frac{3x+\sqrt{4+9x^2}}{2}\right)\right] + C$$
$$= \frac{1}{18}x\sqrt{4+9x^2} - \frac{2}{27}\ln\left(3x+\sqrt{4+9x^2}\right) + C_1.$$

C08S06.022: Let $x = \sin\theta$: $1 - x^2 = \cos^2\theta$, $dx = \cos\theta\,d\theta$. Then

$$K = \int (1-x^2)^{3/2}\,dx = \int \cos^4\theta\,d\theta.$$

You could now use the method in Example 5 of Section 8.4, or—as we do—the result in Problem 54 of Section 8.3: If n is an integer and $n \geqq 2$, then

$$\int \cos^n x\,dx = \frac{(\cos x)^{n-1}\sin x}{n} + \frac{n-1}{n}\int (\cos x)^{n-2}\,dx.$$

Thus

$$K = \frac{1}{4}\cos^3\theta\,\sin\theta + \frac{3}{4}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{4}\cos^3\theta\,\sin\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\,\sin\theta + \frac{1}{2}\int \theta\,d\theta\right) = \frac{1}{4}\sin\theta\,\cos^3\theta + \frac{3}{8}\sin\theta\,\cos\theta + \frac{3}{8}\theta + C.$$

A reference triangle with acute angle θ, opposite side x, and hypotenuse 1 has adjacent side $\sqrt{1-x^2}$. Therefore

$$K = \frac{1}{4}x(1-x^2)^{3/2} + \frac{3}{8}x(1-x^2)^{1/2} + \frac{3}{8}\arcsin x + C$$
$$= \frac{1}{8}x(1-x^2)^{1/2}(2-2x^2+3) + \frac{3}{8}\arcsin x + C = \frac{1}{8}x(5-x^2)\sqrt{1-x^2} + \frac{3}{8}\arcsin x + C.$$

C08S06.023: Let $x = \tan u$: $dx = \sec^2 u\,du$, $1 = x^2 = \sec^2 u$. Hence

$$I = \int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{\sec^3 u}\sec^2 u\,du = \int \cos u\,du = \sin u + C.$$

A reference triangle with acute angle u, opposite side z, and adjacent side 1 has hypotenuse $\sqrt{1+x^2}$. Therefore

$$I = \frac{x}{\sqrt{1+x^2}} + C.$$

C08S06.024: Let $x = 2\sin u$: $4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u$, $dx = 2\cos u\,du$. Thus

$$J = \int \frac{1}{(4-x^2)^2}\,dx = \int \frac{2\cos u}{16\cos^4 u}\,du = \frac{1}{8}\int \sec^3 u\,du = \frac{1}{16}\sec u\,\tan u + \frac{1}{16}\ln|\sec u + \tan u| + C.$$

A reference triangle with acute angle u, opposite side x, and hypotenuse 2 has adjacent side $\sqrt{4-x^2}$. Thus

$$J = \frac{1}{16} \cdot \frac{2x}{4 - x^2} + \frac{1}{16} \ln \left| \frac{2 + x}{\sqrt{4 - x^2}} \right| + C$$
$$= \frac{x}{8(4 - x^2)} + \frac{1}{32} \ln \left| \frac{(2 + x)^2}{(2 + x)(2 - x)} \right| + C = \frac{x}{8(4 - x^2)} + \frac{1}{32} \ln \left| \frac{2 + x}{2 - x} \right| + C.$$

C08S06.025: Let $x = 2 \sin \theta$: $4 - x^2 = 4 - 4 \sin^2 \theta = 4 \cos^2 \theta$, $dx = 2 \cos \theta \, d\theta$. Therefore

$$K = \int \frac{1}{(4 - x^2)^3} \, dx = \int \frac{2 \cos \theta}{64 \cos^6 \theta} \, d\theta = \frac{1}{32} \int \sec^5 \theta \, d\theta.$$

From Example 6 in Section 8.3, we know that if n is an integer and $n \geqq 2$, then

$$\int \sec^n x \, dx = \frac{(\sec x)^{n-2} \tan x}{n - 1} + \frac{n - 2}{n - 1} \int (\sec x)^{n-2} \, dx.$$

Hence, beginning with $n = 5$, we see that

$$K = \frac{1}{32} \left[\frac{1}{4} \sec^3 \theta \, \tan \theta + \frac{3}{4} \int \sec^3 \theta \, d\theta \right]$$
$$= \frac{1}{32} \left[\frac{1}{4} \sec^3 \theta \, \tan \theta + \frac{3}{4} \left(\frac{1}{2} \sec \theta \, \tan \theta + \frac{1}{2} \int \sec \theta \, d\theta \right) \right]$$
$$= \frac{1}{32} \left(\frac{1}{4} \sec^3 \theta \, \tan \theta + \frac{3}{8} \sec \theta \, \tan \theta + \frac{3}{8} \ln | \sec \theta + \tan \theta | \right) + C$$
$$= \frac{1}{128} \sec^3 \theta \, \tan \theta + \frac{3}{256} \sec \theta \, \tan \theta + \frac{3}{256} \ln | \sec \theta + \tan \theta | + C.$$

A reference triangle with acute angle θ, opposite side x, and hypotenuse 2 has adjacent side $\sqrt{4 - x^2}$. So

$$K = \frac{1}{512} \left[4 \cdot \frac{8x}{(4 - x^2)^2} + 6 \cdot \frac{2x}{4 - x^2} + 6 \ln \left| \frac{x + 2}{\sqrt{4 - x^2}} \right| \right] + C$$
$$= \frac{1}{512} \left[\frac{32x}{(4 - x^2)^2} + \frac{12x}{4 - x^2} + 3 \ln \left| \frac{(x + 2)^2}{4 - x^2} \right| \right] + C = \frac{1}{512} \left[\frac{32x}{(4 - x^2)^2} + \frac{12x}{4 - x^2} + 3 \ln \left| \frac{2 + x}{2 - x} \right| \right] + C.$$

C08S06.026: Let $x = \frac{3}{2} \tan u$: $4x^2 + 9 = 9 \tan^2 3u + 9 = 9 \sec^2 u$ and $dx = \frac{3}{2} \sec^2 u \, du$. Thus

$$I = \int \frac{1}{(4x^2 + 9)^3} \, dx = \int \frac{1}{9^3 \sec^6 u} \cdot \frac{3}{2} \sec^2 u \, du = \frac{1}{486} \int \cos^4 u \, du.$$

Now use the recursion/reduction formula in Problem 54 of Section 8.3. The result:

$$I = \frac{1}{486} \left[\frac{1}{4} \cos^3 u \, \sin u + \frac{3}{4} \int \cos^2 u \, du \right]$$
$$= \frac{1}{486} \left[\frac{1}{4} \cos^3 u \, \sin u + \frac{3}{4} \left(\frac{1}{2} \cos u \, \sin u + \frac{1}{2} \int 1 \, du \right) \right]$$
$$= \frac{1}{486} \left(\frac{1}{4} \sin u \, \cos^3 u + \frac{3}{8} \sin u \, \cos u + \frac{3}{8} u \right) + C.$$

A reference triangle with acute angle u, opposite side $2x$, and adjacent side 3 has hypotenuse $\sqrt{9+4x^2}$. Hence

$$I = \frac{1}{486}\left[\frac{1}{4}\cdot\frac{54x}{(9+4x^2)^2} + \frac{3}{8}\cdot\frac{6x}{9+4x^2} + \frac{3}{8}\arctan\left(\frac{2x}{3}\right)\right] + C$$
$$= \frac{27}{4\cdot 243}\cdot\frac{x}{(9+4x^2)^2} + \frac{9}{8\cdot 243}\cdot\frac{x}{9+4x^2} + \frac{3}{16\cdot 243}\arctan\left(\frac{2x}{3}\right) + C$$
$$= \frac{x}{36(9+4x^2)^2} + \frac{x}{216(9+4x^2)} + \frac{1}{1296}\arctan\left(\frac{2x}{3}\right) + C.$$

C08S06.027: Let$x = \dfrac{3}{4}\tan\theta$: $ds = \dfrac{3}{4}\sec^2\theta\,d\theta$, $9+16x^2 = 9+9\tan^2 x^3 = 9\sec^2\theta$. Hence

$$I = \int \sqrt{9+16x^2}\,dx = \int (3\sec\theta)\cdot\frac{3}{4}\sec^2\theta\,d\theta = \frac{9}{4}\int \sec^3\theta\,d\theta = \frac{9}{8}\left(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C.$$

A reference triangle with acute angle θ, opposite side $4x$, and adjacent side 3 has hypotenuse $\sqrt{9+16x^2}$. Thus

$$I = \frac{9}{8}\left[\frac{4x\sqrt{9+16x^2}}{9} + \ln\left(\frac{4x\sqrt{9+16x^2}}{3}\right)\right] + C = \frac{1}{2}x\sqrt{9+16x^2} + \frac{9}{8}\ln\left(4x+\sqrt{9+16x^2}\right) + C_1.$$

C08S06.028: Let $x = \frac{3}{4}\tan u$: $9+16x^2 = 9+9\tan^2 u = 9\sec^2 u$, $ds = \frac{3}{4}\sec^2 u\,du$. Thus

$$I = \int (9+16x^2)^{32}\,dx = \int (27\sec^2 u)\cdot\frac{3}{4}\sec^2 u\,du = \frac{81}{4}\int \sec^5 u\,du.$$

Then, as in the solution of Problem 25, we have

$$I = \frac{81}{4}\left(\frac{1}{4}\sec^3 u\,\tan u + \frac{3}{8}\sec u\,\tan u + \frac{3}{8}|\sec u + \tan u|\right) + C.$$

A reference triangle with acute angle u, opposite side $4x$, and adjacent side 3 has hypotenuse $\sqrt{9+16x^2}$. Therefore

$$I = \frac{81}{4}\left[\frac{1}{4}\cdot\frac{4x(9+16x^2)^{3/2}}{81} + \frac{3}{8}\cdot\frac{4x(9+16x^2)^{1/2}}{9} + \frac{3}{8}\ln\left(\frac{4x+\sqrt{9+16x^2}}{3}\right)\right] + C$$
$$= \frac{1}{4}x(9+16x^2)^{3/2} + \frac{27}{8}x\sqrt{9+16x^2} + \frac{243}{32}\ln\left(4x+\sqrt{9+16x^2}\right) + C_1.$$

C08S06.029: Let $x = 5\sec\theta$: $dx = 5\sec\theta\,\tan\theta\,d\theta$, $x^2-25 = 25\sec^2\theta - 25 = 25\tan^2\theta$. So

$$J = \int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}\cdot 5\sec\theta\,\tan\theta\,d\theta$$
$$= 5\int \tan^2\theta\,d\theta = 5\int (\sec^2\theta - 1)\,d\theta = 5(\tan\theta - \theta) + C.$$

A reference triangle with acute angle θ, adjacent side 5, and hypotenuse x has opposite side $\sqrt{x^2-25}$. Therefore

$$J = 5\left[\frac{\sqrt{x^2-25}}{5} - \operatorname{arcsec}\left(\frac{x}{5}\right)\right] + C = \sqrt{x^2-25}\; - 5\operatorname{arcsec}\left(\frac{x}{5}\right) + C$$

$$= \sqrt{x^2-25}\; + 5\arctan\left(\frac{5}{\sqrt{x^2-25}}\right) + C = \sqrt{x^2-25}\; - 5\arctan\left(\frac{\sqrt{x^2-25}}{5}\right) + C.$$

C08S06.030: Let $x = \dfrac{4}{3}\sec u$: $dx = \dfrac{4}{3}\sec u\,\tan u\,du$, $9x^2 - 16 = 16\sec^2 u - 16 = 16\tan^2 u$. Consequently

$$K = \int \frac{\sqrt{9x^2-16}}{x}\,dx = \int \frac{4\tan u}{\frac{4}{3}\sec u}\cdot\frac{4}{3}\sec u\,\tan u\,du$$

$$= 4\int \tan^2 u\,du = 4\int (\sec^2 u - 1)\,du = 4(\tan u\, - u) + C.$$

A reference triangle with acute angle u, adjacent side 4, and hypotenuse $3x$ has opposite side $\sqrt{9x^2-16}$. Therefore

$$K = 4\left[\frac{\sqrt{9x^2-16}}{4} - \operatorname{arcsec}\left(\frac{3x}{4}\right)\right] + C$$

$$= \sqrt{9x^2-16}\; - 4\operatorname{arcsec}\left(\frac{3x}{4}\right) + C = \sqrt{9x^2-16}\; - 4\arctan\left(\frac{\sqrt{9x^2-16}}{4}\right) + C.$$

C08S06.031: Let $x = \sec\theta$: $x^2 - 1 = \sec^2\theta\, - 1$, $ds = \sec\theta\,\tan\theta\,d\theta$. Then

$$I = \int x^2\sqrt{x^2-1}\;dx = \int (\sec^2\theta)(\tan\theta)(\sec\theta\,\tan\theta)\,d\theta = \int \sec^3\theta\,\tan^2\theta\,d\theta = \int (\sec^5\theta - \sec^3\theta)\,d\theta.$$

Use the result in Example 6 of Section 8.3 (if you haven't memorized it by now!): If n is an integer and $n \geqq 2$, then

$$\int \sec^n x\,dx = \frac{(\sec x)^{n-2}\tan x}{n-1} + \frac{n-2}{n-1}\int (\sec x)^{n-2}\,dx.$$

Thus

$$I = \int (\sec^5\theta - \sec^3\theta)\,d\theta = \frac{1}{4}\sec^3\theta\,\tan\theta + \frac{3}{4}\int \sec^3\theta\,d\theta - \int \sec^3\theta\,d\theta$$

$$= \frac{1}{4}\sec^3\theta\,\tan\theta - \frac{1}{4}\left(\frac{1}{2}\sec\theta\,\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C$$

$$= \frac{1}{4}\sec^3\theta\,\tan\theta - \frac{1}{8}\sec\theta\,\tan\theta - \frac{1}{8}\ln|\sec\theta + \tan\theta| + C.$$

A reference triangle with acute angle θ, adjacent side 1, and hypotenuse x has opposite side $\sqrt{x^2-1}$, and therefore

$$I = \frac{1}{4}x^3\sqrt{x^2-1}\; - \frac{1}{8}x\sqrt{x^2-1} - \frac{1}{8}\ln\left|x + \sqrt{x^2-1}\right| + C.$$

Additional algebraic simplifications of the answer are possible but not normally required (except possibly to reconcile the answer with that of a computer algebra system such as *Mathematica*, *Maple*, or *Derive*).

C08S06.032: Let $x = \dfrac{3}{2}\sec u$: $dx = \dfrac{3}{2}\sec u \tan u\, du$, $\sqrt{4x^2 - 9} = \sqrt{9\sec^2 u - 9} = 3\tan u$. Thus

$$I = \int \frac{x^2}{\sqrt{4x^2-9}}\,dx = \int \frac{\frac{9}{4}\sec^2 u}{3\tan u}\cdot \frac{3}{2}\sec u \tan u\,du$$

$$= \frac{9}{8}\int \sec^3 u\,du = \frac{9}{16}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + C.$$

A reference triangle with acute angle u, adjacent side 3, and hypotenuse $2x$ has opposite side $\sqrt{4x^2-9}$. Therefore

$$I = \frac{9}{16}\left(\frac{2x\sqrt{4x^2-9}}{9} + \ln\left|\frac{2x+\sqrt{4x^2-9}}{3}\right|\right) + C = \frac{1}{8}x\sqrt{4x^2-9} + \frac{9}{16}\ln\left|2x+\sqrt{4x^2-9}\right| + C_1.$$

C08S06.033: Let $x = \dfrac{1}{2}\sec\theta$: $dx = \dfrac{1}{2}\sec\theta\tan\theta\,d\theta$, $(4x^2-1)^{3/2} = (\sec^2\theta - 1)^{3/2} = \tan^3\theta$. Thus

$$J = \int \frac{1}{(4x^2-1)^{3/2}}\,dx = \int \frac{1}{\tan^3\theta}\cdot\frac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{2\sin\theta} + C = -\frac{1}{2}\csc\theta + C.$$

A reference triangle with acute angle θ, adjacent side 1, and hypotenuse $2x$ has opposite side $\sqrt{4x^2-1}$. Therefore

$$J = -\frac{1}{2}\cdot\frac{2x}{\sqrt{4x^2-1}} + C = -\frac{x}{\sqrt{4x^2-1}} + C.$$

C08S06.034: Let $x = \dfrac{3}{2}\sec u$: $dx = \dfrac{3}{2}\sec u\tan u\,du$, $4x^2 - 9 = 9\sec^2 u - 9 = 9\tan^2 u$. So

$$K = \int \frac{1}{x^2\sqrt{4x^2-9}}\,dx = \int \frac{\frac{3}{2}\sec u\tan u}{\left(\frac{9}{4}\sec^2 u\right)(3\tan u)}du = \frac{2}{9}\int\cos u\,du = \frac{2}{9}\sin u + C.$$

A reference triangle with acute angle u, adjacent side 3, and hypotenuse $2x$ has opposite side $\sqrt{4x^2-9}$. Therefore

$$K = \frac{2}{9}\cdot\frac{\sqrt{4x^2-9}}{2x} + C = \frac{\sqrt{4x^2-9}}{9x} + C.$$

C08S06.035: Let $x = \left(\sqrt{5}\right)\sec u$: $x^2 - 5 = 5\sec^2 u - 5 = 5\tan^2 u$, $dx = \left(\sqrt{5}\right)\sec u\tan u\,du$. Therefore

$$I = \int\frac{\sqrt{x^2-5}}{x^2}\,dx = \int\frac{\left(\sqrt{5}\right)\tan u}{5\sec^2 u}\cdot\left(\sqrt{5}\right)\sec u\tan u\,du = \int\frac{\tan^2 u}{\sec u}\,du$$

$$= \int\frac{\sin^2 u}{\cos u}\,du = \int\frac{1-\cos^2 u}{\cos u}\,du = \int(\sec u - \cos u)\,du = \ln|\sec u + \tan u| - \sin u + C.$$

A reference triangle with acute angle u, adjacent side $\sqrt{5}$, and hypotenuse x has opposite side $\sqrt{x^2-5}$. Thus

$$I = \ln\left|\frac{x+\sqrt{x^2-5}}{\sqrt{5}}\right| - \frac{\sqrt{x^2-5}}{x} + C = \ln\left|x+\sqrt{x^2-5}\right| - \frac{\sqrt{x^2-5}}{x} + C_1.$$

C08S06.036: $4x^2 - 5 = 5\sec^2 u - 5 = 5\tan^2 u$ if $2x = \left(\sqrt{5}\right)\sec u$. So let

$$x = \frac{\sqrt{5}}{2}\sec u; \quad dx = \frac{\sqrt{5}}{2}\sec u\,\tan u\,du.$$

Then

$$J = \int (4x^2-5)^{3/2}\,dx = \int \left(5^{3/2}\tan^3 u\right)\cdot\frac{\sqrt{5}}{2}\sec u\,\tan u\,du = \frac{25}{2}\int \tan^4 u\,\sec u\,du$$
$$= \frac{25}{2}\int\left(\sec^4 u - 2\sec^2 u + 1\right)\sec u\,du = \frac{25}{2}\int\left(\sec^5 u - 2\sec^3 u + \sec u\right)du.$$

From Example 6 in Section 8.3: If n is an integer and $n \geqq 2$, then

$$\int \sec^n x\,dx = \frac{(\sec x)^{n-2}\tan x}{n-1} + \frac{n-2}{n-1}\int(\sec x)^{n-2}\,dx.$$

Therefore

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x\,\tan x + \frac{3}{4}\int\sec^3 x\,dx$$
$$= \frac{1}{4}\sec^3 x\,\tan x + \frac{3}{8}\sec u\,\tan u + \frac{3}{8}\ln|\sec u + \tan u| + C;$$
$$2\int\sec^3 u\,du = \sec u\,\tan u + \ln|\sec u + \tan u| + C;$$
$$\int \sec u\,du = \ln|\sec u + \tan u| + C.$$

Thus

$$J = \int\left(\sec^5 u - 2\sec^3 u + \sec u\right)du = \frac{1}{4}\sec^3 u\,\tan u - \frac{5}{8}\sec u\,\tan u + \frac{3}{8}\ln|\sec u + \tan u| + C.$$

A reference triangle with acute angle u, adjacent side $\sqrt{5}$, and hypotenuse $2x$ has opposite side $\sqrt{4x^2-5}$. Therefore

$$J = \frac{25}{2}\left(\frac{1}{4}\cdot\frac{8x^3}{5\sqrt{5}}\cdot\frac{\sqrt{4x^2-5}}{\sqrt{5}} - \frac{5}{8}\cdot\frac{2x\sqrt{4x^2-5}}{5} + \frac{3}{8}\ln\left|\frac{2x+\sqrt{4x^2-5}}{\sqrt{5}}\right|\right) + C$$
$$= x^3\sqrt{4x^2-5} - \frac{25}{8}x\sqrt{4x^2-5} + \frac{75}{16}\ln\left|2x+\sqrt{4x^2-5}\right| + C_1.$$

C08S06.037: Let $x = 5\sinh\theta$. Then $25 + x^2 = 25 + 25\sinh^2\theta = 25\cosh^2\theta$ and $dx = 5\cosh\theta\,d\theta$. So

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \int \frac{5\cosh\theta}{5\cosh\theta}\,d\theta = \theta + C = \sinh^{-1}\left(\frac{x}{5}\right) + C.$$

C08S06.038: Let $x = \sinh\theta$. Then $1 + x^2 = \cosh^2\theta$ and $dx = \cosh\theta\,d\theta$. Next we will first use Eq. (11) of Section 7.6, then Eq. (9):

$$\int \sqrt{1+x^2}\,dx = \int \cosh^2\theta\,d\theta = \frac{1}{2}\int (\cosh 2\theta + 1)\,d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sinh 2\theta + \theta\right) + C = \frac{1}{2}(\sinh\theta\,\cosh\theta + \theta) + C = \frac{1}{2}\left(x\sqrt{1+x^2} + \sinh^{-1}x\right) + C.$$

C08S06.039: Let $x = 2\cosh\theta$. Then $x^2 - 4 = 4\cosh^2\theta - 4 = 4\sinh^2\theta$ and $dx = 2\sinh\theta\,d\theta$. So—at one point using Eq. (5) of Section 7.6—

$$\int \frac{\sqrt{x^2-4}}{x^2}\,dx = \int \frac{2\sinh\theta}{4\cosh^2\theta}\cdot 2\sinh\theta\,d\theta = \int \tanh^2\theta\,d\theta = \int \left(1 - \operatorname{sech}^2\theta\right)d\theta$$

$$= \theta - \tanh\theta + C = \theta - \frac{\sinh\theta}{\cosh\theta} + C = \cosh^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{x^2-4}}{x} + C.$$

C08S06.040: Let $x = \frac{1}{3}\sinh u$: $dx = \frac{1}{3}\cosh u\,du$, $1 + 9x^2 = 1 + \sinh^2 u = \cosh^2 u$. Thus

$$\int \frac{1}{\sqrt{1+9x^2}}\,dx = \int \frac{\frac{1}{3}\cosh u}{\cosh u}\,du = \frac{1}{3}u + C = \frac{1}{3}\sinh^{-1}(3x) + C.$$

C08S06.041: We will use Eqs. (9), (12), and (10) of Section 7.6. Let $x = \sinh\theta$. Then $1 + x^2 = \cosh^2\theta$ and $dx = \cosh\theta\,d\theta$. Hence

$$\int x^2\sqrt{1+x^2}\,dx = \int \sinh^2\theta\,\cosh^2\theta\,d\theta = \frac{1}{4}\int (2\sinh\theta\,\cosh\theta)^2\,d\theta = \frac{1}{4}\int (\sinh 2\theta)^2\,d\theta$$

$$= \frac{1}{8}\int (\cosh 4\theta - 1)\,d\theta = \frac{1}{8}\left(\frac{1}{4}\sinh 4\theta - \theta\right) + C = \frac{1}{8}\left(\frac{1}{2}\sinh 2\theta\,\cosh 2\theta - \theta\right) + C$$

$$= \frac{1}{8}\left[(\sinh\theta\,\cosh\theta)(\cosh^2\theta + \sinh^2\theta) - \theta\right] + C$$

$$= \frac{1}{8}\left(\sinh^3\theta\,\cosh\theta + \sinh\theta\,\cosh^3\theta - \theta\right) + C$$

$$= \frac{1}{8}x^3\sqrt{1+x^2} + \frac{1}{8}x(1+x^2)^{3/2} - \frac{1}{8}\sinh^{-1}x + C$$

$$= \frac{1}{8}x^3\sqrt{1+x^2} + \frac{1}{8}x(1+x^2)\sqrt{1+x^2} - \frac{1}{8}\sinh^{-1}x + C$$

$$= \frac{\sqrt{1+x^2}}{8}(x^3 + x^3 + x) - \frac{1}{8}\sinh^{-1}x + C$$

$$= \frac{1}{8}\left[x(2x^2+1)\sqrt{1+x^2} - \sinh^{-1}x\right] + C.$$

C08S06.042: First solve the equation of the ellipse for

$$y = \frac{b}{a}\sqrt{a^2 - x^2}\,.$$

Then its area is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\ dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\ dx$$
$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)\right]_0^a = \frac{4b}{a}\cdot\frac{a^2}{2}\arcsin(1) = 2ab\cdot\frac{\pi}{2} = \pi ab.$$

C08S06.043: The area of triangle OAC in Fig. 8.6.8 is

$$\frac{1}{2}xy = \frac{1}{2}x\sqrt{a^2 - x^2}\,.$$

The area of the region ABC is (by Example 2)

$$\int_x^a \sqrt{a^2 - u^2}\ du = \left[\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right)\right]_x^a = \frac{a^2}{2}\cdot\arcsin(1) - \frac{x}{2}\sqrt{a^2 - x^2} - \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right).$$

The area A of sector OBC is therefore their sum:

$$A = \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right).$$

But $x = a\cos\theta$, so

$$A = \frac{\pi a^2}{4} - \frac{a^2}{2}\arcsin(\cos\theta) = \frac{\pi a^2}{4} - \frac{a^2}{2}\left(\frac{\pi}{2} - \theta\right) = \frac{\pi a^2}{4} - \frac{\pi a^2}{4} + \frac{1}{2}a^2\theta = \frac{1}{2}a^2\theta.$$

C08S06.044: Given $y = x^2$, we have $ds = \sqrt{1 + (dy/dx)^2}\ dx = \sqrt{1 + 4x^2}\ dx$. So the length in question is

$$L = \int_0^1 \sqrt{1 + 4x^2}\ dx.$$

Let $x = \frac{1}{2}\tan\theta$: $1 + 4x^2 = 1 + \tan^2\theta = \sec^2\theta$, $dx = \dfrac{1}{2}\sec^2\theta\ d\theta$. Hence

$$L = \int_{x=0}^1 \frac{1}{2}\sec^3\theta\ d\theta = \frac{1}{4}\Big[\sec\theta\ \tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{x=0}^1$$
$$= \frac{1}{4}\left[2x\sqrt{1 + 4z^2} + \ln\left(2x + \sqrt{1 + 4x^2}\right)\right]_0^1 = \frac{1}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 1.4789428575.$$

C08S06.045: Given $y = x^2$, we have $ds = \sqrt{1 + (dy/dx)^2}\ dx = \sqrt{1 + 4x^2}\ dx$. So the surface area of revolution around the x-axis is

$$A = \int_0^1 2\pi x^2\sqrt{1 + 4x^2}\ dx.$$

Let $x = \dfrac{1}{2}\tan\theta$: $1 + 4x^2 = \sec^2\theta$, $dx = \dfrac{1}{2}\sec^2\theta\,d\theta$. So

$$A = \int_{x=0}^{1} 2\pi\left(\frac{1}{4}\tan^2\theta\right)(\sec\theta)\left(\frac{1}{2}\sec^2\theta\right)\,d\theta = \frac{\pi}{4}\int_{x=0}^{1}\sec^3\theta\,\tan^2\theta\,d\theta.$$

Now

$$\begin{aligned}\int \sec^3\theta\,\tan^2\theta\,d\theta &= \int\left(\sec^5\theta - \sec^4\theta\right)\,d\theta\\ &= \frac{1}{4}\sec^3\theta\,\tan\theta + \frac{3}{8}\sec\theta\,\tan\theta + \frac{3}{8}\ln|\sec\theta+\tan\theta| - \frac{1}{2}\sec\theta\,\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C\\ &= \frac{1}{4}\sec^3\theta\,\tan\theta - \frac{1}{8}\sec\theta\,\tan\theta - \frac{1}{8}\ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

A reference triangle with acute angle θ, opposite side $2x$, and adjacent side 1 has hypotenuse $\sqrt{1+4x^2}$. Therefore

$$\begin{aligned}S &= \frac{\pi}{4}\left[\frac{1}{2}x(1+4x^2)^{3/2} - \frac{1}{4}x\sqrt{1+4x^2} - \frac{1}{8}\ln\left(2x+\sqrt{1+4x^2}\right)\right]_0^1\\ &= \frac{\pi}{4}\left[\frac{1}{2}\cdot 5\sqrt{5} - \frac{1}{4}\sqrt{5} - \frac{1}{8}\ln\left(2+\sqrt{5}\right)\right] = \frac{\pi}{32}\left[18\sqrt{5} - \ln\left(2+\sqrt{5}\right)\right] \approx 3.8097297049.\end{aligned}$$

C08S06.046: The length of one arch of the sine curve is

$$S = \int_0^{\pi}\sqrt{1+\cos^2 x}\;dx.$$

To obtain the length of the upper half of the ellipse, take

$$y = \sqrt{2-2x^2}, \qquad -1 \leqq x \leqq 1.$$

Then $\dfrac{dy}{dx} = -\dfrac{2x}{\sqrt{2-2x^2}}$, so—after algebraic simplification—the arc length is

$$E = \int_{-1}^{1}\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}\;dx.$$

Let $x = \cos u$. Then

$$E = \int_{\pi}^{0}\frac{\sqrt{1+\cos^2 u}}{\sqrt{1-\cos^2 u}}(-\sin u)\;du = \int_0^{\pi}\sqrt{1+\cos^2 u}\;du = S.$$

C08S06.047: Given $y = \ln x$, it follows that the arc length element is $ds = \dfrac{1}{x}\sqrt{x^2+1}\;dx$, so the arc length in question is

$$L = \int_1^2 \frac{1}{x}\sqrt{x^2+1}\;dx.$$

The substitution $x = \sinh u$ can be made to work, but we prefer to use $x = \tan u$. This results in the definite integral

$$L = \int_{x=1}^{2} (\csc u + \sec u \, \tan u) \; du = \Big[\ln |\csc u - \cot u| + \sec u \Big]_{x=1}^{2}$$

$$= \left[\ln \left| \frac{-1 + \sqrt{1+x^2}}{x} \right| + \sqrt{1+x^2} \right]_1^2 = \ln \left(\frac{-1+\sqrt{5}}{2} \right) - \ln \left(-1 + \sqrt{2} \right) + \sqrt{5} - \sqrt{2}$$

$$= \ln \left(\frac{1}{\sqrt{2} - 1} \right) - \ln \left(\frac{2}{\sqrt{5} - 1} \right) + \sqrt{5} - \sqrt{2} = \ln \left(\sqrt{2} + 1 \right) - \ln \left(\frac{\sqrt{5} + 1}{2} \right) + \sqrt{5} - \sqrt{2}$$

$$= \ln \left(\sqrt{2} + 1 \right) - \ln \left(\sqrt{5} + 1 \right) + \ln 2 + \sqrt{5} - \sqrt{2} \approx 1.222016177.$$

C08S06.048: $A = \displaystyle\int_1^2 2\pi x \frac{\sqrt{x^2+1}}{x} \; dx = 2\pi \int_1^2 \sqrt{x^2+1} \; dx.$

The substitution $x = \tan u$ transforms the antidifferentiation problem into

$$2\pi \int \sec^3 z \; dz = \pi \left(\sec z \, \tan z + \ln |\sec z + \tan z| \right) + C = \pi \left(x\sqrt{x^2+1} + \ln \left| x + \sqrt{x^2+1} \right| \right) + C.$$

Substitution of the limits $x = 1$ and $x = 2$ yields the answer:

$$A = \pi \left[2\sqrt{5} - \sqrt{2} + \ln \left(2 + \sqrt{5} \right) - \ln \left(1 + \sqrt{2} \right) \right] \approx 11.37314434.$$

C08S06.049: First solve for $y = [a^2 - (x-b)^2]^{1/2}$. Then

$$\frac{dy}{dx} = \frac{-(x-b)}{[a^2 - (x-b)^2]^{1/2}}, \qquad \text{and so} \qquad 1 + \left(\frac{dy}{dx} \right)^2 = \frac{a^2}{a^2 - (x-b)^2}.$$

Therefore the surface area of the torus is

$$S = 2 \int_{b-a}^{b+a} 2\pi x \; ds = 4\pi a \int_{b-a}^{b+a} \frac{x}{\sqrt{a^2 - (x-b)^2}} \; dx.$$

The substitution we want should produce

$$a^2 - (x-b)^2 = a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta,$$

so we choose $x = b + a \sin \theta$. Then $dx = a \cos \theta \; d\theta$,

$$\sin \theta = \frac{x-b}{a}, \qquad \text{and} \qquad \theta = \arcsin \left(\frac{x-b}{a} \right).$$

Before proceeding, note that when $x = b + a$, we have

$$a \cos \theta = [a^2 - (x-b)^2]^{1/2} = 0 \qquad \text{and} \qquad \theta = \arcsin(1) = \frac{\pi}{2},$$

and when $x = b - a$,

$$a \cos \theta = [a^2 - (x-b)^2]^{1/2} = 0 \qquad \text{and} \qquad \theta = \arcsin(-1) = -\frac{\pi}{2}.$$

Consequently

$$
\begin{aligned}
S &= 4\pi a \int_{x=b-a}^{b+a} \frac{b + a\sin\theta}{a\cos\theta} \cdot a\cos\theta \, d\theta \\
&= 4\pi a \Big[b\theta - a\cos\theta \Big]_{x=b-a}^{b+a} = 4\pi a \left[b \cdot \frac{\pi}{2} - b \cdot \left(-\frac{\pi}{2} \right) - 0 + 0 \right] = 4\pi^2 ab.
\end{aligned}
$$

C08S06.050: The area is

$$
A = \int_0^4 \sqrt{9 + x^2} \, dx.
$$

Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta \, d\theta$ and $9 + x^2 = 9 + 9\tan^2\theta = 9\sec^2\theta$. Then

$$
A = \int_0^4 9\sec^3\theta \, d\theta = \frac{9}{2} \Big[\sec\theta \, \tan\theta + \ln |\sec\theta + \tan\theta \,| \Big]_{x=0}^{4}.
$$

A reference triangle with acute angle θ, opposite side x, and adjacent side 3 has hypotenuse $\sqrt{9 + x^2}$. Thus

$$
\begin{aligned}
A &= \frac{9}{2} \left[\frac{x\sqrt{x^2+9}}{9} + \ln \left(\frac{x + \sqrt{9+x^2}}{3} \right) \right]_0^4 \\
&= \frac{9}{2} \left(\frac{20}{9} + \ln 3 \right) = 10 + \frac{9}{2} \ln 3 = \frac{20 + 9\ln 3}{2} \approx 14.9437552990.
\end{aligned}
$$

C08S06.051: $A = 4\pi \displaystyle\int_0^{\pi/2} (\sin x)\sqrt{1 + \cos^2 x} \, dx$. With $u = \cos x$ and $du = -\sin x \, dx$, we obtain

$$
A = 4\pi \int_0^1 \sqrt{1 + u^2} \, du.
$$

To find the antiderivative, we let $u = \sinh z$, $du = \cosh z \, dz$. Then we obtain

$$
\begin{aligned}
A &= 4\pi \int_{u=0}^{u=1} \cosh^2 z \, dz = \Big[2\pi (z + \sinh z \, \cosh z) \Big]_{u=0}^{1} \\
&= 2\pi \left[\sinh^{-1} u + u\sqrt{1 + u^2} \, \right]_0^1 = 2\pi \left(\sinh^{-1}(1) + \sqrt{2} \, \right) \\
&= 2\pi \Big[\sqrt{2} + \ln \left(1 + \sqrt{2} \right) \Big] \approx 14.4236.
\end{aligned}
$$

C08S06.052: First we solve the equation of the ellipse for

$$
y = \frac{b}{a} (a^2 - x^2)^{1/2}. \tag{1}
$$

Thus

$$
\frac{dy}{dx} = \frac{b}{a} \cdot \frac{1}{2} (a^2 - x^2)^{1/2} \cdot (-2x) = \frac{-bx}{a(a^2 - x^2)^{1/2}},
$$

so that

$$1+\left(\frac{dy}{dx}\right)^{1/2}=1+\frac{b^2x^2}{a^2(a^2-x^2)}=\frac{a^4-a^2x^2+b^2x^2}{a^2(a^2-x^2)}.$$

Equation (1) also gives the radius of the circle of revolution "at" x, so the surface area of revolution around the x-axis is

$$A=2\int_0^a 2\pi\cdot\frac{b}{a}(a^2-x^2)^{1/2}\cdot\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx.$$

Let

$$x=\frac{a^2\sin u}{\sqrt{a^2-b^2}},\qquad\text{so that}\qquad dx=\frac{a^2}{\sqrt{a^2-b^2}}\cdot\cos u\,du.$$

Then

$$a^4-(a^2-b^2)x^2=a^2-(a^2-b^2)\cdot\frac{a^4\sin^2 u}{a^2-b^2}=a^4(1-\sin^2 u)=a^4\cos^2 u. \tag{2}$$

Hence

$$A=\frac{4\pi b}{a^2}\int_{x=0}^a(a^2\cos u)\cdot\frac{a^2\cos u}{\sqrt{a^2-b^2}}\,du=\frac{4\pi a^2 b}{\sqrt{a^2-b^2}}\int_{x=0}^a\cos^2 u\,du=\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\Big[u+\sin u\,\cos u\Big]_{x=0}^a.$$

Then, by Eq. (2), $\cos u=\dfrac{\sqrt{a^2-(a^2-b^2)x^2}}{a^2}$. So

$$\begin{aligned}A&=\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\left[\arcsin\left(\frac{x\sqrt{a^2-b^2}}{a^2}\right)+\frac{x\sqrt{a^2-b^2}}{a^2}\cdot\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a^2}\right]_0^a\\&=\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\left[\arcsin\left(\frac{\sqrt{a^2-b^2}}{a}\right)+\frac{\sqrt{a^2-b^2}}{a}\cdot\frac{\sqrt{a^2b^2}}{a^2}\right].\end{aligned}$$

Let $c=\sqrt{a^2-b^2}$. Then

$$A=\frac{2\pi a^2 b}{c}\left[\frac{bc}{a^2}+\arcsin\left(\frac{c}{a}\right)\right]=2\pi ab\left[\frac{b}{a}+\frac{a}{c}\arcsin\left(\frac{c}{a}\right)\right].$$

As $b\to a^+$, $c\to 0$ and $\arcsin(c/a)\to c/a$. So $\displaystyle\lim_{b\to a^+}A=2\pi a^2(1+1)=4\pi a^2$.

C08S06.053: This is the case in which $0<a<b$. First we solve the equation of the ellipse for

$$y=\frac{b}{a}(a^2-x^2)^{1/2}. \tag{1}$$

Thus

$$\frac{dy}{dx}=\frac{b}{a}\cdot\frac{1}{2}(a^2-x^2)^{1/2}\cdot(-2x)=\frac{-bx}{a(a^2-x^2)^{1/2}},$$

so that

$$1 + \left(\frac{dy}{dx}\right)^{1/2} = 1 + \frac{b^2x^2}{a^2(a^2-x^2)} = \frac{a^4 - a^2x^2 + b^2x^2}{a^2(a^2-x^2)}.$$

Equation (1) also gives the radius of the circle of revolution "at" x, so the surface area of revolution around the x-axis is

$$A = 2\int_0^a 2\pi \cdot \frac{b}{a}(a^2-x^2)^{1/2} \cdot \frac{\sqrt{a^4+(b^2-a^2)x^2}}{a\sqrt{a^2-x^2}}\,dx = \frac{4\pi b}{a^2}\int_0^a \sqrt{a^4+(b^2-a^2)x^2}\,dx.$$

Let

$$x = \frac{a^2\tan u}{\sqrt{b^2-a^2}}, \qquad \text{so that} \qquad dx = \frac{a^2\sec^2 u}{\sqrt{b^2-a^2}}\,du.$$

Then

$$a^4 + (b^2-a^2)x^2 = a^2 + (b^2-a^2)\cdot\frac{a^4\tan^2 u}{b^2-a^2} = a^4(1+\tan^2 u) = a^2\sec^2 u. \tag{2}$$

Thus

$$\begin{aligned} A &= \frac{4\pi b}{a^2}\int_{x=0}^a (a^2\sec^2 u)\cdot\frac{a^2\sec^2 u}{\sqrt{b^2-a^2}}\,du = \frac{4\pi a^2 b}{\sqrt{b^2-a^2}}\int_{x=0}^a \sec^3 u\,du \\ &= \frac{4\pi a^2 b}{\sqrt{b^2-a^2}}\left[\frac{1}{2}\big(\sec u\,\tan u + \ln|\sec u + \tan u|\big)\right]_{x=0}^a. \end{aligned}$$

Now $\tan u = \dfrac{x\sqrt{b^2-a^2}}{a^2}$, and by Eq. (2),

$$\sec u = \frac{\sqrt{a^4+(b^2-a^2)x^2}}{a^2}.$$

So

$$\begin{aligned} A &= \frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\left[\frac{x\sqrt{b^2-a^2}}{a^2}\cdot\frac{\sqrt{a^4+(b^2-a^2)x^2}}{a^2} + \ln\left|\frac{\sqrt{a^4+(b^2-a^2)x^2}}{a^2} + \frac{x\sqrt{b^2-a^2}}{a^2}\right|\right]_{x=0}^a \\ &= \frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\left[\frac{\sqrt{b^2-a^2}}{a}\cdot\frac{\sqrt{a^2b^2}}{a^2} + \ln\left(\frac{\sqrt{a^2b^2}}{a^2} + \frac{\sqrt{b^2-a^2}}{a}\right)\right]. \end{aligned}$$

Let $c = \sqrt{b^2-a^2}$. Then

$$A = \frac{2\pi a^2 b}{c}\left[\frac{c}{a}\cdot\frac{b}{a} + \ln\left(\frac{b}{a}+\frac{c}{a}\right)\right] = 2\pi ab\left[\frac{b}{a} + \frac{a}{c}\ln\left(\frac{b+c}{a}\right)\right].$$

Now let $b \to a^+$. Then

$$\frac{b+c}{a} \approx 1 + \frac{c}{a}, \qquad \text{so that} \qquad \ln\left(\frac{b+c}{a}\right) \approx \ln\left(1+\frac{c}{a}\right) \approx \frac{c}{a}.$$

So

$$\frac{b}{a} \to 1 \qquad \text{and} \qquad \frac{a}{c}\ln\left(\frac{b+c}{a}\right) \to 1$$

as $b \to a^+$. Therefore as $b \to a^+$, $A \to 2\pi ab(1+1) = 4\pi ab$.

C08S06.054: Given: $y = -1 + 2(x-1)^{1/2}$. Then

$$\frac{dy}{dx} = (x-1)^{-1/2}, \qquad \text{so} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x-1} = \frac{x}{x-1}.$$

Therefore the length of the graph of y is

$$L = \int_2^5 \left(\frac{x}{x-1}\right)^{1/2} dx.$$

Let $x = \sec^2\theta$, so that $dx = 2\sec^2\theta\,\tan\theta\,d\theta$. Hence

$$L = \int_{x=2}^5 \frac{\sec\theta}{\tan\theta}\cdot 2\sec^2\theta\,\tan\theta\,d\theta = 2\int_{x=2}^5 \sec^3\theta\,d\theta = \Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{x=2}^5.$$

A reference triangle with acute angle θ, adjacent side 1, and hypotenuse $\sqrt{x}$ has opposite side $\sqrt{x-1}$. Therefore

$$L = \left[\sqrt{x^2 - x} + \ln\left|\sqrt{x} + \sqrt{x-1}\,\right|\right]_2^5 = 2\sqrt{5} - \sqrt{2} + \ln\left(2+\sqrt{5}\right) - \ln\left(1+\sqrt{2}\right) \approx 3.6201842808.$$

C08S06.055: Given: $y = -1 + 2(x-1)^{1/2}$. Then

$$\frac{dy}{dx} = (x-1)^{-1/2}, \qquad \text{so} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x-1} = \frac{x}{x-1}.$$

Therefore the cost is

$$C = \int_2^5 \sqrt{x}\left(\frac{x}{x-1}\right)^{1/2} dx = \int_2^5 \frac{x}{\sqrt{x-1}}\,dx.$$

Let $x - 1 = u^2$. Then $x = 1 + u^2$ and $dx = 2u\,du$. Hence

$$\begin{aligned} C &= \int_{x=2}^5 \frac{1+u^2}{u}\cdot 2u\,du = 2\int_{x=2}^5 (1+u^2)\,du = 2\left[u + \frac{1}{3}u^3\right]_{x=2}^5 \\ &= 2\left[\sqrt{x-1} + \frac{1}{3}(x-1)^{1/3}\right]_2^5 = 2\left(2 - 1 + \frac{8}{3} - \frac{1}{3}\right) = \frac{20}{3} \qquad \text{(million dollars)}. \end{aligned}$$

C08S06.056: If $y = \frac{1}{20}x^2$, then

$$\frac{dy}{dx} = \frac{1}{10}x, \qquad \text{so that} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{100} = \frac{100 + x^2}{100}.$$

Thus the arc length element in this problem is $ds = \frac{1}{10}\sqrt{100 + x^2}\,dx$. A segment of the string, above the point $(x, 0)$ and of length ds, originally at ground level, is lifted to the height $y = \frac{1}{20}x^2$ (even though its original position was not $(x, 0)$). Hence the work done in lifting the string is

$$W = \int_0^{100} \frac{1}{16}\cdot\frac{1}{20}x^2\cdot\frac{1}{10}\sqrt{100 + x^2}\,dx = \frac{1}{3200}\int_0^{100} x^2\sqrt{100 + x^2}\,dx.$$

Let $x = 10\tan\theta$. Then $100 + x^2 = 100\sec^2\theta$ and $dx = 10\sec^2\theta\, d\theta$. Therefore

$$
\begin{aligned}
W &= \frac{1}{3200}\int_{x=0}^{100}(100\tan^2\theta)(10\sec\theta)(10\sec^2\theta)\,d\theta = \frac{100}{32}\int_{x=0}^{100}\sec^3\theta\,\tan^2\theta\,d\theta \\
&= \frac{25}{8}\int_{x=0}^{100}(\sec^5\theta - \sec^3\theta)\,d\theta = \frac{25}{8}\left[\frac{1}{4}\sec^3\theta\,\tan\theta - \frac{1}{8}\sec\theta\,\tan\theta - \frac{1}{8}\ln|\sec\theta + \tan\theta|\right]_{x=0}^{100}.
\end{aligned}
$$

A reference triangle with acute angle θ, opposite side x, and adjacent side 10 has hypotenuse $\sqrt{x^2+100}$. Therefore

$$
\begin{aligned}
W &= \frac{25}{8}\left[\frac{1}{4}\cdot\frac{x(x^2+100)^{3/2}}{10000} - \frac{1}{8}\cdot\frac{x(x^2+100)^{1/2}}{100} - \frac{1}{8}\ln\left(\frac{x+\sqrt{x^2+100}}{10}\right)\right]_0^{100} \\
&= \left[\frac{1}{12800}x(x^2+100)(x^2+100)^{1/2} - \frac{1}{256}x(x^2+100)^{1/2} - \frac{25}{64}\ln\left(\frac{x+\sqrt{x^2+100}}{10}\right)\right]_0^{100} \\
&= \left[\frac{x^3(x^2+100)^{1/2}}{12800} + \frac{x(x^2+100)^{1/2}}{128} - \frac{x(x^2+100)^{1/2}}{256} - \frac{25}{64}\ln\left(\frac{x+\sqrt{x^2+100}}{10}\right)\right]_0^{100} \\
&= \frac{1000000\sqrt{100\cdot 101}}{12800} + \frac{100\sqrt{100\cdot 101}}{256} - \frac{25}{64}\ln\left(\frac{100+\sqrt{100\cdot 101}}{10}\right) \\
&= \frac{3125}{4}\sqrt{101} + \frac{125}{32}\sqrt{101} - \frac{25}{64}\ln\left(10+\sqrt{101}\right) = \frac{25125}{32}\sqrt{101} - \frac{25}{64}\ln\left(10+\sqrt{101}\right) \\
&= \frac{1}{64}\left[50250\sqrt{101} - 25\ln\left(10+\sqrt{101}\right)\right] \approx 7889.5514748057
\end{aligned}
$$

inch-pounds, about 657.4626229005 ft • lb.

Section 8.7

C08S07.001: $\displaystyle\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx = \arctan(x+2)+C.$

C08S07.002: $\displaystyle\int \frac{2x+5}{x^2+4x+5}\,dx = \int \left(\frac{2x+4}{x^2+4x+5}+\frac{1}{x^2+4x+5}\right)dx = \ln(x^2+4x+5)+\arctan(x+2)+C$

(the antiderivative of the second fraction was computed in the solution of Problem 1).

C08S07.003: $\displaystyle\int \frac{5-3x}{x^2+4x+5}\,dx = -\frac{3}{2}\int \frac{2x+4-\frac{22}{3}}{x^2+4x+5}\,dx = -\frac{3}{2}\ln(x^2+4x+5)+11\int \frac{1}{x^2+4x+5}\,dx$

$\displaystyle = -\frac{3}{2}\ln(x^2+4x+5)+11\arctan(x+2)+C \qquad \text{(see Problem 1).}$

C08S07.004: We will obtain $x^2+4x+5=(x+2)^2+1=1+\tan^2\theta=\sec^2\theta$ if we let $x=-2+\tan\theta$. If so, then $dx=\sec^2\theta\,d\theta$, $x+1=-1+\tan\theta$, and $\tan\theta=x+2$. Thus

$$I=\int \frac{-1+\tan\theta}{\sec^4\theta}\sec^2\theta\,d\theta=\int(-1+\tan\theta)\cos^2\theta\,d\theta=\int(-\cos^2\theta+\sin\theta\,\cos\theta)\,d\theta$$

$$=\int\left(\sin\theta\,\cos\theta-\frac{1+\cos 2\theta}{2}\right)d\theta=\frac{1}{2}\sin^2\theta-\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta+C=\frac{1}{2}\sin^2\theta-\frac{1}{2}\theta-\frac{1}{2}\sin\theta\,\cos\theta+C.$$

A reference triangle with acute angle θ, opposite side $x+2$, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+4x+5}$. Therefore

$$I=\frac{1}{2}\cdot\frac{(x+2)^2}{x^2+4x+5}-\frac{1}{2}\arctan(x+2)-\frac{1}{2}\cdot\frac{x+2}{x^2+4x+5}+C$$

$$=\frac{1}{2}\cdot\frac{x^2+4x+4-x-2}{x^2+4x+5}-\frac{1}{2}\arctan(x+2)+C=\frac{x^2+3x+2}{2(x^2+4x+5)}-\frac{1}{2}\arctan(x+2)+C.$$

C08S07.005: $3-2x-x^2=-(x^2+2x-3)=-(x^2+2x+1-4)=4-(x+1)^2-4-4\sin^2\theta=4\cos^2\theta$ if we (and we do) let $x+1=2\sin\theta$. Then $x=-1+2\sin\theta$, $dx=2\cos\theta\,d\theta$, and $\sin\theta=\frac{1}{2}(x+1)$. So

$$\int\frac{1}{\sqrt{3-2x-x^2}}\,dx=\int\frac{1}{2\cos\theta}\cdot 2\cos\theta\,d\theta=\int 1\,d\theta=\theta+C=\arcsin\left(\frac{x+1}{2}\right)+C.$$

C08S07.006: The same substitution as in the solution of Problem 5 yields

$$J=\int\frac{x+3}{\sqrt{3-2x-x^2}}\,dx=\int\frac{2+2\sin\theta}{2\cos\theta}\cdot 2\cos\theta\,d\theta=2\theta-2\cos\theta+C.$$

A reference triangle with acute angle θ, opposite side $x+1$, and hypotenuse 2 has adjacent side of length $\sqrt{3-2x-x^2}$. Therefore

$$J=2\arcsin\left(\frac{x+1}{2}\right)-\sqrt{3-2x-x^2}+C.$$

C08S07.007: $3-2x-x^2=-(x^2+2x-3)=-(x^2+2x+1-4)=4-(x+1)^2-4-4\sin^2\theta=4\cos^2\theta$ if we (and we do) let $x+1=2\sin\theta$. Then $x=-1+2\sin\theta$, $dx=2\cos\theta\,d\theta$, and $\sin\theta=\frac{1}{2}(x+1)$. So

$$\int x\sqrt{3-2x-x^2}\ dx = \int (-1+2\sin\theta)\cdot(2\cos\theta)\cdot(2\cos\theta)\ d\theta = 4\int (2\cos^2\theta\ \sin\theta - \cos^2\theta)\ d\theta$$

$$= 4\int \left(2\cos^2\theta\ \sin\theta - \frac{1+\cos 2\theta}{2}\right)\ d\theta = 4\left(-\frac{2}{3}\cos^2\theta - \frac{1}{2}\theta - \frac{1}{2}\sin\theta\ \cos\theta\right) + C$$

$$= -\frac{1}{3}(2\cos\theta)^3 - 3\theta - 2\sin\theta\ \cos\theta + C$$

$$= -\frac{1}{3}(3-2x-x^2)^{3/2} - 2\arcsin\left(\frac{x+1}{2}\right) - \frac{x+1}{2}\sqrt{3-2x-x^2}\ + C$$

$$= -2\arcsin\left(\frac{x+1}{2}\right) - \left[\frac{1}{3}(3-2x-x^2) + \frac{1}{2}(x+1)\right]\sqrt{3-2x-x^2}\ + C$$

$$= -2\arcsin\left(\frac{x+1}{2}\right) - \frac{1}{6}\left[2(3-2x-x^2) + 3(x+1)\right]\sqrt{3-2x-x^2}\ + C$$

$$= -2\arcsin\left(\frac{x+1}{2}\right) - \frac{1}{6}\big(-2x^2-4x+6+3x+x\big)\sqrt{3-2x-x^2}\ + C$$

$$= -2\arcsin\left(\frac{x+1}{2}\right) + \frac{1}{6}(2x^2+x-9)\sqrt{3-2x-x^2}\ + C.$$

C08S07.008: $4x^2+4x-3 = 4x^2+4x+1-4 = (2x+1)^2-4 = 4\sec^2\theta\ - 4 = 4\tan^2\theta$ if we (and we do) let $2\sec\theta = 2x+1$. Thus $x = \sec\theta\ - \frac{1}{2}$, $\sec\theta = x+\frac{1}{2}$, and $dx = \sec\theta\ \tan\theta\ d\theta$. Therefore

$$K = \int \frac{1}{4x^2+4x-3}\ dx = \int \frac{1}{4\tan^2\theta}\cdot\sec\theta\ \tan\theta\ d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\ d\theta$$

$$= \frac{1}{4}\int \csc\theta\ d\theta = \frac{1}{4}\ln|\csc\theta - \cot\theta| + C.$$

A reference triangle with acute angle θ, adjacent side 2, and hypotenuse $2x+1$ has opposite side of length $\sqrt{4x^2+4x-3}$. Therefore

$$K = \frac{1}{4}\ln\left|\frac{2x-1}{\sqrt{4x^2+4x-3}}\right| + C = \frac{1}{8}\ln\left(\frac{4x^2-4x+1}{4x^2+4x-3}\right) + C = \frac{1}{8}\ln\left|\frac{2x-1}{2x+3}\right| + C.$$

C08S07.009: $4x^2+4x-3 = 4x^2+4x+1-4 = (2x+1)^2-4 = 4\sec^2\theta\ - 4 = 4\tan^2\theta$ if we (and we do) let $2\sec\theta = 2x+1$. Thus $x = \sec\theta\ - \frac{1}{2}$, $\sec\theta = x+\frac{1}{2}$, and $dx = \sec\theta\ \tan\theta\ d\theta$. Thus

$$I = \int \frac{3x+2}{4x^2+4x-3}\ dx = \int \frac{3\sec\theta + \frac{1}{2}}{4\tan^2\theta}\cdot\sec\theta\ \tan\theta\ d\theta = \frac{1}{8}\int \frac{6\sec\theta\ + 1}{\tan\theta}\cdot\sec\theta\ d\theta$$

$$= \frac{1}{8}\int \left(6\cdot\frac{\sec^2\theta}{\tan\theta} + \frac{\sec\theta}{\tan\theta}\right) d\theta = \frac{1}{8}\int \left(6\cdot\frac{\sec^2\theta}{\tan\theta} + \csc\theta\right) d\theta$$

$$= \frac{1}{8}\big(6\ln|\tan\theta| + \ln|\csc\theta - \cot\theta|\big) + C = \frac{3}{4}\ln|\tan\theta| + \frac{1}{8}\ln|\csc\theta - \cot\theta| + C.$$

A reference triangle with acute angle θ, adjacent side 2, and hypotenuse $2x+1$ has opposite side of length $\sqrt{4x^2+4x-2}$. Therefore

$$\begin{aligned}
I &= \frac{3}{4}\ln\left|\frac{\sqrt{4x^2+4x-3}}{2}\right| + \frac{1}{8}\ln\left|\frac{2x+1-2}{\sqrt{4x^2+4x-3}}\right| + C \\
&= \frac{3}{4}\ln\sqrt{4x^2+4x-3}\; + \frac{1}{8}\ln|2x-1| - \frac{1}{8}\ln\sqrt{4x^2+4x-3}\; + C_1 \\
&= \frac{5}{16}\ln|4x^2+4x-3| + \frac{1}{8}\ln|2x-1| + C_1 = \frac{7}{16}\ln|2x-1| + \frac{5}{16}\ln|2x+3| + C_1.
\end{aligned}$$

C08S07.010: $4x^2+4x-3 = 4x^2+4x+1-4 = (2x+1)^2-4 = 4\sec^2\theta - 4 = 4\tan^2\theta$ if we (and we do) let $2\sec\theta = 2x+1$. Thus $x = \sec\theta - \frac{1}{2}$, $\sec\theta = x+\frac{1}{2}$, and $dx = \sec\theta\,\tan\theta\,d\theta$. Thus

$$\begin{aligned}
J &= \int \sqrt{4x^2+4x-3}\;dx = \int 2\tan\theta\,\sec\theta\,\tan\theta\,d\theta \\
&= 2\int(\sec^3\theta - \sec\theta)\,d\theta = \sec\theta\,\tan\theta - \ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

A reference triangle with acute angle θ, adjacent side 2, and hypotenuse $2x+1$ has opposite side of length $\sqrt{4x^2+4x-2}$. Therefore

$$\begin{aligned}
J &= \frac{1}{4}(2x+1)\sqrt{4x^2+4x-3} - \ln\left|\frac{2x+1+\sqrt{4x^2+4x-3}}{2}\right| + C \\
&= \frac{2x+1}{4}\sqrt{4x^2+4x-3} - \ln\left|2x+1+\sqrt{4x^2+4x-3}\,\right| + C_1.
\end{aligned}$$

C08S07.011: $x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+9 = 9+9\tan^2\theta = 9\sec^2\theta$ if we (and we do) let $3\tan\theta = x+2$; that is, $x = -2+3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, and $\tan\theta = \frac{1}{3}(x+2)$. Then

$$\int\frac{1}{x^2+4x+13}\,dx = \int\frac{1}{9\sec^2\theta}\cdot 3\sec^2\theta\,d\theta = \int\frac{1}{3}\,d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C.$$

C08S07.012: $2x-x^2 = -(x^2-2x) = 1-(x^2-2x+1) = 1-(x-1)^2 = 1-\sin^2\theta = \cos^2\theta$ if we let $\sin\theta = x-1$. That is, $x = 1+\sin\theta$, so that $dx = \cos\theta\,d\theta$ and $\theta = \arcsin(x-1)$. Then

$$\int\frac{1}{\sqrt{2x-x^2}}\,dx = \int\frac{1}{\cos\theta}\cdot\cos\theta\,d\theta = \theta + C = \arcsin(x-1) + C.$$

C08S07.013: $3+2x-x^2 = -(x^2-2x-3) = -(x^2-2x+1-4) = 4-(x-1)^2 = 4-4\sin^2\theta = 4\cos^2\theta$ if $2\sin\theta = x-1$; that is, $x = 1+2\sin\theta$, $dx = 2\cos\theta\,d\theta$, and $\sin\theta = \frac{1}{2}(x-1)$. Therefore

$$K = \int\frac{1}{3+2x-x^2}\,dx = \int\frac{1}{4\cos^2\theta}\cdot 2\cos\theta\,d\theta = \frac{1}{2}\int\sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.$$

A reference triangle with acute angle θ, opposite side $x-1$, and hypotenuse 2 has adjacent side of length $\sqrt{3+2x-x^2}$. Thus

$$K = \frac{1}{2}\left|\frac{x+1}{\sqrt{3+2x-x^2}}\right| + C = \frac{1}{4}\ln\left|\frac{(x+1)^2}{(x+1)(x-3)}\right| + C = \frac{1}{4}\ln\left|\frac{x+1}{x-3}\right| + C.$$

C08S07.014: $8+2x-x^2 = -(x^2-2x+8) = 9-(x-1)^2 = 9-9\sin^2\theta = 9\cos^2\theta$ if $x-1 = 3\sin\theta$; that is, $x = 1+3\sin\theta$, $ds = 3\cos\theta\,d\theta$, and $\sin\theta = \frac{1}{3}(x-1)$. So

$$I = \int x\sqrt{8+2x-x^2}\,dx = \int (1+3\sin\theta)(3\cos\theta)(3\cos\theta)\,d\theta = 9\int (3\sin\theta\,\cos^2\theta + \cos^2\theta)\,d\theta$$

$$= 9\int\left(3\sin\theta\,\cos^2\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = 9\left(-\cos^2\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$

$$= \frac{9}{2}\big(-2\cos^3\theta + \theta + \sin\theta\,\cos\theta\big) + C.$$

A reference triangle with acute angle θ, opposite side $x-1$, and hypotenuse 3 has adjacent side of length $\sqrt{8+2x-x^2}$. Therefore

$$I = \frac{9}{2}\left[-2\cdot\frac{(8+2x-x^2)^{3/2}}{27} + \arcsin\left(\frac{x-1}{3}\right) + \frac{(x-1)\sqrt{8+2x-x^2}}{9}\right] + C$$

$$= \frac{x-1}{2}\sqrt{8+2x-x^2} - \frac{8+2x-x^2}{3}\sqrt{8+2x-x^2} + \frac{9}{2}\arcsin\left(\frac{x-1}{3}\right) + C$$

$$= \left[\frac{1}{2}(x-1) + \frac{1}{3}(x^2-2x-8)\right]\sqrt{8+2x-x^2} + \frac{9}{2}\arcsin\left(\frac{x-1}{3}\right) + C$$

$$= \frac{1}{6}(2x^2-x-19)\sqrt{8+2x-x^2} + \frac{9}{2}\arcsin\left(\frac{x-1}{2}\right) + C.$$

C08S07.015: $x^2+2x+2 = (x+1)^2+1 = 1+\tan^2\theta = \sec^2\theta$ if $x+1 = \tan\theta$; that is, $x = -1+\tan\theta$, $dx = \sec^2\theta\,d\theta$, and $\tan\theta = x+1$. Therefore

$$\int\frac{2x-5}{x^2+2x+2}\,dx = \int\frac{-2+2\tan\theta-5}{\sec^2\theta}\cdot\sec^2\theta\,d\theta = 2\ln|\sec\theta| - 7\theta + C$$

$$= 2\ln\sqrt{x^2+2x+2} - 7\arctan(x+1) + C = \ln(x^2+2x+2) - 7\arctan(x+1) + C.$$

C08S07.016: $4x^2+4x-15 = (2x+1)^2-16 = 16\sec^2 u - 16 = 16\tan^2 u$ if $2x+1 = 4\sec u$. Thus we let $x = \frac{1}{2}(-1+4\sec u)$; then $dx = 2\sec u\,\tan u\,du$ and $\sec u = \frac{1}{4}(2x+1)$. Hence

$$J = \int\frac{2x-1}{4x^2+4x-15}\,dx = \int\frac{4\sec u\,-2}{16\tan^2 u}\cdot 2\sec u\,\tan u\,du = \frac{1}{8}\int\frac{4\sec^2 u\,\tan u\,-2\sec u\,\tan u}{\tan^2 u}\,du$$

$$= \frac{1}{8}\int\left(\frac{4\sec^2 u}{\tan u} - \frac{2\sec u}{\tan u}\right)du = \frac{1}{8}\int\left(\frac{4\sec^2 u}{\tan u} - 2\csc u\right)dx$$

$$= \frac{1}{8}\big(4\ln|\tan u| - 2\ln|\csc u - \cot u|\big) + C = \frac{1}{2}\ln|\tan u| - \frac{1}{4}\ln|\csc u - \cot u| + C.$$

A reference triangle with acute angle u, adjacent side 4, and hypotenuse $2x+1$ has opposite side of length $\sqrt{4x^2+4x-15}$. Therefore

$$J = \frac{1}{2}\ln\left(\frac{\sqrt{4x^2+4x-15}}{4}\right) - \frac{1}{4}\ln\left|\frac{2x+1-4}{\sqrt{4x^2+4x-15}}\right| + C$$

$$= \frac{1}{4}\ln|4x^2+4x-15| + \frac{1}{8}\ln|4x^2+4x-15| - \frac{1}{4}\ln|2x-3| + C_1$$

$$= \frac{3}{8}\ln|4x^2+4x-15| - \frac{1}{4}\ln|2x-3| + C_1 = \frac{3}{8}\ln|2x+5| + \frac{1}{8}\ln|2x-3| + C_1.$$

C08S07.017: $5+12x-9x^2 = -(9x^2-12x-5) = -(9x^2-12x+4-9) = 9-(3x-2)^2 = 9-9\sin^2 u = 9\cos^2 u$ if $3x-2 = 3\sin u$. So let $x = \frac{2}{3}+\sin u$, so that $dx = \cos u\,du$ and $\sin u = \frac{1}{3}(3x-2)$. Then

$$\int \frac{x}{\sqrt{5+12x-9x^2}}\,dx = \int \frac{\frac{2}{3}+\sin u}{3\cos u}\cdot\cos u\,du = \int\left(\frac{2}{3}+\frac{1}{3}\sin u\right)du$$

$$= \frac{2}{9}u - \frac{1}{3}\cos u + C = \frac{2}{9}\arcsin\left(\frac{3x-2}{3}\right) - \frac{1}{9}\sqrt{5+12x-9x^2} + C.$$

C08S07.018: $9x^2+12x+8 = (3x+2)^2+4 = 4\tan^2 u + 4 = 4\sec^2 u$ if $2\tan u = 3x+2$. Hence let $x = \frac{1}{3}(2\tan u - 2)$; then $dx = \frac{2}{3}\sec^2 u\,du$ and $\tan u = \frac{1}{2}(3x+2)$. Therefore

$$K = \int (3x-2)\sqrt{9x^2+12x+8}\,dx = \int (2\tan u - 4)(2\sec u)\cdot\frac{2}{3}\sec^2 u\,du$$

$$= \frac{8}{3}\int (\sec^3 u\,\tan u - 2\sec^3 u)\,du = \frac{8}{3}\int (\sec^2 u\,\sec u\,\tan u - 2\sec^3 u)\,du$$

$$= \frac{8}{3}\left(\frac{1}{3}\sec^3 u - \sec u\,\tan u - \ln|\sec u + \tan u|\right) + C.$$

A reference triangle with acute angle u, opposite side $3x+2$, and adjacent side 2 has hypotenuse of length $\sqrt{9x^2+12x+8}$. Therefore

$$K = \frac{8}{3}\left(\frac{1}{3}\cdot\frac{(9x^2+12x+8)^{3/2}}{8} - \frac{(3x+2)\sqrt{9x^2+12x+8}}{4} - \ln\left|\frac{3x+2+\sqrt{9x^2+12x+8}}{2}\right|\right) + C$$

$$= \frac{1}{9}(9x^2+12x+8)\sqrt{9x^2+12x+8} - \frac{2}{3}(3x+2)\sqrt{9x^2+12x+8}$$

$$- \frac{8}{3}\ln\left|3x+2+\sqrt{9x^2+12x+8}\right| + C_1$$

$$= \frac{1}{9}\left[9x^2+12x+8-6(3x+2)\right]\sqrt{9x^2+12x+8} - \frac{8}{3}\ln\left|3x+2+\sqrt{9x^2+12x+8}\right| + C_1$$

$$= \frac{1}{9}(9x^2-6x-4)\sqrt{9x^2+12x+8} - \frac{8}{3}\ln\left|3x+2+\sqrt{9x^2+12x+8}\right| + C_1.$$

C08S07.019: $9+16x-4x^2 = -(4x^2-16x-9) = 25-(2x-4)^2 = 25-25\sin^2 u = 25\cos^2 u$ if $2x-4 = 5\sin u$. So let $x = \frac{1}{2}(4+5\sin u)$. Then $dx = \frac{5}{2}\cos u\,du$ and $2\sin u = \frac{1}{5}(2x-4)$. Thus

$$\int (7-2x)\sqrt{9+16x-4x^2}\,dx = \int (7-4-5\sin u)(5\cos u)\cdot\frac{5}{2}\cos u\,du = \frac{1}{2}\int (3-5\sin u)(25\cos^2 u)\,du$$

$$= \frac{1}{2}\int (75\cos^2 u - 125\sin u\,\cos^2 u)\,du = \frac{25}{2}\int \left(3\cdot\frac{1+\cos 2u}{2} - 5\sin u\,\cos^2 u\right)\,du$$

$$= \frac{25}{4}\int \big(3 + 3\cos 2u - 10\sin u\,\cos^2 u\big)\,du = \frac{25}{4}\left(3u + 3\sin u\,\cos u + \frac{10}{3}\cos^3 u\right) + C$$

$$= \frac{25}{4}\left[3\arcsin\left(\frac{2x-4}{5}\right) + 3\cdot\frac{2x-4}{5}\cdot\frac{\sqrt{9+16x-4x^2}}{5} + \frac{10}{3}\cdot\frac{(9+16x-4x^2)^{3/2}}{125}\right] + C$$

$$= \frac{75}{4}\arcsin\left(\frac{2x-4}{5}\right) + \frac{3(x-2)}{2}\sqrt{9+16x-4x^2} + \frac{1}{6}(9+16x-4x^2)\sqrt{9+16x-4x^2} + C$$

$$= \frac{75}{4}\arcsin\left(\frac{2x-4}{5}\right) + \left[\frac{9}{6}(x-2) + \frac{1}{6}(9+16x-4x^2)\right]\sqrt{9+16x-4x^2} + C$$

$$= \frac{75}{4}\arcsin\left(\frac{2x-4}{5}\right) + \frac{1}{6}\big(-4x^2+25x-9\big)\sqrt{9+16x-4x^2} + C$$

$$= \frac{75}{4}\arcsin\left(\frac{2x-4}{5}\right) - \frac{4x^2-25x+9}{6}\sqrt{9+16x-4x^2} + C.$$

C08S07.020: $x^2+2x+5 = (x+1)^2+4 = 4\tan^2 u + 4 = 4\sec^2 u$ if we (and we do) let $x = -1+2\tan u$; thus $dx = 2\sec^2 u\,du$ and $\tan u = \frac{1}{2}(x+1)$. And so

$$I = \int \frac{2x+3}{\sqrt{x^2+2x+5}}\,dx = \int \frac{1+4\tan u}{2\sec u}\cdot 2\sec^2 u\,du = \int (\sec u + 4\sec u\,\tan u)\,du$$

$$= 4\sec u + \ln|\sec u + \tan u| + C = 4\cdot\frac{\sqrt{x^2+2x+5}}{2} + \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C$$

$$= 2\sqrt{x^2+2x+5} + \ln\left(x+1+\sqrt{x^2+2x+5}\,\right) + C_1.$$

C08S07.021: $6x - x^2 = -(x^2-6x) = 9-(x^2-6x+9) = 9-(x-3)^2 = 9-9\sin^2\theta = 9\cos^2\theta$ if $3\sin\theta = x-3$, so we let $x = 3+3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$, $\sin\theta = \frac{1}{3}(x-3)$, and

$$I = \int \frac{x+4}{(6x-x^2)^{3/2}}\,dx = \int \frac{7+3\sin\theta}{27\cos^3\theta}\cdot 3\cos\theta\,d\theta = \frac{1}{9}\int \frac{7+3\sin\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{9}\int (7\sec^2\theta + 3\sec\theta\,\tan\theta)\,d\theta = \frac{7}{9}\tan\theta + \frac{1}{3}\sec\theta + C$$

$$= \frac{7}{9}\cdot\frac{x-3}{\sqrt{6x-x^2}} + \frac{1}{\sqrt{6x-x^2}} + C = \frac{7x-12}{\sqrt{6x-x^2}} + C.$$

To obtain the last line, we used a reference triangle with acute angle θ, opposite side $x-3$, and hypotenuse 3, which therefore has adjacent side of length $\sqrt{6x-x^2}$.

C08S07.022: Let $x = \tan u$. Then $1+x^2 = 1+\tan^2 u = \sec^2 u$ and $dx = \sec^2 u\,du$. Hence

$$J = \int \frac{x-1}{(x^2+1)^2}\,dx = \int \frac{-1+\tan u}{\sec^4 u}\cdot\sec^2 u\,du = \int (-1+\tan u)\cos^2 u\,du$$

$$= \int \left(\sin u\,\cos u - \frac{1+\cos 2u}{2}\right)\,du = \frac{1}{2}\sin^2 u - \frac{1}{2}u - \frac{1}{2}\sin u\,\cos u + C.$$

A reference triangle with acute angle u, opposite side x, and adjacent side 1 has hypotenuse of length $\sqrt{1+x^2}$. Therefore

$$J = \frac{1}{2}\cdot\frac{x^2}{1+x^2} - \frac{1}{2}\arctan x - \frac{1}{2}\cdot\frac{x}{1+x^2} + C = \frac{x^2-x}{2(x^2+1)} - \frac{1}{2}\arctan x + C$$

$$= \frac{1}{2}\left(1 - \frac{x+1}{x^2+1} - \arctan x\right) + C = -\frac{1}{2}\left(\frac{x+1}{x^2+1} + \arctan x\right) + C_1.$$

C08S07.023: $4x^2+12x+13 = 4x^2+12x+9+4 = (2x+3)^2+4 = 4+4\tan^2 u = 4\sec^2 u$ if $2\tan u = 2x+3$. Hence we let $x = \frac{1}{2}(-3+2\tan u)$, so that $dx = \sec^2 u\, du$ and $\tan u = \frac{1}{2}(2x+3)$. Then

$$K = \int \frac{2x+3}{(4x^2+12x+13)^2}\,dx = \int \frac{2\tan u\,\sec^2 u}{16\sec^4 u}\,du$$

$$= \frac{1}{8}\int \tan u\,\cos^2 u\,du = \frac{1}{8}\int \sin u\,\cos u\,du = \frac{1}{16}\sin^2 u + C.$$

A reference triangle with acute angle u, opposite side $2x+3$, and adjacent side 2 has hypotenuse of length $\sqrt{4x^2+12x+13}$. Therefore

$$K = \frac{1}{16}\cdot\frac{(2x+3)^2}{4x^2+12x+13} + C = \frac{1}{16}\cdot\frac{4x^2+12x+13-4}{4x^2+12x+13} + C$$

$$= \frac{1}{16}\left(1 - \frac{4}{4x^2+12x+13}\right) + C = -\frac{1}{4(4x^2+12x+13)} + C_1.$$

C08S07.024: Let $x = \sin u$: $1-x^2 = 1-\sin^2 u = \cos^2 u$, $dx = \cos u\,du$. Then

$$I = \int \frac{x^3}{(1-x^2)^4}\,dx = \int \frac{\sin^3 u\,\cos u}{\cos^8 u}\,du = \int \sec^4 u\,\tan^3 u\,du$$

$$= \int \left(\sec^5 u - \sec^3 u\right)\sec u\,\tan u\,du = \frac{1}{6}\sec^6 u - \frac{1}{4}\sec^4 u + C.$$

A reference triangle with acute angle u, opposite side x, and hypotenuse 1 has adjacent side of length $\sqrt{1-x^2}$. Therefore

$$I = \frac{1}{6}\cdot\frac{1}{(1-x^2)^3} - \frac{1}{4}\cdot\frac{1}{(1-x^2)^2} + C = \frac{1}{(1-x^2)^3}\left(\frac{1}{6} - \frac{1}{4}(1-x^2)\right) + C$$

$$= \frac{1}{12(1-x^2)^3}\big(2 - 3(1-x^2)\big) + C = \frac{3x^2-1}{12(1-x^2)^3} + C.$$

Alternatively, the partial fractions decomposition

$$\frac{x^3}{(1-x^2)^4} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{G}{(x+1)^3} + \frac{H}{(x+1)^4}$$

yields

$$
\begin{aligned}
&A(x^7+x^6-3x^5-3x^4+3x^3+3x^2-x-1)+B(x^6+2x^5-x^4-4x^3-x^2+2x+1)\\
&\quad +C(x^5+3x^4+2x^3-2x^2-3x-1)+D(x^4+4x^3+6x^2+4x+1)\\
&\quad +E(x^7-x^6-3x^5+3x^4+3x^3-3x^2-x+1)+F(x^6-2x^5-x^4+4x^3-x^2-2x+1)\\
&\quad +G(x^5-3x^4+2x^3+2x^2-3x+1)+H(x^4-4x^3+6x^2-4x+1)=x^3.
\end{aligned}
$$

It now follows that

$$
\begin{aligned}
A+E&=0, & A+B-E+F&=0,\\
-3A+2B+C-3E-2F+G&=0, & -3A-B+3C+D+3E-F-3G+H&=0,\\
3A-4B+2C+4D+3E+4F+2G-4H&=1, & 3A-B-2C+6D-3E-F+2G+6H&=0,\\
-A+2B-3C+4D-E-2F-3G-4H&=0, & -A+B-C+D+E+F+G+H&=0.
\end{aligned}
$$

This system has the unique solution

$$
A=0,\quad B=-\frac{1}{32},\quad C=\frac{1}{16},\quad D=\frac{1}{16},\quad E=0,\quad F=\frac{1}{32},\quad G=\frac{1}{16},\quad H=-\frac{1}{16}.
$$

Therefore

$$
\int\frac{x^3}{(1-x^2)^4}\,dx=\frac{1}{96}\left[\frac{3}{x-1}-\frac{3}{(x-1)^2}-\frac{2}{(x-1)^3}-\frac{3}{x+1}-\frac{3}{(x+1)^2}+\frac{2}{(x+1)^3}\right]+C. \tag{1}
$$

If you combine the six terms in the brackets on the right-hand side, you will obtain exactly the same answer as that yielded by the method of trigonometric substitution.

The simple substitution $u=1-x^2$, for which $du=-2x\,dx$ and $x^2=1-u$, yields

$$
\begin{aligned}
I&=-\frac{1}{2}\int\frac{1-u}{u^4}\,du=\frac{1}{2}\int\left(u^{-3}-u^{-4}\right)\,du=\frac{1}{2}\left(\frac{1}{3}u^{-3}-\frac{1}{2}u^{-2}\right)+C=\frac{1}{6u^3}-\frac{1}{4u^2}+C\\
&=\frac{1}{6(1-x^2)^3}-\frac{1}{4(1-x^2)^2}+C=\frac{1}{12}\left[\frac{2}{(1-x^2)^3}-\frac{3(1-x^2)}{(1-x^2)^3}\right]+C=\frac{3x^2-1}{12(1-x^2)^3}+C.
\end{aligned}
$$

Integration by parts is also successful. Let

$$
\begin{aligned}
u&=x^2 &&\text{and} && dv=x(1-x^2)^{-4}\,dx. && \text{Then}\\
du&=2x\,dx &&\text{and} && v=\frac{1}{6}(1-x^2)^{-3}. && \text{Thus}
\end{aligned}
$$

$$
\begin{aligned}
I=\frac{x^2}{6(1-x^2)^3}-\frac{1}{3}\int x(1-x^2)^{-3}\,dx&=\frac{x^2}{6(1-x^2)^3}-\frac{1}{12}(1-x^2)^{-2}+C\\
&=\frac{2x^2-(1-x^2)}{12(1-x^2)^3}+C=\frac{3x^2-1}{12(1-x^2)^3}+C.
\end{aligned}
$$

The hyperbolic substitution $x=\tanh u$, for which $dx=\operatorname{sech}^2 u\;du$ and $1-x^2=1-\tanh^2 u=\operatorname{sech}^2 u$, produces

$$
\begin{aligned}
I &= \int \frac{\tanh^3 u}{\operatorname{sech}^8 u}\ \operatorname{sech}^2 u\ du = \int \frac{1-\operatorname{sech}^2 u}{\operatorname{sech}^7 u}\ \operatorname{sech} u\ \tanh u\ du \\
&= \int \left[(\operatorname{sech} u)^{-7} - (\operatorname{sech} u)^{-5}\right] \operatorname{sech} u\ \tanh u\ du \\
&= \frac{1}{6\operatorname{sech}^6 u} - \frac{1}{4\operatorname{sech}^4 u} + C = \frac{1}{6(1-x^2)^3} - \frac{1}{4(1-x^2)^2} + C.
\end{aligned}
$$

Finally, *Derive* 2.56 returns the answer in essentially the same form as in Eq. (1) here, while *Mathematica* 3.0 returns the antiderivative in the form

$$
-\frac{1}{6(-1+x^2)^3} - \frac{1}{4(-1+x^2)^2},
$$

which is essentially the same as the very first result we obtained by the method of trigonometric substitution. Using *Maple* V version 5.1, we proceeded as follows:

```
int((x∧3)/(1 - x∧2)∧4, x);
```

$$
\frac{1}{96}\left[\frac{3}{x-1} - \frac{3}{(x-1)^2} - \frac{2}{(x-1)^3} - \frac{3}{x+1} - \frac{3}{(x+1)^2} + \frac{2}{(x+1)^3}\right] + C
$$

(after factoring out the constant $\frac{1}{96}$ and adding C). Then the command

```
factor(%);
```

yielded, in effect,

$$
\int \frac{x^3}{(1-x^2)^4}\ dx\ =\ \frac{3x^2-1}{12(1-x^2)^3}\ +\ C.
$$

C08S07.025: $x^2+x+1 = x^2+x+\frac{1}{4}+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right] = \frac{3}{4}\left(1+\tan^2 u\right) = \frac{3}{4}\sec^2 u$ provided that

$$
\tan u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right).
$$

Therefore we let

$$
x = \frac{-1+\sqrt{3}\ \tan u}{2}: \qquad dx = \frac{\sqrt{3}}{2}\sec^2 u\ du, \quad \tan u = \frac{\sqrt{3}}{2}\sec^2 u\ du.
$$

Thus

$$
\begin{aligned}
J &= \int \frac{3x-1}{x^2+x+1}\ dx = \int \frac{\frac{1}{2}\left(-3+3\sqrt{3}\ \tan u\right)-1}{\frac{3}{4}\sec^2 u}\cdot\frac{\sqrt{3}}{2}\sec^2 u\ du \\
&= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\int\left(-5+3\sqrt{3}\ \tan u\right) du = \frac{\sqrt{3}}{3}\left(-5u+3\sqrt{3}\ \ln|\sec u|\right)+C.
\end{aligned}
$$

A reference triangle with acute angle u, opposite side $2x+1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{x^2+x+1}$. Therefore

$$J = \frac{\sqrt{3}}{3}\left[-5\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right) + 3\sqrt{3}\,\ln\left(\frac{2\sqrt{x^2+x+1}}{\sqrt{3}}\right)\right] + C$$

$$= -\frac{5\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right) + \frac{3}{2}\ln(x^2+x+1) + C_1.$$

C08S07.026: The same substitution as in the solution of Problem 25 yields

$$K = \int \frac{3x-1}{(x^2+x+1)^2}\,dx = \int \frac{\frac{1}{2}\left(-3+3\sqrt{3}\,\tan u\right)-1}{\frac{9}{16}\sec^4 u}\cdot\frac{\sqrt{3}}{2}\sec^2 u\,du$$

$$= \frac{16}{9}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\int\frac{-5+3\sqrt{3}\,\tan u}{\sec^2 u}\,du = \frac{4\sqrt{3}}{9}\int\left(-5\cdot\frac{1+\cos 2u}{2}+3\sqrt{3}\,\sin u\,\cos u\right)du$$

$$= \frac{4\sqrt{3}}{9}\left[-\frac{5}{2}(u+\sin u\,\cos u)+\frac{3\sqrt{3}}{2}\sin^2 u\right]+C = \frac{2\sqrt{3}}{9}\left(-5u-5\sin u\,\cos u+3\sqrt{3}\,\sin^2 u\right)+C.$$

Then the reference triangle of the solution of Problem 25 yields

$$K = \frac{2\sqrt{3}}{9}\left[-5\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right)-\frac{5\sqrt{3}\,(2x+1)}{4(x^2+x+1)}+\frac{3\sqrt{3}\,(2x+1)^2}{4(x^2+x+1)}\right]+C$$

$$= -\frac{10\sqrt{3}}{9}\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right)-5\cdot\frac{\sqrt{3}}{9}\cdot\frac{\sqrt{3}}{2}\cdot\frac{2x+1}{x^2+x+1}+\frac{\sqrt{3}}{9}\cdot\frac{3\sqrt{3}}{2}\cdot\frac{(2x+1)^2}{x^2+x+1}+C$$

$$= -\frac{10\sqrt{3}}{9}\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right)+\frac{2x+1}{2(x^2+x+1)}\left(2x+1-\frac{5}{3}\right)+C$$

$$= -\frac{10\sqrt{3}}{9}\arctan\left(\frac{\sqrt{3}}{3}[2x+1]\right)+\frac{(2x+1)(3x-1)}{3(x^2+x+1)}+C.$$

C08S07.027: The method of partial fractions yields

$$\frac{1}{(x^2-4)^2} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2} = \frac{1}{32}\left[-\frac{1}{x-2}+\frac{2}{(x-2)^2}+\frac{1}{x+2}+\frac{2}{(x+2)^2}\right].$$

Therefore

$$I = \int\frac{1}{(x^2-4)^2}\,dx = \frac{1}{32}\ln\left|\frac{x+2}{x-2}\right| - \frac{x}{8(x^2-4)}+C.$$

Alternatively, if we let $x = 2\sec u$, then $x^2-4 = 4\sec^2 u - 4 = 4\tan^2 u$ and $dx = 2\sec u\,\tan u\,du$. Then

$$I = \int\frac{2\sec u\,\tan u}{16\tan^4 u}\,du = \frac{1}{8}\int\frac{\sec u}{\tan^3 u}\,du = \frac{1}{8}\int\frac{\cos^2 u}{\sin^3 u}\,du = \frac{1}{8}\int\frac{1-\sin^2 u}{\sin^3 u}\,du = \frac{1}{8}\int\left(\csc^3 u - \csc u\right)du.$$

Then Formulas 15 and 29 from the endpapers of the text yield

$$I = \left(-\frac{1}{2}\csc u\,\cot u - \frac{1}{2}\ln|\csc u - \cot u\,| \right) + C = -\frac{1}{16}\big(\csc u\,\cot u + \ln|\csc u - \cot u\,|\big) + C.$$

A reference triangle with acute angle u, adjacent side 2, and hypotenuse side x has opposite side of length $\sqrt{x^2-4}$. Therefore

$$\begin{aligned} I &= -\frac{1}{16}\left(\frac{2x}{x^2-4} + \ln\left| \frac{x-2}{\sqrt{x^2-4}} \right| \right) + C \\ &= -\frac{x}{8(x^2-4)} + \frac{1}{32}\ln\left| \frac{x^2-4}{(x-2)^2} \right| + C = \frac{1}{32}\ln\left| \frac{x+2}{x-2} \right| - \frac{x}{8(x^2-4)} + C. \end{aligned}$$

C08S07.028: $x - x^2 = -(x^2-x) = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{1}{4}\sin^2 u = \frac{1}{4}\cos^2 u$ if we let $x = \frac{1}{2}(1+\sin u)$, so that $2x-1 = \sin u$, $dx = \frac{1}{2}\cos u\,du$, and $u = \arcsin(2x-1)$. Then

$$J = \int (x-x^2)^{3/2}\,dx = \int \left(\frac{1}{8}\cos^3 u\right)\left(\frac{1}{2}\cos u\right)\,du = \frac{1}{16}\int \cos^4 u\,du.$$

The reduction formula in Problem 54 of Section 8.3 now yields

$$\begin{aligned} J &= \frac{1}{16}\left(\frac{1}{4}\cos^3 u\,\sin u + \frac{3}{4}\int \cos^2 u\,du \right) \\ &= \frac{1}{16}\left(\frac{1}{4}\cos^3 u\,\sin u + \frac{3}{4}\left[\frac{1}{2}\sin u\,\cos u + \frac{1}{2}u \right] \right) + C = \frac{1}{64}\sin u\,\cos^3 u + \frac{3}{128}\sin u\,\cos u + \frac{3}{128}u + C. \end{aligned}$$

A reference triangle with acute angle u, opposite side $2x-1$, and hypotenuse 1 has adjacent side of length $2\sqrt{x-x^2}$, and therefore

$$\begin{aligned} J &= \frac{1}{128}\big[2\cdot(2x-1)\cdot 8(x-x^2)^{3/2} + 3\cdot(2x-1)\cdot 2\sqrt{x-x^2} + 3\arcsin(2x-1) \big] + C \\ &= \frac{1}{128}\big[2(2x-1)(x-x^2)^{1/2}\big\{8(x-x^2)+3\big\} + 3\arcsin(2x-1) \big] + C \\ &= \frac{1}{128}\big[2(2x-1)(3+8x-8x^2)\sqrt{x-x^2} + 3\arcsin(2x-1) \big] + C. \end{aligned}$$

C08S07.029: The partial fractions decomposition

$$\frac{x^2+1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

yields the equation $Ax^2 + Ax + A + Bx^2 + Cx = x^2 + 1$, so that

$$A + B = 1, \qquad A + C = 0, \qquad \text{and} \qquad A = 1.$$

It follows that $B = 0$ and $C = 1$. Hence

$$\frac{x^2+1}{x(x^2+x+1)} = \frac{1}{x} - \frac{1}{x^2+x+1}.$$

Now $x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2 u + \frac{3}{4} = \frac{3}{4}\sec^2 u$ if $\frac{1}{2}\sqrt{3}\,\tan u = x + \frac{1}{2}$, so we let

$$x = \frac{1}{2}\left(-1 + \sqrt{3}\ \tan u\right). \qquad \text{Then} \quad dx = \frac{1}{2}\sqrt{3}\ \sec^2 u\ du \quad \text{and} \quad \tan u = \frac{1}{3}\sqrt{3}\,(2x+1).$$

Thus

$$\int \frac{1}{x^2+x+1}\ dx = \int \frac{\frac{1}{2}\sqrt{3}\ \sec^2 u}{\frac{3}{4}\sec^2 u}\ du = \frac{2u\sqrt{3}}{3} + C = \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\,[2x+1]\right) + C.$$

Therefore

$$\int \frac{x^2+1}{x(x^2+x+1)}\ dx = \int \left(\frac{1}{x} - \frac{1}{x^2+x+1}\right) dx = \ln|x| - \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\,[2x+1]\right) + C.$$

C08S07.030: Let $x = \tan u$. Then $x^2+1 = 1+\tan^2 u = \sec^2 u$, $dx = \sec^2 u\ du$, and $u = \arctan x$. Hence

$$\begin{aligned} J &= \int \frac{x^2+2}{(x^2+1)^2}\ dx = \int \frac{1+\sec^2 u}{\sec^4 u}\cdot \sec^2 u\ du = \int \frac{1+\sec^2 u}{\sec^2 u}\ du = \int (1+\cos^2 u)\ du \\ &= \int \left(1 + \frac{1+\cos 2u}{2}\right) du = \int \left(\frac{3}{2} + \frac{1}{2}\cos 2u\right) du = \frac{3}{2}u + \frac{1}{4}\sin 2u + C = \frac{1}{2}(3u + \sin u\ \cos u) + C. \end{aligned}$$

A reference triangle with acute angle u, opposite side x, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+1}$. Therefore

$$J = \frac{1}{2}\left(3\arctan x + \frac{x}{x^2+1}\right) + C.$$

C08S07.031: Because $x^4 - 2x^2 + 1 = (x^2-1)^2$, we let $x = \sec\theta$. Then $(x^2-1)^2 = (\sec^2\theta - 1)^2 = \tan^4\theta$ and $dx = \sec\theta\ \tan\theta\ d\theta$. Thus

$$\begin{aligned} K &= \int \frac{2x^2+3}{x^4-2x^2+1}\ dx = \int \frac{3+2\sec^2\theta}{\tan^4\theta}\cdot \sec\theta\ \tan\theta\ d\theta = \int \left(\frac{3\sec\theta}{\tan^3\theta} + \frac{2\sec^3\theta}{\tan^3\theta}\right) d\theta \\ &= \int \left(\frac{3\cos^2\theta}{\sin^3\theta} + \frac{2}{\sin^3\theta}\right) d\theta = \int \left(\frac{3(1-\sin^2\theta)}{\sin^3\theta} + 2\csc^3\theta\right) d\theta = \int \left(5\csc^3\theta - 3\csc\theta\right) d\theta. \end{aligned}$$

Formulas 15 and 29 of the endpapers of the text now yield

$$\begin{aligned} K &= 5\left(-\frac{1}{2}\csc\theta\ \cot\theta + \frac{1}{2}\ln|\csc\theta - \cot\theta\,|\right) - 3\ln|\csc\theta - \cot\theta\,| + C \\ &= -\frac{5}{2}\csc\theta\ \cot\theta - \frac{1}{2}\ln|\csc\theta - \cot\theta\,| + C. \end{aligned}$$

A reference triangle with acute angle θ, adjacent side 1, and hypotenuse x has opposite side of length $\sqrt{x^2-1}$. Therefore

$$\begin{aligned} K &= -\frac{5}{2}\cdot\frac{x}{x^2-1} - \frac{1}{2}\ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + C \\ &= -\frac{5x}{2(x^2-1)} + \frac{1}{4}\ln\left|\frac{x^2-1}{(x-1)^2}\right| + C = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{5x}{2(x^2-1)} + C. \end{aligned}$$

C08S07.032: The partial fractions decomposition

$$\frac{x^2+4}{(x^2+1)^2(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+2}$$

leads to the equation

$$(Ax+B)(x^4+3x^2+2) + (Cx+D)(x^2+2) + (Ex+F)(x^4+2x+1) = x^2+4,$$

and thus to

$$A(x^5+3x^3+2x) + B(x^4+3x^2+2) + C(x^3+2x) + D(x^2+2) + E(x^5+2x^3+x) + F(x^4+2x+2+1) = x^2+4.$$

The resulting simultaneous equations are not merely six equations in six unknowns; they are two sets of three equations in three unknowns:

$$\begin{aligned} A+E&=0, & \qquad & & B+F&=0,\\ 3A+C+2E&=0, & \qquad &\text{and} & 3B+D+2F&=1,\\ 2A+2C+E&=0. & \qquad & & 2B+2D+F&=4. \end{aligned}$$

It follows easily that $A = C = E = 0$ and that $B = -2,\ D = 3,$ and $F = 2$. Thus

$$\frac{x^2+4}{(x^2+1)^2(x^2+2)} = \frac{-2}{x^2+1} + \frac{3}{(x^2+1)^2} + \frac{2}{x^2+2}.$$

The substitution $x = \tan u,\ x^2+1 = \sec^2 u,\ dx = \sec^2 u\ du$ then yields

$$\begin{aligned} \int \frac{3}{(x^2+1)^2}\,dx &= \int \frac{3\sec^2 u}{\sec^4 u}\,du = 3\int \cos^2 u\,du = 3\int \frac{1+\cos 2u}{2}\,du = 3\left(\frac{1}{2}u + \frac{1}{4}\sin 2u\right) + C\\ &= \frac{3}{2}(u + \sin u\,\cos u) + C. \end{aligned}$$

A reference triangle with acute angle u, opposite side x, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+1}$. Therefore

$$\int \frac{3}{(x^2+1)^2}\,dx = \frac{3}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C.$$

Next,

$$\int \frac{2}{x^2+2}\,dx = \int \frac{1}{\left(x/\sqrt{2}\right)^2+1}\,dx = \sqrt{2}\int \frac{1/\sqrt{2}}{1+\left(x/\sqrt{2}\,\right)^2}\,dx = \sqrt{2}\ \arctan\left(\frac{\sqrt{2}}{2}x\right) + C.$$

Finally assembling all these results, we have

$$\begin{aligned} \int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx &= -2\arctan x + \frac{3}{2}\arctan x + \frac{3x}{2(x^2+1)} + \sqrt{2}\ \arctan\left(\frac{\sqrt{2}}{2}x\right) + C\\ &= \frac{3x}{2(x^2+1)} - \frac{1}{2}\arctan x + \sqrt{2}\ \arctan\left(\frac{\sqrt{2}}{2}x\right) + C. \end{aligned}$$

C08S07.033: $x^2+2x+5=(x+1)^2+4=4\tan^2 u+4=4\sec^2 u$ if $2\tan u=x+1$, so we let $x=-1+\tan u$. Then $dx=2\sec^2 u\,du$ and $\tan u=\frac{1}{2}(x+1)$. Therefore

$$\begin{aligned}I&=\int\frac{3x+1}{(x^2+2x+5)^2}\,dx=\int\frac{-3+6\tan u+1}{16\sec^4 u}\cdot 2\sec^2 u\,du=\frac{1}{8}\int(-2+6\tan u)\cos^2 u\,du\\&=\frac{1}{8}\int(6\sin u\,\cos u\,-1-\cos 2u)\,du=\frac{1}{8}\,(3\sin^2 u\,-u-\sin u\,\cos u)+C.\end{aligned}$$

A reference triangle with acute angle u, opposite side $x+1$, and adjacent side 2 has hypotenuse of length $\sqrt{x^2+2x+5}$. Therefore

$$I=\frac{1}{8}\left[3\cdot\frac{(x+1)^2}{x^2+2x+5}-\arctan\left(\frac{x+1}{2}\right)-\frac{2(x+1)}{x^2+2x+5}\right]+C=\frac{3x^2+4x+1}{8(x^2+2x+5)}-\frac{1}{8}\arctan\left(\frac{x+1}{2}\right)+C.$$

C08S07.034: $x^2+2x+2=(x+1)^2+1=1+\tan^2 u=\sec^2 u$ if $x+1=\tan u$, so we let $x=-1+\tan u$. Then $dx=\sec^2 u\,du$ and $u=\arctan(x+1)$. But before substitution, we divide the denominator of the integrand into its numerator to find that

$$\frac{x^3-2x}{x^2+2x+2}=x-2+\frac{4}{x^2+2x+2}.$$

Then the trigonometric substitution given here yields

$$\int\frac{4}{x^2+2x+2}\,dx=\int\frac{4\sec^2 u}{\sec^2 u}\,du=4u+C_1=4\arctan(x+1)+C_1.$$

Therefore

$$\int\frac{x^3-2x}{x^2+2x+2}\,dx=\frac{1}{2}x^2-2x+4\arctan(x+1)+C.$$

C08S07.035: The substitution $u=a\tan\theta$ entails $a^2+u^2=a^2\sec^2\theta$ and $du=a\sec^2\theta\,d\theta$. Thereby we find that

$$\int\frac{1}{(a^2+u^2)^n}\,du=\int\frac{a\sec^2\theta}{(a^2\sec^2\theta)^n}\,d\theta=\int\frac{a\sec^2\theta}{a^{2n}(\sec\theta)^{2n}}\,d\theta=\frac{1}{a^{2n-1}}\int(\cos\theta)^{2n-2}\,d\theta.$$

C08S07.036: The substitution $u=a\sin\theta$, $a^2-u^2=a^2\cos^2\theta$, $du=a\cos\theta\,d\theta$ yields

$$\int\frac{1}{(a^2-u^2)^n}\,du=\int\frac{a\cos\theta}{(a^2\cos^2\theta)^n}\,d\theta=\int\frac{a\cos\theta}{a^{2n}(\cos\theta)^{2n}}\,d\theta=\frac{1}{a^{2n-1}}\int(\sec\theta)^{2n-1}\,d\theta.$$

C08S07.037: $x^2-2x+5=(x-1)^2+4=4+4\tan^2 u=4\sec^2 u$ if $2\tan u=x-1$, therefore we let $x=1+2\tan u$. Then $dx=2\sec^2 u\,du$ and $\tan u=\frac{1}{2}(x-1)$. Thus the area is

$$\begin{aligned}A&=\int_0^5\frac{1}{x^2-2x+5}\,dx=\int_{x=0}^5\frac{2\sec^2 u}{4\sec^2 u}\,du=\int_{x=0}^5\frac{1}{2}\,du=\left[\frac{1}{2}u\right]_{x=0}^5=\left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_0^5\\&=\frac{1}{2}\left[\arctan 2-\arctan\left(-\frac{1}{2}\right)\right]=\frac{1}{2}\left[\arctan 2+\arctan\left(\frac{1}{2}\right)\right]=\frac{1}{2}\left(\arctan 2+\frac{\pi}{2}-\arctan 2\right)=\frac{\pi}{4}.\end{aligned}$$

C08S07.038: We use the substitution developed in the solution of Problem 37. The volume obtaining by rotating the region around the y-axis is

$$V = \int_0^5 \frac{2\pi x}{x^2 - 2x + 5}\,dx = 2\pi \int_{x=0}^5 \frac{1 + 2\tan u}{4\sec^2 u} \cdot 2\sec^2 u\,du$$
$$= \pi \int_{x=0}^5 (1 + 2\tan u)\,du = \pi \Big[u - 2\ln|\cos u| \Big]_{x=0}^5.$$

A reference triangle with acute angle u, opposite side $x-1$, and adjacent side 2 has hypotenuse of length $\sqrt{x^2 - 2x + 5}$. Consequently

$$V = \pi \left[\arctan\left(\frac{x-1}{2}\right) + 2\ln\frac{\sqrt{x^2-2x+5}}{2} \right]_0^5$$
$$= \pi \left[\arctan 2 - \arctan\left(-\frac{1}{2}\right) + 2\ln\left(\sqrt{5}\right) - 2\ln\left(\frac{\sqrt{5}}{2}\right) \right]$$
$$= \pi \left[\frac{\pi}{2} + 2\ln\left(\frac{2\sqrt{5}}{\sqrt{5}}\right) \right] = \frac{\pi}{2}(\pi + 4\ln 2) \approx 9.2899743812.$$

C08S07.039: $x^2 - 2x + 5 = (x-1)^2 + 4 = 4 + 4\tan^2 u = 4\sec^2 u$ if $2\tan u = x - 1$, therefore we let $x = 1 + 2\tan u$. Then $dx = 2\sec^2 u\,du$ and $\tan u = \frac{1}{2}(x-1)$. Thus the volume of revolution around the x-axis is

$$V = \pi \int_0^5 \frac{1}{(x^2-2x+5)^2}\,dx = \pi \int_{x=0}^5 \frac{2\sec^2 u}{16\sec^4 u}\,du = \frac{\pi}{8}\int_{x=0}^5 \frac{1+\cos 2u}{2}\,du = \frac{\pi}{16}\Big[u + \sin u\,\cos u \Big]_{x=0}^5.$$

A reference triangle with acute angle u, opposite side $x-1$, and adjacent side 2 has hypotenuse of length $\sqrt{x^2 - 2x + 5}$. Therefore

$$V = \frac{\pi}{16}\left[\arctan\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5} \right]_0^5 = \frac{\pi}{16}\left[\arctan 2 - \arctan\left(-\frac{1}{2}\right) + \frac{8}{20} + \frac{2}{5} \right]$$
$$= \frac{\pi}{16}\left[\arctan 2 + \arctan\left(\frac{1}{2}\right) + \frac{4}{5} \right] = \frac{\pi}{16}\left(\frac{\pi}{2} + \frac{4}{5}\right) = \frac{5\pi^2 + 8\pi}{160} \approx 0.4655047702.$$

C08S07.040: $4x^2 - 20x + 29 = (2x-5)^2 + 4 = 4\tan^2 u + 4 = 4\sec^2 u$ if $2\tan u = 2x - 5$. Hence we let $x = \frac{5}{2} + \tan u$. Then $\tan u = \frac{1}{2}(2x-5)$ and $dx = \sec^2 u\,du$. Therefore the area of the region is

$$A = \int_1^4 \frac{1}{4x^2 - 20x + 29}\,dx = \int_{x=1}^4 \frac{1}{4\sec^2 u} \cdot \sec^2 u\,du = \left[\frac{1}{4}u \right]_{x=1}^4$$
$$= \left[\frac{1}{4}\arctan\left(\frac{2x-5}{2}\right) \right]_1^4 = \frac{1}{4}\left[\arctan\left(\frac{3}{2}\right) - \arctan\left(-\frac{3}{2}\right) \right] = \frac{1}{2}\arctan\left(\frac{3}{2}\right) \approx 0.4913968616.$$

C08S07.041: $4x^2 - 20x + 29 = (2x-5)^2 + 4 = 4\tan^2 u + 4 = 4\sec^2 u$ if $2\tan u = 2x - 5$. Hence we let $x = \frac{5}{2} + \tan u$. Then $\tan u = \frac{1}{2}(2x-5)$ and $dx = \sec^2 u\,du$. Therefore the volume generated by rotating the given region around the y-axis is

$$V = \int_1^4 \frac{2\pi x}{4x^2 - 20x + 29}\,dx = \pi \int_{x=1}^4 \frac{\frac{5}{2} + \tan u}{4\sec^2 u} \cdot \sec^2 u\,du = \frac{\pi}{2}\left[\frac{5}{2}u + \ln|\sec u|\right]_{x=1}^4.$$

A reference triangle with acute angle u, opposite side $2x - 5$, and adjacent side 2 has hypotenuse of length $\sqrt{4x^2 - 20x + 29}$. Therefore

$$V = \frac{\pi}{2}\left[\frac{5}{2}\arctan\left(\frac{2x-5}{2}\right) + \ln\left(\frac{\sqrt{4x^2 - 20x + 29}}{2}\right)\right]_1^4$$
$$= \frac{\pi}{2}\left[\frac{5}{2}\arctan\left(\frac{3}{2}\right) - \frac{5}{2}\arctan\left(-\frac{3}{2}\right) + \frac{1}{4}\ln(13) - \frac{1}{4}\ln(13)\right] = \frac{5\pi}{2}\arctan\left(\frac{3}{2}\right) \approx 7.7188438524.$$

C08S07.042: The substitution used in the solutions of Problems 40 and 41 yield the volume of revolution around the x-axis to be

$$V = \pi \int_1^4 \frac{1}{(4x^2 - 20x + 29)^2}\,dx = \pi \int_{x=1}^4 \frac{\sec^2 u}{16\sec^4 u}\,du$$
$$= \frac{\pi}{16}\int_{x=1}^4 \cos^2 u\,du = \frac{\pi}{32}\int_{x=1}^4 (1 + \cos 2u)\,du = \frac{\pi}{32}\Big[u + \sin u\,\cos u\Big]_{x=1}^4.$$

A reference triangle with acute angle u, opposite side $2x - 5$, and adjacent side 2 has hypotenuse of length $\sqrt{4x^2 - 20x + 29}$. Therefore

$$V = \frac{\pi}{32}\left[\arctan\left(\frac{2x-5}{2}\right) + \frac{4x - 10}{4x^2 - 20x + 29}\right]_1^4 = \frac{\pi}{32}\left[\arctan\left(\frac{3}{2}\right) - \arctan\left(-\frac{3}{2}\right) + \frac{12}{13}\right]$$
$$= \frac{\pi}{32}\left[2\arctan\left(\frac{3}{2}\right) + \frac{12}{13}\right] = \frac{\pi}{16}\left(\frac{6}{13} + \arctan\frac{3}{2}\right) \approx 0.2835939613.$$

C08S07.043: Given $(4x+4)^2 + (4y-19)^2 = 377$, implicit differentiation yields

$$8(4x+4) + 8(4y-19) = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = -\frac{4x+4}{4y-19}.$$

Thus

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(4x+4)^2}{(4y-19)^2} = \frac{(4x+4)^2 + (4y-19)^2}{377 - (4x+4)^2} = \frac{377}{377 - (4x+4)^2}.$$

Therefore the length of the road is

$$L = \int_0^3 \frac{\sqrt{377}}{\sqrt{377 - (4x+4)^2}}\,dx.$$

Now $377 - (4x+4)^2 = 377 - 16(x+1)^2 = 377 - 377\sin^2 u = 377\cos^2 u$ if $377\sin^2 u = 16(x+1)^2$. So we let $x = -1 + \frac{1}{4}\left(\sqrt{377}\,\sin u\right)$. Then $dx = \frac{1}{4}\sqrt{377}\,\cos u\,du$, and this substitution yields

$$L = \int_{x=0}^{3} \frac{\sqrt{377}}{\sqrt{377}\,\cos u} \cdot \frac{\sqrt{377}}{4} \cos u\, du = \int_{x=0}^{3} \frac{\sqrt{377}}{4}\, du = \left[\frac{\sqrt{377}}{4} u \right]_{x=0}^{3}$$

$$= \left[\frac{\sqrt{377}}{4} \arcsin\left(\frac{4x+4}{\sqrt{377}} \right) \right]_0^3 = \frac{\sqrt{377}}{4} \left[\left(\arcsin \frac{16}{\sqrt{377}} \right) - \arcsin\left(\frac{4}{\sqrt{377}} \right) \right].$$

It is easy to show that $\arcsin x - \arcsin y = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2} \right)$. Therefore (after a bit of arithmetic) you can also show that

$$L = \frac{\sqrt{377}}{4} \arcsin\left(\frac{260}{377} \right) \approx 3.6940487219 \quad \text{(miles)}.$$

For an alternative approach, the equation of the road may be put into the form

$$(x+1)^2 + \left(y - \tfrac{19}{4}\right)^2 = \frac{377}{16},$$

so the given curve joining $A(0,\,0)$ with $B(3,\,2)$ is an arc of a circle having center at $C\left(-1,\,\frac{19}{4}\right)$ and radius $\frac{1}{4}\sqrt{377}$. The straight line segment AB has length $\sqrt{13}$, so the law of cosines may be used to find the angle θ between the two radii CA and CB:

$$13 = \frac{377}{16} + \frac{377}{16} - 2 \cdot \frac{377}{16} \cos\theta.$$

Hence the length of the circular arc AB is simply the product of the radius of the circle with the angle θ (in radians):

$$\frac{\sqrt{377}}{4} \arccos\left(\frac{273}{377} \right) \approx 3.6940487219 \quad \text{(miles)}.$$

C08S07.044: If the integrand in the solution of Problem 43 is multiplied by $10/(1+x)$, then the resulting integral will give the total cost C of the road. Therefore

$$C = \int_0^3 \frac{10\sqrt{377}}{(x+1)\sqrt{377 - (4x+4)^2}}\, dx.$$

The substitution used in the solution of Problem 43 yields

$$C = \int_{x=0}^{3} \frac{10\sqrt{377}}{\frac{1}{4}\left(\sqrt{377}\,\sin u\right)\sqrt{377}\,\cos u} \cdot \frac{\sqrt{377}}{4} \cos u\, du = \int_{x=0}^{3} 10 \csc u\, du = 10 \Big[\ln|\csc u - \cot u| \Big]_{x=0}^{3}.$$

A reference triangle with acute angle u, opposite side $4(x+1)$, and hypotenuse $\sqrt{377}$ has adjacent side of length $\sqrt{377 - (4x+4)^2}$. Therefore

$$C = 10 \left[\ln\left(\frac{\sqrt{377} - \sqrt{377 - (4x+4)^2}}{4x+4} \right) \right]_0^3 = 10 \left(\ln \frac{\sqrt{377} - \sqrt{121}}{16} - \ln \frac{\sqrt{377} - \sqrt{361}}{4} \right)$$

$$= 10 \ln\left(\frac{\sqrt{377} - 11}{16} \cdot \frac{4}{\sqrt{377} - 19} \right) = 10 \ln \frac{\sqrt{377} - 11}{4\left(\sqrt{377} - 19\right)} \approx 16.197962748565 \quad \text{(million dollars)}.$$

For part (b), the straight road from (0, 0) to (3, 2) follows the graph of $y = \frac{2}{3}x$, for which

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4}{9} = \frac{13}{9}.$$

Thus the cost of the straight road will be

$$S = \int_0^3 \frac{\sqrt{13}}{3} \cdot \frac{10}{x+1}\,dx = \frac{10\sqrt{13}}{3}\left[\ln(x+1)\right]_0^3 = \frac{10\sqrt{13}}{3}\ln 4 = \frac{20\sqrt{13}}{3}\ln 2 \approx 16.661184673015$$

million dollars.

C08S07.045: The equation

$$\frac{3x+2}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}$$

leads to $A(x^2+2x+2) + B(x^2-x) + C(x-1) = 3x+12$, and thereby to

$$A + B = 0, \qquad 2A - B + C = 3, \qquad 2A - C = 2.$$

Thus

$$\frac{3x+2}{(x-1)(x^2+2x+2)} = \frac{1}{x-1} - \frac{x}{x^2+2x+2}.$$

Now $x^2 + 2x + 2 = (x+1)^2 + 1 = 1 + \tan^2\theta = \sec^2\theta$ provided that $x + 1 = \tan\theta$. Therefore we let $x = -1 + \tan\theta$, and thus $dx = \sec^2\theta\,d\theta$ and $\theta = \arctan(x+1)$. Hence

$$J = \int \frac{x}{(x+1)^2+1}\,dx = \int \frac{-1+\tan\theta}{\sec^2\theta}\cdot\sec^2\theta\,d\theta = -\theta + \ln|\sec\theta| + C.$$

A reference triangle with acute angle θ, opposite side $x+1$, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+2x+2}$, and therefore

$$J = -\arctan(x+1) + \ln\sqrt{x^2+2x+2} + C = \frac{1}{2}\ln(x^2+2x+2) - \arctan(x+1) + C.$$

In conclusion,

$$\int \frac{3x+2}{x^3+x^2-2}\,dx = \ln|x-1| - \frac{1}{2}(x^2+2x+2) + \arctan(x+1) + C.$$

C08S07.046: We begin with the partial fractions decomposition

$$\frac{1}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4},$$

which leads to $A(x^2-2x+4) + B(x^2+2x) + C(x+2) = 1$. Thus

$$A + B = 0, \qquad -2A + 2B + C = 0, \qquad \text{and} \qquad 4A + 2C = 1.$$

It follows that $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = \frac{1}{3}$. Therefore

$$\frac{1}{x^3+8} = \frac{1}{12}\left(\frac{1}{x+2} + \frac{-x+4}{x^2-2x+4}\right).$$

Now $x^2 - 2x + 4 = (x-1)^2 + 3 = 3\tan^2 u + 3 = 3\sec^2 u$ provided that $\sqrt{3}\,\tan u = x - 1$, so we let $x = 1 + \sqrt{3}\,\tan u$. Then $dx = \sqrt{3}\,\sec^2 u\,du$ and $\tan u = (x-1)/\sqrt{3}$. Thus

$$\int \frac{-x+4}{x^2-2x+4}\,dx = \int \frac{-1-\sqrt{3}\,\tan u + 4}{3\sec^2 u}\cdot\sqrt{3}\,\sec^2 u\,du = \int \frac{3-\sqrt{3}\,\tan u}{\sqrt{3}}\,du$$

$$= \int \left(\sqrt{3}\,-\tan u\right)du = u\sqrt{3}\,+\ln|\cos u| + C.$$

Therefore

$$\int \frac{1}{x^3+8}\,dx = \frac{1}{12}\left(\ln|x+2| + \sqrt{3}\,\arctan\frac{x-1}{\sqrt{3}} + \ln\left|\frac{\sqrt{3}}{\sqrt{x^2-2x+4}}\right|\right) + C$$

$$= \frac{1}{12}\left(\ln|x+2| + \sqrt{3}\,\arctan\left(\frac{\sqrt{3}}{3}[x-1]\right) + \frac{1}{2}\ln 3 - \frac{1}{2}\ln(x^2-2x+4)\right] + C$$

$$= \frac{1}{24}\left[2\ln|x+2| + 2\sqrt{3}\,\arctan\left(\frac{\sqrt{3}}{3}[x-1]\right) - \ln(x^2-2x+4)\right] + C_1.$$

C08S07.047: Division of denominator into numerator reveals that

$$\frac{x^4+2x^2}{x^3-1} = x + \frac{2x^2+x}{(x-1)(x^2+x+1)}. \tag{1}$$

The partial fraction decomposition of the last term in Eq. (1) has the form

$$\frac{2x^2+x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

and it follows that $A(x^2+x+1) + B(x^2-x) + C(x-1) = 2x^2+x$, and thus

$$A+B=2, \quad A-B+C=1, \quad \text{and} \quad A-C=0.$$

Therefore $A = B = C = 1$, and so

$$\frac{2x^2+x}{x^3-1} = \frac{1}{x-1} + \frac{x+1}{x^2+x+1}.$$

Now write

$$\frac{x+1}{x^2+x+1} = \frac{1}{2}\cdot\frac{2x+2}{x^2+x+1} = \frac{1}{2}\left(\frac{2x+1}{x^2+x+1} + \frac{1}{x^2+x+1}\right).$$

Then

$$K = \int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + r^2}\,dx$$

where $r = \frac{1}{2}\sqrt{3}$. Let $u = x + \frac{1}{2}$. Then use integral formula 17 from the endpapers of the text:

$$K = \int \frac{1}{u^2+r^2}\,du = \frac{1}{r}\arctan\left(\frac{u}{r}\right) + C = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

Finally put all this work together to obtain

$$J = \frac{1}{2}x^2 + \ln|x-1| + \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

C08S07.048: If

$$x^4+1 = (x^2+ax+1)(x^2+bx+1) = x^4+(a+b)x^3+(ab+2)x^2+(a+b)x+1,$$

then $a+b=0$ and $ab+2=0$, so $b=-a$ and $a^2=b^2=2$. Therefore $a=\pm\sqrt{2}$ and $b=\mp\sqrt{2}$. In either case,

$$x^4+1 = (x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1).$$

Thus we have the partial fractions decomposition

$$\frac{x^2+1}{x^4+1} = \frac{Ax+B}{x^2+x\sqrt{2}+1} + \frac{Cx+D}{x^2-x\sqrt{2}+1},$$

and it follows that

$$A(x^3+x^2\sqrt{2}+x) + B(x^2-x\sqrt{2}+1) + C(x^3+x\sqrt{2}+x) + D(x^2+x\sqrt{2}+1) = x^2+1.$$

Therefore

$$\begin{aligned} A+C=0, &\qquad -A\sqrt{2}+B+C\sqrt{2}+E=1,\\ A-B\sqrt{2}+C+D\sqrt{2}=0, &\qquad B+D=1. \end{aligned}$$

It follows that $A=C=0$ and $B=D=\frac{1}{2}$. Therefore

$$\frac{x^2+1}{x^4+1} = \frac{1}{2}\left(\frac{1}{x^2+x\sqrt{2}+1} + \frac{1}{x^2-x\sqrt{2}+1}\right).$$

Let $r=\frac{1}{2}\sqrt{2}$. Then

$$x^2+2rx+1 = (x+r)^2+1-r^2 = (x+r)^2+\frac{1}{2} = \frac{1}{2}\tan^2 u + \frac{1}{2} = \frac{1}{2}\sec^2 u = r^2\sec^2 u$$

if $x+r=r\tan u$. So we let $x=-r+r\tan u$. Then $dx = r\sec^2 u\,du$, and thus

$$\begin{aligned} \int \frac{1}{x^2+x\sqrt{2}+1}\,dx &= \int \frac{r\sec^2 u}{r^2\sec^2 u}\,du = \int \frac{1}{r}\,du = \frac{u}{r}+C_1\\ &= \frac{1}{r}\arctan\left(\frac{x+r}{r}\right)+C_1 = \sqrt{2}\,\arctan\left(x\sqrt{2}+1\right)+C_1. \end{aligned}$$

Next, $x^2-x\sqrt{2}+1 = (x-r)^2+1-r^2 = (x-r)^2+\frac{1}{2} = (x-r)^2+r^2 = r^2\tan^2 v + r^2 = r^2\sec^2 v$ if $x-r=r\tan v$, so we let $x=r+r\tan v$; $dx = r\sec^2 v\,dv$, so

$$\begin{aligned} \int \frac{1}{x^2-rx+1}\,dx &= \int \frac{r\sec^2 v}{r^2\sec^2 v}\,dv = \int \frac{1}{r}\,dv = \frac{v}{r}+C_2\\ &= \frac{1}{r}\arctan\left(\frac{x-r}{r}\right)+C_2 = \sqrt{2}\,\arctan\left(x\sqrt{2}\,+1\right)+C_2. \end{aligned}$$

Therefore

$$
\begin{aligned}
\int_0^1 \frac{x^2+1}{x^4+1}\,dx &= \frac{1}{2}\int_0^1 \left(\frac{1}{x^2+x\sqrt{2}+1}+\frac{1}{x^2-x\sqrt{2}+1}\right)dx \\
&= \frac{1}{2}\Big[\sqrt{2}\arctan\left(x\sqrt{2}\ +1\right)+\sqrt{2}\ \arctan\left(x\sqrt{2}\ -1\right)\Big]_0^1 \\
&= \frac{\sqrt{2}}{2}\Big[\arctan\left(\sqrt{2}+1\right)+\arctan\left(\sqrt{2}-1\right)-\arctan(1)-\arctan(-1)\Big] \\
&= \frac{\sqrt{2}}{2}\Big[\arctan\left(\sqrt{2}\ +1\right)+\arctan\left(\sqrt{2}\ -1\right)\Big] \\
&= \frac{\sqrt{2}}{2}\left[\arctan\left(\sqrt{2}\ +1\right)+\arctan\left(\frac{1}{\sqrt{2}\ +1}\right)\right] = \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} = \frac{\pi\sqrt{2}}{4} \approx 1.1107207345.
\end{aligned}
$$

C08S07.049: The trial factorization

$$x^4+x^2+1=(x^2+ax+1)(x^2+bx+1)=x^4+(a+b)x^3+(ab+2)x^2+(a+b)x+1$$

yields $a+b=0$ and $ab+2=1$. Hence $b=-a$ and $ab=-1$, so that $a=\pm1$ and $b=\mp1$. Either way,

$$x^4+x^2+1=(x^2+x+1)(x^2-x+1).$$

The partial fractions decomposition

$$\frac{2x^3+3x}{x^4+x^2+1}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

then yields

$$A(x^3-x^2+x)+B(x^2-x+1)+C(x^3+x^2+x)+D(x^2+x+1)=2x^3+3x,$$

and thus

$$
\begin{aligned}
A+C&=2, & -A+B+C+D&=0,\\
A-B+C+D&=3, & B+D&=0.
\end{aligned}
$$

These equations are easily solved for $A=C=1$, $B=-\frac{1}{2}$, and $D=\frac{1}{2}$. Therefore

$$\frac{2x^3+3x}{x^4+x^2+1}=\frac{1}{2}\left(\frac{2x-1}{x^2+x+1}+\frac{2x+1}{x^2-x+1}\right).$$

Next, $x^2+x+1=x^2+x+\frac{1}{4}+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\tan^2u+\frac{32}{4}=\frac{3}{4}\sec^2u$ if $x+\frac{1}{2}=\frac{1}{2}\sqrt{3}\ \tan u$. Therefore we let

$$x=-\frac{-1+\sqrt{3}\ \tan u}{2},\qquad \text{and so}\qquad dx=\frac{\sqrt{3}}{2}\sec^2u\,du\qquad\text{and}\qquad \tan u=\frac{2x+1}{\sqrt{3}}.$$

Thus

$$J_1 = \int \frac{2x-1}{x^2+x+1}\,dx = \int \frac{-1+\sqrt{3}\,\tan u - 1}{\frac{3}{4}\sec^2 u} \cdot \frac{\sqrt{3}}{2}\sec^2 u\,du$$

$$= \frac{2\sqrt{3}}{3}\int (-2+\sqrt{3}\,\tan u)\,du = -\frac{4\sqrt{3}}{3}u + 2\ln|\sec u| + C.$$

A reference triangle with acute angle u, opposite side $2x+1$, and adjacent side 3 has hypotenuse of length $2\sqrt{x^2+x+1}$. Therefore

$$J_1 = -\frac{4\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + 2\ln\left|\frac{2\sqrt{x^2+x+1}}{3}\right| + C = -\frac{4\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \ln(x^2+x+1) + C_1.$$

Similarly, $x^2 - x + 1 = x^2 - x + \frac{1}{4} + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2 v + \frac{3}{4} = \frac{3}{4}\sec^2 v$ if $x - \frac{1}{2} = \frac{1}{2}\sqrt{3}\,\tan v$. Therefore we let

$$x = \frac{1+\sqrt{3}\,\tan v}{2}, \qquad \text{so that} \qquad dx = \frac{\sqrt{3}}{2}\sec^2 v\,dv \qquad \text{and} \qquad \tan v = \frac{2x-1}{\sqrt{3}}.$$

Hence

$$J_2 = \int \frac{2x+1}{x^2-x+1}\,dx = \int \frac{1+\sqrt{3}\,\tan v + 1}{\frac{3}{4}\sec^2 v} \cdot \frac{\sqrt{3}}{2}\sec^2 v\,dv$$

$$= \frac{2\sqrt{3}}{3}\int (2+\sqrt{3}\,\tan v)\,dv = \frac{2\sqrt{3}}{3}\left(2v + \sqrt{3}\,\ln|\sec v|\right) + C = \frac{4\sqrt{3}}{3}v + 2\ln|\sec v| + C.$$

A reference triangle with acute angle v, opposite side $2x-1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{x^2-x+1}$. Therefore

$$J_2 = \frac{4\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \ln(x^2-x+1) + C_2.$$

Thus

$$\int \frac{2x^3+3x}{x^4+x^2+1}\,dx = \frac{1}{2}(J_1+J_2) + C$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2}\ln(x^2-x+1) - \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln(x^2+x+1) + C.$$

Using the result in Problem 64 of Section 7.5, the antiderivative can be further simplified to

$$\frac{1}{2}\ln(x^4+x^2+1) - \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{2x^2+1}\right) + C.$$

C08S07.050: The factorization

$$x^4 - 2x^3 + 4x - 4 = (x^2-2)(x^2-2x+2) = \left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)(x^2-2x+2)$$

yields the partial fractions decomposition

$$\frac{16(x-1)}{x^4-2x^3+4x-4} = \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}} + \frac{Cx+D}{x^2-2x+2} = \frac{2}{x-\sqrt{2}} + \frac{2}{x+\sqrt{2}} - \frac{4(x-2)}{x^2-2x+2}.$$

Next note that

$$\frac{x-2}{x^2-2x+2} = \frac{x-1}{x^2-2x+2} - \frac{1}{1+(x-1)^2}.$$

Therefore

$$\int \frac{4(x-2)}{x^2-2x+2}\,dx = 2\ln(x^2-2x+2) - 4\arctan(x-1) + C.$$

Consequently

$$\begin{aligned}
\int_0^1 \frac{16(x-1)}{x^4-2x^3+4x-4}\,dx &= \Big[2\ln\left(x-\sqrt{2}\right) + 2\ln\left(x+\sqrt{2}\right) - 2\ln(x^2-2x+2) + 4\arctan(x-1)\Big]_0^1 \\
&= \Big[2\ln|x^2-2| - 2\ln(x^2-2x+2) + 4\arctan(x-1)\Big]_0^1 \\
&= 2\ln 1 - 2\ln 2 - 2\ln 1 + 2\ln 2 + 4\arctan(0) - 4\arctan(-1) = -4\cdot\left(-\frac{\pi}{4}\right) = \pi.
\end{aligned}$$

C08S07.051: *Mathematica* 3.0 gives the partial fraction decomposition

$$\frac{7x^4+28x^3+50x^2+67x+23}{(x-1)(x^2+2x+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2} + \frac{Dx+E}{(x^2+2x+2)^2} = \frac{7}{x-1} - \frac{6x-5}{(x^2+2x+2)^2}.$$

Next, $x^2+2x+2 = (x+1)^2+1 = 1+\tan^2 u = \sec^2 u$ if $x+1 = \tan u$. Hence we let

$$x = -1+\tan u, \qquad \text{so that} \qquad dx = \sec^2 u\,du.$$

Then

$$\begin{aligned}
K = \int \frac{6x-5}{(x^2+2x+2)^2}\,dx &= \int \frac{-6+6\tan u-5}{\sec^4 u}\cdot\sec^2 u\,du \\
&= \int(-11\cos^2 u + 6\sin u\,\cos u)\,du = -\frac{11}{2}(u+\sin u\,\cos u) + 3\sin^2 u + C.
\end{aligned}$$

A reference triangle with acute angle u, opposite side $x+1$, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+2x+2}$. Therefore

$$K = -\frac{11}{2}\arctan(x+1) - \frac{11}{2}\cdot\frac{x+1}{x^2+2x+2} + 3\cdot\frac{(x+1)^2}{x^2+2x+2} + C.$$

Therefore

$$\int \frac{7x^4+28x^3+50x^2+67x+23}{(x-1)(x^2+2x+2)^2}\,dx = 7\ln|x-1| + \frac{11}{2}\arctan(x+1) + \frac{11(x+1)}{2(x^2+2x+2)} - \frac{3(x+1)^2}{x^2+2x+2} + C.$$

Mathematica 3.0 and *Maple* V version 5.1 both return the antiderivative in the form

$$7 \ln|x-1| + \frac{11}{2} \arctan(x+1) + \frac{11x+17}{2(x^2+2x+2)}.$$

Ignoring C, the difference between their answer and the first is

$$\frac{11(x+1)}{2(x^2+2x+2)} - \frac{3(x^2+2x+1)}{x^2+2x+2} - \frac{11x+17}{2(x^2+2x+2)} = -3,$$

a constant (of course).

C08S07.052: *Mathematica* 3.0 gives the partial fraction decomposition

$$\frac{35+84x+55x^2-x^3+5x^4-4x^5}{(x^2+1)^2(x^2+6x+10)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+6x+10} = \frac{7x+3}{(x^2+1)^2} - \frac{4x-5}{x^2+6x+10}.$$

Let $x = \tan u$. Then $dx = \sec^2 u\, du$ and $x^2+1 = \sec^2 u$. Thus

$$\begin{aligned} I_1 &= \int \frac{7x+3}{(x^2+1)^2}\, dx = \int \frac{7\tan u + 3}{\sec^4 u} \cdot \sec^2 u\, du \\ &= \int \left[7 \sin u \cos u + \frac{3}{2}(1+\cos 2u) \right] du = \frac{7}{2}\sin^2 u + \frac{3}{2}(u + \sin u \cos u) + C. \end{aligned}$$

A reference triangle with acute angle u, opposite side x, and adjacent side 1 has hypotenuse of length $\sqrt{1+x^2}$. Therefore

$$I_1 = \frac{7}{2} \cdot \frac{x^2}{1+x^2} + \frac{3}{2}\arctan x + \frac{3}{2} \cdot \frac{x}{1+x^2} + C_1.$$

Next, $x^2+6x+10 = (x+3)^2+1 = \tan^2 v + 1 = \sec^2 v$ if $x+3 = \tan v$. Hence we let

$$x = -3 + \tan v, \qquad \text{so that} \qquad dx = \sec^2 v\, dv.$$

Then

$$I_2 = \int \frac{4x-5}{x^2+6x+10}\, dx = \int \frac{-12+4\tan v - 5}{\sec^2 v} \cdot \sec^2 v\, dv = -17v + 4\ln|\sec v| + C.$$

A reference triangle with acute angle v, opposite side $x+3$, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+6x+10}$. Therefore

$$I_2 = -17\arctan(x+3) + 4\ln\sqrt{x^2+6x+10} + C_2 = -17\arctan(x+3) + 2\ln(x^2+6x+10) + C_2.$$

Therefore

$$\begin{aligned} &\int \frac{35+84x+55x^2-x^3+5x^4-4x^5}{(x^2+1)^2(x^2+6x+10)}\, dx = I_1 - I_2 \\ &\qquad = \frac{7x^2}{2(x^2+1)} + \frac{3}{2}\arctan x + \frac{3x}{2(x^2+1)} + 17\arctan(x+3) - 2\ln(x^2+6x+10) + C. \end{aligned}$$

Mathematica 3.0 returns for the antiderivative

$$\frac{3x}{2(x^2+1)} - \frac{7}{2(x^2+1)} + \frac{3}{2}\arctan x + 17\arctan(x+3) - 2\ln(x^2+6x+10).$$

Ignoring C, the difference between the two answers is

$$\frac{7x^2}{2(x^2+1)} - \frac{7}{2(x^2+1)} = \frac{7(x^2+1)}{2(x^2+1)} = \frac{7}{2},$$

a constant.

C08S07.053: Given:

$$h(x) = \frac{32x^5 + 16x^4 + 19x^3 - 98x^2 - 107x - 15}{(x^2 - 2x - 15)(4x^2 + 4x + 5)^2}.$$

First we factor: $x^2 - 2x - 15 = (x-5)(x+3)$. Then *Mathematica* 3.0 yields the partial fraction decomposition

$$h(x) = \frac{A}{x-5} + \frac{B}{x+3} + \frac{Cx+D}{4x^2+4x+5} + \frac{Ex+F}{(4x^2+4x+5)^2} = \frac{7}{8(x-5)} + \frac{9}{8(x+3)} + \frac{3x-4}{(4x^2+4x+5)^2}.$$

Then $4x^2 + 4x + 5 = (2x+1)^2 + 4 = 4\tan^2 u + 4 = 4\sec^2 u$ if $2\tan u = 2x+1$, so we let

$$x = \frac{-1 + 2\tan u}{2}. \qquad \text{Then} \qquad dx = \sec^2 u\, du \qquad \text{and} \qquad \tan u = \frac{2x+1}{2}.$$

Hence

$$\begin{aligned} K &= \int \frac{3x-4}{(4x^2+4x+5)^2}\, dx = \int \frac{-\frac{3}{2} + 3\tan u - 4}{16\sec^4 u} \cdot \sec^2 u\, du = \frac{1}{32}\int \frac{-11 + 6\tan u}{\sec^2 u}\, du \\ &= \frac{1}{32}\int \left[-\frac{11}{2}(1 + \cos 2u) + 6\sin u\, \cos u \right] du = \frac{1}{64}\big(-11u - 11\sin u\, \cos u + 6\sin^2 u \big) + C. \end{aligned}$$

A reference triangle with acute angle u, opposite side $2x+1$, and adjacent side 2 has hypotenuse of length $\sqrt{4x^2+4x+5}$. Therefore

$$\begin{aligned} K &= \frac{1}{64}\left[-11\arctan\left(\frac{2x+1}{2}\right) - 11 \cdot \frac{2(2x+1)}{4x^2+4x+5} + \frac{6(2x+1)^2}{4x^2+4x+5} \right] + C \\ &= -\frac{11}{64}\arctan\left(\frac{2x+1}{2}\right) - \frac{11}{32} \cdot \frac{2x+1}{4x^2+4x+5} + \frac{3}{32} \cdot \frac{(2x+1)^2}{4x^2+4x+5} + C. \end{aligned}$$

Therefore

$$\int h(x)\, dx = \frac{7}{8}\ln|x-5| + \frac{9}{8}\ln|x+3| - \frac{11}{64}\arctan\left(\frac{2x+1}{2}\right) - \frac{11}{32} \cdot \frac{2x+1}{4x^2+4x+5} + \frac{3}{32} \cdot \frac{(2x+1)^2}{4x^2+4x+5} + C.$$

Mathematica 3.0 returns for the antiderivative

$$\frac{7}{8}\ln|x-5| + \frac{9}{8}\ln|x+3| - \frac{11}{64}\arctan\left(\frac{2x+1}{2}\right) - \frac{22x+23}{32(4x^2+4x+5)}.$$

Ignoring C, the two answers differ by

$$\frac{22x+23}{32(4x^2+4x+5)} - \frac{22x+11}{32(4x^2+4x+5)} + \frac{3(4x^2+4x+1)}{32(4x^2+4x+5)} = \frac{12x^2+12x+15}{32(4x^2+4x+5)} = \frac{3}{32},$$

a constant.

C08S07.054: Given:

$$j(x) = \frac{63x^5 + 302x^4 + 480x^3 + 376x^2 - 240x - 300}{(x^2+6x+10)^2(4x^2+4x+5)^2}.$$

Mathematica 3.0 yields the partial fraction decomposition

$$\begin{aligned} j(x) &= \frac{Ax+B}{x^2+6x+10} + \frac{Cx+D}{(x^2+6x+10)^2} + \frac{Ex+F}{4x^2+4x+5} + \frac{Gx+H}{(4x^2+4x+5)^2} \\ &= \frac{4(x+3)}{(x^2+6x+10)^2} - \frac{x+6}{(4x^2+4x+5)^2}. \end{aligned}$$

Then

$$J_1 = \int \frac{4x+12}{(x^2+6x+10)^2}\,dx = 2\int \frac{2x+6}{(x^2+6x+10)^2}\,dx = -\frac{2}{x^2+6x+10} + C_1.$$

Next, as in the solution of Problem 53, we let

$$x = \frac{-1+2\tan v}{2}, \quad \text{so that} \quad dx = \sec^2 v\,dv, \quad dx = \sec^2 v\,dv, \quad \text{and} \quad 4x^2+4x+5 = 4\sec^2 v.$$

Then

$$\begin{aligned} J_2 &= \int \frac{x+6}{(4x^2+4x+5)^2}\,dx = \int \frac{-\frac{1}{2}+\tan v+6}{16\sec^4 v}\cdot \sec^2 v\,dv = \frac{1}{32}\int (11\cos^2 v + 2\sin v\,\cos v)\,dv \\ &= \frac{1}{64}\int \big[11(1+\cos 2v) + 4\sin v\,\cos v\big]\,dv = \frac{1}{64}\big(11v + 11\sin v\,\cos v + 2\sin^2 v\big) + C. \end{aligned}$$

A reference triangle with acute angle $v,$ opposite side $2x+1,$ and adjacent side 2 has hypotenuse of length $\sqrt{4x^2+4x+5}\,.$ Therefore

$$\begin{aligned} J_2 &= \frac{1}{64}\left[11\arctan\left(\frac{2x+1}{2}\right) + 11\cdot\frac{2(2x+1)}{4x^2+4x+5} + 2\cdot\frac{(2x+1)^2}{4x^2+4x+5}\right] + C \\ &= \frac{11}{64}\arctan\left(\frac{2x+1}{2}\right) + \frac{11}{32}\cdot\frac{2x+1}{4x^2+4x+5} + \frac{1}{32}\cdot\frac{(2x+1)^2}{4x^2+4x+5} + C. \end{aligned}$$

Thus

$$\begin{aligned} &\int j(x)\,dx = J_1 - J_2 \\ &\qquad = -\frac{2}{x^2+6x+10} - \frac{11}{64}\arctan\left(\frac{2x+1}{2}\right) - \frac{11}{32}\cdot\frac{2x+1}{4x^2+4x+5} - \frac{1}{32}\cdot\frac{(2x+1)^2}{4x^2+4x+5} + C. \end{aligned}$$

Mathematica 3.0 gives the antiderivative in essentially the form

$$-\frac{2}{10+6x+x^2} - \frac{22x+7}{32(5+4x+4x^2)} - \frac{11}{64}\arctan\left(\frac{2x+1}{2}\right).$$

Ignoring $C,$ the difference between the two answers is

$$-\frac{22x+7}{32(4x^2+4x+5)} + \frac{22x+11}{32(4x^2+4x+5)} + \frac{4x^2+4x+1}{32(4x^2+4x+5)} = \frac{4x^2+4x+5}{32(4x^2+4x+5)} = \frac{1}{32},$$

a constant.

C08S07.055: *Mathematica* 3.0 reports that the antiderivative is

$$\frac{2b-5a+(b-2a)x}{2(x^2+4x+5)}+\frac{(b-2a)\arctan(x+2)}{2}+C.$$

The inverse tangent term will disappear if $b=2a$. Hence choose $a \neq 0$ and $b=2a$; then the antiderivative will become

$$\frac{2b-5a}{2(x^2+4x+5)}+C \;=\; -\frac{a}{2(x^2+4x+5)}+C.$$

C08S07.056: *Mathematica* 3.0 reports that the antiderivative is

$$\frac{10a-5b+2c+(3a-2b+c)x}{2(x^2+4x+5)}+\frac{(5a-2b+c)\arctan(x+2)}{2}+C.$$

The inverse tangent term will disappear if $c=2b-5a$. Hence choose a and b arbitrary (but not both zero) and let $c=2b-5a$. The integrand will take the form

$$-\frac{2ax+b}{2(x^2+4x+5)}+C.$$

C07S07.057: *Mathematica* 3.0 reports that the antiderivative is

$$\begin{aligned}&2(b+c-4a)\arctan(x+1)+2(11a-4b+c)\arctan(x+2)\\&\qquad\qquad-\frac{2a-3b+2c}{10}\cdot\big[\ln(x^2+2x+2)-\ln(x^2+4x+5)\big]+C.\end{aligned}$$

In order for the inverse tangent and logarithmic terms to drop out, we impose the conditions

$$\begin{aligned}-4a+b+c &= 0;\\ 11a-4b+c &= 0;\\ 2a-3b+2c &= 0.\end{aligned}$$

Mathematica reports that the only solution is $a=b=c=0$. Thus there are *no* nonzero coefficients that yield a rational function as the antiderivative.

C08S07.058: *Mathematica* 3.0 reports that the antiderivative is

$$\begin{aligned}&\frac{1}{8}\left[\frac{-30a+20b-10c+4d+(-4a+6b-4c+2d)x}{(x^2+4x+5)^2}\right.\\&\qquad\left.+\frac{(13b-6c+3d)(x+2)-2a(32+15x)}{x^2+4x+5}+(-30a+13b-6c+3d)\arctan(x+2)\right]+C\end{aligned}$$

The inverse tangent term will drop out provided that

$$d=\frac{30a-13b+6c}{3}. \tag{1}$$

Hence choose $a,\ b,$ and c arbitrary but not all zero and let Eq. (1) determine d. Then the antiderivative will be a rational function. For example, if $a = b = c = 3$, then $d = 23$, and the antiderivative is

$$\frac{5x+4}{(x^2+4x+5)^2} - \frac{3}{2(x^2+4x+5)} + C.$$

Section 8.8

C08S08.001: The integral converges because

$$\int_2^\infty x^{-3/2}\,dx = \lim_{z\to\infty}\left[-2x^{-1/2}\right]_2^z = \frac{2}{\sqrt{2}} - \lim_{z\to\infty}\frac{2}{\sqrt{z}} = \sqrt{2}.$$

Computer algebra programs generally have little difficulty with improper integrals. The *Maple* V version 5.1 command

```
int(x∧(−3/2), x=2..infinity);
```

and the *Mathematica* 3.0 command

```
Integrate[ x∧(−3/2), { x, 2, Infinity } ];
```

both immediately produce the response $\sqrt{2}$. In *Derive* 2.56 (which is menu-driven), the commands

`Author` [Enter]
`x∧(−3/2)` [Enter]
`Calculus` [Enter]
`Integrate` [Enter]
`Lower Limit` 2 `Upper Limit` `inf` [Enter]

produce the display

$$\int_2^\infty x^{-3/2}\,dx;$$

then the command `Evaluate` produces the immediate result $\sqrt{2}$.

C08S08.002: This improper integral diverges:

$$\int_1^\infty x^{-2/3}\,dx = \lim_{z\to\infty}\left[3x^{1/3}\right]_1^z = -3 + \lim_{z\to\infty} 3z^{1/3} = +\infty.$$

C08S08.003: Divergent: $\displaystyle\int_0^4 x^{-3/2}\,dx = \lim_{z\to0^+}\left[2x^{-1/2}\right]_z^4 = -\frac{2}{\sqrt{4}} + \lim_{z\to0^+}\frac{2}{\sqrt{z}} = +\infty.$

C08S08.004: Convergent: $\displaystyle\int_0^8 x^{-2/3}\,dx = \lim_{z\to0^+}\left[3x^{1/3}\right]_z^8 = 6 - 0 = 6.$

C08S08.005: Divergent: $\displaystyle\int_1^\infty \frac{1}{x+1}\,dx = \lim_{z\to\infty}\left[\ln(x+1)\right]_1^z = -\ln 2 + \lim_{z\to\infty}\ln(z+1) = +\infty.$

C08S08.006: Divergent: $\displaystyle\int_3^\infty (x+1)^{-1/2}\,dx = \lim_{z\to\infty}\left[2\sqrt{x+1}\right]_3^z = -4 + \lim_{z\to\infty}\sqrt{x+1} = +\infty.$

C08S08.007: Convergent: $\displaystyle\int_5^\infty (x-1)^{-3/2}\,dx = \lim_{z\to\infty}\left[-2(x-1)^{-1/2}\right]_5^\infty = \frac{2}{\sqrt{4}} - \lim_{z\to\infty}\frac{2}{\sqrt{z-1}} = 1.$

C08S08.008: Convergent: $\displaystyle\int_0^4 (4-x)^{-1/2}\,dx = \lim_{z\to 4^-}\Big[-2(4-x)^{1/2}\Big]_0^z = 2\cdot\sqrt{4} - 2\cdot\sqrt{0} = 4.$

C08S08.009: Divergent: $\displaystyle\int_0^9 (9-x)^{-3/2}\,dx = \lim_{z\to 9^-}\Big[2(9-x)^{-1/2}\Big]_0^z = -\frac{2}{\sqrt{9}} + \lim_{z\to 9^-}\frac{2}{\sqrt{9-z}} = +\infty.$

C08S08.010: Divergent: $\displaystyle\int_0^3 \frac{1}{(x-3)^2}\,dx = \lim_{z\to 3^-}\left[\frac{1}{3-x}\right]_0^z = +\infty.$

C08S08.011: Convergent: $\displaystyle\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx = \lim_{z\to -\infty}\left[-\frac{1}{2(x+1)^2}\right]_z^{-2} = -\frac{1}{2} + \lim_{z\to-\infty}\frac{1}{2(z+1)^2} = -\frac{1}{2}.$

C08S08.012: Divergent: $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx = \lim_{Z\to -\infty}\Big[-2\sqrt{4-x}\,\Big]_z^{0} = -2\cdot\sqrt{4} + \lim_{z\to-\infty} 2\sqrt{4-z} = +\infty.$

C08S08.013: Convergent: $\displaystyle\int_{-1}^{8} x^{-1/3}\,dx = \lim_{z\to 8^-}\left[\frac{3}{2}x^{2/3}\right]_{-1}^{z} = -\frac{3}{2} + \lim_{z\to 8^-}\left(\frac{3}{2}z^{2/3}\right) = -\frac{3}{2} + 6 = \frac{9}{2}.$

C08S08.014: Convergent: $\displaystyle\int_{-4}^{4} \frac{1}{(x+4)^{2/3}}\,dx = \lim_{z\to -4^+}\Big[3(x+4)^{1/3}\Big]_z^{4} = 6 - 0 = 6.$

C08S08.015: Divergent: $\displaystyle\int_{2}^{\infty} (x-1)^{-1/3}\,dx = \lim_{z\to\infty}\left[\frac{3}{2}(x-1)^{2/3}\right]_2^{z} = +\infty.$

C08S08.016: Convergent:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x}{(x^2+4)^{3/2}}\,dx &= \int_{-\infty}^{0} x(x^2+4)^{-3/2}\,dx + \int_0^{\infty} x(x^2+4)^{-3/2}\,dx\\ &= \lim_{z\to-\infty}\left[-\frac{1}{\sqrt{x^2+4}}\right]_z^0 + \lim_{w\to\infty}\left[-\frac{1}{\sqrt{x^2+4}}\right]_0^w = -\frac{1}{2} + 0 - 0 + \frac{1}{2} = 0.\end{aligned}$$

C08S08.017: Divergent:

$$\int_{-\infty}^{\infty}\frac{x}{x^2+4}\,dx = \int_{-\infty}^{0}\frac{x}{x^2+4}\,dx + \int_0^{\infty}\frac{x}{x^2+4}\,dx = \lim_{z\to-\infty}\left[\frac{1}{2}\ln(x^2+4)\right]_z^0 + \lim_{w\to\infty}\left[\frac{1}{2}\ln(x^2+4)\right]_0^w,$$

and neither of the last two limits exists.

C08S08.018: Convergent: $\displaystyle\int_0^{\infty} e^{-(x+1)}\,dx = \lim_{z\to\infty}\Big[-e^{-(x+1)}\Big]_0^z = 0 + e^{-1} = \frac{1}{e}.$

C08S08.019: Convergent: $\displaystyle\int_0^1 \frac{\exp\left(\sqrt{x}\,\right)}{\sqrt{x}}\,dx = \lim_{z\to 0^+}\Big[2\exp\left(\sqrt{x}\,\right)\Big]_z^1 = 2e - 2 = 2(e-1).$

C08S08.020: Divergent:

$$\int_0^2\frac{x}{x^2-1}\,dx = \int_0^1\frac{x}{x^2-1}\,dx + \int_1^2\frac{x}{x^2-1}\,dx = \lim_{z\to 1^-}\left[\frac{1}{2}\ln|x^2-1|\right]_0^z + \lim_{w\to 1^+}\left[\frac{1}{2}\ln|x^2-1|\right]_w^2,$$

and neither of the last two limits exists (both are $-\infty$).

C08S08.021: Convergent: $\displaystyle\int_0^\infty xe^{-3x}\,dx = \lim_{z\to\infty}\left[-\frac{3x+1}{9}e^{-3x}\right]_0^z = \frac{1}{9}$. To find the antiderivative:

$$\text{Let}\quad u = x \quad\text{and}\quad dv = e^{-3x}\,dx.$$

$$\text{Then}\quad du = dx \quad\text{and}\quad v = -\frac{1}{3}e^{-3x}.\quad \text{Hence}$$

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C = -\frac{3x+1}{9}e^{-3x} + C.$$

C08S08.022: Convergent: $\displaystyle\int_{-\infty}^2 e^{2x}\,dx = \lim_{z\to-\infty}\left[\frac{1}{2}e^{2x}\right]_z^2 = \frac{1}{2}e^4 - 0 = \frac{1}{2}e^4 \approx 27.2990750166.$

C08S08.023: Convergent: $\displaystyle\int_0^\infty x\exp\left(-x^2\right)\,dx = \lim_{z\to\infty}\left[-\frac{1}{2}\exp\left(-x^2\right)\right]_0^z = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$

C08S08.024: Given: $\displaystyle I = \int_{-\infty}^\infty |x|\exp\left(-x^2\right)\,dx = \int_{-\infty}^0 (-x)\exp\left(-x^2\right)\,dx + \int_0^\infty x\exp\left(-x^2\right)\,dx.$

Now substitute $u = -x,\ x = -u,$ and $dx = -du$ in the second of the three integrals, noting that $-\infty < x \leqq 0$ corresponds to $0 \leqq u < +\infty$. Thus

$$\begin{aligned} I &= \int_\infty^0 (-u)\exp\left(-u^2\right)\,du + \int_0^\infty x\exp\left(-x^2\right)\,dx \\ &= \int_0^\infty u\exp\left(-u^2\right)\,du + \int_0^\infty x\exp\left(-x^2\right)\,dx = 2\int_0^\infty x\exp\left(-x^2\right)\,dx = 2\cdot\frac{1}{2} = 1 \end{aligned}$$

by the result in Problem 23. Thus the improper integral I converges.

C08S08.025: Convergent: $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{z\to\infty}\Big[\arctan x\Big]_0^z = \lim_{z\to\infty}\arctan z = \frac{\pi}{2}.$

C08S08.026: Divergent: $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{z\to\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_0^z = \lim_{z\to\infty}\frac{1}{2}\ln(1+z^2) = +\infty.$

C08S08.027: $\displaystyle\int_0^{2n\pi}\cos x\,dx = 0$ if n is a positive integer, but

$$\int_0^{2n\pi+(\pi/2)}\cos x\,dx = \Big[\sin x\Big]_0^{2n\pi+(\pi/2)} = 1.$$

Therefore $\displaystyle\lim_{z\to\infty}\int_0^z \cos x\,dx$ does not exist; this improper integral diverges.

C08S08.028: If n is a positive integer, then

$$\int_0^{2n\pi}\sin^2 x\,dx = \int_0^{2n\pi}\frac{1-\cos 2x}{2}\,dx = \left[\frac{1}{2}(x - \sin x\,\cos x)\right]_0^{2n\pi} = n\pi.$$

Therefore the given improper integral diverges because $\displaystyle\lim_{z\to\infty}\int_0^z \sin^2 x\,dx = +\infty.$

C08S08.029: Divergent: $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{z\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^z = \lim_{z\to\infty}\frac{1}{2}(\ln z)^2 = +\infty.$

(But the divergence is relatively slow: $\displaystyle\int_1^{10^9} \frac{\ln x}{x}\,dx \approx 214.7268734744.$)

C08S08.030: Divergent: $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{z\to\infty}\Big[\ln(\ln x)\Big]_2^z = +\infty.$

(But the divergence is extremely slow: $\displaystyle\int_2^{10^9} \frac{1}{x\ln x}\,dx \approx 3.3977699432.$)

C08S08.031: Convergent: $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{z\to\infty}\left[-\frac{1}{\ln x}\right]_2^z = \frac{1}{\ln 2}.$

C08S08.032: Convergent: $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{z\to\infty}\left[-\frac{1+\ln x}{x}\right]_1^z = 1-\left(\lim_{z\to\infty}\frac{1+\ln z}{z}\right) = 1-\lim_{z\to\infty}\frac{1}{z} = 1.$

The next-to-last equality results from an application of l'Hôpital's rule. The antiderivative was obtained by integration by parts:

$$\text{Let}\quad u = \ln x \qquad \text{and} \quad dv = \frac{1}{x^2}\,dx;$$

$$\text{then}\quad du = \frac{1}{x}\,dx \qquad \text{and} \quad v = -\frac{1}{x}.$$

Therefore $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x} - \frac{1}{x}\ln x + C.$

C08S08.033: Convergent: $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{z\to 0^+}\Big[2\sqrt{\sin x}\Big]_z^{\pi/2} = 2-0=2.$

C08S08.034: Divergent: $\displaystyle\int_0^{\pi/2} \frac{\sin x}{(\cos x)^{3/2}}\,dx = \lim_{z\to\pi/2^-}\left[\frac{3}{(\cos x)^{1/3}}\right]_0^z = +\infty.$

C08S08.035: First note that—by l'Hôpital's rule—

$$\lim_{z\to 0^+} z\ln z = \lim_{z\to 0^+}\frac{\ln z}{\dfrac{1}{z}} = \lim_{z\to 0^+}\frac{\dfrac{1}{z}}{-\dfrac{1}{z^2}} = \lim_{z\to 0^+}(-z) = 0.$$

Therefore the given improper integral converges because

$$\int_0^1 \ln x\,dx = \lim_{z\to 0^+}\Big[-x + x\ln x\Big]_z^1 = -1 + \lim_{z\to 0^+}(z - z\ln z) = -1+0-0 = -1.$$

See Example 1 of Section 8.3 for the computation of the antiderivative.

C08S08.036: Divergent: $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx = \lim_{z\to 0^+}\left[\frac{1}{2}(\ln x)^2\right]_z^1 = -\left[\lim_{z\to 0^+}\frac{1}{2}(\ln z)^2\right] = -\infty.$

C08S08.037: Let $u = \ln x$ and $dv = \dfrac{1}{x^2}\,dx$. Then $du = \dfrac{1}{x}\,dx$; choose $v = -\dfrac{1}{x}$. Hence

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

Therefore the given improper integral diverges because

$$\int_0^1 \frac{\ln x}{x^2}\,dx = \lim_{z\to 0^+}\left[-\frac{1}{x}(1+\ln x)\right]_z^1 = -1 + \left(\lim_{z\to 0^+}\frac{1+\ln z}{z}\right) = -\infty.$$

(If you use l'Hôpital's rule on the last limit, you'll get the *wrong answer!*)

C08S08.038: Let $u = e^{-x}$ and $dv = \cos x\,dx$. Then $du = -e^{-x}\,dx$; choose $v = \sin x$. Then

$$K = \int e^{-x}\cos x\,dx = e^{-x}\sin x + \int e^{-x}\sin x\,dx.$$

Now let $u = e^{-x}$ and $dv = \sin x\,dx$. Then $du = -e^{-x}\,dx$; choose $v = -\cos x$. Therefore

$$\begin{aligned} K &= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx;\\ 2K &= e^{-x}(\sin x - \cos x) + 2C;\\ K &= \int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + C. \end{aligned}$$

Therefore the given improper integral converges because

$$\int_0^\infty e^{-x}\cos x\,dx = \lim_{z\to\infty}\left[\frac{1}{2}e^{-x}(\sin x - \cos x)\right]_0^z = \frac{1}{2} + \left[\lim_{z\to\infty}\frac{e^{-z}(\sin z - \cos z)}{2}\right] = \frac{1}{2}.$$

C08S08.039: The first improper integral diverges because

$$\int_0^1 \frac{1}{x+x^2}\,dx = \int_0^1\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \lim_{z\to 0^+}\left[\ln\left|\frac{x}{x+1}\right|\right]_z^1 = -\ln 2 - \left(\lim_{z\to 0^+}\ln\frac{z}{z+1}\right) = +\infty.$$

The second integral converges because

$$\int_1^\infty \frac{1}{x+x^2}\,dx = -\ln\frac{1}{2} + \left(\lim_{z\to\infty}\ln\frac{z}{z+1}\right) = 0 + \ln 2 = \ln 2.$$

C08S08.040: First, the method of partial fractions yields

$$\frac{1}{x^2+x^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{1}{x^2} - \frac{1}{x^2+1}.$$

The first improper integral diverges because

$$\int_0^1 \frac{1}{x^2+x^4}\,dx = \lim_{z\to 0^+}\left[-\frac{1}{x} - \arctan x\right]_z^1 = +\infty.$$

The second integral converges because

$$\int_1^\infty \frac{1}{x^2+x^4}\,dx = \lim_{z\to\infty}\left[-\frac{1}{x} - \arctan x\right]_1^z = -\frac{\pi}{2} + 1 + \frac{\pi}{4} = \frac{4-\pi}{4} \approx 0.2146018366.$$

C08S08.041: Let $x = u^2$. Then $dx = 2u\ du$, and thus

$$\int \frac{1}{x^{1/2}+x^{3/2}}\ dx = \int \frac{2u}{x+x^2}\ du = \int \frac{2}{1+u^2}\ du = 2\arctan u + C = 2\arctan\sqrt{x}\ + C.$$

Hence both improper integrals converge, because

$$\int_0^1 \frac{1}{x^{1/2}+x^{3/2}}\ dx = \lim_{z\to 0^+}\Big[2\arctan\sqrt{x}\Big]_z^1 = 2\cdot\frac{\pi}{4} - 2\cdot 0 = \frac{\pi}{2}$$

and

$$\int_1^\infty \frac{1}{x^{1/2}+x^{3/2}}\ dx = \lim_{z\to\infty}\Big[2\arctan\sqrt{x}\Big]_1^z = 2\cdot\frac{\pi}{2} - 2\cdot\frac{\pi}{4} = \frac{\pi}{2}.$$

C08S08.042: Let $x = u^3$; then $dx = 3u^2\ du$. Hence

$$\int \frac{1}{x^{2/3}+x^{4/3}}\ dx = \int \frac{3u^2}{u^2+u^4}\ du = \int \frac{3}{1+u^2}\ du = 3\arctan u\ + C = 3\arctan\left(x^{1/3}\right) + C.$$

Therefore

$$\int_0^1 \frac{1}{x^{2/3}+x^{4/3}}\ dx = \lim_{z\to 0^+}\left[3\arctan\left(x^{1/3}\right)\right]_z^1 = 3\arctan(1) - 3\arctan(0) = \frac{3\pi}{4}$$

and

$$\int_1^\infty \frac{1}{x^{2/3}+x^{4/3}}\ dx = \lim_{z\to\infty}\left[3\arctan\left(x^{1/3}\right)\right]_1^z = \frac{3\pi}{2} - \frac{3\pi}{4} = \frac{3\pi}{4}.$$

Thus both improper integrals converge.

C08S08.043: If $k = 1$, then

$$\int_0^1 \frac{1}{x^k}\ dx = \lim_{z\to 0^+}\Big[\ln x\Big]_z^1 = +\infty.$$

If $k \neq 1$, then

$$\int_0^1 x^{-k}\ dx = \lim_{x\to 0^+}\left[\frac{x^{1-k}}{1-k}\right]_z^1 = \frac{1}{1-k} - \left(\lim_{z\to 0^+}\frac{z^{1-k}}{1-k}\right) = \begin{cases} \dfrac{1}{1-k} & \text{if } k < 1, \\ +\infty & \text{if } k > 1. \end{cases}$$

Therefore the given improper integral converges precisely when $k < 1$.

C08S08.044: If $k = 1$, then

$$\int_1^\infty \frac{1}{x^k}\ dx = \lim_{z\to\infty}\Big[\ln x\Big]_1^z = +\infty.$$

If $k \neq 1$, then

$$\int_1^\infty x^{-k}\,dx = \lim_{z\to\infty}\left[\frac{x^{1-k}}{1-k}\right]_1^z = -\frac{1}{1-k} + \left(\lim_{z\to\infty}\frac{z^{1-k}}{1-k}\right) = \begin{cases} +\infty & \text{if } k < 1, \\ \dfrac{1}{k-1} & \text{if } k > 1. \end{cases}$$

Therefore the given improper integral converges exactly when $k > 1$.

C08S08.045: If $k = -1$, then

$$\int_0^1 x^k \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{z\to 0^+}\left[\frac{1}{2}(\ln x)^2\right]_z^1 = -\left[\lim_{z\to 0^+}\frac{1}{2}(\ln z)^2\right] = -\infty.$$

If $k \neq -1$, then use integration by parts:

$$\text{Let}\quad u = \ln x \qquad \text{and}\quad dv = x^k\,dx;$$

$$\text{then}\quad du = \frac{1}{x}\,dx \qquad \text{and}\quad v = \frac{x^{k+1}}{k+1}.$$

Hence

$$\begin{aligned}\int x^k \ln x\,dx &= \frac{x^{k+1}\ln x}{k+1} - \int \frac{x^k}{k+1}\,dx \\ &= \frac{x^{k+1}\ln x}{k+1} - \frac{x^{k+1}}{(k+1)^2} + C = \frac{x^{k+1}}{(k+1)^2}\left[(k+1)(\ln x)\,-1\right] + C.\end{aligned}$$

Therefore if $k \neq -1$, then

$$\begin{aligned}I = \int_0^1 x^k \ln x\,dx &= \lim_{z\to 0^+}\left[\frac{x^{k+1}}{(k+1)^2}\left\{(k+1)(\ln x)\,-1\right\}\right]_z^1 \\ &= -\frac{1}{(k+1)^2} - \lim_{z\to 0^+}\left[\frac{z^{k+1}}{(k+1)^2}\left\{(k+1)(\ln z)\,-1\right\}\right].\end{aligned}$$

Suppose first that $k < -1$. Write $k = -1-\epsilon$ where $\epsilon > 0$. Then

$$I = \int_0^1 \frac{\ln x}{x^{1+\epsilon}}\,dx.$$

But $0 < x \leqq 1$, so $x^{1+\epsilon} = x\cdot x^\epsilon \leqq x$. Therefore

$$\frac{\ln x}{x^{1+\epsilon}} \leqq \frac{\ln x}{x}.$$

Therefore I diverges to $-\infty$ if $k \leqq -1$.

Now suppose that $k > -1$. Recall that

$$I = -\frac{1}{(k+1)^2} - \lim_{z\to 0^+}\left[\frac{z^{k+1}}{(k+1)^2}\left\{(k+1)(\ln z)\,-1\right\}\right].$$

Now

$$\lim_{z\to 0^+} z^{k+1}\ln z = \lim_{z\to 0^+} z^{\epsilon}\ln z \qquad (\text{where } \epsilon > 0)$$

$$= \lim_{z\to 0^+} \frac{\ln z}{z^{-\epsilon}} = \lim_{z\to 0^+} \frac{1}{-\epsilon z^{-\epsilon-1} z} = -\frac{1}{\epsilon}\left(\lim_{z\to 0^+} \frac{1}{z^{-\epsilon}}\right) = -\left(\lim_{z\to 0^+} \frac{z^{\epsilon}}{\epsilon}\right) = 0.$$

Also,

$$\lim_{z\to 0^+} z^{k+1} = \lim_{z\to 0^+} z^{\epsilon} = 0.$$

Therefore $I = -\dfrac{1}{(k+1)^2}$ if $k > -1$.

Consequently $\displaystyle\int_0^1 x^k \ln x\,dx$ converges exactly when $k > -1$, and its value for such k is $-\dfrac{1}{(k+1)^2}$.

C08S08.046: Let $f(k) = \displaystyle\int_1^{\infty} \frac{1}{x(\ln x)^k}\,dx$. Then

$$f(-1) = \int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{z\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^z = \lim_{z\to\infty} \frac{1}{2}(\ln z)^2 = +\infty.$$

If $k < -1$, then write $k = -1-\epsilon$ where $\epsilon > 0$. Then

$$f(k) = f(-1-\epsilon) = \int_1^{\infty} \frac{(\ln x)^{1+\epsilon}}{x}\,dx \geqq \int_1^{\infty} \frac{\ln x}{x}\,dx = +\infty.$$

Therefore the given improper integral diverges to $+\infty$ if $k \leqq -1$.

If $k > -1$, then $-k < 1$. So

$$f(k) = \int_1^{\infty} \frac{(\ln x)^{-k}}{x}\,dx = \lim_{z\to\infty}\left[\frac{(\ln x)^{-k+1}}{-k+1}\right]_1^z = \lim_{z\to\infty} \frac{(\ln z)^{1-k}}{1-k}. \tag{1}$$

If $k < 1$, then $-1 < k < 1$, so $0 < 1-k < 2$. Therefore the integral in Eq. (1) diverges to $+\infty$ in this case. If $k = 1$, then

$$f(1) = \int_1^{\infty} \frac{1}{x\ln x}\,dx = \Big[\ln(\ln x)\Big]_1^{\infty} = +\infty.$$

Finally, if $k > 1$, then $1-k < 0$. Therefore, by Eq. (1),

$$f(k) = \lim_{z\to\infty}\left[\frac{(\ln x)^{1-k}}{1-k}\right]_1^z = \lim_{z\to\infty} \frac{1}{(1-k)(\ln z)^{k-1}} - \lim_{w\to 1^+} \frac{1}{(1-k)(\ln w)^{k-1}} = +\infty.$$

In conclusion, $\displaystyle\int_1^{\infty} \frac{1}{x(\ln x)^k}\,dx$ diverges to $+\infty$ for *all* real numbers k.

C08S08.047: Given: $\Gamma(t) = \displaystyle\int_0^{\infty} x^{t-1}e^{-x}\,dx$ for $t > 0$. Thus

$$\Gamma(t+1) = \int_0^{\infty} x^t e^{-x}\,dx.$$

Let $u = x^t$ and $dv = e^{-x}\,dx$. Then $du = tx^{t-1}\,dx$ and $v = -e^{-x}$. Thus

$$\Gamma(t+1) = \Big[-(x^t e^{-x})\Big]_{x=0}^{\infty} + t\int_0^{\infty} x^{t-1}e^{-x}\,dx = -\left(\lim_{z\to\infty}\frac{z^t}{e^z}\right) + 0 + t\Gamma(t) = t\Gamma(t).$$

To evaluate the last limit, we used the result in Problem 61 of Section 7.2.

C08S08.048: Prove that $\Gamma(n+1) = n!$ for every positive integer n.

Proof: By Example 5 of Section 8.8, $\Gamma(1+1) = \Gamma(2) = 1!$, so the theorem holds when $n = 1$. Assume that $\Gamma(k+1) = k!$ for some integer $k \geqq 1$. Then

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = k!(k+1) = (k+1)!.$$

Thus whenever the theorem holds for the positive integer k, it also holds for $k+1$. Therefore, by induction, $\Gamma(n+1) = n!$ for every positive integer n. ◀

C08S08.049: The area is

$$A = \int_1^{\infty}\frac{1}{x}\,dx = \lim_{z\to\infty}\Big[\ln x\Big]_1^z = \lim_{z\to\infty}\ln z = +\infty.$$

C08S08.050: The volume is

$$V = \int_1^{\infty}\frac{\pi}{x^2}\,dx = \lim_{z\to\infty}\left[-\frac{\pi}{x}\right]_1^z = \pi - \left(\lim_{z\to\infty}\frac{\pi}{z}\right) = \pi.$$

C08S08.051: Because $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$, we have arc length element

$$ds = \left(1 + \frac{1}{x^4}\right)^{1/2}dx = \frac{\sqrt{x^4+1}}{x^2}\,dx.$$

Therefore the area of the surface of Gabriel's horn satisfies the inequality

$$S = \int_1^{\infty} 2\pi\cdot\frac{1}{x}\cdot\frac{\sqrt{x^4+1}}{x^2}\,dx = 2\pi\left(\lim_{z\to\infty}\int_1^z\frac{\sqrt{x^4+1}}{x^3}\,dx\right)$$

$$\geqq 2\pi\left(\lim_{z\to\infty}\int_1^z\frac{\sqrt{x^4}}{x^3}\,dx\right) = 2\pi\left(\lim_{z\to\infty}\Big[\ln x\Big]_1^z\right) = +\infty.$$

C08S08.052: First,

$$\int_0^{\infty}\frac{1+x}{1+x^2}\,dx = \lim_{z\to\infty}\left[\arctan x + \frac{1}{2}\ln(1+x^2)\right]_0^z = \lim_{z\to\infty}\left[\arctan z + \frac{1}{2}\ln(1+z^2)\right] = +\infty.$$

Therefore $\displaystyle\int_{-\infty}^{\infty}\frac{1+x}{1+x^2}\,dx$ diverges. But

$$\lim_{t\to\infty}\int_{-t}^{t}\frac{1+x}{1+x^2}\,dx = \lim_{t\to\infty}\left[\arctan x + \frac{1}{2}\ln(1+x^2)\right]_{-t}^{t}$$

$$= \lim_{t\to\infty}\left[\arctan t - \arctan(-t) + \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln(1+t^2)\right] = \lim_{t\to\infty}2\arctan t = \pi.$$

This technique of assigning a "plausible" (or, perhaps, "balanced") value to a divergent improper integral of the form

$$\int_{-\infty}^{\infty} f(x)\,dx \tag{1}$$

is the evaluation of the so-called *Cauchy principal value* of the integral in Eq. (1).

C08S08.053: We will use the definition

$$\Gamma(t) = \int_{x=0}^{\infty} x^{t-1}e^{-x}\,dx$$

for $t > 0$ and the result of Problem 48 to the effect that $\Gamma(n+1) = n!$ if n is a positive integer. For fixed nonnegative integers m and n, let

$$J(m,\,n) = \int_0^1 x^m(\ln x)^n\,dx.$$

Then

$$J(m,\,0) = \int_0^1 x^m\,dx = \frac{1}{m+1} = \frac{0!(-1)^0}{(m+1)^1} = \frac{n!(-1)^n}{(m+1)^{n+1}}$$

where $n = 0$. Therefore

$$J(m,\,n) = \frac{n!(-1)^n}{(m+1)^{n+1}}$$

if $n = 0$ and $m \geqq 0$. Assume that

$$J(m,\,k) = \frac{k!(-1)^k}{(m+1)^{k+1}}$$

for some integer $k \geqq 0$ and all integers $m \geqq 0$. Then

$$J(m,\,k+1) = \int_0^1 x^m(\ln x)^{k+1}\,dx.$$

Let $u = (\ln x)^{k+1}$ and $dv = x^m\,dx$. Then

$$du = \frac{(k+1)(\ln x)^k}{x}\,dx \qquad \text{and} \qquad v = \frac{x^{m+1}}{m+1}.$$

Therefore

$$J(m,\,k+1) = \left[\frac{(\ln x)^{k+1}x^{m+1}}{m+1}\right]_0^1 - \frac{k+1}{m+1}\int_0^1 x^m(\ln x)^k\,dx.$$

The value of the evaluation bracket is zero because

$$\lim_{x\to 0^+} \frac{(\ln x)^{k+1}}{\dfrac{1}{x^{m+1}}} = \left(\lim_{x\to 0^+} \frac{\ln x}{x^{(m+1)/(k+1)}}\right)^{k+1} = \left(\lim_{x\to 0^+} \frac{k+1}{(m+1)x^{(m+1)/(k+1)}}\right)^{k+1} = 0^{k+1} = 0.$$

Therefore

$$J(m,\,k+1) = -\frac{k+1}{m+1}\cdot J(m,\,k) = -\frac{k+1}{m+1}\cdot\frac{k!(-1)^k}{(m+1)^{k+1}} = \frac{(-1)^{k+1}(k+1)!}{(m+1)^{k+2}}.$$

Therefore, by induction,

$$\int_0^1 x^m(\ln x)^n\,dx = \frac{n!(-1)^n}{(m+1)^{n+1}}$$

for all positive integers m and n.

C08S08.054: As we saw in Section 8.8, the present value of $10+t$ thousand dollars t years in the future is $(10+t)e^{-t/10}$ thousand dollars if the interest rate is 10%. So the total present value of the perpetuity is

$$P = \int_0^\infty (10+t)e^{-t/10}\,dt.$$

Let $u = 10+t$ and $dv = e^{-t/10}\,dt$. Then $du = dt$ and we may choose $v = -10e^{-t/10}$. Thus

$$P = \Big[-10(10+t)e^{-t/10}\Big]_0^\infty + 10\int_0^\infty e^{-t/10}\,dt = 100 - 100\Big[e^{-t/10}\Big]_0^\infty = 200;$$

that is, \$200,000.

C08S08.055: We assume that $a > 0$. A short segment of the rod "at" position $x \geqq 0$ and of length dx has mass $\delta\,dx$, and thereby exerts on m the force

$$\frac{Gm\delta}{(x+a)^2}\,dx.$$

Therefore the total force exerted by the rod on m is

$$F = \int_0^\infty \frac{Gm\delta}{(x+a)^2}\,dx = \left[-\frac{Gm\delta}{x+a}\right]_0^\infty = \frac{Gm\delta}{a}.$$

What if $a = 0$? Then the total force is

$$F = \int_0^\infty \frac{Gm\delta}{x^2}\,dx = \left[-\frac{Gm\delta}{x}\right]_0^\infty = \lim_{z\to 0^+}\frac{Gm\delta}{z} = +\infty.$$

You will obtain the same result if $a < 0$.

C08S08.056: A small segment of the rod "at" location y and with length dy has mass $\delta\,dy$, so exerts on m the force

$$\frac{Gm\delta}{a^2+y^2}\,dy.$$

The vertical components of all such forces cancel (perhaps the reason that Cauchy developed the idea of the principal value of certain improper integrals—see the solution to Problem 52), so the total force exerted by the rod on the mass m is the sum (i.e., integral) of the horizontal components of such forces:

$$F = \int_{y=-\infty}^\infty \frac{Gm\delta\cos\theta}{a^2+y^2}\,dy = Gm\delta\int_{-\infty}^\infty \frac{a}{(a^2+y^2)^{3/2}}\,dy = 2Gma\delta\int_0^\infty \frac{1}{(a^2+y^2)^{3/2}}\,dy.$$

Then a trigonometric substitution, or integral formula 52 of the endpapers, yields

$$F = 2Gma\delta\left[\frac{y}{a^2\sqrt{a^2+y^2}}\right]_0^\infty = \frac{2Gm\delta}{a}\left(\lim_{y\to\infty}\frac{y}{\sqrt{a^2+y^2}}\right) = \frac{2Gm\delta}{a}\left(\lim_{y\to\infty}\frac{1}{\sqrt{1+(a/y)^2}}\right) = \frac{2Gm\delta}{a}.$$

C08S08.057: In the integral

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2}e^{-x}\,dx$$

we substitute $x = u^2$, so that $dx = 2u\,du$. Then

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{u}\cdot e^{-u^2}\cdot 2u\,du = 2\int_0^\infty e^{-u^2}\,du = 2\int_0^\infty e^{-x^2}\,dx.$$

C08S08.058: We will use $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. The volume of revolution around the x-axis is

$$V = \int_0^\infty \pi\left[\exp\left(-x^2\right)\right]^2\,dx = \pi\int_0^\infty \exp\left(-2x^2\right)\,dx.$$

Let $u = x\sqrt{2}$, so that $x = u/\sqrt{2}$ and $dx = \left(1/\sqrt{2}\,\right)\,du$. Then

$$V = \pi\int_0^\infty \frac{\sqrt{2}}{2}\exp\left(-u^2\right)\,du = \frac{\pi\sqrt{2}}{2}\cdot\frac{\sqrt{\pi}}{2} = \frac{\pi^{3/2}\sqrt{2}}{4} \approx 1.968701243.$$

C08S08.059: The volume of revolution around the y-axis is

$$V = \int_0^\infty 2\pi x\exp\left(-x^2\right)\,dx = \pi\Big[-\exp\left(-x^2\right)\Big]_0^\infty = \pi.$$

C08S08.060: We will use the result of Problem 47, $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, and Eq. (9), which tells us that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. If $n = 1$, then

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^n}\sqrt{\pi}.$$

So the desired result holds when $n = 1$. Assume that

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{1\cdot3\cdot5\cdots(2k-1)}{2^k}\sqrt{\pi}$$

for some integer $k \geqq 1$. Then

$$\begin{aligned}\Gamma\left(k+1+\frac{1}{2}\right) &= \left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\\ &= \frac{2k+1}{2}\cdot\frac{1\cdot3\cdot5\cdots(2k-1)}{2^k}\sqrt{\pi} = \frac{1\cdot3\cdot5\cdots(2k-1)\cdot(2k+1)}{2^{k+1}}\sqrt{\pi}.\end{aligned}$$

Therefore, by induction, the desired result holds for every integer $n \geqq 1$.

C08S08.061: Part (a): Suppose that $k > 1$ and let

$$I = \int_0^\infty x^k \exp\left(-x^2\right)\,dx.$$

Let $u = x^{k-1}$ and $dv = x\exp\left(-x^2\right)\,dx$. Then

$$du = (k-1)x^{k-2}\,dx \qquad \text{and} \qquad v = -\frac{1}{2}\exp\left(-x^2\right).$$

Therefore

$$I = \left[-\frac{x^{k-1}}{2}\exp\left(-x^2\right)\right]_0^\infty + \frac{k-1}{2}\int_0^\infty x^{k-2}\exp\left(-x^2\right)\,dx.$$

The evaluation bracket is zero by Problem 62 of Section 7.2. This concludes the proof in part (a).

Part (b): Now suppose that n is a positive integer. If $n = 1$, then

$$\int_0^\infty x^{n-1}\exp\left(-x^2\right)\,dx = \int_0^\infty \exp\left(-x^2\right)\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

by Eq. (9) of the text. Assume that for some integer $k \geqq 1$,

$$\int_0^\infty x^{k-1}\exp\left(-x^2\right)\,dx = \frac{1}{2}\Gamma\left(\frac{k}{2}\right)$$

and *in addition* that

$$\int_0^\infty x^{m-1}\exp\left(-x^2\right)\,dx = \frac{1}{2}\Gamma\left(\frac{m}{2}\right)$$

for every integer m such that $1 \leqq m \leqq k$. Then

$$\begin{aligned}\int_0^\infty x^k\exp\left(-x^2\right)\,dx &= \frac{k-1}{2}\int_0^\infty x^{k-2}\exp\left(-x^2\right)\,dx\\ &= \frac{k-1}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{k-1}{2}\right) = \frac{1}{2}\Gamma\left(1+\frac{k-1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).\end{aligned}$$

Therefore, by induction,

$$\int_0^\infty x^{n-1}\exp\left(-x^2\right)\,dx = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

for every positive integer n.

C08S08.062: Answer:

$$P = \int_0^\infty 10000e^{-3t/50}\,dt = -\frac{500000}{3}\left[e^{-3t/50}\right]_0^\infty = \frac{500000}{3};$$

that is, \$166,666.67.

C08S08.063: Substitute $t = \dfrac{x}{\sqrt{2}}$ in the given interval. This routinely gives Eq. (10).

C08S08.064: Begin with

$$\operatorname{erf}\left(\frac{u}{\sqrt{2}}\right)$$

as given in Eq. (15), then use the substitution of the previous solution. This leads routinely to Eq. (16).

C08S08.065: We defined

$$I_b = \int_0^b x^5 e^{-x}\,dx$$

and asked *Mathematica* 3.0 to evaluate I_b for increasing large positive values of b. The results:

b	I_b (approximately)
10	111.949684454516
20	119.991370939137
30	119.999997291182
40	119.999999999505
50	119.999999999999933
60	119.99999999999999999258

(Results were essentially the same with *Maple* V version 5.1.) It seems very likely that

$$k = \lim_{b\to\infty} \frac{1}{60} I_b = 2.$$

Indeed, because

$$\int x^5 e^{-x}\,dx = -(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)e^{-x} + C,$$

it follows that

$$I_b = 120 - \frac{b^5 + 5b^4 + 20b^3 + 60b^2 + 120b + 12}{e^b} \to 120$$

as $b \to +\infty$, and this proves that $k = 2$.

C08S08.066: We defined

$$k(b) = \frac{\pi}{\displaystyle\int_0^b \frac{\sin x}{x}\,dx} \tag{1}$$

and asked *Mathematica* 3.0 to evaluate $k(b)$ for various increasing large values of b. Here are the results:

b	$k(b)$ (approximately)
10	1.8944114397618596

20	2.0291357941544347
40	1.9795980537404538
80	1.9980480443011534
160	1.9922776386463579
320	2.0035967916498463
640	2.0012584221020270
1280	1.9998026067119984
2560	1.9995416983267274
5120	2.0001739448256683
10240	1.9999973452978946

As in the previous problem, it seems likely that $k = 2$. Unlike the previous problem, it appears that $k(b)$ oscillates around the limiting value $k = 2$ as $b \to +\infty$. This is plausible given the oscillatory behavior of the denominator in Eq. (1). Methods of Laplace transforms can be used to prove that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

(exactly), and—given that result—this proves that $k = 2$.

C08S08.067: We defined

$$k_b = \frac{\pi}{\left(\sqrt{2}\right) \cdot \displaystyle\int_0^b \frac{1}{x^2+2}\,dx}$$

and asked *Mathematica* 3.0 to evaluate k_b for various large increasing values of b. Here are the results:

b	k_b (approximately)
10	2.1964469705919963
20	2.0941114933683164
40	2.0460327426191207
80	2.0227515961720504
160	2.0113173416565956
320	2.0056428162376127
640	2.0028174473301223
1280	2.0014077338319199
2560	2.0007036195035935

5120	2.0003517479044710
10240	2.0001758584911264

There is good evidence that $k = 2$. In fact, because

$$I_b = \int_0^b \frac{1}{x^2+2}\,dx = \left[\frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\right]_0^b = \frac{1}{\sqrt{2}}\arctan\left(\frac{b}{\sqrt{2}}\right),$$

it follows that

$$\lim_{b\to\infty}\frac{\sqrt{2}}{\pi}I_b = \frac{\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{2}}\cdot\frac{\pi}{2} = \frac{1}{2},$$

and therefore $k = 2$ exactly.

C08S08.068: Because

$$\int_0^\infty \frac{1-e^{-3x}}{x}\,dx \geqq \int_0^\infty \frac{1}{x}\,dx \geqq \int_1^\infty \frac{1}{x}\,dx = \Big[\ln x\Big]_1^\infty = +\infty,$$

no such integer k can exist.

C08S08.069: We defined

$$k(b) = \frac{\pi}{e\cdot\displaystyle\int_0^b \exp\left(-x^2\right)\cos 2x\,dx}$$

and asked *Mathematica* 3.0 to evaluate $k(b)$ for some large positive values of b. The results:

$$k(10) = k(100) \approx 3.5449077018110320545963349666 8229$$

to the number of decimal places shown. We conclude that there is no such integer k.

C08S08.070: We defined

$$k(b) = \frac{\pi}{\left(\sqrt{2}\right)\cdot\displaystyle\int_0^\infty \sin(x^2)\,dx}$$

and asked *Mathematica* 3.0 to approximate $k(b)$ for some large positive values of b. The results:

$$k(10^2) \approx 1.9849201201159858,$$

$$k(10^6) \approx 2.0000012629663674, \quad \text{and}$$

$$k(10^9) \approx 2.0000000001888944.$$

It seems plausible that $k = 2$. Can you *prove* it?

C08S08.071: In the notation of this section, we have $\mu = 100$ and $\sigma = 15$. For part (a), we let $a = 10/\sigma$. Then

$$P = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \operatorname{erf}\left(\frac{\sqrt{2}}{3}\right) \approx 0.4950149249.$$

Thus just under 50% of students have IQs between 90 and 110. For part (b), we let $a = 25/\sigma$. Then

$$P = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{5}{3\sqrt{2}}\right)\right] \approx 0.0477903523.$$

Thus just under 5% of students have IQs of 125 or higher.

C08S08.072: Let $\mu = 69$ and $\sigma = 3$. For part (a), we let

$$a = -\frac{2}{\sigma} = -\frac{2}{3} \quad \text{and} \quad b = \frac{3}{\sigma} = 1.$$

Then we compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}}{3}\right) \approx 0.5888522085.$$

Thus just under 59% of adult males are between $5'\ 7''$ and $6'$. Part (b): Let $a = 7/\sigma = 7/3$. Then

$$P = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{7}{3\sqrt{2}}\right)\right] \approx 0.0098153286.$$

Therefore just under 1% of adult males are $6'\ 4''$ or taller.

C08S08.073: Here we take $p = q = 1/2$ and $N = 1000$; we have $\mu = Np = 450$ and $\sigma = \sqrt{Npq} = 15$. So we let $a = 25/\sigma = 5/3$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \operatorname{erf}\left(\frac{5}{3\sqrt{2}}\right) \approx 0.9044192954.$$

Thus there is over a 90% probability of 425 to 475 heads. For part (b), we let $a = 50/\sigma = 10/3$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{5\sqrt{2}}{3}\right)\right] \approx 0.0004290603.$$

Thus there is only a probability of 0.04%—les than one chance in 2000—of 500 or more heads.

C08S08.074: We are given $p = 3/5$, $q = 2/5$, and $N = 600$. Then $\mu = Np = 360$ and $\sigma = \sqrt{Npq} = 12$. Part (a): We take $a = 15/\sigma = 5/4$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \operatorname{erf}\left(\frac{5}{4\sqrt{2}}\right) \approx 0.7887004527.$$

Thus the probability of obtaining 345 to 375 heads is just under 0.79 (sometimes called a 79% probability). For part (b), we let $b = 10/15$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-b} \exp\left(-\tfrac{1}{2}x^2\right)\,dx = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{\sqrt{2}}{3}\right)\right] \approx 0.2524925375.$$

Thus there is just over a 25% probability of 350 or fewer heads.

C08S08.075: Here we have $p = q = 1/2$ and $N = 50$. Then $\mu = Np = 25$ and $\sigma = \sqrt{Npq} = 5/\sqrt{2}$. Part (a): We let $a = 5/\sigma = \sqrt{2}$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} \exp\left(-\tfrac{1}{2}x^2\right) dx = \frac{1}{2}\left[1 - \operatorname{erf}(1)\right] \approx 0.0786496035.$$

Thus there is slightly more than a 1 in 13 chance of passing by pure guessing. Part (b): We let $a = 10/\sigma$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} \exp\left(-\tfrac{1}{2}x^2\right) dx = \frac{1}{2}\left[1 - \operatorname{erf}(2)\right] \approx 0.0023388675.$$

Thus there is less than 1 chance in 425 of making a C by pure guessing.

C08S08.076: Here we take $p = 0.99, q = 0.01$, and $N = 500$. In the notation of Section 7.8, $\mu = NP = 495$ and $\sigma = \sqrt{Npq} = \frac{3}{10}\sqrt{55} \approx 2.22486$. We let $a = 5/\sigma = \frac{10}{33}\sqrt{55} \approx 2.24733$ and compute

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} \exp\left(-\tfrac{1}{2}x^2\right) dx = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{5}{33}\sqrt{10}\right)\right] \approx 0.0123093807.$$

Therefore there is only about one chance in 81 that ten or more are defective in a batch of 500.

C08S08.077: Let $p = 0.55$, $q = 0.45$, and $N = 750$. In the notation of Section 7.8, we have $\mu = Np = 412.5$ and $\sigma = \sqrt{Npq} = \frac{3}{4}\sqrt{330} \approx 13.62443$. Now 59% of the $N = 750$ voters amounts to 442.5, so we let

$$a = \frac{442.5 - \mu}{\sigma} = \frac{4}{33}\sqrt{330} \approx 2.20183$$

and evaluate

$$P = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp\left(-\tfrac{1}{2}x^2\right) dx = \operatorname{erf}\left(\frac{4}{33}\sqrt{165}\right) \approx 0.9723295720.$$

Thus there is a 97.23% probability that between 51% and 59% will say that they are Democratic voters, and thus that between 41% and 49% will say that they are Republican voters.

C08S08.078: For each integer $n \geqq 0$, let

$$I_n = \int_1^{\infty} \frac{(\ln x)^n}{x^2}\, dx.$$

Then

$$I_0 = \int_1^{\infty} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1 = 0!,$$

and therefore $I_n = n!$ if $n = 0$. Next, suppose that $I_k = k!$ for some integer $k \geqq 0$. Then

$$I_{k+1} = \int_1^{\infty} \frac{(\ln x)^{k+1}}{x^2}\, dx.$$

Integrate by parts: Let

$$u = (\ln x)^{k+1}, \qquad dv = \frac{1}{x^2}\, dx.$$

$$\text{Then} \quad du = \frac{(k+1)(\ln x)^k}{x}\, dx, \qquad v = -\frac{1}{x}.$$

Thus

$$I_{k+1} \;=\; \left[-\frac{(\ln x)^{k+1}}{x}\right]_1^{\infty} + (k+1)\int_1^{\infty} \frac{(\ln x)^k}{x^2}\,dx \;=\; 0 + (k+1)I_k = (k+1)I_k = (k+1)(k!) = (k+1)!.$$

Therefore, by induction, $I_n = n!$ for each integer $n \geqq 0$.

Chapter 8 Miscellaneous Problems

C08S0M.001: The substitution $x = u^2,\ dx = 2u\ du$ yields

$$\int \frac{1}{(1+x)\sqrt{x}}\ dx = \int \frac{2u}{(1+u^2)u}\ du = 2\arctan\sqrt{x} + C.$$

C08S0M.002: Substitute $u = 1 + \tan t$ or simply $u = \tan t$, if necessary. But clearly

$$\int \frac{\sec^2 t}{1+\tan t}\ dt = \ln|1 + \tan t| + C.$$

C08S0M.003: $\displaystyle\int \sin x\ \sec x\ dx = \int \frac{\sin x}{\cos x}\ dx = -\ln|\cos x| + C = \ln|\sec x| + C.$

C08S0M.004: $\displaystyle\int \frac{\csc x\ \cot x}{1+\csc^2 x}\ dx = -\arctan(\csc x) + C = \frac{\pi}{2} - \arctan(\csc x) + C_1 = \arctan(\sin x) + C_1.$

C08S0M.005: $\displaystyle\int \frac{\tan\theta}{\cos^2\theta}\ d\theta = \int (\cos\theta)^{-3}\sin\theta\ d\theta = \frac{1}{2}(\cos\theta)^{-2} + C = \frac{1}{2}\sec^2\theta + C.$

C08S0M.006: $\displaystyle\int \csc^4 x\ dx = \int \left(\csc^2 x + \cot^2 x\ \csc^2 x\right)\ dx = -\cot x - \frac{1}{3}\cot^3 x + C.$

C08S0M.007: Let $u = x$ and $dv = \tan^2 x\ dx = (\sec^2 x - 1)\,dx$. Then $du = dx$ and $v = -x + \tan x$. Thus

$$\begin{aligned}\int x\tan^2 x\ dx &= x\tan x - x^2 + \int (x - \tan x)\ dx\\ &= x\tan x - x^2 + \frac{1}{2}x^2 + \ln|\cos x| + C = x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C.\end{aligned}$$

C08S0M.008: Let $u = x^2,\ dv = \cos^2 x\ dx = \dfrac{1+\cos 2x}{2}\ dx$. Then $du = 2x\ dx$ and $v = \dfrac{1}{2}x + \dfrac{1}{2}\sin x\ \cos x$. Thus

$$\begin{aligned}I = \int x^2\cos^2 x\ dx &= \frac{1}{2}x^3 + \frac{1}{2}x^2\sin x\ \cos x - \int (x^2 + x\sin x\ \cos x)\,dx\\ &= \frac{1}{2}x^3 - \frac{1}{3}x^3 + \frac{1}{2}x^2\sin x\ \cos x - \int x\sin x\ \cos x\ dx.\end{aligned}$$

Now let $u = x,\ dv = \sin x\ \cos x\ dx$. Then $du = dx$ and $v = \dfrac{1}{2}\sin^2 x$. Therefore

$$\begin{aligned}I &= \frac{1}{6}x^3 + \frac{1}{2}x^2\sin x\ \cos x - \frac{1}{2}x\sin^2 x + \int \frac{1}{2}\sin^2 x\ dx\\ &= \frac{1}{6}x^3 + \frac{1}{2}x^2\sin x\ \cos x - \frac{1}{2}x\sin^2 x + \frac{1}{4}\int (1 - \cos 2x)\ dx\\ &= \frac{1}{6}x^3 + \frac{1}{2}x^2\sin x\ \cos x - \frac{1}{2}x\sin^2 x + \frac{1}{4}x - \frac{1}{4}\sin x\ \cos x + C.\end{aligned}$$

C08S0M.009: Let $u = x^3$ and $dv = x^2(2 - x^3)^{1/2}\ dx$. Then $du = 3x^2\ dx$ and $v = -\dfrac{2}{9}(2 - x^3)^{3/2}$. So

$$\int x^5(2-x^3)^{1/2}\,dx = -\frac{2}{9}x^3(2-x^3)^{3/2} + \frac{2}{3}\int x^2(2-x^3)^{3/2}\,dx = -\frac{2}{9}x^3(2-x^3)^{3/2} - \frac{4}{45}(2-x^3)^{5/2} + C.$$

C08S0M.010: Let $x = 2\tan u$. Then $x^2 + 4 = 4 + 4\tan^2 u = 4\sec^2 u$ and $dx = 2\sec^2 u\,du$. Hence

$$J = \int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2 u}{2\sec u}\,du = \ln|\sec u + \tan u| + C.$$

A reference triangle with acute angle u, opposite side x, and adjacent side 2 has hypotenuse of length $\sqrt{x^2+4}$. Therefore

$$J = \ln\left(\frac{x+\sqrt{x^2+4}}{2}\right) + C = \ln\left(x+\sqrt{x^2+4}\right) + C_1.$$

C08S0M.011: Let $x = 5\tan u$. Then $25 + x^2 = 25 + 25\tan^2 u = 25\sec^2 u$ and $dx = 5\sec^2 u\,du$. Thus

$$\begin{aligned} K = \int \frac{x^2}{\sqrt{25+x^2}}\,dx &= \int \frac{125\tan^2 u\,\sec^2 u}{5\sec u}\,du \\ &= 25\int(\sec^3 u - \sec u)\,du = 25\left(\frac{1}{2}\sec u\,\tan u - \frac{1}{2}\ln|\sec u + \tan u|\right) + C. \end{aligned}$$

The antiderivative is a consequence of integral formulas 14 and 28 of the endpapers of the text. Next, a reference triangle with acute angle u, opposite side x, and adjacent side 5 has hypotenuse of length $\sqrt{25-x^2}$. Therefore

$$K = \frac{25}{2}\left[\frac{x\sqrt{25+x^2}}{25} - \ln\left(\frac{x+\sqrt{25+x^2}}{5}\right)\right] + C = \frac{1}{2}x\sqrt{25+x^2} - \frac{25}{2}\ln\left(x+\sqrt{25+x^2}\right) + C_1.$$

When we simplify an answer by allowing a constant such as $\frac{25}{2}\ln 5$ to be absorbed by the constant C of integration, we will generally indicate this by replacing C with C_1, as in this solution.

C08S0M.012: Here, $\sqrt{4-\sin^2 x} = \sqrt{4-4\sin^2 u} = \sqrt{4\cos^2 u} = 2\cos u$ if $\sin x = 2\sin u$. So we let $x = \arcsin(2\sin u)$, and thus

$$\cos x\,dx = 2\cos u\,du \quad\text{and}\quad u = \arcsin\left(\frac{\sin x}{2}\right).$$

Therefore

$$I = \int(\cos x)\sqrt{4-\sin^2 x}\,dx = \int(2\cos u)(2\cos u)\,du = 2\int(1+\cos 2u)\,du = 2(u + \sin u\,\cos u) + C.$$

A reference triangle with acute angle u, opposite side $\sin x$, and hypotenuse 2 has adjacent side of length $\sqrt{4-\sin^2 x}$. Therefore

$$I = 2\arcsin\left(\frac{\sin x}{2}\right) + (\sin x)\cdot\frac{\sqrt{4-\sin^2 x}}{2} + C.$$

C08S0M.013: Complete the square:

$$x^2 - x + 1 = x^2 - x + \frac{1}{4} + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4} + \frac{3}{4}\tan^2 u = \frac{3}{4}\sec^2 u$$

if $x - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\tan u$. So we let

$$x = \frac{1 + \sqrt{3}\,\tan u}{2}; \quad dx = \frac{\sqrt{3}}{2}\sec^2 u\,du \quad \text{and} \quad \tan u = \frac{2x-1}{\sqrt{3}}.$$

Therefore

$$J = \int \frac{1}{x^2 - x + 1}\,dx = \frac{\sqrt{3}}{2}\int \frac{\sec^2 u}{\frac{3}{4}\sec^2 u}\,du = \frac{2\sqrt{3}}{3}u + C.$$

A reference triangle with acute angle u, opposite side $2x-1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{x^2 - x + 1}$. Therefore

$$J = \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}[2x-1]\right) + C.$$

C08S0M.014: We first complete the square:

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4} + \frac{3}{4}\tan^2 u = \frac{3}{4}\sec^2 u$$

if $x + \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\tan u$. So we let

$$x = \frac{-1 + \sqrt{3}\,\tan u}{2}; \quad dx = \frac{\sqrt{3}}{2}\sec^2 u\,du \quad \text{and} \quad \tan u = \frac{2x+1}{\sqrt{3}}.$$

Consequently

$$K = \int \sqrt{x^2 + x + 1}\,dx = \int \left(\frac{\sqrt{3}}{2}\sec u\right)\left(\frac{\sqrt{3}}{2}\sec^2 u\right)du = \frac{3}{8}\left(\sec u\,\tan u + \ln|\sec u + \tan u|\right) + C.$$

A reference triangle with acute angle u, opposite side $2x+1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{x^2 + x + 1}$. Therefore

$$\begin{aligned} K &= \frac{3}{8}\left[\frac{2(2x+1)\sqrt{x^2+x+1}}{3} + \ln\left(\frac{2x+1+2\sqrt{x^2+x+1}}{\sqrt{3}}\right)\right] + C \\ &= \frac{1}{4}(2x+1)\sqrt{x^2+x+1} + \frac{3}{8}\ln\left(2x+1+2\sqrt{x^2+x+1}\right) + C_1. \end{aligned}$$

C08S0M.015: Given: $\displaystyle\int \frac{5x+31}{3x^2 - 4x + 11}\,dx$. First complete the square in the denominator:

$$3x^2 - 4x + 11 = \frac{1}{3}(9x^2 - 12x + 33) = \frac{1}{3}\left([3x-2]^2 + 29\right) = \frac{1}{3}\left(29\tan^2\theta + 29\right) = \frac{29}{3}\sec^2\theta$$

if $3x - 2 = \sqrt{29}\,\tan\theta$. So let

$$x = \frac{2+\sqrt{29}\,\tan\theta}{3}; \quad \text{then} \quad dx = \frac{\sqrt{29}}{3}\sec^2\theta \quad \text{and} \quad \tan\theta = \frac{3x-2}{\sqrt{29}}.$$

Then

$$\begin{aligned}
I &= \int \frac{\frac{10}{3} + \frac{5}{3}\sqrt{29}\,\tan\theta + \frac{93}{3}}{\frac{29}{3}\sec^2\theta} \cdot \frac{\sqrt{29}}{3}\sec^2\theta\, d\theta \\
&= \frac{\sqrt{29}}{87}\int \left(103 + 5\sqrt{29}\,\tan\theta\right) d\theta = \frac{\sqrt{29}}{87}\left(103\theta + 5\sqrt{29}\,\ln|\sec\theta|\right) + C.
\end{aligned}$$

A reference triangle with acute angle θ, opposite side $3x-2$, and adjacent side $\sqrt{29}$ has hypotenuse of length $\sqrt{9x^2-12x+33}$. Therefore

$$\begin{aligned}
I &= \frac{\sqrt{29}}{87}\cdot 103 \cdot \arctan\left(\frac{3x-2}{\sqrt{29}}\right) + \frac{5\cdot 29}{87}\ln\left|\frac{\sqrt{3}\,\sqrt{3x^2-4x+11}}{\sqrt{29}}\right| + C \\
&= \frac{\sqrt{29}}{87}\cdot 103 \cdot \arctan\left(\frac{3x-2}{\sqrt{29}}\right) + \frac{5}{3}\ln\left(\sqrt{3x^2-4x+11}\right) + C_1 \\
&= \frac{103\sqrt{29}}{87}\arctan\left(\frac{3x-2}{\sqrt{29}}\right) + \frac{5}{6}\ln(3x^2-4x+11) + C_1.
\end{aligned}$$

C08S0M.016: Division of denominator into numerator yields

$$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}.$$

Therefore $\displaystyle\int \frac{x^4+1}{x^2+1}\,dx = \frac{1}{3}x^3 - x + 2\arctan x + C.$

C08S0M.017: $\displaystyle\int (x^4+x^7)^{1/2}\,dx = \int x^2(1+x^3)^{1/2}\,dx = \frac{2}{9}(1+x^3)^{3/2} + C.$

C08S0M.018: The substitution $x = u^2$, $dx = 2u\,du$ yields

$$\int \frac{\sqrt{x}}{1+x}\,dx = \int \frac{2u^2}{1+u^2}\,du = \int \left(2 - \frac{2}{1+u^2}\right) du = 2u - 2\arctan u + C = 2\sqrt{x} - 2\arctan\sqrt{x} + C.$$

C08S0M.019: We use integral formula 16 of the endpapers and the substitution $u = \sin x$, $du = \cos x\,dx$:

$$\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx = \int \frac{1}{\sqrt{4-u^2}}\,du = \arcsin\left(\frac{u}{2}\right) + C = \arcsin\left(\frac{\sin x}{2}\right) + C.$$

C08S0M.020: $\displaystyle\int \frac{\cos 2x}{\cos x}\,dx = \int \frac{2\cos^2 x - 1}{\cos x}\,dx = \int (2\cos x - \sec x)\,dx = 2\sin x - \ln|\sec x + \tan x| + C.$

C08S0M.021: Let $u = \ln(\cos x)$. Then $du = -\dfrac{\sin x}{\cos x}\,dx = -\tan x\,dx$, and thus

$$\int \frac{\tan x}{\ln(\cos x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\ln(\cos x)| + C.$$

C08S0M.022: First note that $\sqrt{1-x^4} = \sqrt{1-\sin^2 u} = \cos u$ if $x^2 = \sin u$. So we let

$$x = \sqrt{\sin u}: \quad 2x\,dx = \cos u\,du, \quad x^6 = \sin^3 u.$$

Thus

$$I = \int \frac{x^7}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{x^6}{\sqrt{1-x^4}} \cdot 2x\,dx = \frac{1}{2}\int \frac{\sin^3 u}{\cos u} \cdot \cos u\,du$$
$$= \frac{1}{2}\int (1-\cos^2 u)\sin u\,du = -\frac{1}{2}\cos u + \frac{1}{6}\cos^3 u + C.$$

A reference triangle with acute angle u, opposite side x^2, and hypotenuse 1 has adjacent side of length $\sqrt{1-x^4}$. Hence

$$I = -\frac{1}{2}(1-x^4)^{1/2} + \frac{1}{6}(1-x^4)^{3/2} + C = \frac{1}{6}(1-x^4)^{1/2}(1-x^4-3) + C = -\frac{1}{6}(x^4-2)\sqrt{1-x^4} + C.$$

C08S0M.023: Let $u = \ln(1+x)$, $dv = dx$. Then $du = \dfrac{1}{1+x}\,dx$, and we let $v = 1+x$. Then

$$\int \ln(1+x)\,dx = (1+x)\ln(1+x) - \int 1\,dx = (1+x)\ln(1+x) - x + C.$$

The choice $v = x$ will produce an answer that appears different: $x\ln(1+x) - x + \ln(1+x) + C$.

C08S0M.024: Let $u = \operatorname{arcsec} x$ and $dv = x\,dx$. Then

$$du = \frac{1}{|x|\sqrt{x^2-1}}\,dx \quad \text{and} \quad v = \frac{1}{2}x^2 = \frac{1}{2}|x|\cdot|x|.$$

Then

$$J = \int x \operatorname{arcsec} x\,dx = \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}\int \frac{|x|}{\sqrt{x^2-1}}\,dx.$$

Thus

$$J = \begin{cases} \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}(x^2-1)^{1/2} + C_1 & \text{if} \quad x > 1, \\ \\ \frac{1}{2}x^2 \operatorname{arcsec} x + \frac{1}{2}(x^2-1)^{1/2} + C_2 & \text{if} \quad x < -1. \end{cases}$$

Therefore $J = \dfrac{1}{2}x^2 \operatorname{arcsec} x - \dfrac{|x|}{2x}(x^2-1)^{1/2} + C$.

C08S0M.025: Let $x = 3\tan u$: $dx = 3\sec^2 u\,du$ and $\sqrt{x^2+9} = \sqrt{9\tan^2 u + 9} = 3\sec u$. Thus

$$K = \int \sqrt{x^2+9}\,dx = \int 9\sec^3 u\,du = \frac{9}{2}(\sec u\,\tan u + \ln|\sec u + \tan u|) + C.$$

A reference triangle with acute angle u, opposite side x, and adjacent side 3 has hypotenuse of length $\sqrt{x^2+9}$. Thus

$$K = \frac{9}{2}\left[\frac{x\sqrt{x^2+9}}{9} + \ln\left(\frac{x+\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{2}x\sqrt{x^2+9} + \frac{9}{2}\ln\left(x + \sqrt{x^2+9}\right) + C_1.$$

A hyperbolic substitution will yield the antiderivative in the form $K = \dfrac{1}{2}x\sqrt{x^2+9} + \dfrac{9}{2}\sinh^{-1}\left(\dfrac{x}{3}\right) + C.$

C08S0M.026: Let $x = 2\sin u$. Then $dx = 2\cos u\,du$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\cos u$. Thus

$$J = \int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^2 u\,\cos u}{2\cos u}\,du = 2\int (1-\cos 2u)\,du = 2(u - \sin u\,\cos u) + C.$$

A reference triangle with acute angle u, opposite side x, and hypotenuse 2 has adjacent side of length $\sqrt{4-u^2}$. Thus

$$J = 2\arcsin\left(\frac{x}{2}\right) - 2\cdot\frac{x\sqrt{4-x^2}}{4} + C = 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C.$$

C08S0M.027: Note that $2x - x^2 = -(x^2-2x) = 1-(x-1)^2 = 1-\sin^2 u = \cos^2 u$ if $x-1 = \sin u$. Hence we let $x = 1+\sin u$, so that $dx = \cos u\,du$. Then

$$I = \int \sqrt{2x-x^2}\,dx = \int \cos^2 u\,du = \frac{1}{2}\int (1+\cos 2u)\,du = \frac{1}{2}\,(u + \sin u\,\cos u) + C.$$

A reference triangle with acute angle u, opposite side $x-1$, and hypotenuse 1 has adjacent side of length $\sqrt{2x-x^2}$. Therefore

$$I = \frac{1}{2}\arcsin(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} + C.$$

Mathematica 3.0 returns an answer that differs only in that $\arcsin(x-1)$ is replaced with $-\arcsin(1-x)$.

C08S0M.028: The partial fractions decomposition of the integrand has the form

$$\frac{4x-2}{x^3-x} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1}.$$

This yields the equation $A(x^2+x) + B(x^2-1) + C(x^2-x) = 4x-2$, and thus the simultaneous equations

$$A+B+C=0, \quad A-C=4, \quad \text{and} \quad -B=-2.$$

It follows that $B=2$, $A=1$, and $C=-3$. Therefore

$$\int \frac{4x-2}{x^3-x}\,dx = \ln|x-1| + 2\ln|x| - 3\ln|x+1| + C = \ln\left|\frac{x^3-x^2}{(x+1)^3}\right| + C.$$

C08S0M.029: First divide denominator into numerator to obtain

$$\frac{x^4}{x^2-2} = x^2+2+\frac{4}{x^2-2}.$$

Then the partial fractions decomposition of the last term yields

$$\frac{4}{x^2-2} = \frac{A}{x+\sqrt{2}} + \frac{B}{x-\sqrt{2}}$$

and thus the equation $A\left(x-\sqrt{2}\right) + B\left(x+\sqrt{2}\right) = 4$. Consequently

$$A+B=0 \quad \text{and} \quad -A\sqrt{2} + B\sqrt{2} = 4,$$

and it follows that $A = -\sqrt{2}$ and $B = \sqrt{2}$. Therefore

$$\int \frac{x^4}{x^2-2}\,dx = \frac{1}{3}x^3 + 2x - \sqrt{2}\,\ln\left|x+\sqrt{2}\right| + \sqrt{2}\,\ln\left|x-\sqrt{2}\right| + C = \frac{1}{3}x^3 + 2x + \sqrt{2}\,\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C.$$

Mathematica 3.0 and *Maple* V version 5.1 apparently prefer hyperbolic substitutions to the method of partial fractions; they both yield instead the equivalent answer

$$2x + \frac{x^3}{3} - 2\sqrt{2}\,\tanh^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

(remember that most computer algebra programs omit the "$+\,C$").

C08S0M.030: The substitution $u = \sec x$ yields $du = \sec x\,\tan x\,dx$, and thus

$$\int \frac{\sec x\,\tan x}{\sec x + \sec^2 x}\,dx = \int \frac{1}{u+u^2}\,du = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)du$$

$$= \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sec x}{1+\sec x}\right| + C = \ln\left|\frac{1}{1+\cos x}\right| + C = C - \ln|1+\cos x|.$$

Alternatively,

$$\int \frac{\sec x\,\tan x}{\sec x + \sec^2 x}\,dx = \int \frac{\tan x}{1+\sec x}\,dx = \int \frac{\sin x}{1+\cos x}\,dx = C - \ln|1+\cos x|.$$

Mathematica 3.0 returns the antiderivative in a form equivalent to

$$-\frac{4}{\sec x + \sec^2 x}\,(\sec^2 x)\left[\cos^2\left(\frac{x}{2}\right)\right]\ln\left(\cos\left(\frac{x}{2}\right)\right) + C.$$

Maple V version 5.1, by contrast, yields

$$\int \frac{sec x\,\tan x}{\sec x\,+\,\sec^2 x}\,dx \;=\; \ln\frac{\sec x}{1+\sec x} + C,$$

essentially the same as the answer we obtained "by hand."

C08S0M.031: First, $x^2 + 2x + 2 = 1 + (x+1)^2 = 2 + \tan^2 u = \sec^2 u$ if $\tan u = x + 1$. Hence we let $x = -1 + \tan u$, so that $dx = \sec^2 u\,du$ and

$$J = \int \frac{x}{(x^2+2x+2)^2} = \int \frac{-1+\tan u}{\sec^4 u}\cdot \sec^2 u\,du = \int (-\cos^2 u + \sin u\,\cos u)\,du$$

$$= \int \left(\sin u\,\cos u - \frac{1+\cos 2u}{2}\right)du = \frac{1}{2}\sin^2 u - \frac{1}{2}u - \frac{1}{2}\sin u\,\cos u + C.$$

Then a reference triangle with acute angle u, opposite side $x+1$, and adjacent side 1 has hypotenuse of length $\sqrt{x^2+x+2}$. Therefore

$$J = \frac{1}{2}\cdot\frac{(x+1)^2}{x^2+2x+2} - \frac{1}{2}\arctan(x+1) - \frac{1}{2}\cdot\frac{x+1}{x^2+2x+2} + C = \frac{x^2+x}{2(x^2+2x+2)} - \frac{1}{2}\arctan(x+1) + C.$$

Mathematica 3.0, *Derive* 2.56, and *Maple* V version 5.1 all yield instead the equivalent result

$$J = -\frac{x+2}{2(x^2+x+2)} - \frac{1}{2}\arctan(x+1) + C.$$

C08S0M.032: Let $u = x^{12}$. Then $du = 12u^{11}\ du$ and $u = x^{1/12}$. Hence

$$I = \int \frac{x^{1/3}}{x^{1/2}+x^{1/4}}\ dx = \int \frac{12u^{15}}{u^6+u^3}\ du = 12\int \frac{u^{12}}{u^3+1}\ du.$$

The method of partial fractions then yields

$$\frac{u^{12}}{u^3+1} = u^9 - u^6 + u^3 - 1 + \frac{1}{3}\left(\frac{1}{u+1} - \frac{u-2}{u^2-u+1}\right).$$

To find the antiderivative of the last term, note that

$$u^2 - u + 1 = \left(u - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2\theta + \frac{3}{4} = \frac{3}{4}\sec^2\theta$$

if $u - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\tan\theta$. So we let

$$u = \frac{1+\sqrt{3}\ \tan\theta}{2}: \quad du = \frac{\sqrt{3}}{2}\sec^2\theta\ d\theta \quad \text{and} \quad \tan\theta = \frac{2u-1}{\sqrt{3}}.$$

Therefore

$$\begin{aligned}\int \frac{u-2}{u^2-u+1}\ du &= \int \frac{\frac{1}{2}\left(-3+\sqrt{3}\ \tan\theta\right)}{\frac{3}{4}\sec^2\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\ d\theta\\ &= \frac{\sqrt{3}}{3}\int(-3+\sqrt{3}\ \tan\theta)\ d\theta = \frac{\sqrt{3}}{3}\left(-3\theta+\sqrt{3}\ \ln|\sec\theta|\right)+C.\end{aligned}$$

A reference triangle with acute angle θ, opposite side $2u-1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{u^2-u+1}$. Therefore

$$\begin{aligned}\int \frac{u-2}{u^2-u+1}\ du &= \frac{\sqrt{3}}{3}\left[-3\arctan\left(\frac{2u-1}{\sqrt{3}}\right)+\sqrt{3}\ \ln\left|\frac{2\sqrt{u^2-u+1}}{\sqrt{3}}\right|\right]+C\\ &= -\sqrt{3}\ \arctan\left(\frac{2u-1}{\sqrt{3}}\right)+\frac{1}{2}\ln(u^2-u+1)+C_1.\end{aligned}$$

Therefore

$$\begin{aligned}I &= 12\int\frac{u^{12}}{u^3+1}\ du\\ &= 12\left[\frac{1}{10}u^{10}-\frac{1}{7}u^7+\frac{1}{4}u^4-u+\frac{1}{3}\ln|u+1|+\frac{\sqrt{3}}{3}\ \arctan\left(\frac{2u-1}{\sqrt{3}}\right)-\frac{1}{6}\ln(u^2-u+1)\right]+C\\ &= \frac{6}{5}u^{10}-\frac{12}{7}u^7+3u^4-12u+4\ln|u+1|+4\sqrt{3}\ \arctan\left(\frac{2u-1}{\sqrt{u}}\right)-4\ln(u^2-u+1)+C\\ &= \frac{6}{5}x^{5/6}-\frac{12}{7}x^{7/12}+3x^{1/3}-12x^{1/12}+4\ln(1+x^{1/12})\\ &\qquad +4\sqrt{3}\ \arctan\left(\frac{2x^{1/12}-1}{\sqrt{3}}\right)-2\ln(x^{1/6}-x^{1/12}+1)+C.\end{aligned}$$

C08S0M.033: We use the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$:

$$\int \frac{1}{1+\cos 2\theta}\,d\theta = \frac{1}{2}\int \frac{2}{1+\cos 2\theta}\,d\theta = \frac{1}{2}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{2}\int \sec^2\theta\,d\theta = \frac{1}{2}\tan\theta + C.$$

C08S0M.034: $\displaystyle\int \frac{\sec x}{\tan x}\,dx = \int \csc x\,dx = \ln|\csc x - \cot x| + C.$

C08S0M.035: $\displaystyle\int \sec^3 x\,\tan^3 x\,dx = \int (\sec^2 x)(\sec^2 x - 1)\sec x\,\tan x\,dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

C08S0M.036: Integration by parts: Let $u = \arctan x$ and $dv = x^2\,dx$. Then

$$du = \frac{1}{1+x^2}\,dx; \qquad \text{choose} \qquad v = \frac{1}{3}x^3.$$

Thus

$$\int x^2 \arctan x\,dx = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int\left(x - \frac{x}{x^2+1}\right)dx = \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C.$$

C08S0M.037: It's almost always wise to develop a reduction formula for problems of this sort. Suppose that n is a positive integer. Let

$$I_n = \int x(\ln x)^n\,dx.$$

Integration by parts: Let $u = (\ln x)^n$ and $dv = x\,dx$. Then

$$du = \frac{n(\ln x)^{n-1}}{x}\,dx \qquad \text{and} \qquad v = \frac{1}{2}x^2.$$

Therefore

$$I_n = \int x(\ln x)^n\,dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}\,dx.$$

And thus

$$\begin{aligned}
I_3 = \int x(\ln x)^3\,dx &= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\int x(\ln x)^2\,dx \\
&= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\left[\frac{1}{2}x^2(\ln x)^2 - \int x(\ln x)\,dx\right] \\
&= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\left[\frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx\right] \\
&= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2\ln x - \frac{3}{8}x^2 + C \\
&= \frac{x^2}{8}\left[4(\ln x)^3 - 6(\ln x)^2 + 6\ln x - 3\right] + C.
\end{aligned}$$

C08S0M.038: By Eq. (44) of Section 7.6,

$$J = \int \frac{1}{x\sqrt{x^2+1}}\,dx = -\operatorname{csch}^{-1}|x| + C.$$

Alternatively, or if you need the answer expressed in a more familiar form, let $x = \tan u$. Then $dx = \sec^2 u\,du$ and $1 + x^2 = \sec^2 u$. Then

$$J = \int \frac{\sec^2 u}{\sec u\,\tan u}\,du = \int \frac{\sec u}{\tan u}\,du = \int \csc u\,du = \ln|\csc u - \cot u| + C.$$

Then a reference triangle with acute angle u, opposite side x, and adjacent side 1 has hypotenuse of length $\sqrt{1+u^2}$, and therefore

$$J = \ln\left|\frac{-1+\sqrt{1+x^2}}{x}\right| + C = \ln\left(\frac{-1+\sqrt{1+x^2}}{|x|}\right) + C = \ln\left(-1+\sqrt{1+x^2}\right) - \ln|x| + C.$$

And for a second variation—for those of us who memorized a different form of the antiderivative of the cosecant function—

$$J = \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C = \ln|x| - \ln\left(1+\sqrt{1+x^2}\right) + C.$$

C08S0M.039: Let $u = e^x$. Then $du = e^x\,dx$ and $x = \ln u$. Therefore

$$K = \int e^x\sqrt{1+e^{2x}}\,dx = \int \sqrt{1+u^2}\,du.$$

Now let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $1+u^2 = \sec^2\theta$. So

$$K = \int \sec^3\theta\,d\theta = \frac{1}{2}\big(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\big) + C.$$

Then a reference triangle with acute angle θ, opposite side u, and adjacent side 1 has hypotenuse of length $\sqrt{1+u^2}$. Hence

$$K = \frac{1}{2}\left[u\sqrt{1+u^2} + \ln\left(u+\sqrt{1+u^2}\right)\right] + C = \frac{1}{2}e^x\sqrt{1+e^{2x}} + \frac{1}{2}\ln\left(e^x+\sqrt{1+e^{2x}}\right) + C.$$

Mathematica 3.0, with its well-known penchant for using hyperbolic functions, returns the equivalent

$$K = \frac{1}{2}e^x\sqrt{1+e^{2x}} + \frac{1}{2}\sinh^{-1}(e^x) + C.$$

Maple V version 5.1 yields the same antiderivative, except that *Maple* writes $(e^x)^2$ instead of e^{2x}.

C08S0M.040: First, $4x - x^2 = -(x^2-4x) = 4-(x-2)^2 = 4-4\sin^2 u = 4\cos^2 u$ if $x - 2 = 2\sin u$. Therefore we let

$$x = 2 + 2\sin u; \quad \text{then} \quad dx = 2\cos u\,du, \quad \sin u = \frac{x-2}{2}.$$

A reference triangle for this substitution has acute angle u, opposite side $x-2$, and hypotenuse 2. Hence its adjacent side has length $\sqrt{4x-x^2}$; thus

$$\int \frac{x}{\sqrt{4x - x^2}}\,dx = \int \frac{2 + 2\sin u}{2\cos u} \cdot 2\cos u\,du = 2u - 2\cos u + C = 2\arcsin\left(\frac{x-2}{2}\right) - \sqrt{4x - x^2} + C.$$

C08S0M.041: Let $x = 3\sec u$. Then $\sqrt{x^2 - 9} = \sqrt{9\sec^2 u - 9} = \sqrt{9\tan^2 u} = 3\tan u$; moreover, $dx = 3\sec u \tan u\ du$. A reference triangle for this substitution has acute angle u, hypotenuse x, and adjacent side 3, thus opposite side of length $\sqrt{x^2 - 9}$. Therefore

$$\int \frac{1}{x^3\sqrt{x^2-9}}\,dx = \int \frac{3\sec u\ \tan u}{(27\sec^3 u)(3\tan u)}\,du = \frac{1}{27}\int \frac{1+\cos 2u}{2} = \frac{1}{54}(u + \sin u\ \cos u) + C$$

$$= \frac{1}{54}\left[\operatorname{arcsec}\left(\frac{x}{3}\right) + \frac{3\sqrt{x^2-9}}{x^2}\right] + C = \frac{1}{54}\operatorname{arcsec}\left(\frac{x}{3}\right) + \frac{\sqrt{x^2-9}}{18x^2} + C.$$

There is a technical point that we have glossed over too many times not to mention. In our substitution, $\sqrt{9\tan^2 u} = 3\tan u$ is true only if $\tan u \geqq 0$. Nevertheless, our antiderivative is correct for all x such that $|x| > 3$, including values of x for which $\tan u < 0$. We confess to a pragmatic approach to such problems. If the substitution is "legal" only for certain values of the variables, use it anyway. Find the antiderivative, then verify its validity for *all* meaningful values of the variable by differentiation. Almost always, you will find that an antiderivative valid for an interval of values of the variables is valid for all meaningful values of the variables.

Maple V version 5.1 returns for the antiderivative the equivalent

$$\int \frac{1}{x^3\sqrt{x^2-9}}\,dx = \frac{\sqrt{-9+x^2}}{18x^2} - \frac{1}{54}\arctan\left(\frac{3}{\sqrt{-9+x^2}}\right) + C,$$

as does *Mathematica* 3.0.

C08S0M.042: Do *not* use the method of partial fractions! (No one wants to solve seventeen equations in seventeen unknowns, even if they *are* linear equations!) Let $u = 7x + 1$. Then

$$x = \frac{u-1}{7} \quad \text{and} \quad dx = \frac{1}{7}\,du.$$

Therefore

$$\int \frac{x}{(7x+1)^{17}}\,dx = \frac{1}{7}\int \frac{u-1}{u^{17}} \cdot \frac{1}{7}\,du = \frac{1}{49}\int (u^{-16} - u^{-17})\,du = \frac{1}{49}\left(\frac{u^{-16}}{16} - \frac{u^{-15}}{15}\right) + C$$

$$= \frac{1}{49}\left[\frac{1}{16(7x+1)^{16}} - \frac{1}{15(7x+1)^{15}}\right] + C = \frac{1}{49(7x+1)^{16}}\left(\frac{1}{16} - \frac{7x+1}{15}\right) + C$$

$$= \frac{1}{49(7x+1)^{16}} \cdot \frac{15 - 112x - 16}{240} + C = -\frac{112x+1}{11760(7x+1)^{16}} + C.$$

Note: Just for fun, we had *Mathematica* write the seventeen equations in seventeen unknowns to solve for the coefficients A, B, ..., Q resulting from trying to find the partial fractions decomposition

$$\frac{x}{(7x+1)^{17}} = \frac{A}{7x+1} + \frac{B}{(7x+1)^2} + \frac{C}{(7x+1)^3} + \cdots + \frac{P}{(7x+1)^{16}} + \frac{Q}{(7x+1)^{17}}.$$

It turns out that the matrix of coefficient is upper triangular, so the equations are actually fairly easy to solve, although they are quite imposing. For example, one of these equations is

$$9421331920A + 5299499205B + 2826399576C + 1413199788D$$

$$+ 652246056E + 271769190F + 98825160G + 29647548H + 6588344I + 823543J = 0.$$

The partial fractions decomposition turns out to be simple:

$$\frac{x}{(7x+1)^{17}} = \frac{1}{7(7x+1)^{16}} - \frac{1}{(7x+1)^{17}}.$$

A quick way to obtain this result:

$$\frac{x}{(7x+1)^{17}} = \frac{1}{7} \cdot \frac{7x+1-1}{(7x+1)^{17}} = \frac{1}{7}\left[\frac{7x+1}{(7x+1)^{17}} - \frac{1}{(7x+1)^{17}}\right].$$

C08S0M.043: The partial fractions decomposition of the integrand has the form

$$\frac{4x^2+x+1}{4x^3+x} = \frac{A}{x} + \frac{Bx+C}{4x^2+1},$$

which leads to $A(4x^2+1) + Bx^2 + Cx = 4x^2 + x + 1,$ and thereby to the simultaneous equations

$$4A + B = 4, \quad C = 1, \quad A = 1$$

which are easily solved for $A = 1,\ B = 0,$ and $C = 1.$ Therefore

$$\int \frac{4x^2+x+1}{4x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{1}{4x^2+1}\right) dx = \ln|x| + \frac{1}{2}\arctan(2x) + C.$$

The easiest way to find the antiderivative of the second fraction is by judicious guesswork.

C08S0M.044: Division of numerator by denominator yields

$$\frac{4x^3-x+1}{x^3+1} = 4 - \frac{x+3}{x^3+1}.$$

Then the method of partial fractions leads to

$$\frac{x+3}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1},$$

and thus to $A(x^2-x+1) + B(x^2+x) + C(x+1) = x+3,$ and so we obtain the simultaneous equations

$$A + B = 0, \quad -A + B + C = 1, \quad \text{and} \quad A + C = 3.$$

These are easy to solve for $A = \frac{2}{3},\ B = -\frac{2}{3},\ C = \frac{7}{3},$ and hence

$$\frac{x+3}{x^3+1} = \frac{1}{3}\left(\frac{2}{x+1} - \frac{2x-7}{x^3-x+1}\right).$$

Now

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2 u + \frac{3}{4} = \frac{3}{4}\sec^2 u \quad \text{if} \quad \frac{\sqrt{3}}{2}\tan u = x - \frac{1}{2}.$$

Hence we let

$$x = \frac{1+\sqrt{3}\ \tan u}{2}, \quad \text{and so} \quad dx = \frac{\sqrt{3}}{2}\sec^2 u\, du \quad \text{and} \quad \tan u = \frac{2x-1}{\sqrt{3}}.$$

Therefore

$$\begin{aligned}\int \frac{2x-7}{x^2-x+1}\,dx &= \int \frac{1+\sqrt{3}\ \tan u - 7}{\frac{3}{4}\sec^2 u}\cdot\frac{\sqrt{3}}{2}\sec^2 u\,du \\ &= \frac{2\sqrt{3}}{3}\int(-6+\sqrt{3}\ \tan u)\,du = \frac{2\sqrt{3}}{3}\big(-6u+\sqrt{3}\ \ln|\sec u|\big)+C.\end{aligned}$$

A reference triangle with acute angle u, opposite side $2x-1$, and adjacent side $\sqrt{3}$ has hypotenuse of length $2\sqrt{x^2-x+1}$. Therefore

$$\begin{aligned}\int \frac{2x-7}{x^2-x+1}\,dx &= \frac{2\sqrt{3}}{3}\left[-6\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+\sqrt{3}\ \ln\left(\sqrt{x^2-x+1}\right)\right]+C \\ &= -4\sqrt{3}\ \arctan\left(\frac{2x-1}{\sqrt{3}}\right)+\ln(x^2-x+1)+C.\end{aligned}$$

Consequently

$$\int \frac{4x^3-x+1}{x^3+1}\,dx = 4x-\frac{2}{3}\ln|x+1|-\frac{4\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+\frac{1}{3}\ln(x^2-x+1)+C.$$

C08S0M.045: $\displaystyle\int \tan^2 x\ \sec x\ dx = \int(\sec^3 x - \sec x)\ dx = \frac{1}{2}\big(\sec x\ \tan x - \ln|\sec x + \tan x|\big)+C.$

C08S0M.046: Here's an easy way to get the required partial fractions decomposition of the integrand:

$$\frac{x^2+2x+2}{(x+1)^3} = \frac{x^2+2x+1}{(x+1)^3}+\frac{1}{(x+1)^3} = \frac{1}{x+1}+\frac{1}{(x+1)^3}.$$

Therefore

$$\int \frac{x^2+2x+2}{(x+1)^3}\,dx = \ln|x+1| - \frac{1}{2(x+1)^2}+C.$$

C08S0M.047: The partial fractions decomposition

$$\frac{x^4+2x+2}{x^5+x^4} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x+1}$$

yields the equation $A(x^4+x^3)+B(x^3+x^2)+C(x^2+x)+D(x+1)+Ex^4 = x^4+2x+2,$ and thereby the simultaneous equations

$$\begin{aligned}A+E&=1,\\ A+B&=0,\\ B+C&=0,\\ C+D&=2,\\ D&=2.\end{aligned}$$

These equations are easy to solve "from the bottom up," and you'll find that $D = 2$, $C = 0$, $B = 0$, $A = 0$, and $E = 1$. Therefore

$$\int \frac{x^4 + 2x + 2}{x^5 + x^4}\,dx = \int \left(\frac{2}{x^4} + \frac{1}{x+1} \right) dx = \ln|x+1| - \frac{2}{3x^3} + C.$$

C08S0M.048: The partial fractions decomposition

$$\frac{8x^2 - 4x + 7}{(x^2+1)(4x+1)} = \frac{A}{4x+1} + \frac{Bx + C}{x^2+1}$$

produces the equation $A(x^2+1) + B(4x^2 + x) + C(4x+1) = 8x^2 - 4x + 7$, and thus the simultaneous equations

$$A + 4B = 8, \quad B + 4C = -4, \quad \text{and} \quad A + C = 7,$$

which are easy to solve for $C = -1$, $A = 8$, and $B = 0$. Therefore

$$\int \frac{8x^2 - 4x + 7}{(x^2+1)(4x+1)}\,dx = \int \left(\frac{8}{4x+1} - \frac{1}{x^2+1} \right) dx = 2\ln|4x+1| - \arctan x + C.$$

C08S0M.049: The partial fractions decomposition of the integrand is

$$\frac{3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1}{(x^3-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2},$$

and thus

$$\begin{aligned} &A(x-1)(x^4 + 2x^3 + 3x^2 + 2x + 1) + B(x^4 + 2x^3 + 3x^2 + 2x + 1) \\ &\qquad + (Cx+D)(x^4 - x^3 - x + 1) + (Ex+F)(x^2 - 2x + 1) = 3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1. \end{aligned}$$

Thus we obtain the following simultaneous equations:

$$\begin{aligned} A \qquad\; + C \qquad\qquad\qquad\quad &= 3, \\ A + B - C + D \qquad\qquad\quad &= -1, \\ A + 2B \qquad - D + E \qquad &= 2, \\ -A + 3B - C \qquad - 2E + F &= -12, \\ -A + 2B + C - D + E - 2F &= -2, \\ -A + B \qquad + D \qquad + F &= 1. \end{aligned}$$

It follows that $A = 1$, $B = -1$, $C = 2$, $D = 1$, $E = 4$, and $F = 2$. Hence

$$\frac{3x^5 - x^4 + 2x^3 - 12x^2 - 2x + 1}{(x^3-1)^2} = \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+x+1} + \frac{2(2x+1)}{(x^2+x+1)^2},$$

and therefore the required antiderivative is

$$\ln|x-1| + \frac{1}{x-1} + \ln(x^2+x+1) - \frac{2}{x^2+x+1} + C.$$

C08S0M.050: Don't even consider the method of partial fractions; the factorization of the denominator is awkward. Instead note that

$$x^4 + 4x^2 + 8 = x^4 + 4x^2 + 4 + 4 = (x^2 + 2)^2 + 4 = 4\tan^2 u + 4 = 4\sec^2 u$$

if $x^2 + 2 = 2\tan u$, so we let $x^2 = -2 + 2\tan u$. Then

$$\tan u = \frac{x^2 + 2}{2}, \quad 2x\,dx = 2\sec^2 u\,du, \quad \text{and} \quad x\,dx = \sec^2 u\,du.$$

Therefore

$$\int \frac{x}{x^4 + 4x^2 + 8}\,dx = \int \frac{\sec^2 u}{4\sec^2 u}\,du = \frac{1}{4}u + C = \frac{1}{4}\arctan\left(\frac{x^2 + 2}{2}\right) + C.$$

C08S0M.051: Another problem in which it is probably wise to develop a reduction formula. If n is a positive integer, let

$$I_n = \int (\ln x)^n\,dx.$$

Then use integration by parts with $u = (\ln x)^n$ and $dv = dx$. Then

$$du = \frac{n(\ln x)^{n-1}}{x}\,dx; \quad \text{choose} \quad v = x.$$

Hence

$$I_n = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx = x(\ln x)^n - nI_{n-1}.$$

So

$$\begin{aligned}
\int (\ln x)^6\,dx &= x(\ln x)^6 - 6\int (\ln x)^5\,dx \\
&= x(\ln x)^6 - 6\left[x(\ln x)^5 - 5\int (\ln x)^4\,dx\right] \\
&= x(\ln x)^6 - 6x(\ln x)^5 + 30\left[x(\ln x)^4 - 4\int (\ln x)^3\,dx\right] \\
&= x(\ln x)^6 - 6x(\ln x)^5 + 30x(\ln x)^4 - 120\left[x(\ln x)^3 - 3\int (\ln x)^2\,dx\right] \\
&= x(\ln x)^6 - 6x(\ln x)^5 + 30x(\ln x)^4 - 120x(\ln x)^3 + 360\left[x(\ln x)^2 - 2\int (\ln x)\,dx\right] \\
&= x(\ln x)^6 - 6x(\ln x)^5 + 30x(\ln x)^4 - 120x(\ln x)^3 + 360x(\ln x)^2 - 720\left[x\ln x - \int 1\,dx\right] \\
&= x(\ln x)^6 - 6x(\ln x)^5 + 30x(\ln x)^4 - 120x(\ln x)^3 + 360x(\ln x)^2 - 720x\ln x + 720x + C.
\end{aligned}$$

C08S0M.052: Let $x = u^3$. Then $dx = 3u^2\,dx$ and $u = x^{1/3}$. Thus

$$\int \frac{(1+x^{2/3})^{3/2}}{x^{1/3}}\,dx = \int \frac{(1+u^2)^{3/2}}{u}\cdot 3u^2\,du$$
$$= \int 3u(1+u^2)^{3/2}\,du = \frac{3}{5}(1+u^2)^{5/2} + C = \frac{3}{5}(1+x^{2/3})^{5/2} + C.$$

C08S0M.053: $\displaystyle\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx = \frac{1}{3}(\arcsin x)^3 + C.$

C08S0M.054: Let $x = u^6$. Then $dx = 6u^5\,du$ and $u = x^{1/6}$. So

$$I = \int \frac{1}{x^{3/2}(1+x^{1/3})}\,dx = \int \frac{6u^5}{u^9(1+u^2)}\,du = \int \frac{6}{u^4(1+u^2)}\,du.$$

Next,

$$\frac{6}{u^4(1+u^2)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u^4} + \frac{Eu+F}{1+u^2}$$

yields $A(u^3+u^5) + B(u^2+u^4) + C(u+u^3) + D(1+u^2) + Eu^5 + Fu^6 = 6$, and thus

$$\begin{aligned} A+E &= 0, & B+F &= 0,\\ A+C &= 0, & B+D &= 0,\\ C &= 0, & D &= 6. \end{aligned}$$

It follows easily that $D = 6$, $C = 0$, $B = -6$, $A = 0$, $F = 6$, and $E = 0$. Hence

$$I = \int \left(-\frac{6}{u^2} + \frac{6}{u^4} + \frac{6}{1+u^2}\right) du$$
$$= \frac{6}{u} - \frac{2}{u^3} + 6\arctan u + C = 6x^{-1/6} - 2x^{-1/2} + 6\arctan\left(x^{1/6}\right) + C.$$

C08S0M.055: Here we have

$$\int \tan^3 z\,dz = \int (\sec^2 z\, - 1)\tan z\,dz = \int (\sec^2 z\,\tan z\, - \tan z)\,dz$$
$$= \frac{1}{2}\tan^2 z + \ln|\cos z| + C = \frac{1}{2}\sec^2 z + \ln|\cos z| + C_1.$$

C08S0M.056: We will use the reduction formula in Problem 54 of Section 7.3.

$$\int \sin^2\omega\,\cos^4\omega\,d\omega = \int (\cos^4\omega - \cos^6\omega)\,d\omega = \int \cos^4\omega\,d\omega - \frac{1}{6}\cos^5\omega\,\sin\omega - \frac{5}{6}\int \cos^4\omega\,d\omega$$
$$= -\frac{1}{6}\cos^5\omega\,\sin\omega + \frac{1}{6}\left[\frac{1}{4}\cos^3\omega\,\sin\omega + \frac{3}{4}\int \cos^2\omega\,d\omega\right]$$
$$= -\frac{1}{6}\cos^5\omega\,\sin\omega + \frac{1}{24}\cos^3\omega\,\sin\omega + \frac{1}{8}\int \frac{1+\cos 2\omega}{2}\,d\omega$$
$$= -\frac{1}{6}\cos^5\omega\,\sin\omega + \frac{1}{24}\cos^3\omega\,\sin\omega + \frac{1}{16}\cos\omega\,\sin\omega + \frac{1}{16}\omega + C.$$

Mathematica 3.0 uses other trigonometric identities to produce its answer,

$$\frac{1}{192}(12\omega + 3\sin 2\omega - 3\sin 4\omega - \sin 6\omega).$$

C08S0M.057: Let $u = \exp(x^2)$. Then $du = 2x\exp(x^2)\,dx$ and $\exp(2x^2) = \left(e^{x^2}\right)^2 = u^2$. Hence

$$\int \frac{x\exp(x^2)}{1+\exp(2x^2)}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\left(\exp(x^2)\right) + C.$$

C08S0M.058: Here we have

$$\int \frac{\cos^3 x}{(\sin x)^{1/2}}\,dx = \int \frac{(1-\sin^2 x)\cos x}{(\sin x)^{1/2}}\,dx = \int \left[(\sin x)^{-1/2}\cos x - (\sin x)^{3/2}\cos x\right]dx$$

$$= 2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} + C = \frac{2}{5}(5-\sin^2 x)\sqrt{\sin x} + C.$$

C08S0M.059: Let $u = x^2$ and $dv = x\exp(-x^2)\,dx$. Then $du = 2x\,dx$; choose $v = -\frac{1}{2}\exp(-x^2)$. Then

$$\int x^3\exp(-x^2)\,dx = -\frac{1}{2}x^2\exp(-x^2) + \int x\exp(-x^2)\,dx$$

$$= -\frac{1}{2}x^2\exp(-x^2) - \frac{1}{2}\exp(-x^2) + C = -\frac{x^2+1}{2}\exp(-x^2) + C.$$

C08S0M.060: First let $x = w^2$. Then $dx = 2w\,dw$, so

$$J = \int \sin\sqrt{x}\,dx = \int 2w\sin w\,dw.$$

Then integrate by parts with $u = 2w$ and $dv = \sin w\,dw$, so that $du = 2\,dw$; choose $v = -\cos w$. Then

$$J = -2w\cos w + \int 2w\,dw = -2w\cos w + 2\sin w + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$$

C08S0M.061: Integrate by parts with $u = \arcsin x$ and $dv = \frac{1}{x^2}\,dx$, so that

$$du = \frac{1}{\sqrt{1-x^2}}\,dx; \quad \text{choose} \quad v = -\frac{1}{x}.$$

Then

$$K = \int \frac{\arcsin x}{x^2}\,dx = -\frac{1}{x}\arcsin x + \int \frac{1}{x\sqrt{1-x^2}}\,dx = -\frac{1}{x}\arcsin x - \operatorname{sech}^{-1}|x| + C.$$

If you prefer to avoid hyperbolic functions, substitute $x = \sin\theta$ in the last integral. With $1 - x^2 = \cos^2\theta$ and $dx = \cos\theta\,d\theta$, you'll get

$$K = -\frac{1}{x}\arcsin x + \int \frac{\cos\theta}{\sin\theta\,\cos\theta}\,d\theta = -\frac{1}{x}\arcsin x + \ln|\csc\theta - \cot\theta| + C.$$

Then a reference triangle with acute angle θ, opposite side x, and hypotenuse 1 has adjacent side of length $\sqrt{1-x^2}$. Therefore

$$K = -\frac{1}{x}\arcsin x + \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C = -\frac{1}{x}\arcsin x + \ln\left(1-\sqrt{1-x^2}\right) - \ln|x| + C.$$

C08S0M.062: Let $x = 3\sec u$. Consequently $\sqrt{x^2-9} = \sqrt{9\sec^2 u - 9} = \sqrt{9\tan^2 u} = 3\tan u$ and $dx = 2\sec u \tan u\ du$. A reference triangle for this substitution has acute angle u, hypotenuse x, and adjacent side 3, so the side opposite u has length $\sqrt{x^2-9}$. Therefore

$$\begin{aligned}\int \sqrt{x^2-9}\ dx &= \int 9\sec u\ \tan^2 u\ du = 9\int(\sec^3 u - \sec u)\ du = \frac{9}{2}\big(\sec u\ \tan u - \ln|\sec u + \tan u|\big) + C\\ &= \frac{9}{2}\left(\frac{x\sqrt{x^2-9}}{9} - \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right|\right) + C = \frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\ln\left(x+\sqrt{x^2-9}\right) + C_1.\end{aligned}$$

C08S0M.063: Let $x = \sin u$. Then $1 - x^2 = 1 - \sin^2 u = \cos^2 u$ and $dx = \cos u\ du$. A reference triangle for this substitution has acute angle u, opposite side x, and hypotenuse 1, thus adjacent side of length $\sqrt{1-x^2}$. Therefore

$$\begin{aligned}\int x^2\sqrt{1-x^2}\ dx &= \int \sin^2 u\ \cos^2 u\ du = \frac{1}{4}\int(2\sin u\ \cos u)^2\ du = \frac{1}{4}\int \sin^2 2u\ du = \frac{1}{8}\int(1-\cos 4u)\ du\\ &= \frac{1}{8}\left(u - \frac{1}{4}\sin 4u\right) + C = \frac{1}{8}u - \frac{1}{16}\sin 2u\ \cos 2u + C\\ &= \frac{1}{8}u - \frac{1}{8}(\sin u\ \cos u)(\cos^2 u - \sin^2 u) + C = \frac{1}{8}\left(u - \sin u\ \cos^3 u + \sin^3 u\ \cos u\right) + C\\ &= \frac{1}{8}\big[\arcsin x - x(1-x^2)^{3/2} + x^3(1-x^2)^{1/2}\big] + C\\ &= \frac{1}{8}x(2x^2-1)\sqrt{1-x^2} + \frac{1}{8}\arcsin x + C.\end{aligned}$$

C08S0M.064: First, $2x - x^2 = 1 - (x-1)^2 = 1 - \sin^2 u = \cos^2 u$ if $x = 1 + \sin u$. We therefore use this substitution with $dx = \cos u\ du$. The result:

$$\begin{aligned}J &= \int x\sqrt{2x-x^2}\ dx = \int(1+\sin u)\cos^2 u\ du\\ &= \int\left(\frac{1+\cos 2u}{2} + \cos^2 u\ \sin u\right)du = \frac{1}{2}u + \frac{1}{2}\sin u\ \cos u - \frac{1}{3}\cos^3 u + C.\end{aligned}$$

A reference triangle for the trigonometric substitution has acute angle u, opposite side $x-1$, and hypotenuse 1. Thus its adjacent side has length $\sqrt{2x-x^2}$, and therefore

$$\begin{aligned}J &= \frac{1}{2}\arcsin(x-1) + \frac{1}{2}(x-1)\sqrt{2x-x^2} - \frac{1}{3}(2x-x^2)^{3/2} + C\\ &= \frac{1}{2}\arcsin(x-1) + (2x-x^2)^{1/2}\left(\frac{x}{2} - \frac{1}{2} - \frac{2x}{3} + \frac{x^2}{3}\right) + C\\ &= \frac{1}{2}\arcsin(x-1) + \frac{1}{6}(2x^2-x-3)\sqrt{2x-x^2} + C.\end{aligned}$$

C08S0M.065: The partial fractions decomposition of the integrand has the form

$$\frac{x-2}{4x^2+4x+1} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2},$$

and therefore $A(2x+1)+B = x-2$. It follows that $A = \frac{1}{2}$ and $B = -\frac{5}{2}$, and thus

$$\int \frac{x-2}{(2x+1)^2}\,dx = \frac{1}{2}\int \left(\frac{1}{2x+1} - \frac{5}{(2x+1)^2}\right)dx = \frac{1}{4}\ln|2x+1| + \frac{5}{4(2x+1)} + C.$$

C08S0M.066: The partial fractions decomposition of the integrand has the form

$$\frac{2x^2-5x-1}{x^3-2x^2-x+2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x+1},$$

so that $A(x^2-1)+B(x^2-x-2)+C(x^2-3x+2) = 2x^2-5x-1$. It follows that

$$\begin{aligned} A+B+C &= 2,\\ -B-3C &= -5,\\ -A-2B+2C &= -1. \end{aligned}$$

We find that $C = 1$, $B = 2$, and $A = -1$. Therefore

$$\int \frac{2x^2-5x-1}{x^3-2x^2-x+2}\,dx = -\ln|x-2| + 2\ln|x-1| + \ln|x+1| + C = \ln\left|\frac{(x-1)^2(x+1)}{x-2}\right| + C.$$

C08S0M.067: $\displaystyle\int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{1}{2}\ln|e^{2x}-1| + C.$

C08S0M.068: Let $u = \sin x$. Then

$$\begin{aligned} J &= \int \frac{\cos x}{\sin^2 x - 3\sin x + 2}\,dx = \int \frac{1}{u^2-3u+2}\,du\\ &= \int \left(\frac{1}{u-2} - \frac{1}{u-1}\right)du = \ln\left|\frac{u-2}{u-1}\right| + C = \ln\left(\frac{2-\sin x}{1-\sin x}\right) + C. \end{aligned}$$

C08S0M.069: The partial fractions decomposition of the integrand has the form

$$\frac{2x^3+3x^2+4}{(x+1)^4} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4}.$$

It follows that $A(x^3+3x^2+3x+1)+B(x^2+2x+1)+C(x+1)+D = 2x^3+3x^2+4$, and thus

$$\begin{aligned} A &= 2,\\ 3A+B &= 3,\\ 3A+2B+C &= 0,\\ A+B+C+D &= 4. \end{aligned}$$

The triangular form of this system of equations makes it easy to solve for $A = 2$, $B = -3$, $C = 0$, and $D = 5$. Therefore

$$\int \frac{2x^3 + 3x^2 + 4}{(x+1)^4}\,dx = 2\ln|x+1| + \frac{3}{x+1} - \frac{5}{3(x+1)^3} + C.$$

C08S0M.070: Let $u = \tan x$. Then $du = \sec^2 x\,dx$ and

$$\begin{aligned}\int \frac{\sec^2 x}{\tan^2 x + 2\tan x + 2}\,dx &= \int \frac{1}{u^2 + 2u + 2}\,du \\ &= \int \frac{1}{1 + (1+u)^2}\,du = \arctan(u+1) + C = \arctan(1 + \tan x) + C.\end{aligned}$$

C08S0M.071: The partial fractions decomposition of the integrand has the form

$$\frac{x^3 + x^2 + 2x + 1}{x^4 + 2x^2 + 1} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2+1)^2},$$

so that $A(x^3 + x) + B(x^2 + 1) + Cx + D = x^3 + x^2 + 2x + 1$. It follows that $A = 1$, $B = 1$, $C = 1$, and $D = 0$. Therefore

$$\int \frac{x^3 + x^2 + 2x + 1}{x^4 + 2x^2 + 1}\,dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx = \frac{1}{2}\ln(x^2+1) + \arctan x - \frac{1}{2(x^2+1)} + C.$$

C08S0M.072: Integration by parts will probably succeed, but we prefer an approach that cannot fail:

$$\begin{aligned}\cos 3x &= \cos(2x + x) = \cos 2x\,\cos x - \sin 2x\,\sin x \\ &= (1 - 2\sin^2 x)\cos x - 2\sin^2 x\,\cos x = \cos x - 4\sin^2 x\,\cos x.\end{aligned}$$

Therefore

$$\int \sin x\,\cos 3x\,dx = \int (\sin x\,\cos x - 4\sin^3 x\,\cos x)\,dx = \frac{1}{2}\sin^2 x - \sin^4 x + C.$$

There are many ways of finding this particular antiderivative, so a variety of answers—any two of which differ by a constant—is possible. For example, *Mathematica* 3.0, *Maple* V Release 5.1, *Derive* 2.56, and the TI-92 all return the antiderivative in the form

$$\int \sin x\,\cos 3x\,dx = \frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x$$

(remember that computer algebra programs normally omit the constant of integration). See Problems 59 through 61 of Section 8.4 (and the preceding instructions) for an explanation of how these computer algebra programs may have computed this antiderivative. By the way, integration by parts (twice) with $u = \cos 3x$ (the first time) yields

$$\int \sin x\,\cos 3x\,dx = \frac{1}{8}(\cos x\,\cos 3x + 3\sin x\,\sin 3x) + C.$$

C08S0M.073: Use integration by parts with $u = x^3$ and $dv = c^2(x^3 - 1)^{1/2}\,dx$. Then

$$du = 3x^2\ dx; \qquad \text{choose} \quad v = \frac{2}{9}(x^3 - 1)^{3/2}.$$

Then

$$\begin{aligned}
K = \int x^5\sqrt{x^3 - 1}\ dx &= \frac{2}{9}x^3(x^3 - 1)^{3/2} - \int \frac{2}{3}x^2(x^3 - 1)^{3/2}\ dx \\
&= \frac{2}{9}x^3(x^3 - 1)^{3/2} - \frac{4}{45}(x^3 - 1)^{5/2} + C = \frac{1}{45}\left[10x^3(x^3 - 1)^{3/2} - 4(x^3 - 1)^{5/2}\right] + C \\
&= \frac{2(x^3 - 1)^{1/2}}{45}\left[5x^3(x^3 - 1) - 2(x^3 - 1)^2\right] + C = \frac{2}{45}(3x^6 - x^3 - 2)\sqrt{x^3 - 1} + C.
\end{aligned}$$

C08S0M.074: Let $u = \ln(x^2 + 2x)$ and $dv = dx$. Then

$$du = \frac{2x + 2}{x^2 + 2x}\ dx; \qquad \text{choose} \quad v = x + 2.$$

Then

$$\begin{aligned}
\int \ln(x^2 + 2x)\ dx &= (x + 2)\ln(x^2 + 2x) - \int \frac{2(x + 1)(x + 2)}{x(x + 2)}\ dx \\
&= (x + 2)\ln(x^2 + 2x) - 2\int \left(1 + \frac{1}{x}\right) dx = (x + 2)\ln(x^2 + 2x) - 2x - 2\ln|x| + C.
\end{aligned}$$

As usual, other forms of the answer abound. *Mathematica* 3.0 and *Maple* V version 5.1, for example, report (in effect) that

$$\int \ln(x^2 + 2x)\ dx = x\ln(x^2 + 2x) + 2\ln(x + 2) - 2x + C.$$

C08S0M.075: $\displaystyle\int \frac{\sqrt{1 + \sin x}}{\sec x}\ dx = \int (1 + \sin x)^{1/2}\cos x\ dx = \frac{2}{3}(1 + \sin x)^{3/2} + C.$

C08S0M.076: Let $u = x^3$. Then $dx = 3u^2\ du$ and $u = x^{1/3}$. Hence

$$\int \frac{1}{x^{2/3}(1 + x^{2/3})}\ dx = \int \frac{3u^2}{u^2(1 + u^2)}\ du = 3\arctan u + C = 3\arctan\left(x^{1/3}\right) + C.$$

C08S0M.077: $\displaystyle\int \frac{\sin x}{\sin 2x}\ dx = \int \frac{\sin x}{2\sin x\ \cos x}\ dx = \frac{1}{2}\int \sec x\ dx = \frac{1}{2}\ln|\sec x + \tan x| + C.$

C08S0M.078: We will use the half-angle formula in Eq. (10) of Appendix C (it also appears inside the front cover of the hardcover edition of the text).

$$\int \sqrt{1 + \cos t}\ dt = \sqrt{2}\int \sqrt{\frac{1 + \cos t}{2}}\ dt = \sqrt{2}\int \sqrt{\cos^2\left(\frac{t}{2}\right)}\ dt = \sqrt{2}\int \left|\cos\frac{t}{2}\right|\ dt.$$

Hence

$$\int \sqrt{1+\cos t}\ dt = \begin{cases} 2\sqrt{2}\sin\dfrac{t}{2} + C & \text{if } 0 \leqq t \leqq \pi, \\[2ex] -2\sqrt{2}\sin\dfrac{t}{2} + C & \text{if } \pi \leqq t \leqq 2\pi. \end{cases}$$

Mathematica 3.0 returns

$$\int \sqrt{1+\cos t}\ dt \ = \ 2(1+\cos t)^{1/2}\ \tan\left(\frac{1}{2}t\right) \ + \ C$$

if you prefer an antiderivative that is not defined "piecewise." *Maple* V version 5.1 yields

$$\int \sqrt{1+\cos t}\ dt \ = \ \frac{\sqrt{2}\ \sqrt{2+2\cos t}}{1+\cos t}\cdot \sin t \ + \ C.$$

C08S0M.079: We multiply numerator and denominator by $\sqrt{1-\sin t}$ and thus obtain

$$\int \sqrt{1+\sin t}\ dt = \int \frac{\sqrt{1-\sin^2 t}}{\sqrt{1-\sin t}}\ dt = \int \frac{\sqrt{\cos^2 t}}{\sqrt{1-\sin t}}\ dt = \int (1-\sin t)^{-1/2}\,|\cos t\,|\ dt.$$

Therefore

$$\int \sqrt{1+\sin t}\ dt = \begin{cases} -2\sqrt{1-\sin t}\ + C & \text{if } \cos t \geqq 0, \\[2ex] 2\sqrt{1-\sin t}\ + C & \text{if } \cos t \leqq 0. \end{cases}$$

Mathematica 3.0 returns (in effect)

$$\int \sqrt{1+\sin t}\ dt = \frac{2\left(\sin\dfrac{t}{2} - \cos\dfrac{t}{2}\right)}{\sin\dfrac{t}{2} + \cos\dfrac{t}{2}}\cdot\sqrt{1+\sin t}\ + C$$

and *Maple* V version 5.1 yields

$$\int \sqrt{1+\sin t}\ dt \ = \ \frac{2(-1+\sin t)\sqrt{1+\sin t}}{\cos t} \ + \ C.$$

C08S0M.080: Let $u = \tan t$. Then $du = \sec^2 t\ dt$, and hence

$$\int \frac{\sec^2 t}{1-\tan^2 t}\ dt = \int \frac{1}{1-u^2}\ du = \frac{1}{2}\int\left(\frac{1}{1+u} + \frac{1}{1-u}\right)du$$

$$= \frac{1}{2}\ln 1+u| - \frac{1}{2}\ln|1-u| + C = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\tan t}{1-\tan t}\right| + C.$$

Alternatively, if you are fond of trigonometric solutions, multiply numerator and denominator in the original integral by $\cos^2 t$ to obtain

$$\int \frac{1}{\cos^2 t - \sin^2 t}\ dt = \int \frac{1}{\cos 2t}\ dt = \int \sec 2t\ dt = \frac{1}{2}\ln|\sec 2t + \tan 2t\,| + C.$$

C08S0M.081: Integrate by parts with $u = \ln(x^2 + x + 1)$ and $dv = dx$. Then

$$du = \frac{2x+1}{x^2+x+1}\,dx; \qquad \text{choose} \quad v = x.$$

Then

$$J = \int \ln(x^2+x+1)\,dx = x\ln(x^2+x+1) - \int \frac{2x^2+x}{x^2+x+1}\,dx = x\ln(x^2+x+1) - \int \left(2 - \frac{x+2}{x^2+x+1}\right) dx.$$

Now

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2 u + \frac{3}{4} = \frac{3}{4}\sec^2 u$$

if $x + \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\tan u$, so we let

$$x = \frac{-1+\sqrt{3}\,\tan u}{2}, \quad dx = \frac{\sqrt{3}}{2}\sec^2 u\,du, \quad \tan u = \frac{2x+1}{\sqrt{3}}.$$

A reference triangle for this substitution has acute angle u, opposite side $2x+1$, and adjacent side $\sqrt{3}$, so its hypotenuse has length $2\sqrt{x^2+x+1}$. Thus

$$\int \frac{x+2}{x^2+x+1}\,dx = \int \frac{\frac{1}{2}\left(3+\sqrt{3}\,\tan u\right)}{\frac{3}{4}\sec^2 u} \cdot \frac{\sqrt{3}}{2}\sec^2 u\,du = \frac{\sqrt{3}}{3}\int \left(3+\sqrt{3}\,\tan u\right) du$$

$$= \frac{\sqrt{3}}{3}\left(3u + \sqrt{3}\,\ln|\sec u|\right) + C = \frac{\sqrt{3}}{3}\left[3\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \sqrt{3}\,\ln\left(\frac{2\sqrt{x^2+x+1}}{\sqrt{3}}\right)\right] + C.$$

Therefore

$$J = x\ln(x^2+x+1) - 2x + \sqrt{3}\,\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\ln(x^2+x+1) + C.$$

C08S0M.082: Let $w = e^x$, so that $dw = e^x\,dx$. Then

$$J = \int e^x \arcsin\left(e^x\right)\,dx = \int \arcsin w\,dw.$$

Then let $u = \arcsin w$ and $dv = dw$. Thus

$$du = \frac{1}{\sqrt{1-w^2}}\,dw; \qquad \text{choose} \quad v = w.$$

Consequently,

$$J = w\arcsin w - \int w(1-w^2)^{-1/2}\,dw = w\arcsin w + (1-w^2)^{1/2} + C = e^x\arcsin\left(e^x\right) + \sqrt{1-e^{2x}} + C.$$

C08S0M.083: Integrate by parts with

$$u = \arctan x \quad \text{and} \quad dv = \frac{1}{x^2}\,dx.$$

Then

$$du = \frac{1}{1+x^2}\,dx; \qquad \text{choose} \quad v = -\frac{1}{x}.$$

Thus

$$\begin{aligned}\int \frac{\arctan x}{x^2}\,dx &= -\frac{1}{x}\arctan x + \int \frac{1}{x(x^2+1)}\,dx \\ &= -\frac{1}{x}\arctan x + \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = -\frac{1}{x}\arctan x + \ln|x| - \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$

C08S0M.084: Let $x = 5\sec u$: $dx = 5\sec u\,\tan u\,du$, $x^2 - 25 = 25\sec^2 u - 25 = 25\tan^2 u$. So

$$\begin{aligned}I = \int \frac{x^2}{\sqrt{x^2-25}}\,dx &= \int \frac{25\sec^2 u}{5\tan u}\cdot 5\sec u\,\tan u\,du \\ &= \int 25\sec^3 u\,du = \frac{25}{2}\big(\sec u\,\tan u + \ln|\sec u + \tan u\,|\big) + C.\end{aligned}$$

A reference triangle for this trigonometric substitution has acute angle u, hypotenuse x, and adjacent side 5, so its opposite side has length $\sqrt{x^2-25}$. Therefore

$$I = \frac{25}{2}\left(\frac{x\sqrt{x^2-25}}{25} + \ln\left|\frac{x+\sqrt{x^2-25}}{5}\right|\right) + C = \frac{x}{2}\sqrt{x^2-25} + \frac{25}{2}\ln\left|x + \sqrt{x^2-25}\right| + C_1.$$

C08S0M.085: We use the idea of the method of partial fractions but avoid the algebra as follows:

$$\frac{x^3}{(x^2+1)^2} = \frac{x^3+x-x}{(x^2+1)^2} = \frac{x(x^2+1)}{(x^2+1)^2} - \frac{x}{(x^2+1)^2} = \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

Therefore

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx = \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C.$$

C08S0M.086: Note that $6x - x^2 = -(x^2-6x) = 9 - (x^2-6x+9) = 9 - (x-3)^3 = 9 - 9\sin^2 u = 9\cos^2 u$ if $x - 3 = 3\sin u$. Hence we let

$$x = 3 + 3\sin u. \qquad \text{Then} \quad dx = 3\cos u\,du \quad \text{and} \quad \sin u = \frac{x-3}{3}.$$

A reference triangle for this substitution has acute angle u, opposite side $x-3$, and hypotenuse 3, thus adjacent side of length $\sqrt{6x-x^2}$. Hence

$$\begin{aligned}\int \frac{1}{x\sqrt{6x-x^2}}\,dx &= \int \frac{3\cos u}{3(1+\sin u)\cdot 3\cos u}\,du = \frac{1}{3}\int \frac{1}{1+\sin u}\,du = \frac{1}{3}\int \frac{1-\sin u}{\cos^2 u}\,du \\ &= \frac{1}{3}\int (\sec^2 u - \sec u\,\tan u)\,du = \frac{1}{3}(\tan u - \sec u) + C = \frac{1}{3}\cdot\frac{x-6}{\sqrt{6x-x^2}} + C \\ &= -\frac{6x-x^2}{3x\sqrt{6x-x^2}} + C = -\frac{\sqrt{6x-x^2}}{3x} + C.\end{aligned}$$

C08S0M.087: First, $x^2+4 = 4\tan^2 u + 4 = 4\sec^2 u$ if $x = 2\tan u$, so that $dx = 2\sec^2 u\ du$ and $\tan u = \frac{1}{2}x$. A reference triangle for this substitution has acute angle u, opposite side x, and adjacent side 2, thus its hypotenuse has length $\sqrt{x^2+4}$. Therefore

$$\int \frac{3x+2}{(x^2+4)^{3/2}}\ dx = \int \frac{2+6\tan u}{8\sec^3 u} \cdot 2\sec^2 u\ du = \frac{1}{2}\int (1+3\tan u)\cos u\ du = \frac{1}{2}\int (\cos u + 3\sin u)\ du$$

$$= \frac{1}{2}(\sin u - 3\cos u) + C = \frac{1}{2}\left(\frac{x}{\sqrt{x^2+4}} - \frac{6}{\sqrt{x^2+4}}\right) + C = \frac{x-6}{2\sqrt{x^2+4}} + C.$$

C08S0M.088: Let $u = \ln x$ and $dv = x^{3/2}\ dx$. Then

$$du = \frac{1}{x}\ dx; \qquad \text{choose} \quad v = \frac{2}{5}x^{5/2}.$$

Then

$$\int x^{3/2}\ln x\ dx = \frac{2}{5}x^{5/2}\ln x - \frac{2}{5}\int x^{3/2}\ dx = \frac{2}{5}x^{5/2}\ln x - \frac{4}{25}x^{5/2} + C = \frac{2}{25}x^{5/2}(5\ln x - 2) + C.$$

C08S0M.089: $\displaystyle\int \frac{\sqrt{1+\sin^2 x}}{\sec x\ \csc x}\ dx = \int (1+\sin^2 x)^{1/2}(\sin x\ \cos x)\ dx = \frac{1}{3}(1+\sin^2 x)^{3/2} + C.$

C08S0M.090: Let $u = \sqrt{\sin x}$. Then

$$du = \frac{1}{2}(\sin x)^{-1/2}\cos x\ dx = \frac{1}{(2\sec x)\sqrt{\sin x}}\ dx.$$

So

$$\int \frac{\exp\left(\sqrt{\sin x}\right)}{(\sec x)\sqrt{\sin x}}\ dx = 2\int e^u du = 2e^u + C = 2\exp\left(\sqrt{\sin x}\right) + C.$$

C08S0M.091: Integration by parts is indicated, but there are several choices. We found that $u = \sin x$ and $dv = xe^x\ dx$ was a bad choice, leading to the correct but complicated antiderivative

$$I = \frac{1}{2}(x-1)e^x\sin x - \frac{1}{2}(x-2)e^x\cos x + \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x + C.$$

A better choice is $u = x$, $dv = e^x\sin x\ dx$. Even so, we need a preliminary computation to find v. We integrate by parts with $p = e^x$ and $dq = \sin x\ dx$. Then $dp = e^x\ dx$ and we may choose $q = -\cos x$. Then

$$v = \int e^x\sin x\ dx = -e^x\cos x + \int e^x\cos x\ dx. \tag{1}$$

We integrate by parts a second time, with $p = e^x$ and $dq = \cos x\ dx$. Then with $dp = e^x\ dx$ and $q = \sin x$ we find that

$$v = -e^x\cos x + e^x\sin x - \int e^x\sin x\ dx = e^x\sin x - e^x\cos x - v,$$

and therefore we may choose $v = \dfrac{1}{2}e^x(\sin x - \cos x)$. Also $du = dx$, and thus

$$I = \int xe^x \sin x\, dx = u \cdot v - \int v\, du = \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int e^x \sin x\, dx + \frac{1}{2}\int e^x \cos x\, dx.$$

Now by Eq. (1),

$$\int e^x \cos x\, dx = e^x \cos x + v = e^x \cos x + \frac{1}{2}e^x(\sin x - \cos x) + C = \frac{1}{2}e^x(\sin x + \cos x) + C.$$

Therefore

$$\begin{aligned} I &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\cdot\frac{1}{2}e^x(\sin x - \cos x) + \frac{1}{2}\cdot\frac{1}{2}e^x(\sin x + \cos x) + C \\ &= \frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x \cos x + C = \frac{1}{2}e^x(x\sin x - x\cos x + \cos x) + C. \end{aligned}$$

Maple V version 5.1 yields essentially the same answer:

$$I \;=\; \frac{1}{2}\big[xe^x \sin x - (x-1)e^x \cos x\big] \;+\; C.$$

C08S0M.092: Integrate by parts with $u = x^{3/2}$ and $dv = x^{1/2}\exp\left(x^{3/2}\right)\, dx$. Then

$$du = \frac{3}{2}x^{1/2}\, dx; \qquad \text{choose} \quad v = \frac{2}{3}\exp\left(x^{3/2}\right).$$

Thus

$$\begin{aligned} \int x^2 \exp\left(x^{3/2}\right)\, dx &= \frac{2}{3}x^{3/2}\exp\left(x^{3/2}\right) - \int x^{1/2}\exp\left(x^{3/2}\right)\, dx \\ &= \frac{2}{3}x^{3/2}\exp\left(x^{3/2}\right) - \frac{2}{3}\exp\left(x^{3/2}\right) + C = \frac{2}{3}\left(x^{3/2} - 1\right)\exp\left(x^{3/2}\right) + C. \end{aligned}$$

C08S0M.093: First integrate by parts with $u = \arctan x$ and $dv = (x-1)^{-3}\, dx$. Then the new integrand will be a rational function of x, to which the method of partial fractions can be applied if necessary. We have

$$du = \frac{1}{1+x^2}\, dx \quad \text{and we choose} \quad v = -\frac{1}{2}(x-1)^{-2}.$$

Then

$$J = \int \frac{\arctan x}{(x-1)^3}\, dx = -\frac{\arctan x}{2(x-1)^2} + \frac{1}{2}\int \frac{1}{(x-1)^2(x^2+1)}\, dx.$$

The partial fractions decomposition

$$\frac{1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

leads to the equation $A(x^3 - x^2 + x - 1) + B(x^2+1) + C(x^3 + 2x^2 + x) + D(x^2 - 2x + 1) = 1,$ and thus to the simultaneous equations

$$\begin{aligned} A + C &= 0, \qquad & -A + B - 2C + D &= 0, \\ A + C - 2D &= 0, \qquad & -A + B + D &= 1. \end{aligned}$$

Then $C = -A$ and $D = 0$, so that $A + B = 0$ and $-A + B = 1$. Hence $B = \frac{1}{2}$, $A = -\frac{1}{2}$, and $C = \frac{1}{2}$. Therefore

$$\frac{1}{2} \cdot \frac{1}{(x-1)^2(x^2+1)} = \frac{1}{4}\left(-\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{x}{x^2+1}\right).$$

So, finally,

$$J = -\frac{\arctan x}{2(x-1)^2} + \frac{1}{4}\left(\frac{1}{2}\ln(x^2+1) - \frac{1}{x-1} - \ln|x-1|\right) + C.$$

C08S0M.094: Integrate by parts with $u = \ln(1+\sqrt{x}\,)$ and $dv = dx$. Then

$$du = \frac{1}{2\sqrt{x}\,(1+\sqrt{x}\,)}\,dx, \qquad \text{but choose} \quad v = x - 1 = \left(\sqrt{x}\,+1\right)\left(\sqrt{x}\,-1\right).$$

Then

$$\int \ln\left(1+\sqrt{x}\,\right)\,dx = (x-1)\ln\left(1+\sqrt{x}\,\right) - \int \frac{(\sqrt{x}\,+1)(\sqrt{x}\,-1)}{2\sqrt{x}\,(1+\sqrt{x}\,)}\,dx$$

$$= (x-1)\ln\left(1+\sqrt{x}\,\right) - \int \frac{\sqrt{x}\,-1}{2\sqrt{x}}\,dx = (x-1)\ln\left(1+\sqrt{x}\,\right) - \frac{1}{2}x + \sqrt{x}\,+C.$$

C08S0M.095: First,

$$3 + 6x - 9x^2 = -(9x^2 - 6x - 3) = -(9x^2 - 6x + 1 - 4) = 4 - (3x-1)^2 = 4 - 4\sin^2 u = 4\cos^2 u$$

if $3x - 1 = 2\sin u$, so we let

$$x = \frac{1 + 2\sin u}{3}; \quad dx = \frac{2}{3}\cos u\,du \quad \text{and} \quad \sin u = \frac{3x-1}{2}.$$

A reference triangle for this substitution has acute angle u, opposite side $3x - 1$, and hypotenuse 2, so its adjacent (to u) side has length $\sqrt{3 + 6x - 9x^2}$. Therefore

$$\int \frac{2x+3}{\sqrt{3+6x-9x^2}}\,dx = \int \frac{\frac{2}{3}\left(1 + 2\sin u + \frac{9}{2}\right)}{2\cos u} \cdot \frac{2}{3}\cos u\,du = \frac{2}{9}\int\left(\frac{11}{2} + 2\sin u\right)du$$

$$= \frac{11}{9}u - \frac{4}{9}\cos u + C = \frac{11}{9}\arcsin\left(\frac{3x-1}{2}\right) - \frac{2}{9}\sqrt{3+6x-9x^2}\,+C.$$

C08S0M.096: Let $u = e^x$: $du = e^x\,dx = u\,dx$, so $dx = \dfrac{1}{u}\,du$. Therefore

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du = \operatorname{arcsec}|u| + C = \operatorname{arcsec}(e^x) + C.$$

To "visualize" $\operatorname{arcsec}(e^x)$, consider a right triangle with acute angle u, adjacent side 1, and hypotenuse e^x. Then the side of this triangle opposite the angle u has length $\sqrt{e^{2x}-1}$, and therefore an alternative form of the antiderivative in this problem is

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \arctan\left(\sqrt{e^{2x}-1}\,\right) + C.$$

C08S0M.097: The method of partial fractions can be avoided with the substitution $u = x - 1$, so that $x = u + 1$ and $dx = du$. Then

$$\begin{aligned}\int \frac{x^4}{(x-1)^2}\,dx &= \int \frac{(u+1)^4}{u^2}\,du = \frac{u^4 + 4u^3 + 6u^2 + 4u + 1}{u^2}\,du \\ &= \int \left(u^2 + 4u + 6 + \frac{4}{u} + \frac{1}{u^2}\right) du = \frac{1}{3}u^3 + 2u^2 + 6u + 4\ln|u| - \frac{1}{u} + C \\ &= \frac{1}{3}(x-1)^3 + 2(x-1)^2 + 6(x-1) + 4\ln|x-1| - \frac{1}{x-1} + C \\ &= \frac{1}{3}x^3 + x^2 + 3x - \frac{1}{x-1} + 4\ln|x-1| + C_1\end{aligned}$$

where $C_1 = C + \dfrac{13}{3}$.

If you used the method of partial fractions, you should have found that

$$\frac{x^4}{(x-1)^2} = x^2 + 2x + 3 + \frac{4}{x-1} + \frac{1}{(x-1)^2}.$$

C08S0M.098: Integrate by parts with $u = \arctan\left(\sqrt{x}\,\right)$ and $dv = x^{3/2}\,dx$. Then

$$du = \frac{1}{2\sqrt{x}\,(1+x)}\,dx; \quad \text{choose} \quad v = \frac{2}{5}x^{5/2}. \quad \text{Then}$$

$$\begin{aligned}\int x^{3/2}\arctan\left(\sqrt{x}\,\right)dx &= \frac{2}{5}x^{5/2}\arctan\left(\sqrt{x}\,\right) - \frac{1}{5}\int \frac{x^2}{x+1}\,dx \\ &= \frac{2}{5}x^{5/2}\arctan\left(\sqrt{x}\,\right) - \frac{1}{5}\int \left(x - 1 + \frac{1}{x+1}\right) dx \\ &= \frac{2}{5}x^{5/2}\arctan\left(\sqrt{x}\,\right) - \frac{1}{10}x^2 + \frac{1}{5}x - \frac{1}{5}\ln|x+1| + C.\end{aligned}$$

C08S0M.099: Integrate by parts with $u = \operatorname{arcsec}\left(\sqrt{x}\,\right)$ and $dv = dx$. Then

$$du = \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2x\sqrt{x-1}}\,dx; \quad \text{choose} \quad v = x.$$

Then

$$\int \operatorname{arcsec}\left(\sqrt{x}\,\right)dx = x\operatorname{arcsec}\left(\sqrt{x}\,\right) - \int \frac{1}{2}(x-1)^{-1/2}\,dx = x\operatorname{arcsec}\left(\sqrt{x}\,\right) - \sqrt{x-1} + C.$$

C08S0M.100: Let

$$u^2 = \frac{1-x^2}{1+x^2}.$$

Then

$$(1+x^2)u^2 = 1-x^2; \qquad u^2 + x^2u^2 = 1-x^2;$$

$$x^2 + x^2u^2 = 1-u^2; \qquad x^2 = \frac{1-u^2}{1+u^2}.$$

Therefore

$$2x\ dx = \frac{(1+u^2)(-2u)-(2u)(1-u^2)}{(1+u^2)^2}\ du = -\frac{4u}{(1+u^2)^2}\ du.$$

Thus

$$K = \int x\sqrt{\frac{1-x^2}{1+x^2}}\ dx = -\int \frac{2u^2}{(1+u^2)^2}\ du.$$

The partial fractions decomposition of the integrand is

$$-\frac{2u^2}{(1+u^2)^2} = -\frac{2}{1+u^2} + \frac{2}{(1+u^2)^2},$$

and a trigonometric substitution would be required to antidifferentiate the last fraction (unless you find it in a table of integrals), so one might as well avoid the partial fractions approach by using a trigonometric substitution to begin with. Thus we let $u = \tan\theta$, so that $du = \sec^2\theta\ d\theta$. Then

$$K = \int \frac{-2\tan^2\theta}{\sec^4\theta}\ \sec^2\theta\ d\theta = -\int 2\sin^2\theta\ d\theta = -\int (1-\cos 2\theta)\ d\theta = -\theta + \sin\theta\ \cos\theta + C.$$

A reference triangle for this trigonometric substitution has acute angle θ, opposite side u, and adjacent side 1. So the hypotenuse of this triangle has length $\sqrt{1+u^2}$, and therefore

$$K = -\arctan u + \frac{u}{1+u^2} + C = -\arctan\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) + \frac{1}{1+\dfrac{1-x^2}{1+x^2}} \cdot \sqrt{\frac{1-x^2}{1+x^2}}\ + C$$

$$= -\arctan\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) + \frac{1+x^2}{2} \cdot \sqrt{\frac{1-x^2}{1+x^2}}\ + C = -\arctan\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) + \frac{1}{2}\sqrt{1-x^4}\ + C.$$

C07S0M.101: If $y = \cosh x$, then

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x,$$

so that $ds = \cosh x\ dx$. We will also use Eq. (11) of Section 7.6 to find that the surface area of revolution is

$$A = \int_0^1 2\pi\cosh^2 x\ dx = \pi\int_0^1 (1+\cosh 2x)\ dx = \pi\left[x + \frac{1}{2}\sinh 2x\right]_0^1$$

$$= \pi\left(1 + \frac{e^2 - e^{-2}}{4}\right) = \frac{\pi}{4}\left(4 + e^2 - e^{-2}\right) \approx 8.83865166003373.$$

C08S0M.102: If $y = e^{-x}$, then

$$1+\left(\frac{dy}{dx}\right)^2 = 1+e^{-2x}, \quad \text{so that} \quad ds = \sqrt{1+e^{-2x}}\ dx.$$

Therefore the arc length in question is

$$L = \int_0^1 \sqrt{1+e^{-2x}}\ dx.$$

Let $u = e^{-x}$. Then $du = -e^{-x}\ dx = -u\ dx$, so that $dx = -\dfrac{1}{u}\ du$. Therefore

$$L = \int_{x=0}^1 -\frac{\sqrt{1+u^2}}{u}\ du.$$

Now let $u = \tan\theta$. Then $du = \sec^2\theta\ d\theta$, so that

$$\begin{aligned}
L &= -\int_{x=0}^1 \frac{\sec\theta}{\tan\theta}\cdot\sec^2\theta\ d\theta = -\int_{x=0}^1 \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\ d\theta\\
&= -\int_{x=0}^1 (\csc\theta + \sec\theta\,\tan\theta)\ d\theta = \Big[-\sec\theta - \ln|\csc\theta - \cot\theta|\Big]_{x=0}^1.
\end{aligned}$$

A reference triangle for the trigonometric substitution has acute angle θ, opposite side u, and adjacent side 1. Hence the hypotenuse of that triangle has length $\sqrt{1+u^2}$, and consequently

$$\begin{aligned}
L &= -\left[\sqrt{1+u^2} + \ln\left(\frac{\sqrt{1+u^2}-1}{u}\right)\right]_{x=0}^1 = \left[\sqrt{1+u^2} + \ln\left(\sqrt{1+u^2}-1\right) - \ln u\right]_{u=1/e}^1\\
&= \sqrt{2} + \ln\left(\sqrt{2}-1\right) - \sqrt{1+e^{-2}} - \ln\left(\sqrt{1+e^{-2}}-1\right) - 1 \approx 1.192701401972.
\end{aligned}$$

C08S0M.103: Given $y = e^{-x}$,

$$\frac{dy}{dx} = -e^{-x}, \qquad \text{so that} \quad ds = \sqrt{1+e^{-2x}}\ dx.$$

Hence the surface area of revolution is

$$A_t = \int_0^t 2\pi e^{-x}\sqrt{1+e^{-2x}}\ dx.$$

Let $u = e^{-x}$. Then $du = -e^{-x}\ dx$, and so

$$A_t = \int_{x=0}^t -2\pi\sqrt{1+u^2}\ du.$$

Next, let $u = \tan\theta$. Then $du = \sec^2\theta\ d\theta$ and $\sqrt{1+u^2} = \sec\theta$. Therefore

$$\begin{aligned}
A_t &= -2\pi\int_{x=0}^t \sec^3\theta\ d\theta = -\pi\Big[\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_{x=0}^t\\
&= -\pi\left[u\sqrt{1+u^2} + \ln\left(u+\sqrt{1+u^2}\right)\right]_{x=0}^t = -\pi\left[e^{-x}\sqrt{1+e^{-2x}} + \ln\left(e^{-x}+\sqrt{1+e^{-2x}}\right)\right]_0^t\\
&= \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right) - e^{-t}\sqrt{1+e^{-2t}} - \ln\left(e^{-t}+\sqrt{1+e^{-2t}}\right)\right].
\end{aligned}$$

Clearly

$$\lim_{t\to\infty} A_t = \pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] \approx 7.211799724207.$$

C08S0M.104: Given $y = \dfrac{1}{x}$, we have arc length element

$$ds = \sqrt{1+\frac{1}{x^4}}\ dx = \frac{\sqrt{x^4+1}}{x^2}\ dx.$$

Hence the surface area of revolution is

$$A_t = \int_1^t \frac{2\pi}{x}\cdot\frac{\sqrt{x^4+1}}{x^2}\ dx = \int_1^t \frac{2\pi x\sqrt{x^4+1}}{x^4}\ dx.$$

Let $x = \sqrt{\tan u}$. Then $x^2 = \tan u$ and $2x\ dx = \sec^2 u\ du$. For later reference, also note that $1+x^4 = \sec^2 u$, so that $\sec u = \sqrt{1+x^4}$. Thus

$$\begin{aligned}
A_t &= \pi\int_{x=1}^t \frac{\sec^3 u}{\tan^2 u}\ du = \pi\int_{x=1}^t \frac{1+\tan^2 u}{\tan^2 u}\cdot\sec u\ du = \pi\int_{x=1}^t \left(\frac{\sec u}{\tan^2 u} + \sec u\right) du\\
&= \pi\int_{x=1}^t \left(\frac{\cos u}{\sin^2 u} + \sec u\right) du = \pi\Big[-\frac{1}{\sin u} + \ln|\sec u + \tan u|\Big]_{x=1}^t\\
&= \pi\left[-\frac{\sqrt{1+x^4}}{x^2} + \ln\left(x^2+\sqrt{1+x^4}\right)\right]_1^t = \pi\left[\sqrt{2} - \ln\left(1+\sqrt{2}\right) - \frac{\sqrt{1+t^4}}{t^2} + \ln\left(t^2+\sqrt{1+t^4}\right)\right].
\end{aligned}$$

Therefore

$$\lim_{t\to\infty} A_t = \pi\left[\sqrt{2} - \ln\left(1+\sqrt{2}\right) - 1 + \lim_{t\to\infty}\ \ln\left(t^2+\sqrt{1+t^4}\right)\right] = +\infty.$$

Also see Problem 51 in Section 8.8.

C08S0M.105: Given $y = (x^2-1)^{1/2}$, we have

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}},$$

so the arc length element is

$$ds = \sqrt{1+\frac{x^2}{x^2-1}}\ dx = \left(\frac{2x^2-1}{x^2-1}\right)^{1/2}\ dx.$$

We will use integral formula 44 from the endpapers to help us find that the surface area of revolution is

$$\begin{aligned}
A &= \int_1^2 2\pi(x^2-1)^{1/2}\left(\frac{2x^2-1}{x^2-1}\right)^{1/2}\ dx = 2\pi\int_1^2 \sqrt{2x^2-1}\ dx = 2\pi\sqrt{2}\ \int_1^2 \left(x^2-\frac{1}{2}\right)^{1/2}\ dx\\
&= 2\pi\sqrt{2}\ \left[\frac{x}{2}\left(x^2-\frac{1}{2}\right)^{1/2} - \frac{1}{4}\ln\left(x+\left(x^2-\frac{1}{2}\right)^{1/2}\right)\right]_1^2 \qquad \text{(using integral formula 44)}\\
&= 2\pi\sqrt{2}\ \left[\sqrt{\frac{7}{2}} - \frac{1}{4}\ln\left(2+\sqrt{\frac{7}{2}}\right) - \frac{1}{2}\sqrt{\frac{1}{2}} + \frac{1}{4}\ln\left(1+\sqrt{\frac{1}{2}}\right)\right] \approx 11.663528688558.
\end{aligned}$$

C08S0M.106: Let $u = (\ln x)^n$ and $dv = x^m\,dx$ (m and n are positive integers). Then

$$du = \frac{n(\ln x)^{n-1}}{x}\,dx \quad \text{and we choose} \quad v = \frac{x^{m+1}}{m+1}.$$

Then

$$\int x^m(\ln x)^n\,dx = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$$

In particular,

$$\begin{aligned}
\int x^3(\ln x)^3\,dx &= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{4}\int x^3(\ln x)^2\,dx \\
&= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{4}\left[\frac{1}{4}x^4(\ln x)^2 - \frac{2}{4}\int x^3\ln x\,dx\right] \\
&= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{16}x^4(\ln x)^2 + \frac{3}{8}\left[\frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx\right] \\
&= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{16}x^4(\ln x)^2 + \frac{3}{32}x^4\ln x - \frac{3}{128}x^4 + C.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_1^e x^3(\ln x)^3\,dx &= \frac{1}{4}e^4 - \frac{3}{16}e^4 + \frac{3}{32}e^4 - \frac{3}{128}e^4 + \frac{3}{128} \\
&= \frac{32-24+12-3}{128}\cdot e^4 + \frac{3}{128} = \frac{17e^4+3}{128} \approx 7.274754301277.
\end{aligned}$$

C08S0M.107: Let $u = (\sin x)^{m-1}$ and $dv = (\cos x)^n \sin x\,dx$ (m and n are integers with $n \geqq 0$ and $m \geqq 2$). Then

$$du = (m-1)(\sin x)^{m-2}\cos x\,dx \quad \text{and} \quad v = -\frac{1}{n+1}(\cos x)^{n+1}.$$

Therefore

$$\begin{aligned}
I = \int (\sin x)^m(\cos x)^n\,dx &= -\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^{n+2}\,dx \\
&= -\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^n(1-\sin^2 x)\,dx \\
&= -\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^n\,dx - \frac{m-1}{n+1}I.
\end{aligned}$$

Thus

$$\left(\frac{m-1}{n+1}+1\right)I = -\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^n\,dx;$$

$$\frac{m+n}{n+1}I = -\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^n\,dx;$$

and, finally,

$$\begin{aligned}
I &= -\frac{n+1}{m+n}\cdot\frac{1}{n+1}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{n+1}{m+n}\cdot\frac{m-1}{n+1}\int(\sin x)^{m-2}(\cos x)^n\ dx\\
&= -\frac{1}{m+n}(\sin x)^{m-1}(\cos x)^{n+1} + \frac{m-1}{m+n}\int(\sin x)^{m-2}(\cos x)^n\ dx.
\end{aligned}$$

C08S0M.108: We will use the reduction formula in Problem 107 and also the reduction formula in Problem 54 of Section 8.3; viz.,

$$\int(\cos x)^n\ dx = \frac{1}{n}(\cos x)^{n-1}\sin x + \frac{n-1}{n}\int(\cos x)^{n-2}\ dx$$

if n is an integer and $n \geqq 2$. Then

$$\begin{aligned}
\int \sin^6 x\ \cos^5 x\ dx &= -\frac{1}{11}\sin^5 x\ \cos^6 x + \frac{5}{11}\int \sin^4 x\ \cos^5 x\ dx\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x + \frac{5}{11}\left[-\frac{1}{9}\sin^3 x\cos^6 x + \frac{1}{3}\int \sin^2 x\ \cos^5 x\ dx\right]\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x + \frac{5}{33}\int \sin^2 x\ \cos^5 x\ dx\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x + \frac{5}{33}\left[-\frac{1}{7}\sin x\ \cos^6 x + \frac{1}{7}\int \cos^5 x\ dx\right]\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x - \frac{5}{231}\sin x\cos^6 x + \frac{5}{231}\int \cos^5 x\ dx\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x - \frac{5}{231}\sin x\cos^6 x\\
&\qquad + \frac{5}{231}\left[\frac{1}{5}\cos^4 x\ \sin x + \frac{4}{5}\int \cos^3 x\ dx\right]\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x - \frac{5}{231}\sin x\cos^6 x\\
&\qquad + \frac{1}{231}\cos^4 x\ \sin x + \frac{4}{231}\left[\frac{1}{3}\cos^2 x\ \sin x + \frac{2}{3}\int \cos x\ dx\right]\\
&= -\frac{1}{11}\sin^5 x\ \cos^6 x - \frac{5}{99}\sin^3 x\ \cos^6 x - \frac{5}{231}\sin x\cos^6 x\\
&\qquad + \frac{1}{231}\cos^4 x\ \sin x + \frac{4}{693}\cos^2 x\ \sin x + \frac{8}{693}\sin x + C.
\end{aligned}$$

Therefore $\displaystyle\int_0^{\pi/2} \sin^6 x\ \cos^5 x\ dx = \frac{8}{693} \approx 0.011544011544.$

C08S0M.109: We need a result in Problem 58 of Section 8.3: If n is a positive integer, then

$$\int_0^{\pi/2}(\sin x)^{2n}\ dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}.$$

The area in question is

$$A = 2\int_0^2 x^{5/2}\sqrt{2-x}\ dx.$$

Let $x = 2\sin^2\theta$. Then $\sqrt{2-x}\ = \sqrt{2}\ \cos\theta$ and $dx = 4\sin\theta\,\cos\theta\,d\theta$. Therefore

$$\begin{aligned}
A &= 2\int_0^{\pi/2}(2^{5/2}\sin^5\theta)\left(\sqrt{2}\ \cos\theta\right)(4\sin\theta\,\cos\theta)\ d\theta = 64\int_0^{\pi/2}\sin^6\theta\,\cos^2\theta\ d\theta\\
&= 64\int_0^{\pi/2}\left[\sin^6\theta - \sin^8\theta\right]\ d\theta = 64\left[\left(\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right) - \left(\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\right]\\
&= 64\cdot\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{8} = \frac{5\pi}{4} \approx 3.9269908169872415480783 04.
\end{aligned}$$

C08S0M.110: If $0 < t < 1$, then $0 < 1 - t < 1$, so

$$\frac{t^4(1-t)^4}{1+t^2} > 0.$$

Therefore

$$0 < \int_0^1 \frac{t^4(1-t)^4}{1+t^2}\ dt.$$

Division of denominator into numerator yields

$$\frac{t^4(1-t)^4}{1+t^2} = t^6 - 4t^5 + 5t^4 - 4t^2 + 4 - \frac{4}{t^2+1}.$$

Consequently

$$\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\ dt = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi.$$

C08S0M.111: First,

$$\begin{aligned}
\int_0^1 t^4(1-t)^4\ dt &= \int_0^1 (t^8 - 4t^7 + 6t^6 - 4t^5 + t^4)\ dt\\
&= \frac{1}{9} - \frac{1}{2} + \frac{6}{7} - \frac{2}{3} + \frac{1}{5} = \frac{70 - 315 + 540 - 420 + 126}{630} = \frac{1}{630}.
\end{aligned}$$

But $\dfrac{1}{2} \leqq \dfrac{1}{t^2+1} \leqq 1$ if $0 \leqq t \leqq 1$. Therefore

$$\begin{aligned}
\frac{1}{1260} &< \int_0^1 \frac{t^4(1-t)^4}{t^2+1}\ dt < \frac{1}{630};\\
\frac{1}{1260} &< \frac{22}{7} - \pi < \frac{1}{630};\\
-\frac{1}{630} &< \pi - \frac{22}{7} < -\frac{1}{1260};\\
\frac{22}{7} - \frac{1}{630} &< \pi < \frac{22}{7} - \frac{1}{1260}.
\end{aligned}$$

C08S0M.112: Given $y = \dfrac{4}{5}x^{5/4}$:

$$\frac{dy}{dx} = x^{1/4}, \quad \text{so} \quad ds = \sqrt{1+\sqrt{x}}\ dx.$$

Therefore the length of the given curve is

$$L = \int_0^1 \sqrt{1+\sqrt{x}}\ dx.$$

Let $x = \tan^4\theta$. Then $dx = 4\tan^3\theta\,\sec^2\theta\,d\theta$ and $\sqrt{1+\sqrt{x}}\ = \sqrt{1+\tan^2\theta}\ = \sec\theta$. Hence

$$L = \int_0^{\pi/4} 4\tan^3\theta\sec^3\theta\,d\theta = 4\int_0^{\pi/4}(\sec^5\theta - \sec^3\theta)\tan\theta\,d\theta = 4\int_0^{\pi/4}(\sec^4\theta - \sec^2\theta)\,\sec\theta\,\tan\theta\,d\theta$$

$$= 4\left[\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right]_0^{\pi/4} = 4\left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3}\right) = \frac{8}{15}\left(1+\sqrt{2}\right) \approx 1.287580566599.$$

C08S0M.113: Given $y = \dfrac{4}{3}x^{3/4}$,

$$\frac{dy}{dx} = x^{-1/4}, \quad \text{so} \quad ds = \sqrt{1+x^{-1/2}}\ dx.$$

Therefore the length of the given curve is

$$L = \int_1^4 \sqrt{1+x^{-1/2}}\ dx.$$

Now $1 + x^{-1/2} = 1 + \tan^2 u = \sec^2 u$ if $x^{-1/2} = \tan^2 u$; that is, if $x^{1/2} = \cot^2 u$. So we let $x = \cot^4 u$; then $dx = -4\cot^3 u\,\csc^2 u\,du$, and this substitution yields

$$L = \int_{x=1}^4 (-4\sec u)(\cot^3 u\,\csc^3 u)\,du = -4\int_{x=1}^4 \frac{\cos^3 u}{\sin^5 u\,\cos u}\,du$$

$$= 4\int_{x=1}^4 \frac{\sin^2 u - 1}{\sin^5 u}\,du = 4\int_{x=1}^4 (\csc^3 u - \csc^5 u)\,du.$$

Now we need to pause to develop a reduction formula. Suppose that n is an integer and $n \geqq 3$. Let

$$J_n = \int \csc^n x\,dx.$$

Now integrate by parts: Let $u = (\csc x)^{n-2}$ and $dv = \csc^2 x\,dx$. Then

$$du = -(n-2)(\csc x)^{n-3}(\csc x\,\cot x)\,dx; \qquad \text{choose } v = -\cot x.$$

Then

$$J_n = -(\csc x)^{n-2}\cot x - (n-2)\int(\csc x)^{n-2}\cot^2 x\,dx$$

$$= -(\csc x)^{n-2}\cot x - (n-2)\int(\csc x)^{n-2}(\csc^2 x - 1)\,dx$$

$$= -(\csc x)^{n-2}\cot x - (n-2)J_n + (n-2)\int(\csc x)^{n-2}\,dx.$$

Thus

$$(n-1)J_n = -(\csc x)^{n-2} \cot x + (n-2)\int (\csc x)^{n-2}\, dx,$$

and, finally,

$$J_n = \int \csc^n x\, dx = -\frac{1}{n-1}(\csc x)^{n-2} \cot x + \frac{n-2}{n-1}\int (\csc x)^{n-2}\, dx.$$

With the aid of this reduction formula, we find that

$$\int \csc^3 x\, dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\int \csc x\, dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

and

$$\begin{aligned}\int \csc^5 x\, dx &= -\frac{1}{4}\csc^3 x\,\cot x + \frac{3}{4}\int \csc^3 x\, dx\\ &= -\frac{1}{4}\csc^3 x\,\cot x - \frac{3}{8}\csc x\,\cot x + \frac{3}{8}\ln|\csc x - \cot x| + C.\end{aligned}$$

Therefore

$$\begin{aligned}L &= 4\left[\frac{1}{4}\csc^3 u\,\cot u - \frac{1}{8}\csc u\,\cot u + \frac{1}{8}\ln|\csc u - \cot u|\right]_{x=1}^{4}\\ &= \left[x^{1/4}\left(1+\sqrt{x}\,\right)^{3/2} - \frac{1}{2}x^{1/4}\left(1+\sqrt{x}\,\right)^{1/2} + \frac{1}{2}\ln\left(\sqrt{1+\sqrt{x}}\; - x^{1/4}\right)\right]_1^4\\ &= 3\sqrt{6} - \frac{1}{2}\sqrt{6} + \frac{1}{2}\ln\left(\sqrt{3} - \sqrt{2}\right) - 2\sqrt{2} + \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(\sqrt{2} - 1\right)\\ &= \frac{5}{2}\sqrt{6} - \frac{3}{2}\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}\right) = \frac{5}{2}\sqrt{6} - \frac{3}{2}\sqrt{2} + \frac{1}{2}\ln\left[\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}+1\right)\right]\\ &= \frac{1}{2}\left[5\sqrt{6} - 3\sqrt{2} + \ln\left(\sqrt{6} + \sqrt{3} - 2 - \sqrt{2}\right)\right] \approx 3.869982889518.\end{aligned}$$

C08S0M.114: Let $y(t)$ denote the depth of water (in feet) in the tank at time t (in minutes) and let $V(t)$ denote the volume of water in the tank (in cubic feet) at time t. Similar triangles show that

$$V(t) = \frac{1}{3}\pi\cdot\frac{1}{4}[y(t)]^3; \qquad \text{that is,} \qquad V = \frac{\pi}{12}y^3. \tag{1}$$

We are also given

$$\frac{dV}{dt} = 50 - 10\sqrt{y}; \qquad V(0) = 0, \quad y(0) = 0.$$

By Eq. (1) and the chain rule,

$$\frac{dV}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt} = 50y - 10y^{1/2},$$

so that

$$\frac{y^2}{5-y^{1/2}}\,dy = \frac{40}{\pi}\,dt. \tag{2}$$

Let $u = 5 - y^{1/2}$. Then $y^{1/2} = 5 - u$, $y = (5-u)^2$, and $dy = -2(5-u)\,du$. Hence Eq. (2) takes the form

$$-\frac{2}{u}(5-u)^5\,du = \frac{40}{\pi}\,dt; \quad \text{that is,} \quad \frac{(5-u)^5}{u}\,du = -\frac{20}{\pi}\,dt.$$

Therefore

$$\frac{3125 - 3125u + 1250u^2 - 250u^3 + 25u^4 - u^5}{u}\,du = -\frac{20}{\pi}t\,dt.$$

Now we antidifferentiate and find that

$$F(u) = 3125\ln u - 3125u + 625u^2 - \frac{250}{3}u^3 + \frac{25}{4}u^4 - \frac{1}{5}u^5 = C - \frac{20}{\pi}t.$$

When $y = 0$, $y = 0$ and $u = 5$. Hence

$$C = 3125\ln 5 - \frac{85625}{12}.$$

Moreover, $y = 9$ when $u = 2$, at which time

$$t = \frac{\pi}{20}\left[C - F(2)\right] = \frac{\pi}{20}\left(3125\ln\frac{5}{2} - \frac{56247}{20}\right) \approx 8.0202562539 \quad \text{(minutes)}.$$

C08S0M.115: Let $u = e^x$: $du = e^x\,dx = u\,dx$, so $dx = \dfrac{1}{u}\,du$. Hence

$$\int \frac{1}{1+e^x+e^{-x}}\,dx = \int \frac{1}{u(1+u+u^{-1})}\,du = \int \frac{1}{u^2+u+1}\,du = \int \frac{1}{\left(u+\frac{1}{2}\right)^2+\frac{3}{4}}\,du$$

$$= \frac{4}{3}\int \frac{1}{\left(\dfrac{2u+1}{\sqrt{3}}\right)^2+1}\,du = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2u+1}{\sqrt{3}}\right) + C = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2e^x+1}{\sqrt{3}}\right) + C.$$

This substitution will always succeed in integrals of this ilk because you'll always obtain a rational function of u after making the substitution.

C08S0M.116: The only real root of the equation $x^3 + x + 1 = 0$ is $r \approx -0.6823278038$. Division of $x - r$ into $x^3 + x + 1$ yields the quotient $x^2 + rx + 1 + r^2$. The partial fractions decomposition of the integrand has the form

$$\frac{1}{x^3+x+1} = \frac{A}{x-r} + \frac{Bx+C}{x^2+rx+1+r^2}.$$

It follows that $A(x^2 + rx + 1 + r^2) + B(x^2 - rx) + C(x - r) = 1$, and thus that

$$A + B = 0, \quad rA - rB + C = 0, \quad \text{and} \quad -rC = 1.$$

It is easy to solve these equations for

$$A = \frac{1}{3r^2+1}, \quad B = -\frac{1}{3r^2+1}, \quad \text{and} \quad C = -\frac{2r}{3r^2+1}.$$

Therefore

$$\int_0^1 \frac{1}{x^3+x+1} = \frac{1}{3r^2+1}\int_0^1 \left(\frac{1}{x-r} - \frac{x+2r}{x^2+rx+1+r^2}\right) dx.$$

Let $s = 1 + r^2$. Then

$$\begin{aligned} Q = \int \frac{1}{x^2+rx+s}\,dx &= \int \frac{1}{x^2+rx+\frac{1}{4}r^2+s-\frac{1}{4}r^2}\,dx \\ &= \int \frac{1}{\left(x+\frac{1}{2}r\right)^2+\frac{1}{2}(4s-r^2)}\,dx = \int \frac{1}{\left(x+\frac{1}{2}r\right)^2+\omega^2}\,dx \end{aligned}$$

where $\omega^2 = \frac{1}{4}(4s - r^2)$. Hence

$$Q = \frac{1}{\omega}\int \frac{\dfrac{1}{\omega}}{\left(\dfrac{2x+r}{2\omega}\right)^2+1}\,dx = \frac{1}{\omega}\arctan\left(\frac{2x+r}{2\omega}\right)+C = \frac{2}{\sqrt{4s-r^2}}\arctan\left(\frac{2x+r}{\sqrt{4s-r^2}}\right)+C.$$

With $s = r^2 + 1$, we find that $4s - r^2 = 3r^2 + 4$, and so

$$Q = \frac{2}{\sqrt{3r^2+4}}\arctan\left(\frac{2x+r}{\sqrt{3r^2+4}}\right)+C.$$

Therefore

$$\begin{aligned} \int \frac{x+2r}{x^2+rx+1+r^2}\,dr &= \frac{1}{2}\int \frac{2x+r}{x^2+rx+1+r^2}\,dx + \frac{3r}{2}\int \frac{1}{x^2+rx+1+r^2}\,dx \\ &= \frac{1}{2}\ln(x^2+rx+1+r^2) + \frac{3r}{\sqrt{3r^2+4}}\arctan\left(\frac{2x+r}{\sqrt{3r^2+4}}\right)+C. \end{aligned}$$

Thus

$$\int_0^1 \frac{1}{x^3+x+1}\,dx = \frac{1}{3r^2+1}\left[\ln(x-r) - \frac{1}{2}\ln(x^2+rx+1+r^2) - \frac{3r}{\sqrt{3r^2+4}}\arctan\left(\frac{2x+r}{\sqrt{3r^2+4}}\right)\right]_0^1$$

$$\begin{aligned} &= \frac{1}{3r^2+1}\left[\ln(1-r) - \ln(-r) - \frac{1}{2}\ln(r^2+r+2) + \frac{1}{2}\ln(r^2+1)\right. \\ &\qquad \left. - \frac{3r}{\sqrt{3r^2+4}}\arctan\left(\frac{2+r}{\sqrt{3r^2+4}}\right) + \frac{3r}{\sqrt{3r^2+4}}\arctan\left(\frac{r}{\sqrt{3r^2+4}}\right)\right] \approx 0.630319322412. \end{aligned}$$

C08S0M.117: $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx = -\ln\left(1+e^{-x}\right)+C.$

Mathematica 3.0 returns the antiderivative $x - \ln(1+e^x)$.

C08S0M.118: The substitution $u = x^4 + x^2$, $du = 2(x + 2x^3)\,dx$ yields

$$\int \frac{x+2x^3}{(x^4+x^2)^3}\,dx = \frac{1}{2}\int \frac{1}{u^3}\,du = -\frac{1}{4u^2}+C = -\frac{1}{4(x^4+x^2)^2}+C.$$

In case you need to know, the partial fractions decomposition of the integrand is

$$\frac{x+2x^3}{(x^4+x^2)^3} = \frac{1}{x^5} - \frac{1}{x^3} + \frac{x}{(1+x^2)^2} + \frac{x}{(1+x^2)^3}.$$

C08S0M.119: Let $u = \tan\theta$, so that $du = \sec^2\theta\, d\theta$, $\theta = \arctan u$, $1 + u^2 = \sec^2\theta$, and

$$d\theta = \frac{1}{1+u^2}\, du.$$

Thus

$$H = \int \sqrt{\tan\theta}\, d\theta = \int \frac{u^{1/2}}{1+u^2}\, du.$$

Now let $u = x^2$, so that $x = u^{1/2}$ and $du = 2x\, dx$. Then

$$H = \int \frac{2x^2}{1+x^4}\, dx.$$

The partial fractions decomposition of the last integrand has the form

$$\frac{2x^2}{x^4+1} = \frac{Ax+B}{x^2 - x\sqrt{2}\, + 1} + \frac{Cx+D}{x^2 + x\sqrt{2}\, + 1},$$

and we find that

$$A(x^3 + x^2\sqrt{2}\, + x) + B(x^2 + x\sqrt{2}\, + 1) + C(x^3 - x\sqrt{2}\, + x) + D(x^2 - x\sqrt{2} + 1) = 2x^2.$$

Thus we obtain the simultaneous equations

$$A + C = 0, \qquad A\sqrt{2} + B - C\sqrt{2} + D = 2,$$
$$A + B\sqrt{2} + C - D\sqrt{2} = 0, \qquad B + D = 0.$$

Then it's easy to solve for $B = D = 0$, $A = \frac{1}{2}\sqrt{2}$, and $C = -\frac{1}{2}\sqrt{2}$. Therefore

$$\frac{2x^2}{1+x^4} = \frac{\sqrt{2}}{2}\left(\frac{x}{x^2 - x\sqrt{2}\, + 1} - \frac{x}{x^2 + x\sqrt{2}\, + 1}\right).$$

Now let $r = \frac{1}{2}\sqrt{2}$. Then

$$\frac{x}{x^2 - 2rx + 1} = \frac{1}{2} \cdot \frac{2x - 2r}{x^2 - 2rx + 1} + \frac{r}{x^2 - 2rx + 1}.$$

It's easy to antidifferentiate the second fraction; you'll obtain $\frac{1}{2}\ln(x^2 - 2rx + 1) + C$. For the last fraction, a trigonometric substitution is one technique that will succeed:

$$x^2 - 2rx + 1 = x^2 - 2rx + r^2 + 1 - r^2 = (x-r)^2 + \frac{1}{2} = \frac{1}{2}\tan^2\omega + \frac{1}{2} = \frac{1}{2}\sec^2\omega$$

if $x - r = r\tan\omega$. So we let

$$x = r + r\tan\omega, \quad dx = r\sec^2\omega\, d\omega, \quad \tan\omega = \frac{x-r}{r} = 2rx - 1 = x\sqrt{2}\, - 1.$$

Then

$$\int \frac{r}{x^2 - 2rx + 1}\,dx = r\int \frac{r}{\frac{1}{2}\sec^2\omega}\cdot \sec^2\omega\,d\omega = 2r^2\omega + C = \omega + C = \arctan\left(x\sqrt{2}\,-1\right) + C.$$

The case of $x^2 + 2rx + 1$ is handled similarly—only a few sign changes—and the result is that

$$H = \frac{\sqrt{2}}{4}\left[2\arctan\left(x\sqrt{2}\,-1\right) + 2\arctan\left(x\sqrt{2}\,+1\right) + \ln\left(x^2 - x\sqrt{2}\,+1\right) - \ln\left(x^2 + x\sqrt{2}\,+1\right)\right] + C.$$

Therefore, because $x = \sqrt{u}\, = \sqrt{\tan\theta}\,,$ we finally obtain

$$\begin{aligned} H &= \int \sqrt{\tan\theta}\,d\theta \\ &= \frac{\sqrt{2}}{4}\left[2\arctan\left(-1+\sqrt{2\tan\theta}\,\right) + 2\arctan\left(1+\sqrt{2\tan\theta}\,\right)\right. \\ &\qquad\qquad \left. + \ln\left(\tan\theta - \sqrt{2\tan\theta}\,+1\right) - \ln\left(\tan\theta + \sqrt{2\tan\theta}+1\right)\right] + C. \end{aligned}$$

C08S0M.120: If

$$u^n = \frac{ax+b}{cx+d}, \qquad \text{then} \qquad x = \frac{b - du^n}{cu^n - a}.$$

Moreover,

$$dx = \frac{ndu^{n-1}(a - cu^n) - ncu^{n-1}(b - du^n)}{(cu^n - a)^2}\,du.$$

Thus if $p(x)$ is a polynomial, then

$$p(x)\left(\frac{ax+b}{cx+d}\right)^{1/n}\,dx = (u^n)^{1/n}\,p\left(\frac{b-du^n}{cu^n - a}\right)\cdot\frac{ndu^{n-1}(a - cu^n) - ncu^{n-1}(b-du^n)}{(cu^n - a)^2}\,du$$

is a rational function of u.

C08S0M.121: Let $u^2 = 3x - 2$. Then

$$x = \frac{u^2+2}{3}, \quad dx = \frac{2}{3}u\,du, \quad \text{and} \quad u = (3x-2)^{1/2}.$$

Hence

$$\begin{aligned}\int x^3\sqrt{3x-2}\,dx &= \int \frac{1}{27}(u^2+2)^3\cdot u\cdot\frac{2}{3}u\,du = \frac{2}{81}\int(u^8+6u^6+12u^4+8u^2)\,du\\
&= \frac{2}{81}\left(\frac{1}{9}u^9+\frac{6}{7}u^7+\frac{12}{5}u^5+\frac{8}{3}u^3\right)+C\\
&= \frac{2}{729}(3x-2)^{9/2}+\frac{4}{189}(3x-2)^{7/2}+\frac{8}{135}(3x-2)^{5/2}+\frac{16}{243}(3x-2)^{3/2}+C\\
&= \frac{1}{25515}\Big[70(3x-2)^{9/2}+540(3x-2)^{7/2}+1512(3x-2)^{5/2}+1680(3x-2)^{3/2}\Big]+C\\
&= \frac{2\sqrt{3x-2}}{25515}\Big[35(3x-2)^4+270(3x-2)^3+756(3x-2)^2+840(3x-2)\Big]+C\\
&= \frac{2(3x-2)^{3/2}}{25515}(945x^3+540x^2+288x+128)+C.\end{aligned}$$

Mathematica 3.0 obtains the equivalent

$$\int x^3\sqrt{3x-2}\,dx = \frac{\sqrt{3x-2}}{25515}(5670x^4-540x^3-432x^2-384x-512)+C,$$

and *Maple* V version 5.1 gives the antiderivative in the form shown in the third line of the display here.

C08S0M.122: Let $u^3=x^2+1$. Then $u=(x^2+1)^{1/3},\ x^2=u^3-1,\ 2x\,dx=3u^2\,du,$ and $x\,dx=\frac{3}{2}u^2\,du.$ Hence

$$\begin{aligned}\int x^3(x^2+1)^{1/3}\,dx &= \int(u^3-1)\cdot u\cdot\frac{3}{2}u^2\,du = \frac{3}{2}\int(u^6-u^3)\,du = \frac{3}{2}\left(\frac{1}{7}u^7-\frac{1}{4}u^4\right)+C\\
&= \frac{3}{56}\left(4u^7-7u^4\right)+C = \frac{3u^4}{56}\left(4u^3-7\right)+C = \frac{3(x^2+1)^{4/3}}{56}\left[4(x^2+1)-7\right]+C\\
&= \frac{3}{56}(4x^2-3)(x^2+1)^{4/3}+C = \frac{3}{56}(4x^4+x^2-3)(x^2+1)^{1/3}+C.\end{aligned}$$

C08S0M.123: Let $u^3=x^2-1$. Then $3u^2\,du=2x\,dx,\ x\,dx=\frac{3}{2}u^2\,du,\ x^2=u^3+1,$ and $(x^2-1)^{4/3}=u^4.$ Thus

$$\begin{aligned}\int\frac{x^3}{(x^2-1)^{4/3}}\,dx &= \int\frac{u^3+1}{u^4}\cdot\frac{3}{2}u^2\,du = \frac{3}{2}\int\left(u+\frac{1}{u^2}\right)du = \frac{3}{2}\left(\frac{1}{2}u^2-\frac{1}{u}\right)+C = \frac{3}{4}\left(u^2-\frac{2}{u}\right)+C\\
&= \frac{3}{4}\cdot\frac{u^3-2}{u}+C = \frac{3(u^3-2)}{4u}+C = \frac{3(x^2-1-2)}{4(x^2-1)^{1/3}}+C = \frac{3(x^2-3)}{4(x^2-1)^{1/3}}+C.\end{aligned}$$

C08S0M.124: Let $u^2=x-1$. Then $u=(x-1)^{1/2},\ x=u^2+1,$ and $dx=2u\,du.$ So

$$\int x^2(x-1)^{3/2}\,dx = \int (u^2+1)^2 \cdot u^3 \cdot 2u\,du = 2\int (u^8+2u^6+u^4)\,du = \frac{2}{9}u^9 + \frac{4}{7}u^7 + \frac{2}{5}u^5 + C$$

$$= \frac{2u^5}{315}(35u^4+90u^2+63) + C = \frac{2(x-1)^{5/2}}{315}\Big[35(x-1)^2 + 90(x-1) + 63\Big] + C$$

$$= \frac{2(x-1)^{5/2}}{315}(35x^2 - 70x + 35 + 90x - 90 + 63) + C$$

$$= \frac{2(x-1)^{5/2}}{315}(35x^2+20x+8) + C = \frac{2\sqrt{x-1}}{315}(x^2-2x+1)(35x^2+20x+8) + C$$

$$= \frac{2\sqrt{x-1}}{315}(35x^4 - 50x^3 + 3x^2 + 4x + 8) + C.$$

C08S0M.125: Let $u^2 = x^3+1$. Then $u = (x^3+1)^{1/2}$, $x^3 = u^2 - 1$, $3x^2\,dx = 2u\,du$, and $x^2\,dx = \frac{2}{3}u\,du$. Therefore

$$\int \frac{x^5}{\sqrt{x^3+1}}\,dx = \int \frac{x^3}{(x^3+1)^{1/2}} \cdot x^2\,dx = \int \frac{u^2-1}{u} \cdot \frac{2}{3}u\,du = \frac{2}{3}\int (u^2-1)\,du = \frac{2}{3}\left(\frac{1}{3}u^3 - u\right) + C$$

$$= \frac{2}{9}(u^3 - 3u) + C = \frac{2u}{9}(u^2-3) + C = \frac{2}{9}(x^2-2)\sqrt{x^3+1} + C.$$

C08S0M.126: Let $u^3 = x^4+1$. Then $x^4 = u^3-1$, $4x^3\,dx = 3u^2\,du$, $x^3\,dx = \frac{3}{4}u^2\,du$, $x = (u^3-1)^{1/4}$, and $u = (x^4+1)^{1/3}$. Thus

$$\int x^7(x^4+1)^{1/3}\,dx = \int x^4(x^4+1)^{1/3}\,x^3\,dx = \int (u^3-1)\cdot u \cdot \frac{3}{4}u^2\,du = \frac{3}{4}\int (u^6-u^3)\,du$$

$$= \frac{3}{4}\left(\frac{1}{7}u^7 - \frac{1}{4}u^4\right) + C = \frac{3}{112}(4u^7 - 7u^4) + C = \frac{3u^4}{112}(4u^3-7) + C$$

$$= \frac{3(x^4+1)^{4/3}}{112}(4x^4-3) + C = \frac{3}{112}(4x^8+x^4-3)(x^4+1)^{1/3} + C.$$

C08S0M.127: Let

$$u^2 = \frac{1+x}{1-x}. \quad \text{Then} \quad x = \frac{u^2-1}{u^2+1} \quad \text{and} \quad dx = \frac{2u(u^2+1) - 2u(u^2-1)}{(u^2+1)^2}\,du = \frac{4u}{(u^2+1)^2}\,du.$$

Therefore

$$J = \int \left(\frac{1+x}{1-x}\right)^{1/2} dx = \int \frac{4u^2}{(u^2+1)^2}\,du.$$

The method of partial fractions requires us to integrate $4(u^2+1)^{-2}$ with a trigonometric substitution, so we might as well go directly to the trigonometry. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$. A reference triangle for this substitution has acute angle θ, opposite side u, and adjacent side 1, and therefore has hypotenuse of length $\sqrt{1+u^2}$. Therefore

$$J = \int \frac{4\tan^2\theta}{\sec^4\theta} \cdot \sec^2\theta \, d\theta = \int 4\sin^2\theta \, d\theta = \int 2(1-\cos 2\theta) \, d\theta = 2(\theta - \sin\theta\,\cos\theta) + C$$

$$= 2\left(\arctan u - \frac{u}{u^2+1}\right) + C = 2\left[\arctan\left(\frac{1+x}{1-x}\right)^{1/2} - \frac{\left(\dfrac{1+x}{1-x}\right)^{1/2}}{\dfrac{1+x}{1-x}+1}\right] + C$$

$$= 2\left[\arctan\left(\frac{1+x}{1-x}\right)^{1/2} - \frac{1}{2}(1-x)\left(\frac{1+x}{1-x}\right)^{1/2}\right] + C = 2\arctan\left(\frac{1+x}{1-x}\right)^{1/2} - \sqrt{1-x^2} + C.$$

C08S0M.128: Let $u = x^2+1$. Then $x = u^2-1$, $dx = 2u\,du$, and $u = (x+1)^{1/2}$. Thus

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u^2-1}{u} \cdot 2u\,du = 2\int (u^2-1)\,du = 2\left(\frac{1}{3}u^3 - u\right) + C$$

$$= \frac{2u}{3}(u^2-3) + C = \frac{2}{3}(x-2)\sqrt{x+1} + C.$$

C08S0M.129: Let $u^3 = x+1$. Then $u = (x+1)^{1/3}$, $x = u^3-1$, and $dx = 3u^2\,du$. Thus

$$K = \int \frac{(x+1)^{1/3}}{x}\,dx = \int \frac{u}{u^3-1} \cdot 3u^2\,du = 3\int \frac{u^3}{u^3-1}\,du = 3\int \left(1 + \frac{1}{u^3-1}\right)du.$$

The partial fractions decomposition of the last fraction has the form

$$\frac{1}{u^3-1} = \frac{A}{u-1} + \frac{Bu+C}{u^2+u+1}.$$

Thus we find that $A(u^2+u+1) + b(u^2-u) + C(u-1) = 1$, and thus we obtain the simultaneous equations

$$A+B=0, \quad A-B+C=0, \quad \text{and} \quad A-C=1.$$

It follows that $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = -\frac{2}{3}$. Thus

$$\frac{1}{u^3-1} = \frac{1}{3}\left(\frac{1}{u-1} - \frac{u+2}{u^2+u+1}\right).$$

Now

$$u^2+u+1 = \left(u+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2\theta + \frac{3}{4} = \frac{3}{4}\sec^2\theta \quad \text{if} \quad \frac{\sqrt{3}}{2}\tan\theta = u + \frac{1}{2}.$$

Therefore we let

$$u = \frac{-1+\sqrt{3}\,\tan\theta}{2}; \quad \text{thus} \quad \tan\theta = \frac{2u+1}{\sqrt{3}}, \quad du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta.$$

Note that a reference triangle for this substitution has acute angle θ, opposite side $2u+1$, and adjacent side $\sqrt{3}$, thus hypotenuse of length $2\sqrt{u^2+u+1}$. Therefore

$$\int \frac{u+2}{u^2+u+1}\,du = \frac{1}{2}\int \frac{3+\sqrt{3}\,\tan\theta}{\frac{3}{4}\sec^2\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = \frac{\sqrt{3}}{3}\int \left(3+\sqrt{3}\,\tan\theta\right)d\theta$$

$$= \frac{\sqrt{3}}{3}\left(3\theta+\sqrt{3}\,\ln|\sec\theta|\right)+C = \frac{\sqrt{3}}{3}\left[3\arctan\left(\frac{2u+1}{\sqrt{3}}\right)+\sqrt{3}\,\ln\left(\sqrt{u^2+u+1}\right)\right]+C_1$$

$$= \sqrt{3}\,\arctan\left(\frac{2u+1}{\sqrt{3}}\right)+\frac{1}{2}\ln(u^2+u+1)+C_1.$$

Therefore

$$K = 3\int\left(1+\frac{1}{u^3-1}\right)du = \int\left(3+\frac{1}{u-1}-\frac{u+2}{u^2+u+1}\right)du$$

$$= 3u+\ln|u-1|-\sqrt{3}\,\arctan\left(\frac{2u+1}{\sqrt{3}}\right)-\frac{1}{2}\ln(u^2+u+1)+C$$

$$= 3(x+1)^{1/3}+\ln\left|(x+1)^{1/3}-1\right|$$

$$-\sqrt{3}\,\arctan\left(\frac{2(x+1)^{1/3}+1}{\sqrt{3}}\right)-\frac{1}{2}\ln\left((x+1)^{2/3}+(x+1)^{1/3}+1\right)+C.$$

Mathematica 3.0 returns

$$K = 3(x+1)^{1/3}-\frac{3(1+x^{-1})^{2/3}}{2(1+x)^{2/3}}\cdot\text{Hypergeometric2F1}\left[\frac{2}{3},\,\frac{2}{3},\,\frac{5}{3},\,-\frac{1}{x}\right]$$

(the generalized hypergeometric function is discussed more fully in some of the solutions in Chapter 11). *Derive* 2.56 and *Maple* V version 5.1 yield, with minor algebraic changes, the answer we obtained "by hand."

C08S0M.130: The substitution $x = u^2$ yields

$$I = \int\sqrt{1+\sqrt{x}}\,dx = \int 2u\sqrt{1+u}\,du,$$

thus requiring a second substitution of the form $1+u=z$ or $1+u=z^2$. A better substitution would be $x=(u-1)^2$, so that $dx = 2(u-1)\,du$, $u-1=x^{1/2}$, and $u = 1+x^{1/2}$. Thereby we find that

$$I = \int 2u^{1/2}(u-1)\,du = 2\int\left(u^{3/2}-u^{1/2}\right)du = 2\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C = \frac{4}{15}\left(3u^{5/2}-5u^{3/2}\right)+C$$

$$= \frac{4u^{3/2}}{15}(3u-5)+C = \frac{4\left(1+\sqrt{x}\,\right)^{3/2}}{15}\left(3\sqrt{x}-2\right)+C = \frac{4}{15}\left(3x+\sqrt{x}-2\right)\sqrt{1+\sqrt{x}}+C.$$

C08S0M.131: The substitution $u^2 = 1+e^x$ entails $e^{2x} = u^2-1$, $2x = \ln(u^2-1)$,

$$x = \frac{1}{2}\ln(u^2-1), \quad \text{and} \quad dx = \frac{1}{2}\cdot\frac{2u}{u^2-1}\,du = \frac{u}{u^2-1}\,du.$$

Therefore

$$\int \sqrt{1+e^{2x}}\ dx = \int \frac{u^2}{u^2-1}\ du = \int \left(1 + \frac{1}{u^2-1}\right) du = \int \left(1 + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{u+1}\right) du$$

$$= \frac{1}{2}\int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du = \frac{1}{2}\left(2u + \ln\left|\frac{u-1}{u+1}\right|\right) + C = \sqrt{1+e^{2x}} + \frac{1}{2}\ln\left|\frac{-1+\sqrt{1+e^{2x}}}{1+\sqrt{1+e^{2x}}}\right| + C.$$

C08S0M.132: Given $y = \dfrac{2}{3}x^{3/2}$, we have

$$\frac{dy}{dx} = x^{1/2}, \qquad \text{so} \qquad ds = \sqrt{1+x}\ dx.$$

Thus the surface area of revolution around the x-axis of the given curve is

$$A = \int_3^8 \frac{4\pi}{3} x^{3/2}(1+x)^{1/2}\ dx.$$

Let $u = x^2$. Then $dx = 2u\ du$, and therefore

$$A = \int_{x=3}^8 \frac{4\pi}{3} \cdot u^3 \cdot (1+u^2)^{1/2} \cdot 2u\ du = \frac{8\pi}{3}\int_{x=3}^8 u^4(1+u^2)^{1/2}\ du.$$

Now let $u = \tan\theta$, so that $du = \sec^2\theta\ d\theta$ and $1 + u^2 = \sec^2\theta$. Thus

$$A = \frac{8\pi}{3}\int_{x=3}^8 \tan^4\theta\ \sec^3\theta\ d\theta = \frac{8\pi}{3}\int_{x=3}^8 (\sec^2\theta\ - 1)^2\ \sec^3\theta\ d\theta = \frac{8\pi}{3}\int_{x=3}^8 (\sec^7\theta - 2\sec^5\theta + \sec^3\theta)\ d\theta.$$

Now use the result in Example 6 of Section 8.3: If n is an integer and $n \geqq 2$, then

$$\int (\sec x)^n\ dx = \frac{1}{n-1}(\sec x)^{n-2}\ \tan x + \frac{n-2}{n-1}\int (\sec x)^{n-2}\ dx.$$

Thus we find that

$$\int \sec^3\theta\ d\theta = \frac{1}{2}(\sec\theta\ \tan\theta + \ln|\sec\theta + \tan\theta\,|) + C;$$

$$\int \sec^5\theta\ d\theta = \frac{1}{4}\sec^3\theta\ \tan\theta + \frac{3}{4}\int \sec^3\theta\ d\theta$$
$$= \frac{1}{4}\sec^3\theta\ \tan\theta + \frac{3}{8}\sec\theta\ \tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta\,| + C;$$

$$\int \sec^7\theta\ d\theta = \frac{1}{6}\sec^5\theta\ \tan\theta + \frac{5}{6}\int \sec^5\theta\ d\theta$$
$$= \frac{1}{6}\sec^6\theta\ \tan\theta + \frac{5}{24}\sec^3\theta\ \tan\theta + \frac{5}{16}\sec\theta\ \tan\theta + \frac{5}{16}\ln|\sec\theta + \tan\theta\,| + C.$$

Therefore

$$A = \frac{8\pi}{3}\int_{x=3}^8 (\sec^7\theta - 2\sec^5\theta + \sec^3\theta)\ d\theta$$

$$= \frac{8\pi}{3}\left[\frac{1}{6}\sec^5\theta\,\tan\theta - \frac{7}{24}\sec^3\theta\,\tan\theta + \frac{1}{16}\sec\theta\,\tan\theta + \frac{1}{16}\ln|\sec\theta+\tan\theta|\right]_{x=3}^{8}$$

$$= \frac{\pi}{3}\left[\frac{4}{3}u(1+u^2)^{5/2} - \frac{7}{3}u(1+u^2)^{3/2} + \frac{1}{2}u(1+u^2)^{1/2} + \frac{1}{2}\ln\left|u+(1+u^2)^{1/2}\right|\right]_{u=\sqrt{3}}^{2\sqrt{2}}$$

$$= \frac{\pi}{3}\left[\frac{4}{3}\cdot 243\cdot 2\sqrt{2} - \frac{7}{3}\cdot 27\cdot 2\sqrt{2} + \frac{1}{2}\cdot 3\cdot 2\sqrt{2} + \frac{1}{2}\ln\left(3+2\sqrt{2}\right)\right.$$

$$\left. - \frac{4}{3}\cdot 32\cdot\sqrt{3} + \frac{7}{3}\cdot 8\cdot\sqrt{3} - \frac{1}{2}\cdot 2\sqrt{3} - \frac{1}{2}\ln\left(2+\sqrt{3}\right)\right]$$

$$= \frac{\pi}{3}\left[525\sqrt{2} - 25\sqrt{3} + \frac{1}{2}\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\right]$$

$$= \frac{\pi}{6}\left[1050\sqrt{2} - 50\sqrt{3} + \ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\right] \approx 732.3929447915.$$

C08S0M.133: The area is

$$A = 2\int_0^1 x\sqrt{1-x}\;dx.$$

Let $u = 1-x$. Then $x = 1-u$ and $dx = -du$. Hence

$$A = -2\int_{u=1}^{0}(1-u)u^{1/2}\,du = 2\int_0^1 (u^{1/2}-u^{3/2})\,du = 2\left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1 = 2\left(\frac{2}{3}-\frac{2}{5}\right) = \frac{8}{15}.$$

C08S0M.134: The area is

$$A = 2\int_0^1 x\left(\frac{1-x}{1+x}\right)^{1/2}\,dx.$$

Let

$$u^2 = \frac{1-x}{1+x}. \quad \text{Then} \quad x = \frac{1-u^2}{1+u^2} \quad \text{and} \quad dx = -\frac{4u}{(1+u^2)^2}\,du$$

(as in the solution of Problem 127). Hence

$$A = -2\int_{u=1}^{0}\frac{1-u^2}{1+u^2}\cdot\frac{4}{(1+u^2)^2}\cdot u\,du = 8\int_0^1 \frac{u^2(1-u^2)}{(1+u^2)^3}\,du.$$

Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $1+u^2 = \sec^2\theta$. Hence

$$A = 8\int_{u=0}^{2}\frac{(\tan^2\theta)(1-\tan^2\theta)}{\sec^6\theta}\,\sec^2\theta\,d\theta = 8\int_{u=0}^{1}(\tan^2\theta - \tan^4\theta)\cos^4\theta\,d\theta$$

$$= 8\int_{u=0}^{1}(\sin^2\theta\,\cos^2\theta - \sin^4\theta)\,d\theta = 8\int_{u=0}^{1}\left[(\sin^2\theta)(1-\sin^2\theta) - \sin^4\theta\right]\,d\theta$$

$$= 8\int_{u=0}^{1}(\sin^2\theta - 2\sin^4\theta)\,d\theta.$$

Now we use the reduction formula in Problem 53 of Section 8.3: If n is an integer and $n \geqq 2$, then

$$\int \sin^n x \, dx = -\frac{1}{n}(\sin x)^{n-1} \cos x + \frac{n-1}{n} \int (\sin x)^{n-2} \, dx.$$

Therefore

$$\int \sin^2 \theta \, d\theta = -\frac{1}{2} \sin\theta \, \cos\theta + \frac{1}{2}\theta + C;$$

$$\begin{aligned}\int \sin^4 \theta \, d\theta &= -\frac{1}{4} \sin^3 \theta \, \cos\theta + \frac{3}{4} \int \sin^2 \theta \, d\theta \\ &= -\frac{1}{4} \sin^3 \theta \, \cos\theta - \frac{3}{8} \sin\theta \, \cos\theta + \frac{3}{8}\theta + C.\end{aligned}$$

Thus

$$A = 8\left[\frac{1}{2} \sin^2 \theta \, \cos\theta + \frac{1}{4} \sin\theta \, \cos\theta - \frac{1}{4}\theta\right]_0^{\pi/4} = 4\left(\frac{\sqrt{2}}{2}\right)^4 + 2\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{\pi}{2} = \frac{4-\pi}{2} \approx 0.429203673205.$$

C08S0M.135: Using the recommended substitution, we find that

$$\begin{aligned}\int \frac{1}{1+\cos\theta} \, d\theta &= \int \frac{1}{1 + \dfrac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} \, du = \int \frac{2}{1+u^2+1-u^2} \, du \\ &= u + C = \tan\frac{\theta}{2} + C = \frac{1-\cos\theta}{\sin\theta} + C = \frac{\sin\theta}{1+\cos\theta} + C.\end{aligned}$$

C08S0M.136: Using the recommended substitution, we obtain

$$\begin{aligned}\int \frac{1}{5+4\cos\theta} \, d\theta &= \int \frac{1}{5 + 4 \cdot \dfrac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} \, du = \int \frac{2}{5+5u^2+4-4u^2} \, du = \frac{2}{9+u^2} \, du \\ &= \frac{2}{9} \int \frac{1}{1 + \left(\dfrac{u}{3}\right)^2} \, du = \frac{2}{3} \arctan\left(\frac{u}{3}\right) + C = \frac{2}{3} \arctan\left(\frac{1}{3} \tan\frac{\theta}{2}\right) + C \\ &= \frac{2}{3} \arctan\left(\frac{1-\cos\theta}{3\sin\theta}\right) + C = \frac{2}{3} \arctan\left(\frac{\sin\theta}{3+3\cos\theta}\right) + C.\end{aligned}$$

C08S0M.137: The recommended substitution yields

$$\begin{aligned}\int \frac{1}{1+\sin\theta} \, d\theta &= \int \frac{1}{1 + \dfrac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} \, du = \int \frac{2}{1+u^2+2u} \, du = \int 2(u+1)^{-2} \, du = -\frac{2}{u+1} + C \\ &= -\frac{2}{1+\tan\dfrac{\theta}{2}} + C = -\frac{2\sin\theta}{1+\sin\theta-\cos\theta} + C = -\frac{2+2\cos\theta}{1+\sin\theta+\cos\theta} + C.\end{aligned}$$

C08S0M.138: The recommended substitution yields

$$\int \frac{1}{(1-\cos\theta)^2}\,d\theta = \int \frac{1}{\left(1-\dfrac{1-u^2}{1+u^2}\right)^2}\cdot\frac{2}{1+u^2}\,du = \int \frac{(1+u^2)^2}{(1+u^2-1-u^2)^2}\cdot\frac{2}{1+u^2}\,du = \int \frac{2(1+u^2)}{4u^4}\,du$$

$$= \frac{1}{2}\int (u^{-4}+u^{-2})\,du = \frac{1}{2}\left(-\frac{1}{3u^3}-\frac{1}{u}\right)+C = -\frac{1}{6\tan^3\dfrac{\theta}{2}} - \frac{1}{2\tan\dfrac{\theta}{2}}+C$$

$$= -\frac{1}{6\cdot\left(\dfrac{1-\cos\theta}{\sin\theta}\right)^3} - \frac{1}{2\cdot\dfrac{1-\cos\theta}{\sin\theta}} + C = -\frac{\sin^3\theta}{6(1-\cos\theta)^3} - \frac{\sin\theta}{2(1-\cos\theta)}+C.$$

C08S0M.139: The substitution $u = \tan\dfrac{\theta}{2}$ yields

$$I = \int \frac{1}{\sin\theta+\cos\theta}\,d\theta = \int \frac{1}{\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int \frac{2}{2u+1-u^2}\,du.$$

Now

$$-u^2+2u+1 = -(u^2-2u-1) = -(u^2-2u+1-2) = 2-(u-1)^2 = 2-2\sin^2 w = 2\cos^2 w$$

if $u-1=\sqrt{2}\,\sin w$, so we let $u = 1+\sqrt{2}\,\sin w$, so that

$$du = \sqrt{2}\,\cos w\,dw \qquad\text{and}\qquad \sin w = \frac{u-1}{\sqrt{2}}.$$

Thus

$$I = \int \frac{2}{2u+1-u^2}\,du = \int \frac{2}{2\cos^2 w}\cdot\sqrt{2}\,\cos w\,dw = \sqrt{2}\,\ln|\sec w+\tan w\,|+C.$$

A reference triangle for the trigonometric substitution has acute angle w, opposite side $u-1$, and hypotenuse $\sqrt{2}$. Therefore its adjacent side has length $\sqrt{2u+1-u^2}$, and thus

$$I = \sqrt{2}\,\ln\left|\frac{\sqrt{2}\,+u-1}{\sqrt{2u+1-u^2}}\right|+C$$

$$= \sqrt{2}\,\left[\ln\left(-1+\sqrt{2}\,+\frac{1-\cos\theta}{\sin\theta}\right) - \frac{1}{2}\ln\left(1+\frac{2-2\cos\theta}{\sin\theta}-\frac{(1-\cos\theta)^2}{\sin^2\theta}\right)\right]+C.$$

C08S0M.140: The recommended substitution $u = \tan\dfrac{\phi}{2}$ yields

$$J = \int \frac{1}{2+\sin\phi+\cos\phi}\,d\phi = \int \frac{1}{2+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du = \int \frac{2}{2+2u^2+2u+1-u^2}\,du$$

$$= \int \frac{2}{u^2+2u+3}\,du.$$

Now $u^2+2u+3 = (u+1)^2+2 = 2\tan^2 w+2 = 2\sec^2 w$ if $u+1=\sqrt{2}\,\tan w$. Hence we let

$$u = -1 + \sqrt{2}\,\tan w. \quad \text{Then} \quad du = \sqrt{2}\,\sec^2 w\, dw \quad \text{and} \quad \tan w = \frac{u+1}{\sqrt{2}}.$$

Therefore

$$J = \int \frac{2}{2\sec^2 w} \cdot \sqrt{2}\,\sec^2 w\, dw = w\sqrt{2} + C = \sqrt{2}\,\arctan\left(\frac{u+1}{\sqrt{2}}\right) + C$$

$$= \sqrt{2}\,\arctan\left(\frac{1+\tan\dfrac{\phi}{2}}{\sqrt{2}}\right) + C = \sqrt{2}\,\arctan\left(\frac{1+\sin\phi-\cos\phi}{\sqrt{2}\,\sin\phi}\right) + C.$$

C08S0M.141: The substitution $u = \tan\dfrac{\theta}{2}$ yields

$$K = \int \frac{\sin\theta}{2+\cos\theta}\, d\theta = \int \frac{2u}{2(1+u^2)+(1-u^2)} \cdot \frac{2}{1+u^2}\, du = \int \frac{4u}{(u^2+3)(u^2+1)}\, du.$$

Next, the partial fractions decomposition

$$\frac{4u}{(u^2+1)(u^2+3)} = \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2+3}$$

leads to the equation $A(u^3+3u) + B(u^2+3) + C(u^3+3) + D(u^2+1) = 4u$, and thus to the system

$$A + C = 0, \qquad B + D = 0,$$
$$3A + C = 4, \qquad 3B + D = 0$$

with solution $A = 2$, $C = -2$, $B = D = 0$. Therefore

$$K = \int \left(\frac{2u}{u^2+1} - \frac{2u}{u^2+3}\right) du = \ln(u^2+1) - \ln(u^2+3) + C = \ln\left(\frac{u^2+1}{u^2+3}\right) + C$$

$$= \ln\left(\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2+3\sin^2\theta}\right) + C = \ln\left(\frac{1-2\cos\theta+1}{1-2\cos\theta+1+2\sin^2\theta}\right) + C$$

$$= \ln\left(\frac{2-2\cos\theta}{2+2\sin^2\theta-2\cos\theta}\right) + C = \ln\left(\frac{1-\cos\theta}{1+\sin^2\theta-\cos\theta}\right) + C$$

$$= \ln\left(\frac{1-\cos\theta}{2-\cos\theta-\cos^2\theta}\right) + C = \ln\left(\frac{1-\cos\theta}{(1-\cos\theta)(2+\cos\theta)}\right) + C = -\ln(2+\cos\theta) + C.$$

C08S0M.142: The substitution $u = \tan\dfrac{\theta}{2}$ yields

$$J = \int \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}\, d\theta = \int \frac{2u-(1-u^2)}{2u+(1-u^2)} \cdot \frac{2}{1+u^2}\, du = -2\int \frac{u^2+2u-1}{(u^2+1)(u^2-2u-1)}\, du.$$

The last denominator is not completely factored, but there is a slim chance that if we ignore this difficulty we may be able to obtain a suitable partial fractions decomposition anyway. It's worth a try because the factorization $u^2 - 2u - 1 = \left(u - 1 + \sqrt{2}\,\right)\left(u - 1 - \sqrt{2}\,\right)$ would certainly discourage most people from trying the method of partial fractions at all. So let's see what happens. It will now be essential to check the resulting decomposition—if any—to see if the algebra is valid. We try:

$$\frac{u^2+2u-1}{(u^2+1)(u^2-2u-1)} = \frac{Au+B}{u^2+1} + \frac{Cu+D}{u^2-2u-1},$$

which leads to the equation

$$A(u^3-2u^2-u)+B(u^2-2u-1)+C(u^3+3)+D(u^2+1) = u^2+2u-1.$$

Thus we obtain the system

$$A+C=0, \qquad -2A+B+D=1,$$

$$-A-2B+C=2, \qquad -B+D=-1,$$

and we find one solution—we hope it's the only solution—to be $A=-1$, $B=0$, $C=1$, $D=-1$. This produces the *tentative* decomposition

$$\frac{u^2+2u-1}{(u^2+1)(u^2-2u-1)} = -\frac{u}{u^2+1} + \frac{u-1}{u^2-2u-1},$$

and it is with great relief that we verify that it holds for all real u (other than the two zeros of the last denominator). Therefore

$$J = -2\left[-\frac{1}{2}\ln(u^2+1)+\frac{1}{2}\ln|u^2-2u-1|\right]+C = \ln\left(1+\tan^2\frac{\theta}{2}\right)-\ln\left|\tan^2\frac{\theta}{2}-2\tan\frac{\theta}{2}-1\right|+C$$

$$= \ln\left(1+\left[\frac{1-\cos\theta}{\sin\theta}\right]^2\right)-\ln\left|\left(\frac{1-\cos\theta}{\sin\theta}\right)^2-2\cdot\frac{1-\cos\theta}{\sin\theta}-1\right|+C$$

$$= \ln\left(\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin^2\theta}\right)-\ln\left|\frac{1-2\cos\theta+\cos^2\theta-2\sin\theta+2\sin\theta\,\cos\theta-\sin^2\theta}{\sin^2\theta}\right|+C$$

$$= \ln\left|\frac{1-2\cos\theta+1}{1-2\cos\theta+\cos^2\theta-2\sin\theta+2\sin\theta\,\cos\theta-\sin^2\theta}\right|+C$$

$$= \ln\left|\frac{2-2\cos\theta}{(1-\cos\theta)^2-2\sin\theta+2\sin\theta\,\cos\theta-1+\cos^2\theta}\right|+C$$

$$= \ln\left|\frac{2(1-\cos\theta)}{(1-\cos\theta)^2-(2\sin\theta)(1-\cos\theta)-(1-\cos\theta)(1+\cos\theta)}\right|+C$$

$$= \ln\left|\frac{2}{1-\cos\theta-2\sin\theta-1-\cos\theta}\right|+C = -\ln|\sin\theta+\cos\theta|+C.$$

C08S0M.143: The substitution $u = \tan\dfrac{\theta}{2}$ yields

$$\int \sec\theta\,d\theta = \int \frac{1}{\cos\theta}\,d\theta = \int \frac{1+u^2}{1-u^2}\cdot\frac{2}{1+u^2}\,du = \int \frac{2}{1-u^2}\,du = \int \left(\frac{1}{1+u}+\frac{1}{1-u}\right)du$$

$$= \ln\left|\frac{1+u}{1-u}\right| + C = \ln\left|\frac{1+\tan\dfrac{\theta}{2}}{1-\tan\dfrac{\theta}{2}}\right| + C = \ln\left|\frac{1+\left(\dfrac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}}{1-\left(\dfrac{1-\cos\theta}{1+\cos\theta}\right)^{1/2}}\right| + C$$

$$= \ln\left|\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right| + C = \ln\left|\frac{1+\cos\theta+2\sqrt{1-\cos^2\theta}+1-\cos\theta}{1+\cos\theta-(1-\cos\theta)}\right| + C$$

$$= \ln\left|\frac{2+2\sin\theta}{2\cos\theta}\right| + C = \ln|\sec\theta+\tan\theta| + C.$$

C08S0M.144: The substitution $u = \tan\dfrac{\theta}{2}$ yields

$$\int \csc\theta\,d\theta = \int \frac{1}{\sin\theta}\,d\theta = \int \frac{1+u^2}{2u}\cdot\frac{2}{1+u^2}\,du = \ln|u| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$$

$$= \ln\left|\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right| + C = \frac{1}{2}\ln\left|\frac{(1-\cos\theta)^2}{1-\cos^2\theta}\right| + C = \ln\left|\frac{1-\cos\theta}{\sin\theta}\right| + C = \ln|\csc\theta-\cot\theta| + C.$$

Section 9.1

C09S01.001: Separate the variables:

$$\frac{dy}{dx} = 2y; \qquad \frac{1}{y}\,dy = 2\,dx;$$

$$\ln y = C + 2x; \qquad y(x) = Ae^{2x}.$$

$$3 = y(1) = Ae^2: \qquad A = 3e^{-2}.$$

Therefore $y(x) = 3\exp(2x-2) = 3e^{2x-2}$.

C09S01.002: Given: $\dfrac{dy}{dx} = -3y,\ y(5) = -10$:

$$\frac{1}{y}\,dy = -3\,dx; \qquad \ln y = C - 3x;$$

$$y(x) = Ae^{-3x}. \qquad -10 = y(5) = Ae^{-15};$$

$$A = -10e^{15}. \qquad y(x) = -10\exp(15-3x).$$

C09S01.003: Given: $\dfrac{dy}{dx} = 2y^2,\ y(7) = 3$:

$$-\frac{1}{y^2}\,dy = -2\,dx; \qquad \frac{1}{y} = C - 2x;$$

$$y(x) = \frac{1}{C-2x}. \qquad 3 = y(7) = \frac{1}{C-14}:$$

$$C = \frac{43}{3}. \qquad y(x) = \frac{1}{\frac{43}{3} - 2x} = \frac{3}{43-6x}.$$

C09S01.004: Given: $\dfrac{dy}{dx} = \dfrac{7}{y},\ y(0) = 6$:

$$2y\,dy = 14\,dx; \qquad y^2 = C + 14x.$$

$$36 = [y(0)]^2 = C: \qquad C = 36.$$

$$y^2 = 36 + 14x: \qquad y(x) = \sqrt{36+14x}\,.$$

We chose the nonnegative square root in the last step because $y(0) > 0$.

C09S01.005: Given: $\dfrac{dy}{dx} = 2y^{1/2},\ y(0) = 9$:

$$y^{-1/2}\,dy = 2\,dx; \qquad 2y^{1/2} = C + 2x;$$

$$y^{1/2} = A + x; \qquad y(x) = (A+x)^2.$$

$$y(0) = 9: \qquad A = 3.$$

Therefore $y(x) = (x+3)^2$.

C09S01.006: Given: $\dfrac{dy}{dx} = 6y^{2/3},\ y(1) = 8$:

$$y^{-2/3}\,dy = 6\,dx; \qquad 3y^{1/3} = 6x + C.$$

$$y(1) = 8: \qquad C = 0.$$

$$y^{1/3} = 2x; \qquad y(x) = 8x^3.$$

C09S01.007: Given: $\dfrac{dy}{dx} = 1 + y,\ y(0) = 5$:

$$\frac{1}{1+y}\,dy = 1\,dx; \qquad \ln(1+y) = x + C;$$

$$1 + y = Ae^x; \qquad y(x) = Ae^x - 1.$$

$$5 = y(0) = A - 1: \qquad A = 6.$$

Therefore $y(x) = 6e^x - 1$.

C09S01.008: Given: $\dfrac{dy}{dx} = (2+y)^2,\ y(5) = 3$:

$$-\frac{1}{(y+2)^2}\,dy = -1\,dx; \qquad \frac{1}{y+2} = C - x;$$

$$y(x) = -2 + \frac{1}{C-x}. \qquad 3 = y(5) = -2 + \frac{1}{C-5};$$

$$\frac{1}{C-5} = 5; \qquad C = \frac{26}{5}.$$

Therefore

$$y(x) = -2 + \frac{1}{\frac{26}{5} - x} = -2 + \frac{5}{26-5x} = \frac{-52+10x+5}{26-5x} = \frac{10x-47}{26-5x}.$$

C09S01.009: Given: $\dfrac{dy}{dx} = e^{-y},\ y(0) = 2$:

$$e^y\,dy = 1\,dx; \qquad e^y = x + C;$$

$$y(x) = \ln(x+C). \qquad 2 = y(0) = \ln C:$$

$$C = e^2. \qquad y(x) = \ln(x+e^2).$$

C09S01.010: Given: $\dfrac{dy}{dx} = 2\sec y,\ y(0) = 0$:

$$\cos y\,dy = 2\,dx; \qquad \sin y = 2x + C.$$

$$y(0) = 0: \qquad 0 = 0 + C;$$

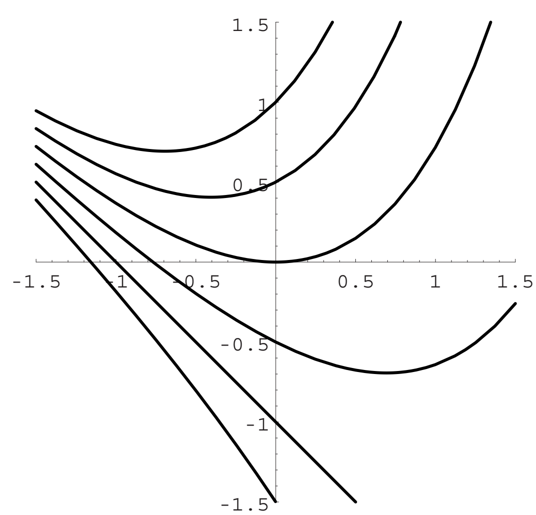
$$C = 0; \qquad \sin y = 2x.$$

Therefore $y(x) = \sin^{-1}(2x)$.

C09S01.011: If the slope of $y = g(x)$ at the point (x, y) is the sum of x and y, then we expect $y = g(x)$ to be a solution of the differential equation

$$\frac{dy}{dx} = x + y. \tag{1}$$

Some solutions of this differential equation with initial conditions $y(0) = -1.5,\ -1,\ -0.5,\ 0,\ 0.5,$ and 1 are shown next. The figure is constructed with the same scale on the x- and y-axes, so you can confirm with a ruler that the solution curves agree with the differential equation in (1).

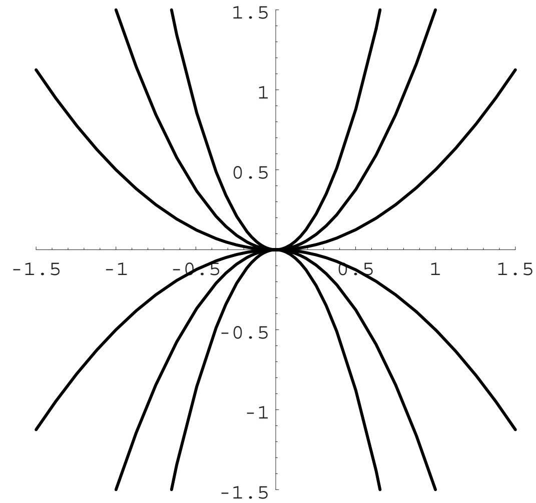


C09S01.012: If the line tangent to the graph of $y = g(x)$ at the point (x, y) meets the x-axis at the point $(x/2, 0)$, then $y = g(x)$ should be a solution of the differential equation

$$\frac{dy}{dx} = \frac{y - 0}{x - \frac{1}{2}x} = \frac{2y}{x}. \tag{1}$$

The general solution of the equation in (1) is $y(x) = Cx^2$. Some particular solutions (with $c = -3.5, -1.5, -0.5, 0.5, 1.5,$ and 3.5) are shown next. The figure was constructed with the same scale on the x- and y-axes,

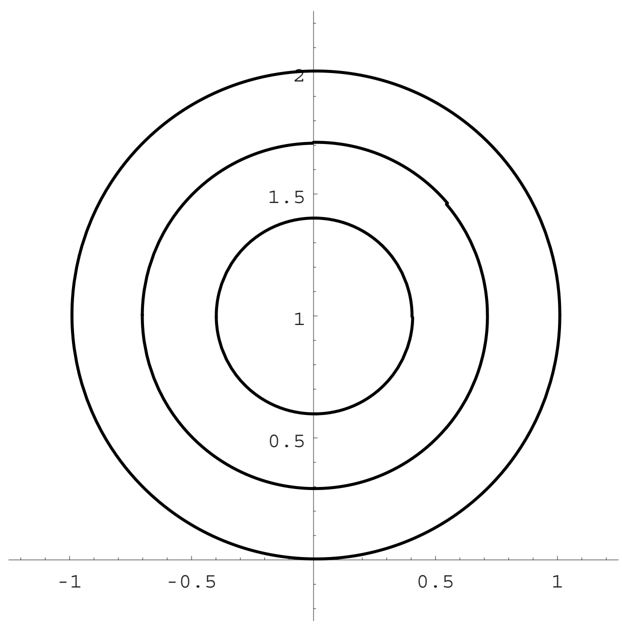
so you can confirm with a ruler that the solution curves agree with the differential equation in (1).



C09S01.013: If every straight line normal to the graph of $y = g(x)$ passes through the point $(0, 1)$, then we expect that $y = g(x)$ will be a solution of the differential equation

$$\frac{dy}{dx} = \frac{x}{1 - y}. \tag{1}$$

The general solution of this equation is implicitly defined by $x^2 + (y - 1)^2 = C$ where $C > 0$. Some solution curves for $C = 0.16$, $C = 0.5$, and $C = 1$ are shown next. Note that two functions are solutions for each value of C. Because the figure was constructed with the same scale on the x- and y-axes, you can confirm with a ruler that the solution curves agree with the differential equation in (1).



C09S01.014: Suppose that the graph of $y = g(x)$ is normal to every curve of the form $y = kx^2$ where they meet. The trick here is to eliminate k (basically because we have no control over its value). We can eliminate k by using the fact that $D_x(k) = 0$. Thus if $y = kx^2$, then

$$\frac{y}{x^2} = k, \qquad \text{and thus} \qquad \frac{dy}{dx} = \frac{2y}{x}$$

by implicit differentiation and subsequent simplification. If the graph of $y = g(x)$ is normal to the graph of $y = kx^2$ where they meet, then

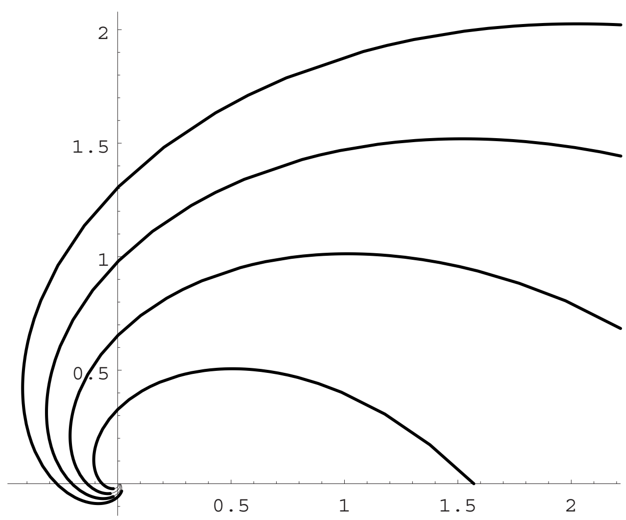
$$g'(x) = \frac{dy}{dx} = -\frac{x}{2y}.$$

The general solution of this equation is implicitly defined by the equation $x^2 + 2y^2 = C$.

C09S01.015: If, for each (x, y) on the graph of $y = g(x)$, the line tangent to the graph at that point passes through $(-y, x)$, then we expect for $y = g(x)$ to satisfy the differential equation

$$\frac{dy}{dx} = \frac{y - x}{y + x}. \tag{1}$$

The substitution of the new dependent variable $u = y/x$ leads to the general solution of this equation—see the discussion of *Homogeneous Equations* in Section 1.6 of Edwards and Penney: *Differential Equations: Computing and Modeling* (2nd edition, Prentice Hall, 2000). Some solution curves are shown next; note than none appears to be the graph of a single function.



C09S01.016: $\dfrac{dP}{dt} = k\sqrt{P}$.

C09S01.017: $\dfrac{dv}{dt} = -kv^2$ (where k is a positive constant).

C09S01.018: $\dfrac{dv}{dt} = k(250 - v)$ (where k is a positive constant).

C09S01.019: $\dfrac{dN}{dt} = k(P - N)$ (where k is a positive constant).

C09S01.020: $\dfrac{dN}{dt} = kN(P - N)$ (where k is a positive constant).

C08S01.021: The principal (in dollars) at time t (in years) is $A(t) = 1000e^{(0.08)t}$. Therefore $A'(t) = 80e^{(0.08)t}$. So the answers to the two questions in Problem 21 are these: $A'(5) = \$119.35$ and $A'(20) = \$396.24$.

C09S01.022: Take $t = 0$ (years) as 1970. Then the population at time t is given by $P(t) = 25000e^{kt}$ where k is the growth rate of the population. We are given $P(10) = 30000$, so

$$\begin{aligned} 25000e^{10k} &= 30000; & e^{10k} &= \frac{6}{5}; \\ 10k &= \ln\frac{6}{5}; & k &= \frac{1}{10}\ln\frac{6}{5}. \end{aligned}$$

Therefore in the year 2010 the population will be

$$P(40) = 25000\exp\left(\frac{40}{10}\ln\frac{6}{5}\right) = 25000\exp\left(\ln\left(\frac{6}{5}\right)^4\right) = 25000\cdot\left(\frac{6}{5}\right)^4 = 51840.$$

C09S01.023: Let $P(t)$ be the number of bacteria present at time t (in hours), with initial number $P_0 = P(0)$. Then $P(t) = P_0e^{kt}$ where k is a constant. We are given $P(10) = 6P_0$, and therefore

$$P_0e^{10k} = 6P_0; \qquad 10k = \ln 6; \qquad k = \frac{1}{10}\ln 6.$$

If T is the doubling time, then

$$\begin{aligned} &P(T) = 2P_0 = P_0\exp\left(\frac{T}{10}\ln 6\right); \\ &\frac{T}{10}\ln 6 = \ln 2; \\ &T = \frac{10\ln 2}{\ln 6} \approx 3.8685280723. \end{aligned}$$

Thus the doubling time is approximately 3 h 52 min.

C09S01.024: Suppose that the skull was formed at time $t = 0$ (in years). Then the amount of ^{14}C it contains at time t will be

$$N(t) = N_0e^{-kt},$$

where $N_0 = N(0)$ is the initial amount and k is the decay constant for ^{14}C. We have already seen that

$$k = \frac{\ln 2}{5700} \approx 0.0001216047685,$$

although the half-life $\tau = 5700$ is so imprecisely known that we really should carry no more than four significant figures; we'll take $k = 0.0001216$. We find the value $t = T$ corresponding to the present by solving

$$\begin{aligned} N(T) &= \frac{1}{6}N_0; & N_0e^{-kT} &= \frac{1}{6}N_0; \\ e^{kT} &= 6; & T &= \frac{1}{k}\ln 6 = \frac{5700\ln 6}{\ln 2} \approx 14734. \end{aligned}$$

Answer: The skull is between 14500 and 15000 years old.

C09S01.025: We take $t = 0$ (in years) corresponding to the time of formation of the relic and let t denote its age. Then the amount of ^{14}C it contains at time t is $N(t) = N_0 e^{-kt}$, where $k \approx 0.0001216$. Therefore

$$\left(5.0 \times 10^{10}\right) e^{-kt} = 4.6 \times 10^{10}, \qquad \text{so} \qquad t \approx \frac{0.0833816}{k} \approx 686.$$

We conclude that the relic is between 650 and 700 years old and is probably not authentic (although it may be much older if it recently has been contaminated with "modern" carbon).

C09S01.026: At time t (in years) the investment will grow to $A(t) = 5000e^{it}$ where $i = 0.06$. Thus the answer is $A(18) = 5000\exp(1.08) = \$14723.40$.

C09S01.027: After t years the fine will grow to $A(t) = 30e^{it}$ (cents) where $i = 0.05$. Thus the answer is $30\exp(0.05 \cdot 100) = 30e^5 = 4452$ cents; that is, \$44.52.

C09S01.028: If $C(t)$ is the concentration of the drug at time t (hours), then $C(t) = C_0 e^{-kt}$ where $k = \frac{1}{5}\ln 2$. We require C_0 so large that $C(1) = 45 \cdot 50 = 2250$. Thus $C_0 e^{-k} \geqq 2250$; that is, $C_0 \geqq 2250e^k \approx 2584.57$ (mg).

C09S01.029: Let $S(t)$ represent the sales t weeks after advertising is discontinued and let $S(0) = S_0$. Then for some constant λ,

$$\frac{dS}{dt} = -\lambda S, \quad \text{so} \quad S(t) = S_0 e^{-\lambda t}.$$

Because $S(1) = (0.95)S_0 = S_0 e^{-\lambda}$, $\lambda = \ln \dfrac{20}{19}$. Therefore at time $t = T$, when sales have declined to 75% of the initial rate,

$$S(T) = \tfrac{3}{4}S_0 = S_0 e^{-\lambda T}: \quad e^{\lambda T} = \tfrac{4}{3}; \quad T = \frac{\ln(\frac{4}{3})}{\lambda} = \frac{\ln(\frac{4}{3})}{\ln(\frac{20}{19})} \approx 5.608.$$

So the company plans to resume advertising about 5.6 weeks (about 39 days) after cessation of advertising.

C09S01.030: Let L denote the number of words on the basic list at time $t = 0$ (in years) corresponding to the year A.D. 1400. The number at time $t \geqq 0$ is then given by $Q(t) = Le^{-kt}$. We are given

$$Q(1000) = (0.23)L = Le^{-1000k}, \qquad \text{so} \qquad k = (0.001)\ln\left(\frac{1}{0.23}\right) \approx 0.001469676.$$

In 2000 we would expect that about the fraction $e^{-600k} \approx 0.414$ of the words in the basic list in 1400—about 41.4% of them—would still be in use.

As a matter of independent interest, 87% of the words in the *Prologue* to the *Canterbury Tales* are still in use. You are invited to speculate about the reason for the apparent discrepancy.

C09S01.031: Let Q denote the amount of radioactive cobalt remaining at time t (in years), with the occurrence of the accident set at time $t = 0$. Then

$$Q = Q_0 e^{-(t\ln 2)/(5.27)}.$$

If T is the number of years until the level of radioactivity has dropped to a hundredth of its initial value, then

$$\frac{1}{100} = e^{-(T \ln 2)/(5.27)}, \qquad \text{so that} \qquad T = (5.27)\,\frac{\ln 100}{\ln 2},$$

approximately 35 years.

C09S01.032: Let $Q(t)$ denote the amount of ^{238}U in the mineral body at time t (in years), with the supposition that the mineral body was formed at time $t = 0$. Then

$$Q(t) = Q_0 e^{-kt} \qquad \text{where} \qquad k = \frac{\ln 2}{\tau};$$

τ denotes the half-life of ^{238}U, about 4.51×10^9 years. Let $t = T$ correspond to the present, so that

$$\frac{Q(T)}{Q_0 - Q(T)} = 0.9.$$

Therefore

$$\begin{aligned}
\frac{e^{-kT}}{1 - e^{-kT}} &= 0.9;\\
e^{-kT} &= 0.9 - (0.9)e^{-kT};\\
(1.9)e^{-kT} &= 0.9;\\
e^{kT} &= \frac{19}{9};\\
T &= \frac{1}{k}\ln\left(\frac{19}{9}\right) \approx 4.86 \times 10^9.
\end{aligned}$$

Answer: The cataclysm occurred approximately 4.86×10^9 years ago.

C09S01.033: Of course we set up coordinates so that $t = 0$ corresponds to 12 noon. (a) $P(t) = 49e^{kt}$, and

$$294 = 49e^k: \qquad e^k = 6; \quad k = \ln 6.$$

Therefore $P(t) = 49\exp(t\ln 6) = 49\exp(\ln 6^t) = 49 \cdot 6^t$. (b) At 1:40 P.M. we have $t = \frac{5}{3}$, so the number of bacteria present at that time is

$$P\left(\tfrac{5}{3}\right) = 49 \cdot 6^{5/3} \approx 971.$$

(c) If $P(t) = 20000$, then

$$49 \cdot 6^T = 20000; \qquad 6^T = \frac{20000}{49};$$

$$T\ln 6 = \ln\frac{20000}{49}; \qquad T = \frac{1}{\ln 6}\cdot\ln\frac{20000}{49} \approx 3.355175377985.$$

Answer: The clock time then will be approximately 3:21:19 P.M.

C09S01.034: (a) $A(t) = 10e^{kt}$. Also $30 = A\left(\frac{15}{2}\right) = 10e^{15k/2}$, so

$$e^{15k/2} = 3: \qquad k = \frac{2}{15}\ln 3 = \ln\left(3^{2/15}\right).$$

Therefore $A(t) = 10(e^k)^t = 10 \cdot 3^{2t/15}$. (b) After five years, we have $A(5) = 10 \cdot 3^{2/3} \approx 20.8008382305$. (c) $A(T) = 100$ when

$$A(T) = 10 \cdot 3^{2T/15}: \qquad 3^{2T/15} = 10; \quad T = \frac{15}{2} \cdot \frac{\ln 10}{\ln 3}.$$

Thus the pollution level will reach 100 pu in approximately 15.7192745572 years.

C09S01.035: (a) $A(t) = 15e^{-kt}$; $10 = A(5) = 15e^{-5k}$, so

$$\frac{3}{2} = e^{5k}: \qquad k = \frac{1}{5}\ln\frac{3}{2}.$$

Therefore

$$A(t) = 15\exp\left(-\frac{t}{5}\ln\frac{3}{2}\right) = 15 \cdot \left(\frac{3}{2}\right)^{-t/5} = 15 \cdot \left(\frac{2}{3}\right)^{t/5}.$$

(b) After 8 months we have

$$A(8) = 15 \cdot \left(\frac{2}{3}\right)^{8/5} \approx 7.8405261683.$$

(c) $A(T) = 1$ when

$$\left(\frac{2}{3}\right)^{T/5} = \frac{1}{15}; \qquad \left(\frac{2}{3}\right)^{T} = \left(\frac{1}{15}\right)^{5};$$

$$T\ln\frac{2}{3} = 5\ln\frac{1}{15}; \qquad T = 5 \cdot \frac{\ln\left(\frac{1}{15}\right)}{\ln\left(\frac{2}{3}\right)} = \frac{5\ln 15}{\ln(3/2)}.$$

Answer: It will be safe to return after approximately 33.394368 months.

C09S01.036: If $L(t)$ denotes the number of human language families at time t (in years), then $L(t) = e^{kt}$ where k is a positive constant. To find k:

$$1.5 = L(6000) = e^{6000k}, \quad \text{so that} \quad k = \frac{1}{6000}\ln\frac{3}{2}.$$

If "now" corresponds to time $t = T$, then we are given $L(T) = 3300$. Therefore

$$e^{kT} = 3300; \qquad kT = \ln 3300;$$

$$T = \frac{1}{k}\ln 3300 = \frac{6000\ln 3300}{\ln(3/2)}.$$

Therefore $T \approx 119887.175278$, suggesting that the original human language was first spoken about 120,000 years ago.

C09S01.037: If $L(t)$ denotes the number of Native American language families at time t (in years), then $L(t) = e^{kt}$ where k is a positive constant. To find k, we use the given data

$$\frac{3}{2} = L(6000) = e^{6600k}: \qquad k = \frac{1}{6000}\ln\frac{3}{2}.$$

If $t = T$ corresponds to the present time, then $150 = L(T) = e^{kT}$, so that $kT = \ln 150$. Therefore

$$T = \frac{1}{k}\ln 150 = \frac{6000\ln 150}{\ln(3/2)} \approx 74146.483047.$$

This analysis suggests that the ancestors of today's Native Americans first arrived in the western hemisphere about 74000 years ago.

C09S01.038: If $U(t)$ denotes the number (in millions) of internet users at time t (in years, with $t = 0$ corresponding to 1998), then $U(t) = 40e^{kt}$ where k is a positive constant. Moreover,

$$80 = U(100) = 40e^{100k}, \quad \text{so that} \quad e^{100k} = 2: \quad k = \frac{\ln 2}{100}.$$

Thus virtually everybody will be an internet user at that time T for which $U(T) = 6000$:

$$40e^{kT} = 6000; \qquad e^{kT} = 150; \qquad T = \frac{1}{k}\ln 150 = \frac{100\ln 150}{\ln 2}.$$

Because $T \approx 722.8818690496$, which is about 1.98 years, it follows that virtually everyone is using the internet today, and *that's* why it takes graphics so long to download.

C09S01.039: Let $y(t)$ denote the height of the water in the tank (in feet) at time t (in hours). Then $y(0) = 9$ and $y(1) = 4$. By Eq. (29) of Section 9.1, we have

$$\frac{dy}{dt} = -ky^{1/2},$$

and it follows that $y^{-1/2}\,dy = -k\,dt$, so that $2y^{1/2} = C - kt$ where C is a constant. The condition $y(0) = 9$ now yields $C = 6$, so that $y^{1/2} = 3 - \frac{1}{2}kt$. The condition $y(1) = 4$ then yields $2 = 3 - \frac{1}{2}k$, so that $k = 2$. Hence $y(t) = (3-t)^2$ for $0 \leqq t \leqq 3$. Therefore $y(t) = 0$ when $t = 3$, and thus it takes three hours total for the tank to drain completely.

C09S01.040: Let $y(t)$ denote the depth of water in the tank (in feet) at time t (in *seconds*). (We must use units of feet and seconds because we are using $g = 32$ ft/s^2.) Then the volume of water in the tank at time t will be $V(t) = 9\pi y(t)$. By Eq. (27) of Section 9.1, we have

$$\frac{dV}{dt} = -a\sqrt{2gy} = -8ay^{1/2}$$

where $a = \frac{1}{144}\pi$ is the area of the hole at the bottom of the tank (note the conversion of inches into feet). Now

$$\frac{dV}{dt} = -\frac{\pi}{144}\cdot 8y^{1/2},$$

so

$$9\pi\frac{dy}{dt} = -\frac{\pi}{18}y^{1/2};$$

$$y^{-1/2}\,dy = -\frac{1}{162}\,dt;$$

$$2y^{1/2} = C - \frac{t}{162}.$$

Now the condition $y(0) = 9$ yields $C = 6$, and therefore

$$y(t) = \left(3 - \frac{t}{324}\right)^2, \qquad 0 \leqq t \leqq 972.$$

Hence the tank is empty when $t = 972$ (s); that is, it requires 16 min 12 s for the tank to drain completely.

C09S01.041: We will use Eq. (28) of Section 9.1, which allows us to work in units of feet and hours (instead of seconds) because the conversion of the gravitational acceleration g into such units is taken care of by the proportionality constant c there. Let $y(t)$ denote the depth of the water in the tank at time t. If $r(t)$ is the radius of the circular surface of the water then, using similar triangles, we have

$$\frac{r(t)}{y(t)} = \frac{5}{16}, \qquad \text{so that} \qquad r(t) = \frac{5}{16}\,y(t).$$

Hence the volume of water in the tank at time t will be

$$V(t) = \frac{1}{3}\pi \cdot \frac{25}{144}\,[y(t)]^3,$$

and therefore (with the aid of Eq. (28))

$$\begin{aligned}
\frac{dV}{dt} &= \frac{25\pi}{144}y^2\frac{dy}{dt} = -\,cy^{1/2};\\
y^{3/2}\,dy &= -\frac{144c}{25\pi}\,dt = -k\,dt \qquad \text{(where } k \text{ is constant);}\\
\frac{2}{5}y^{5/2} &= C - kt.
\end{aligned}$$

Therefore $y(t) = (A - Bt)^{2/5}$ for some constants A and B. Because $16 = y(0) = A^{2/5}$, we see that $A = 1024$, so that $y(t) = (1024 - Bt)^{2/5}$. Moreover, $9 = y(1) = (1024 - B)^{2/5}$, and it follows that $B = 781$. Therefore

$$y(t) = (1024 - 781t)^{2/5}.$$

Consequently the tank will be empty when

$$t = \frac{1024}{781} \approx 1.31114 \quad \text{(h);}$$

that is, in a little less than 1 h 19 min.

C09S01.042: Let r denote the radius of the tank and h its height. Let $y(t)$ denote the height of water in the tank at time t, $0 \leqq t \leqq T$, so that $y(0) = h$ and $y(T) = 0$. Let $V(t)$ be the volume of water in the tank at time t, so that $V(t) = \pi r^2 y(t)$. We will use Eq. (28) of Section 9.1, so that the units of distance and time are not important:

$$\frac{dV}{dt} = -cy^{1/2}.$$

Thus

$$\frac{dV}{dt} = \pi r^2 \frac{dy}{dt} = -cy^{1/2};$$

$$y^{-1/2}\,dy = -2k\,dt \qquad (k \text{ constant});$$

$$2y^{1/2} = C - 2kt;$$

$$2h^{1/2} = C - 2k \cdot 0 = C;$$

$$2y^{1/2} = 2h^{1/2} - 2kt;$$

$$y^{1/2} = h^{1/2} - kt.$$

Now $0 = y(T) = \left(h^{1/2} - kT\right)^2$, and it follows that $k = h^{1/2}/T$. Therefore

$$y(t) = \left(h^{1/2} - h^{1/2}\frac{t}{T}\right)^2 = h\left(1 - \frac{t}{T}\right)^2;$$

$$V(t) = \pi r^2 y(t) = \pi r^2 h\left(1 - \frac{t}{T}\right)^2 = V_0\left(1 - \frac{t}{T}\right)^2$$

for $0 \leqq t \leqq T$, as we were to show.

C09S01.043: Let $y(t)$ be the height of water in the tank (in feet) at time t (in hours), with $t = 0$ corresponding to 12 noon. Then $y(0) = 12$ and $y(1) = 6$. When the height of water in the tank is h, then—by the method of cross sections (section 6.2)—the volume of water in the tank will be

$$V = \int_0^h \pi y^{3/2}\,dy = \left[\frac{2}{5}\pi y^{5/2}\right]_0^h = \frac{2}{5}\pi h^{5/2}.$$

Therefore at time t, the volume of water in the tank will be

$$V(t) = \frac{2}{5}\pi\,[y(t)]^{5/2}.$$

We will use Eq. (28) of Section 9.1 because the constant of proportionality there will allow us to use hours and feet for our units rather than seconds and feet. Thus we find that

$$\frac{dV}{dt} = \pi\,[y(t)]^{3/2} \cdot \frac{dy}{dt} = -cy^{1/2};$$

$$\pi y\,dy = -c\,dt;$$

$$\frac{\pi}{2}y^2 = A - ct;$$

$$y(t) = (B - Ct)^{1/2}.$$

The condition $12 = y(0) = \sqrt{B}$ now yields $B = 144$ (not -144; look at the last equation in the display), so that $y(t) = (144 - Ct)^{1/2}$. Next, $6 = y(1) = (144 - C)^{1/2}$, so that $C = 108$. Therefore $y(t) = \sqrt{144 - 108t}$. The tank will be empty when $y(t) = 0$, which will occur when $t = \frac{4}{3}$ (h). Answer: The tank will be empty at 1:20 P.M.

C09S01.044: Among other things, we need to find the radius of the hole at the bottom of this tank, so we must use Eq. (27) of Section 9.1 and work with feet and seconds (because we plan to use $g = 32$ ft/s^2). We

also let $t = 0$ correspond to 12 noon. If h is the height of water in the tank, then (by the method of cross sections) the volume of water it contains will be

$$V = \int_0^h \pi y\, dy = \frac{1}{2}\pi h^2.$$

Therefore if the height of water at time t is $y(t)$, we have $V(t) = \frac{1}{2}\pi\,[y(t)]^2$. This and Eq. (27) yield

$$\begin{aligned}
&\frac{dV}{dt} = \pi y \frac{dy}{dt} = -2\sqrt{2gy} = -8ay^{1/2};\\
&y^{1/2}\, dy = -8r^2\, dt \qquad (r \text{ is the radius of the hole});\\
&\frac{2}{3}y^{3/2} = C_1 - 8r^2 t;\\
&y^{3/2} = C - 12r^2 t;\\
&y(t) = (C - 12r^2 t)^{2/3}.
\end{aligned}$$

The condition $y(0) = 4$ implies that $C = 8$, so that $y(t) = (8 - 12r^2 t)^{2/3}$. The depth of water in the tank at 1 P.M. is known to be 1 foot, so—converting one hour into seconds—

$$\begin{aligned}
&1 = y(3600);\\
&8 - (12r^2)(3600) = 1;\\
&r^2 = \frac{21}{9 \cdot 4 \cdot 3600};\\
&r = \frac{\sqrt{21}}{360} \quad \text{(ft)}.
\end{aligned}$$

Answers: (a) At time t in seconds, the depth of the water will be

$$y(t) = \left(8 - \frac{7t}{3600}\right)^{2/3} \quad \text{(ft)};$$

at time t in hours, the depth of the water will be $y(t) = (8 - 7t)^{2/3}$ (ft). (b) The tank will be empty when $y(t) = 0$; that is, when $t = \frac{8}{7}$ (h); in other words, at approximately 1:08:34 P.M. (c) The radius of the bottom hole is $\frac{1}{30}\sqrt{21} \approx 0.152753$ in.

C09S01.045: Set up a coordinate system in which one end of the tank lies in the xy-plane with its lowest point at the origin, thus bounded by the circle with equation $x^2 + (y - 3)^2 = 9$. A horizontal cross section of the tank "at" location y ($0 \leqq y \leqq 6$) has width $2\sqrt{6y - y^2}$ and length 5, so if the depth of xylene in the tank is h, then its volume is

$$V = \int_0^h 10(6y - y^2)^{1/2}\, dy.$$

Thus if the depth of xylene in the tank is y, then its volume is given by

$$V(y) = \int_0^y 10(6u - u^2)^{1/2}\, du.$$

Hence by the chain rule and the fundamental theorem of calculus,

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = 10(6y - y^2)^{1/2} \frac{dy}{dt}.$$

We will use Eq. (27) of Section 9.1,

$$\frac{dV}{dt} = -a\sqrt{2gy}\,,$$

in which we take $g = 32$ ft/s^2; a is the area of the hole in the bottom of the tank, so—converting inches into feet—we have $a = \pi/144$ (ft^2). As usual, $y = y(t)$ is the depth of water in the tank at time t (distances will be measured in feet and time in seconds). Thus

$$10(6y - y^2)^{1/2} \frac{dy}{dt} = -a\sqrt{2gy} = -8ay^{1/2} = -\frac{\pi}{18} y^{1/2};$$

$$(6 - y)^{1/2}\, dy = -\frac{\pi}{180}\, dt;$$

$$\frac{2}{3}(6 - y)^{3/2} = C + \frac{\pi t}{180}.$$

Now $y(0) = 3$, so $C = \frac{2}{3}(3^{3/2}) = 2\sqrt{3}$. Therefore

$$\frac{2}{3}(6 - y)^{3/2} = 2\sqrt{3} + \frac{\pi t}{180}.$$

Hence $y = 0$ when

$$t = \frac{180}{\pi}\left(4\sqrt{6} - 2\sqrt{3}\right) \approx 362.9033$$

(seconds); that is, just a little less than 6 min 3 s.

C09S01.046: We will use units of centimeters and seconds in this solution. Set up a coordinate system in which the tank has its lowest point at the origin and its vertical diameter lying on the y-axis, so that the equation of the cross section of the tank in the xy-plane will be $x^2 + (y - 25)^2 = 625$. If the liquid in the tank has depth y, then the radius of its circular surface will be $x = (50y - y^2)^{1/2}$, so in Eq. (26) of Section 9.1 we take

$$A(y) \;=\; \pi(50y - y^2), \quad a \;=\; \frac{\pi}{4}, \quad \text{and} \quad g = 980.$$

Thus we obtain

$$A(y)\, dy \;=\; -14a\sqrt{10y}\; dt; \qquad \text{that is,}$$

$$\pi(50y - y^2)\, dy \;=\; -\frac{7\pi\sqrt{10}}{2}\, y^{1/2}\, dt;$$

$$\left(50y^{1/2} - y^{3/2}\right)\, dy \;=\; -\frac{7\sqrt{10}}{2}\, dt;$$

$$\frac{100}{3}\, y^{3/2} - \frac{2}{5}\, y^{5/2} \;=\; C - \frac{7t\sqrt{10}}{2}.$$

When $t = 0$, we have $y = 50$. It follows that

$$C = \frac{10000}{3}\sqrt{2}\,.$$

The tank is empty when $y = 0$, which leads to the equation

$$t = \frac{2}{7\sqrt{10}}\,C = \frac{4000\sqrt{5}}{21} \approx 425.9177$$

seconds, about 7 min 5.9 s.

C09S01.047: Let $h = f^{-1}$, let a be the area of the hole, and let $y(t)$ be the depth of water in the tank at time t. We use $c = 1$, $g = 32$, and $\dfrac{dy}{dt} = -\dfrac{1}{10800}$ (feet per second). Then by Eq. (26) of Section 9.1,

$$-\frac{1}{10800}\,A(y) = -a\sqrt{64y}$$

where $A(y) = \pi\,[h(y)]^2$. Therefore

$$[h(y)]^2 = \frac{86400}{\pi}ay^{1/2},$$

and thus

$$[h(y)]^4 = \frac{7464960000}{\pi^2}\,a^2y.$$

Finally, because $y = f(x)$ and $x = h(y)$, we have

$$f(x) = \frac{\pi^2}{(864000)^2}\,x^4.$$

Now $f(1) = 4$, so $86400a = \pi/2$; it follows that

$$a = \frac{\pi}{172800} = \pi r^2$$

where r is the radius of the hole. Therefore

$$r = \frac{1}{240\sqrt{3}} \quad \text{(feet).} \qquad \text{That is,} \quad r \approx 0.02887 \quad \text{(in.)}$$

Section 9.2

C09S02.001: The following sequence of commands in *Mathematica* 3.0 will generate the slope field and the solution curves through the given points. Begin with the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where

```
f[x_, y_] := −y − Sin[x]
```

Then set up the viewing window $a \leqq x \leqq b,\ c \leqq y \leqq d$:

```
a = −3;  b = 3;  c = −3;  d = 3;
```

The unit vectors that comprise the components of the short line segments tangent to the solution curves—those that form the slope field—are these:

```
u[x_, y_] := 1/Sqrt[1 + (f[x,y])∧2]

v[x_, y_] := f[x,y]/Sqrt[1 + (f[x,y])∧2]
```

The next commands construct the slope field.

```
Needs["Graphics`PlotField`"]

dfield = PlotVectorField[ { u[x,y], v[x,y] }, { x, a, b }, { y, c, d },
    HeadWidth → 0, HeadLength → 0, PlotPoints → 19,
    PlotRange → {{ a, b, }, { c, d }}, Axes → True,
    Frame → True, AspectRatio → 1 ];
```

To set up the first initial point and solution curve:

```
x0 = −2.5;  y0 = 2.0;
point1 = Graphics[ { PointSize[0.025], Point[ { x0, y0 } ] } ];
soln = NDSolve[ { y′[x] == f[x, y[x]], y[x0] == y0 }, y[x], { x, a, b } ];
soln[[1, 1, 2]];
curve1 = Plot[ soln[[1, 1, 2]], { x, a, b },
    PlotStyle → { Thickness[0.0065] } ];
```

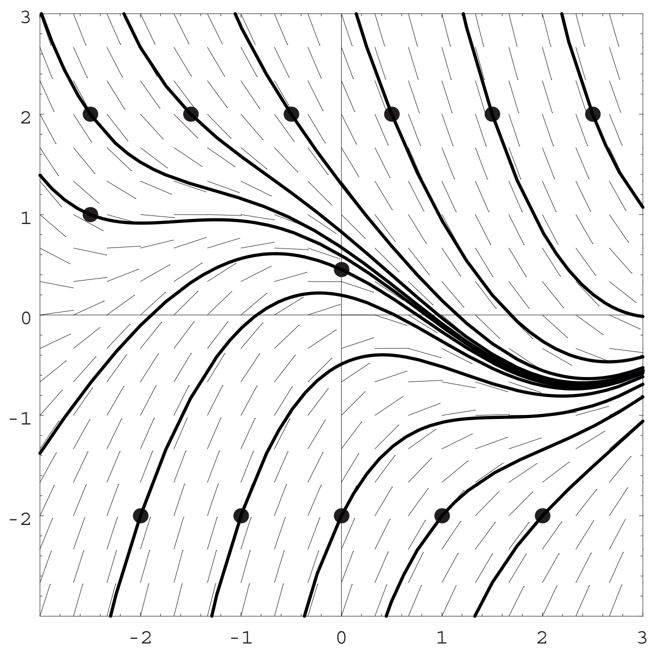
The last option may be omitted; it's used to thicken the solution curve to make it more visible. Repeat the last sequence of commands with the remaining initial points; for example,

```
x7 = −2.5;  y7 = 1.0;
point7 = Graphics[ { PointSize[0.025], Point[ { x7, y7 } ] } ];
soln = NDSolve[ { y′[x] == f[x, y[x]], y[x0] == y0 }, y[x], { x, a, b } ];
soln[[1, 1, 2]];
curve7 = Plot[ soln[[1, 1, 2]], { x, a, b },
    PlotStyle → { Thickness[0.0065] } ];
```

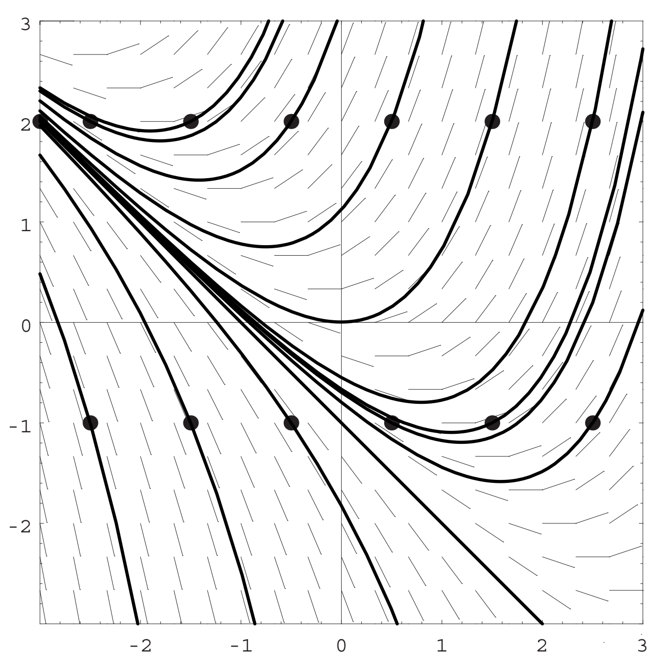
The final version of the figure can be generated by the *Mathematica* command

```
Show[ dfield, point1, curve1, point2, curve2, point3, curve3, point4, curve4,
    point5, curve5, point6, curve6, point7, curve7, point8, curve8, point9, curve9,
    point10, curve10, point11, curve11, point12, curve12, point13, curve13 ];
```

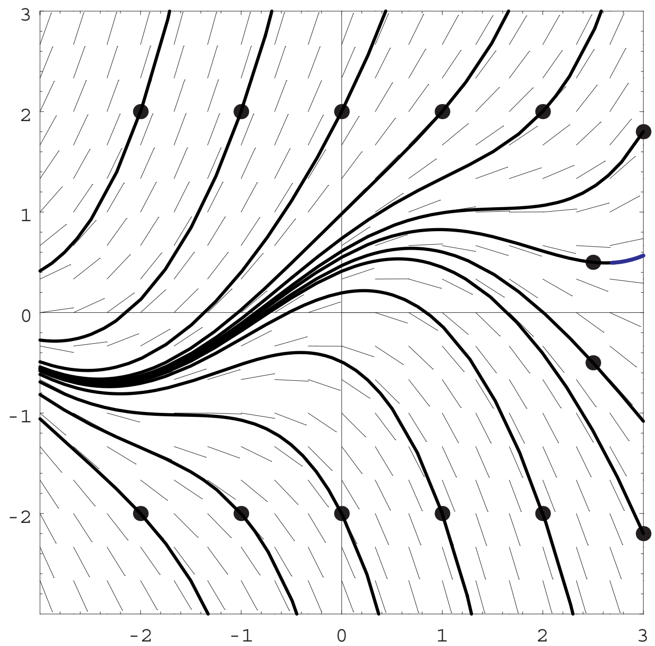
The resulting figure is next.



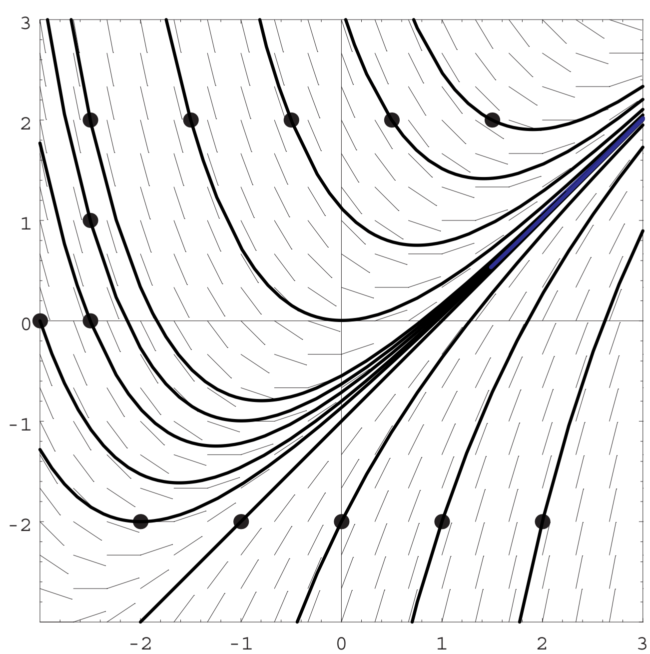
C09S02.002: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x, y)$ and the initial points (x_i, y_i) and the viewing window. The resulting figure is next.



C09S02.003: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x, y)$ and the initial points (x_i, y_i) and the viewing window. The resulting figure is next.

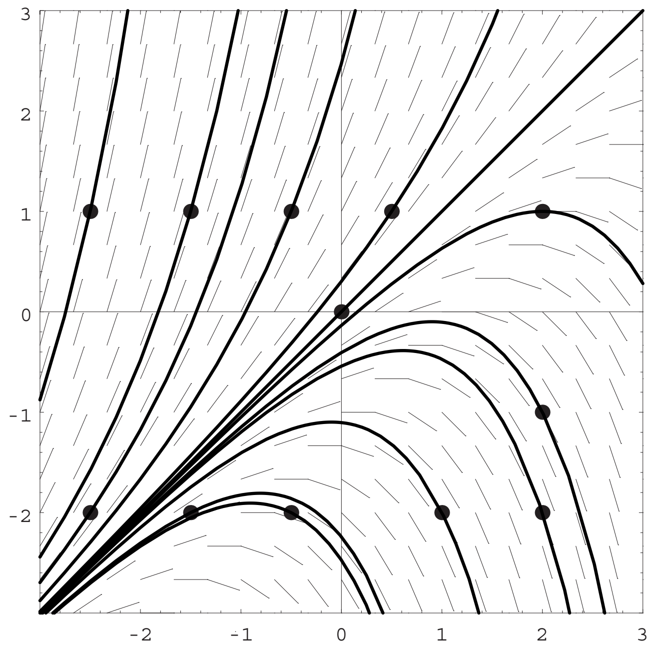


C09S02.004: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x, y)$ and the initial points (x_i, y_i) and the viewing window. The resulting figure is next.

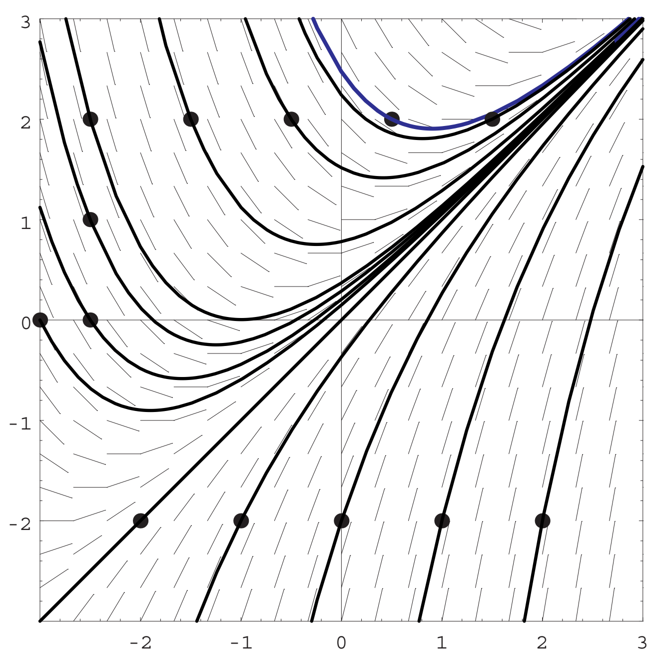


C09S02.005: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x, y)$

and the initial points $(x_i,\, y_i)$ and the viewing window. The resulting figure is next.

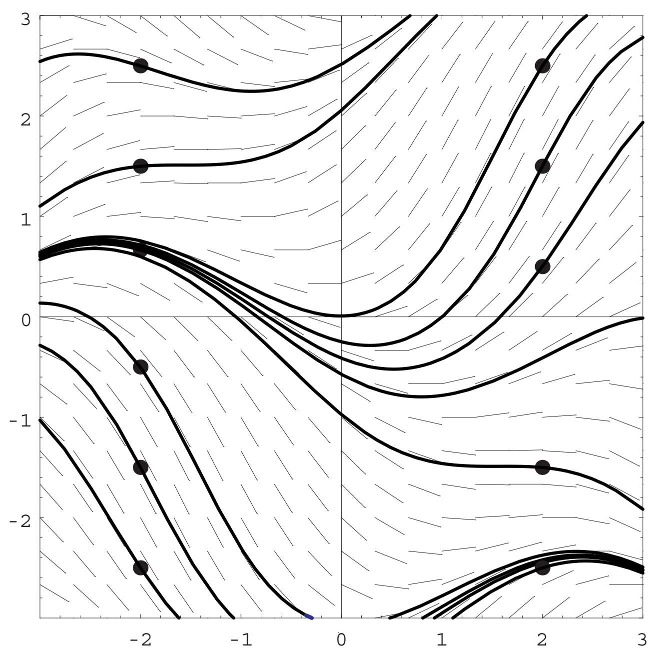


C09S02.006: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x,\, y)$ and the initial points $(x_i,\, y_i)$ and the viewing window. The resulting figure is next.

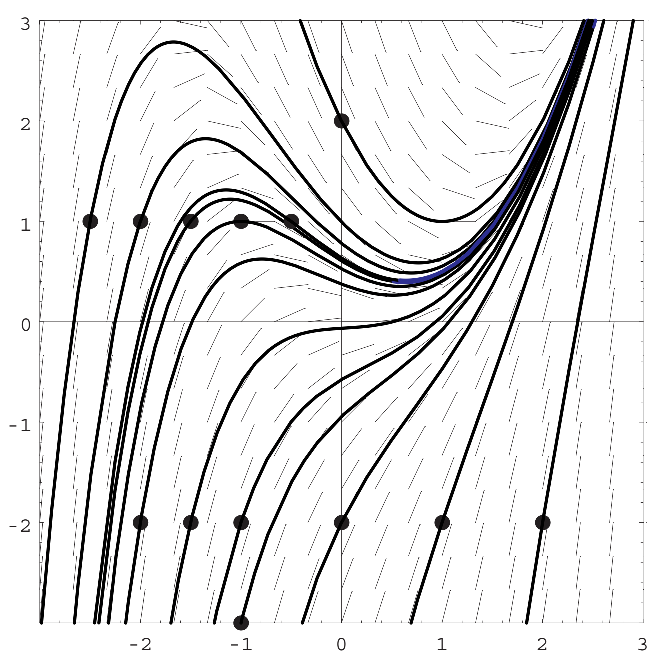


C09S02.007: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x,\, y)$

and the initial points $(x_i,\, y_i)$ and the viewing window. The resulting figure is next.

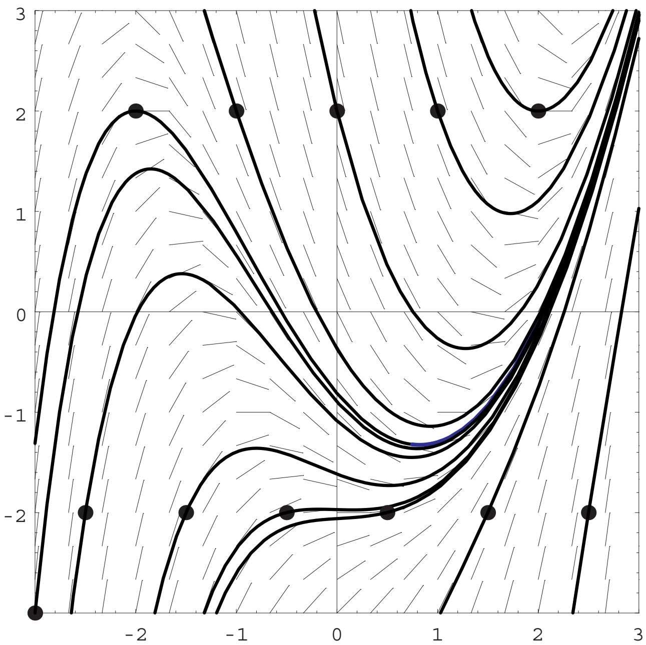


C09S02.008: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x,\, y)$ and the initial points $(x_i,\, y_i)$ and the viewing window. The resulting figure is next.

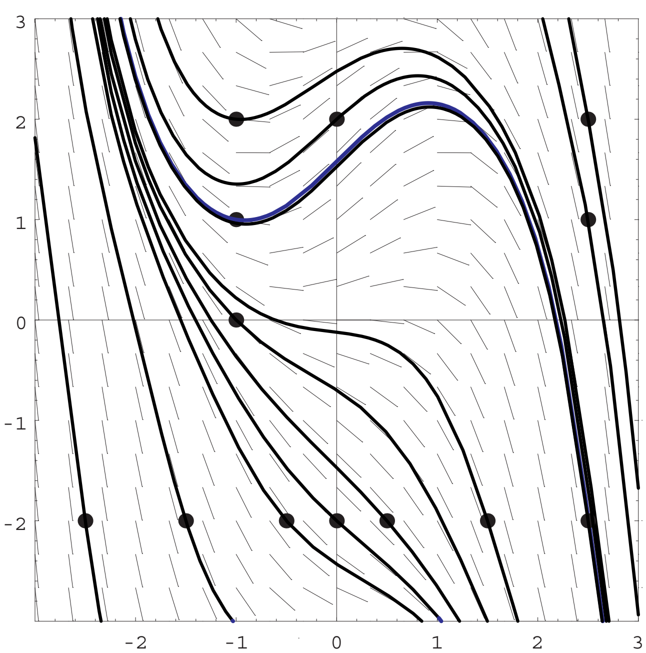


C09S02.009: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x,\, y)$

and the initial points (x_i, y_i) and the viewing window. The resulting figure is next.



C09S02.010: Follow the template given in the solution of Problem 1, with the obvious changes to $f(x, y)$ and the initial points (x_i, y_i) and the viewing window. The resulting figure is next.



C09S02.011: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)	x	y $(h = 0.1)$	y (true)
0.25	1.500	1.558	0.1	1.800	1.810
0.50	1.125	1.213	0.2	1.620	1.637
			0.3	1.458	1.482
			0.4	1.312	1.341
			0.5	1.181	1.213

C09S02.012: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)	x	y $(h = 0.1)$	y (true)
0.25	0.750	0.824	0.1	0.600	0.611
0.50	1.125	1.359	0.2	0.720	0.746
			0.3	0.864	0.911
			0.4	1.037	1.113
			0.5	1.244	1.359

C09S02.013: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)	x	y $(h = 0.1)$	y (true)
0.25	1.500	1.568	0.1	1.200	1.210
0.50	2.125	2.297	0.2	1.420	1.443
			0.3	1.662	1.700
			0.4	1.928	1.984
			0.5	2.221	2.297

C09S02.014: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)	x	y $(h = 0.1)$	y (true)
0.25	0.750	0.808	0.1	0.900	0.910
0.50	0.625	0.713	0.2	0.820	0.837
			0.3	0.758	0.782
			0.4	0.712	0.741
			0.5	0.681	0.713

C09S02.015: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)
0.25	1.000	0.966
0.50	0.938	0.851

x	y $(h = 0.1)$	y (true)
0.1	1.000	0.995
0.2	0.990	0.979
0.3	0.969	0.950
0.4	0.936	0.908
0.5	0.889	0.851

C09S02.016: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)
0.25	2.000	1.879
0.50	1.750	1.558

x	y $(h = 0.1)$	y (true)
0.1	2.000	1.980
0.2	1.960	1.922
0.3	1.882	1.828
0.4	1.769	1.704
0.5	1.627	1.558

C09S02.017: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)
0.25	3.000	2.953
0.50	2.859	2.647

x	y $(h = 0.1)$	y (true)
0.1	3.000	2.997
0.2	2.991	2.976
0.3	2.955	2.920
0.4	2.875	2.814
0.5	2.737	2.647

C09S02.018: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)
0.25	0.250	0.223
0.50	0.445	0.405

x	y $(h = 0.1)$	y (true)
0.1	0.100	0.095
0.2	0.190	0.182
0.3	0.273	0.262
0.4	0.349	0.336
0.5	0.420	0.405

C09S02.019: Results, rounded to three places:

x	y $(h = 0.25)$	y (true)
0.25	1.125	1.134
0.50	1.267	1.287

x	y $(h = 0.1)$	y (true)
0.1	1.050	1.051
0.2	1.103	1.105
0.3	1.158	1.162
0.4	1.216	1.223
0.5	1.278	1.287

C09S02.020: Results, rounded to three places:

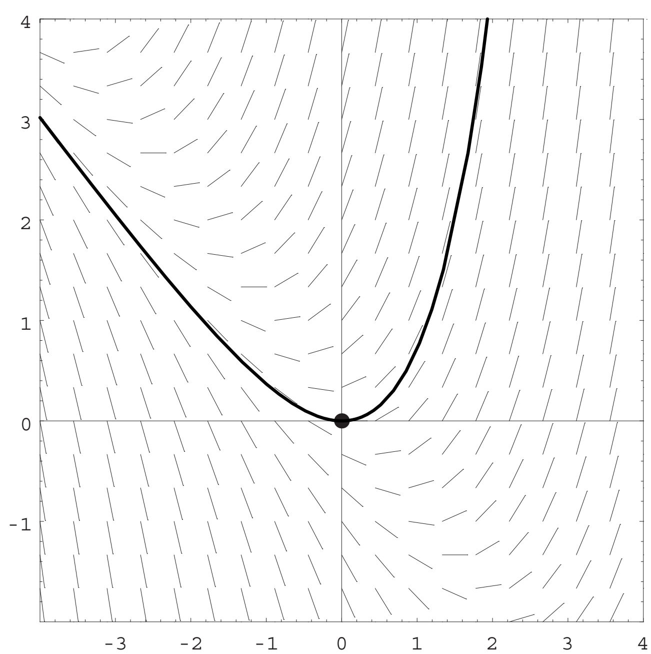
x	y $(h = 0.25)$	y (true)
0.25	1.000	1.067
0.50	1.125	1.333

x	y $(h = 0.1)$	y (true)
0.1	1.000	1.051
0.2	1.103	1.101
0.3	1.062	1.099
0.4	1.129	1.190
0.5	1.231	1.333

C09S02.021: We follow the template given in the solution of Problem 1 for using *Mathematica* 3.0 to generate both the slope field and the desired solution curve. The result is shown at the end of this solution. The exact solution of the initial value problem

$$\frac{dy}{dx} = x + y, \qquad y(0) = 0$$

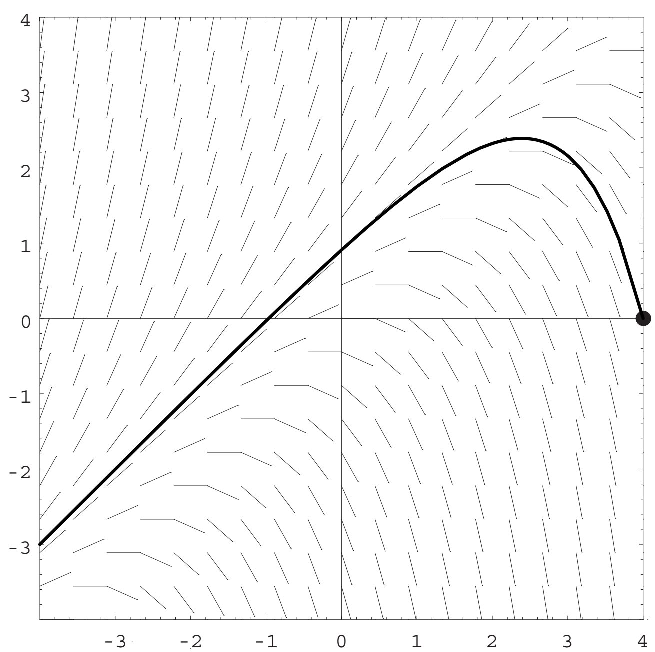
is $y(x) = e^x - x - 1$, and $y(-4) = e^{-4} + 3 \approx 3.018316$, in agreement with the following figure.



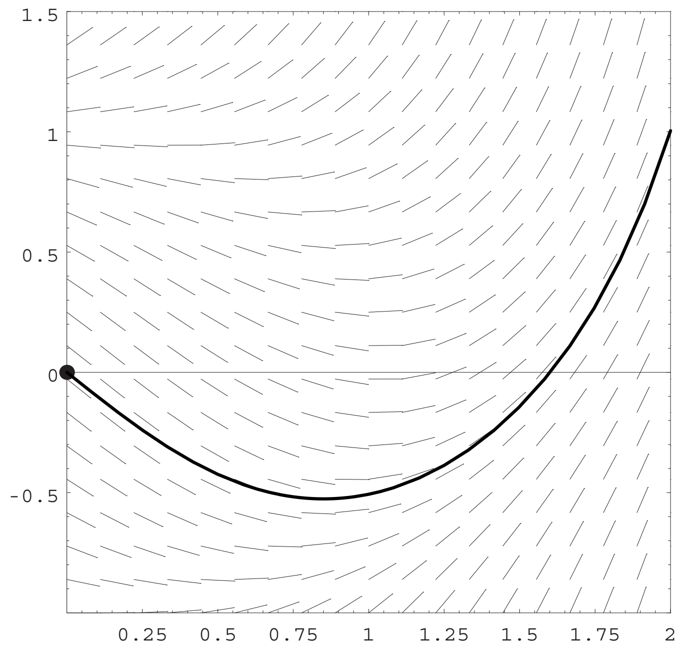
C09S02.022: We followed the template given in the solution of Problem 1 to have *Mathematica* 3.0 generate the slope field and solution curve; the result is shown following this solution. The exact solution of the initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(4) = 0$$

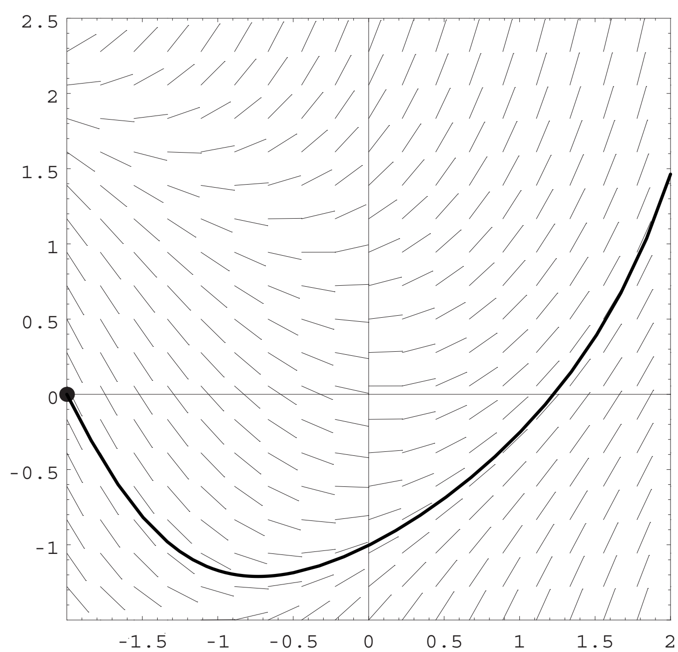
is $y(x) = x + 1 - 5e^{x-4}$, and $y(-4) = -5e^{-8} - 3 \approx -3.001677$.



C09S02.023: We followed the template given in the solution of Problem 1. Thus *Mathematica* 3.0 generated the figure shown at the conclusion of this solution. In the solution of Problem 27 we find that $y(2) \approx 1.0044$ (although the last digit is in question).



C09S02.024: We followed the template in the solution of Problem 1 for using *Mathematica* 3.0 to generate the slope field and solution curve. The result is shown following this solution. In Problem 28 we find that $y(2) \approx 1.4633$ (although the last digit is in question).



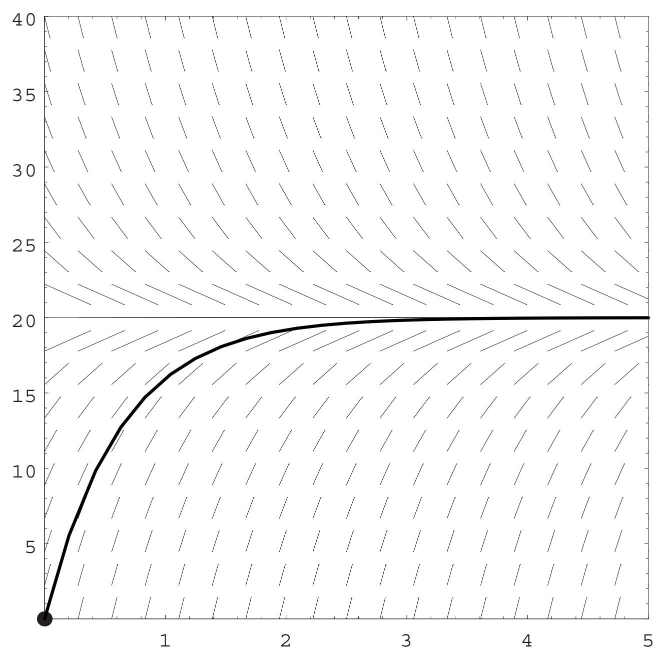
C09S02.025: We constructed the slope field and solution curve as in the solution of Problem 1. The result is shown after this solution. The exact solution of the given initial value problem is

$$v(t) = 20\left[1 - \exp\left(-\frac{8t}{5}\right)\right],$$

so—in accord with the figure—the limiting velocity is 20 ft/s. Landing with this velocity is about the same as landing after jumping off a wall 6.25 feet high, so the landing is perfectly survivable (but be sure to relax and bend your knees). A "strategically located haystack" would nevertheless be welcome. Your velocity will be 95% of your limiting velocity when

$$t = \frac{5}{8}\ln 20 \approx 1.872333$$

seconds.



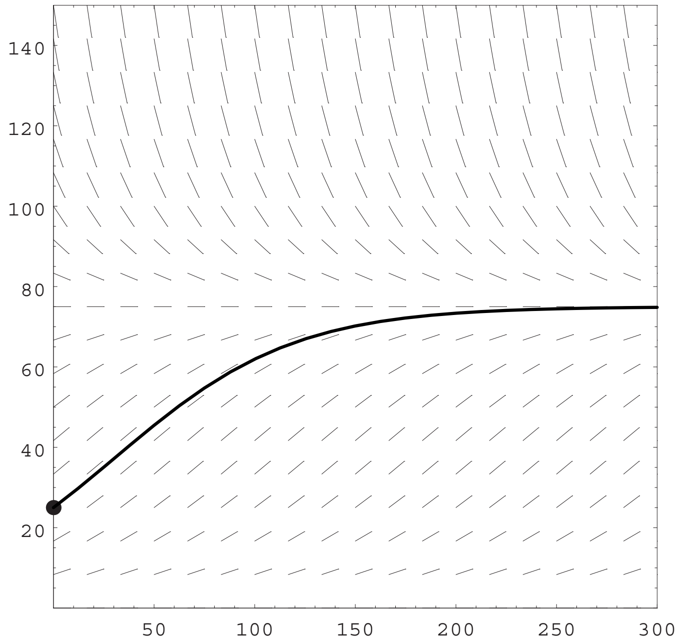
C08S02.026: We constructed the slope field and solution curve as in the solution of Problem 1. The result follows this solution. The exact solution of the given initial value problem is

$$P(t) = \frac{75}{1 + 2\exp\left(-\dfrac{9t}{400}\right)}, \tag{1}$$

and it follows that the doubling time is

$$t = \frac{400}{9}\ln 4 \approx 61.613083$$

months. It is also clear—both from the figure and from Eq. (1)—that the limiting population is 75 deer.



C09S02.027: With $f(x, y) = x^2 + y^2 - 1$, we began with the initial values $x = 0$, $y = 0$, $h = 0.1$, $k = 2$, and $n = 0$ and executed the *Mathematica* 3.0 command

```
While[ n < 10*k, { n = n + 1, y = y + h*f[x,y], x = x + h,
  If[IntegerQ[n/k], Print[ { x, y } ]] } ]
```

We repeated with $h = 0.01$ and $k = 20$, again with $h = 0.001$ and $k = 200$, again with $h = 0.0001$ and $k = 2000$, and once more with $h = 0.00001$ and $k = 20000$. The results are shown next.

	$h = 0.1$	$h = 0.01$	$h = 0.001$	$h = 0.0001$	$h = 0.00001$
x	y	y	y	y	y
0.2	− 0.19800	− 0.19513	− 0.19479	− 0.19475	− 0.19475
0.4	− 0.37267	− 0.36119	− 0.35999	− 0.35987	− 0.35985
0.6	− 0.49817	− 0.47591	− 0.47367	− 0.47345	− 0.47343
0.8	− 0.55948	− 0.52741	− 0.52423	− 0.52391	− 0.52388
1.0	− 0.55135	− 0.51131	− 0.50734	− 0.50694	− 0.50690
1.2	− 0.47281	− 0.42533	− 0.42056	− 0.42008	− 0.42003
1.4	− 0.32094	− 0.26359	− 0.25772	− 0.25713	− 0.25707
1.6	− 0.08538	− 0.01026	− 0.00238	− 0.00158	− 0.00150
1.8	0.26121	0.37239	0.38466	0.38591	0.38603
2.0	0.77724	0.99768	1.00172	1.00417	1.00442

It appears that, to three places, $y(2) \approx 1.004$. We did not pursue further accuracy because the next-to-last column required over 12 seconds to execute and print and the last column required over 121 seconds. Linear extrapolation suggests that an additional column would require over an hour; we prefer to use numerical methods more sophisticated than Euler's method for such problems.

C09S02.028: With $f(x, y) = x + \frac{1}{2}y^2$, we began with the initial values $x = -2$, $y = 0$, $h = 0.1$, $k = 4$, and $n = 0$ and executed the *Mathematica* 3.0 command

```
While[ n < 10*k, { n = n + 1, y = y + h*f[x,y], x = x + h,
  If[IntegerQ[n/k], Print[ { x, y } ]] } ]
```

We repeated with $h = 0.01$ and $k = 40$, again with $h = 0.001$ and $k = 400$, again with $h = 0.0001$ and $k = 4000$, and once more with $h = 0.00001$ and $k = 40000$. The results are shown next.

	$h = 0.1$	$h = 0.01$	$h = 0.001$	$h = 0.0001$	$h = 0.00001$
x	y	y	y	y	y
-1.6	-0.71477	-0.68846	-0.68584	-0.68558	-0.68556
-1.2	-1.13094	-1.08353	-1.07893	-1.07847	-1.07843
-0.8	-1.26303	-1.21363	-1.20880	-1.20832	-1.20827
-0.4	-1.20867	-1.16655	-1.16235	-1.16193	-1.16189
0.0	-1.04232	-1.00761	-1.00408	-1.00372	-1.00369
0.4	-0.79852	-0.76778	-0.76459	-0.76427	-0.76424
0.8	-0.48290	-0.45117	-0.44784	-0.44751	-0.44747
1.2	-0.07749	-0.03692	-0.03261	-0.03218	-0.03214
1.6	0.46934	0.53659	0.54391	0.54465	0.54473
2.0	1.29001	1.44354	1.46131	1.46311	1.46329

It appears that, to three places, $y(2) \approx 1.463$. We did not pursue further accuracy because the next-to-last column required over 25 seconds to execute and print and the last column required over 252 seconds. Linear extrapolation suggests that an additional column would require well over an hour; we prefer to use numerical methods more sophisticated than Euler's method for such problems. (Hardware: Power Macintosh 7600/120 with 64Mb RAM running System 9.0.)

C09S02.029: Direct substitution verifies that both $y_1(x) \equiv 1$ and $y_2(x) = \cos x$ satisfy the given initial value problem. This does not contradict the existence-uniqueness theorem of Section 9.2 because

$$\frac{\partial}{\partial y}\left(-\sqrt{1-y^2}\right) = \frac{y}{\sqrt{1-y^2}}$$

is not continuous at the point $(0, 1)$, so the theorem does not guarantee uniqueness of a solution passing through that point.

C09S02.030: By inspection, $y_1(x) \equiv 0$ is one solution. If another is not apparent by inspection, separation of variables yields the second solution $y_2(x) = x^3$. The existence-uniqueness theorem is not contradicted because

$$\frac{\partial}{\partial y}\left(3y^{2/3}\right) = \frac{1}{y^{1/3}}$$

is not continuous at $(x, y) = (0, 0)$.

C09S02.031: If $a \geqq 0$ and $y(x) = (x-a)^3$, then

$$\frac{dy}{dx} = 3(x-a)^2 = 3\left[(x-a)^3\right]^{2/3} = 3\left[y(x)\right]^{2/3}.$$

Therefore $y(x) = (x-a)^{2/3}$ is a solution of the given differential equation on $[0, +\infty)$. Consequently, if $a \geqq 0$, then

$$y(x) = \begin{cases} x^3 & \text{if } x \leqq 0, \\ 0 & \text{if } 0 \leqq x \leqq a, \\ (x-a)^3 & \text{if } a \leqq x \end{cases}$$

is a solution of the given initial value problem. (It is easy to verify that $y(x)$ is differentiable for all x.) This does not contradict the existence-uniqueness theorem because that theorem guarantees existence and uniqueness only on some open interval containing $x = -1$; here, uniqueness fails only at the "distant" point $x = 0$.

C09S02.032: It is clear that $y(x) = kx$ satisfies the given differential equation:

$$x\,\frac{dy}{dx} = kx = y(x) \quad \text{and} \quad y(0) = 0$$

if k is any constant. Therefore the initial value problem

$$x\,\frac{dy}{dx} = y, \qquad y(0) = 0$$

has infinitely many solutions passing through the point $(0, 0)$.

Section 9.3

C09S03.001: Given: $\dfrac{dy}{dx} = 2x\sqrt{y}$. Then

$$\begin{aligned} y^{-1/2}\,dy &= 2x\,dx; \\ 2y^{1/2} &= x^2 + C; \\ y^{1/2} &= \frac{x^2 + C}{2}; \\ y(x) &= \left(\frac{x^2 + C}{2}\right)^2. \end{aligned}$$

C09S03.002: Given: $\dfrac{dy}{dx} = 2xy^2$. Then

$$\begin{aligned} y^{-2}\,dy &= 2x\,dx; \\ y^{-1} &= C - x^2; \\ y(x) &= \frac{1}{C - x^2}. \end{aligned}$$

C09S03.003: Given: $\dfrac{dy}{dx} = x^2y^2$. Then

$$\begin{aligned} y^{-2}\,dy &= x^2\,dx; \\ y^{-1} &= C - \frac{1}{3}x^3; \\ y(x) &= \frac{1}{C - \frac{1}{3}x^3} = \frac{3}{K - x^3} \end{aligned}$$

where $K = 3C$ is a constant.

C09S03.004: Given: $\dfrac{dy}{dx} = (xy)^{3/2}$. Then

$$\begin{aligned} y^{-3/2}\,dy &= x^{3/2}\,dx; \\ -2y^{-1/2} &= \frac{2}{5}x^{5/2} + C_1; \\ y^{-1/2} &= -\frac{1}{5}x^{5/2} + C_2 \qquad (C_2 \text{ is a constant}); \\ y^{1/2} &= -\frac{1}{\frac{1}{5}x^{5/2} - C_2} = -\frac{5}{x^{5/2} + C_3} \qquad (C_3 \text{ is a constant}); \\ y(x) &= \left(\frac{5}{x^{5/2} + C_3}\right)^2. \end{aligned}$$

C09S03.005: Given: $\dfrac{dy}{dx} = 2x(y-1)^{1/2}$. Then

$$(y-1)^{-1/2}\,dy = 2x\,dx;$$
$$2(y-1)^{1/2} = x^2 + C;$$
$$(y-1)^{1/2} = \frac{x^2+C}{2};$$
$$y - 1 = \left(\frac{x^2+C}{2}\right)^2;$$
$$y(x) = 1 + \left(\frac{x^2+C}{2}\right)^2.$$

C09S03.006: Given: $\dfrac{dy}{dx} = 4x^3(y-4)^2$. Then

$$(y-4)^{-2}\,dy = 4x^3\,dx; \qquad (y-4)^{-1} = C - x^4;$$
$$y - 4 = \frac{1}{C-x^4}; \qquad y(x) = 4 + \frac{1}{C-x^4}.$$

C09S03.007: Given: $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$. Then

$$(1+\sqrt{y}\,)\,dy = \left(1+\sqrt{x}\,\right)\,dx; \qquad y + \frac{2}{3}y^{3/2} = x + \frac{2}{3}x^{3/2} + C.$$

It is possible to solve explicitly for $y(x)$. To see the explicit form, enter the *Mathematica* command

```
DSolve[ y'[x] == (1 + Sqrt[x])/(1 + Sqrt[y[x]]), y[x], x ]
```

and be prepared for about 32 lines of output.

C09S03.008: Given $\dfrac{dy}{dx} = \dfrac{x+x^3}{y+y^3}$. Then

$$(y+y^3)\,dy = (x+x^3)\,dx; \qquad \frac{1}{2}y^2 + \frac{1}{4}y^4 = \frac{1}{2}x^2 + \frac{1}{4}x^4 + C.$$

It is possible to solve explicitly for $y(x)$, but probably better not to do so, as there are ambiguities of sign involving the square roots. The result is, however, not too complicated:

$$y(x) = \pm\sqrt{-1 \pm \sqrt{(x^2+1)^2 + C}\,}\,.$$

C09S03.009: Given: $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2(3y^2+1)}$. Then

$$(3y^2+1)\,dy = \left(1 + \frac{1}{x^2}\right)dx; \qquad y^3 + y = x - \frac{1}{x} + C.$$

It is possible to solve explicitly for $y(x)$. Enter the *Mathematica* command

```
DSolve[ y'[x] == (x^2 + 1)/(x^2*(3*(y[x])^2 + 1)), y[x], x ]
```

to see the result.

C09S03.010: Given: $\dfrac{dy}{dx} = \dfrac{(x^3-1)y^3}{x^2(2y^3-3)}$. Then

$$\frac{2y^3-3}{y^3}\,dy = \frac{x^3-1}{x^2}\,dx; \qquad (2-3y^{-3})\,dy = (x-x^{-2})\,dx;$$

$$2y + \frac{3}{2}y^{-2} = \frac{1}{2}x^2 + \frac{1}{x} + C; \qquad 4xy^3 + 3x = x^3y^2 + 2y^2 + 2Cxy^2.$$

It is possible to solve explicitly for $y(x)$ using any of various computer algebra programs, but the results are rather complicated.

C09S03.011: Given: $\dfrac{dy}{dx} = y^2$, $y(0) = 1$. Then

$$y^{-2}\,dy = dx; \qquad y^{-1} = C - x; \qquad y(x) = \frac{1}{C-x}.$$

Then the initial condition yields

$$1 = y(0) = \frac{1}{C}, \qquad \text{and thus} \qquad y(x) = \frac{1}{1-x}.$$

C09S03.012: Given: $\dfrac{dy}{dx} = y^{1/2}$, $y(0) = 4$. Then

$$y^{-1/2}\,dy = dx; \qquad 2y^{1/2} = x + C; \qquad y^{1/2} = \frac{x+C}{2};$$

$$y(x) = \left(\frac{x+C}{2}\right)^2.$$

The last equation and the initial condition tell us only that $C^2 = 16$, but the third equation tells us that

$$\sqrt{4} = [y(0)]^{1/2} = \frac{0+C}{2}, \qquad \text{so that} \qquad C = 4.$$

Therefore $y(x) = \left(\dfrac{x+4}{2}\right)^2$.

C09S03.013: Given: $\dfrac{dy}{dx} = \dfrac{1}{4y^3}$, $y(0) = 1$. Then

$$4y^3\,dy = dx; \qquad y^4 = x + C; \qquad 1^4 = [y(0)]^4 = 0 + C;$$

$$C = 1; \qquad [y(x)]^4 = x + 1; \qquad y(x) = (x+1)^{1/4}.$$

We take the positive root in the last step because $y(0) > 0$.

C09S03.014: Given: $\dfrac{dy}{dx} = \dfrac{1}{x^2y}$, $y(1) = 2$. Then

$$2y\,dy = 2x^{-2}\,dx; \qquad y^2 = C - \frac{2}{x}, \qquad 2^2 = [y(1)]^2 = C - 2;$$

$$C = 6; \qquad y^2 = 6 - \frac{2}{x}; \qquad y(x) = \sqrt{6 - 2x^{-1}}\,.$$

We took the positive square root in the last step because $y(1) > 0$.

C09S03.015: Given: $\dfrac{dy}{dx} = \sqrt{xy^3}\,,\ y(0) = 4.$ Then

$$y^{-3/2}\,dy = x^{1/2}\,dx; \qquad 3y^{-3/2}\,dy = 3x^{1/2}\,dx; \qquad 6y^{-1/2} = C_1 - 2x^{3/2};$$

$$y^{-1/2} = C_2 - \frac{1}{3}x^{3/2}; \qquad y^{1/2} = \frac{1}{C_2 - \frac{1}{3}x^{3/2}}; \qquad y^{1/2} = \frac{3}{C - x^{3/2}};$$

$$2 = [y(0)]^{1/2} = \frac{3}{C}; \qquad C = \frac{3}{2}; \qquad y(x) = \frac{9}{\left(\frac{3}{2} - x^{3/2}\right)^2};$$

$$y(x) = \frac{36}{\left(3 - 2x^{3/2}\right)^2}.$$

C09S03.016: Given: $\dfrac{dy}{dx} = \dfrac{x}{y},\ y(3) = 5.$ Then

$$y\,dy = x\,dx; \qquad 2y\,dy = 2x\,dx; \qquad y^2 = x^2 + C;$$

$$25 = [y(3)]^2 = 9 + C; \qquad C = 16; \qquad y^2 = x^2 + 16;$$

$$y(x) = \sqrt{x^2 + 16}.$$

We took the positive root in the last step because $y(3) > 0$.

C09S03.017: Given: $\dfrac{dy}{dx} = -\dfrac{x}{y},\ y(12) = -5.$ Then

$$y\,dy = -x\,dx; \qquad 2y\,dy = -2x\,dx; \qquad y^2 = C - x^2;$$

$$25 = [y(12)]^2 = C - 144; \qquad C = 169; \qquad y^2 = 169 - x^2;$$

$$y(x) = -\sqrt{169 - x^2}.$$

We took the negative root in the last step because $y(12) < 0$.

C09S03.018: Given: $y^2\dfrac{dy}{dx} = x^2 + 2x + 1,\ y(1) = 2.$ Thus

$$3y^2\,dy = 3(x+1)^2\,dx; \qquad y^3 = (x+1)^3 + C; \qquad 2^3 = [y(1)]^3 = (1+1)^3 + C;$$

$$C = 0; \qquad y^3 = (x+1)^3; \qquad y(x) = x + 1.$$

C09S03.019: Given: $\dfrac{dy}{dx} = 3x^2y^2 - y^2,\ y(0) = 1.$ Then

$$y^{-2}\,dy = (3x^2 - 1)\,dx; \qquad y^{-1} = x - x^3 + C; \qquad y = \frac{1}{x - x^3 + C};$$

$$1 = y(0) = \frac{1}{C}; \qquad y(x) = \frac{1}{x - x^3 + 1}.$$

C09S03.020: Given: $\dfrac{dy}{dx} = 2xy^3(2x^2+1)$, $y(1) = 1$. Then

$$y^{-3}\,dy = (4x^3+2x)\,dx; \qquad -\frac{1}{2}y^{-2} = x^4 + x^2 + C_1; \qquad y^{-2} = C - 2x^2 - 2x^4;$$

$$y^2 = \frac{1}{C - 2x^2 - 2x^4}; \qquad 1 = [y(1)]^2 = \frac{1}{C-4}; \qquad C = 5;$$

$$y^2 = \frac{1}{5 - 2x^2 - 2x^4}; \qquad y(x) = \frac{1}{\sqrt{5 - 2x^2 - 2x^4}}.$$

We took the positive root in the last step because $y(1) > 0$.

C09S03.021: Given: $\dfrac{dy}{dx} = y + 1$, $y(0) = 1$.

$$\int \frac{dy}{y+1} = \int 1\,dx; \qquad \ln(y+1) = x + C;$$

$$y + 1 = e^{x+C} = Ae^x; \qquad y(x) = Ae^x - 1;$$

$$1 = y(0) = A - 1; \qquad A = 2.$$

Answer: $y(x) = 2e^x - 1$.

C09S03.022: Given: $\dfrac{dy}{dx} = 2 - y$, $y(0) = 3$.

$$\int \frac{dy}{y-2} = \int (-1)\,dx; \qquad \ln(y-2) = C - x; \qquad y - 2 = e^{C-x} = Ae^{-x};$$

$$y(x) = 2 + Ae^{-x}; \qquad 3 = y(0) = 2 + A; \qquad y(x) = 2 + e^{-x}.$$

C09S03.023: Given: $\dfrac{dy}{dx} = 2y - 3$, $y(0) = 2$.

$$\int \frac{2\,dy}{2y-3} = \int 2\,dx; \qquad \ln(2y-3) = 2x + C; \qquad 2y - 3 = e^{2x+C} = Ae^{2x};$$

$$y(x) = \frac{Ae^{2x}+3}{2}; \qquad 2 = y(0) = \frac{A+3}{2}; \qquad y(x) = \frac{e^{2x}+3}{2}.$$

C09S03.024: Given: $\dfrac{dy}{dx} = \dfrac{1}{4} - \dfrac{y}{16} = \dfrac{4-y}{16}$, $y(0) = 20$.

$$\int \frac{dy}{y-4} = \int -\frac{1}{16}\,dx; \qquad \ln(y-4) = C - \frac{x}{16}; \qquad y - 4 = e^{C-(x/16)} = Ae^{-x/16};$$

$$y(x) = 4 + Ae^{-x/16}; \qquad 20 = y(0) = 4 + A; \qquad y(x) = 4 + 16e^{-x/16}.$$

C09S03.025: Given: $\dfrac{dx}{dt} = 2(x-1)$, $x(0) = 0$.

$$\int \frac{dx}{x-1} = \int 2\,dt; \qquad \ln(x-1) = 2t + C; \qquad x - 1 = e^{2t+C} = Ae^{2t};$$

$$x(t) = 1 + Ae^{2t}; \qquad 0 = x(0) = 1 + A; \qquad x(t) = 1 - e^{2t}.$$

C09S03.026: Given: $\dfrac{dx}{dt} = 2 - 3x,\ \ x(0) = 4.$

$$\int \frac{3\,dx}{3x-2} = \int (-3)\,dt; \qquad \ln(3x-2) = C - 3t; \qquad 3x - 2 = e^{C-3t} = Ae^{-3t};$$

$$x(t) = \frac{1}{3}\left(2 + Ae^{-3t}\right); \qquad 4 = x(0) = \frac{1}{3}(2 + A); \qquad x(t) = \frac{1}{3}\left(2 + 10e^{-3t}\right).$$

C09S03.027: Given: $\dfrac{dx}{dt} = 5(x+2),\ \ x(0) = 25.$

$$\int \frac{dx}{x+2} = \int 5\,dt; \qquad \ln(x+2) = 5t + C; \qquad x + 2 = e^{5t+C} = Ae^{5t};$$

$$x(t) = Ae^{5t} - 2; \qquad 25 = x(0) = A - 2; \qquad x(t) = 27e^{5t} - 2.$$

C09S03.028: Given: $\dfrac{dx}{dt} = -3 - 4x,\ \ x(0) = -5.$

$$\int \frac{4\,dx}{4x+3} = \int (-4)\,dt; \qquad \ln(4x+3) = C - 4t; \qquad 4x + 3 = e^{C-4t} = Ae^{-4t};$$

$$x(t) = \frac{1}{4}\left(Ae^{-4t} - 3\right); \qquad -5 = x(0) = \frac{1}{4}(A - 3); \qquad x(t) = -\frac{1}{4}\left(17e^{-4t} + 3\right).$$

C09S03.029: Given: $\dfrac{dv}{dt} = 10(10 - v),\ \ v(0) = 0.$

$$\int \frac{dv}{v-10} = \int (-10)dt; \qquad \ln(v - 10) = C - 10t; \qquad v - 10 = e^{C-10t} = Ae^{-10t};$$

$$v(t) = 10 + Ae^{-10t}; \qquad 0 = v(0) = 10 + A; \qquad v(t) = 10\left(1 - e^{-10t}\right).$$

C09S03.030: Given: $\dfrac{dv}{dt} = -5(10 - v),\ \ v(0) = -10.$

$$\int \frac{dv}{v-10} = \int 5\,dt; \qquad \ln(v - 10) = 5t + C; \qquad v - 10 = Ae^{5t};$$

$$v(t) = 10 + Ae^{5t}; \qquad -10 = v(0) = 10 + A; \qquad v(t) = 10 - 20e^{5t}.$$

C09S03.031: Let the population at time t (in years) be $Q(t)$; $t = 0$ corresponds to the year 1990. From the data given in the problem, we know that $\dfrac{dQ}{dt} = (0.04)Q + 50000$; $Q(0) = 1{,}500{,}000$.

$$25\,\frac{dQ}{dt} = Q + 1{,}250{,}000;$$

$$\frac{1}{Q + 1{,}250{,}000} \cdot \frac{dQ}{dt} = \frac{1}{25};$$

$$\ln(Q + 1{,}250{,}000) = (0.04)t + C;$$

$$Q(t) + 1{,}250{,}000 = Ke^{t/25}.$$

Now from the condition $Q(0) = 1{,}500{,}000$ it follows that $1{,}500{,}000 + 1{,}250{,}000 = K$, so

$$Q(t) + 1{,}250{,}000 = 2{,}750{,}000e^{t/25}.$$

In the year 2010, we have $Q(20) = -1{,}250{,}000 + 2{,}750{,}000e^{0.8} \approx 4{,}870{,}238$, so the population in the year 2010 will be approximately 4.87 million people.

C09S03.032: Let $h(t)$ denote the temperature (in °F) of the cake at time t (in minutes). By Newton's law of cooling, we have $h'(t) = k \cdot [h(t) - A]$ where k is a constant and $A = 70$ is the ambient temperature. Thus

$$\int \frac{dh}{h-A} = \int k\,dt; \qquad \ln(h-A) = C_1 + kt; \qquad h - A = Ce^{kt}.$$

Thus $h(t) = 70 + Ce^{kt}$. The initial condition $h(0) = 210$ implies that $210 = 70 + C$, so $C = 140$. Thus $h(t) = 70 + 140e^{kt}$. We are also given $h(30) = 140$, so that

$$140 = 70 + 140e^{30k}; \qquad 70 = 140e^{30k};$$

$$\frac{1}{2} = e^{30k}; \qquad k = -\frac{\ln 2}{30}.$$

Now $h(t) = 100$ when $100 = 70 + 140e^{kt}$, so that

$$e^{kt} = \frac{3}{14}, \qquad \text{and so} \qquad t = \frac{1}{k}\ln\frac{3}{14} = \frac{30\ln(14/3)}{\ln 2} \approx 66.67177264.$$

Answer: The cake will be at 100°F about one hour and seven minutes after it is removed from the oven.

C09S03.033: One effective way to derive a differential equation is to estimate the changes that take place in the dependent variable over a short interval $[t,\ t+\Delta t]$ where t is the independent variable. In this problem t is measured in months, and the change in the principal balance from time t to time $t + \Delta t$ is

$$P(t+\Delta t) - P(t) \ \approx\ rP(t)\,\Delta t - c\,\Delta t.$$

The reason is that the interest added to the principal balance is $rP(t)\,\Delta t$ and the monthly payment decreases the principal by $c\,\Delta t$. Thus

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} \ \approx\ rP(t) - c. \tag{1}$$

The errors in this approximation will approach zero as $\Delta t \to 0$, and when we evaluate the limits of both sides of the approximation in (1) we obtain

$$\frac{dP}{dt} = rP - c, \qquad P(0) = P_0.$$

C09S03.034: First we solve the initial value problem derived in Problem 33.

$$\int \frac{r\,dP}{rP-c} = \int r\,dt; \qquad \ln(rP-c) = C + rt; \qquad rP - c = Ae^{rt};$$

$$P(t) = \frac{1}{r}\left(c + Ae^{rt}\right); \qquad P_0 = P(0) = \frac{1}{r}\,(c+A); \qquad P(t) = \frac{c + (rP_0 - c)\,e^{rt}}{r}.$$

In Problem 34, the loan is to be paid off in 36 months, and thus $P(36) = 0$. We use this information to solve for the monthly payment c:

$$\begin{aligned}
&\frac{c + (rP_0 - c)e^{36r}}{r} = 0;\\
&c\left(1 - e^{36r}\right) + rP_0 e^{36r} = 0;\\
&c = \frac{rP_0 e^{36r}}{e^{36r} - 1}.
\end{aligned}$$

In part (a), we substitute $P_0 = 3600$ and $r = 0.01$ (the 12% annual rate converted to the monthly rate of 1%, then converted to a decimal) and find that $c = \$119.08$. In part (b), we substitute $r = 0.015$ instead and find that $c = \$129.42$.

C09S03.035: Let $P = P(t)$ denote the number of people who have heard the rumor after t days. Then

$$\begin{aligned}
&\frac{dP}{dt} = k(100000 - P); \qquad \int \frac{dP}{P - 100000} = \int (-k)\,dt;\\
&\ln(P - 100000) = C - kt; \qquad P - 100000 = Ae^{-kt};\\
&P(t) = 100000 - Ae^{-kt}.
\end{aligned}$$

We assume that $P(0) = 0$, so that $A = 100000$ and thus $P(t) = 100000\left(1 - e^{-kt}\right)$. Next, $P(7) = 10000$, so

$$\begin{aligned}
&100000\left(1 - e^{-7k}\right) = 10000; \qquad 1 - e^{-7k} = \frac{1}{10};\\
&e^{-7k} = \frac{9}{10}; \qquad e^{7k} = \frac{10}{9}; \qquad k = \frac{1}{7}\ln\frac{10}{9}.
\end{aligned}$$

Half the population of the city will have heard the rumor when $P(T) = 50000$, so that

$$100000\left(1 - e^{-kT}\right) = 50000; \qquad 1 - e^{-kT} = \frac{1}{2}; \qquad e^{kT} = 2; \qquad T = \frac{\ln 2}{k} \approx 46.05169435.$$

Therefore half the population will have heard the rumor 46 days after it begins.

C09S03.036: Here we have $P_0 = 280$ (million),

$$\beta = \frac{17}{1000} = 0.017, \qquad \delta = \frac{7}{1000} = 0.007,$$

$k = \beta - \delta = 0.01$, $I = 1.5$, and $I/k = 150$. By Eq. (15),

$$P(t) = 280e^{(0.01)t} + 150\left[e^{(0.01)t} - 1\right].$$

In the year 2020 we therefore have $P(20) \approx 375.2$ (million). The increase in the population is 95.2 million; natural growth accounts for

$$280e^{(0.01)(20)} - 280 \approx 62.0$$

million and immigration accounts for the remaining 33.2 million.

C09S03.037: Assuming that you begin with nothing, the value of the account $P(t)$ (in thousands of dollars, at time t in years) satisfies $P(0) = P_0 = 0$. If I is your yearly investment, which we assume is made continuously (well approximated by equal monthly deposits), then

$$\begin{aligned}
\frac{dP}{dt} &= \frac{1}{10}P(t) + I; & 10\frac{dP}{dt} &= P + 10I;\\
\frac{10}{P+10I}\,dP &= dt; & \frac{1}{P+10I}\,dP &= \frac{1}{10}\,dt;\\
\ln(P+10I) &= \frac{1}{10}t + C; & P + 10I &= A\exp\left(\frac{t}{10}\right);\\
P(t) &= -10I + A\exp\left(\frac{t}{10}\right). & 0 = P(0) &= A - 10I:\\
A &= 10I. & P(t) &= 10I\left[-1+\exp\left(\frac{t}{10}\right)\right].
\end{aligned}$$

Thus the account will grow to a value of 5,000,000 in 30 years when

$$5000 = 10I\left[-1+\exp\left(\frac{30}{10}\right)\right]: \qquad I = \frac{500}{e^3-1} \approx 26.19785.$$

Hence your monthly investment should be $I/12 \approx 2.18315$; that is, approximately \$2183.15 per month. It is of interest to note that your total investment will be \$654,946.21 and that the accrued interest will be \$4,345,053.79. You should now recompute the "real" answer to this problem under the assumption that your interest income will be subject to federal, state, and local taxes. Don't forget to compute the total tax you expect to pay over the 30 years of investing.

C09S03.038: Set up your coordinate system with time t in hours and with $t = 0$ corresponding to the time of death. Let $T(t)$ denote the temperature of the body (in °F) at time $t \geqq 0$. Then the solution of the initial value problem

$$\frac{dT}{dt} = k(70 - T), \qquad T(0) = 98.6$$

is

$$T(t) = 70 + (28.6)e^{-kt}.$$

If $t = a$ at 12 noon, then

$$\begin{aligned}
T(a) &= 70 + (28.6)e^{-ka} = 80 \qquad \text{and}\\
T(a+1) &= 70 + (28.6)e^{-k(a+1)} = 75.
\end{aligned}$$

Hence

$$(28.6)e^{-ka} = 10 \qquad \text{and} \qquad (28.6)e^{-ka}e^{-k} = 5.$$

It follows that $k = \ln 2$, and the first of the previous two equations then yields

$$a = \frac{\ln(2.86)}{\ln 2} \approx 1.516$$

(in hours), so the death occurred at about 10:29 A.M.

C09S03.039: Given: $\dfrac{dN}{dt} = k(10000 - N)$, with time t measured in months.

$$-\frac{dN}{10000 - N} = -k\,dt;$$

$$\ln(10000 - N) = C_1 - kt;$$

$$10000 - N = Ce^{-kt}.$$

On January 1, $t = 0$ and $N = 1000$. On April 1, $t = 3$ and $N = 2000$. On October 1, $t = 9$; we want to determine the value of N then.

$$9000 = Ce^0 = C, \quad \text{so} \quad N(t) = 10000 - 9000e^{-kt}.$$

$$2000 = 10000 - 9000e^{-3k}, \quad \text{so} \quad 8 = 9e^{-3k}.$$

Therefore $k = \frac{1}{3}\ln\left(\frac{9}{8}\right)$. So $N(9) = 1000 - 9000e^{-9k} = 1000\left(10 - 9e^{-3\ln(9/8)}\right) \approx 3679$.

C09S03.040: $\dfrac{dx}{dt} = k(100000 - x(t))$:

$$\frac{-dx}{100000 - x} = -k\,dt;$$

$$\ln(100000 - x) = C_1 - kt;$$

$$100000 - x = Ce^{-kt};$$

$$x(t) = 100000 - Ce^{-kt}.$$

On March 1, $t = 0$ and $x = 20000$. On March 15, $t = 14$ and $x = 60000$.

$$20000 = 100000 - C, \text{ so } C = 80000.$$

$$x(t) = 10000\left(10 - 8e^{-kt}\right).$$

$$60000 = x(14) = 10000(10 - 8e^{-14k}). \text{ so } 6 = 10 - 8e^{-14k}.$$

Solve for $k = \frac{1}{14}\ln 2$.

(a) $x(t) = 10000(10 - 8e^{-kt})$ where $k = \frac{1}{14}\ln 2$.

(b) $x(T) = 80000$: Solve $10 - 8e^{-kT} = 2$ for T: $T = \dfrac{1}{k}\ln 4 = 28$. So 80000 people will be infected on March 29.

(c) $\lim\limits_{t\to\infty} N(t) = 100000$: Eventually everybody gets the flu.

C09S03.041: Let $t = 0$ when it began to snow, with $t = t_0$ at 7:00 A.M. Let $x(t)$ denote the distance traveled by the snowplow along the road, so that $x(t_0) = 0$. If $y = ct$ is the depth of the snow at time t, w is the width of the road, and $v = x'(t)$ is the velocity of the snowplow, then "plowing at a constant rate" means that the product wyv is constant. Hence $x(t)$ satisfies the differential equation

$$k\frac{dx}{dt} = \frac{1}{t}$$

where k is a positive constant. The solution for which $x(t_0) = 0$ satisfies the equation

$$t = t_0 e^{kx}.$$

We are given $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, and it follows that

$$t_0 + 1 = t_0 e^{2k} \qquad \text{and} \qquad t_0 + 3 = t_0 e^{4k}.$$

Elimination of t_0 yields the equation

$$e^{4k} - 3e^{2k} + 2 = 0; \qquad \text{that is,} \qquad \left(e^{2k} - 1\right)\left(e^{2k} - 2\right) = 0.$$

Thus it follows (because $k > 0$) that $e^{2k} = 2$. Hence $t_0 + 1 = 2t_0$, and so $t_0 = 1$. Therefore it began to snow at 6:00 A.M.

C09S03.042: Let $t = 0$ when it began to snow, with $t = t_0$ at 7:00 A.M. Let $x(t)$ denote the distance traveled by the snowplow along the road, so that $x(t_0) = 0$. If $y = ct$ is the depth of the snow at time t, w is the width of the road, and $v = x'(t)$ is the velocity of the snowplow, then "plowing at a constant rate" means that the product wyv is constant. Hence $x(t)$ satisfies the differential equation

$$k\frac{dx}{dt} = \frac{1}{t}$$

where k is a positive constant. The solution for which $x(t_0) = 0$ satisfies the equation

$$t = t_0 e^{kx}.$$

We are given $x = 4$ when $t = t_0 + 1$ and $x = 7$ when $t = t_0 + 2$, and it follows that

$$t_0 + 1 = t_0 e^{4k} \qquad \text{and} \qquad t_0 + 2 = t_0 e^{7k} \tag{1}$$

at 8:00 A.M. and 9:00 A.M., respectively. Elimination of t_0 yields the equation

$$2e^{4k} - e^{7k} - 1 = 0,$$

which we solve (using Newton's method) for $k \approx 0.08276$. With this value of k we finally solve either of the equations in (1) for $t_0 \approx 2.5483$ (h), about 2 h 33 min. Thus it began to snow at about 4:27 A.M.

C09S03.043: Substitution of $v = dy/dx$ in the differential equation for $y = y(x)$ yields

$$a\frac{dv}{dx} = \sqrt{1 + v^2}\,,$$

and separation of variables then yields

$$\begin{aligned} \frac{1}{\sqrt{1+v^2}}\,dv &= \frac{1}{a}\,dx; \\ \sinh^{-1} v &= \frac{x}{a} + C_1; \\ \frac{dy}{dx} &= \sinh\left(\frac{x}{a} + C_1\right). \end{aligned}$$

Because $y(0) = 0$, it follows that $C_1 = 0$, and therefore

$$\begin{aligned}\frac{dy}{dx} &= \sinh\left(\frac{x}{a}\right);\\ y(x) &= a\cosh\left(\frac{x}{a}\right) + C.\end{aligned}$$

Of course the (vertical) position of the x-axis may be adjusted so that $C = 0$, and the units in which T and ρ are measured may be adjusted so that $a = 1$. In essence, then, the shape of the hanging cable is the graph of $y = \cosh x$.

Section 9.4

C09S04.001: $\rho(x) = \exp\left(\int 1\,dx\right) = e^x$:

$$e^x \frac{dy}{dx} + e^x y = 2e^x; \qquad e^x y(x) = 2e^x + C;$$

$$y(x) = 2 + Ce^{-x}. \qquad 0 = y(0) = C + 2:$$

$$y(x) = 2(1 - e^{-x}).$$

C09S04.002: $\rho(x) = e^{-2x}$:

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = 3; \qquad e^{-2x}y(x) = 3x + C; \qquad y(x) = (3x + C)e^{2x}.$$

$$0 = y(0) = C: \qquad y(x) = 3xe^{2x}.$$

C09S04.003: $\rho(x) = \exp\left(\int 3\,dx\right) = e^{3x}$:

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y(x) = 2x; \qquad e^{3x}y(x) = x^2 + C; \qquad y(x) = e^{-3x}(x^2 + C).$$

C09S04.004: $\rho(x) = \exp\left(\int -2x\,dx\right) = \exp\left(-x^2\right)$:

$$\exp\left(-x^2\right)\frac{dy}{dx} - 2x\exp\left(-x^2\right) = 1; \qquad \exp\left(-x^2\right)y(x) = x + C; \qquad y(x) = (x + C)\exp\left(x^2\right).$$

C09S04.005: Given: $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3,\ \ y(1) = 5$:

$$\rho(x) = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\ln x) = x^2.$$

Therefore

$$x^2\frac{dy}{dx} + 2xy = 3x^2; \qquad x^2y(x) = x^3 + C; \qquad y(x) = x + \frac{C}{x^2}.$$

$$5 = y(1) = 1 + C: \qquad C = 4. \qquad y(x) = x + \frac{4}{x^2}.$$

C09S04.006: Given: $\dfrac{dy}{dx} + \dfrac{5}{x}y = 7x,\ \ y(2) = 5$:

$$\rho(x) = \exp\left(\int \frac{5}{x}\,dx\right) = \exp(5\ln x) = x^5.$$

Therefore

$$x^5\frac{dy}{dx} + 5x^4y = 7x^6; \qquad x^5y(x) = x^7 + C; \qquad y(x) = x^2 + \frac{C}{x^5}.$$

$$5 = y(2) = 4 + \frac{C}{32}: \qquad C = 32. \qquad y(x) = x^2 + \frac{32}{x^5}.$$

C09S04.007: Given: $\dfrac{dy}{dx} + \dfrac{1}{2x}y = 5x^{-1/2}$.

$$\rho(x) = \exp\left(\int \frac{1}{2x}\,dx\right) = \exp\left(\frac{1}{2}\ln x\right) = x^{1/2}.$$

Therefore

$$x^{1/2}\frac{dy}{dx} + \frac{1}{2}x^{-1/2}y = 5; \qquad x^{1/2}y(x) = 5x + C; \qquad y(x) = 5x^{1/2} + Cx^{-1/2}.$$

C09S04.008: Given: $\dfrac{dy}{dx} + \dfrac{1}{3x}y = 4$.

$$\rho(x) = \exp\left(\int \frac{1}{3x}\,dx\right) = \exp\left(\frac{1}{3}\ln x\right) = x^{1/3}.$$

Then

$$x^{1/3}\frac{dy}{dx} + \frac{1}{3}x^{-2/3}y = 4x^{1/3}; \qquad x^{1/3}y(x) = 3x^{4/3} + C; \qquad y(x) = 3x + Cx^{-1/3}.$$

C09S04.009: Given $\dfrac{dy}{dx} - \dfrac{1}{x}y = 1$, an integrating factor is

$$\rho(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp(-\ln x) = \frac{1}{x}.$$

Thus

$$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}\cdot y(x) = \frac{1}{x}; \qquad \frac{1}{x}\cdot y(x) = C + \ln x; \qquad y(x) = Cx + x\ln x.$$

$$7 = y(1) = C: \qquad y(x) = 7x + x\ln x.$$

C09S04.010: Given $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{9}{2}x^2$, an integrating factor is

$$\rho(x) = \exp\left(\int -\frac{3}{2x}\,dx\right) = \exp\left(-\frac{3}{2}\ln x\right) = x^{-3/2}.$$

Thus

$$x^{-3/2}\frac{dy}{dx} - \frac{3}{2}x^{-5/2}y(x) = \frac{9}{2}x^{1/2}; \qquad x^{-3/2}y(x) = 3x^{3/2} + C; \qquad y(x) = 3x^3 + Cx^{3/2}.$$

C09S04.011: Given: $x\dfrac{dy}{dx} + (1-3x)y = 0$, $y(1) = 0$, write the equation in the form

$$\frac{dy}{dx} + \left(\frac{1}{x} - 3\right)y = 0.$$

Then an integrating factor is

$$\rho(x) = \exp\left(\int \left[\frac{1}{x} - 3\right] dx\right) = xe^{-3x}.$$

Therefore

$$xe^{-3x}\frac{dy}{dx} + \left(e^{-3x} - 3xe^{-3x}\right)\cdot y = 0; \qquad xe^{-3x}y(x) = C; \qquad y(x) = \frac{C}{x}e^{3x}.$$

$$0 = y(0) = 0: \qquad y(x) \equiv 0.$$

C09S04.012: Given $\dfrac{dy}{dx} + \dfrac{3}{x}y = 2x^4,\ y(2) = 1,$ an integrating factor is

$$\rho(x) = \exp\left(\int \frac{3}{x}\,dx\right) = x^3.$$

Therefore

$$x^3\frac{dy}{dx} + 3x^2y(x) = 2x^7; \qquad x^3y(x) = \frac{1}{4}x^8 + C; \qquad y(x) = \frac{1}{4}x^5 + \frac{C}{x^3}.$$

$$1 = y(2) = 8 + \frac{C}{8}: \qquad C = -56. \qquad y(x) = \frac{1}{4}x^5 - \frac{56}{x^3}.$$

C09S04.013: $\rho(x) = \exp\left(\displaystyle\int 1\,dx\right) = e^x$:

$$e^x\frac{dy}{dx} + e^xy(x) = e^{2x}; \qquad e^xy(x) = C + \frac{1}{2}e^{2x}; \qquad y(x) = Ce^{-x} + \frac{1}{2}e^x.$$

$$1 = y(0) = C + \frac{1}{2}: \qquad C = \frac{1}{2}. \qquad y(x) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x.$$

C09S04.014: Given: $\dfrac{dy}{dx} - \dfrac{3}{x}y = x^2,\ y(1) = 10,$ an integrating factor is

$$\rho(x) = \exp\left(\int -\frac{3}{x}\,ds\right) = x^{-3}.$$

Thus

$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = x^{-1}; \qquad x^{-3}y(x) = C + \ln x; \qquad y(x) = x^3(C + \ln x).$$

$$10 = y(1) = C: \qquad y(x) = 10x^3 + x^3\ln x.$$

C09S04.015: An integrating factor is $\rho(x) = \exp\left(\displaystyle\int 2x\,dx\right) = \exp(x^2)$. Hence

$$\exp(x^2)\frac{dy}{dx} + 2x\exp(x^2)y(x) = x\exp(x^2); \qquad y(x)\exp(x^2) = C + \frac{1}{2}\exp(x^2);$$

$$y(x) = \frac{1}{2} + C\exp(-x^2). \qquad -2 = y(0) = \frac{1}{2} + C:$$

$$C = -\frac{5}{2}. \qquad y(x) = \frac{1 - 5\exp(-x^2)}{2}.$$

C09S04.016: Given $\dfrac{dy}{dx} + (\cos x)y = \cos x$, $y(\pi) = 2$. An integrating factor is

$$\rho(x) = \exp\left(\int \cos x\, dx\right) = \exp(\sin x).$$

Therefore

$$\exp(\sin x)\frac{dy}{dx} + (\cos x \exp(\sin x)y(x) = (\cos x)\exp(\sin x); \qquad y(x)\exp(\sin x) = C + \exp(\sin x);$$

$$y(x) = 1 + C\exp(-\sin x). \qquad 2 = y(\pi) = 1 + C:$$

$$C = 1. \qquad y(x) = 1 + \exp(-\sin x).$$

C09S04.017: $\dfrac{dy}{dx} + \dfrac{1}{1+x}y = \dfrac{\cos x}{1+x}$, so an integrating factor is

$$\rho(x) = \exp\left(\int \frac{1}{1+x}\, dx\right) = 1 + x.$$

Therefore

$$(1+x)\frac{dy}{dx} + y = \cos x; \qquad (1+x)y(x) = C + \sin x; \qquad y(x) = \frac{C + \sin x}{1+x}.$$

$$1 = y(0) = C: \qquad y(x) = \frac{1 + \sin x}{1+x}.$$

C09S04.018: Given $x\dfrac{dy}{dx} - 2y = x^3\cos x$:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x, \quad \text{so} \quad \rho(x) = \exp\left(\int -\frac{2}{x}\, dx\right) = \frac{1}{x^2}.$$

Thus

$$\frac{1}{x^2}\cdot\frac{dy}{dx} - \frac{2}{x^3}\cdot y(x) = \cos x; \qquad \frac{1}{x^2}\cdot y(x) = C + \sin x; \qquad y(x) = Cx^2 + x^2\sin x.$$

C09S04.019: An integrating factor is

$$\rho(x) = \exp\left(\int \cot x\, dx\right) = \exp(\ln\sin x) = \sin x.$$

Hence

$$(\sin x)\frac{dy}{dx} + (\cos x)y(x) = \sin x\,\cos x; \qquad y(x)\sin x = C + \frac{1}{2}\sin^2 x; \qquad y(x) = C\csc x + \frac{1}{2}\sin x.$$

Mathematica 3.0 yields

```
DSolve[ y'[x] + y[x]*Cot[x] == Cos[x], y[x], x]
```

$$y(x) = C_1 \csc x - \frac{1}{2}\cos x \cot x.$$

C09S04.020: Given: $\dfrac{dy}{dx} - (1+x)y = 1+x,\ y(0) = 0$. An integrating factor is

$$\rho(x) = \exp\left(\int (-1-x)\,dx\right) = \exp\left(-x - \tfrac{1}{2}x^2\right).$$

Thus

$$\frac{dy}{dx}\exp\left(-x-\tfrac{1}{2}x^2\right) - (1+x)y(x)\exp\left(-x-\tfrac{1}{2}x^2\right) = (1+x)\exp\left(-x-\tfrac{1}{2}x^2\right);$$

$$y(x)\exp\left(-x-\tfrac{1}{2}x^2\right) = C - \exp\left(-x-\tfrac{1}{2}x^2\right); \qquad y(x) = C\exp\left(x+\tfrac{1}{2}x^2\right) - 1.$$

But $y(0) = 0 = C - 1$, and therefore

$$y(x) = \exp\left(x+\tfrac{1}{2}x^2\right) - 1.$$

C09S04.021: An integrating factor for the equation $\dfrac{dy}{dx} - 2xy = 1$ is

$$\rho(x) = \exp\left(\int -2x\,dx\right) = \exp\left(-x^2\right),$$

which yields

$$\exp\left(-x^2\right)\frac{dy}{dx} - 2x\exp\left(-x^2\right)y(x) = \exp\left(-x^2\right);$$

$$\exp\left(-x^2\right)y(x) = \int_0^x \exp\left(-t^2\right)\,dt + C = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C;$$

$$y(x) = \left[\exp\left(x^2\right)\right]\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C\right].$$

C09S04.022: An integrating factor for the differential equation

$$\frac{dy}{dx} - \frac{1}{2x}y = \cos x$$

is

$$\rho(x) = \exp\left(\int -\frac{1}{2x}\,dx\right) = \exp\left(-\tfrac{1}{2}\ln x\right) = x^{-1/2},$$

which yields

$$x^{-1/2}\frac{dy}{dx} - \frac{1}{2}x^{-3/2}y(x) = \frac{\cos x}{\sqrt{x}};$$

$$x^{-1/2}y(x) = \int_1^x \frac{\cos t}{\sqrt{t}}\,dt;$$

$$y(x) = \sqrt{x}\int_1^x \frac{\cos t}{\sqrt{t}}\,dt.$$

The usefulness of such an expression can be seen by the fact that *Mathematica* 3.0 reports that

$$y(x) = \sqrt{2\pi x}\left[\texttt{FresnelC}\left(\sqrt{\frac{2x}{\pi}}\right) - \texttt{FresnelC}\left(\sqrt{\frac{2}{\pi}}\right)\right],$$

thus making it easy to sketch the graph of $y(x)$ or to print out a table of its values at selected values of x. Here, $\texttt{FresnelC}(z)$ is the *Fresnel cosine integral*, given by

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt,$$

which has a variety of applications in mathematical physics, particularly in diffraction theory, which is important in the design of lenses for cameras and telescopes.

C09S04.023: Let $A(t)$ denote the amount of salt (in kilograms) in the tank at time t (in seconds) and measure volume in liters. Then

$$\frac{dA}{dt} = -\frac{1}{200}A(t), \quad A(0) = 100,$$

and the solution of this familiar initial value problem is

$$A(t) = 100\exp\left(-\frac{t}{200}\right).$$

Hence $A(t) = 10$ when $t = 200\ln 10 \approx 460.517$ (s), about 7 min 40.517 s.

C09S04.024: We use units of millions of cubic feet and days. Let $P(t)$ denote the amount of pollutant in the reservoir at time t. Then

$$\frac{dP}{dt} = \frac{1}{4} - \frac{500}{8000}P(t) = \frac{1}{4} - \frac{1}{16}P(t), \qquad P(0) = 20.$$

The integrating factor $\rho(t) = e^{-t/16}$ yields the solution $P(t) = 4 + 16e^{-t/16}$. Finally, $P(t) = 8$ when $t = 32\ln 2 \approx 22.18$. Answer: After about 22.18 days.

C09S04.025: Substitute of $V = 1640$ km^3 and $r = 410$ km^3/y in the last equation in the solution of Example 5 yields

$$t = \frac{V}{r}\ln 4 = 4\ln 4 \approx 5.5452$$

(years).

C09S04.026: The volume of liquid V (in gallons) in the tank at time t (in minutes) is $V(t) = 60 - t$. Let $x(t)$ denote the number of pounds of salt in the tank at time t. Then

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t}, \quad x(0) = 0.$$

We write the differential equation in the form

$$\frac{dx}{dt} + \frac{3}{60-t}x = 2$$

and compute the integrating factor

$$\rho(t) = \exp\left(\int \frac{3}{60-t}\,dt\right) = \exp\left(-3\ln(60-t)\right) = \frac{1}{(60-t)^3}.$$

Thus

$$\frac{1}{(60-t)^3}\cdot\frac{dx}{dt} + \frac{3}{(60-t)^4}x(t) = \frac{2}{(60-t)^3};$$

$$\frac{1}{(60-t)^3}x(t) = \frac{1}{(60-t)^2} + C;$$

$$x(t) = 60 - t + C\cdot(60-t)^3.$$

Then the initial condition $x(0) = 0$ yields $3600C = -1$, and thus the answer in part (a) is

$$x(t) = 60 - t - \frac{(60-t)^3}{3600}.$$

In part (b), we have

$$\frac{dx}{dt} = -1 + \frac{(60-t)^2}{1200},$$

so that $x'(t) = 0$ when $t = 60 \pm 20\sqrt{3}$. We reject the larger root because the tank is empty when $t = 60$, and hence the maximum value of $x(t)$ occurs when $t = 60 - 20\sqrt{3} \approx 25.36$ (min). The maximum amount of salt ever in the tank is therefore

$$x\left(60 - 20\sqrt{3}\right) = \frac{40\sqrt{3}}{3} \approx 23.094$$

(pounds).

C09S04.027: If $V(t)$ is the volume of brine (in gallons) in the tank at time t (in minutes), then it's easy to see that $V(t) = 100 + 2t$. Let $x(t)$ denote the number of pounds of salt in the tank at time t. Then

$$\frac{dx}{dt} = 5 - \frac{3x}{100+2t}, \qquad x(0) = 50.$$

An integrating factor is

$$\rho(t) = \exp\left(\int \frac{3}{100+2t}\,dt\right) = \exp\left(\tfrac{3}{2}\ln(100+2t)\right) = (100+2t)^{3/2},$$

and thereby the differential equation takes the form

$$(100+2t)^{3/2}\frac{dx}{dt} + 3(100+2t)^{1/2}x(t) = 5(100+2t)^{3/2};$$

$$(100+2t)^{3/2}x(t) = (100+2t)^{5/2} + C;$$

$$x(t) = 100 + 2t + \frac{C}{(100+2t)^{3/2}}.$$

Then the initial condition $x(0) = 50$ yields $C = -50000$, and therefore

$$x(t) = 100 + 2t - \frac{50000}{(100+2t)^{3/2}}.$$

The tank is full when $t = 150$, and at that time the tank will contain

$$x(150) = \frac{1575}{4} = 393.75$$

pounds of salt.

C09S04.028: Part (a): We have

$$\frac{dx}{dt} = -\frac{5x}{100} = -\frac{1}{20}x(t), \quad x(0) = 50,$$

and it follows immediately that $x(t) = 50e^{-t/20}$. Part (b): The input and output rates of salt (in pounds) with respect to tank 2 are

$$\frac{5}{100}x(t) \qquad \text{and} \qquad \frac{5}{200}y(t),$$

respectively. Therefore

$$\frac{dy}{dt} = \frac{5}{2}e^{-t/20} - \frac{1}{40}y(t); \qquad \text{that is,}$$
$$\frac{dy}{dt} + \frac{1}{40}y(t) = \frac{5}{2}e^{-t/20}.$$

An integrating factor for the last equation is

$$\rho(t) = \exp\left(\int \frac{1}{40}\,dt\right) = e^{t/40},$$

and we thereby obtain

$$e^{t/40}\frac{dy}{dt} + \frac{1}{40}e^{t/40}y(t) = \frac{5}{2}e^{-t/40};$$
$$e^{t/40}y(t) = C - 100e^{-t/40};$$
$$y(t) = Ce^{-t/40} - 100e^{-t/20}.$$

Then the initial condition $y(0) = 50$ yields $C = 150$, so the answer in part (b) is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

Part (c): First,

$$\frac{dy}{dt} = \frac{5}{4}e^{-t/20}\left(3e^{t/40} - 4\right);$$

$y'(t) = 0$ when $t = 40\ln\frac{4}{3} \approx 11.507283$ (min), about 11 min 30.437 s. Thus the maximum amount of salt ever in tank 2 is

$$y\left(40\ln\tfrac{4}{3}\right) = \frac{225}{4} = 56.25$$

(pounds).

C09S04.029: Part (a):

$$
\begin{aligned}
A(t+\Delta t) &\approx A(t) + (0.12)(30e^{t/20})\,\Delta t + (0.06)A(t);\\
\frac{A(t+\Delta t)-A(t)}{\Delta t} &\approx (3.6)e^{t/20} + (0.06)A(t);\\
\frac{dA}{dt} - (0.06)A(t) &= (3.6)e^{t/20} = (3.6)e^{(0.05)t}, \quad A(0) = 0.
\end{aligned}
$$

An integrating factor for this linear equation is

$$\rho(t) = \exp\left(\int (-0.06)\,dt\right) = e^{(-0.06)t}.$$

Thus

$$
\begin{aligned}
&e^{(-0.06)t}\frac{dA}{dt} - (0.06)e^{(-0.06)t}A(t) = (3.6)e^{(-0.01)t};\\
&e^{(-0.06)t}A(t) = C = 360e^{(-0.01)t};\\
&A(t) = Ce^{(0.06)t} - 360e^{(0.05)t}.
\end{aligned}
$$

Now $0 = A(0) = C - 360$, so

$$A(t) = 360\left[e^{(0.06)t} - e^{(0.05)t}\right].$$

Part (b): $A(40) = 360\left(e^{12/5} - e^2\right) \approx 1308.28330$. Units are in thousand of dollars, so this amounts to \$1,308,283.30 (less taxes).

C09S04.030: The mass of the hailstone at time t is

$$m = \frac{4}{3}\pi\delta r^3 = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi k^3 t^3.$$

Hence (by Newton's second law of motion)

$$\frac{d}{dt}(mv) = mg;$$

that is,

$$
\begin{aligned}
m(t)\frac{dv}{dt} + v(t)\frac{dm}{dt} &= mg; & \frac{4}{3}\pi k^3 t^3\frac{dv}{dt} + 4\pi k^3 t^2 v(t) &= \frac{4}{3}\pi k^3 t^3 g;\\
t^3\frac{dv}{dt} + 3t^2 v(t) &= t^3 g; & t\frac{dv}{dt} + 3v &= gt.
\end{aligned}
$$

Thus we obtain the linear initial value problem

$$\frac{dv}{dt} + \frac{3}{t}v = g, \qquad v(0) = 0$$

with integrating factor

$$\rho(t) = \exp\left(\int \frac{3}{t}\,dt\right) = \exp(3\ln t) = t^3.$$

This yields

$$t^3 \frac{dv}{dt} + 3t^2 v = t^3 g, \qquad \text{and thus}$$

$$t^3 v(t) = \frac{1}{4} t^4 g + C.$$

The initial condition implies that $C = 0$, and hence $v(t) = \frac{1}{4} gt$. The desired conclusion then follows immediately because

$$\frac{dv}{dt} = \frac{1}{4} g.$$

C09S04.031: Let $v(t)$ denote the velocity of the sports car (in kilometers per hour) at time t (in seconds). We thereby obtain the linear initial value problem

$$\frac{dv}{dt} = k(250 - v), \qquad v(0) = 0.$$

This equation is easy to solve by the method of separation of variables, but we choose to solve it as a linear differential equation:

$$\frac{dv}{dt} + kv = 250k.$$

An integrating factor is

$$\rho(t) = \exp\left(\int k\,dt\right) = e^{kt},$$

and thus

$$e^{kt} \frac{dv}{dt} + ke^{kt} v = 250ke^{kt};$$

$$e^{kt} v(t) = 250e^{kt} + C;$$

$$v(t) = 250 + Ce^{-kt}.$$

The initial condition yields $C = -250$, so that

$$v(t) = 250\left(1 - e^{-kt}\right).$$

We are also given $v(10) = 100$, and thus

$$100 = 250\left(1 - e^{-10k}\right); \qquad \text{thus} \qquad k = \frac{1}{10} \ln \frac{5}{3}.$$

To find when $v(t) = 200$, we solve

$$200 = 250\left(1 - e^{-kt}\right); \qquad \frac{4}{5} = 1 - e^{-kt};$$

$$e^{-kt} = \frac{1}{5}; \qquad t = \frac{\ln 5}{k} = \frac{10 \ln 5}{\ln(5/3)}.$$

Answer: About 31.506601 seconds.

C09S04.032: Part (a): We could more easily solve the differential equation

$$\frac{dv}{dt} = -kv$$

by the method of separation of variables, but we demonstrate here the use of the integrating factor $\rho(t) = e^{kt}$:

$$e^{kt}\frac{dv}{dt} + ke^{kt}v(t) = 0; \qquad e^{kt}v(t) = C;$$

$$v(t) = Ce^{-kt}. \qquad 0 = v_0 = v(0) = C:$$

$$v(t) = v_0e^{-kt}.$$

Then the position function may be derived as follows:

$$x(t) = -\frac{v_0}{k}e^{-kt} + C_1; \qquad x_0 = x(0) = C_1 - \frac{v_0}{k}:$$

$$C_1 = \frac{v_0}{k} + x_0; \qquad x(t) = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right).$$

Part (b): First we compute

$$\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k}$$

(because $k > 0$). Therefore the total distance traveled by the body is $\dfrac{v_0}{k} < +\infty$.

C09S04.033: We use the formulas given in the statement of Problem 32:

$$v(t) = v_0e^{-kt} \quad \text{and} \quad x(t) = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right).$$

The initial condition $v_0 = 40$ and the information $v(10) = 20$ yields

$$20 = v(10) = 40e^{-10k}; \qquad e^{10k} = 2; \qquad k = \frac{1}{10}\ln 2.$$

In the solution of Problem 32 we saw that the total distance traveled by the motorboat is

$$\left[\lim_{t\to\infty} x(t)\right] - x_0 = \frac{v_0}{k},$$

and in this case we have

$$\frac{v_0}{k} = \frac{400}{\ln 2} \approx 577.078 \quad \text{(ft)}.$$

C09S04.034: Here we have

$$\frac{dv}{dt} = -kv^2; \qquad v(0) = v_0, \quad x(0) = x_0. \tag{1}$$

The differential equation in (1) is not linear, but we can solve it by the method of separation of variables:

$$-\frac{1}{v^2}\,dv = k\,dt; \qquad \frac{1}{v} = kt + C;$$

$$v(t) = \frac{1}{kt+C}. \qquad v_0 = v(0) = \frac{1}{C}:$$

$$v(t) = \frac{1}{kt+(1/v_0)}; \qquad v(t) = \frac{v_0}{kv_0t+1}.$$

Then

$$x(t) = \frac{1}{k}\ln(kv_0t+1) + C_1; \qquad x_0 = x(0) = C_1; \qquad x(t) = x_0 + \frac{1}{k}\ln(kv_0t+1).$$

Note that $x(t) \to +\infty$ as $t \to +\infty$.

C09S04.035: The formulas in Problem 34 for velocity and position (distance traveled) are

$$v(t) = \frac{v_0}{kv_0t+1} \quad \text{and} \quad x(t) = x_0 + \frac{1}{k}\ln(kv_0t+1).$$

From the data given in Problem 33, we have $v_0 = 40$ and $v(10) = 20$. Thus

$$20 = v(10) = \frac{40}{1+400k}; \qquad 20 + 8000k = 40;$$

$$8000k = 20; \qquad k = \frac{1}{400}.$$

Therefore

$$v(t) = \frac{40}{1+(t/10)} = \frac{400}{t+10}.$$

To find the total distance traveled, we may assume that $x_0 = 0$. Then

$$x(t) = 400\ln\left(1+\frac{t}{10}\right), \qquad \text{so that}$$

$$x(60) = 400\ln 7 \approx 778.364 \quad \text{(ft)}.$$

C09S04.036: Let $v(t)$ denote the velocity of the body, $v_0 = v(0)$ its initial velocity, $x(t)$ its position (distance traveled) at time t, and $x(0) = x_0$ its initial position. The differential equation of motion,

$$\frac{dv}{dt} = -kv^{3/2},$$

is not linear, but we can solve it by the method of separation of variables:

$$-v^{3/2}\,dv = k\,dt; \qquad 2v^{-1/2} = kt + C \quad (C > 0);$$

$$v(t) = \frac{4}{(kt+C)^2}. \qquad v_0 = v(0) = \frac{4}{C^2}:$$

$$C = \frac{2}{\sqrt{v_0}}.$$

Therefore

$$v(t) = \frac{4}{\left(kt + \dfrac{2}{\sqrt{v_0}}\right)^2} = \left(\frac{2\sqrt{v_0}}{kt\sqrt{v_0}\, + 2}\right)^2 = \frac{4v_0}{(2 + kt\sqrt{v_0}\,)^2}.$$

A single antidifferentiation gives

$$x(t) = C_0 - \frac{4\sqrt{v_0}}{k\,(2 + kt\sqrt{v_0}\,)};$$

$$x_0 = x(0) = C_1 - \frac{2\sqrt{v_0}}{k}: \qquad C_1 = x_0 + \frac{2\sqrt{v_0}}{k}.$$

Therefore

$$\begin{aligned} x(t) &= x_0 + \frac{2\sqrt{v_0}}{k} - \frac{2\sqrt{v_0}}{k}\left(\frac{2}{2 + kt\sqrt{v_0}}\right) \\ &= x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{2 + kt\sqrt{v_0}}\right). \end{aligned}$$

Finally,

$$\lim_{t\to\infty} x(t) = x_0 + \frac{2\sqrt{v_0}}{k},$$

so the total distance the body coasts is $\dfrac{2}{k}\sqrt{v_0} < +\infty$.

C09S04.037: With the usual notation of this section, we have

$$\frac{dv}{dt} = 10 - \frac{1}{10}v; \qquad v(0) = v_0 = 0, \quad x(0) = x_0 = 0.$$

An integration factor for this linear equation is $\rho(t) = e^{t/10}$, and thus (in the usual way)

$$e^{t/10}v(t) = 100e^{t/10} + C; \qquad v(t) = 100 + Ce^{-t/10};$$

$$0 = v(0) = 100 + C: \qquad v(t) = 100\left(1 - e^{-t/10}\right).$$

Part (a): It is clear that the limiting velocity of the car is 100 ft/s. Part (b):

$$x(t) = C_1 + 100t + 1000e^{-t/10}; \qquad 0 = x(0) = C_1 + 1000;$$

$$x(t) = 100t - 1000\left(1 - e^{-t/10}\right).$$

The car reaches 90% of its limiting velocity at that time t for which

$$v(t) = 90: \qquad 1 - e^{-t/10} = \frac{9}{10}; \qquad e^{-t/10} = \frac{1}{10}.$$

Therefore $t = 10\ln 10 \approx 23.025851$ (s). The distance the car travels from rest until that time is

$$x(10\ln 10) = -900 + 1000\ln 10 \approx 1402.585093 \quad \text{(ft)}.$$

C09S04.038: In the standard notation of this section, we have

$$\frac{dv}{dt} = 10 - \frac{1}{1000}v^2; \qquad v_0 = v(0) = 0, \quad x_0 = x(0) = 0.$$

One option at this point is the use of a computer algebra program to solve this separable equation. The *Mathematica* 3.0 command

```
DSolve[ { v'[t] == 10 - (1/1000)*(v[t])^2, v[0] == 0 }, v[t], t ]
```

will produce a solution; alternatively, separation of variables yields

$$1000\frac{dv}{dt} = 10000 - v^2 = (100+v)(100-v); \qquad \frac{1}{(100+v)(100-v)}\,dv = \frac{1}{1000}\,dt;$$

$$\frac{1}{200}\left(\frac{1}{100+v} + \frac{1}{100-v}\right)dv = \frac{1}{1000}\,dt; \qquad \left(\frac{1}{100+v} + \frac{1}{100-v}\right)dv = \frac{1}{5}\,dt;$$

$$\ln\frac{100+v}{100-v} = C + \frac{1}{5}t; \qquad \frac{100+v}{100-v} = Ae^{t/5}.$$

$$0 = v(0); \quad 1 = Ae^0; \quad A = 1: \qquad 100 + v = 100e^{t/5} - e^{t/5}v(t);$$

$$(1+e^{t/5})v(t) = 100(e^{t/5}-1); \qquad v(t) = 100\cdot\frac{e^{t/5}-1}{e^{t/5}+1}.$$

Part (a): Clearly $v(t) \to 100$ as $\to +\infty$. Part (b): We next solve $v(t) = 90$:

$$\frac{100+90}{100-90} = e^{t/5}; \qquad e^{t/5} = 1900; \qquad t = 5\ln 1900 \approx 37.748046$$

(seconds). Finally,

$$x(t) = C_1 - 100t + 1000\ln(1+e^{t/5}).$$

Without loss of generality, we assume that $x(0) = x_0 = 0$, and it follows that $C_1 = -1000\ln 2$, so that

$$\begin{aligned} x(t) &= -1000\ln 2 + 1000\ln(1+e^{t/5}) - 100t \\ &= -100t + 1000\ln\frac{1+e^{t/5}}{2}. \end{aligned}$$

Hence the distance traveled by the car while attaining 90% of its limiting velocity is

$$x(5\ln 1900) = 1000\ln\frac{1901}{2} - 500\ln 1900 \approx 3082.183579 \quad \text{(ft)}.$$

C09S04.039: We are to solve the initial value problem

$$\frac{dv}{dt} = 5 - \frac{1}{10}v, \quad v(0) = 0.$$

The usual integrating factor $\rho(t) = e^{t/10}$ yields the solution $v(t) = 50(1 - e^{-t/10})$, and it is clear that $v(t) \to 50$ as $t \to +\infty$.

C09S04.040: The mass of a drum is

$$m = \frac{640}{32} = 20$$

slugs. We also have $B = 8 \cdot (62.5) = 500$ (lb) and $F_R = -v$ (pounds). Hence the differential equation given in the statement of Problem 40 takes the form

$$20\frac{dv}{dt} = -640 + 500 - v = -140 - v.$$

(Note that we have taken the upward direction as the positive direction.) The *Mathematica* 3.0 command

```
DSolve[ { 20*v'[t] == -140 - v[t], v[0] == 0 }, v[t], t ]
```

then yields the solution $v(t) = 140(e^{-(0.05)t} - 1)$. An integration (with $y(0) = 0$) produces the position (depth) function

$$y(t) = 2800(e^{-(0.05)t} - 1) - 140t,$$

and solution of the equation $v(t) = -75$ (ft/s) yields

$$t = 20 \ln \frac{28}{13} \approx 15.35$$

(seconds). Then $y(15.35) \approx -648.31$, so the maximum safe depth is just over 648 feet.

C09S04.041: Here we have $y_0 = 0$, $v_0 = 49$, and $v_\tau = -g/\rho = -245$. Thus the velocity and position (altitude) functions are

$$v(t) = 294e^{-t/25} - 245 \qquad \text{and}$$
$$y(t) = 7350 - 245t - 7350e^{-t/25}.$$

If the maximum height occurs at time t_m, then we solve $v(t_m) = 0$ and find that

$$t_m = 25 \ln \frac{294}{245} \approx 4.5580389198,$$

and hence the maximum height is $y(t_m) \approx 108.2804646370$ (meters). Impact occurs when $y(t) = 0$; that is, when

$$7350 - 245t - 7350e^{-t/25} = 0.$$

A few iterations of Newton's method with initial "guess" $t_0 = 10$ yields the solution $t = t_i \approx 9.4109499312$. The impact speed will be

$$|v(t_i)| \approx 43.2273093261 \quad \text{(m/s)}.$$

C09S04.042: Set up a coordinate system in which $y = 0$ at the level of the hovering helicopter and the downward direction is positive. Part (a): In the usual way we find that $v(t) = 200(1 - e^{-4t/25})$. Part (b): Clearly $v(t) \to 200$ as $t \to +\infty$. Part (c): Integration yields the position function

$$y(t) = C + 200t + 1250e^{-4t/25},$$

and the initial condition $y(0) = 0$ implies that $C = -1250$. To find when the ball reaches the ground, we use Newton's method to solve $y(t) = 3000$ and find that the time of descent is approximately 21.0340853733 seconds.

C09S04.043: Beginning with

$$\frac{dv}{dt} = -g - kv, \quad v(0) = v_0,$$

the integrating factor $\rho(t) = e^{kt}$ yields

$$e^{kt}\frac{dv}{dt}ke^{kt}v(t) = -ge^{kt}; \qquad e^{kt}v(t) = C - \frac{g}{k}e^{kt};$$

$$v(t) = Ce^{-kt} - \frac{g}{k}. \qquad v_0 = v(0) = C - \frac{g}{k}:$$

$$C = v_0 + \frac{g}{k}.$$

Therefore

$$v(t) = \left(v_0 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k} = v_0e^{-kt} + \frac{g}{k}(e^{-kt} - 1).$$

Finally, the limiting velocity of the projectile is

$$v_\tau = \lim_{t\to\infty} v(t) = -\frac{g}{k}.$$

C09S04.044: We assume the usual meanings of the symbols and work in the "usual" coordinate system. Before opening the parachute, we have

$$\frac{dv}{dt} = -32 - (0.15)v; \qquad v(0) = 0, \quad y(0) = 10000.$$

It follows in the usual way that

$$v(t) \approx (213.333)(e^{-(0.15)t} - 1) \qquad \text{and so} \qquad v(20) \approx -202.712 \quad \text{(ft/s)}.$$

Next,

$$y(t) \approx 11422.2 - (1422.22)e^{-(0.15)t} - (213.333)t; \qquad y(20) \approx 7084.75 \quad \text{(ft)}.$$

After opening the parachute, we have

$$\frac{dv}{dt} = -32 - (1.5)v, \qquad v(0) \approx -202.712, \quad y(0) \approx 7084.75.$$

Thus

$$v(t) \approx -21.3333 - (181.379)e^{-(1.5)t} \qquad \text{and}$$

$$y(t) \approx 6964.83 + (120.919)e^{-(1.5)t} - (21.3333)t.$$

A few iterations of Newton's method reveals that $y(t) = 0$ when $t \approx 326.476$. Hence when the parachute opens at time $t = 20$, the woman's altitude is about 7085 feet. Her total time of descent is approximately $20+326.5 = 346.5$ seconds, about 5 min 46.5 s. Her impact speed will be approximately $|v(326.476)| \approx 21.33$ ft/s, about 15 mi/h. This is the same impact speed you'd experience if you jumped (with no parachute, of course) from the top of a wall a little over 7 ft high (a little less than 2.2 m).

Section 9.5

C09S05.001: First note that $\dfrac{1}{x(1-x)} = \dfrac{1}{x} + \dfrac{1}{1-x}$. Then

$$\int \frac{1}{x(1-x)}\,dx = \int 1\,dt; \qquad \ln\frac{x}{1-x} = t + C_1;$$

$$\frac{x}{1-x} = Ce^t; \qquad \frac{2}{1-2} = \frac{x(0)}{1-x(0)} = C = -2;$$

$$\frac{x}{1-x} = -2e^t; \qquad x = 2(x-1)e^t;$$

$$(1-2e^t)x = -2e^t; \qquad x(t) = \frac{2e^t}{2e^t-1} = \frac{2}{2-e^{-t}}.$$

C09S05.002: First note that $\dfrac{1}{x(10-x)} = \dfrac{1}{10}\left(\dfrac{1}{x} + \dfrac{1}{10-x}\right)$. Thus

$$\int \frac{1}{10x-x^2}\,dx = \int 1\,dt; \qquad \int \left(\frac{1}{x} + \frac{1}{10-x}\right)dx = 10t + C_1;$$

$$\ln\frac{x}{10-x} = 10t + C_1; \qquad \frac{x}{10-x} = Ce^{10t};$$

$$\frac{1}{10-1} = \frac{x(0)}{10-x(0)} = C; \qquad C = \frac{1}{9};$$

$$\frac{x}{10-x} = \frac{1}{9}e^{10t}; \qquad 9x = 10e^{10t} - e^{10t}x;$$

$$(9+e^{10t})x = 10e^{10t}; \qquad x(t) = \frac{10e^{10t}}{9+e^{10t}} = \frac{10}{1+9e^{-10t}}.$$

C09S05.003: First note that $\dfrac{1}{1-x^2} = \dfrac{1}{2}\left(\dfrac{1}{1-x} + \dfrac{1}{1+x}\right)$. Hence

$$\int \frac{1}{1-x^2}\,dx = \int 1\,dt; \qquad \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx = 2t + C_1;$$

$$\ln\frac{1+x}{1-x} = 2t + C_1; \qquad \frac{1+x}{1-x} = Ce^{2t};$$

$$\frac{1+3}{1-3} = C = -2; \qquad (1+x) = -2e^{2t}(1-x);$$

$$(1-2e^{2t})x = -(1+2e^{2t}); \qquad x(t) = \frac{2e^{2t}+1}{2e^{2t}-1}.$$

C09S05.004: First note that $\dfrac{1}{9-4x^2} = \dfrac{1}{6}\left(\dfrac{1}{3+2x} + \dfrac{1}{3-2x}\right)$. Thus

$$\frac{1}{6}\int\left(\frac{1}{3+2x}+\frac{1}{3-2x}\right)dx=\int 1\ dt;$$

$$\frac{1}{2}\ln(3+2x)-\frac{1}{2}(3-2x)=6t+C_1;$$

$$\ln\frac{3+2x}{3-2x}=12t+C_2; \qquad \frac{3+2x}{3-2x}=Ce^{12t}; \qquad \frac{3+0}{3-0}=C=1;$$

$$3+2x=(3-2x)e^{12t}; \qquad 2x+(2x)e^{12t}=3e^{12t}-3; \qquad x(t)=\frac{3(e^{12t}-1)}{2(e^{12t}+1)}.$$

C09S05.005: First note that

$$\frac{1}{x(5-x)} \ = \ \frac{1}{5}\left(\frac{1}{x}+\frac{1}{5-x}\right).$$

Therefore

$$\left(\frac{1}{x}+\frac{1}{5-x}\right)dx \ = \ 15\ dt; \qquad \ln\left|\frac{x}{5-x}\right| \ = \ 15t+C;$$

$$\left|\frac{x}{5-x}\right| \ Ae^{15t} \qquad \text{(where } A=e^C>0\text{);}$$

$$x(0) \ = \ 8, \quad \text{so} \quad A=\frac{8}{3}. \qquad \text{Also } \ x>0 \ \text{ and } \ 5-x<0.$$

Hence

$$\frac{x}{x-5} \ = \ \frac{8}{3}e^{15t}; \qquad 3x \ = \ 8xe^{15t}-40e^{15t};$$

$$x(t) \ = -\frac{40e^{15t}}{3-8e^{15t}}; \qquad x(t) \ = \ \frac{40}{8-3e^{-15t}}.$$

C09S05.006: We are to solve

$$\frac{1}{x(x-5)}\ dx \ = \ 3\ dt, \qquad x(0) \ = \ 2.$$

Thus

$$\left(\frac{x}{x-5}-\frac{1}{x}\right)dx \ = \ 15\ dt; \qquad \ln\left|\frac{x-5}{x}\right| \ = \ 15t+C;$$

$$\left|\frac{x-5}{x}\right| \ = \ Ae^{15t} \qquad (A=e^C>0).$$

$$x>0 \quad \text{and} \quad x-5<0, \qquad \text{so} \quad \frac{5-x}{x} \ = \ Ae^{15t}.$$

$$x(0) \ = \ 2, \quad \text{so} \quad A \ = \ \frac{3}{2}: \qquad \frac{5-x}{x} \ = \ \frac{3}{2}e^{15t};$$

$$10 - 2x = 3xe^{15t}; \qquad x(t) = \frac{10}{2 + 3e^{15t}}.$$

C09S05.007: We are to solve

$$\left(\frac{1}{x} + \frac{1}{7-x}\right) dx = 28\,dt, \qquad x(0) = 11.$$

$$\left|\frac{x}{7-x}\right| = 28t + C; \qquad \left|\frac{x}{7-x}\right| = Ae^{28t} \qquad (A = e^C > 0).$$

$$x(0) = 11: \qquad A = \frac{11}{4}, \quad x > 0, \quad 7 - x < 0.$$

$$4x = 11(x-7)e^{28t}; \qquad 4x - 11xe^{28t} = -77e^{28t};$$

$$x(t) = \frac{77e^{28t}}{11e^{28t} - 4}; \qquad x(t) = \frac{77}{11 - 4e^{-28t}}.$$

C09S05.008: Given:

$$\frac{1}{x(x-13)}\,dx = 7\,dt, \qquad x(0) = 17.$$

$$\left(\frac{1}{x-13} - \frac{1}{x}\right) dx = 91\,dt; \qquad \ln\left|\frac{x-13}{x}\right| = 91t + C;$$

$$\left|\frac{x-13}{x}\right| = Ae^{91t}. \qquad x(0) = 17:$$

$$A = \frac{4}{17}, \quad x > 0, \quad x - 13 > 0. \qquad \frac{x-13}{x} = \frac{4}{17}e^{91t};$$

$$17x - 221 = 4xe^{91t}; \qquad x(t) = \frac{221}{17 - 4e^{91t}}.$$

C09S05.009: Given:

$$\frac{dP}{dt} = kP^{1/2}; \qquad P(0) = 100, \quad P'(0) = 20.$$

Separation of variables yields

$$P^{-1/2}\,dP = k\,dt; \qquad 2P^{1/2} = C + kt.$$

$$20 = 2(P_0)^{1/2} = C: \qquad P(t) = \left(10 + \tfrac{1}{2}kt\right)^2.$$

$$20 = P'(0) = k \cdot 10: \quad k = 2; \qquad P(t) = (10 + t)^2.$$

Therefore in one year there will be $P(12) = 22^2 = 484$ rabbits.

C09S05.010: If the death rate δ is proportional to $P^{-1/2}$ (and $\beta = 0$), then

$$\frac{dP}{dt} = -kP^{-1/2} \cdot P = -kP^{1/2}; \qquad P(0) = 900, \quad P(6) = 441.$$

Separation of variables yields

$$P^{-1/2}\,dP = -k\,dt; \qquad 2P^{1/2} = C - kt.$$

$$60 = C: \quad 2P^{1/2} = 60 - kt. \qquad 2\cdot 21 = 60 - 6k: \quad k = 3.$$

$$2P^{1/2} = 60 - 3t; \qquad P(t) = \left(\frac{60 - 3t}{2}\right)^2.$$

Clearly $P(t) = 0$ when $t = 20$. Answer: 20 weeks.

C09S05.011: If $\beta = aP^{-1/2}$ and $\delta = bP^{-1/2}$, then

$$\frac{dP}{dt} = (a-b)P^{-1/2}\cdot P = kP^{1/2}$$

where $k = a - b$. In part (b) we will use the information that $P_0 = P(0) = 100$ and $P(6) = 169$. For Part (a):

$$P^{-1/2}\,dP = k\,dt; \qquad 2P^{1/2} = C + kt \qquad (C > 0);$$

$$P(t) = \left(\tfrac{1}{2}kt + \tfrac{1}{2}C\right)^2. \qquad P_0 = \left(\tfrac{1}{2}C\right)^2: \quad \tfrac{1}{2}C = \sqrt{P_0} \quad \text{(because } C > 0\text{)}.$$

$$\text{Therefore} \quad P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\right)^2.$$

Part (b): Here we have

$$P(t) = \left(\tfrac{1}{2}kt + 10\right)^2.$$

Thus

$$196 = P(6) = (3k+10)^2; \qquad 3k + 10 = \pm 13;$$

$$k = 1 \quad \text{(because } k > 0\text{)}. \qquad P(t) = \left(10 + \tfrac{1}{2}t\right)^2.$$

Thus after one year there will be $P(12) = 16^2 = 256$ fish in the lake.

C09S05.012: Here we have

$$\frac{dP}{dt} = kP^2; \qquad P(0) = 12 \quad \text{(in 1988)}, \quad P(10) = 24 \quad \text{(in 1998)}.$$

Thus

$$-\frac{1}{P^2}\,dP = -k\,dt; \qquad \frac{1}{P} = C - kt;$$

$$P(t) = \frac{1}{C - kt}. \qquad 12 = P(0) = \frac{1}{C}:$$

$$P(t) = \frac{12}{1 - 12kt}. \qquad 24 = P(10) = \frac{12}{1 - 120k}:$$

$$k = \frac{1}{240}. \qquad\qquad P(t) = \frac{240}{20 - t}, \quad 0 \leqq t < 20.$$

There will be four dozen alligators in the swamp when $P(t) = 48$: $t = 15$; that is, in the year 2003. Because $P(t) \to +\infty$ as $t \to 20^-$, "doomsday" occurs in the year 2008.

C09S05.013: The birth rate is $\beta = aP(t)$ and the death rate is $\delta = bP(t)$ where $a > b > 0$. Thus

$$\frac{dP}{dt} = [aP(t) - bP(t)] \cdot P(t) = kP^2$$

where $k = a - b > 0$. As usual, let $P_0 = P(0)$. Then

$$-\frac{1}{P^2}\,dP = -k\,dt; \qquad \frac{1}{P} = C - kt; \qquad P(t) = \frac{1}{C - kt}.$$

Part (a):

$$P_0 = P(0) = \frac{1}{C}, \quad \text{so} \quad P(t) = \frac{P_0}{1 - kP_0t}.$$

Part (b): With $P_0 = 6$, $P(10) = 9$, and t measured in months:

$$P(t) = \frac{6}{1 - 6kt}; \qquad\qquad 9 = P(10) = \frac{6}{1 - 60k};$$

$$k = \frac{1}{180}; \qquad\qquad P(t) = \frac{180}{30 - t}.$$

Because $P(t) \to +\infty$ as $t \to 30^-$, "doomsday" occurs when $t = 30$ months.

C09S05.014: The birth rate is $\beta = aP(t)$ and the death rate is $\delta = bP(t)$ where $b > a > 0$. Thus

$$\frac{dP}{dt} = [aP(t) - bP(t)] \cdot P(t) = -kP^2$$

where $k = b - a > 0$. As usual, let $P_0 = P(0)$. Then

$$-\frac{1}{P^2}\,dP = k\,dt; \qquad \frac{1}{P} = C + kt; \qquad P(t) = \frac{1}{C + kt}.$$

Then

$$P_0 = P(0) = \frac{1}{C}, \quad \text{so} \quad P(t) = \frac{P_0}{1 + kP_0t}.$$

The rabbit population dies out in the long run: Because k and P_0 are positive,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{P_0}{1 + kP_0t} = 0.$$

C09S05.015: Measure P in millions and t in years, with $t = 0$ corresponding to the year 1940. Given: $P(0) = 100$, $P'(0) = 1$, and

$$\frac{dP}{dt} = kP(200 - P) \qquad (k \text{ constant}). \tag{1}$$

Note that $\dfrac{1}{P(200-P)} = \dfrac{1}{200}\left(\dfrac{1}{P} + \dfrac{1}{200-P}\right)$. Thus

$$\int \frac{1}{P(200-P)}\,dP = \int k\,dt; \qquad \int \left(\frac{1}{P} + \frac{1}{200-P}\right) dP = 200kt + C_1;$$

$$\ln\frac{P}{200-P} = 200kt + C_1; \qquad \frac{P}{200-P} = Ce^{200kt}$$

$$\frac{100}{100} = Ce^0 = C; \qquad \frac{P}{200-P} = e^{200kt}.$$

By Eq. (1), $1 = P'(0) = k\cdot 100(200-100) = 10000k$, so $k = 1/10000$. Therefore

$$\frac{P}{200-P} = e^{t/50}; \qquad \frac{200-P}{P} = e^{-t/50};$$

$$\frac{200}{P} = 1 + e^{-t/50}; \qquad P(t) = \frac{200}{1+e^{-t/50}}.$$

Thus the population in the year 2000 (corresponding to $t = 60$) will be

$$P(60) = \frac{200}{1+e^{-6/5}} \approx 153.7 \quad \text{(million)}.$$

C09S05.016: Given:

$$N'(t) = kN(t)(15000 - N(t)) \quad (k \ \text{constant}), \quad N(0) = 5000, \quad N'(0) = 500.$$

First note that $\dfrac{1}{N(15000-N)} = \dfrac{1}{15000}\left(\dfrac{1}{N} + \dfrac{1}{15000-N}\right)$. So

$$\int \left(\frac{1}{N} + \frac{1}{15000-N}\right) dN = \int 15000k\,dt; \qquad \ln\frac{N}{15000-N} = C_1 + 15000kt;$$

$$\frac{N}{15000-N} = Ce^{15000kt} \qquad \frac{5000}{10000} = C;$$

$$\frac{N}{15000-N} = \frac{1}{2}e^{15000kt}; \qquad 500 = N'(0) = k\cdot 5000\cdot 10000;$$

$$k = \frac{1}{100000}; \qquad \frac{N}{15000-N} = \frac{1}{2}e^{3t/20};$$

$$N = 7500e^{3t/20} - \frac{1}{2}e^{3t/20}N; \qquad N + \tfrac{1}{2}Ne^{3t/20} = 7500e^{3t/20};$$

$$\left(1 + \tfrac{1}{2}e^{3t/20}\right)N = 7500e^{3t/20}; \qquad N(t) = \frac{7500e^{3t/20}}{1+\frac{1}{2}e^{3t/20}};$$

$$N(t) = \frac{15000e^{3t/20}}{2+e^{3t/20}}; \qquad N(t) = \frac{15000}{1+2e^{-3t/20}}.$$

Now we solve $N(T) = 10000$:

$$1 + 2e^{-3T/20} = \frac{3}{2}; \qquad e^{-3T/20} = \frac{1}{4};$$

$$e^{3T/20} = 4; \qquad 3T = 20\ln 4;$$

and therefore $T = \frac{20}{3}\ln 4 \approx 9.242$. Thus it will require a little more than nine additional days for another 5000 people to contract this disease.

C09S05.017: Given:

$$\frac{dx}{dt} = \frac{4}{5}x - \frac{1}{250}x^2 = \frac{200x - x^2}{250}; \quad x(0) = 50.$$

First note that $\dfrac{1}{200x - x^2} = \dfrac{1}{x(200-x)} = \dfrac{1}{200}\left(\dfrac{1}{x} + \dfrac{1}{200-x}\right)$. Then

$$\frac{1}{200}\int\left(\frac{1}{x} + \frac{1}{200-x}\right)dx = \int \frac{1}{250}\,dt; \qquad \ln\frac{x}{200-x} = \frac{4}{5}t + C_1;$$

$$\frac{x}{200-x} = Ce^{4t/5}; \qquad \frac{1}{3} = Ce^0 = C;$$

$$\frac{x}{200-x} = \frac{1}{3}e^{4t/5}; \qquad x = \frac{200}{3}e^{4t/5} - \frac{1}{3}e^{4t/5}x;$$

$$\left(1 + \frac{1}{3}e^{4t/5}\right)x = \frac{200}{3}e^{4t/5}; \qquad x(t) = \frac{200e^{4t/5}}{3 + e^{4t/5}} = \frac{200}{1 + 3e^{-4t/5}}.$$

Part (a): We need to solve $x(T) = 100$:

$$100 = \frac{200}{1 + 3e^{-4T/5}}; \qquad 1 + 3e^{-4T/5} = 2;$$

$$e^{-4T/5} = \frac{1}{3}; \qquad \frac{4}{5}T = \ln 3.$$

Thus $T = \dfrac{5}{4}\ln 3 \approx 1.373$ (seconds).

Part (b): As $t \to +\infty$, $x(t) \to 200$. So there is no "maximum" amount of salt that will dissolve, but for all practical purposes, the maximum is 200 g. (The amount that dissolves becomes arbitrarily close to, but remains always less than, 200 g.)

C09S05.018: With $P(t)$ measuring the number of squirrels at time t (in months), we are given

$$\frac{dP}{dt} = \frac{1}{1000}P^2 - kP, \quad P(0) = 100, \quad P'(0) = 8$$

(where k is a constant). Substitution of these numerical data in the differential equation yields

$$8 = P'(0) = \frac{10000}{1000} - 100k,$$

so that $100k = 10 - 8 = 2$: $k = \dfrac{1}{50}$. Next,

$$\frac{dP}{dt} = \frac{1}{1000}P^2 - \frac{1}{50}P = \frac{P(P-20)}{1000}.$$

Note that $\dfrac{1}{P(P-20)} = \dfrac{1}{20}\left(\dfrac{1}{P-20} - \dfrac{1}{P}\right)$. Thus

$$\int \frac{1}{P(P-20)}\,dP = \int \frac{1}{1000}\,dt;$$

$$\frac{1}{20}\int\left(\frac{1}{P-20} - \frac{1}{P}\right)dP = \frac{1}{1000}t + C_2;$$

$$\ln\frac{P-20}{P} = \frac{1}{50}t + C_1; \qquad \frac{P-20}{P} = Ce^{t/50};$$

$$1 - \frac{20}{P} = Ce^{t/50}; \qquad \frac{20}{P} = 1 - Ce^{t/50}$$

$$P(t) = \frac{20}{1-Ce^{t/50}}; \qquad 100 = P(0) = \frac{20}{1-C};$$

$$C = \frac{4}{5}; \qquad P(t) = \frac{20}{1-\frac{4}{5}e^{t/50}}.$$

We need to find the value of T for which $P(T) = 200$:

$$200 = \frac{20}{1-\frac{4}{5}e^{T/50}}; \qquad 1 - \frac{4}{5}e^{T/50} = \frac{20}{200} = \frac{1}{10};$$

$$\frac{4}{5}e^{T/50} = \frac{9}{10}; \qquad e^{T/50} = \frac{9}{8};$$

$T = 50\ln\dfrac{9}{8} \approx 5.889$ (months).

C09S05.019: We are given an animal population $P(t)$ at time t (in years) such that

$$\frac{dP}{dt} = kP^2 - \frac{1}{100}P; \qquad P(0) = 200, \quad P'(0) = 2$$

where k is a constant. Substitution of the numerical data in the differential equation yields

$$2 = P'(0) = 40000k - 2, \quad \text{so that} \quad k = \frac{1}{10000}.$$

Thus

$$\frac{dP}{dt} = \frac{P^2 - 100}{10000} = \frac{P(P-100)}{10000}.$$

Because $\dfrac{1}{P(P-100)} = \dfrac{1}{100}\left(\dfrac{1}{P-100} - \dfrac{1}{P}\right)$, we have

$$\frac{1}{100}\int\left(\frac{1}{P-100} - \frac{1}{P}\right)dP = \int\frac{1}{10000}\,dt.$$

Therefore

$$\ln \frac{P-100}{P} = C_1 + \frac{1}{100}t; \qquad \frac{P-100}{P} = Ce^{t/100};$$

$$\frac{200-100}{200} = C = \frac{1}{2}; \qquad 1 - \frac{100}{P} = \frac{1}{2}e^{t/100};$$

$$\frac{100}{P} = 1 - \frac{1}{2}e^{t/100}; \qquad P(t) = \frac{100}{1-\frac{1}{2}e^{t/100}} = \frac{200}{2-e^{t/100}}.$$

Part (a): We need to solve $P(T) = 1000$:

$$1000 = P(t) = \frac{200}{2-e^{T/100}}; \qquad 2 - e^{T/100} = \frac{1}{5};$$

$$e^{T/100} = \frac{9}{5}; \qquad T = 100 \ln \frac{9}{5}.$$

Answer: In approximately 58.779 years. Part (b): Doomsday will occur when the denominator in $P(t)$ is zero; that is, when $e^{t/100} = 2$, so that $t = 100 \ln 2$. Answer: In approximately 69.315 years.

C09S05.020: Part (a): We are given a population $x(t)$ of alligators at time t (in months) satisfying the initial value problem

$$\frac{dx}{dt} = \frac{1}{10000}x^2 - \frac{1}{100}x = \frac{x^2 - 100x}{10000}, \quad x(0) = 25.$$

Because $\dfrac{1}{x(100-x)} = \dfrac{1}{100}\left(\dfrac{1}{x-100} - \dfrac{1}{x}\right)$, we have

$$\frac{1}{100}\int \left(\frac{1}{x-100} - \frac{1}{x}\right) dx = \int \frac{1}{10000}\, dt; \qquad \ln \frac{x-100}{x} = C_1 + \frac{1}{100}t;$$

$$\frac{x-100}{x} = Ce^{t/100}; \qquad 1 - \frac{100}{x} = Ce^{t/100};$$

$$\frac{100}{x} = 1 - Ce^{t/100}; \qquad x(t) = \frac{100}{1-Ce^{t/100}}.$$

The initial condition $x(0) = 25$ now yields $C = -3$. Therefore

$$x(t) = \frac{100}{1+3e^{t/100}} \to 0 \quad \text{as} \quad t \to +\infty.$$

Part (b): If the initial condition is $x(0) = 125$, then $C = \frac{1}{3}$, so that

$$x(t) = \frac{100}{1-\frac{1}{3}e^{t/100}} = \frac{300}{3-e^{t/100}}.$$

Now $x(t) \to +\infty$ as $t \to (100 \ln 3)^-$, so doomsday occurs after approximately 109.861 (months).

C09S05.021: If we write $P' = bP(a/b - P)$ we see that $M = a/b$. Hence

$$\frac{B_0 P_0}{D_0} = \frac{(aP_0)P_0}{bP_0^2} = \frac{a}{b} = M.$$

Note also (for Problems 22 and 23) that $a = B_0/P_0$ and $b = D_0P_0^2 = k$. (—C.H.E.)

C09S05.022: The relations in Problem 21 given $k = 1/2400$ and $M = 160$. The solution is $P(t) = 19200/(120 + 40e^{-t/5})$. We find that $P = 0.95M$ after about 27.69 months. (—C.H.E.)

C09S05.023: The relations in Problem 21 give $k = 1/2400$ and $M = 180$. The solution is $P(t) = 43200/(240 - 60e^{-3t/80})$. We find that $P = 1.05M$ after about 44.22 months. (—C.H.E.)

C09S05.024: If we write $P' = aP(P - b/a)$ we see that $M = b/a$. Hence

$$\frac{D_0P_0}{B_0} = \frac{(bP_0)P_0}{aP_0^2} = \frac{b}{a} = M.$$

Note also (for Problems 25 and 26) that $b = D_0/P_0$ and $a = B_0/P_0^2 = k$. (—C.H.E.)

C09S05.025: The relations in Problem 24 give $k = 1/1000$ and $M = 90$. The solution is

$$P(t) = \frac{9000}{100 - 10e^{9t/10}}.$$

We find that $P = 10M$ after about 24.41 months. (—C.H.E.)

C09S05.026: The relations in Problem 24 given $k = 1/1100$ and $M = 120$. The solution is

$$P(t) = \frac{13200}{110 + 10e^{6t/55}}.$$

We find that $P = 0.1M$ after about 42.12 months. (—C.H.E.)

C09S05.027: We work in thousands of persons, so $M = 100$ for the total fixed population. We substitute $M = 100$, $P'(0) = 1$, and $P_0 = 50$ in the logistic equation, and thereby obtain

$$1 = k(50)(100 - 50), \qquad \text{so} \qquad k = 0.0004.$$

If t denotes the number of days until 80 thousand people have heard the rumor, then Eq. (7) gives

$$80 = \frac{50 \times 100}{50 + (100 - 50)e^{-0.04t}},$$

so that t is approximately 34.66. Thus the rumor will have spread to 80% of the population in a little less than 35 days. (—C.H.E.)

C09S05.028: Proceeding as in Example 3 in the text, we solve the equations

$$25.00k(M - 25.00) = 3/8, \qquad 47.54k(M - 47.54) = 1/2$$

for $M = 100$ and $k = 0.0002$. Then Eq. (7) gives the population function

$$P(t) = \frac{2500}{25 + 75e^{-0.02t}}.$$

We find that $P = 75$ when $t = 50 \ln 9 \approx 110$, that is, in 2035 A.D. (—C.H.E.)

C09S05.029: The solution of the initial value problem given in the statement of Problem 29 is

$$P(t) = \frac{1}{\dfrac{1489}{313500} + \dfrac{341881}{1358500}\exp\left(-\dfrac{627t}{20000}\right)} \approx \frac{1}{0.0047496013 + (0.0276636323)e^{-(0.03135)t}}. \tag{1}$$

Part (a): The year 1930 corresponds to $t = 140$, for which the equation in (1) predicts $P(140) \approx 127.008$ (million). Part (b):

$$\lim_{t\to\infty} P(t) = \frac{313500}{1489} \approx 210.544 \quad \text{(million)}.$$

Part (c): The following table gives the year, the population predicted by Eq. (1), and the actual U.S. population (in millions) from the 1992 *World Almanac and Book of Facts* (New York: Pharos Books, 1991, pp. 74–75). The population data are rounded.

Year	Predicted population	Actual population
1790	3.900	3.930
1800	5.300	5.308
1810	7.185	7.240
1820	9.708	9.638
1830	13.061	12.861
1840	17.471	17.063
1850	23.193	23.192
1860	30.499	31.443
1870	39.616	38.558
1880	50.690	50.189
1890	63.707	62.980
1900	78.427	76.212
1910	94.362	92.228
1920	110.819	106.022
1930	127.008	123.203
1940	142.191	132.165
1950	155.803	151.326
1960	167.525	179.323
1970	177.272	203.302

1980	185.146	226.542
1990	191.358	248.710
2000	196.169	*281.422

*Note: This datum is from `www.census.gov/main/www/cen2000/html`, where the U.S. population on April 1, 2000 is given as 281,421,906.

C09S05.030: Any way you look at it, you should see that, the larger the parameter $k > 0$ is, the faster the logistic population $P(t)$ approaches its limiting population M. (—C.H.E.)

C09S05.031: We begin with

$$\frac{dP}{dt} = kP(M - P), \qquad P(0) = P_0. \tag{6}$$

Thus

$$\frac{1}{P(M-P)}\,dP = k\,dt; \qquad \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = k\,dt;$$

$$\ln\left|\frac{P}{M-P}\right| = kMt + C; \qquad \left|\frac{P}{M-P}\right| = Ae^{kMt} \qquad (A = e^C > 0);$$

$$\frac{P}{M-P} = Be^{kMt} \qquad (B = \pm A).$$

For later use, we note at this point that $B = \dfrac{P_0}{M - P_0}$. Next,

$$P = MBe^{kMt} - PBE^{kMt};$$

$$P(t) = \frac{MBe^{kMt}}{1 + Be^{kMt}} = \frac{MB}{e^{-kMt} + B} = \frac{\dfrac{MP_0}{M-P_0}}{e^{-kMt} + \dfrac{P_0}{M-P_0}}.$$

Therefore

$$P(t) = \frac{MP_0}{(M - P_0)e^{-kMt} + P_0}. \tag{7}$$

C09S05.032: Part (a): We begin with

$$\frac{dP}{dt} = kP(P - M), \qquad P(0) = P_0. \tag{13}$$

Then

$$\frac{1}{P(P-M)}\,dP = k\,dt; \qquad \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right)dP = k\,dt;$$

$$\ln\left|\frac{P-M}{P}\right| = kMt + C; \qquad \left|\frac{P-M}{P}\right| = Ae^{kMt} \qquad (A = e^C > 0);$$

$$\frac{P-M}{P} = Be^{kMt} \qquad (B = \pm A).$$

We note for later use that $B = \dfrac{P_0 - M}{P_0}$. Thus

$$\begin{aligned}
P - M &= BPe^{kMt};\\
P - BPe^{kMt} &= M;\\
P(t) &= \frac{M}{1 - Be^{kMt}} = \frac{MP_0}{P_0 - P_0Be^{kMt}}\\
&= \frac{MP_0}{P_0 - (P_0 - M)e^{kMt}} = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}.
\end{aligned}$$

Part (b): If $P_0 < 0$, then—assuming that M and k are positive—

$$\lim_{t\to 0} P(t) = 0.$$

C09S05.033: We begin with

$$\frac{dP}{dt} = kP(M - P) \tag{3}$$

and differentiate both sides with respect to t (using the chain rule on the right-hand side). Thus

$$\begin{aligned}
\frac{d^2P}{dP^2} &= \left\{\frac{d}{dP}\left[kP(M-P)\right]\right\}\cdot\frac{dP}{dt} = k(M - P - P)\cdot kP(M-P)\\
&= k^2P(M-P)(M-2P) = 2k^2P(P-M)\left(P - \tfrac{1}{2}M\right).
\end{aligned}$$

The conclusions stated in Problem 33 are now clear.

C09S05.034: We begin with

$$\frac{dy}{dx} = \frac{bxy - qy}{px - axy}. \tag{20}$$

Thus

$$\begin{aligned}
\frac{dy}{dx} &= \frac{bx - q}{p - ay}\cdot\frac{y}{x}; &\qquad \frac{p - ay}{y}\,dy &= \frac{bx - q}{x}\,dx;\\
(p\ln y) - ay &= bx - (q\ln x) + A; &\qquad (\ln y^p) - ay &= bx - (\ln x^q) + A;\\
y^pe^{-ay} &= Be^{bx}x^{-q} \quad (B = e^A > 0); &\qquad \frac{y^p}{e^{ay}} &= C\cdot\frac{e^{bx}}{x^q} \quad (C = \pm A).
\end{aligned}$$

Therefore

$$x^qy^p = Ce^{bx}e^{ay}. \tag{21}$$

Section 9.6

C09S06.001: Characteristic equation: $r^2 - 7r + 10 = (r-2)(r-5) = 0$: $r_1 = 2$, $r_2 = 5$;

$$y(x) = c_1 e^{2x} + c_2 e^{5x}.$$

C09S06.002: Char. Eqtn.: $r^2 + 2r - 15 = (r-3)(r+5) = 0$: $r_1 = 3$, $r_2 = -5$; $y(x) = c_1 e^{3x} + c_2 e^{-5x}$.

C09S06.003: C. E.: $4r^2 - 4r - 3 = (2r+1)(2r-3) = 0$: $r_1 = -\frac{1}{2}$, $r_2 = \frac{3}{2}$; $y(x) = c_1 e^{-x/2} + c_2 e^{3x/2}$.

C09S06.004: C. E.: $12r^2+13r+3 = (4r+3)(3r+1) = 0$: $r_1 = -\frac{3}{4}$, $r_2 = -\frac{1}{3}$; $y(x) = c_1 e^{-3x/4}+c_2 e^{-x/3}$.

C09S06.005: C. E.: $r^2 + 4r + 1 = 0$:

$$r = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}\,;$$

$y(x) = c_1 \exp\left(\left[-2+\sqrt{3}\,\right]x\right) + c_2 \exp\left(\left[-2-\sqrt{3}\,\right]x\right)$.

C09S06.006: Characteristic equation: $4r^2 - 4r - 19 = 0$:

$$r = \frac{4 \pm \sqrt{16+304}}{8} = \frac{1}{2} \pm \sqrt{5}\,;$$

$y(x) = c_1 \exp\left(\dfrac{1+2\sqrt{5}}{2}\,x\right) + c_2 \exp\left(\dfrac{1-2\sqrt{5}}{2}\,x\right)$.

C09S06.007: C. E.: $4r^2 + 12r + 9 = 0$; $(2r+3)^2 = 0$; $r_1 = r_2 = -\frac{3}{2}$. Hence

$$y(x) = (c_1 + c_2 x)e^{-3x/2}.$$

C09S06.008: C. E.: $9r^2 - 30r + 25 = 0$; $(3r-5)^2 = 0$; $r_1 = r_2 = \frac{5}{3}$. Hence

$$y(x) = (c_1 + c_2 x)e^{5x/3}.$$

C09S06.009: C. E.: $25r^2 - 20r + 4 = 0$; $(5r-2)^2 = 0$; $r_1 = r_2 = \frac{2}{5}$. Therefore

$$y(x) = (c_1 + c_2 x)e^{2x/5}.$$

C09S06.010: C. E.: $49r^2 + 126r + 81 = 0$; $(7r+9)^2 = 0$; $r_1 = r_2 = -\frac{9}{7}$. Therefore

$$y(x) = (c_1 + c_2 x)e^{-9x/7}.$$

C09S06.011: Characteristic equation: $r^2 + 6r + 13 = 0$:

$$r = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2i.$$

Therefore $y(x) = e^{-3x}\,(c_1 \cos 2x + c_2 \sin 2x)$.

C09S06.012: Characteristic equation: $r^2 - 10r + 74 = 0$:

$$r = \frac{10 \pm \sqrt{100 - 296}}{2} = 5 \pm 7i.$$

Hence $y(x) = e^{5x}(c_1 \cos 7x + c_2 \sin 7x)$.

C09S06.013: Characteristic equation: $9r^2 + 6r + 226 = 0$:

$$r = \frac{-6 \pm \sqrt{36 - 8136}}{18} = \frac{-6 \pm 90i}{18} = -\frac{1}{3} \pm 5i;$$

$y(x) = e^{-x/3}(c_1 \cos 5x + c_2 \sin 5x)$.

C09S06.014: Characteristic equation: $9r^2 + 90r + 226 = 0$:

$$r = \frac{-90 \pm \sqrt{8100 - 8136}}{18} = \frac{-90 \pm 6i}{18} = -5 \pm \frac{1}{3}i;$$

$y(x) = e^{-5x}\left(c_s \cos \dfrac{x}{3} + c_2 \sin \dfrac{x}{3}\right)$.

C09S06.015: Characteristic equation: $2r^2 - 11r + 12 = 0$:

$$r = \frac{11 \pm \sqrt{121 - 96}}{4} = \frac{11 \pm 5}{4}.$$

Thus the general solution is $y(x) = c_1 e^{3x/2} + c_2 e^{4x}$. Also

$$y'(x) = \frac{3}{2} c_1 e^{3x/2} + 4c_2 e^{4x};$$

$$5 = y(0) = c_1 + c_2;$$

$$15 = y'(0) = \frac{3}{2} c_1 + 4c_2.$$

Therefore $c_1 = 2$, $c_2 = 3$, and $y(x) = 2e^{3x/2} + 3e^{4x}$.

C09S06.016: Characteristic equation: $r^2 - 2r - 35 = 0$; $(r + 5)(r - 7) = 0$. Thus the general solution is $y(x) = c_1 e^{-5x} + c_2 e^{7x}$. Also

$$y'(x) = -5c_1 e^{-5x} + 7c_2 e^{7x};$$

$$12 = y(0) = c_1 + c_2;$$

$$0 = y'(0) = -5c_1 + 7c_2.$$

Therefore $c_1 = 7$, $c_2 = 5$, and $y(x) = 7e^{-5x} + 5e^{7x}$.

C09S06.017: The roots of the characteristic equation are $r_1 = 7$ and $r_2 = 11$; the solution of the given initial value problem is $y(x) = 9e^{7x} - 5e^{11x}$.

C09S06.018: The roots of the characteristic equation are $r_1 = -\frac{2}{3}$ and $r_2 = \frac{3}{4}$; the solution of the given initial value problem is $y(x) = -6e^{-2x/3} + 8e^{3x/4}$.

C09S06.019: The roots of the characteristic equation are $r_1 = r_2 = -11$; the solution of the given initial value problem is $y(x) = (2 - 3x)e^{-11x}$.

C09S06.020: The roots of the characteristic equation are $r_1 = r_2 = -\frac{7}{3}$; the solution of the given initial value problem is $y(x) = (3x + 6)e^{-7x/3}$.

C09S06.021: The roots of the characteristic equation are $r_1 = 5i$ and $r_2 = -5i$; the solution of the given initial value problem is $y(x) = 7\cos 5x + 2\sin 5x$.

C09S06.022: The roots of the characteristic equation are $r_1 = \frac{10}{3}i$ and $r_2 = -\frac{10}{3}i$; the solution of the given initial value problem is

$$y(x) = 99\cos\frac{10x}{3} + 30\sin\frac{10x}{3}.$$

C09S06.023: The roots of the characteristic equation are $r_1 = -2 + 4i$ and $r_2 = -2 - 4i$; the solution of the given initial value problem is $y(x) = e^{-2x}(9\cos 4x + 7\sin 4x)$.

C09S06.024: The roots of the characteristic equation are $r_1 = -5 + 9i$ and $r_2 = -5 - 9i$; the solution of the given initial value problem is $y(x) = e^{-5x}(11\cos 9x + 5\sin 9x)$.

C09S06.025: The roots of the characteristic equation are $r_1 = -\frac{1}{2} + 5i$ and $r_2 = -\frac{1}{2} - 5i$; the solution of the given initial value problem is $y(x) = e^{-x/2}(10\cos 5x + 6\sin 5x)$.

C09S06.026: The roots of the characteristic equation are $r_1 = -\frac{1}{10} + 10i$ and $r_2 = -\frac{1}{10} - 10i$; the solution of the given initial value problem is $y(x) = e^{-x/10}(30\cos 10x - 3\sin 10x)$.

C09S06.027: The roots of the characteristic equation are $r_1 = 0$ and $r_2 = -10$:

$$r(r + 10) = 0; \qquad r^2 + 10r = 0; \qquad y'' + 10y' = 0.$$

C09S06.028: The roots of the characteristic equation are $r_1 = 10$ and $r_2 = -10$:

$$(r - 10)(r + 10) = 0; \qquad r^2 - 100 = 0; \qquad y'' - 100y = 0.$$

C09S06.029: The roots of the characteristic equation are $r_1 = r_2 = -10$:

$$(r + 10)^2 = 0; \qquad r^2 + 20r + 100 = 0; \qquad y'' + 20y' + 100y = 0.$$

C09S06.030: The roots of the characteristic equation are $r_1 = 10$ and $r_2 = 100$:

$$(r - 10)(r - 100) = 0; \qquad r^2 - 110r + 1000 = 0; \qquad y'' - 110y' + 1000y = 0.$$

C09S06.031: The roots of the characteristic equation are $r_1 = r_2 = 0$:

$$(r - 0)(r - 0) = 0; \qquad r^2 = 0; \qquad y'' = 0.$$

C09S06.032: The roots of the characteristic equation are $r_1 = 1 + \sqrt{2}$ and $r_2 = 1 - \sqrt{2}$:

$$\left(r - 1 - \sqrt{2}\,\right)\left(r - 1 + \sqrt{2}\,\right) = 0; \qquad r^2 - 2r - 1 = 0; \qquad y'' - 2y' - y = 0.$$

C09S06.033: The roots of the characteristic equation are $r_1 = -5 + \frac{1}{5}i$ and $r_2 = -5 - \frac{1}{5}i$:

$$\left(r + 5 - \frac{1}{5}i\right)\left(r + 5 + \frac{1}{5}i\right) = 0; \qquad r^2 + 10r + \frac{626}{25} = 0; \qquad 25y'' + 250y' + 626y = 0.$$

C09S06.034: The roots of the characteristic equation are $r_1 = -\frac{1}{5} + 5i$ and $r_2 = -\frac{1}{5} - 5i$:

$$\left(r + \frac{1}{5} - 5i\right)\left(r + \frac{1}{5} + 5i\right) = 0; \qquad r^2 + \frac{2}{5}r + \frac{626}{25} = 0; \qquad 25y'' + 10y' + 626y = 0.$$

C09S06.035: The characteristic equation is $r^2 + 25 = 0$, and hence the general solution is

$$y(x) = c_1 \cos 5x + c_2 \sin 5x.$$

Part (a): The condition $y(0) = 0$ yields $c_1 = 0$, and hence $y(x) = c_2 \sin 5x$. But the second condition $y(\pi) = 0$ is satisfied for every choice of the constant c_2. Moreover, if so, then

$$y'' + 25y = -25c_2 \sin x + 25c_2 \sin x \equiv 0$$

for every choice of the constant c_2, and therefore the given *boundary value problem* has infinitely many solutions, one for every choice of the constant c_2. Part (b): The condition $y(0) = 0$ yields $c_1 = 0$, and hence $y(x) = c_2 \sin 5x$. But the second condition

$$0 = y(3) = c_2 \sin 15 \approx (0.6502878401)c_2$$

is satisfied only if $c_2 = 0$. Therefore the given *boundary value problem* has at most the trivial solution $y(x) \equiv 0$ (and substitution verifies that, indeed, this is a solution). Hence the problem has no nontrivial solutions. The point of this problem is to draw a very sharp distinction between second-order initial value problems, with initial conditions $y(a) = b_0,\ y'(a) = b_1$ given at the *same* abscissa, and second-order *boundary value problems*, which typically have values of y (and/or y') imposed at two different values of the abscissa. In particular, the vital existence-uniqueness theorem stated in Section 8.6 does not hold for such boundary value problems.

C09S06.036: Without loss of generality we assume that $a = 1$. The characteristic equation $ar^2 + br + c = 0$ has solution(s)

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Case 1: $b^2 - 4c = k^2$ where $k > 0$. Note that $0 < k < b$. Then the roots of the characteristic equation are

$$r_1 = -\frac{b + k}{2} < 0 \qquad \text{and} \qquad r_2 = -\frac{b - k}{2} < 0.$$

Therefore $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \to 0$ as $x \to +\infty$.

Case 2: $b^2 - 4c = 0$. Then the roots of the characteristic equation are

$$r_1 = r_2 = -\frac{b}{2} = -k < 0 \qquad \text{where} \qquad k > 0.$$

The general solution of the differential equation is then

$$y(x) \;=\; (c_1 + c_2 x)e^{-kx} \;=\; \frac{c_1 + c_2 x}{e^{kx}},$$

and therefore $y(x) \;\to\; 0$ as $x \;\to\; +\infty$.

Case 3: $b^2 - 4c = -k^2 < 0$ where $k > 0$. The roots of the characteristic equation are

$$r_1,\; r_2 \;=\; \frac{-b \pm ik}{2},$$

and the general solution is $y(x) \;=\; e^{-bx/2}(c_1 \cos kx + c_2 \sin kx)$. Because $-b/2 \;<\; 0$, $|\cos kx| \;\leqq\; 1$, and $|\sin kx| \leqq 1$ for all x and all $k > 0$, it now follows that $y(x) \;\to\; 0$ as $x \;\to\; +\infty$.

Section 9.7

C09S07.001: Given: $2x'' + 50x = 0$; $x(0) = 4$, $x'(0) = 15$. The general solution and its derivative are

$$x(t) = A\cos 5t + B\sin 5t \qquad \text{and}$$
$$x'(t) = -5A\sin 5t + 5B\cos 5t.$$

The initial conditions yield $A = 4$ and $B = 3$. In the notation is Section 9.7, we have

$$C = \sqrt{16+9} = 5, \quad \cos\alpha = \frac{4}{5}, \quad \text{and} \quad \sin\alpha = \frac{3}{5}.$$

Therefore $x(t) = 5\cos\left(5t - \tan^{-1}\frac{3}{4}\right) \approx 5\cos(5t - 0.64350111)$.

C09S07.002: Given: $3x'' + 48x = 0$; $x(0) = -6$, $x'(0) = 32$. The general solution and its derivative are

$$x(t) = A\cos 4t + B\sin 4t \qquad \text{and}$$
$$x'(t) = -4A\sin 4t + 4B\cos 4t.$$

The initial conditions yield $A = -6$ and $B = 8$. Hence

$$x(t) = -6\cos 4t + 8\sin 4t.$$

In the notation of Section 9.7,

$$C = \sqrt{36+64} = 10, \quad \cos\alpha = -\frac{3}{5}, \quad \text{and} \quad \sin\alpha = \frac{4}{5}.$$

Thus the phase angle α lies in the second quadrant, and so $\alpha = \pi + \tan^{-1}\left(-\frac{4}{3}\right) = \pi - \tan^{-1}\frac{4}{3}$. Thus

$$x(t) = 10\cos\left(4t - \pi + \tan^{-1}\frac{4}{3}\right) \approx 10\cos(4t - 2.21429744).$$

C09S07.003: Given: $4x'' + 36x = 0$; $x(0) = -5$, $x'(0) = -36$. Then the general solution and its derivative are

$$x(t) = A\cos 3t + B\sin 3t \qquad \text{and}$$
$$x'(t) = -3A\sin 3t + 4B\cos 3t.$$

Then the initial conditions yields $A = -5$ and $B = -12$, so the solution is

$$x(t) = -5\cos 3t - 12\sin 3t.$$

In the notation of Section 9.7 we have

$$C = \sqrt{25+144} = 13, \quad \cos\alpha = -\frac{5}{13}, \quad \text{and} \quad \sin\alpha = -\frac{12}{13}.$$

Thus the phase angle α lies in the third quadrant; $\alpha = \pi + \tan^{-1}\left(\frac{12}{5}\right)$. Hence

$$x(t) = 13\cos\left(3t - \pi - \tan^{-1}\frac{12}{5}\right) \approx 13\cos(3t - 4.31759786).$$

C09S07.004: Given: $5x'' + 80x = 0$, $x(0) = 15$, $x'(0) = -32$. The general solution and its derivative are

$$x(t) = A\cos 4t + B\sin 4t \quad \text{and}$$

$$x'(t) = -4A\sin 4t + 4B\cos 4t.$$

The initial conditions yield $A = 15$ and $B = -8$. In the notation of Section 9.7, we have

$$C = \sqrt{225 + 64} = 17, \quad \cos\alpha = \frac{15}{17}, \quad \text{and} \quad \sin\alpha = -\frac{5}{17}.$$

Thus the phase angle is $\alpha = 2\pi + \tan^{-1}\left(-\frac{8}{15}\right)$, and therefore

$$x(t) = 17\cos\left(4t - 2\pi + \tan^{-1}\tfrac{8}{15}\right) \approx 17\cos(4t + 0.48995733).$$

C09S07.005: Given: $x'' + 6x' + 8x = 0$. The characteristic equation is

$$r^2 + 6r + 8 = (r+2)(r+4) = 0$$

with roots -2 and -4. In the notation of Section 9.7, $c^2 = 9 > 8 = 4km$, so the motion is overdamped. The general solution is $x(t) = Ae^{-2t} + Be^{-4t}$, and the initial conditions yield $A = 4$ and $B = -2$. Hence the solution is

$$x(t) = 4e^{-2t} - 2e^{-4t}.$$

C09S07.006: Given: $x'' + 10x' + 21x = 0$. The characteristic equation is

$$r^2 + 10r + 21 = (r+3)(r+7) = 0,$$

with roots -3 and -7. In the notation of Section 9.7, $c^2 = 100 > 84 = 4km$, so the resulting motion is overdamped. The general solution is $x(t) = Ae^{-3t} + Be^{-7t}$, and the initial conditions yield $A = 4$ and $B = -2$. Hence

$$x(t) = 4e^{-3t} - 2e^{-7t}.$$

C09S07.007: The characteristic equation is

$$r^2 + 8r + 16 = (r+4)^2 = 0,$$

with repeated roots $r_1 = r_2 = -4$. In the notation of Section 9.7, $c^2 - 4km = 64 - 64 = 0$, so the motion is critically damped. The general solution is

$$x(t) = (A + Bt)e^{-4t},$$

and the initial conditions yield $A = 5$ and $B = 10$. Hence

$$x(t) = (10t + 5)e^{-4t}.$$

C09S07.008: The characteristic equation is $r^2 + 6r + 25 = 0$, with roots

$$r_1,\ r_2\ =\ \frac{-6 \pm \sqrt{36-100}}{2}\ =\ \frac{-6 \pm 8i}{2}\ =\ -3 \pm 4i.$$

In the notation of Section 9.7, $c^2 = 144 < 400 = 4km$, so the motion is underdamped. The general solution and its derivative are

$$\begin{aligned} x(t)\ &=\ e^{-3t}(A\cos 4t + B\sin 4t) \qquad \text{and} \\ x'(t)\ &=\ e^{-3t}(4B\cos 4t - 4A\sin 4t) - 3e^{-3t}(A\cos 4t + B\sin 4t), \end{aligned}$$

and the given initial conditions yield $A = 0$ and $B = -2$. Hence the solution is

$$x(t)\ =\ -2e^{-3t}\sin 4t\ =\ -2e^{-3t}\cos\left(4t - \frac{\pi}{2}\right).$$

C09S07.009: The differential equation has characteristic equation $r^2 + 8r + 20 = 0$, with roots

$$r_1,\ r_2\ =\ \frac{-8 \pm \sqrt{64-80}}{2}\ =\ \frac{-8 \pm 4i}{2}\ =\ -4 \pm 2i.$$

Thus the differential equation has general solution

$$x(t)\ =\ e^{-4t}(A\cos 2t\ +\ B\sin 2t).$$

Because (in the notation of Section 9.7) $c^2 = 256 < 320 = 4km$, the motion is underdamped. The initial conditions yield $A = 5$ and $B = 12$, so one form of the solution is

$$x(t)\ =\ e^{-4t}(5\cos 2t\ +\ 12\sin 2t).$$

Continuing the notation of Section 9.7, we have

$$C\ =\ \sqrt{A^2 + B^2}\ =\ 13, \quad \cos\alpha\ =\ \frac{5}{13}, \quad \text{and} \quad \sin\alpha\ =\ \frac{12}{13},$$

and hence $\alpha = \tan^{-1}\frac{12}{5}$. Thus the solution may also be written in the form

$$x(t)\ =\ 13e^{-4t}\cos\left(2t - \tan^{-1}\frac{12}{5}\right)\ \approx\ 13e^{-4t}\cos(2t - 1.17600521).$$

C09S07.010: Given: $x'' + 10x' + 125x = 0$. The associated characteristic equation is $r^2 + 10r + 125 = 0$, with roots

$$r_1,\ r_2\ =\ \frac{-10 \pm \sqrt{100-500}}{2}\ =\ -5 \pm 10i.$$

In the notation of Section 9.7, we have $c^2 - 4km = 100 - 500 < 0$, so the motion is underdamped. The general solution of the differential equation and its derivative are

$$\begin{aligned} x(t)\ &=\ e^{-5t}(A\cos 10t + B\sin 10t) \qquad \text{and} \\ x'(t)\ &=\ e^{-5t}(10B\cos 10t - 10A\sin 10t) - 5e^{-5t}(A\cos 10t + B\sin 10t). \end{aligned}$$

The initial conditions yield $A = 6$ and $B = 8$, and hence one form of the solution is

$$x(t) = e^{-5t}(6\cos 10t + 8\sin 10t).$$

Continuing the notation of Section 9.7, we have

$$C = \sqrt{A^2 + B^2} = 10, \quad \cos\alpha = \frac{3}{5}, \quad \text{and} \quad \sin\alpha = \frac{4}{5}.$$

Therefore $\alpha = \tan^{-1}\dfrac{4}{3}$, and therefore

$$x(t) = 10e^{-5t}\cos\left(10t - \tan^{-1}\frac{4}{3}\right) \approx 10e^{-5t}\cos(10t - 0.92729522).$$

C09S07.011: The associated homogeneous equation is $x'' + 9x = 0$, which has the complementary solution $x_c(t) = c_1\cos 3t + c_2\sin 3t$. A particular solution has the form

$$x_p(t) = A\cos 2t + B\sin 2t,$$

and substitution into the original nonhomogeneous equation yields

$$-4A\cos 2t - 4B\sin 2t + 9A\cos 2t + 9B\sin 2t = 10\cos 2t,$$

so that $A = 2$ and $B = 0$. Hence the general solution of the nonhomogeneous equation, and its derivative, are

$$x(t) = c_1\cos 3t + c_2\sin 3t - 2\cos 2t \quad \text{and}$$

$$x'(t) = 3c_2\cos 3t - 3c_1\sin 3t + 4\sin 2t.$$

The initial conditions $x(0) = x'(0) = 0$ then yield $c_1 = 2$ and $c_2 = 0$. Therefore

$$x(t) = 2\cos 2t - 2\cos 3t.$$

C09S07.012: The associated homogeneous equation has characteristic equation $r^2 + 4 = 0$, and hence the complementary solution is

$$x_c(t) = c_1\cos 2t + c_2\sin 2t.$$

A particular solution of the nonhomogeneous equation has the form $x_p(t) = A\cos 3t + B\sin 3t$, and it follows easily that

$$x_p''(t) + 4x_p(t) = -9A\cos 3t - 9B\sin 3t + 4A\cos 3t + 4B\sin 3t = 5\sin 3t.$$

Thus $A = 0$ and $B = -1$. Hence the general solution of the original equation is

$$x(t) = c_1\cos 2t + c_2\sin 2t - \sin 3t.$$

The initial conditions $x(0) = x'(0) = 0$ then yield $c_1 = 0$ and $c_2 = \frac{3}{2}$. Hence the solution of the given initial value problem is

$$x(t) = \frac{3}{2}\sin 2t - \sin 3t.$$

C09S07.013: The associated homogeneous equation has characteristic equation $r^2 + 100 = 0$ and thus complementary solution

$$x_c(t) = c_1 \cos 10t + c_2 \sin 10t.$$

The given nonhomogeneous equation has particular solution of the form $x_p(t) = A\cos 5t + B\sin 5t$, and substitution yields

$$x_p''(t) + 100x_p(t) = 75A\cos 5t + 75B\sin 5t = 300\sin 5t,$$

so that $A = 0$ and $B = 4$. Therefore the nonhomogeneous equation has general solution

$$x(t) = c_1 \cos 10t + c_2 \sin 10t + 4\sin 5t.$$

The initial conditions $x(0) = x'(0) = 0$ yield $c_1 = 0$ and $c_2 = -2$, so the solution of the original initial value problem is

$$x(t) = 4\sin 5t - 2\sin 10t.$$

C09S07.014: The characteristic equation of the associated homogeneous equation is $r^2 + 25 = 0$, so the complementary solution is $x_c(t) = c_1 \cos 5t + c_2 \sin 5t$. Moreover, a particular solution has the form $x_p(t) = A\cos 4t + B\sin 4t$, and substitution yields

$$x_p''(t) + 25x_p(t) = 9A\cos 4t + 9B\sin 4t = 90\cos 4t,$$

so that $A = 10$ and $B = 0$. Hence the original differential equation has general solution

$$x(t) = c_1 \cos 5t + c_2 \sin 5t + 10\cos 4t.$$

The initial conditions $x(0) = 25$, $x'(0) = 10$ then yield $c_1 = 15$ and $c_2 = 2$. Therefore the general solution of the given initial value problem is

$$x(t) = 15\cos 5t + 2\sin 5t + 10\cos 4t.$$

In the notation of Section 9.7, we have

$$C = \sqrt{225 + 4} = \sqrt{229}, \quad \cos\alpha = \frac{15}{\sqrt{229}}, \quad \text{and} \quad \sin\alpha = \frac{2}{\sqrt{229}}.$$

Therefore the general solution may also be expressed in the form

$$x(t) = 10\cos 4t + \sqrt{229}\,\cos\left(5t - \tan^{-1}\frac{2}{15}\right) \approx 10\cos 4t + \sqrt{229}\,\cos(5t - 0.13255153).$$

C09S07.015: The steady periodic solution has the form

$$x_{sp}(t) = A\cos 3t + B\sin 3t$$

and satisfies the given differential equation; therefore

$$-9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + 4A\cos 3t + 4B\sin 3t = 130\cos 3t,$$

and it follows that A and B are solutions of the simultaneous equations

$$\begin{aligned} -5A + 12B &= 130, \\ -5B - 12A &= 0, \qquad \text{so that} \end{aligned}$$

$$A = -\frac{50}{13} \qquad \text{and} \qquad B = \frac{120}{13}.$$

Therefore

$$x_{sp}(t) = -\frac{50}{13}\cos 3t + \frac{120}{13}\sin 3t.$$

In the notation of Section 9.7, we have

$$C = \sqrt{A^2 + B^2} = 10, \qquad \cos\alpha = -\frac{5}{13}, \qquad \text{and} \qquad \sin\alpha = \frac{12}{13}.$$

Consequently $C = 1$, $\alpha = \pi - \tan^{-1}\left(\frac{12}{5}\right)$, and

$$x_{sp}(t) = 10\cos\left(3t - \pi + \tan^{-1}\tfrac{12}{5}\right) \approx 10\cos(3t - 1.96558745).$$

C09S07.016: The steady periodic solution has the form $x_{sp}(t) = A\cos 5t + B\sin 5t$ and satisfies the equation

$$-25A\cos 5t - 25B\sin 5t - 15A\sin 5t + 15B\cos 5t + 5A\cos 5t + 5B\sin 5t = -150\cos 5t.$$

Hence the coefficients A and B satisfy the simultaneous equations

$$\begin{aligned} -20A + 15B &= -500, \\ -20B - 15B &= 0. \end{aligned}$$

It follows that $A = 16$ and $B = -12$, so $x_{sp}(t) = 16\cos 5t - 12\sin 5t$. In the notation of Section 9.7,

$$C = \sqrt{A^2 + B^2} = 20, \qquad \cos\alpha = \frac{4}{5}, \qquad \text{and} \qquad \sin\alpha = -\frac{3}{5}.$$

Therefore $\tan\alpha = -\frac{3}{4}$, and hence

$$x_{sp}(t) = 20\cos\left(5t + \tan^{-1}\tfrac{3}{4}\right) \approx 20\cos(5t + 0.64350111).$$

C09S07.017: The associated homogeneous equation has characteristic equation $r^2 + 4r + 5 \ ,= 0$, with roots

$$r_1,\ r_2 = \frac{-4 \pm \sqrt{4 - 20}}{2} = -2 \pm i,$$

so the complementary solution has the form $x_c(t) = e^{-2t}(c_1\cos t + c_2\sin t)$. The steady periodic solution hs the form $x_{sp}(t) = A\cos 3t + B\sin 3t$, and substitution in the original differential equation yields

$$-9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + 5A\cos 3t + 5B\sin 3t = 40\cos 3t;$$

$$-4A + 12B = 40,$$
$$-4B - 12A = 0.$$

Therefore $A = -1$ and $B = 3$, so that $x_{sp}(t) = 3\sin 3t - \cos 3t$. The general solution $x(t)$ of the original differential equation is thus

$$x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + 3\sin 3t - \cos 3t; \qquad \text{moreover}$$
$$x'(t) = e^{-2t}(c_2 \cos t - c_1 \sin t - 2c_1 \cos t - 2c_2 \sin t) + 9\cos 3t + 3\sin 3t.$$

The initial conditions $x(0) = x'(0) = 0$ next yield

$$c_1 - 1 = 0,$$
$$c_2 - 2c_1 + 9 = 0,$$

and thus $c_1 = 1$ and $c_2 = -7$. Hence the general solution of the given initial value problem is

$$x(t) = e^{-2t}(\cos t - 7\sin t) + 3\sin 3t - \cos 3t.$$

The transient solution is $x_{tr}(t) = e^{-2t}(\cos t - 7\sin t)$. In the notation of Section 9.7, we have

$$C = \sqrt{1+49} = 5\sqrt{2}, \qquad \cos\alpha = \frac{1}{5\sqrt{2}}, \qquad \text{and} \qquad \sin\alpha = -\frac{7}{5\sqrt{2}}.$$

Hence $\alpha = 2\pi - \tan^{-1}(7)$. Therefore the transient solution may be expressed in the form

$$x_{tr}(t) = 5e^{-2t}\sqrt{2}\,\cos(t - 2\pi + \tan^{-1} 7) = 5e^{-2t}\sqrt{2}\,\cos(t + \tan^{-1} 7) \approx 5e^{-2t}\sqrt{2}\,\cos(t + 1.42889927).$$

The steady periodic solution is $x_{sp}(t) = 3\sin 3t - \cos 3t$. In the notation of Section 9.7,

$$C = \sqrt{10}, \qquad \cos\alpha = -\frac{1}{\sqrt{10}}, \qquad \text{and} \qquad \sin\alpha = \frac{3}{\sqrt{10}}.$$

Therefore $\alpha = \pi + \tan^{-1}(-3) = \pi - \tan^{-1}(3)$. Hence

$$x_{sp}(t) = \sqrt{10}\,\cos(3t - \pi + \tan^{-1} 3) \approx \sqrt{10}\,\cos(3t - 1.89254688).$$

C09S07.018: The roots of the characteristic equation are $-4 \pm 3i$. The complementary and particular solutions are

$$x_c(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) \qquad \text{and} \qquad x_p(t) = A\cos t + B\sin t.$$

Substitution of the latter in the original differential equation yields $A = 1$ and $B = 22$. Hence $x_p(t) = \cos t + 22\sin t$, and the general solution of the original differential equation is

$$x(t) = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t) + \cos t + 22\sin t; \qquad \text{moreover,}$$
$$x'(t) = e^{-4t}(3c_2 \cos 3t - 3c_1 \sin 3t - 4c_1 \cos 3t - 4c_2 \sin 3t) - \sin t + 22\cos t.$$

The initial condition $x(0) = 5$ implies that $c_1 = 4$, and the condition $x'(0) = 0$ then yields $c_2 = -2$. Hence the given initial value problem has solution

$$x(t) = e^{-4t}(4\cos 3t - 2\sin 3t) + \cos t + 22\sin t.$$

The transient solution is $x_{tr}(t) = e^{-4t}(4\cos 3t - 2\sin 3t)$. In this notation of Section 9.7, we have

$$C = \sqrt{16+4} = 2\sqrt{5}, \qquad \cos\alpha = \frac{2}{\sqrt{5}}, \qquad \text{and} \qquad \sin\alpha = -\frac{1}{\sqrt{5}}.$$

Therefore

$$x_{tr}(t) = 2e^{-4t}\sqrt{5}\,\cos\left(3t + 2\pi - \arctan\tfrac{1}{2}\right) \approx 2e^{-4t}\sqrt{5}\,\cos(3t - 0.46364761).$$

The steady periodic solution is $x_{sp}(t) = \cos t + 22\sin t$. Again using the notation of Section 9.7, we have

$$C = \sqrt{485}, \qquad \cos\alpha = \frac{1}{\sqrt{485}}, \qquad \text{and} \qquad \sin\alpha = \frac{22}{\sqrt{485}}.$$

Therefore $\tan\alpha = 22$ and

$$x_{sp}(t) = \sqrt{485}\,\cos(t - \arctan 22) \approx \sqrt{485}\,\cos(t - 1.52537305).$$

C09S07.019: Equation (8) yields frequency 2 rad/s; that is, $\dfrac{1}{\pi}$ Hz. The period is π s.

C09S07.020: With $m = \frac{3}{4}$, $k = 48$, and $c = 0$, we find that the frequency is $\omega_0 = \sqrt{k/m} = 8$ rad/s and the period is $T = 2\pi/\omega_0 = \pi/4$ s.

C09S07.021: The spring constant is $k = 15/0.2 = 75$ N/m. The solution of $3x'' + 75x = 0$ with $x(0) = 0$ and $x'(0) = -10$ is $x(t) = -2\sin 5t$. Thus the amplitude is 2 m, the frequency is 5 rad/s, and the period is $2\pi/5$ s. (—C.H.E.)

C09S07.022: Part (a): With $m = \frac{1}{4}$ (kg) and $k = 9/(0.25) = 36$ (N/m) we find that $\omega_0 = 12$ (rad/s). The solution of $x'' + 144x = 0$ with $x(0) = 1$ and $x'(0) = -5$ is

$$\begin{aligned} x(t) &= \cos 12t - \frac{5}{12}\sin 12t \\ &= \frac{13}{12}\left(\frac{12}{13}\cos 12t - \frac{5}{13}\sin 12t\right) = \frac{13}{12}\cos(12t - \alpha) \end{aligned}$$

where $\alpha = 2\pi - \tan^{-1}\left(\frac{5}{12}\right) \approx 0.58883942$. Part (b): Here we have

$$C = \frac{13}{12} \approx 1.08333333 \text{ (ft)} \quad \text{and} \quad T = \frac{2\pi}{12} \approx 0.52359878 \text{ (s)}. \qquad \text{(—C.H.E.)}$$

C09S07.023: Following the suggestion in the statement of the problem, we have

$$mx'' + cx' + kx = F(t) + mg; \qquad x(0) = x_0, \quad x'(0) = v_0. \tag{1}$$

Note that $kx_0 = mg$. Hence if we let $y(t) = x(t) - x_0$, then $y(0) = x_0 - x_0 = 0$ and $y'(0) = x'(0) = v_0$. Thus substitution in the equations in (1) yields

$$\begin{aligned} my'' + cy' + ky + kx_0 &= F(t) + kx_0; \qquad \text{that is,} \\ my'' + cy' + ky &= F(t); \qquad y(0) = 0, \quad y'(0) = v_0. \end{aligned}$$

C09S07.024: Newton's second law $F = a$ here takes the form

$$\rho\pi r^2 h x'' = \rho\pi r^2 hg - \pi r^2 xg; \qquad \text{that is,}$$
$$x'' + \frac{g}{\rho h}x = g.$$

The solution of this equation for which $x(0) = x'(0) = 0$ is

$$x(t) = \rho h(1 - \cos\omega_0 t)$$

where $\omega_0 = \sqrt{g/(\rho h)}$. With the given numerical values of ρ, h, and g, the amplitude of oscillation is $\rho h = 100$ cm and the period is

$$p = 2\pi\sqrt{\frac{\rho h}{g}} \approx 2.007090 \quad \text{(s)}. \qquad \text{(—C.H.E.)}$$

C09S07.025: The fact that the buoy weighs 100 lb means that $mg = 100$, so that $m = 100/32 = 3.125$ slugs. The weight of water is 62.4 lb/ft^3, so the equation $F = ma$ of Newton's second law of motion takes the form

$$\frac{100}{32}x'' = 100 - (62.4)\pi r^2 x.$$

It follows that the circular frequency ω of the buoy is given by

$$\omega^2 = \frac{32 \cdot (62.4) \cdot \pi r^2}{100}.$$

But the fact that the period of the buoy is $\rho = 2.5$ s means that $\omega = 2\pi/(2.5)$. Equating these two results yields $r \approx 0.3173201415$ ft, approximately 3.8078 in. (—C.H.E.)

C09S07.026: Part (a): Substitution of $M_r = (r/R)^3 M$ in $R_r = -GM_r m/r^2$ yields

$$F_r = -\frac{GMm}{R^3}r.$$

Part (b): Because $GM/R^3 = g/R$, the equation $mr'' = F_r$ yields the differential equation

$$r'' + \frac{g}{R}r = 0. \qquad (1)$$

Part (c): The solution Eq. (1) for which $r(0) = R$ and $r'(0) = 0$ is $r(t) = R\cos\omega_0 t$ where $\omega_0 = \sqrt{g/R}$. Hence, with $g = 32.2$ ft/s^2 and $R = 3960 \cdot 5280$ ft, we find that the period of the simple harmonic motion of the particle is

$$p = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g}} \approx 5063.0998$$

(seconds), approximately 84 min 23.0998 s. (—C.H.E.)

C09S07.027: Part (a):

$$x(t) = 50\left(e^{-2t/5} - e^{-t/2}\right).$$

Part (b): $x'(t) = 0$ when

$$25e^{-t/2} - 20e^{-2t/5} = 5e^{-2t/5}\left(5e^{-t/10} - 4\right) = 0;$$

$$t = 10\ln\frac{5}{4} \approx 2.23143551.$$

Hence the greatest distance that the mass travels to the right is

$$x\left(10\ln\frac{5}{4}\right) = \frac{512}{125} = 4.096. \qquad \text{(—C.H.E.)}$$

C09S07.028: Part (a):

$$x(t) = e^{-t/5}(20\cos 3t + 15\sin 3t) = 25e^{-t/5}\cos(3t - \alpha)$$

where $\alpha = \tan^{-1}\frac{3}{4} \approx 0.64350111$. Part (b): It follows that the oscillations are "bounded" by the curves $x = \pm 25e^{-t/5}$ and that the pseudoperiod of oscillation is $T = 2\pi/3$ (because $\omega = 3$). (—C.H.E.)

C09S07.029: Part (a): With $m = \frac{12}{32} = \frac{3}{8}$ slug, $c = 3$ lb-s/ft, and $k = 24$ lb/ft, the differential equation takes the form

$$3x'' + 24x' + 129x = 0.$$

The solution satisfying $x(0) = 1$ and $x'(0) = 0$ is

$$\begin{aligned} x(t) &= e^{-4t}\left(\cos 4t\sqrt{3} + \frac{1}{\sqrt{3}}\sin 4t\sqrt{3}\right) \\ &= \frac{2}{\sqrt{3}}e^{-4t}\left(\frac{\sqrt{3}}{2}\cos 4t\sqrt{3} + \frac{1}{2}\sin 4t\sqrt{3}\right) = \frac{2}{\sqrt{3}}e^{-4t}\cos\left(4t\sqrt{3} - \frac{\pi}{6}\right). \end{aligned}$$

Part (b): The time-varying amplitude is $2/\sqrt{3} \approx 1.1547$ ft, the frequency is $4\sqrt{3} \approx 6.9282$ rad/s, and the phase angle is $\pi/6$. (—C.H.E.)

C09S07.030: Part (a): With $m = 100$ slugs we get $\omega = \sqrt{k/100}$. But we are given that

$$\omega = (80 \text{ cycles/min})(2\pi)(1 \text{ min}/60 \text{ s}) = 8\pi/3,$$

and equating the two values yields $k \approx 7018$ lb/ft. Part (b): With $\omega_1 = 2\pi(78/60)$ s^{-1}, Eq. (18) in the text yields $c \approx 372.314$ lb/(ft/s). Hence $p = c/2m \approx 1.8615$. Finally $e^{-pt} = 0.01$ gives $t \approx 2.47$ s.
(—C.H.E.)

C09S07.031: In the case of critical damping, we have

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = -\frac{c}{2m} = -p.$$

The general solution of the differential equation and its derivative are

$$x(t) = (c_1 + c_2 t)e^{-pt} \qquad \text{and}$$

$$x'(t) = (c_1 - pc_1 t - pc_2)e^{-pt}.$$

The initial conditions yield $x_0 = c_2$ and $v_0 = c_1 - pc_2$, and it follows that $c_1 = px_0 + v_0$. Therefore

$$x(t) = (px_0t + v_0t + x_0)e^{-pt}.$$

C09S07.032: See Problem 31. If $t > 0$ and $x(t) = 0$, then

$$\begin{aligned} x_0 + v_0t + px_0t &= 0: \\ t &= -\frac{x_0}{v_0 + px_0}. \end{aligned}$$

Hence the equation $x(t) = 0$ has a positive solution if and only if x_0 and $v_0 + px_0$ have opposite signs.

C09S07.033: See Problem 31. If $x(t)$ has a local extremum for $t > 0$, then $x'(t) = 0$ for some $t > 0$. Thus

$$\begin{aligned} x'(t) &= (px_0 + v_0 - p^2x_0t - pv_0t - px_0)e^{-pt} = 0; \\ v_0 - p^2x_0t - pv_0t &= 0; \\ t &= \frac{v_0}{p(px_0 + v_0)}. \end{aligned}$$

Because $p > 0$, a positive solution t of $x'(t) = 0$ exists if and only if v_0 and $px_0 + v_0$ have the same sign.

C09S07.034: Because this is the overdamped case, we know that $c^2 > 4km$. The characteristic equation $mr^2 + cr + k = 0$ has roots

$$\begin{aligned} r &= \frac{-c \pm \sqrt{c^2 - 4km}}{2m} \\ &= -\frac{c}{2m} \pm \sqrt{\frac{c^2 - 4km}{4m^2}} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}; \\ r_1 &= -p + \sqrt{p^2 - \omega_0^2}, \quad r_2 = -p - \sqrt{p^2 - \omega_0^2}. \end{aligned}$$

Note that

$$\gamma = \frac{r_1 - r_2}{2} = \frac{-p + \sqrt{p^2 - \omega_0^2} + p + \sqrt{p^2 - \omega_0^2}}{2} = \sqrt{p^2 - \omega_0^2}.$$

The solution of the differential equation and its derivative are

$$\begin{aligned} x(t) &= c_1e^{r_1t} + c_2e^{r_2t} \qquad \text{and} \\ x'(t) &= c_1r_1e^{r_1t} + c_2r_2e^{r_2t}. \end{aligned}$$

The initial conditions yield

$$x_0 = x(0) = c_1 + c_2 \quad \text{and} \quad v_0 = x'(0) = c_1r_1 + c_2r_2,$$

and it follows that

$$c_1 = \frac{r_2x_0 - v_0}{r_2 - r_1} = \frac{v_0 - r_2x_0}{2\gamma} \quad \text{and} \quad c_2 = \frac{r_1x_0 - v_0}{r_1 - r_2} = \frac{r_1x_0 - v_0}{2\gamma}.$$

Therefore

$$x(t) = \frac{1}{2\gamma}\left[(v_0 - r_2x_0)e^{r_1t} - (v_0 - r_1x_0)e^{r_2t}\right].$$

C09S07.035: The motion is overdamped, so we know that $c^2 > 4km$. Substitute $x_0 = 0$ in the solution in Problem 34 to find that

$$x(t) = \frac{1}{2\gamma}\left(v_0e^{r_1t} - v_0e^{r_2t}\right) = \frac{v_0}{\gamma}\cdot\frac{e^{r_1t} - e^{r_2t}}{2}$$

where

$$r_1 = -p + \sqrt{p^2 - \omega_0^2}\,, \qquad r_2 = -p - \sqrt{p^2 - \omega_0^2}\,, \qquad \text{and} \qquad p = \frac{c}{2m}.$$

Thus $r_1 = -p + \gamma$ and $r_2 = -p - \gamma$. Hence

$$e^{r_1t} - e^{r_2t} = \exp(-pt)\left[e^{\gamma t} - e^{-\gamma t}\right] = 2\exp(-pt)\sinh\gamma t.$$

Therefore

$$x(t) = \frac{v_0}{\gamma}e^{-pt}\sinh\gamma t.$$

C09S07.036: By Problem 34 and its solution.

$$x(t) = \frac{1}{2\gamma}\left[(v_0 - r_2x_0)e^{r_1t} - (v_0 - r_1x_0)e^{r_2t}\right]$$

where

$$\gamma = \frac{r_1 - r_2}{2} > 0 \qquad \text{and} \qquad r_2 < r_1 < 0.$$

The condition $x(t) = 0$ is equivalent to

$$\begin{aligned}
(v_0 - r_2x_0)e^{r_1t} &= (v_0 - v_1x_0)e^{r_2t};\\
(v_0 - r_2x_0)e^{(r_1-r_2)t} &= v_0 - r_1x_0;\\
e^{(r_1-r_2)t} &= \frac{v_0 - r_1x_0}{v_0 - r_2x_0};\\
t &= \frac{1}{r_1 - r_2}\ln\frac{v_0 - r_1x_0}{v_0 - r_2x_0}.
\end{aligned}$$

Now $r_1 - r_2 > 0$, so there is only a single solution for t. If, moreover, there is to a single positive solution, then

$$\frac{v_0 - r_1x_0}{v_0 - r_2x_0} > 1.$$

It now follows that there is no positive solution for t except in the following two (exclusive) cases, in each of which there is exactly one positive solution:

$$v_0 > r_2 x_0 \quad \text{and} \quad x_0 < 0;$$

$$v_0 < r_2 x_0 \quad \text{and} \quad x_0 > 0.$$

C09S07.037: With $m = 1,\ c = 0,\ k = 9,\ F_0 = 60,$ and $\omega = 3$, we have

$$x'' + 9x = 60\cos 3t;$$

$$x'(0) = x'(0) = 0.$$

Part (a): If there is a solution of the form $x(t) = A\cos 3t + B\sin 3t,$ then

$$x''(t) = -9A\cos 3t - 9B\sin 3t.$$

Hence $x'' + 9x = 0$ for all t. So there can be no such solution. Part (b): If $x_p(t) = 10t\sin 3t,$ then

$$\begin{aligned} x_p'(t) &= 10\sin 3t + 30t\cos 3t; \\ x_p''(t) &= 60\cos 3t - 90t\sin 3t; \\ x_p''(t) + 9x_p(t) &= 60\cos 3t - 90t\sin 3t + 90t\sin 3t = 60\cos 3t. \end{aligned}$$

Chapter 9 Miscellaneous Problems

C09S0M.001: Given $\dfrac{dy}{dx} = 2x + \cos x$, $y(0) = 0$:

$$\begin{aligned} y(x) &= x^2 + \sin x + C; \\ 0 &= y(0) = C; \\ y(x) &= x^2 + \sin x. \end{aligned}$$

A *Mathematica* command for solving this initial value problem is

```
DSolve[ { y'[x] == 2*x + Cos[x], y[0] == 0 }, y[x], x ]
```

and other computer algebra systems, such as *Maple*, *Derive*, and MATLAB, use similar commands.

C09S0M.002: Given $\dfrac{dy}{dx} = 3x^{1/2} + x^{-1/2}$, $y(1) = 10$:

$$\begin{aligned} y(x) &= 2x^{3/2} + 2x^{1/2} + C; \\ 10 &= y(1) = 4 + C; \\ y(x) &= 2x^{3/2} + 2x^{1/2} + 6. \end{aligned}$$

C09S0M.003: Given: $\dfrac{dy}{dx} = (y+1)^2$.

$$\begin{aligned} \frac{dy}{(y+1)^2} &= 1\ dx; \\ -\frac{1}{y+1} &= x + C; \\ y + 1 &= -\frac{1}{x+C}; \\ y(x) &= -1 - \frac{1}{x+C}. \end{aligned}$$

C09S0M.004: Given: $\dfrac{dy}{dx} = (y+1)^{1/2}$.

$$(y+1)^{-1/2}\ dy = 1\ dx; \quad 2(y+1)^{1/2} = x + C; \quad y + 1 = \left(\frac{x+C}{2}\right)^2; \quad y(x) = -1 + \left(\frac{x+C}{2}\right)^2.$$

C09S0M.005: Given: $\dfrac{dy}{dx} = 3x^2y^2$, $y(0) = 1$.

$$y^{-2}\ dy = 3x^2\ dx; \quad -(y^{-1}) = x^3 + C; \quad y(x) = -\frac{1}{x^3+C}.$$

But $1 = y(0) = -\dfrac{1}{C}$, and therefore $y(x) = \dfrac{1}{1-x^3}$.

C09S0M.006: Given: $\dfrac{dy}{dx} = x^{1/3}y^{1/3}$, $y(1) = 1$.

$$4y^{-1/3}\,dy = 4x^{1/3}\,dx; \quad 6y^{2/3} = 3x^{4/3} + C; \quad y(x) = \left(\frac{3x^{4/3} + C}{6}\right)^{3/2}.$$

But $1 = y(1) = \left(\dfrac{3 + C}{6}\right)^{3/2}$, so $C = 3$. Therefore $y(x) = \left(\dfrac{x^{4/3} + 1}{2}\right)^{3/2}$.

A computer algebra system may return the solution in a quite different form. For example, the *Mathematica* command

```
DSolve[ { y′[x] == (x∗y[x])∧(1/3), y[1] = 1 }, y[x], x ]
```

returns the particular solution

$$y(x) = \frac{\sqrt{1 + 3x^{4/3} + 3x^{8/3} + x^4}}{2\sqrt{2}}.$$

C09S0M.007: Given: $x^2y^2\dfrac{dy}{dx} = 1$.

$$3y^2\,dy = 3x^{-2}dx; \quad y^3 = -3x^{-1} + C; \quad y(x) = \left(C - \frac{3}{x}\right)^{1/3}.$$

C09S0M.008: Given: $x^{1/2}y^{1/2}\dfrac{dy}{dx} = 1$.

$$3y^{1/2}\,dy = 3x^{-1/2}\,dx; \quad 2y^{3/2} = 6x^{1/2} + 2C; \quad y(x) = \left(3\sqrt{x} + C\right)^{2/3}.$$

C09S0M.009: Given: $\dfrac{dy}{dx} = y^2\cos x$, $y(0) = 1$.

$$y^{-2}\,dy = (\cos x)\,dx; \quad -(y^{-1}) = C + \sin x; \quad y(x) = -\frac{1}{C + \sin x}.$$

But $1 = y(0) = -\dfrac{1}{C}$, so $C = -1$. Therefore $y(x) = \dfrac{1}{1 - \sin x}$.

C09S0M.010: Given: $\dfrac{dy}{dx} = y^{1/2}\sin x$, $y(0) = 4$.

$$y^{-1/2}\,dy = (\sin x)\,dx; \quad 2y^{1/2} = C - \cos x; \quad y(x) = \left(\frac{C - \cos x}{2}\right)^2.$$

Then we impose the condition $y(0) = 4$ on the *second* of the preceding equations:

$$2\sqrt{4} = C - \cos 0; \quad C - 1 = 4; \quad C = 5.$$

Therefore $y(x) = \left(\dfrac{5 - \cos x}{2}\right)^2$.

C09S0M.011: Given: $\dfrac{dy}{dx} = \dfrac{y^2\,(1 - \sqrt{x}\,)}{x^2\left(1 - \sqrt{y}\,\right)}$.

$$\frac{1-y^{1/2}}{y^2}\,dy = \frac{1-x^{1/2}}{x^2}\,dx;$$

$$(y^{-2}-y^{-3/2})\,dy = (x^{-2}-x^{-3/2})\,dx;$$

$$\frac{1}{y}-\frac{2}{\sqrt{y}} = \frac{1}{x}-\frac{2}{\sqrt{x}}+C.$$

You should leave the solution in this (implicitly defined) form because it's troublesome to solve for y explicitly as a function of x. *Mathematica* finds two solutions:

$$y(x) = \frac{x-2x^{3/2}+2x^2+Cx^2 \pm 2\sqrt{x^3-2x^{7/2}+x^4+Cx^4}}{1-4x^{1/2}+4x+2Cx-4Cx^{3/2}+C^2x^2}.$$

C09S0M.012: Given: $\dfrac{dy}{dx} = \dfrac{y^{1/2}(x+1)^3}{x^{1/2}(y+1)^3}$.

$$\frac{(y+1)^3}{y^{1/2}}\,dy = \frac{(x+1)^3}{x^{1/2}}\,dx;$$

$$(y^{5/2}+3y^{3/2}+3y^{1/2}+y^{-1/2})\,dy = (x^{5/2}+3x^{3/2}+3x^{1/2}+x^{-1/2})\,dx;$$

$$\frac{2}{7}y^{7/2}+\frac{6}{5}y^{5/2}+2y^{3/2}+2y^{1/2} = \frac{2}{7}x^{7/2}+\frac{6}{5}x^{5/2}+2x^{3/2}+2x^{1/2}+C.$$

C09S0M.013: The equation is linear, with solution $y(x) = Cx^3 + x^3\ln x$. (—C.H.E.)

C09S0M.014: The equation is separable, with solution $y(x) = \dfrac{x}{3-Cx-x\ln x}$. (—C.H.E.)

C09S0M.015: The equation is separable, with solution $y(x) = C\exp\left(\dfrac{1-x}{x^3}\right)$. (—C.H.E.)

C09S0M.016: The equation is separable, with solution $y(x) = \dfrac{x}{1+Cx+2x\ln x}$. (—C.H.E.)

C09S0M.017: The equation is linear, with solution $y(x) = \dfrac{C+\ln x}{x^2}$. (—C.H.E.)

C09S0M.018: The equation is separable, with solution $y(x) = \tan\left(C+x+\dfrac{1}{3}x^2\right)$. (—C.H.E.)

C09S0M.019: The equation is linear, with solution $y(x) = (x^3+C)e^{-3x}$. (—C.H.E.)

C09S0M.020: The equation is separable, with solution $y(x) = \dfrac{x^2}{x^5+Cx^2+1}$. (—C.H.E.)

C09S0M.021: The equation is linear, with solution $y(x) = 2x^{-3/2}+Cx^{-3}$. (—C.H.E.)

C09S0M.022: The equation is linear, with solution $y(x) = \dfrac{C+\ln(x-1)}{x+1}$. (—C.H.E.)

C09S0M.023: The equation is separable, with solution $y(x) = \dfrac{x^{1/2}}{6x^2+Cx^{1/2}+2}$. (—C.H.E.)

C09S0M.024: The equation is linear, with solution $y(x) = \dfrac{x^3 + 3x^2 + 3x + C}{(x+1)^2}$. (—C.H.E.)

C09S0M.025: Given:

$$\frac{dy}{dx} = e^x + y; \qquad \text{that is,} \qquad \frac{dy}{dx} - y = e^x. \tag{1}$$

This is a linear differential equation with integrating factor $\rho(x) = e^{-x}$. Multiplication of both sides of the second equation in (1) by $\rho(x)$ yields

$$\begin{aligned} &e^{-x}\frac{dy}{dx} - e^{-x}y = 1; \\ &e^{-x}y = x + C; \\ &y(x) = (x + C)e^x. \end{aligned}$$

C09S0M.026: The equation is linear, with solution $y(x) = \dfrac{e^{2x} + C}{x}$. (—C.H.E.)

C09S0M.027: As a separable equation:

$$\frac{1}{y+7}\,dy = 3x^2\,dx; \qquad \ln(y+7) = x^3 + C_1;$$

$$y + 7 = \exp\left(x^3 + C_1\right) = C\exp\left(x^3\right); \qquad y(x) = -7 + C\exp\left(x^3\right).$$

As the linear equation $\dfrac{dy}{dx} - 3x^2y = 21x^2$:

$$\text{Integrating factor:} \quad \rho(x) = \exp\left(\int (-3x^2)\,dx\right) = \exp\left(-x^3\right).$$

Thus

$$y(x)\cdot\exp\left(-x^3\right) = \int 21x^2\exp\left(-x^3\right)\,dx = -7\exp\left(-x^3\right) + C.$$

Therefore $y(x) = -7 + C\exp\left(x^3\right)$.

C09S0M.028: As a separable equation:

$$\frac{1}{y+1}\,dy = \frac{2x}{x^2+1}\,dx; \qquad \ln(y+1) = C_1 + \ln(x^2+1);$$

$$y + 1 = \exp\left(C_1 + \ln(x^2+1)\right) = C(x^2+1); \qquad y(x) = -1 + C(x^2+1).$$

In “standand” linear form

$$\frac{dy}{dx} - \frac{2x}{x^2+1}y = \frac{2x}{x^2+1},$$

the equation has integrating factor

$$\rho(x) = \exp\left(\int -\frac{2x}{x^2+1}\,dx\right) = \frac{1}{x^2+1}.$$

Hence

$$\frac{1}{x^2+1}y = \int \frac{2x}{(x^2+1)^2}\,dx = -\frac{1}{x^2+1} + C.$$

Therefore $y(x) = -1 + C(x^2+1) = Ax^2 + B$.

C09S0M.029: First note that $\dfrac{1}{x^2+5x+6} = \dfrac{1}{x+2} - \dfrac{1}{x+3}$. So

$$\int \frac{1}{x^2+5x+6}\,dx = \int 1\,dt; \qquad \int\left(\frac{1}{x+2} - \frac{1}{x+3}\right)dx = t + C_1;$$

$$\ln\frac{x+2}{x+3} = t + C_1; \qquad \frac{x+2}{x+3} = Ce^t;$$

$$\frac{7}{8} = Ce^0 = C; \qquad 8(x+2) = 7(x+3)e^t;$$

$$(8-7e^t)x = 21e^t - 16; \qquad x(t) = \frac{21e^t - 16}{8-7e^t}.$$

C09S0M.030: First note that $\dfrac{1}{2x^2+x-15} = \dfrac{1}{11}\left(\dfrac{2}{2x-5} - \dfrac{1}{x+3}\right)$. Then

$$\frac{1}{11}\int\left(\frac{2}{2x-5} - \frac{1}{x+3}\right)dx = \int 1\,dt; \qquad \ln\frac{2x-5}{x+3} = 11t + C_1;$$

$$\frac{2x-5}{x+3} = Ce^{11t}; \qquad C = \frac{15}{13};$$

$$\frac{2x-5}{x+3} = \frac{15}{13}e^{11t}; \qquad 26x - 65 = (15x+45)e^{11t};$$

$$26x - 15e^{11t}x = 45e^{11t} + 65; \qquad x(t) = \frac{45e^{11t}+65}{26-15e^{11t}}.$$

C09S0M.031: Let τ denote the half-life of potassium, so that τ is approximately 1.28×10^9. Measure time t also in years, with $t = 0$ corresponding to the time when the rock contained only potassium, and with $t = T$ corresponding to the present. Then at time $t = 0$, the amount of potassium was $Q(0)$ and no argon was present. At present, the amount of potassium is $Q(T)$ and the amount of argon is $A(T)$, where $A(t)$ is the amount of argon in the rock at time t. Now

$$Q(t) = Q_0 e^{-(t\ln 2)/\tau}, \quad \text{so that} \quad A(t) = \tfrac{1}{9}\left(Q_0 - Q(t)\right).$$

We also are given the observation that $A(T) = Q(T)$. Thus

$$Q_0 - Q(t) = 9Q(T), \quad \text{and so} \quad Q(T) = (0.1)Q_0 = Q_0 e^{-(T\ln 2)/\tau}.$$

Therefore

$$\ln 10 = \frac{T}{\tau}\ln 2 \quad \text{and thus} \quad T = \frac{\ln 10}{\ln 2}(1.28 \times 10^9) \approx 4.2521 \times 10^9.$$

Thus the rock is approximately 4.25×10^9 years old.

C09S0M.032: Let $T(t)$ denote the temperature of the buttermilk (in degrees Celsius) at time t (in minutes); let A denote the temperature on the front porch. Assume that the buttermilk is placed on the porch at time $t = 0$. Then Newton's law of cooling takes the form

$$\frac{dT}{dt} = k(T - A),$$

but in this special case we have $A = 0$, so that the temperature undergoes "natural decay," much like radioactive decay. Thus, as in the derivation of Eq. (5), we have $T(t) = T_0 e^{kt}$, although $k < 0$ in this problem. We know that $T_0 = T(0) = 0$, so that $T(t) = 25e^{kt}$; moreover, the information that $T(20) = 15$ yields

$$25e^{20k} = 15, \quad \text{so that} \quad k = \frac{1}{20} \ln\left(\frac{3}{5}\right).$$

Finally, we solve $T(t) = 5$:

$$25e^{5k} = 5; \quad e^{-kt} = 5; \quad t = -\frac{\ln 5}{k} = -\frac{20 \ln 5}{\ln\left(\frac{3}{5}\right)} \approx 63.01.$$

Answer: The buttermilk will be at 5°C about 1 hour and 3 minutes after placing it on the porch.

C09S0M.033: First, $\dfrac{dA}{dt} = -kA$, so $A(t) = A_0 e^{-kt}$.

$$\frac{3}{4}A_0 = A_0 e^{-k}, \quad \text{so} \quad k = \ln \frac{4}{3}.$$

Also

$$\frac{1}{2}A_0 = A_0 e^{-kT}$$

where T is the time required for half the sugar to dissolve. So

$$\frac{\ln 2}{k} = T = \frac{\ln 2}{\ln(\frac{4}{3})} \approx 2.40942 \quad \text{(minutes)},$$

so half of the sugar is dissolved in about 2 minutes and 25 seconds.

C09S0M.034: $\dfrac{dI}{dx} = -(1.4)I$, so $I(x) = I_0 e^{-(1.4)x}$.

(a) $I(x) = \frac{1}{2}I_0$: $I_0 e^{-(1.4)x} = \frac{1}{2}I_0$; $e^{(1.4)x} = 2$; $x = \dfrac{\ln 2}{1.4} \approx 0.495$ (meters).

(b) $I(10) = I_0 e^{-(1.4)(10)} \approx (0.000000832)\, I_0$; that is, about $\dfrac{1}{1202600} I_0$.

(c) $I(x) = (0.01)I_0$: As in part (a), $x = \dfrac{\ln 100}{1.4} \approx 3.29$ (meters).

C09S0M.035: We begin with the equation $p(x) = (29.92)e^{-x/5}$.

(a) $p\left(\frac{10000}{5280}\right) \approx 20.486$ (inches); $p\left(\frac{30000}{5280}\right) \approx 9.604$ (inches).

(b) If x is the altitude in question, then we must solve

$$15 = (29.92)e^{-x/5}, \quad \text{and thus} \quad x = 5 \ln\left(\frac{29.92}{15}\right) \approx 3.4524 \quad \text{(miles)},$$

approximately 18230 feet.

(c) According to *Trails Illustrated* Topo Map 322 "Denali National Park and Preserve" (Trails Illustrated, Evergreen, CO, 1990, 1993) and other sources, the summit of Mt. McKinley is 20320 ft above sea level. We evaluate

$$p\left(\tfrac{20320}{5280}\right) = (29.92)\exp\left(\frac{20320}{5 \cdot 5280}\right) \approx 13.8575$$

to find that the atmospheric pressure at the summit is about 13.86 inches of mercury. For an engrossing story of an ascent of this peak, see Ruth Anne Kocour's *Facing the Extreme* (with Michael Hodgson, New York: St. Martin's Press, 1998).

C09S0M.036: The differential equation and initial condition are

$$\frac{dA}{dt} = -kA; \qquad A(0) = A_0 = 10S$$

with time t measured in days. We solve the differential equation:

$$\begin{aligned}
&\frac{1}{A} \cdot \frac{dA}{dt} = -k;\\
&\ln A = -kt + C;\\
&A(t) = e^{-kt+C} = e^C e^{-kt} = C_1 e^{-kt}
\end{aligned}$$

where $C_1 = e^C$ is a constant. Next, $A(0) = 10S = C_1$, and therefore

$$A(t) = 10Se^{-kt}.$$

To find k, we use the information that $A(100) = 7S$:

$$10Se^{-100k} = 7S; \quad e^{100k} = \frac{10}{7}; \quad k = \frac{1}{100}\ln\frac{10}{7}.$$

To find when it is safe to return to the contaminated area, we solve $A(T) = S$:

$$10Se^{-kT} = S; \quad e^{kT} = 10; \quad T = \frac{1}{k}\ln 10 = \frac{100\ln 10}{\ln\left(\frac{10}{7}\right)} \approx 646.$$

It will be safe to return to the contaminated area 646 days after the accident.

C09S0M.037: The decay constant k satisfies the equation $140k = \ln 2$, and so $k = (\ln 2)/140$. Measuring radioactivity as a multiple of the "safe level" 1, it is then $P(t) = 5e^{-kt}$ with t measured in days. When we solve $P(t) = 1$, we find that $t \approx 325.07$, so the room should be safe to enter in a little over 325 days.

C09S0M.038: The projected revenues are $r(t) = (1.85)e^{(0.03)t}$ and the projected budget is $b(t) = 2e^{kt}$ for some constant k (values of both functions are in billions of dollars; remember that in the U.S., a billion is a *thousand* million). To have a balanced budget in seven years, we solve $r(7) = b(7)$:

$$(1.85)e^{0.21} = 2e^{7k}; \quad 7k = \ln\left((0.925)e^{0.21}\right) = 0.21 + \ln(0.925); \quad k = \frac{0.21 + \ln(0.925)}{7} \approx 0.01886264.$$

So the annual budget increase should be approximately 1.886%.

C09S0M.039: The characteristic equation

$$6r^2 - 19r + 15 = (2r-3)(3r-5) = 0$$

has the real distinct roots $r_1 = \frac{3}{2}$ and $r_2 = \frac{5}{3}$. Hence the general solution of the differential equation is

$$\begin{aligned} y(x) &= Ae^{3x/2} + Be^{5x/3}, \qquad \text{for which} \\ y'(x) &= \frac{3}{2}Ae^{3x/2} + \frac{5}{3}Be^{5x/3}. \end{aligned}$$

The initial conditions yield

$$\begin{aligned} 13 &= y(0) = A + B \qquad \text{and} \\ 21 &= y'(0) = \frac{3}{2}A + \frac{5}{3}B, \end{aligned}$$

and it quickly follows that $A = 4$ and $B = 9$. Hence $y(x) = 4e^{3x/2} + 9e^{5x/3}$.

C09S0M.040: Alternatives to the manual methods demonstrated in the previous solution include use of a computer algebra system such as *Mathematica* 3.0:

```
Solve[ 50*r∧2 − 5*r − 28 == 0, r ]
```

$\{\{\, r \to -\frac{7}{10} \,\},\ \{\, r \to \frac{4}{5} \,\}\}$

```
y[x_] := a*Exp[ 4*x/5 ] + b*Exp[ −7*x/10 ]
```

```
Solve[ { y[0] == 25, y′[0] == −10 }, { a, b } ]
```

$\{\{\, a \to 5 \,\},\ \{\, b \to 20 \,\}\}$

Therefore the solution of the given initial value problem is $y(x) = 5e^{4x/5} + 20e^{-7x/10}$. For an even shorter solution, execute the command

```
DSolve[ { 50*y′′[x] - 5*y′[x] - 28*y[x] == 0, y[0] == 25, y′[0] == −10 }, y[x], x ]
```

—which returns the solution in the form $y(x) = e^{-7x/10}(20 + 5e^{3x/2})$.

C09S0M.041: The characteristic equation

$$121r^2 + 154r + 49 = (11r+7)^2 = 0$$

has the repeated root $r_1 = r_2 = -\frac{7}{11}$, and thus the given equation has general solution

$$\begin{aligned} y(x) &= (Ax + B)e^{-7x/11}, \qquad \text{for which} \\ y'(x) &= \left(A - \tfrac{7}{11}Ax - \tfrac{7}{11}B\right)e^{-7x/11}. \end{aligned}$$

The initial conditions then yield $A = 17$ and $B = 11$, and hence the solution of the given initial value problem is $y(x) = (17x + 11)e^{-7x/11}$.

C09S0M.042: The characteristic equation

$$169r^2 - 130r + 25 = (13r-5)^2 = 0$$

has the repeated roots $r_1 = r_2 = \frac{5}{13}$. Thus the given equation has general solution

$$y(x) = (Ax+B)e^{5x/13}, \qquad \text{for which}$$
$$y'(x) = \left(A + \tfrac{5}{13}Ax + \tfrac{5}{13}B\right)e^{5x/13}.$$

It follows from the initial conditions that $A = 29$ and $B = 26$. Hence the solution of the original initial value problem is $y(x) = (29x+26)e^{5x/13}$.

C09S0M.043: First we solve the characteristic equation:

$$100r^2 + 20r + 10001 = 0; \qquad 100r^2 + 20r + 1 = -10000;$$
$$(10r+1)^2 = (100i)^2; \qquad 10r + 1 = \pm 100i;$$
$$r = -\frac{1}{10} \pm 10i.$$

So the general solution of the differential equation may be expressed in the form

$$y(x) = e^{-x/10}(A\cos 10x + B\sin 10x), \qquad \text{for which}$$
$$y'(x) = e^{-x/10}\left(10B\cos 10x - 10A\sin 10x - \frac{1}{10}A\cos 10x - \frac{1}{10}B\sin 10x\right).$$

Then the initial conditions yield $A = 10$ and $B = 1$, so the solution of the given initial value problem is

$$y(x) = e^{-x/10}(10\cos 10x + \sin 10x)$$

.

C09S0M.044: First we solve the characteristic equation:

$$100r^2 + 2000r + 10001 = 0; \qquad (10r+100)^2 = -1;$$
$$10r + 100 = \pm i; \qquad r = -10 \pm \frac{1}{10}i.$$

Therefore the given differential equation has general solution

$$y(x) = e^{-10x}\left(A\cos\frac{x}{10} + B\sin\frac{x}{10}\right), \qquad \text{for which}$$
$$y'(x) = e^{-10x}\left(\frac{1}{10}B\cos\frac{x}{10} - \frac{1}{10}A\sin\frac{x}{10} - 10A\cos\frac{x}{10} - 10B\sin\frac{x}{10}\right).$$

The initial conditions yield $A = 1$ and $B = 10$. Hence the solution of the original initial value problem may be written in the form

$$y(x) = e^{-10x}\left(\cos\frac{x}{10} + 10\sin\frac{x}{10}\right).$$

The solutions of Problems 43 and 44 demonstrate vividly some of the effects on an underdamped system of increasing the damping coefficient—quicker damping, increased pseudoperiod.

C09S0M.045: Part (a): With N in thousands (of transistors) and t in years, we have $N(t) = 29e^{rt}$.

Part (b): In 1993 we have $t = 14$. So

$$31000 = N(14) = 29e^{14r}; \qquad 14r = \ln\frac{31000}{29}; \qquad r = \frac{1}{14}\ln\frac{31000}{29} \approx 0.498174761.$$

Expressed as a percentage, the annual growth rate is about 49.8%. Part (c): Let τ denote the "doubling time" and let $N_0 = N(0)$. Then from the equation $N(\tau) = 2N_0 = N_0e^{r\tau}$ we find that

$$\tau = \frac{\ln 2}{r} = \frac{14\ln 2}{\ln\frac{31000}{29}} \approx 1.39137354$$

years. Thus the doubling time is 12τ, about 16.7 months. Part (d): In the year 2001 we have $t = 22$, so in that year the typical microcomputer would be expected to contain

$$N(22) = 29e^{22r} \approx 1{,}668{,}007.855$$

thousand transistors; that is, about 1668 million transistors. In American English, that's about 1.668 billion transistors; in British English, it's about 1.668 thousand million transistors (a British "billion" is a *million* millions).

C09S0M.046: An atom of ^{14}C weighs about $w = 2.338 \times 10^{-23}$ grams. If we take the half-life of ^{14}C to be $\tau = 5700$ years, then at least $70{,}000{,}000/\tau$ half-lives have elapsed since the demise of the dinosaur. Working backwards from "today," 5700 years ago we would expect to find two atoms of ^{14}C, $2 \cdot 5700$ years ago there would be four such atoms, and so on. So the weight of the ^{14}C in the living dinosaur would be at least

$$2^{12000}\,w \approx 5.3554 \times 10^{3589}$$

grams. By comparison, the earth weighs about 5.988×10^{27} grams. So even if no other elements were present in the dinosaur's body, it would have weighed well over 10^{3560} times as much as the earth. In fact, its weight would have been an extremely large multiple of the total mass of the universe!

C09S0M.047: The differential equation leads to

$$P^{-1/2}\,dP = -k\,dt; \qquad 2P^{1/2} = C - kt; \qquad P^{1/2} = \frac{C - kt}{2};$$
$$30 = [P(0)]^{1/2} = \frac{C}{2}; \qquad C = 60; \qquad P^{1/2} = \frac{60 - kt}{2}.$$

Then the information that $P(6) = 441$ yields

$$21 = [P(6)]^{1/2} = 30 - 3k; \qquad k = 3; \qquad P(t) = \left(\frac{60 - 3t}{2}\right)^2.$$

Because we have measured time t in weeks, the answer is that all the fish will die at the end of 20 weeks.

C09S0M.048: Proof: Suppose that $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\,\right)^2$. Then $P(0) = \left(\sqrt{P_0}\,\right)^2 = P_0$ and

$$\frac{dP}{dt} = 2\left(\tfrac{1}{2}kt + \sqrt{P_0}\,\right) \cdot \tfrac{1}{2}k$$

and

$$k\sqrt{P(t)} = k\left(\tfrac{1}{2}kt + \sqrt{P_0}\,\right).$$

Therefore $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\,\right)^2$ is a solution of the initial value problem given here. Moreover, if P is differentiable and satisfies the given initial value problem, then

$$P^{-1/2}\,dP = k\,dt; \qquad 2P^{1/2} = C + kt; \qquad P^{1/2} = \frac{C+kt}{2};$$

$$\sqrt{P_0} = [P(0)]^{1/2} = \frac{C}{2}; \qquad C = 2\sqrt{P_0}; \qquad P^{1/2} = \frac{kt + 2\sqrt{P_0}}{2};$$

$$P(t) = \left(\tfrac{1}{2}kt + \sqrt{P_0}\,\right)^2.$$

Therefore $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\,\right)^2$ is the [unique] solution of the given initial value problem.

C09S0M.049: Given (in effect): $P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\,\right)^2$, $P_0 = 100$ (we take $t = 0$ [years] in 1970 and measure population in thousands), and $P(10) = 121$. Thus

$$P(t) = \left(\tfrac{1}{2}kt + 10\right)^2,$$

and therefore $121 = P(10) = (5k+10)^2$, so that $5k + 10 = \pm 11$. Because $k > 0$, we see that $k = \frac{1}{5}$, and hence

$$P(t) = \left(\tfrac{1}{10}t + 10\right)^2.$$

Thus in the year 2000 the population will be $P(30) = 169$ (thousand). The population will be 200 (thousand) when $P(T) = 200$:

$$\left(\tfrac{1}{10}T + 10\right) = 200; \qquad \tfrac{1}{10}T + 10 = 10\sqrt{2}; \qquad T = 100\left(\sqrt{2} - 1\right) \approx 41.4.$$

Thus the population will reach 200000 in the "year" $1970 + 41.4 = 2011.4$; that is, about May 26, 2011.

C09S0M.050: Given: $\dfrac{dP}{dt} = kP^2$, $P(0) = P_0$ where $k > 0$. Then

$$-\frac{1}{P^2}\,dP = -k\,dt; \qquad \frac{1}{P} = C - kt; \qquad P(t) = \frac{1}{C - kt};$$

$$P_0 = P(0) = \frac{1}{C}; \qquad C = \frac{1}{P_0}; \qquad P(t) = \frac{P_0}{1 - kP_0t}.$$

C09S0M.051: If $P_0 = 2$ and $P(3) = 4$ (time t is measured in months), then

$$P(t) = \frac{2}{1 - 2kt}; \qquad 4 = P(3) = \frac{2}{1 - 6k}; \qquad 1 - 6k = \frac{1}{2};$$

$$k = \frac{1}{12}; \qquad P(t) = \frac{2}{1 - \frac{1}{6}t} = \frac{12}{6 - t}.$$

Answer: $\displaystyle\lim_{t\to 6^-} P(t) = +\infty$.

C09S0M.052: We are given

$$\frac{dv}{dt} = -kv^2, \quad v_0 = v(0) = 40$$

where k is a positive constant. Thus (part(a))

$$-\frac{1}{v^2}\,dv = k\,dt; \qquad \frac{1}{v} = C + kt; \qquad v(t) = \frac{1}{C+kt};$$

$$40 = v(0) = \frac{1}{C}; \qquad C = \frac{1}{40}; \qquad v(t) = \frac{40}{1+40kt}$$

for $t \geqq 0$. For part (b), we have

$$20 = v(10) = \frac{40}{1+400k}; \qquad 1 + 400k = 2; \qquad k = \frac{1}{400};$$

$$v(t) = \frac{40}{1+\frac{1}{10}t} = \frac{400}{10+t}.$$

Hence $v(T) = 5$ when

$$\frac{400}{10+T} = 5; \qquad 10 + T = 80; \qquad T = 70.$$

Answer: After 70 s the boat will slow to a speed of 5 ft/s.

C09S0M.053: First we solve the initial value problem $\dfrac{dP}{dt} = -3P^{1/2}$, $P(0) = 900$, with t measured in weeks:

$$\int P^{-1/2}\,dP = -3\,dt; \qquad 2P^{1/2} = C - 3t; \qquad 2\cdot 30 = C - 3\cdot 0;$$

$$2P^{1/2} = 60 - 3t; \qquad P(t) = \frac{9}{4}\,(20-t)^2\,.$$

So all the fish will die after 20 weeks.

C09S0M.054: Let $x(t)$ denote the position of the race car at time t (the units are meters and seconds) and let $v(t) = x'(t)$ denote its velocity then. First we solve the initial value problem

$$\frac{dv}{dt} = -kv, \quad v(0) = v_0$$

where v_0 denotes the initial velocity of the race car and k is a positive constant. By Theorem 1 of Section 7.5 the solution is $v(t) = v_0e^{-kt}$. Moreover, $v'(0) = -2$, and thus $-2 = -kv_0$, so that $k = 2/v_0$. Next,

$$x(t) = C - \frac{v_0}{k}e^{-kt} = C - \frac{1}{2}v_0^2e^{-2t/v_0}$$

and $0 = x(0) = C - \frac{1}{2}v_0^2$, so that $x(t) = \frac{1}{2}v_0^2\left(1 - e^{-2t/v_0}\right)$. Next, $v(t) \to 0$ as $t \to +\infty$, so that

$$1800 = \lim_{t\to\infty} x(t) = \frac{1}{2}v_0^2,$$

and this implies that $v_0 = 60$ (meters per second), a little over 134 mi/h.

C09S0M.055: Problem 33 in Section 9.3 is to derive the initial value problem

$$\frac{dP}{dt} = rP - c, \quad P(0) = P_0$$

where $P(t)$ is the balance owed at time t (in months), where r is the monthly interest rate (compounded continuously) and c is the monthly payment (assumed made continuously). First we need to solve this initial value problem:

$$\int \frac{r\,dP}{rP-c} = \int r\,dt; \qquad \ln(rP-c) = C + rt; \qquad rP - c = Ae^{rt};$$

$$P(t) = \frac{1}{r}\left(c + Ae^{rt}\right); \qquad P_0 = P(0) = \frac{1}{r}(c+A); \qquad P(t) = \frac{c + (rP_0 - c)\,e^{rt}}{r}.$$

In this problem, the loan is to be paid off in $25 \cdot 12 = 300$ months, and thus $P(300) = 0$. We use this information to solve for the monthly payment c:

$$\frac{c + (rP_0 - c)e^{300r}}{r} = 0;$$

$$c\left(1 - e^{300r}\right) + rP_0e^{300r} = 0;$$

$$c = \frac{rP_0e^{300r}}{e^{300r} - 1}.$$

With $P_0 = 120000$ and $r = 0.08/12$, we find that $c = \$925.21$. With $r = 0.12/12$ we find that $c = \$1262.87$. In the latter case the total of all 300 monthly payments is \$378862.45.

C09S0M.056: First we solve the initial value problem $1000\dfrac{dv}{dt} = 5000 - 100v$, $v(0) = 0$:

$$10\frac{dv}{dt} = 50 - v; \qquad \int \frac{dv}{50-v} = \int \frac{1}{10}\,dt; \qquad -\ln(50-v) = \frac{1}{10}t - C;$$

$$\ln(50-v) = C - \frac{1}{10}t; \qquad 50 - v = Ae^{-t/10}; \qquad v(t) = 50 - Ae^{-t/10};$$

$$0 = v(0) = 50 - A; \qquad v(t) = 50\left(1 - e^{-t/10}\right).$$

Because $v(t) \to 50$ as $t \to +\infty$, the powerboat can attain any speed up to 50 ft/s. (Technically, there is no *maximum* speed, but the boat can reach speeds arbitrarily close to 50 ft/s.)

C09S0M.057: Let $h(t)$ denote the temperature within the freezer (in degrees Celsius) at time t (in hours), with $t = 0$ corresponding to the time the power goes off. By Newton's law of cooling, there is a positive constant k such that

$$\frac{dh}{dt} = k(20-h); \qquad \int \frac{dh}{20-h} = \int k\,dt; \qquad -\ln(20-h) = kt - C;$$

$$\ln(20-h) = C - kt; \qquad 20 - h = Ae^{-kt}; \qquad h(t) = 20 - Ae^{-kt};$$

$$-16 = h(0) = 20 - A; \qquad h(t) = 20 - 36e^{-kt}; \qquad h(7) = -10;$$

$$20 - 36e^{-7k} = -10; \qquad 36e^{-7k} = 30; \qquad k = \frac{1}{7}\ln\frac{6}{5}.$$

Finally we solve $h(T) = 0$ for

$$T = \frac{7 \ln \frac{9}{5}}{\ln \frac{6}{5}} \approx 22.5673076.$$

So the critical temperature will be reached about 22 hours and 34 minutes after the power goes off; that is, at 9:34 P.M. on the following day. The data given here were drawn from a real incident.

C09S0M.058: By Theorem 1 of Section 9.1, the solution of the differential equation given in this problem is $A(t) = A_0 e^{-t/400}$. For part (a), we compute

$$\frac{A(25)}{A_0} = e^{-1/16} \approx 0.9394130628,$$

and the answer is 93.94%. For part (b), we solve $A(T) = \frac{1}{2} A_0$ for $T = 400 \ln 2 \approx 277.2588722240$, so the answer is about 277 years.

C09S0M.059: The differential equation we need to solve is $\dfrac{dv}{dt} = a - \rho v$ with initial condition $v(0) = 0$:

$$\int \frac{-\rho}{a - \rho v}\,dv = \int (-\rho)\,dt; \qquad \ln(a - \rho v) = C - \rho t; \qquad a - \rho v = Ae^{-\rho t};$$

$$a - \rho \cdot 0 = A; \qquad a - \rho v = ae^{-\rho t}; \qquad \rho v = a\left(1 - e^{-\rho t}\right);$$

$$v(t) = \frac{a}{\rho}\left(1 - e^{-\rho t}\right).$$

If $a = 17.6$ and $\rho = 0.1$, then

$$v(10) = 176\left(1 - \frac{1}{e}\right) \approx 111.2532183538$$

feet per second, about 75.854 mi/h. Also $v(t) \to 176$ as $t \to +\infty$, so the limiting velocity is 176 ft/s, exactly 120 mi/h.

C09S0M.060: If the safe limit of radioactivity is S, then the radiation level at time t (months) will be $r(t) = 10Se^{-kt}$ where k is a positive constant. We solve $r(6) = 9S$ for $k = \frac{1}{6} \ln \frac{10}{9}$, and with this value of k we solve $r(T) = S$:

$$10Se^{-kT} = S; \qquad e^{kT} = 10; \qquad T = \frac{\ln 10}{k} = \frac{6 \ln 10}{\ln \frac{10}{9}} \approx 131.126072.$$

We divide by 12 to convert this answer in months to 10.927173 years. It will take just under 11 years for the levels of radiation to drop to the safe limit.

C09S0M.061: Given: $S(t) = 30e^{(0.05)t}$ (t is in years; $t = 0$ corresponds to age 30).

(a) $\Delta A = A(t + \Delta t) - A(t) \approx (0.06)A(t)\,\Delta t + (0.12)S(t)\,\Delta t$.

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = (0.06)A(t) + (0.12)S(t);$$

$$\frac{dA}{dt} + (-0.06)A(t) = (3.6)e^{(0.05)t}.$$

(b) The last equation is a linear first-order differential equation. Our earlier methods yield the solution

$$A(t) = -\frac{3.6}{0.01}\left(e^{(0.05)t} - e^{(0.06)t}\right), \quad \text{so that}$$
$$A(t) = 360\left(e^{(0.06)t} - e^{(0.05)t}\right).$$

Now $A(40) = 360\left(e^{2.4} - e^{2}\right) \approx 1308.28330$. Because the units in this problem are in thousands of dollars, the answer is that the retirement money available will be \$1,308,283.30.

C09S0M.062: Given: $\dfrac{dP}{dt} = \beta_0 e^{-\alpha t}P$, $P(0) = P_0$. Then

$$\frac{1}{P}\,dP = \beta_0 e^{-\alpha t}\,dt; \qquad \ln P = C - \frac{\beta_0}{\alpha}e^{-\alpha t}.$$
$$\ln P_0 = C - \frac{\beta_0}{\alpha}: \qquad C = \frac{\beta_0}{\alpha} + \ln P_0.$$
$$P(t) = \exp\left(\frac{\beta_0}{\alpha} + \ln P_0 - \frac{\beta_0}{\alpha}e^{-\alpha t}\right) = P_0 \exp\left(\frac{\beta_0}{\alpha}\left[1 - e^{-\alpha t}\right]\right).$$

C09S0M.063: If we substitute $P(0) = 10^6$ and $P'(0) = 3 \times 10^5$ into the differential equation

$$P'(t) = \beta_0 e^{-\alpha t}P(t),$$

we find that $\beta_0 = 0.3$. Hence the solution given in Problem 62 is

$$P(t) = P_0 \exp\left(\frac{0.3}{\alpha}\left[1 - e^{-\alpha t}\right]\right).$$

The fact that $P(6) = 2P_0$ now yields the equation

$$f(\alpha) = (0.3)\left(1 - e^{-6\alpha}\right) - \alpha\ln 2 = 0$$

for α. We apply the iterative formula of Newton's method,

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)},$$

with $f'(\alpha) = (1.8)e^{-6\alpha} - \ln 2$ and initial guess $\alpha_0 = 1$. Thereby we find that $\alpha_1 \approx 0.5381$, $\alpha_2 \approx 0.3926$, $\dots$, and $\alpha \approx \alpha_6 \approx 0.39148754$. Therefore the limiting cell population as $t \to +\infty$ is

$$P_0 \exp\left(\frac{\beta_0}{\alpha}\right) \approx (10^6)\exp\left(\frac{0.3}{0.39148754}\right) \approx 2.152 \times 10^6.$$

Therefore the tumor does not grow much further after six months. (—C.H.E.)

C09S0M.064: The characteristic equation has the repeated roots $r_1 = r_2 = -1$. Hence the differential equation has general solution

$$x_1(t) = (At + B)e^{-t}, \qquad \text{for which}$$
$$x_1'(t) = (A - At - B)e^{-t}.$$

It follows that $A = 1$ and $B = 0$, and thus that $x_1(t) = te^{-t}$.

C09S0M.065: If $k = 1 - 10^{-2n}$ where (without loss of generality) n is a positive integer, then the characteristic equation has distinct real roots

$$r_1 = -1 - 10^{-n} \quad \text{and} \quad r_2 = -1 + 10^{-n}.$$

Thus the differential equation has general solution

$$\begin{aligned} x_2(t) &= Ae^{r_1 t} + Be^{r_2 t}, \qquad \text{so that} \\ x_2'(t) &= r_1 Ae^{r_1 t} + r_2 Be^{r_2 t}. \end{aligned}$$

The initial conditions yield the simultaneous equations

$$\begin{aligned} A + B &= 0, \\ r_1 A + r_2 B &= 1, \end{aligned}$$

and it follows that

$$\begin{aligned} x_2(t) &= \frac{1}{r_2 - r_1}\left(e^{r_2 t} - e^{r_1 t}\right) \\ &= \frac{e^{-t}}{2 \cdot 10^{-n}}\left[\exp\left(10^{-n}t\right) - \exp\left(-10^{-n}t\right)\right] = 10^n e^{-t} \sinh\left(10^{-n}t\right). \end{aligned}$$

C09S0M.066: If $k = 1 + 10^{-2n}$ where (without loss of generality) n is a positive integer, then the characteristic equation has complex conjugate roots

$$r_1,\ r_2 = -1 \pm 10^{-n} i.$$

Accordingly, the general solution of the differential equation is

$$\begin{aligned} x_3(t) &= e^{-t}\left(A\cos 10^{-n}t + B\sin 10^{-n}t\right), \qquad \text{for which} \\ x_3'(t) &= e^{-t}\left(10^{-n}B\cos 10^{-n}t - 10^{-n}A\sin 10^{-n}t - A\cos 10^{-n}t - B\sin 10^{-n}t\right). \end{aligned}$$

The initial conditions yield $A = 0$ and $B = 10^n$, and hence

$$x_3(t) = 10^n e^{-t} \sin\left(10^{-n}t\right).$$

C09S0M.067: If $t > 0$ is fixed, then—by l'Hôpital's rule (with $w = 10^{-n}$ as the variable)—

$$\begin{aligned} \lim_{n\to\infty} x_2(t) &= \lim_{n\to\infty} 10^n e^{-t} \sinh\left(10^{-n}t\right) \\ &= \lim_{w\to 0^+} \frac{e^{-t}\sinh wt}{w} = \lim_{w\to 0^+} te^{-t}\cosh wt = te^{-t} = x_1(t). \end{aligned}$$

Similarly,

$$
\begin{aligned}
\lim_{n\to\infty} x_3(t) \;&=\; \lim_{n\to\infty} 10^n e^{-t} \sin\left(10^{-n} t\right) \\
&=\; \lim_{w\to 0^+} \frac{e^{-t}\sin wt}{w} \;=\; \lim_{w\to 0^+} te^{-t}\cos wt \;=\; te^{-t} \;=\; x_1(t).
\end{aligned}
$$

Section 10.1

C10S01.001: The given line with equation $y = -\frac{1}{2}x + \frac{5}{2}$ has slope $-\frac{1}{2}$, so the parallel line through $(1, -2)$ has equation $y + 2 = -\frac{1}{2}(x - 1)$; that is, $x + 2y + 3 = 0$.

C10S01.002: The equation $4y = 3x - 7$ of the given line may be rewritten in the form $y = \frac{3}{4}x - \frac{7}{4}$. Hence every perpendicular line has slope $-\frac{4}{3}$. Thus the perpendicular through the point $(-3, 2)$ has equation $y - 2 = -\frac{4}{3}(x + 3)$; that is, $4x + 3y + 6 = 0$.

C10S01.003: The radius of the circle terminating at $(3, -4)$ has slope $-\frac{4}{3}$. Hence the line L tangent to the circle at that point (because L is perpendicular to that radius) has equation $y + 4 = \frac{3}{4}(x - 3)$; that is, $3x - 4y = 25$.

C10S01.004: If $y^2 = x + 3$, then

$$2y\frac{dy}{dx} = 1, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{2y}.$$

Therefore the slope of the line L tangent to the given curve at $(6, -3)$ is $-\frac{1}{6}$. Hence an equation of L is $y + 3 = -\frac{1}{6}(x - 6)$; that is, $x + 6y + 12 = 0$.

C10S01.005: Given $x^2 + 2y^2 = 6$, we have by implicit differentiation

$$2x + 4y\frac{dy}{dx} = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = -\frac{x}{2y}.$$

Therefore the tangent to the given curve at $(2, -1)$ has slope 1, so the normal to the curve there has slope -1 and thus equation $y + 1 = (-1)(x - 2) = -x + 2$; that is, $x + y = 1$.

C10S01.006: The segment S with endpoints $A(-3, 2)$ and $B(5, -4)$ has slope

$$\frac{-4 - 2}{5 - (-3)} = -\frac{6}{8} = -\frac{3}{4}$$

and its midpoint is $(1, -1)$. Therefore the perpendicular bisector of S has equation $y + 1 = \frac{4}{3}(x - 1)$; that is, $4x - 3y = 7$.

C10S01.007: Given $x^2 + 2x + y^2 = 4$, complete the square in each variable to find that $x^2 + 2x + 1 + y^2 = 5$; that is, $(x + 1)^2 + (y - 0)^2 = 5$. Hence the circle has center $(-1, 0)$ and radius $\sqrt{5}$.

C10S01.008: Given $x^2 + y^2 - 4y = 5$, complete the square in each variable to find that $x^2 + y^2 - 4y + 4 = 9$; that is, $(x - 0)^2 + (y - 2)^2 = 9$. Hence the circle has center $(0, 2)$ and radius 3.

C10S01.009: Given $x^2 + y^2 - 4x + 6y = 3$, complete the square in each variable to find that

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 16; \qquad \text{that is,} \qquad (x - 2)^2 + (y + 3)^2 = 16.$$

Therefore the circle has center $(2, -3)$ and radius 4.

C10S01.010: Given $x^2 + y^2 + 8x - 6y = 0$, complete the square in each variable to find that

$$x^2 + 8x + 16 + y^2 - 6y + 9 = 25; \qquad \text{that is,} \qquad (x + 4)^2 + (y - 3)^2 = 25.$$

Thus the circle has center $(-4,\,3)$ and radius 5.

C10S01.011: Given $4x^2 + 4y^2 - 4x = 3,$ complete the square in each variable as follows:

$$x^2 + y^2 - x = \frac{3}{4}; \qquad x^2 - x + \frac{1}{4} + y^2 = 1; \qquad \left(x - \frac{1}{2}\right)^2 + (y - 0)^2 = 1.$$

Consequently the circle has center $\left(\frac{1}{2},\,0\right)$ and radius 1.

C10S01.012: Given $4x^2 + 4y^2 + 12y = 7,$ complete the square in each variable as follows:

$$x^2 + y^2 + 3y = \frac{7}{4}; \qquad x^2 + y^2 + 3y + \frac{9}{4} = 4; \qquad (x - 0)^2 + \left(y + \frac{3}{2}\right)^2 = 4.$$

Therefore the circle has center $\left(0,\,-\frac{3}{2}\right)$ and radius 2.

C10S01.013: Given $2x^2 + 2y^2 - 2x + 6y = 13,$ complete the square in each variable as follows:

$$x^2 + y^2 - x + 3y = \frac{13}{2}; \qquad x^2 - x + \frac{1}{4} + y^2 + 3y + \frac{9}{4} = 9; \qquad \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 9.$$

Thus the circle has center $\left(\frac{1}{2},\,-\frac{3}{2}\right)$ and radius 3.

C10S01.014: Given $9x^2 + 9y^2 - 12x = 5,$ complete the square in each variable as follows:

$$x^2 + y^2 - \frac{4}{3}x = \frac{5}{9}; \qquad x^2 - \frac{4}{3}x + \frac{4}{9} + y^2 = 1; \qquad \left(x - \frac{2}{3}\right)^2 + (y - 0)^2 = 1.$$

So the circle has center $\left(\frac{2}{3},\,0\right)$ and radius 1.

C10S01.015: Given $9x^2 + 9y^2 + 6x - 24y = 19,$ complete the square in each variable:

$$x^2 + y^2 + \frac{2}{3}x - \frac{8}{3}y = \frac{19}{9}; \qquad x^2 + \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{8}{3}y + \frac{16}{9} = 4; \qquad \left(x + \frac{1}{3}\right)^2 + \left(y - \frac{4}{3}\right)^2 = 4.$$

Therefore the circle has center $\left(-\frac{1}{3},\,\frac{4}{3}\right)$ and radius 2.

C10S01.016: Given $36x^2 + 36y^2 - 48x - 108y = 47,$ complete the square in each variable:

$$x^2 + y^2 - \frac{4}{3}x - 3y = \frac{47}{36}; \qquad x^2 - \frac{4}{3}x + \frac{4}{9} + y^2 - 3y + \frac{9}{4} = \frac{47}{36} + \frac{16}{36} + \frac{81}{36} = \frac{144}{36} = 4;$$
$$\left(x - \frac{2}{3}\right)^2 + \left(y - \frac{3}{2}\right)^2 = 4.$$

So this circle has center $\left(\frac{2}{3},\,\frac{3}{2}\right)$ and radius 2.

C10S01.017: Given $x^2 + y^2 - 6x - 4y = -13,$ complete the square in each variable:

$$x^2 - 6x + 9 + y^2 - 4y + 4 = -13 + 13 = 0; \qquad (x - 3)^2 + (y - 2)^2 = 0.$$

Therefore the graph of the given equation consists of the single point $(3,\,2)$.

C10S01.018: Given $2x^2 + 2y^2 + 6x + 2y = -5,$ complete the square in each variable:

$$x^2 + y^2 + 3x + y = -\frac{5}{2}; \qquad x^2 + 3x + \frac{9}{4} + y^2 + y + \frac{1}{4} = 0; \qquad \left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 0.$$

Therefore the graph of the given equation consists of the single point $\left(-\frac{3}{2}, -\frac{1}{2}\right)$.

C10S01.019: Given $x^2 + y^2 - 6x - 10y = -84$, complete the square in each variable:

$$x^2 - 6x + 9 + y^2 - 10y + 25 = -50; \qquad (x-3)^2 + (y-5)^2 = -50.$$

The graph of the given equation has no points on it.

C10S01.020: Given $9x^2 + 9y^2 - 6x - 6y = -11$, complete the square in each variable:

$$9x^2 - 6x + 1 + 9y^2 - 6x + 1 = -9; \qquad (3x-1)^2 + (3y-1)^2 = -9.$$

There are no points on the graph of the given equation.

C10S01.021: First use the distance formula to find that the radius of the circle is

$$\sqrt{(2-(-1))^2 + (3-(-2))^2} = \sqrt{3^2 + 5^2} = \sqrt{34}\,.$$

Thus an equation of the circle is $(x+1)^2 + (y+2)^2 = 34$.

C10S01.022: Suppose that $P(a,\, a+4)$ is the point where the circle is tangent to the line $y = x + 4$. This line has slope 1, so the radius terminating at P has slope -1. Therefore

$$\frac{a+4+2}{a-2} = -1; \qquad a + 6 = 2 - a; \qquad a = -2.$$

So $P = P(-2,\, 2)$. The distance formula then yields the fact that the radius of the circle is

$$\sqrt{(-2-2)^2 + (2+2)^2} = \sqrt{4^2 + 4^2} = \sqrt{32}\,.$$

Therefore an equation of the circle is $(x-2)^2 + (y+2)^2 = 32$.

C10S01.023: Suppose that $P(a, 2a-4)$ is the point at which the circle is tangent to the line $y = 2x - 4$. This line has slope 2, so the radius terminating at P has slope $-\frac{1}{2}$. Thus

$$\frac{2a-4-6}{a-6} = -\frac{1}{2}; \qquad 4a - 20 = 6 - a; \qquad a = \frac{26}{5}; \qquad 2a - 4 = \frac{32}{5}.$$

Then the distance formula yields the fact that the circle has radius

$$\sqrt{\left(6 - \frac{26}{5}\right)^2 + \left(6 - \frac{32}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{4}{25}} = \frac{2}{5}\sqrt{5}\,.$$

Hence an equation of the circle is $(x-6)^2 + (y-6)^2 = \frac{4}{5}$.

C10S01.024: The circle in question passes through the three points $A(4,\, 6)$, $B(-2,\, -2)$, and $C(5,\, -1)$. The elegant way to solve the problem is long: First find the perpendicular bisectors of AB and AC; that is, write their Cartesian equations. Solve these equations simultaneously to find the coordinates of the center O of the circle. Use the distance formula to find the length of the radius OA of the circle. Finally write

its equation. The alternative method we choose is probably a little shorter. We know that the circle has an equation of the form $(x-a)^2+(y-b)^2=r^2$. Because A, B, and C all lie on the circle, it follows that

$$\begin{aligned}(4-a)^2+(6-b)^2&=r^2,\\(-2-a)^2+(-2-b)^2&=r^2,\qquad\text{and}\\(5-a)^2+(-1-b)^2&=r^2.\end{aligned}$$

That is,

$$\begin{aligned}a^2-8a+16+b^2-12b+36&=r^2,\\a^2+4a+4+b^2+4b+4&=r^2,\qquad\text{and}\\a^2-10a+25+b^2+2b+1&=r^2.\end{aligned}$$

Thus

$$8a+12b-52=a^2+b^2-r^2, \tag{1}$$
$$-4a-4b-8=a^2+b^2-r^2,\qquad\text{and} \tag{2}$$
$$10a-2b-26=a^2+b^2-r^2. \tag{3}$$

So the left-hand sides of Eqs. (1) and (2) are equal, as are the left-hand sides of Eqs. (2) and (3). This means that

$$12a+16b-44=0\qquad\text{and}\qquad 14a+2b-18=0.$$

It follows easily that $a=1$ and $b=2$, and then from Eq. (1) [say] that $r=5$. Therefore an equation of the circle in question is $(x-1)^2+(y-2)^2=25$.

C10S01.025: The distance formula implies that $(x-3)^2+(y-2)^2=(x-7)^2+(y-4)^2$. Therefore

$$\begin{aligned}-6x+9-4y+4&=-14x+49-8y+16;\\8x+4y&=52.\end{aligned}$$

Hence $P(x,\,y)$ satisfies the equation $2x+y=13$. The locus is a straight line; it is in fact the perpendicular bisector of the line segment joining the two given points. We omit its graph because it would occupy space unnecessarily.

C10S01.026: The distance formula implies that the coordinates of $P(x,\,y)$ satisfy

$$\begin{aligned}\sqrt{(x+2)^2+(y-1)^2}&=\frac{1}{2}\sqrt{(x-4)^2+(y+2)^2};\\4(x^2+4x+4+y^2-2y+1)&=x^2-8x+16+y^2+4y+4;\\4x^2+16x+4y^2-8y+20&=x^2-8x+y^2+4y+20;\\3x^2+24x+3y^2-12y&=0;\\x^2+8x+y^2-4y&=0;\end{aligned}$$

$$x^2 + 8x + 16 + y^2 - 4y + 4 = 20;$$

$$(x + 4)^2 + (y - 2)^2 = 20.$$

All of the displayed equations are answers, some better than others; we prefer the last as it allows us to identify the locus of $P(x, y)$ as the circle with center $(-4, 2)$ and radius $2\sqrt{5}$. We omit the sketch of the graph of the locus because it occupies space unnecessarily.

C10S01.027: The distance formula implies that the coordinates of $P(x, y)$ satisfy

$$3\sqrt{(x-5)^2 + (y-10)^2} = \sqrt{(x+3)^2 + (y-2)^2};$$

$$9x^2 - 90x + 9y^2 - 180y + 1125 = x^2 + 6x + y^2 - 4y + 13;$$

$$8x^2 - 96x + 8y^2 - 176y + 1112 = 0;$$

$$x^2 - 12x + y^2 - 22y + 139 = 0;$$

$$x^2 - 12x + 36 + y^2 - 22y + 121 = 157 - 139 = 18;$$

$$(x - 6)^2 + (y - 11)^2 = 18.$$

Therefore the locus of $P(x, y)$ is the circle with center $(6, 11)$ and radius $3\sqrt{2}$. We omit a sketch as it would occupy space unnecessarily.

C10S01.028: The distance formula implies that the coordinates of $P(x, y)$ satisfy

$$x + 3 = \sqrt{(x-3)^2 + y^2};$$

$$x^2 + 6x + 9 = x^2 - 6x + 9 + y^2;$$

$$y^2 = 12x.$$

Thus the locus of $P(x, y)$ is a parabola with vertex $(0, 0)$, axis the positive x-axis, opening to the right. To see it, execute [say] the *Mathematica* command

```
ParametricPlot[ {(t∧2)/12, t}, {t, −5, 5} ];
```

C10S01.029: The distance formula tells us that the point $P(x, y)$ satisfies the equations

$$\sqrt{(x-4)^2 + y^2} + \sqrt{(x+4)^2 + y^2} = 10;$$

$$\sqrt{(x-4)^2 + y^2} = 10 - \sqrt{(x+4)^2 + y^2};$$

$$(x - 4)^2 + y^2 = 100 - 20\sqrt{(x+4)^2 + y^2} + (x + 4)^2 + y^2;$$

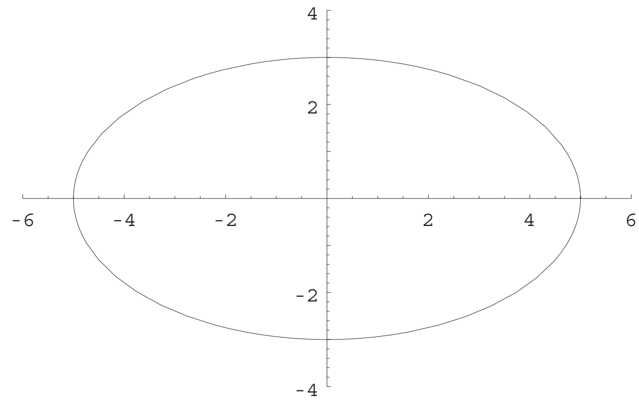
$$20\sqrt{(x+4)^2 + y^2} = 100 + 16x;$$

$$5\sqrt{(x+4)^2 + y^2} = 25 + 4x;$$

$$25x^2 + 200x + 400 + 25y^2 = 625 + 200x + 16x^2;$$

$$9x^2 + 25y^2 = 225.$$

This is an equation of the ellipse with center (0, 0), horizontal major axis of length 10, vertical minor axis of length 6, and intercepts (±5, 0) and (0, ±3). Its graph is shown next.



C10S01.030: The coordinates of $P(x, y)$ must, by the distance formula, satisfy

$$\begin{aligned}
\sqrt{x^2+(y-3)^2}+\sqrt{x^2+(y+3)^2} &= 10;\\
x^2+y^2-6y+9 &= 100-20\sqrt{x^2+(y+3)^2}\ +x^2+y^2+6y+9;\\
20\sqrt{x^2+(y+3)^2} &= 12y+100;\\
5\sqrt{x^2+(y+3)^2} &= 3y+25;\\
25x^2+25y^2+150y+225 &= 9y^2+150y+625;\\
25x^2+16y^2 &= 400.
\end{aligned}$$

This is an equation of the ellipse with center (0, 0), vertical major axis of length 10, horizontal minor axis of length 8, and with intercepts (±4, 0) and (0, ±5). To save space we omit its graph. To see its graph, execute [for example] the *Mathematica* command

```
ParametricPlot[ {4*cos[t], 5*sin[t]}, {t, 0, 2*Pi}, PlotRange -> {{-5,5}, {-6,6} ];
```

C10S01.031: Let $P(a, a^2)$ be a point at which such a line is tangent to the parabola. Then

$$2a=\frac{a^2-1}{a-2};\qquad 2a^2-4a=a^2-1;$$

$$a^2-4a+1=0;\qquad a=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}\,.$$

Thus there are two such lines, one with slope $4+2\sqrt{3}$ and one with slope $4-2\sqrt{3}$. Their equations are

$$y-1=\left(4+2\sqrt{3}\,\right)(x-2)\quad\text{and}\quad y-1=\left(4-2\sqrt{3}\,\right)(x-2).$$

C10S01.032: Let $P(a, a^2)$ be a point where a line through $(-1, 2)$ meets the parabola $y=x^2$ and is normal to it there. The slope of such a line is thus $-1/(2a)$, and thus:

$$-\frac{1}{2a} = \frac{a^2-2}{a+1}; \qquad -a-1 = 2a^3-4a;$$

$$2a^3-3a+1=0; \qquad (a-1)(2a^2+2a-1)=0;$$

$$a = 1 \quad \text{or} \quad a = \frac{-2\pm\sqrt{4+8}}{4} = \frac{-1\pm\sqrt{3}}{2}.$$

It follows that there are three such lines; their slopes are

$$-\frac{1}{2}, \qquad -\frac{1+\sqrt{3}}{2}, \qquad \text{and} \qquad \frac{-1+\sqrt{3}}{2}$$

and their equations may be written as

$$y-2 = -\frac{1}{2}(x+1), \quad y-2 = -\frac{1+\sqrt{3}}{2}(x+1), \quad \text{and} \quad y-2 = \frac{-1+\sqrt{3}}{2}(x+1).$$

Each of these three lines actually meets the parabola in two points but is normal to it at only one of the two. To see the parabola and the lines, execute [for example] the *Mathematica* command

```
Plot[ { x∧2, 2 - (x + 1)/2, 2 - (x + 1)*(1 + Sqrt[3])/2, 2 + (x + 1)*(Sqrt[3] - 1)/2 },
               { x, -3, 3 }, AspectRatio -> Automatic ];
```

C10S01.033: Every line parallel to the line with equation $y = 4x$ also has slope 4. Suppose that such a line is also normal to the graph of $y = 4/x$ at the point P. The tangent at $P(x, y)$ has slope $-4/(x^2)$, so the normal there has slope $x^2/4$, which must also equal 4. It follows that $x = \pm 4$, so there are two such lines, one containing the point $(4, 1)$ and the other containing the point $(-4, -1)$. Equations of these lines are $y - 1 = 4(x-4)$ and $y + 1 = 4(x+4)$.

C10S01.034: Every line parallel to $y = 3x - 5$ has slope 3. If such a line is also tangent to the graph of $y = x^3$ at the point (x, y), then $3x^2 = 3$, and thus $x = \pm 1$. Thus there are two such lines, one tangent to the graph of $y = x^3$ at the point $(1, 1)$, the other tangent at the point $(-1, -1)$. Their equations are $y - 1 = 3(x-1)$ and $y + 1 = 3(x+1)$. Each of these two lines also meets the graph of $y = x^3$ at a second point but is not tangent to the graph at that second point.

C10S01.035: Equation (11) is

$$x^2(1-e^2) - 2p(1+e^2)x + y^2 = -p^2(1-e^2).$$

If $e > 1$, then this equation may be written in the form

$$\begin{aligned}
&x^2 + 2p\frac{e^2+1}{e^2-1}x - \frac{y^2}{e^2-1} = -p^2;\\
&x^2 + 2p\frac{e^2+1}{e-1}x + p^2\left(\frac{e^2+1}{e^2-1}\right)^2 - \frac{y^2}{e^2-1} = -p^2 + p^2\left(\frac{e^2+1}{e^2-1}\right)^2;\\
&\qquad = p^2\left[\left(\frac{e^2+1}{e^2-1}\right)^2 - 1\right] = \frac{e^4+2e^2+1-e^4+2e^2-1}{(e^2-1)^2}p^2 = \frac{4e^2}{(e^2-1)^2}p^2.
\end{aligned}$$

Let

$$h = -p\frac{e^2+1}{e^2-1}.$$

Then Eq. (11) takes the form

$$x^2 - 2hx + h^2 - \frac{y^2}{e^2-1} = \frac{4e^2p^2}{(e^2-1)^2}.$$

Now let

$$a = \frac{2pe}{e^2-1}.$$

Then Eq. (11) simplifies to

$$x^2 - 2hx + h^2 - \frac{y^2}{e^2-1} = a^2;$$

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

Finally, let $b = a\sqrt{e^2-1}$. Then Eq. (11) further simplifies to

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$h = -p\cdot\frac{e^2+1}{e^2-1}, \quad a = \frac{2pe}{e^2-1}, \quad \text{and} \quad b = a\sqrt{e^2-1} = \frac{2pe}{\sqrt{e^2-1}}.$$

Section 10.2

C10S02.001: (a): $\left(\frac{1}{2}\sqrt{2},\,\frac{1}{2}\sqrt{2}\,\right)$; (b): $\left(1,\,-\sqrt{3}\,\right)$; (c): $\left(\frac{1}{2},\,-\frac{1}{2}\sqrt{3}\,\right)$; (d): $(0,\,-3)$; (e): $\left(\sqrt{2},\,-\sqrt{2}\,\right)$; (f): $\left(\sqrt{3},\,-1\right)$; (g): $\left(-\sqrt{3},\,1\right)$.

C10S02.002: (a): $\left(\sqrt{2},\,\frac{5}{4}\pi\right)$ and $\left(-\sqrt{2},\,\frac{1}{4}\pi\right)$; (b): $\left(2,\,-\frac{1}{6}\pi\right)$ and $\left(-2,\,\frac{5}{6}\pi\right)$; (c): $\left(2\sqrt{2},\,\frac{1}{4}\pi\right)$ and $\left(-2\sqrt{2},\,\frac{5}{4}\pi\right)$; (d): $\left(2,\,\frac{2}{3}\pi\right)$ and $\left(-2,\,\frac{5}{3}\pi\right)$; (e): $\left(2,\,-\frac{1}{4}\pi\right)$ and $\left(-2,\,\frac{3}{4}\pi\right)$; (f): $\left(2\sqrt{3},\,\frac{5}{6}\pi\right)$ and $\left(-2\sqrt{3},\,\frac{11}{6}\pi\right)$.

C10S02.003: $r\cos\theta = 4$: $r = 4\sec\theta$.

C10S02.004: $r\sin\theta = 6$: $r = 6\csc\theta,\ 0 < \theta < \pi$.

C10S02.005: $r\cos\theta = 3r\sin\theta$: $\tan\theta = \dfrac{1}{3}$; $\theta = \tan^{-1}\left(\dfrac{1}{3}\right)$.

C10S02.006: $r^2\cos^2\theta + r^2\sin^2\theta = 25$: $r = 5$.

C10S02.007: $r^2\sin\theta\,\cos\theta = 1$: $r^2 = \csc\theta\,\sec\theta$.

C10S02.008: $r^2(\cos^2\theta - \sin^2\theta) = 1$: $r^2 = \dfrac{1}{\cos 2\theta}$; $r^2 = \sec 2\theta$.

C10S02.009: $r\sin\theta = r^2\cos^2\theta$: $r = \sec\theta\,\tan\theta$. Note that no points on the graph are lost when r is cancelled.

C10S02.010: $r(\cos\theta + \sin\theta) = 4$: $r = \dfrac{4}{\cos\theta + \sin\theta}$.

C10S02.011: $r^2 = 9$: $x^2 + y^2 = 9$.

C10S02.012: $\theta = \frac{3}{4}\pi$: $\tan\theta = -1$; $\dfrac{r\sin\theta}{r\cos\theta} = -1$; $y = -x$.

C10S02.013: $r^2 = -5r\cos\theta$: $x^2 + y^2 + 5x = 0$.

C10S02.014: $r = 2\sin\theta\,\cos\theta$: $r^3 = 2(r\sin\theta)(r\cos\theta)$; $\pm(x^2+y^2)^{3/2} = 2xy$; $(x^2+y^2)^3 = 4x^2y^2$.

C10S02.015: $r = 1 - \cos 2\theta$: $r = 2\cdot\dfrac{1-\cos 2\theta}{2} = 2\sin^2\theta$; $r^3 = 2(r\sin\theta)^2$; $(x^2+y^2)^{3/2} = 2y^2$; $(x^2+y^2)^3 = 4y^4$.

C10S02.016: $r = 2 + \sin\theta$:

$$\begin{aligned}
r^2 &= 2r + r\sin\theta;\\
(x^2+y^2)^2 &= \pm 2(x^2+y^2)^{1/2} + y;\\
(x^2+y^2-y)^2 &= 4(x^2+y^2);\\
x^4 + 2x^2y^2 + y^4 - 2x^2y - 2y^3 + y^2 &= 4x^2 + 4y^2;\\
x^4 + 2x^2y^2 + y^4 &= 4x^2 + 3y^2 + 2x^2y + 2y^3.
\end{aligned}$$

C10S02.017: $r = 3\sec\theta$: $r\cos\theta = 3$; $x = 3$.

C10S02.018: $r^2 = \cos 2\theta = \cos^2\theta - \sin^2\theta$: $r^4 = (r\cos\theta)^2 - (r\sin\theta)^2$; $(x^2+y^2)^2 = x^2 - y^2$.

C10S02.019: $x = 2$; $r = 2\sec\theta$.

C10S02.020: $y = 3$; $r = 3\csc\theta$.

C10S02.021: $y + 1 = (-1)\cdot(x-2)$; thus $x + y = 1$. $r(\cos\theta + \sin\theta) = 1$: $r = \dfrac{1}{\cos\theta + \sin\theta}$.

C10S02.022: $y - 2 = x - 4$: $x - y = 2$. $r = \dfrac{2}{\cos\theta - \sin\theta}$.

C10S02.023: $y - 3 = x - 1$: $x - y + 2 = 0$; $r = \dfrac{2}{\sin\theta - \cos\theta}$.

C10S02.024: $(x-3)^2 + y^2 = 9$: $x^2 - 6x + y^2 = 0$. $r^2 = 6r\cos\theta$: $r = 6\cos\theta$.

C10S02.025: $x^2 + (y+4)^2 = 16$: $x^2 + y^2 + 8y = 0$. $r^2 + 8r\sin\theta = 0$: $r + 8\sin\theta = 0$.

C10S02.026: $(x-3)^2 + (y-4)^2 = 25$: $x^2 - 6x + y^2 - 8y = 0$. $r^2 = 6r\cos\theta + 8r\sin\theta$: $r = 6\cos\theta + 8\sin\theta$.

C10S02.027: $(x-1)^2 + (y-1)^2 = 2$: $x^2 - 2x + y^2 - 2y = 0$. $r^2 = 2r(\cos\theta + \sin\theta)$: $r = 2\cos\theta + 2\sin\theta$.

C10S02.028: The radius of the circle is $\sqrt{4^2+3^2} = 5$. Therefore a Cartesian equation of the circle is $(x-5)^2 + (y+2)^2 = 25$; that is, $x^2 - 10x + y^2 + 4y + 4 = 0$. A polar equation of the circle is $r^2 = 10r\cos\theta - 4r\sin\theta - 4$.

C10S02.029: Given: $r^2 = -4r\cos\theta$. Thus $x^2 + 4x + y^2 = 0$; that is, $(x+2)^2 + (y-0)^2 = 4$. This is an equation of the circle with center $(-2, 0)$ and radius 2, and thus it's the one shown in Fig. 9.2.23.

C10S02.030: Given: $r = 5\cos\theta + 5\sin\theta$. Thus $x^2 - 5x + y^2 - 5y = 0$;

$$x^2 - 5x + \frac{25}{4} + y^2 - 5y + \frac{25}{4} = \frac{25}{2};$$

$$\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{5}{2}\sqrt{2}.$$

Therefore we are given the equation of a circle through the origin with center at $\left(\frac{5}{2}, \frac{5}{2}\right)$, the circle shown in Fig. 10.2.21.

C10S02.031: Given: $r = -4\cos\theta + 3\sin\theta$. Thus $r^2 = -4r\cos\theta + 3r\sin\theta$;

$$x^2 + 4x + y^2 - 3y = 0;$$

$$(x+2)^2 + \left(y - \frac{3}{2}\right)^2 = 4 + \frac{9}{4} = \frac{25}{4}.$$

Thus we are given the equation of a circle with center $\left(-2, \frac{3}{2}\right)$, radius $\frac{5}{2}$, and passing through the origin. This is the circle shown in Fig. 10.2.24.

C10S02.032: Given: $r = 8\cos\theta - 15\sin\theta$. Then $r^2 = 8r\cos\theta - 15r\sin\theta$;

$$x^2 - 8x + y^2 + 15y = 0;$$

$$(x-4)^2 + \left(y - \frac{15}{2}\right)^2 = 16 + \frac{225}{4} = \frac{289}{4}.$$

The graph is a circle with center at $\left(4, -\frac{15}{2}\right)$, with radius $\frac{17}{2}$, and passing through the origin. Its graph appears in Fig. 10.2.22.

C10S02.033: First note that $r = 0$ when $6\cos\theta = 8$; that is, never. Also r is maximal when $\theta = 0$, for which $r = 14$; r is minimal when $\theta = \pi$, for which $r = 2$. So the graph of this limaçon is the one shown in Fig. 10.2.26.

C10S02.034: First note that $r = 0$ when $\cos\theta = -1$; that is, when $\theta = \pi$. Also $r \geqq 0$ for all θ. So the graph of this limaçon is the one shown in Fig. 10.2.28.

C10S02.035: The maximum value of r is 14, which occurs when $\theta = 0$; the minimum value of r is -4, which occurs when $\theta = \pi$. So the graph of this limaçon is the one shown in Fig. 10.2.25.

C10S02.036: The maximum value of r is $r(0) = 14$ and the minimum value of r is $r(\pi) = -8$. Hence the graph of this limaçon is the one shown in Fig. 10.2.27.

C10S02.037: Assume that $a^2 + b^2 \neq 0$; that is, that neither a nor b is zero. Suppose that $r = a\cos\theta + b\sin\theta$. Then

$$r^2 = ar\cos\theta + br\sin\theta;$$

$$x^2 - ax + y^2 - by = 0;$$

$$x^2 - ax + \frac{1}{4}a^2 + y^2 - by + \frac{1}{4}b^2 = \frac{a^2 + b^2}{4};$$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

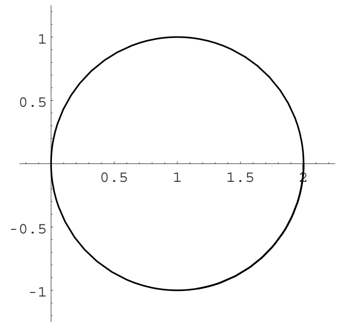
Therefore the graph of $r = a\cos\theta + b\sin\theta$ is a circle with center $\left(\frac{1}{2}a, \frac{1}{2}b\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

C10S02.038: Assume that $0 < a < b$. Suppose that $r = a + b\cos\theta$. Then $r = 0$ when $\cos\theta = -a/b$; that is, when

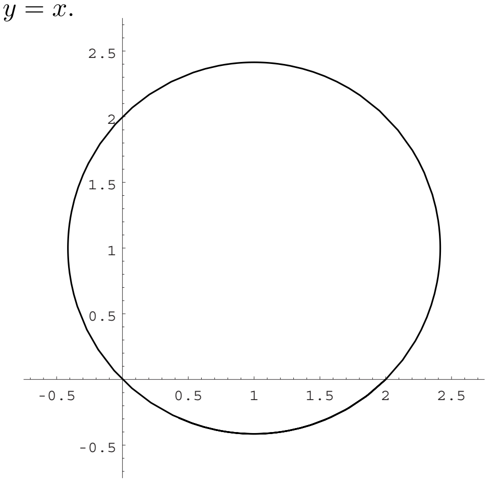
$$\theta = \alpha = \cos^{-1}\left(-\frac{a}{b}\right) \qquad \text{and when} \qquad \theta = \beta = 2\pi - \cos^{-1}\left(-\frac{a}{b}\right).$$

Moreover, $r < 0$ if $\alpha < \theta < \beta$, so the graph has an inner loop. The maximum value of r is $r(0) = a + b$ and the minimum value of r is $r(\pi) = a - b < 0$.

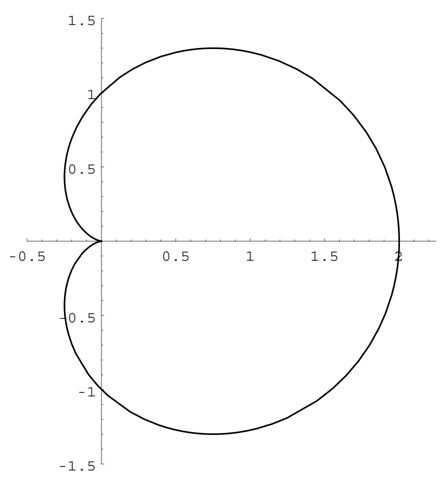
C10S02.039: The graph of the circle with polar equation $r = 2\cos\theta$ is shown next. This graph is symmetric around the x-axis.



C10S02.040: The graph of the circle with polar equation $r = 2\cos\theta + 2\sin\theta$ is shown next. This graph is symmetric around the line $y = x$.

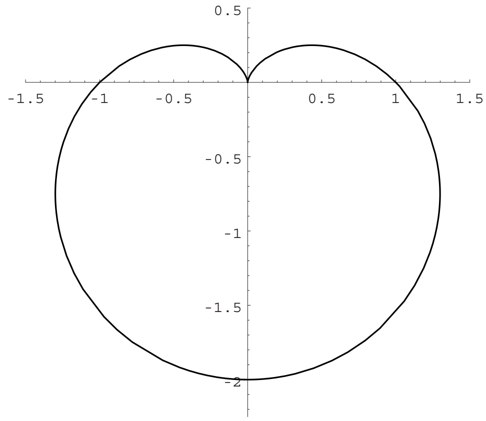


C10S02.041: The graph of the cardioid with polar equation $r = 1 + \cos\theta$ is shown next. This graph is symmetric around the x-axis.

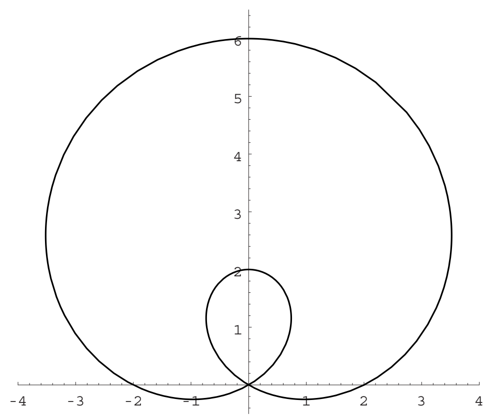


C10S02.042: The graph of the cardioid with polar equation $r = 1 - \sin\theta$ is shown next. This graph is

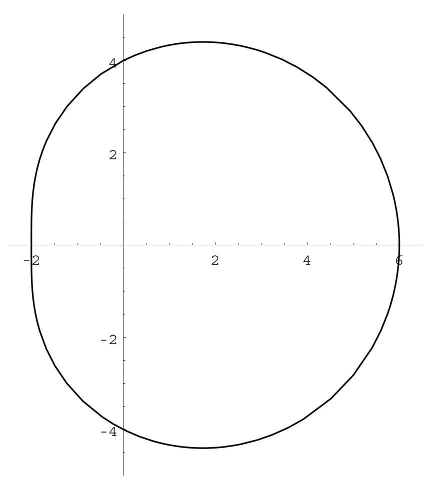
symmetric around the y-axis.



C10S02.043: The graph of the limaçon with polar equation $r = 2 + 4\sin\theta$ is shown next. This graph is symmetric around the y-axis.

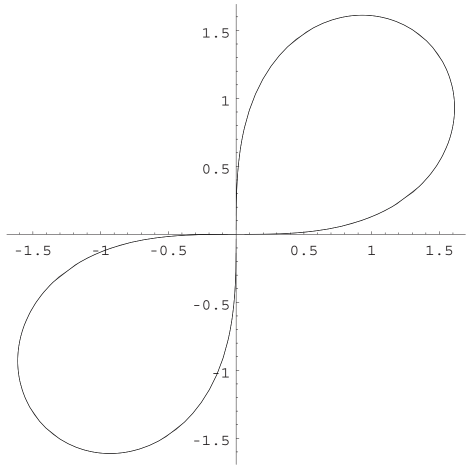


C10S02.044: The graph of the limaçon with polar equation $r = 4 + 2\cos\theta$ is shown next. This graph is symmetric around the x-axis.

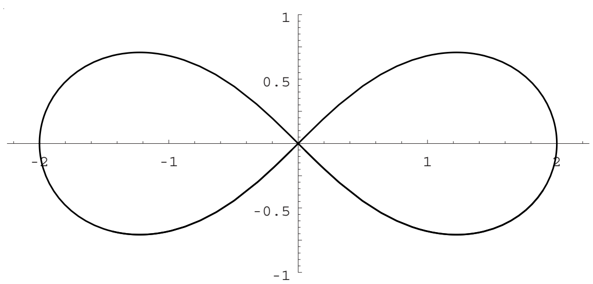


C10S02.045: The graph of the lemniscate with polar equation $r^2 = 4\sin 2\theta$ is shown next. This graph is symmetric around the line $y = x$, around the line $y = -x$, and around the pole (meaning that (x, y) is on

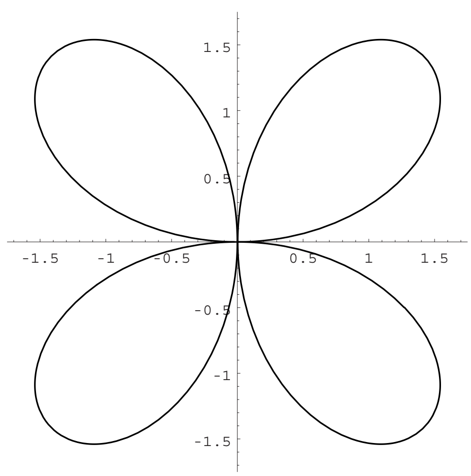
the graph if and only if $(-x,\,-y)$ is on the graph).



C10S02.046: The graph of the lemniscate with polar equation $r^2 = 4\cos 2\theta$ is shown next. This graph is symmetric around both coordinate axes and around the pole.

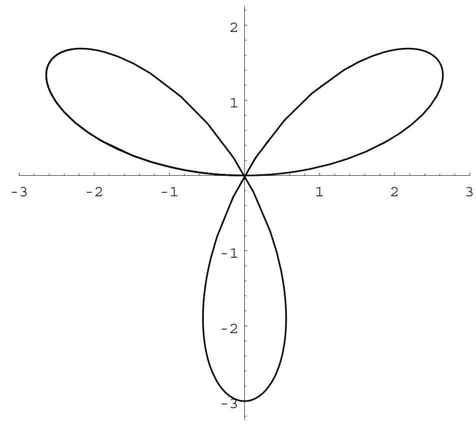


C10S02.047: The graph of the four-leaved rose with polar equation $r = 2\sin 2\theta$ is shown next. This graph is symmetric around both coordinate axes, around both lines $y = x$ and $y = -x$, and around the pole.

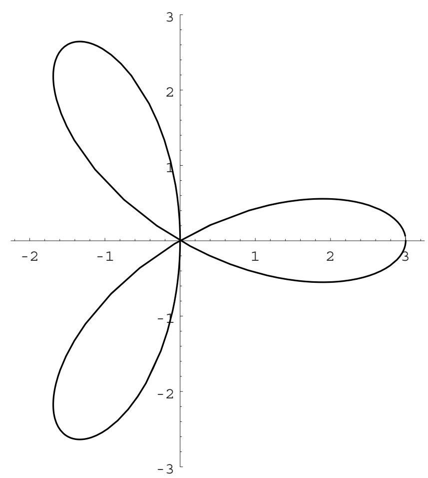


C10S02.048: The graph of the three-leaved rose with polar equation $r = 3\sin 3\theta$ is shown next. This

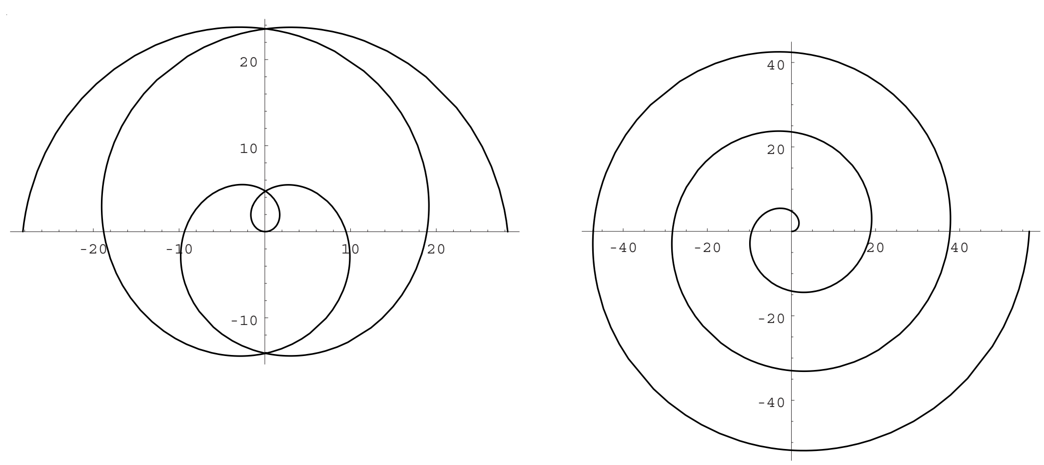
graph is symmetric around the y-axis. It is also invariant under a rotation of $120°$ around the origin.



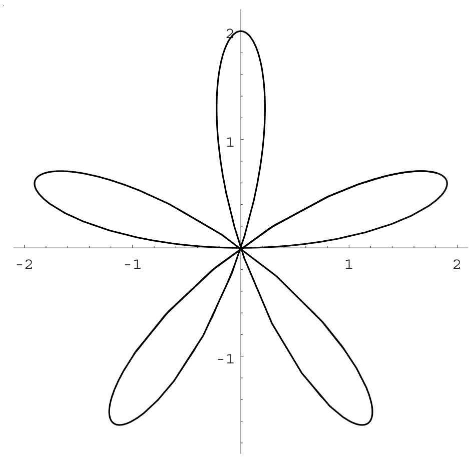
C10S02.049: The graph of the three-leaved rose with polar equation $r = 3\cos 3\theta$ is shown next. This graph is symmetric around the x-axis. It is also unchanged if it is rotated any integral multiple of $120°$ around the origin.



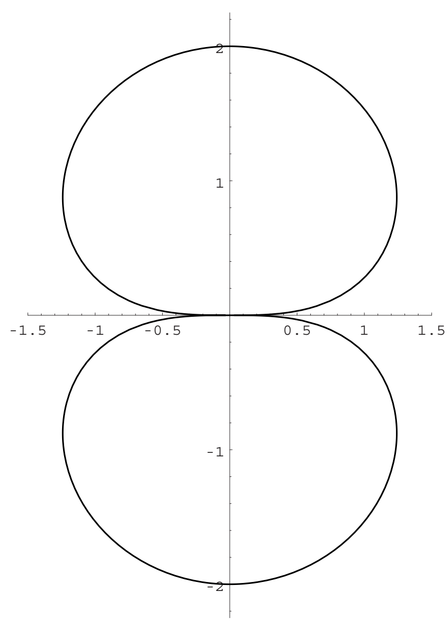
C10S02.050: The graph of the spiral of Archimedes with polar equation $r = 3\theta$ is shown next. The graph on the left shows the spiral for $-3\pi \leqq \theta \leqq 3\pi$. This graph is symmetric around the y-axis. The graph on the right shows the spiral for $0 \leqq \theta \leqq 6\pi$. The latter graph has no symmetries.



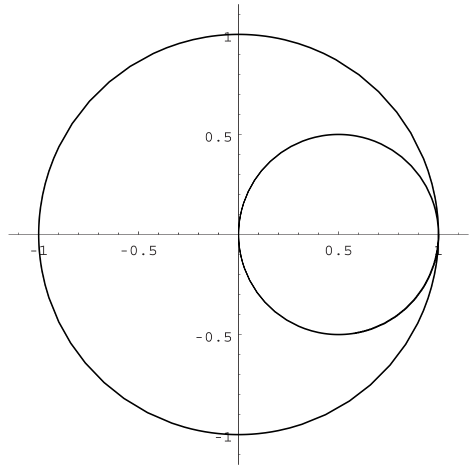
C10S02.051: The graph of the five-leaved rose with polar equation $r = 2\sin 5\theta$ is shown next. This graph is symmetric around the y-axis and is also unchanged if rotated any integral multiple of $72°$ around the origin.



C10S02.052: The graph of the figure eight with polar equation $r^2 = 4\sin\theta$ is shown next. This graph is symmetric around both coordinate axes.

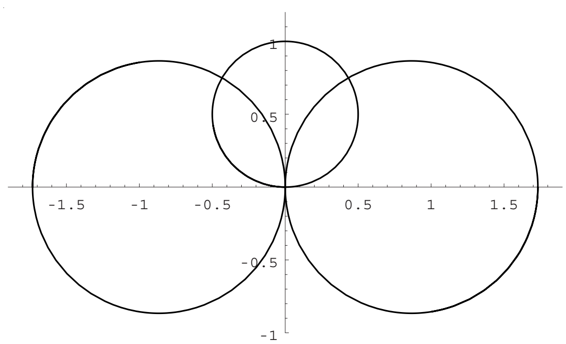


C10S02.053: The graphs of the polar equations $r = 1$ and $r = \cos\theta$ are shown next.



Solving $\cos\theta = 1$ yields $\theta = 0$. This corresponds to the point with polar coordinates $(1,\ 0)$ on both the large circle and the small circle. The graph makes it clear that there are almost certainly no other solutions. A rigorous demonstration that there are no other solutions is possible but would take this discussion too far afield.

C10S02.054: The graphs of the circle with polar equation $r = \sin\theta$ and the "double circle" with polar equation $r^2 = 3\cos^2\theta$ are shown next.



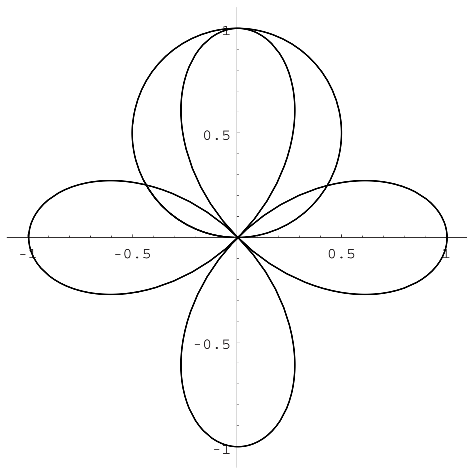
To find some of the points of intersection, we solve

$$\sin^2\theta = 3\cos^2\theta; \quad \tan^2\theta = 3; \quad \tan\theta = \pm\sqrt{3}\,.$$

Thus we obtain $\theta = \pm\frac{1}{3}\pi$, $\theta = \pm\frac{2}{3}\pi$. Thereby we find some of the points of intersection of the two graphs: They meet at the point in the first quadrant with polar coordinates $\left(\frac{1}{2}\sqrt{3},\ \frac{1}{3}\pi\right)$; they meet at the point in the second quadrant with polar coordinates $\left(\frac{1}{2}\sqrt{3},\ \frac{2}{3}\pi\right)$. They also meet at the pole, which lies on the small circle because it has polar coordinates $(0,\ 0)$ and which also lies on the "double circle" because it has polar coordinates $\left(0,\ \frac{1}{2}\pi\right)$. There are no other solutions. To verify this assertion, show that all three curves in the figure are circles, then prove rigorously the lemma to the effect that two circles that do not coincide can meet in no more than two points.

C10S02.055: The graphs of the circle with polar equation $r = \sin\theta$ and the four-leaved rose with polar

equation $r = \cos 2\theta$ are shown next.



We find some of the points where they meet by solving their equations simultaneously as follows:

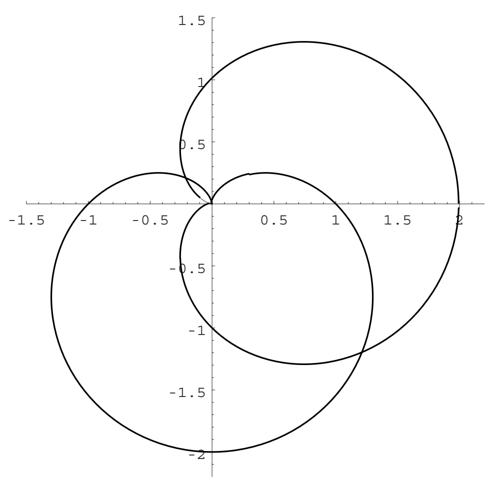
$$\sin\theta = \cos 2\theta; \qquad \sin\theta = 1 - 2\sin^2\theta;$$

$$2\sin^2\theta + \sin\theta - 1 = 0; \qquad (2\sin\theta - 1)(\sin\theta + 1) = 0;$$

$$\sin\theta = \frac{1}{2} \quad \text{or} \quad \sin\theta = -1; \qquad \theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ \text{or}\ \frac{3\pi}{2}.$$

Thus the curves meet at the points with polar coordinates $\left(\frac{1}{2},\ \frac{1}{6}\pi\right)$ and $\left(\frac{1}{2},\ \frac{5}{6}\pi\right)$. They also meet at the point with polar coordinates $\left(-1,\ \frac{3}{2}\pi\right)$ because it also has polar coordinates $\left(1,\ \frac{1}{2}\pi\right)$. Finally, they also meet at the pole because it has polar coordinates $(0,\ 0)$ as well as polar coordinates $\left(0,\ \frac{1}{4}\pi\right)$.

C10S02.056: The graphs of the cardioids with polar equations $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$ are shown next.

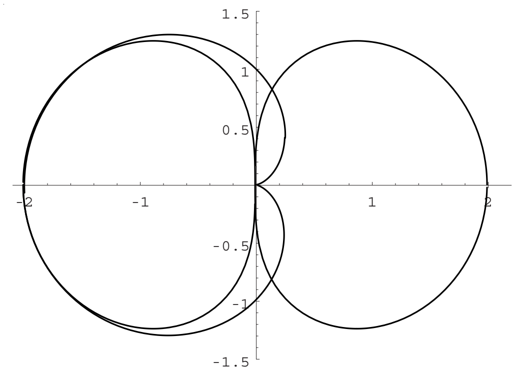


Simultaneous solution of their equations yields $\tan\theta = -1$, which has the two solutions $\theta = \frac{3}{4}\pi$ and $\theta = \frac{7}{4}\pi$ for $0 \leqq \theta \leqq 2\pi$. Thus we find that the cardioids meet at the two points with polar coordinates

$$\left(1 - \frac{1}{2}\sqrt{2},\ \frac{3}{4}\pi\right) \quad \text{and} \quad \left(1 + \frac{1}{2}\sqrt{2},\ \frac{7}{4}\pi\right).$$

They also meet at the pole, which lies on the first cardioid because it has polar coordinates $(0,\ \pi)$ and on the second cardioid because it has polar coordinates $\left(0,\ \frac{1}{2}\pi\right)$. There are no other solutions.

C10S02.057: The graphs of the cardioid with polar equation $r = 1 - \cos\theta$ and the double oval with polar equation $r^2 = 4\cos\theta$ are shown next.



We solve their equations simultaneously as follows:

$$(1 - \cos\theta)^2 = 4\cos\theta; \qquad 1 - 2\cos\theta + \cos^2\theta = 4\cos\theta;$$

$$\cos^2\theta - 6\cos\theta + 1 = 0; \qquad \cos\theta = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}\,;$$

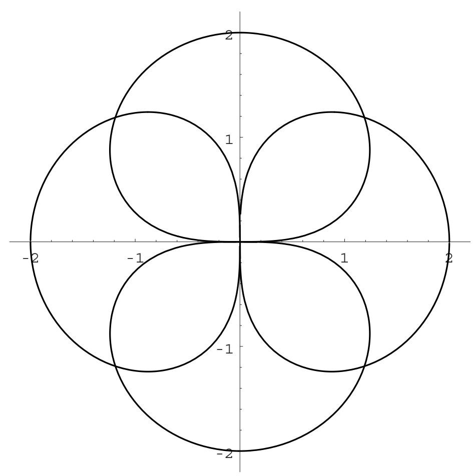
$$\cos\theta = 3 - 2\sqrt{2}\,; \qquad \theta = \pm\cos^{-1}\left(3 - 2\sqrt{2}\,\right).$$

Thus we obtain two of the four points of intersection: the points with polar coordinates

$$\left(2\sqrt{2} - 2,\ \cos^{-1}\left(3 - 2\sqrt{2}\,\right)\right) \qquad \text{and} \qquad \left(2\sqrt{2} - 2,\ -\cos^{-1}\left(3 - 2\sqrt{2}\,\right)\right).$$

The curves also meet at $(0, 0)$ because it also has polar coordinates $\left(0, \frac{1}{2}\pi\right)$. Moreover, they meet at the point $(2, \pi)$ because it also has polar coordinates $(-2, 0)$. There are no other points of intersection.

C10S02.058: The graphs of the two double ovals with polar equations $r^2 = 4\sin\theta$ and $r^2 = 4\cos\theta$ are shown next.



Simultaneous solution of the two equations yields $\sin\theta = \pm\cos\theta$, so that $\tan\theta = \pm 1$. Thus solutions seem to consist of odd integral multiples of $\frac{1}{4}\pi$. Certainly there are no other possibilities for θ. Let's examine the four of these between 0 and 2π.

Case 1: $\theta = \frac{1}{4}\pi$. Then $4\sin\theta = \sqrt{8}$ and $4\cos\theta = \sqrt{8}$. So we find the two points of intersection in the first and third quadrants:

$$\left(8^{1/4},\ \tfrac{1}{4}\pi\right) \qquad \text{and} \qquad \left(-8^{1/4},\ \tfrac{1}{4}\pi\right).$$

Case 2: $\theta = \frac{3}{4}\pi$. Then $4\sin\theta = \sqrt{8}$ but $4\cos\theta = -\sqrt{8}$. So we find that the two points in the second and fourth quadrants

$$\left(8^{1/4},\ \tfrac{3}{4}\pi\right) \qquad \text{and} \qquad \left(-8^{1/4},\ \tfrac{3}{4}\pi\right)$$

satisfy the equation $r^2 = 4\sin\theta$ but not the equation $r^2 = 4\cos\theta$.

Case 3: $\theta = \frac{5}{4}\pi$. Then both $4\sin\theta$ and $4\cos\theta$ are negative, so we find no solutions in this case.

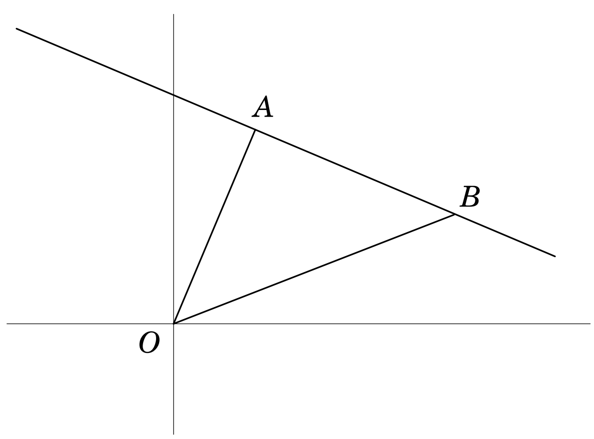
Case 4: $\theta = \frac{7}{4}\pi$. Then $4\cos\theta = 8$ but $4\sin\theta = -8$, so we find that the two points in the second and fourth quadrants

$$\left(8^{1/4},\ \tfrac{7}{4}\pi\right) \qquad \text{and} \qquad \left(-8^{1/4},\ \tfrac{7}{4}\pi\right)$$

satisfy the equation $r^2 = 4\cos\theta$ but not the equation $r^2 = 4\sin\theta$. But re-examine Case 2: The two points from Case 2, which have polar coordinates that satisfy the equation $r^2 = 4\sin\theta$, also have polar coordinates that satisfy the equation $r^2 = 4\cos\theta$. So we have found four solutions after all.

In addition, there is a fifth solution; the pole has coordinates $(0,\ 0)$ and also coordinates $\left(0,\ \frac{1}{4}\pi\right)$, and thus is a simultaneous solution of the original two equations. The four points so obviously solutions when one examines the graph are also solutions for clear algebraic reasons. The origin, also a fifth solution obvious upon inspection of the graph, isn't obtained by ordinary algebraic methods. And there is no sixth solution.

C10S02.059: The following figure shows a typical situation of the sort described in this problem: The point $A(p,\ \alpha)$ lies in the plane, the line L passes through A and is perpendicular to the line segment OA from the pole O to A. Let $B(r,\ \theta)$ be the polar coordinates of a typical point of L. Then OAB is a right triangle.



Moreover, the acute angle of this triangle at O is $\alpha - \theta$, so when we project the hypotenuse of length r onto the side OA of length p, we find that

$$r\cos(\alpha - \theta) = p;$$

$$r(\cos\alpha\ \cos\theta + \sin\alpha\ \sin\theta) = p;$$

$$r = \frac{p}{\cos\alpha\ \cos\theta + \sin\alpha\ \sin\theta}.$$

A rectangular coordinates equation of L can now be obtained from the second displayed equation here as follows:

$$(\cos\alpha)(r\cos\theta) + (\sin\alpha)(r\sin\theta) = p,$$

and therefore L has rectangular equation $x\cos\alpha + y\sin\alpha = p$.

C10S02.060: Given $r = 1 - \cos\theta$, we obtain

$$\begin{aligned} r^2 &= r - r\cos\theta; & r^2 &= r - x;\\ x^2 + y^2 + x &= r; & (x^2+y^2+x)^2 &= x^2+y^2;\\ x^4 + 2x^2y^2 + y^2 + 2x^3 + 2xy^2 + x^2 &= x^2 + y^2; & x^4 + 2x^2y^2 + y^2 + 2x^3 + 2xy^2 &= y^2;\\ (x^2+y^2)^2 + 2x(x^2+y^2) &= y^2. \end{aligned}$$

C10S02.061: Beginning with $a^2(x^2+y^2) = (x^2+y^2-by)^2$, we convert to polar coordinates:

$$\begin{aligned} a^2r^2 &= (r^2 - br\sin\theta)^2; & ar &= \pm(r^2 - br\sin\theta);\\ a &= \pm(r - b\sin\theta); & r - b\sin\theta &= \pm a. \end{aligned}$$

Therefore $r = \pm a + b\sin\theta$. If $|a| = |b|$ and neither is zero, then the graph is a cardioid. If $|a| \neq |b|$ and neither a nor b is zero, then the graph is a limaçon. If either a or b is zero and the other is not, then the graph is a circle. If $a = b = 0$ then the graph consists of the pole alone.

C10S02.062: The graphs of $r = 1 + \cos\theta$ and $r = -1 + \cos\theta$ are identical. Here's why. Suppose that $(p,\,\alpha)$ are the polar coordinates of a point on the first cardioid. Then $(-p,\,\alpha+\pi)$ also lies on the first cardioid. But

$$-1 + \cos(\alpha+\pi) = -1 - \cos\alpha = -(1+\cos\alpha) = -p.$$

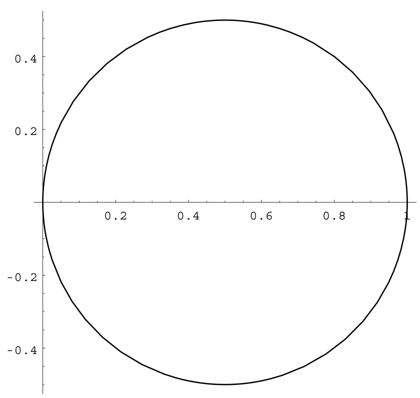
Thus $(-p,\,\alpha+\pi)$ lies on the second cardioid. Therefore $(p,\,\alpha)$ lies on the second cardioid. The argument may be reversed to show that every point on the second cardioid also lies on the first cardioid. That's why their graphs are identical.

C10S02.063: The behavior of the graph of the polar equation $r = \cos(p\theta/q)$ (where p and q are positive integers) depends strongly on the values of p and q. Without loss of generality we may suppose that p and q have no integral factor in common larger than 1. First you should determine what is meant by a "loop" and what is meant by "overlapping loops." The minimum value of the positive integer k required to show the entire graph on the interval $[0,\,k\pi]$ appears in the following table (for the values of p and q we found practical to use). The table is followed by *Mathematica*-generated graphs of $r = \cos(p\theta/q)$ for various values of p and q. The values of p and q are given beneath each graph in the form of the ordered pair $(p,\,q)$. You can probably deduce the way in which the loops depend on p and q with the aid of a little patience and imagination.

In the table, the values of q appear in the first row, in boldface; those of p appear in the first column. The values of k that we deduced in constructing the figures appear in the body of the table.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**13**	**15**
1	1	4	3	8	5	12	7	16					
2	2		6		10		14		18		22	26	30
3	1	4		8	5		7	16		20	11		
4	2		6		10		14		18		22	26	30
5	1	4	3	8		12	7	16	9				
6	2				10		14				22	26	
7	1	4	3	8	5	12		16	9				

And here are the graphs.



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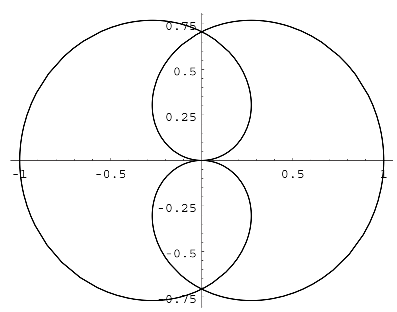

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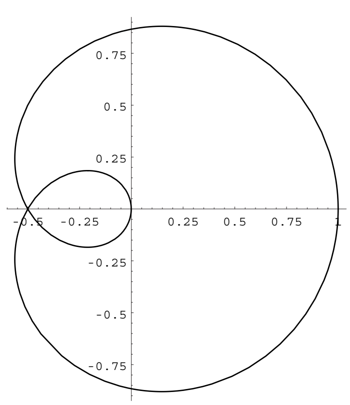

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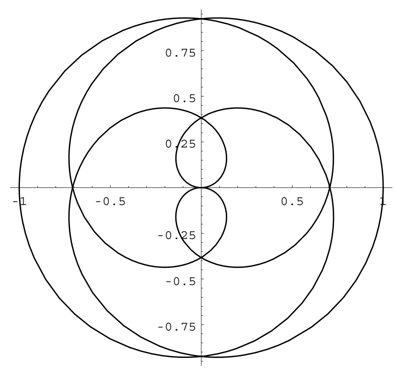

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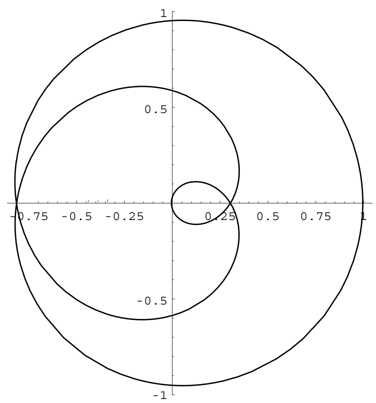

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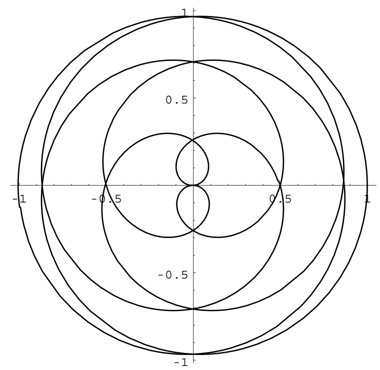

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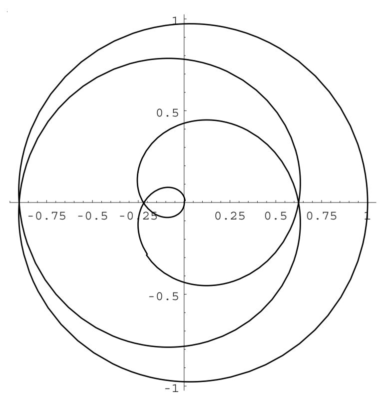

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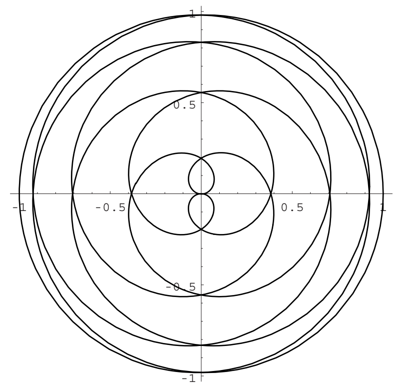

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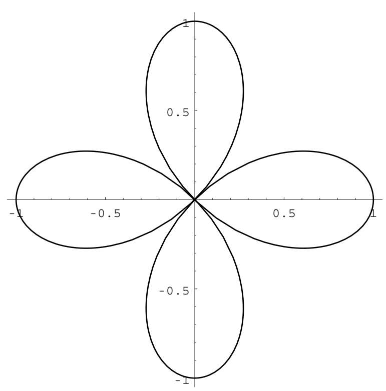

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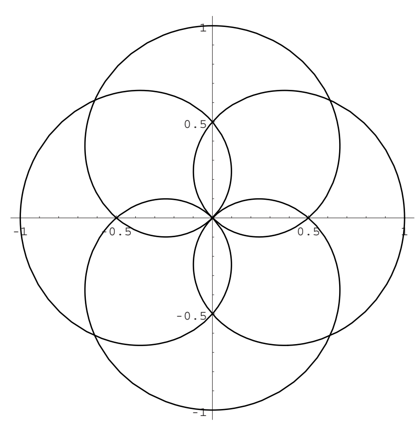

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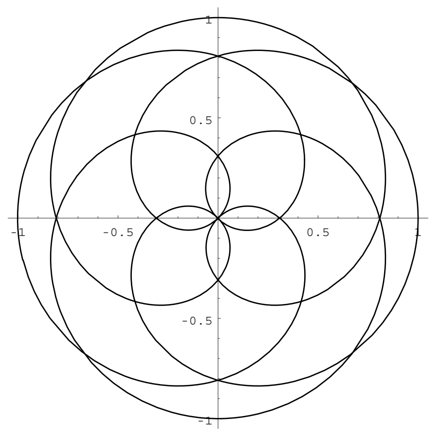

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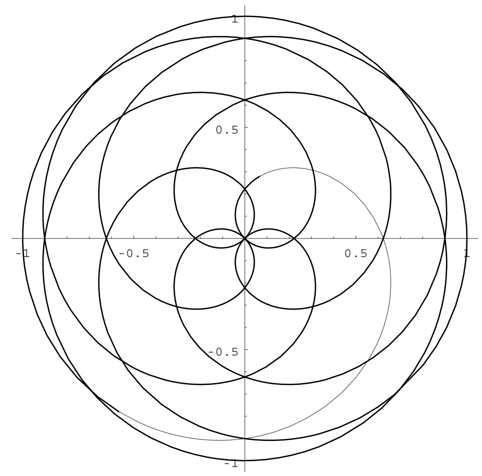

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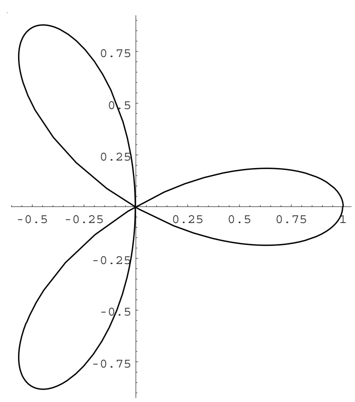

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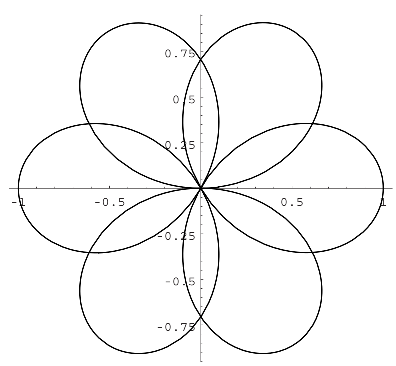

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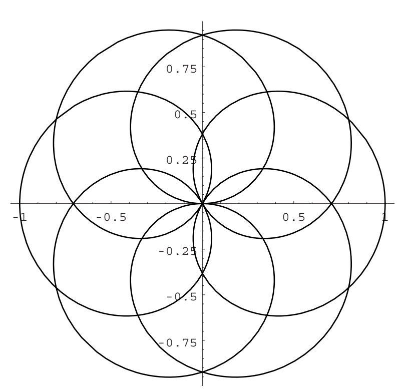

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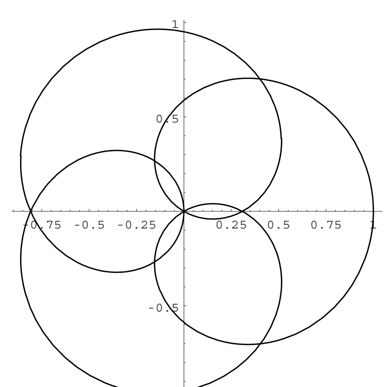

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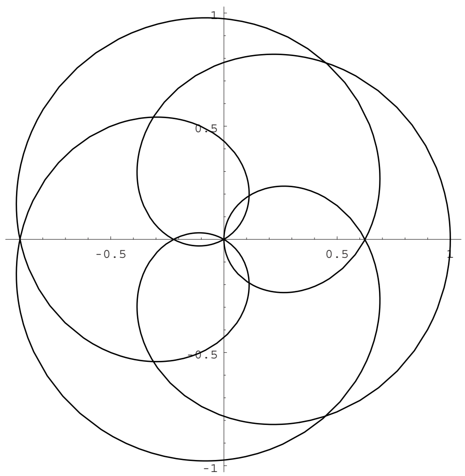

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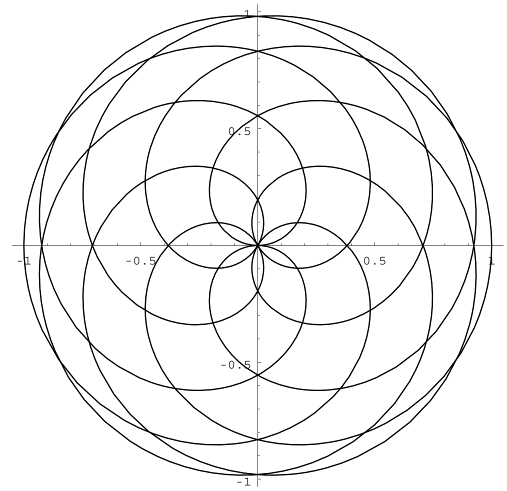

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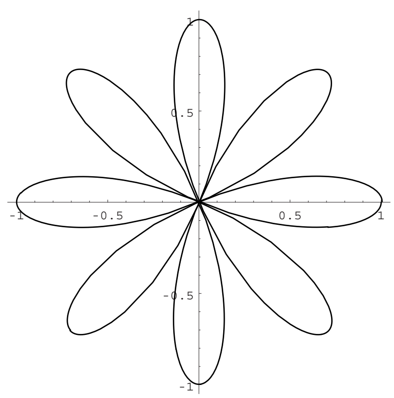

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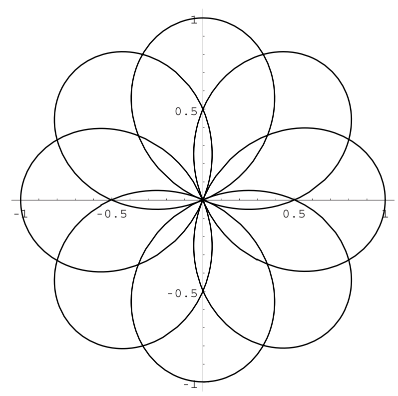

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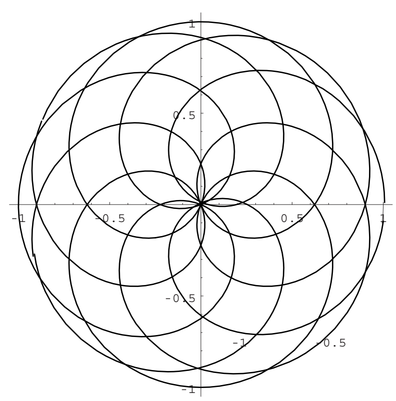

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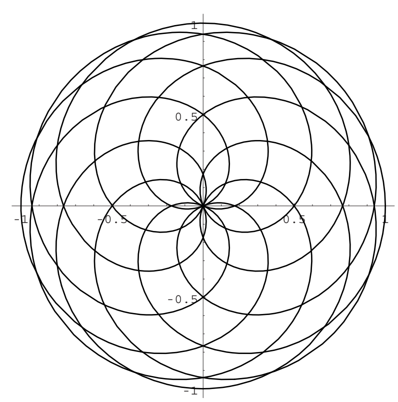

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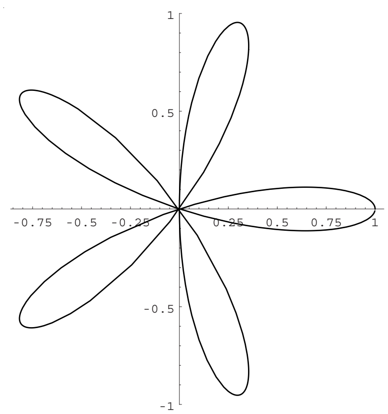

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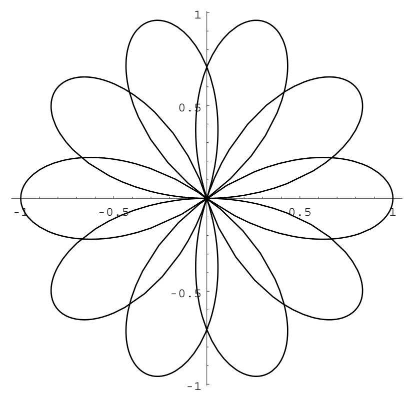

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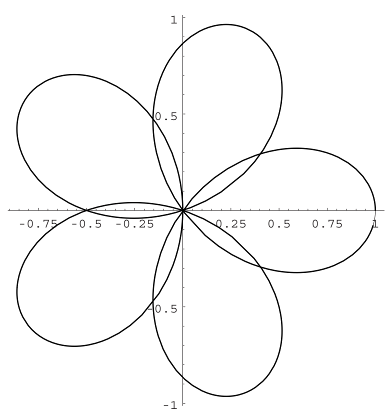

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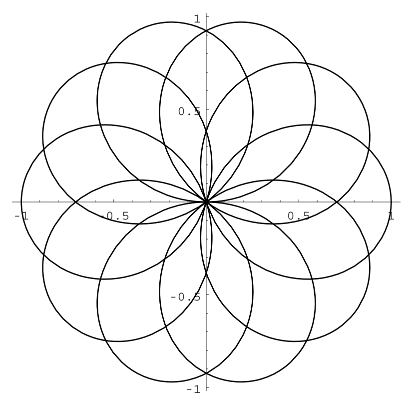

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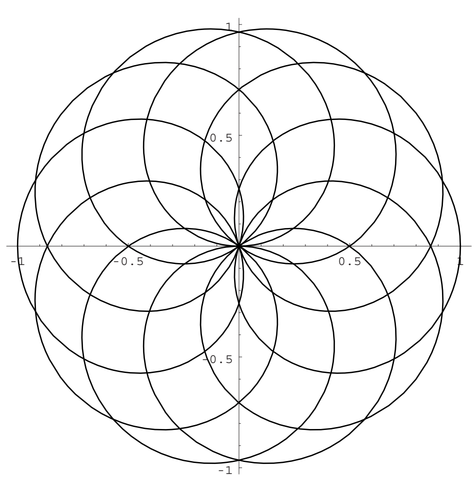

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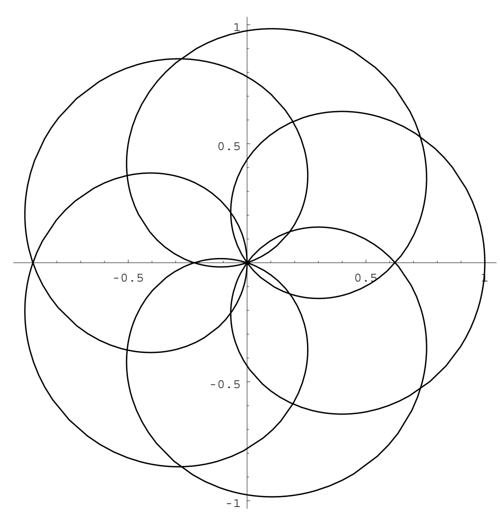

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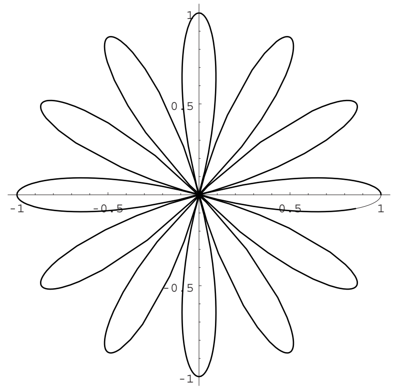

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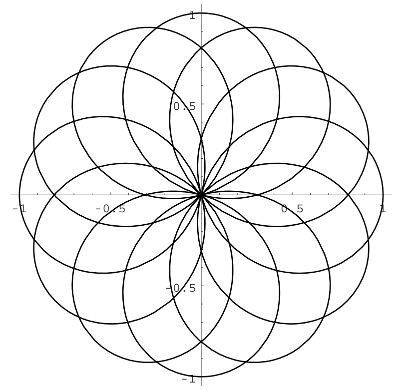
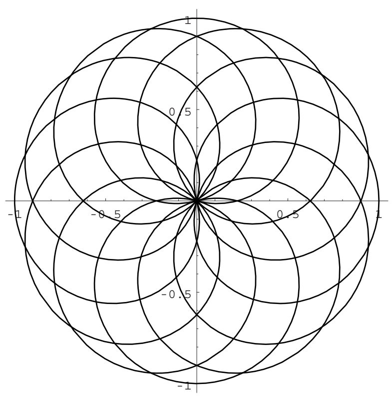
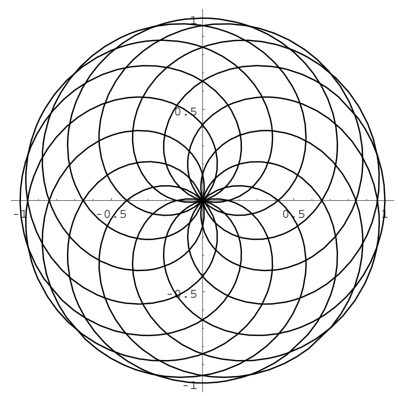


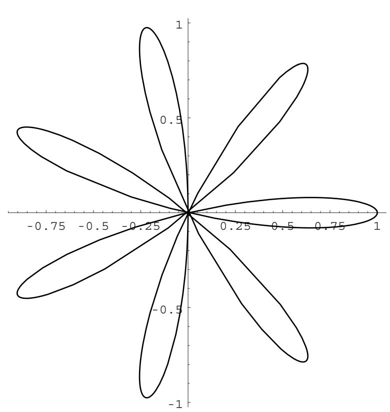
(6,5)



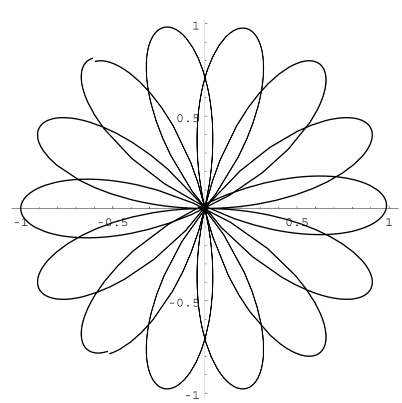
(6,7)



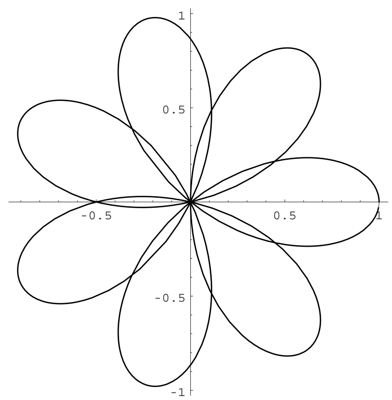
(6,11)



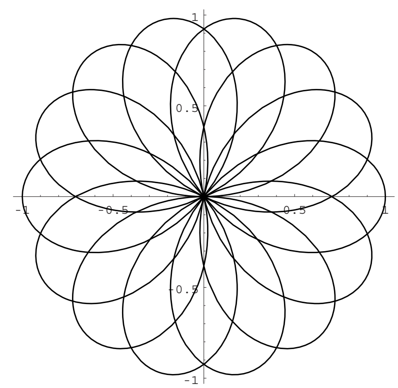
(7,1)



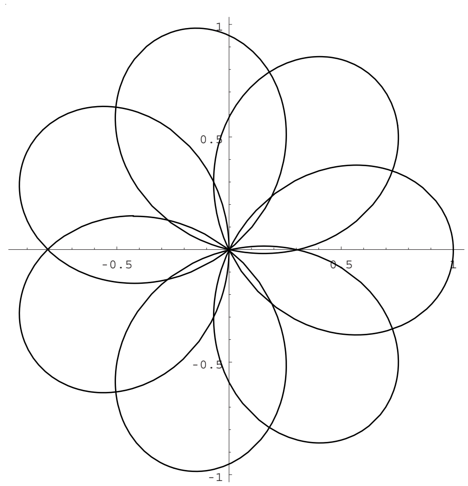
(7,2)



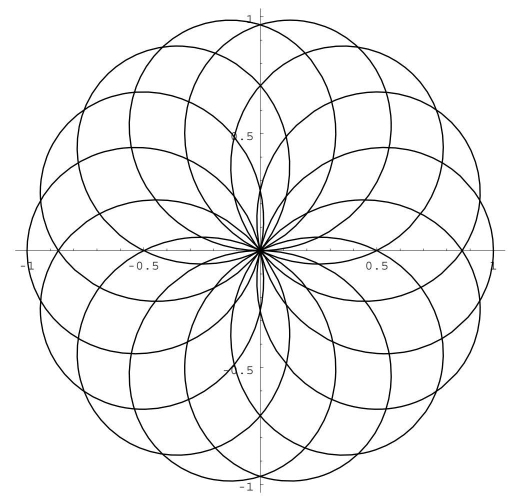
(7,3)



(7,4)

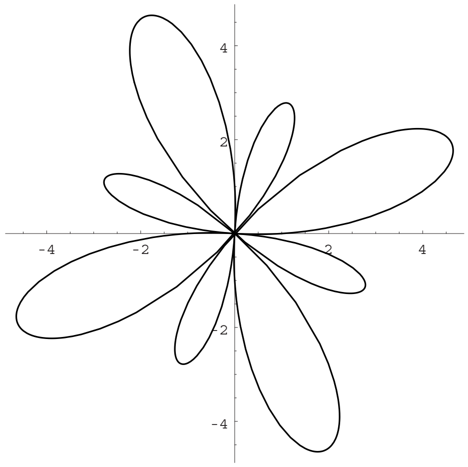


(7,5)

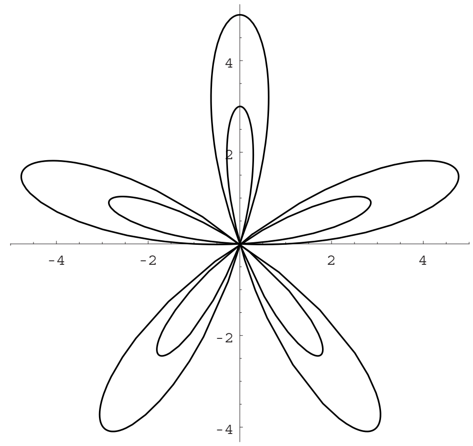


(7,6)

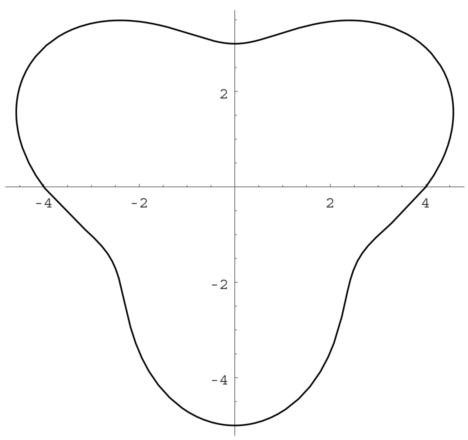
C10S02.064: The graphs of $r = a + b\sin(n\theta)$ are shown next for a few values of a, b, and n. Clearly the quantitative behavior of the graph is independent of the positive numbers a and b, apart from the cases $a > b$, $b = a$, and $a < b$. We show, in order, the cases $(a,\, b,\, n) = (1,\, 4,\, 4)$, $(1,\, 4,\, 5)$, $(4,\, 1,\, 3)$, $(2,\, 2,\, 3)$, and $(2,\, 2,\, 4)$. The last two cases are more subtle than they at first appear—there are cusps, not crossings, at the origin. The obvious generalizations from the conclusions you draw here are indeed valid.



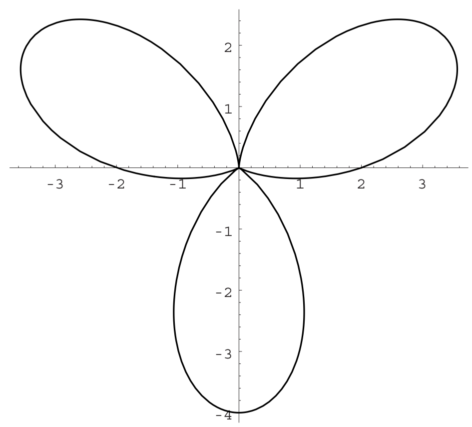
(1,4,4)



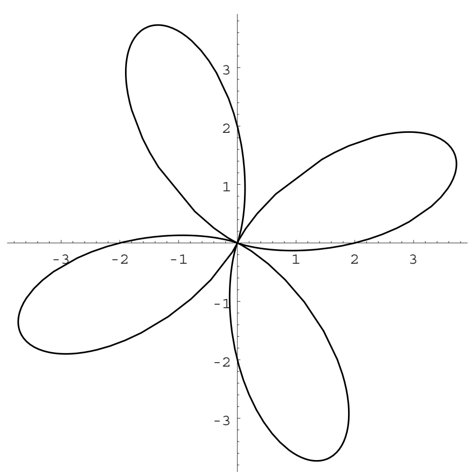
(1,4,5)



(4,1,3)



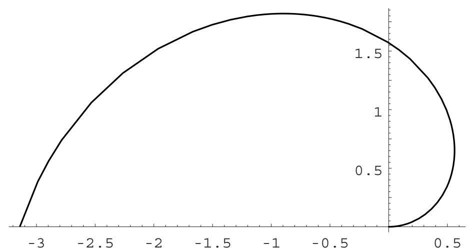
(2,2,3)



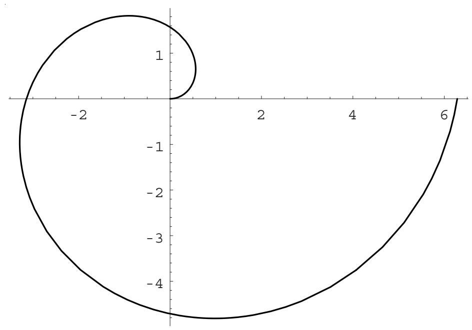
(2,2,4)

Section 10.3

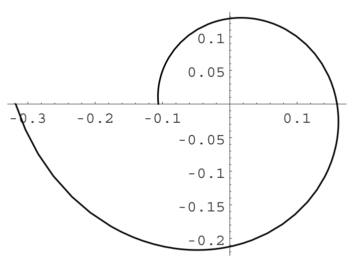
C10S03.001: The graph of $r = \theta,\ 0 \leqq \theta \leqq \pi,$ is shown next.



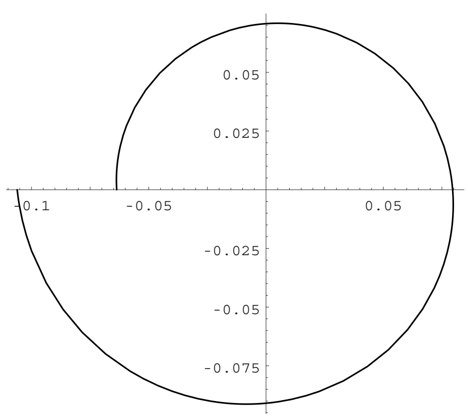
C10S03.002: The graph of $r = \theta,\ 0 \leqq \theta \leqq 2\pi,$ is next.



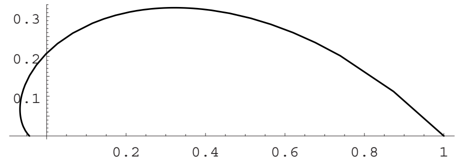
C10S03.003: The graph of $r = 1/\theta,\ \pi \leqq \theta \leqq 3\pi,$ is shown next.



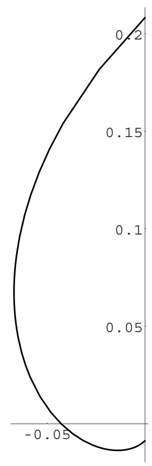
C10S03.004: The graph of the polar equation $r = 1/\theta,\ 3\pi \leqq \theta \leqq 5\pi,$ is next.



C10S03.005: The graph of $r = e^{-\theta}$, $0 \leqq \theta \leqq \pi$, is next.



C10S03.006: The graph of $r = e^{-\theta}$, $\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{3\pi}{2}$, is next.



C10S03.007: Note that the entire curve $r = 2\cos\theta$ is swept out as θ runs through the interval $0 \leqq \theta \leqq \pi$. Thus the area enclosed by this circle is

$$A = \frac{1}{2}\int_0^\pi 4\cos^2\theta\,d\theta = \int_0^\pi (1+\cos 2\theta)\,d\theta = \Big[\theta + \sin\theta\cos\theta\Big]_0^\pi = \pi.$$

The accuracy checkers felt that the area integral should be evaluated over the interval $-\pi/2 \leqq \theta \leqq \pi/2$. These limits will, of course, give the correct answer and are certainly more natural in this problem. On the other hand, the solution shown is correct and has the advantage that trigonometric functions are generally easier to evaluate at integral multiples of π.

C10S03.008: The area enclosed by the circle with polar equation $r = 4\sin\theta$ is

$$A = \frac{1}{2}\int_0^\pi 16\sin^2\theta\,d\theta = 4\int_0^\pi (1-\cos 2\theta)\,d\theta = 4\Big[\theta - \sin\theta\cos\theta\Big]_0^\pi = 4\pi.$$

C10S03.009: The area enclosed by the cardioid with polar equation $r = 1+\cos\theta$ is

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin\theta\cos\theta\right]_0^{2\pi} = \frac{3}{4}\cdot 2\pi = \frac{3}{2}\pi. \end{aligned}$$

C10S03.010: Because $r \geqq 0$ for all θ, the area enclosed by the cardioid with polar equation $r = 2(1-\sin\theta)$ is

$$A = \frac{1}{2}\int_0^{2\pi} 4\left(1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)\,d\theta$$

$$= \int_0^{2\pi} (3 - 4\sin\theta - \cos 2\theta)\,d\theta = \Big[3\theta + 4\cos\theta - \sin\theta\,\cos\theta\Big]_0^{2\pi} = 6\,\pi \approx 18.849555921539.$$

C10S03.011: Because $r > 0$ for all θ, the area enclosed by the limaçon with polar equation $r = 2 - \cos\theta$ is

$$A = \frac{1}{2}\int_0^{2\pi} (2-\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (4 - 4\cos\theta + \cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4 - 4\cos\theta + \frac{1+\cos 2\theta}{2}\right)\,d\theta$$

$$= \frac{1}{2}\left[\frac{9}{2}\theta - 4\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{9}{4}\cdot 2\pi = \frac{9}{2}\pi \approx 14.137166941154.$$

C10S03.012: Because $r > 0$ for all θ, the area enclosed by the limaçon with polar equation $r = 3 + 2\sin\theta$ is

$$A = \frac{1}{2}\int_0^{2\pi} (3+2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} (9 + 12\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \big[9 + 12\sin\theta + 2(1-\cos 2\theta)\big]\,d\theta = \frac{1}{2}\Big[11\theta - 12\cos\theta - \sin 2\theta\Big]_0^{2\pi} = 11\pi \approx 34.557519189488.$$

C10S03.013: Note that the entire circle with polar equation $r = -4\cos\theta$ is swept out as θ runs through any interval of length π. Therefore the area the circle encloses is

$$A = \int_0^{\pi} 8\cos^2\theta\,d\theta = 4\int_0^{\pi}(1+\cos 2\theta)\,d\theta = 4\Big[\theta + \sin\theta\,\cos\theta\Big]_0^{\pi} = 4\pi \approx 12.566370614359.$$

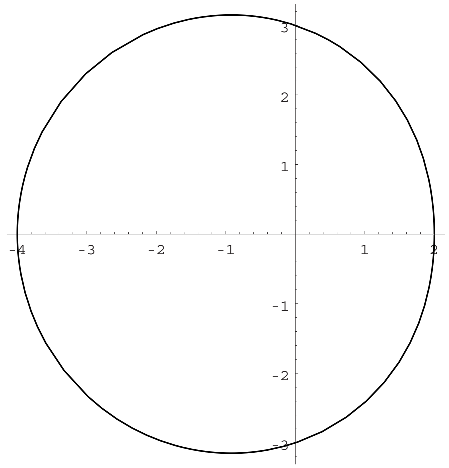
C10S03.014: The area enclosed by the cardioid with polar equation $r = 5(1+\sin\theta)$ is

$$A = \frac{1}{2}\int_0^{2\pi} 25(1 + 2\sin\theta + \sin^2\theta)\,d\theta = \frac{25}{2}\int_0^{2\pi}\left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)\,d\theta$$

$$= \frac{25}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{25}{2}\cdot\frac{3}{2}\cdot 2\pi = \frac{75}{2}\pi \approx 117.809724509617.$$

C10S03.015: The graph of the limaçon with polar equation $r = 3 - \cos\theta$ is shown next. It looks very

much like a circle. Do you see an easy way to deduce that it is *not* a circle?



The area enclosed by this limaçon is

$$A = \frac{1}{2}\int_0^{2\pi} (3 - \cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} \left(9 - 6\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left[\frac{19}{2}\theta - 6\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{19}{2}\pi \approx 29.845130209103.$$

C10S03.016: The *Mathematica* command

```
ParametricPlot[ { (2 + Sin[t] + Cos[t])*Cos[t], (2 + Sin[t] + Cos[t])*Sin[t] },
                  { t, 0, 2*Pi }, AspectRatio → Automatic ];
```

will produce the graph of the curve with the given polar equation $r = 2 + \sin\theta + \cos\theta$. It looks very much like a limaçon, but rotated 45° from the "standard" limaçons shown in the text and elsewhere in this manual. Can you show that it is in fact a limaçon? To find the area A that it encloses, note first that

$$r^2 = 4 + 4\sin\theta + 4\cos\theta + \sin^2\theta + 2\sin\theta\,\cos\theta + \cos^2\theta = 5 + 4\sin\theta + 4\cos\theta + 2\sin\theta\,\cos\theta.$$

Therefore

$$A = \frac{1}{2}\int_0^{2\pi} (5 + 4\sin\theta + 4\cos\theta + 2\sin\theta\,\cos\theta)\,d\theta = \frac{1}{2}\Big[5\theta - 4\cos\theta + 4\sin\theta + \sin^2\theta\Big]_0^{2\pi} = 5\pi \approx 15.707963.$$

C10S03.017: Given: The polar equation $r = 2\cos 2\theta$ of a four-leaved rose. The "loops," or rose petals, are formed by the curve repeatedly passing through the origin at different angles. Examine the graph of this equation in Fig. 10.2.12 (Example 6 of Section 10.2). All we need is to find when $r = 0$; that is, when $\cos 2\theta = 0$. This occurs when θ is an odd integral multiple of $\pi/4$, so we can take for the limits of integration any two consecutive such numbers. Hence the area enclosed by one loop of the rose is

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos^2 2\theta\,d\theta.$$

Using the symmetry of the loop around the x-axis, we can double the area of the upper half of the loop to make the computations slightly simpler:

$$A = \int_0^{\pi/4} 2(1+\cos 4\theta)\,d\theta = 2\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = 2\cdot\frac{\pi}{4} = \frac{\pi}{2} \approx 1.570796326795.$$

C10S03.018: Given: The polar equation $r = 3\sin 3\theta$ of a three-leaved rose (see Fig. 10.3.12). First find when $r = 0$: When θ is any integral multiple of $\pi/3$. Hence the area of one loop is

$$A = \frac{1}{2}\int_0^{\pi/3} 9\sin^2 3\theta\,d\theta = \frac{9}{4}\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta = \frac{9}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{3\pi}{4} \approx 2.3561944902.$$

C10S03.019: Given: The polar equation $r = 2\cos 4\theta$ of an eight-leaved rose (see Fig. 10.3.13). To find the area of one loop, we need to find the limits of integration on θ, which are determined by solving the equation $r = 0$. We find that 4θ will be any odd integral multiple of $\pi/2$, and therefore that θ will be any odd integral multiple of $\pi/8$. We will also double the area of half of one loop to make the computations slightly simpler. Thus the area of one loop is

$$A = \frac{1}{2}\int_{-\pi/8}^{\pi/8} 4\cos^2 4\theta\,d\theta = \int_0^{\pi/8} 2(1+\cos 8\theta)\,d\theta = \left[2\theta + \frac{1}{4}\sin 8\theta\right]_0^{\pi/8} = \frac{\pi}{4} \approx 0.785398163397.$$

C10S03.020: The area of one loop of the five-leaved rose with polar equation $r = \sin 5\theta$ (see Fig. 10.3.14) is

$$A = \frac{1}{2}\int_0^{\pi/5} \sin^2 5\theta\,d\theta = \frac{1}{2}\int_0^{\pi/5} \frac{1}{2}(1-\cos 10\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{20} \approx 0.157079632679.$$

C10S03.021: The lemniscate with polar equation $r^2 = 4\sin 2\theta$ has two loops; its graph can be obtained from the one in Fig. 10.2.13 (Example 7 of Section 10.2) by a $90°$ rotation. To find the area of the loop in the first quadrant, find when $r = 0$: $\sin 2\theta = 0$ when θ is an integral multiple of $\pi/2$. Hence the area of the loop lying in the first quadrant is

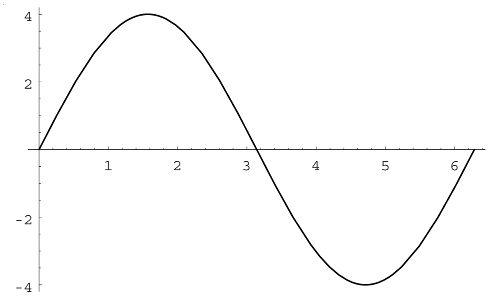
$$A = \frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta\,d\theta = \Big[-\cos 2\theta\Big]_0^{\pi/2} = 1-(-1) = 2.$$

C10S03.022: The lemniscate with polar equation $r^2 = 4\cos 2\theta$ is shown in Fig. 10.3.15. We will find the area of the right-hand loop. First, $r = 0$ when $\cos 2\theta = 0$, so that θ is an odd integral multiple of $\pi/4$. Hence the area of the right-hand loop is

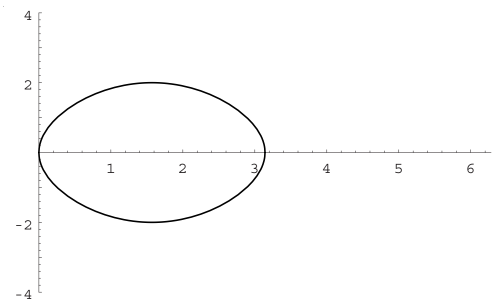
$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos 2\theta\,d\theta = \int_0^{\pi/4} 4\cos 2\theta\,d\theta = \Big[2\sin 2\theta\Big]_0^{\pi/4} = 2.$$

C10S03.023: Given: The polar equation $r^2 = 4\sin\theta$. One way to construct its graph is first to construct

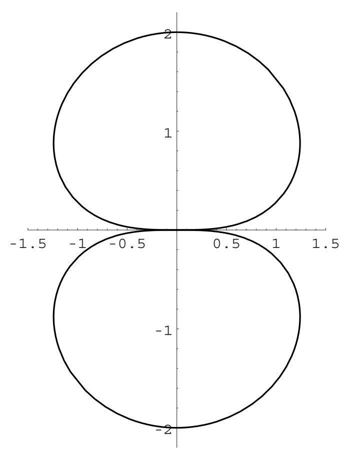
the *Cartesian* graph $y = 4\sin x$, which is shown next for $0 \leqq x \leqq 2\pi$.



Then construct the Cartesian graph of $y = \pm\sqrt{4\sin x}$, shown next, also for $0 \leqq x \leqq 2\pi$. Note that there is no graph for $\pi < x < 2\pi$ but *two* graphs for $0 \leqq x \leqq \pi$.



Finally, use the last graph to construct the polar graph $r^2 = 4\sin\theta$ by sketching both $r = \sqrt{4\sin\theta}$ and $r = -\sqrt{4\sin\theta}$. As θ varies from 0 to π, r begins at 0, increases to a maximum $r = 2$ when $\theta = \pi/2$, then decreases to 0 as θ runs through the values from $\pi/2$ to π. Meanwhile, $-r$ sweeps out the mirror image of the previous curve, and thus we obtain the "double oval" shown in the next figure.



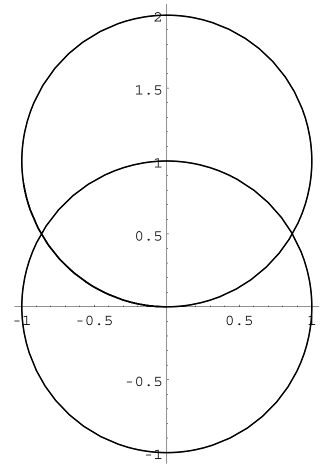
The area of the upper loop is therefore

$$A = \frac{1}{2}\int_0^{\pi} 4\sin\theta\, d\theta = \Big[-2\cos\theta\Big]_0^{\pi} = 2 - (-2) = 4.$$

C10S03.024: The area of one loop of this rose is

$$A = \frac{1}{2}\int_{-\pi/12}^{\pi/12} 36\cos^2 6\theta\, d\theta = 36\int_0^{\pi/12} \frac{1}{2}(1+\cos 12\theta)\, d\theta = 18\left[\theta + \frac{1}{12}\sin 12\theta\right]_0^{\pi/12} = \frac{3\pi}{2}.$$

C10S03.025: See the graph on the right.

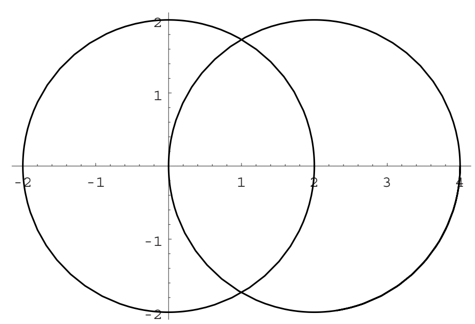


Find the area A of the region both inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$. The circles cross where $2\sin\theta = 1$, thus where $\theta = \pi/6$ and where $\theta = 5\pi/6$. So

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \big[2(1-\cos 2\theta) - 1\big]\,d\theta$$

$$= \frac{1}{2}\Big[\theta - \sin 2\theta\Big]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{2\pi + 3\sqrt{3}}{6} \approx 1.913222954981.$$

C10S03.026: See the graph on the right.

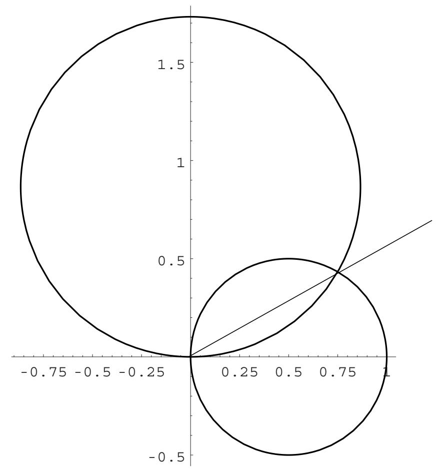


The circles $r = 4\cos\theta$ and $r = 2$ cross where $4\cos\theta = 2$, thus where $\theta = -\pi/3$ and where $\theta = \pi/3$. To find the area A of their intersection, we double the area of its top half; thus

$$A = \int_0^{\pi/3} 4\,d\theta + \int_{\pi/3}^{\pi/2} 16\cos^2\theta\,d\theta = \frac{4\pi}{3} + 4\int_{\pi/3}^{\pi/2} 2(1+\cos 2\theta)\,d\theta = \frac{4\pi}{3} + 4\Big[2\theta + \sin 2\theta\Big]_{\pi/3}^{\pi/2}$$

$$= \frac{4\pi}{3} + 4\left(\pi - \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} + 4\pi - \frac{8\pi}{3} - 2\sqrt{3} = \frac{8\pi - 6\sqrt{3}}{3} \approx 4.91347879.$$

C10S03.027: See the figure on the right.



The two circles $r = \cos\theta$ and $r = \sqrt{3}\,\sin\theta$ cross where

$$\cos\theta = \sqrt{3}\,\sin\theta; \qquad \tan\theta = \frac{\sqrt{3}}{3}; \qquad \theta = \frac{\pi}{6};$$

they also cross at the pole. The region inside both is divided by the ray $\theta = \pi/6$ into a lower region of area B and an upper region of area C. Thus we find the area of the region inside both circles to be $A = B + C$. Now

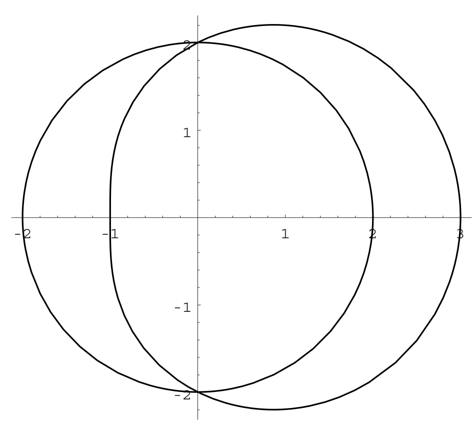
$$B = \frac{1}{2}\int_0^{\pi/6} 3\sin^2\theta\,d\theta = \frac{3}{4}\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta = \frac{3}{4}\Big[\theta - \sin\theta\,\cos\theta\Big]_0^{\pi/6} = \frac{3}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

and

$$C = \frac{1}{2}\int_{\pi/6}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{4}\int_{\pi/6}^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{1}{4}\Big[\theta + \sin\theta\,\cos\theta\Big]_{\pi/6}^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right).$$

Therefore $A = B + C = \dfrac{5\pi - 6\sqrt{3}}{24} \approx 0.221485767606.$

C10S03.028: See the figure on the right.

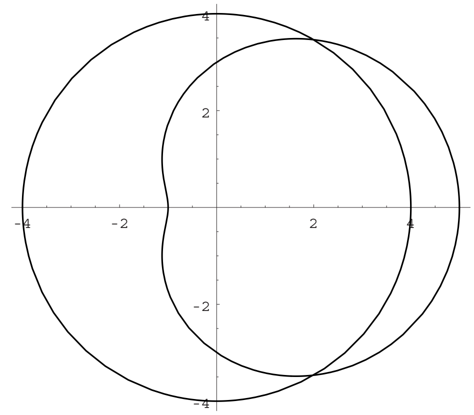


Let A be the area of the region that is both inside the limaçon with polar equation $r = 2 + \cos\theta$ and outside the circle with equation $r = 2$. The curves cross where $2 + \cos\theta = 2$, thus where $\cos\theta = 0$; that is, where $\theta = \pm\pi/2$. Hence

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (4 + 4\cos\theta + \cos^2\theta - 4)\,d\theta = \int_0^{\pi/2} \left(4\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \left[4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = 4 + \frac{\pi}{4} = \frac{16+\pi}{4} \approx 4.785398163397.$$

C10S03.029: See the figure on the right.



Let A denote the area of the region that is both inside the limaçon with polar equation $r = 3 + 2\cos\theta$ and outside the circle with equation $r = 4$. The curves cross where

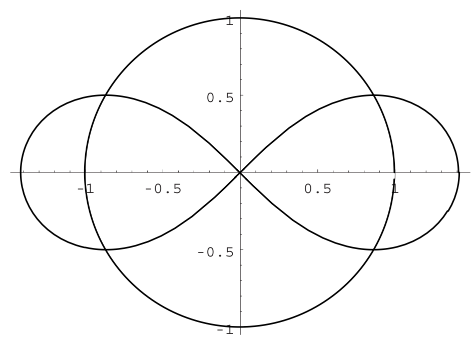
$$3 + 2\cos\theta = 4; \qquad 2\cos\theta = 1; \qquad \cos\theta = \frac{1}{2}; \qquad \theta = \pm\frac{\pi}{3}.$$

Therefore

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(3+2\cos\theta)^2 - 4^2\right] d\theta = \int_0^{\pi/3} \left[9 + 12\cos\theta + 2(1+\cos 2\theta) - 16\right] d\theta$$

$$= \Big[-5\theta + 12\sin\theta + \sin 2\theta\Big]_0^{\pi/3} = -\frac{5\pi}{3} + 12\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{39\sqrt{3} - 10\pi}{6} \approx 6.022342493215.$$

C10S03.030: See the figure on the right.



Let A denote the area of the region that is both within the lemniscate with polar equation $r^2 = 2\cos 2\theta$ and outside the circle with equation $r = 1$. We will use the symmetry of the figure around the y-axis: We will find the area of the half on the right, then double it to find A. The curves cross where $2\cos 2\theta = 1$, thus where $\theta = \pm\pi/6$. To find the area of the half of the figure on the right, we will find the area of its top half and double the result. Thus

$$A = 2 \cdot 2 \cdot \frac{1}{2} \int_0^{\pi/6} (2\cos 2\theta - 1)\,d\theta = 2\Big[(\sin 2\theta) - \theta\Big]_0^{\pi/6} = 2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \frac{3\sqrt{3} - \pi}{3} \approx 0.6848532564.$$

C10S03.031: See Fig. 10.3.16 of the text. The lemniscates $r^2 = \cos 2\theta$ and $r^2 = \sin 2\theta$ cross where $\cos 2\theta = \sin 2\theta$, thus where $\tan 2\theta = 1$; that is, where $\theta = \pi/8$ (and they also cross at the pole). We find the area A within both curves by doubling the area of the half to the right of the y-axis:

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/8} \sin 2\theta\,d\theta + 2 \cdot \frac{1}{2} \int_{\pi/8}^{\pi/4} \cos 2\theta\,d\theta$$

$$= 2\int_0^{\pi/8} \sin 2\theta\,d\theta = \Big[-\cos 2\theta\Big]_0^{\pi/8} = \frac{2-\sqrt{2}}{2} \approx 0.292893218813.$$

C10S03.032: See Fig. 10.3.17 of the text. Given $r = 1 - 2\sin\theta$, we see that $r = 0$ when $\sin\theta = \frac{1}{2}$; that is, when $\theta = \pi/6$ and when $\theta = 5\pi/6$. The small loop is formed when $\frac{1}{6}\pi \leqq \theta \leqq \frac{5}{6}\theta$, where $r \leqq 0$. Let A_2 denote its area. The large loop is formed when $\frac{5}{6}\pi \leqq \theta \leqq \frac{13}{6}\pi$, where $r \geqq 0$. Let A_1 denote its area. Also note that

$$\frac{1}{2}(1-2\sin\theta)^2 = \frac{1}{2}(1 - 4\sin\theta + 4\sin^2\theta) = \frac{1}{2} - 2\sin\theta + 1 - \cos 2\theta = \frac{3}{2} - 2\sin\theta - \cos 2\theta.$$

Therefore

$$A_1 = \int_{5\pi/6}^{13\pi/6} \left(\frac{3}{2} - 2\sin\theta - \cos 2\theta\right) d\theta = \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_{5\pi/6}^{13\pi/6}$$

$$= \frac{1}{4}\left(3\sqrt{3} + 13\pi\right) - \frac{1}{4}\left(5\pi - 3\sqrt{3}\right) = \frac{3\sqrt{3} + 4\pi}{2}$$

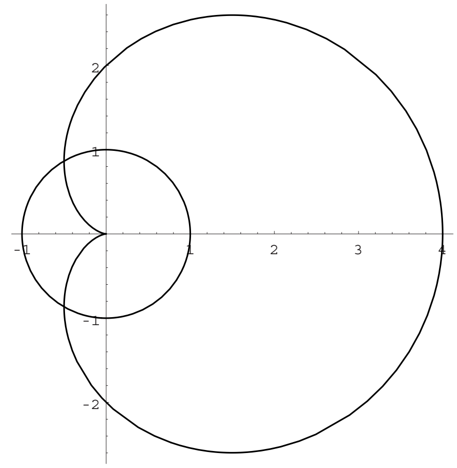
and

$$A_2 = \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{5\pi/6} = \frac{1}{4}\left(5\pi - 3\sqrt{3}\right) - \frac{1}{4}\left(3\sqrt{3} + \pi\right) = \frac{2\pi - 3\sqrt{3}}{2}.$$

Because A_1 measures all of the area within the large loop—including that within the small loop—the area that is both within the large loop of the limaçon and outside its small loop is

$$A = A_1 - A_2 = \pi + 3\sqrt{3} \approx 8.337745076296.$$

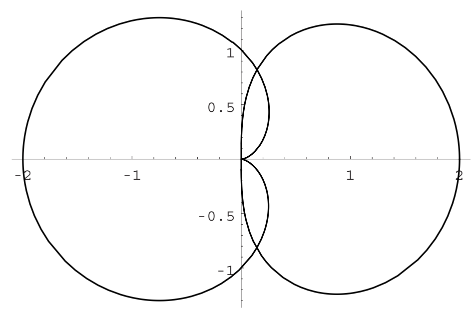
C10S03.033: See the figure on the right.



We are to find the area A of the region that is both inside the cardioid with polar equation $r = 2(1+\cos\theta)$ and outside the circle with equation $r = 1$. The curves cross where $2+2\cos\theta = 1$; it follows that $\cos\theta = -\frac{1}{2}$, so that $\theta = 2\pi/3$ or $\theta = 4\pi/3$. Therefore

$$\begin{aligned}
A &= \frac{1}{2}\int_{4\pi/3}^{8\pi/3} \left[4(1+\cos\theta)^2 - 1\right] d\theta = \frac{1}{2}\int_{4\pi/3}^{8\pi/3} (4 + 8\cos\theta + 4\cos^2\theta - 1)\, d\theta \\
&= \frac{1}{2}\int_{4\pi/3}^{8\pi/3} (3 + 8\cos\theta + 2 + 2\cos 2\theta)\, d\theta = \frac{1}{2}\Big[5\theta + 8\sin\theta + \sin 2\theta\Big]_{4\pi/3}^{8\pi/3} \\
&= \frac{1}{2}\left(\frac{40\pi}{3} + 8\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{20\pi}{3} - 8\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(\frac{20\pi}{3} + 8\sqrt{3} - \sqrt{3}\right) = \frac{1}{2}\left(\frac{20\pi}{3} + 7\sqrt{3}\right) = \frac{20\pi + 21\sqrt{3}}{6} \approx 16.534153338457.
\end{aligned}$$

C10S03.034: See the figure to the right. Note: Only the right-hand loop of the figure-eight curve is shown; the left-hand loop is completely enclosed in the cardioid.



We are to find the area A of the region that is both inside the figure-eight curve $r^2 = 4\cos\theta$ and outside the cardioid $r = 1 - \cos\theta$. All of the region in question lies to the right of the y-axis. Let A_1 be the area of the right-hand loop of the figure-eight curve that lies above the x-axis. Let A_2 be the area of the region in the first quadrant that lies within both the figure-eight curve and the cardioid. Then $A = 2(A_1 - A_2)$. Finding A_1 is easy:

$$A_1 = \frac{1}{2}\int_0^{\pi/2} 4\cos\theta\, d\theta = 2\Big[\sin\theta\Big]_0^{\pi/2} = 2.$$

To find A_2 we need to find where the figure-eight curve and the cardioid meet in the first quadrant. Clearly they meet at the pole. To find the other point, we solve

$$\begin{aligned}
1 - 2\cos\theta + \cos^2\theta &= 4\cos\theta; \\
\cos^2\theta - 6\cos\theta + 1 &= 0; \\
\cos\theta &= \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}\,; \\
\cos\theta &= 3 - 2\sqrt{2}\,; \\
\theta = \alpha &= \arccos\left(3 - 2\sqrt{2}\,\right).
\end{aligned}$$

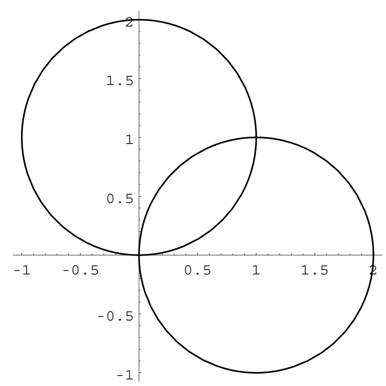
So the two curves cross in the first quadrant where $\theta = \alpha \approx 80.12^\circ$. A reference triangle with acute angle α, adjacent side $x = 3 - 2\sqrt{2}\,$, and hypotenuse 1 has opposite side of length $y = 2\sqrt{3\sqrt{2}\,-4}\,$. Therefore

$$\begin{aligned}
A_2 &= \frac{1}{2}\int_0^\alpha (1-\cos\theta)^2\,d\theta + \frac{1}{2}\int_\alpha^{\pi/2} 4\cos\theta\,d\theta = \frac{1}{2}\int_0^\alpha \left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta + \Big[2\sin\theta\Big]_\alpha^{\pi/2} \\
&= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin\theta\,\cos\theta\right]_0^\alpha + 2 - 2\sin\alpha = \frac{3}{4}\alpha - \sin\alpha + \frac{1}{4}\sin\alpha\,\cos\alpha + 2 - 2\sin\alpha \\
&= \frac{3}{4}\alpha + 2 - 3\sin\alpha + \frac{1}{4}\sin\alpha\,\cos\alpha.
\end{aligned}$$

Therefore

$$\begin{aligned}
A &= 2(A_1 - A_2) = 2\left(2 - \frac{3}{4}\alpha - 2 + 3\sin\alpha - \frac{1}{4}\sin\alpha\,\cos\alpha\right) \\
&= 9\sqrt{3\sqrt{2}\,-4}\, + 2\sqrt{6\sqrt{2}\,-8}\, - \frac{3}{2}\arccos\left(3 - 2\sqrt{2}\,\right) \approx 3.728958744915.
\end{aligned}$$

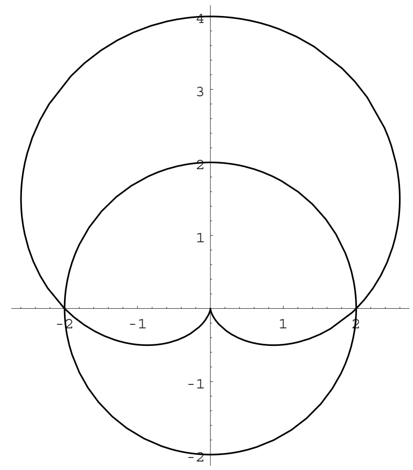
C10S03.035: See the figure to the right.



The two circles with polar equations $r = 2\cos\theta$ and $r = 2\sin\theta$ meet at the pole and where $2\cos\theta = 2\sin\theta$; that is, where $\theta = \pi/4$. So, using symmetry of the figure around the line $y = x$, the area of the region that lies within both circles is

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4} 4\sin^2\theta\,d\theta = \int_0^{\pi/4} 2(1-\cos 2\theta)\,d\theta = \Big[2\theta - \sin 2\theta\Big]_0^{\pi/4} = \frac{\pi-2}{2} \approx 0.570796326795.$$

C10S03.036: See the figure to the right.



Let A denote the area of the region that lies within the cardioid with polar equation $r = 2 + 2\sin\theta$ and outside the circle with equation $r = 2$. These curves cross where $2 = 2 + 2\sin\theta$, so that $\theta = 0$ or $\theta = \pi$. The integrand for finding A is

$$(2 + 2\sin\theta)^2 - 2^2 = 4 + 8\sin\theta + 4\sin^2\theta - 4 = 8\sin\theta + 2(1 - \cos 2\theta).$$

Therefore

$$A = \frac{1}{2}\int_0^{\pi}(8\sin\theta + 2 - 2\cos 2\theta)\,d\theta = \frac{1}{2}\Big[-8\cos\theta + 2\theta - \sin 2\theta\Big]_0^{\pi} = 8 + \pi \approx 11.1415926535898.$$

C10S03.037: See Fig. 10.3.18. Note that the entire circle is generated as θ runs through any interval of length π and that

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\,\cos\theta + \cos^2\theta = 1 + 2\sin\theta\,\cos\theta.$$

Therefore the area enclosed by the circle with polar equation $r = \sin\theta + \cos\theta$ is

$$A = \frac{1}{2}\int_0^{\pi}(1 + 2\sin\theta\,\cos\theta)\,d\theta = \left[\frac{1}{2}\theta + \frac{1}{2}\sin^2\theta\right]_0^{\pi} = \frac{\pi}{2}.$$

To write the equation of this circle in Cartesian form, proceed as follows:

$$r^2 = 4\sin\theta + r\cos\theta;$$

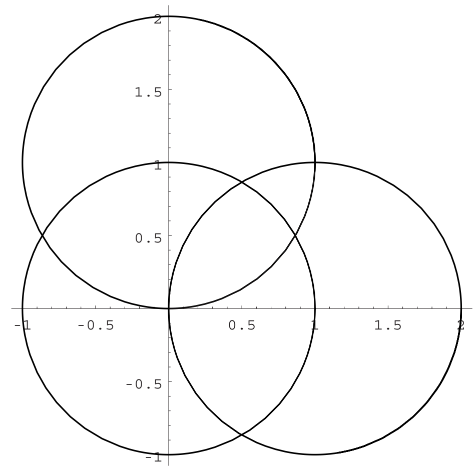
$$x^2 + y^2 = x + y;$$

$$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2};$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

So the figure is, indeed, a circle, and the square of its radius is $\frac{1}{2}$. Therefore the area of this circle is $\pi/2$.

C10S03.038: See the figure on the right.



The circles $r = 1$ and $r = 2\cos\theta$ meet where $\theta = \pi/3$; the circles $r = 1$ and $r = 2\sin\theta$ meet where $\theta = \pi/6$. Hence the area of the region that lies within all three circles is

$$\begin{aligned}
A &= \frac{1}{2}\left[\int_0^{\pi/6}(2\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/3}1^2\,d\theta + \int_{\pi/3}^{\pi/2}(2\cos\theta)^2\,d\theta\right] \\
&= \frac{1}{2}\left[\int_0^{\pi/6}2(1-\cos 2\theta)\,d\theta + \frac{\pi}{6} + \int_{\pi/3}^{\pi/2}2(1+\cos 2\theta)\,d\theta\right] \\
&= \frac{1}{2}\left(\Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} + \Big[2\theta + \sin 2\theta\Big]_{\pi/3}^{\pi/2}\right) + \frac{\pi}{12} = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \pi - \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi}{12} \\
&= \frac{1}{2}\left(\frac{2\pi}{3} - \sqrt{3}\right) + \frac{\pi}{12} = \frac{5\pi}{12} - \frac{\sqrt{3}}{2} = \frac{5\pi - 6\sqrt{3}}{12} \approx 0.442971535211.
\end{aligned}$$

C10S03.039: Part (a):

$$A_1 = \frac{1}{2}\int_0^{2\pi} a^2\theta^2\,d\theta = \left[\frac{1}{6}a^2\theta^3\right]_0^{2\pi} = \frac{4}{3}\pi^3a^2 = \frac{1}{3}\pi(2\pi a)^2.$$

Part (b):

$$A_2 = \frac{1}{2}\int_{2\pi}^{4\pi} a^2\theta^2\,d\theta = \left[\frac{1}{6}a^2\theta^3\right]_{2\pi}^{4\pi} = \frac{28}{3}\pi^3a^2 = \frac{7}{12}\pi(4\pi a)^2.$$

Part (c):

$$R_2 = A_2 - A_1 = \frac{28}{3}\pi^3a^2 - \frac{4}{3}\pi^3a^2 = \frac{24}{3}\pi^3a^2 = 8\pi^3a^2 = 6\cdot\frac{4}{3}\pi^3a^2 = 6A_1.$$

Part (d): If $n \geqq 2$, then

$$A_n = \frac{1}{2}\int_{2(n-1)\pi}^{2n\pi} a^2\theta^2\,d\theta = \left[\frac{1}{6}a^2\theta^3\right]_{2(n-1)\pi}^{2n\pi} = \frac{1}{6}a^2\left[8n^3\pi^3 - 8(n-1)^3\pi^3\right] = \frac{4}{3}\pi^3a^2(3n^2-3n+1),$$

and therefore

$$R_{n+1} = A_{n+1} - A_n = \frac{4}{3}\pi^3 a^2(3n^2 + 6n + 3 - 3n - 3 - 3n^2 + 3n) = \frac{4}{3}\pi^3 a^2 \cdot 6n = 8\pi^3 a^2 n = nR_2.$$

C10S03.040: Use the circles with polar equations $r = a$ and $r = 2a\cos\theta$. They intersect in the first quadrant at the point with polar coordinates $\left(a,\ \frac{1}{3}\pi\right)$. The area of the region that lies within both circles is therefore

$$\begin{aligned}
A &= 2 \cdot \frac{1}{2}\int_0^{\pi/3} a^2\ d\theta + 2 \cdot \frac{1}{2}\int_{\pi/3}^{\pi/2} 4a^2\cos^2\theta\ d\theta = \frac{\pi}{3}a^2 + 2a^2\int_{\pi/3}^{\pi/2}(1 + \cos 2\theta)\ d\theta \\
&= \frac{\pi}{3}a^2 + a^2\Big[2\theta + \sin 2\theta\Big]_{\pi/3}^{\pi/2} = \frac{\pi}{3}a^2 + a^2\left(\pi - \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi - 3\sqrt{3}}{6}a^2 \approx (1.228369698609)a^2.
\end{aligned}$$

C10S03.041: Part (a): The area is

$$\begin{aligned}
A_1 &= \int_0^{2\pi}\frac{1}{2}a^2e^{-2k\theta}\ d\theta - \int_{2\pi}^{4\pi}\frac{1}{2}a^2e^{-2k\theta}\ d\theta = \frac{1}{2}a^2\left[-\frac{e^{-2k\theta}}{2k}\right]_0^{2\pi} - \frac{1}{2}a^2\left[-\frac{e^{-2k\theta}}{2k}\right]_{2\pi}^{4\pi} \\
&= \frac{a^2}{4k}\left(1 - e^{-4k\pi}\right) + \frac{a^2}{4k}\left(e^{-8k\pi} - e^{-4k\pi}\right) = \frac{a^2}{4k}\left(1 - e^{-4k\pi}\right)^2.
\end{aligned}$$

With $k = \frac{1}{10}$ and $a = 1$, we obtain

$$A = \frac{5}{2}\left(1 - e^{-2\pi/5}\right)^2 \approx 1.27945876.$$

Part (b): The area is

$$\begin{aligned}
A_n &= \int_{2(n-1)\pi}^{2n\pi}\frac{1}{2}a^2\exp(-2k\theta)\ d\theta - \int_{2n\pi}^{2(n+1)\pi}\frac{1}{2}a^2\exp(-2k\theta)\ d\theta \\
&= \frac{1}{2}a^2\left[-\frac{\exp(-2k\theta)}{2k}\right]_{2(n-1)\pi}^{2n\pi} - \frac{1}{2}a^2\left[-\frac{\exp(-2k\theta)}{2k}\right]_{2n\pi}^{2(n+1)\pi} \\
&= \frac{a^2}{4k}\left[\exp(-4(n-1)k\pi) - \exp(-4nk\pi) + \exp(-4(n+1)k\pi) - \exp(-4nk\pi)\right] \\
&= \frac{a^2}{4k}\exp(-4(n-1)k\pi)\left[1 - \exp(-4k\pi)\right]^2.
\end{aligned}$$

With $a = 1$ and $k = \frac{1}{10}$, we find that

$$A = \frac{5}{2}e^{-2(n-1)\pi/5}\left(1 - e^{-2\pi/5}\right)^2.$$

C10S03.042: Let $r(\theta) = 2e^{-\theta/10}$. R_1 has area

$$A_1 = 4\pi - \int_0^{2\pi}\frac{1}{2}\left[r(\theta)\right]^2\ d\theta = 4\pi - 10\left(1 - e^{-2\pi/5}\right),$$

R_2 has area

$$A_2 = \int_0^{2\pi} \frac{1}{2} [r(\theta)]^2 \, d\theta - \pi = 10\left(1 - e^{-2\pi/5}\right) - \pi,$$

and, indeed, $A_1 + A_2 = 3\pi$.

C10S03.043: The point of intersection in the second quadrant is located where $\theta = \alpha \approx 2.326839$. Using symmetry, the total area of the shaded region R is approximately

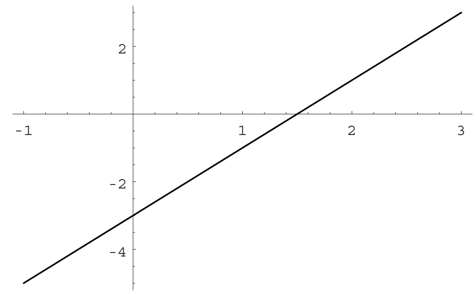
$$2\int_0^{\alpha} \frac{1}{2} \left(e^{-\theta/5}\right)^2 \, d\theta + 2\int_{\alpha}^{\pi} \frac{1}{2} [2(1+\cos\theta)]^2 \, d\theta \approx 1.58069.$$

C10S03.044: The point of intersection in the first quadrant is located where $\theta = \alpha \approx 0.217075400$ and the point of intersection in the second quadrant is located where $\theta = \beta \approx 2.924517254$. So the total area of the shaded region R is approximately

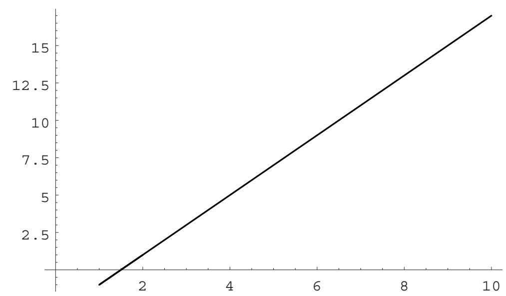
$$\int_{\alpha}^{\beta} \frac{1}{2}(3+\cos 4\theta)^2 \, d\theta + \int_{\beta}^{2\pi+\alpha} \frac{1}{2}(3+3\sin\theta)^2 \, d\theta \approx 17.2661.$$

Section 10.4

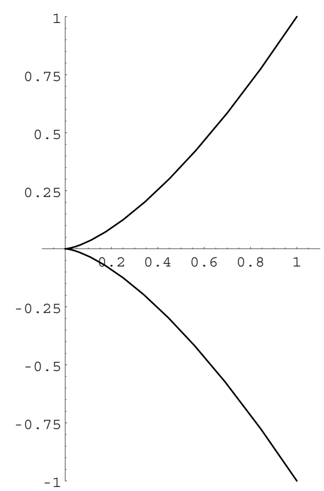
C10S04.001: If $x = t + 1$, then $t = x - 1$, so that $y = 2t - 1 = 2(x - 1) - 1 = 2x - 3$. The graph is next.



C10S04.002: If $x = t^2 + 1$ and $y = 2t^2 - 1$, then $y = 2(x - 1) - 1 = 2x - 3$ with the restriction that $x \geqq 1$. Thus the graph is a "half-line," shown next.

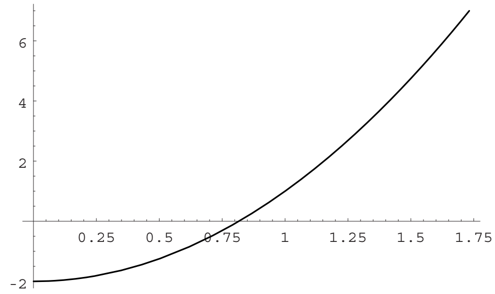


C10S04.003: If $x = t^2$ and $y = t^3$, then $y = \left(t^2\right)^{3/2} = x^{3/2}$; alternatively, $y^2 = x^3$. The graph is shown next.

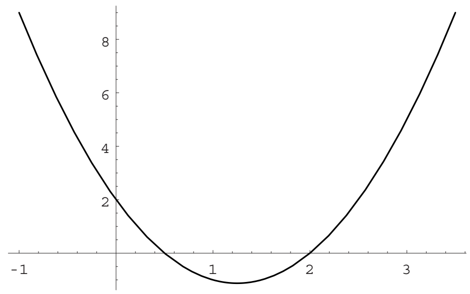


C10S04.004: If $x = t^{1/2}$, then $t = x^2$, so $y = 3t - 2 = 3x^2 - 2$ with the restriction that $x \geqq 0$. The graph

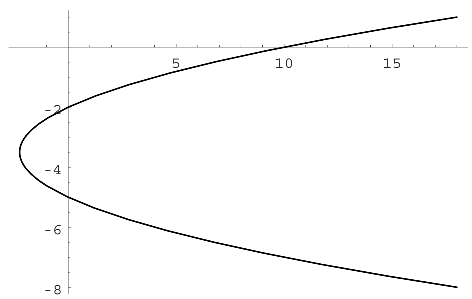
is next.



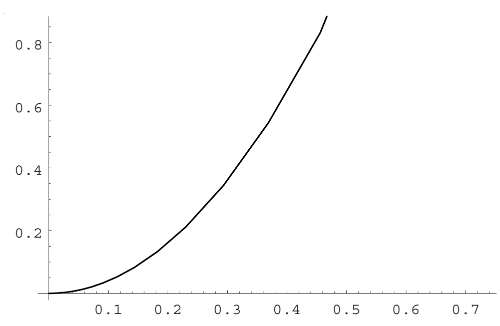
C10S04.005: If $x = t + 1,$ then $t = x - 1,$ so that $y = 2t^2 - t - 1 = 2(x-1)^2 - (x-1) - 1 = 2x^2 - 5x + 2.$ The graph is next.



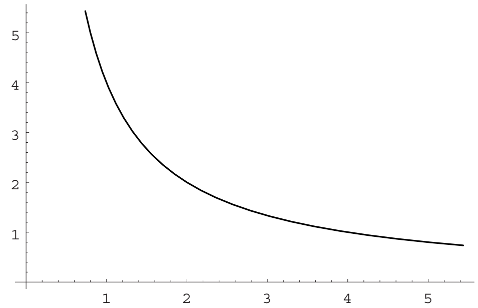
C10S04.006: If $x = t^2 + 3t$ and $y = t - 2$, then $t = y + 2$ and thus $x = (y+2)^2 + 3(y+2) = y^2 + 7y + 10.$ The graph is next.



C10S04.007: If $x = e^t,$ then $y = 4e^{2t} = 4(e^t)^2 = 4x^2$ with the restriction that $x > 0.$ The graph is next.



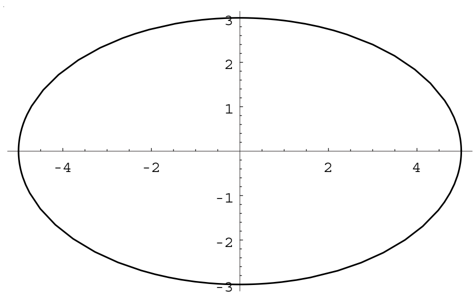
C10S04.008: If $x = 2e^t$, then $y = 2e^{-t} = \dfrac{4}{2e^t} = \dfrac{4}{x}$, $x > 0$. The graph is next.



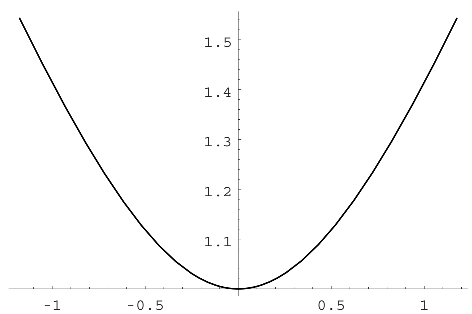
C10S04.009: If $x = 5\cos t$ and $y = 3\sin t$, then

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1; \qquad \text{that is,} \qquad 9x^2 + 25y^2 = 225.$$

The graph of this ellipse is next.



C10S04.010: Given $x = \sinh t$ and $y = \cosh t$, $y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$, and thus $y = \sqrt{1 + x^2}$ (the *positive* square root because $\cosh t > 0$ for all t). The graph is next.

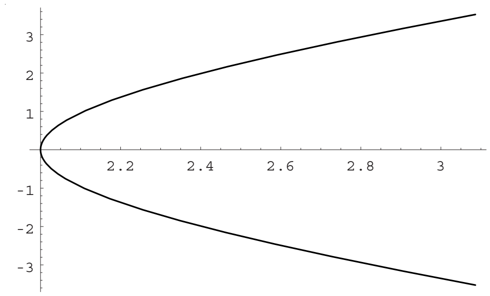


C10S04.011: If $x = 2\cosh t$ and $y = 3\sinh t$, then

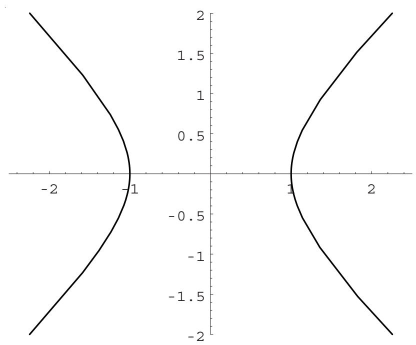
$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = \cosh^2 t - \sinh^2 t = 1; \qquad \text{that is,} \qquad 9x^2 - 4y^2 = 16.$$

But not all points that satisfy the last equation are on the graph, because $x = 2\cosh t \geqq 2$ for all t. Thus

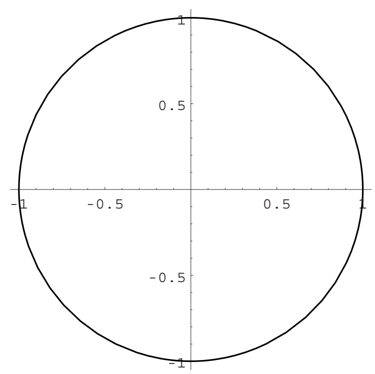
only points on the right half of this hyperbola form the graph of the parametric equations, shown next.



C10S04.012: If $x = \sec t$ and $y = \tan t$, then $1 + y^2 = 1 + \tan^2 t = \sec^2 t = x^2$. The graph is next. Unlike Problem 10, both branches of the hyperbola are present.



C10S04.013: Given $x = \sin 2\pi t$ and $y = \cos 2\pi t$, $0 \leqq t \leqq 1$, it follows that $x^2 + y^2 = 1$. Thus the graph is a circle of radius 1 centered at the origin. As t runs from 0 to 1, the point (x, y) begins at $(0, 1)$ and moves once clockwise around the circle. The graph is next.

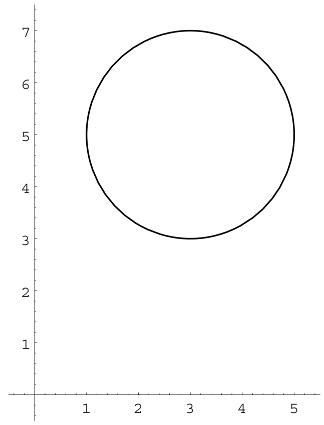


C10S04.014: Given $x = 3 + 2\cos t$ and $y = 5 - 2\sin t$, $0 \leqq t \leqq 2\pi$, we find that

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y-5}{2}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

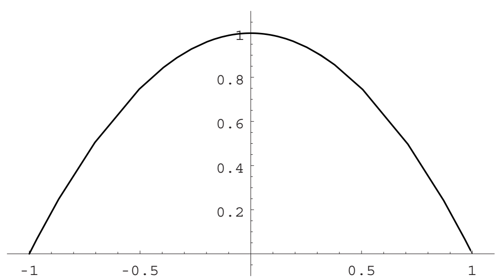
so that $(x-3)^2 + (y-5)^2 = 4$. The graph is a circle of radius 2 with center at $(3, 5)$. As t varies from 0 to 2π, the point (x, y) begins at the point $(5, 5)$ and moves once clockwise around the circle. The graph is

next.



C10S04.015: Given $x = \sin^2 \pi t$ and $y = \cos^2 \pi t$, $0 \leqq t \leqq 2$, it's clear that $x + y = 1$ and that $0 \leqq x \leqq 1$. So the graph is the straight line segment joining (0, 1) and (1, 0). As t varies from 0 to 2, the point (x, y) begins at (0, 1), moves southeast until it reaches (1, 0) when $t = 1$, then moves northwest until it returns to (0, 1) when $t = 2$. We omit the graph to save space.

C10S04.016: Given $x = \cos t$ and $y = \sin^2 t$, $-\pi \leqq t \leqq \pi$, it follows that $x^2 + y = 1$; that is, that $y = 1 - x^2$, $-1 \leqq x \leqq 1$. When $t = -\pi$, the point (x, y) is located at the point $(-1, 0)$. As t increases, (x, y) moves along the parabola from left to right, reaching (1, 0) when $t = 0$. As t continues to increase, the point (x, y) retraces its route, finally returning to its starting point when $t = \pi$. The graph is next.



C10S04.017: Given $x = 2t^2 + 1$, $y = 3t^3 + 2$, we first calculate

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{4t} = \frac{9}{4}t, \qquad \text{so that} \qquad \left.\frac{dy}{dx}\right|_{t=1} = \frac{9}{4}.$$

When $t = 1$, $(x, y) = (3, 5)$, so the tangent line there has equation

$$y - 5 = \frac{9}{4}(x - 3); \qquad 4y - 20 = 9x - 27; \qquad 9x = 4y + 7.$$

Next,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{dx/dt} \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{4t} \cdot \frac{9}{4} = \frac{9}{16t}.$$

The second derivative is positive when $t = 1$, so the graph is concave upward at and near the point (3, 5).

C10S04.018: Given $x = \cos^3 t$ and $y = \sin^3 t$, we find that

$$\frac{dy}{dx} = \frac{3\sin^2 t\,\cos t}{-3\cos^2 t\,\sin t} = -\tan t, \quad \text{so that} \quad \left.\frac{dy}{dx}\right|_{t=\pi/4} = -1.$$

When $t = \pi/4$, $(x,\, y) = \left(\frac{1}{4}\sqrt{2}\,,\ \frac{1}{4}\sqrt{2}\,\right)$, so an equation of the tangent line there is

$$y - \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{4} - x; \qquad x + y = \frac{\sqrt{2}}{2}.$$

Next,

$$\frac{d^2y}{dx^2} = \frac{1}{dx/dt}\cdot\frac{d}{dt}\,(-\tan t) = \frac{-\sec^2 t}{-3\sin t\,\cos^2 t} = \frac{1}{3\sin t\,\cos^4 t},$$

and thus the second derivative is positive when $t = \pi/4$. Hence the graph is concave upward at and near the point of tangency.

C10S04.019: Given $x = t\sin t$ and $y = t\cos t$, we first calculate

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - t\sin t}{\sin t + t\cos t}; \qquad \left.\frac{dy}{dx}\right|_{t=\pi/2} = -\frac{\pi}{2}.$$

Hence an equation of the line tangent to the graph at $(x,\, y) = (\pi/2,\, 0)$ is

$$y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right); \qquad 4y = -\pi(2x - \pi); \qquad 2\pi x + 4y = \pi^2.$$

Next,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1}{dx/dt}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right)\\
&= \frac{1}{\sin t + t\cos t}\cdot\frac{(\sin t + t\cos t)(-2\sin t - t\cos t) - (\cos t - t\sin t)(2\cos t - t\sin t)}{(\sin t + t\cos t)^2}\\
&= -\frac{(\sin t + t\cos t)(2\sin t + t\cos t) + (\cos t - t\sin t)(2\cos t - t\sin t)}{(\sin t + t\cos t)^3}\\
&= -\frac{2\sin^2 t + 3t\sin t\,\cos t + t^2\cos^2 t + 2\cos^2 t - 3t\sin t\,\cos t + t^2\sin^2 t}{(\sin t + t\cos t)^3} = -\frac{t^2 + 2}{(\sin t + t\cos t)^2}.
\end{aligned}$$

Thus

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} = -\left(\frac{\pi^2}{4} + 2\right) < 0,$$

and therefore the graph is concave downward at and near the point of tangency.

C10S04.020: Given: $x = e^t$, $y = e^{-t}$. First,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^t} = -e^{-2t}; \qquad \left.\frac{dy}{dx}\right|_{t=0} = -1.$$

So an equation of the line tangent to the graph at $(x,\, y) = (1,\, 1)$ is $y - 1 = -(x - 1)$; that is, $x + y = 2$. Next,

$$\frac{d^2y}{dx^2} = \frac{1}{e^t} \cdot \frac{2e^{2t}}{e^{4t}} = 2e^{-3t},$$

which is always positive, so the graph is concave upward everywhere.

C10S04.021: Equation (10) tells us that

$$\cot\psi = \frac{1}{r} \cdot \frac{dr}{d\theta}$$

where $0 \leqq \psi \leqq \pi$. Thus, given $r = \exp\left(\theta\sqrt{3}\,\right)$ and the angle $\theta = \pi/2$, we find that

$$\cot\psi = \frac{1}{\exp\left(\theta\sqrt{3}\,\right)} \cdot \left(\sqrt{3}\,\right)\exp\left(\theta\sqrt{3}\,\right) = \sqrt{3}\,.$$

Therefore $\psi = \dfrac{\pi}{6}$.

C10S04.022: Given: $r = \dfrac{1}{\theta}$ and $\theta = 1$.

$$\cot\psi = \theta \cdot \left(-\frac{1}{\theta^2}\right) = -\frac{1}{\theta}.$$

Therefore when $\theta = 1$ we have $\cot\psi = -1$, and thus $\psi = \dfrac{3\pi}{4}$.

C10S04.023: Given $r = \sin 3\theta$ and the angle $\theta = \pi/6$. By Eq. (10) of the text,

$$\cot\psi = \frac{1}{\sin 3\theta} \cdot 3\cos 3\theta = 3\cot 3\theta.$$

Thus when $\theta = \pi/6$, we have $\cot\psi = 3\cot(\pi/2) = 0$, and thus $\psi = \dfrac{\pi}{2}$.

C10S04.024: Given: $r = 1 - \cos\theta$ and the angle $\theta = \pi/3$. Then

$$\cot\psi = \frac{\sin\theta}{1 - \cos\theta}; \qquad (\cot\psi)\bigg|_{\theta=\pi/3} = \frac{\dfrac{\sqrt{3}}{2}}{1 - \dfrac{1}{2}} = \sqrt{3}\,.$$

Therefore $\psi = \dfrac{\pi}{6}$.

C10S04.025: Given $x = t^2$ and $y = t^3 - 3t$,

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}; \qquad \frac{dy}{dx} = 0 \quad \text{when} \quad t = \pm 1.$$

So the graph has horizontal tangents at the point $(1, -2)$ and $(1, 2)$. The graph crosses the x-axis when $t^3 - 3t = 0$: $t = 0$, $t = \pm\sqrt{3}\,$. When $t = 0$ we get a vertical tangent line at $(0, 0)$. When $t = -\sqrt{3}$ the graph passes through the point $(3, 0)$ with slope $-\sqrt{3}\,$; when $t = \sqrt{3}\,$, the graph passes through the *same* point $(3, 0)$ with slope $\sqrt{3}\,$. Therefore there is no line tangent to the graph of the parametric equations at the point $(3, 0)$.

C10S04.026: Given $x = \sin t$ and $y = \sin 2t$,

$$\frac{dy}{dx} = \frac{2\cos 2t}{\cos t}; \qquad \frac{dy}{dx} = 0 \quad \text{when} \quad t = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}.$$

Therefore the graph has a horizontal tangent line at all four of the points $\left(\pm\frac{1}{2}\sqrt{2},\ \pm 1\right)$. The graph crosses the x-axis when $\sin 2t = 0$, thus when $t = 0,\ \pi/2,\ \pi,\ 3\pi/2$. When $t = 0$, the curve passes through the origin with slope 2; when $t = \pi$, it passes through the origin with slope -2. So there is no tangent line at $(0, 0)$. At the other two x-intercepts, $(1, 0)$ and $(-1, 0)$, the tangent line is vertical.

C10S04.027: Given the polar equation $r = 1 + \cos\theta$, we can use θ itself as parameter to obtain

$$x = r\cos\theta = \cos\theta + \cos^2\theta \quad \text{and} \quad y = r\sin\theta = \sin\theta + \sin\theta\,\cos\theta.$$

Thus

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\,\cos\theta}.$$

Next we solve $dy/dx = 0$:

$$\begin{aligned}
&\cos\theta + \cos^2\theta + \cos^2\theta - 1 = 0;\\
&2\cos^2\theta + \cos\theta - 1 = 0;\\
&(2\cos\theta - 1)(\cos\theta + 1) = 0;\\
&\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -1.
\end{aligned}$$

Thus $\theta = \pi/3,\ \pi,\ 5\pi/3$. But we must rule out $\theta = \pi$ because the denominator in dy/dx is zero for that value of θ.

When $\theta = \dfrac{\pi}{3}$, $\quad x = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$ and $y = \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{4} = \dfrac{3\sqrt{3}}{4}$. There is a horizontal tangent.

When $\theta = \dfrac{5\pi}{3}$, $\quad x = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$ and $y = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{4} = -\dfrac{3\sqrt{3}}{4}$. There is a horizontal tangent.

The graph crosses the x-axis when $\cos\theta = -1$ and when $\sin\theta = 0$, so that $\theta = 0$ and $\theta = \pi$. When $\theta = 0$ the tangent line is vertical at the point with Cartesian coordinates $(2, 0)$. What happens if $\theta = \pi$? The derivative is undefined. Nevertheless, something can be done. We use l'Hôpital's rule:

$$\lim_{\theta\to\pi}\frac{dy}{dx} = \lim_{\theta\to\pi}\frac{\sin^2\theta - \cos^2\theta - \cos\theta}{2\sin\theta\,\cos\theta + \sin\theta} = \lim_{\theta\to\pi}\frac{4\sin\theta\,\cos\theta + \sin\theta}{2\cos^2\theta - 2\sin^2\theta + \cos\theta} = \frac{0+0}{2-0-1} = 0.$$

Thus we are justified in stating that the x-axis is tangent to the graph of this cardioid at the point $(0, 0)$.

C10S04.028: Given $r^2 = 4\cos 2\theta$, we will take advantage of the many symmetries of the graph (around both coordinate axes) and work only in the first quadrant, where

$$x = 2(\cos 2\theta)^{1/2}\cos\theta \quad \text{and} \quad y = 2(\cos 2\theta)^{1/2}\sin\theta.$$

Thus we find that

$$\frac{dy}{dx} = \frac{(\cos 2\theta)^{-1/2}(-2\sin 2\theta)\sin\theta + 2(\cos 2\theta)^{1/2}\cos\theta}{(\cos 2\theta)^{-1/2}(-2\sin 2\theta)\cos\theta - (2(\cos 2\theta)^{1/2}\sin\theta}$$

$$= -\frac{2\cos 2\theta\ \cos\theta - 2\sin 2\theta\ \sin\theta}{2\sin 2\theta\ \cos\theta + 2\cos 2\theta\ \sin\theta} = -\frac{\cos(2\theta+\theta)}{\sin(2\theta+\theta)} = -\frac{\cos 3\theta}{\sin 3\theta} = -\cot 3\theta.$$

Hence $dy/dx = 0$ when $\cos 3\theta = 0$; that is, when $\theta = \pi/6$ and when $\theta = \pi/2$ (remember that we are restricting our calculations to the first quadrant). But there is no graph at the latter point, so we find that there is only one horizontal tangent in the first quadrant, where

$$x = 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \qquad \text{and} \qquad y = 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{2}}{2}.$$

Thus there are four points on the graph where the tangent line is horizontal, all four of the points with Cartesian coordinates $\left(\pm\frac{1}{2}\sqrt{6},\ \pm\frac{1}{2}\sqrt{2}\,\right)$. Next, the graph meets the x-axis when $\cos 2\theta = 0$ and when $\sin\theta = 0$, thus when $\theta = 0$ and $\theta = \pi/4$ (still restricting our attention to the first quadrant). Thus we find a vertical tangent at $(2, 0)$ and, by symmetry, another at $(-2, 0)$. There is no tangent line when $\theta = \pi/4$ because the graph passes through the origin twice, once with slope 1 and once with slope -1.

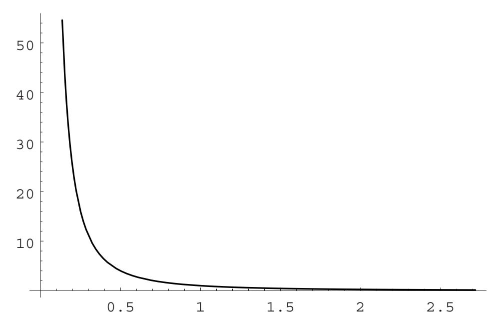
C10S04.029: Given $x = e^{-t}$ and $y = e^{2t}$, we find that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{-e^{-t}} = -2e^{3t}$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{dx/dt}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{-e^{-t}}\cdot\left(-6e^{3t}\right) = 6e^{4t},$$

so the second derivative is positive for all t. Thus the graph of C is concave upward for all t. The graph is shown next; note that there is no graph for $x \leqq 0$ or for $y \leqq 0$.



C10S04.030: Given $x^3 + y^3 = 3xy$, the straight line through the origin with slope $t \geqq 0$ will meet the loop of the folium at only one point in the first quadrant. The line has equation $y = tx$ and thus meets the folium at the point $(x,\ tx)$. Hence

$$x^3 + (tx)^3 = 3x\cdot tx; \qquad x^3 + t^3x^3 = 3tx^2; \qquad (1+t^3)x = 3t.$$

Therefore parametric equations of the loop are

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3}, \qquad 0 \leqq t < +\infty.$$

C10S04.031: If the slope of the curve at $P(x, y)$ is m, then implicit differentiation yields

$$2y\frac{dy}{dx} = 4p; \qquad \frac{dy}{dx} = \frac{2p}{y}; \qquad y = \frac{2p}{m},$$

and thus

$$x = \frac{y^2}{4p} = \frac{4p^2}{4m^2p} = \frac{p}{m^2}, \qquad -\infty < m < +\infty.$$

C10S04.032: Let $r = f(\theta)$ and $r' = f'(\theta)$. Then

$$\tan\alpha = \frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta};$$

$$\tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta};$$

$$\cot\psi = \frac{1+\tan\alpha\,\tan\theta}{\tan\alpha - \tan\theta} = \frac{1 + \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}\cdot\dfrac{\sin\theta}{\cos\theta}}{\dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} - \dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{r'\cos^2\theta - r\sin\theta\,\cos\theta + r'\sin^2\theta + r\sin\theta\,\cos\theta}{r'\sin\theta\,\cos\theta + r\cos^2\theta - r'\sin\theta\,\cos\theta + r\sin^2\theta} = \frac{r'}{r} = \frac{1}{r}\cdot\frac{dr}{d\theta}.$$

C10S04.033: The high point on the circle is $P_0(a\theta, 2a)$ and P has Cartesian coordinates $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$. Therefore the slope of the line containing P_0 and P is

$$\frac{2a - a(1-\cos\theta)}{a\theta - a(\theta - \sin\theta)} = \frac{2-1+\cos\theta}{\theta - \theta + \sin\theta} = \frac{1+\cos\theta}{\sin\theta}.$$

But the slope of the cycloid at the point P is

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{(\sin\theta)(1+\cos\theta)}{1-\cos^2\theta} = \frac{1+\cos\theta}{\sin\theta}.$$

We may conclude that the line containing P_0 and P is tangent to the cycloid at the point P.

C10S04.034: See Fig. 10.4.15. The length of OC is $a-b$, so C has coordinates $x = (a-b)\cos t$, $y = (a-b)\sin t$. The arc length from the point of tangency to $A(a, 0)$ is the same as that to P; denote it by s. Note that $s = ta$. Let α be the angle OCP and θ the angle supplementary to α, so that $\theta = \pi - \alpha$. Then $s = b\theta$, and therefore $ta = b\theta$. The radius b is at the angle $-(\theta - t) = t - \theta$ from the horizontal, so P has coordinates

$$x = (a-b)\cos t + b\cos(t-\theta), \qquad y = (a-b)\sin t + b\sin(t-\theta).$$

Now $\theta = \dfrac{a}{b}t$, so $t - \theta = t - \dfrac{a}{b}t = \dfrac{b-a}{b}t$. Therefore

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right), \qquad y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

C10S04.035: We will need two trigonometric identities before we begin. They are

$$\cos 3t = \cos 2t\,\cos t - \sin 2t\,\sin t = \cos^3 t - \sin^2 t\,\cos t - 2\sin^2 t\,\cos t = \cos^3 t - 3\sin^2 t\,\cos t \tag{1}$$

and

$$\sin 3t = \sin 2t\,\cos t + \cos 2t\,\sin t = 2\sin t\,\cos^2 t + \cos^2 t\,\sin t - \sin^3 t = 3\sin t\,\cos^2 t - \sin^3 t. \tag{2}$$

We begin with the parametric equations

$$x = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right) \tag{3}$$

and

$$y = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right). \tag{4}$$

If $b = \dfrac{a}{4}$, then $\dfrac{a-b}{b} = \dfrac{a - \frac{1}{4}a}{\frac{1}{4}a} = \dfrac{3a}{a} = 3$. Thus Eqs. (3)and (4) become

$$x = \frac{3}{4}a\cos t + \frac{a}{4}\cos 3t = \frac{a}{4}\left(3\cos t + \cos 3t\right) \tag{5}$$

and

$$y = \frac{3}{4}a\sin t - \frac{a}{4}\sin 3t = \frac{a}{4}\left(3\sin t - \sin 3t\right). \tag{6}$$

Then Eqs. (1) and (2) yield

$$x = \frac{a}{4}\left(3\cos t + \cos^3 t - 3\sin^2 t\,\cos t\right) = \frac{a}{4}\left(3\cos t + \cos^3 t - 3\cos t + 3\cos^3 t\right) = a\cos^3 t$$

and

$$y = \frac{a}{4}\left(3\sin t - 3\sin t\,\cos^2 t + \sin^3 t\right) = \frac{a}{4}\left(3\sin t - 3\sin t + 3\sin^3 t + \sin^3 t\right) = a\sin^3 t.$$

C10S04.036: Part (a): If $x = a\cos^3 t$ and $y = a\sin^3 t$, then

$$x^{2/3} + y^{2/3} = a^{2/3}(\cos^2 t + \sin^2 t) = a^{2/3}.$$

So every point of the hypocycloid lies on the graph $x^{2/3} + y^{2/3} = a^{2/3}$. But $\cos^3 t$ and $\sin^3 t$ take on all values from -1 to 1 (and it follows that x and y take on values in all four quadrants), so the hypocycloid is the entire graph $x^{2/3} + y^{2/3} = a^{2/3}$. (This argument can be strengthed by considering a ray from the origin making the angle t ($0 \leqq t < 2\pi$) with the nonnegative x-axis and examining its intersection with the astroid.)

Part (b): Next,

$$\frac{dy}{dx} = \frac{3a\sin^2 t\,\cos t}{-3a\cos^2 t\,\sin t} = -\tan t$$

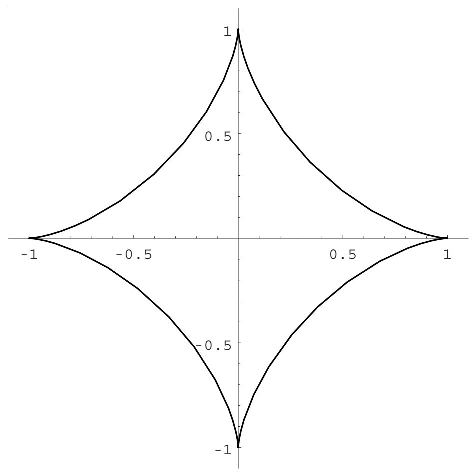
and, consequently, $dx/dy = -\cot t$. So $dy/dx = 0$ at every integral multiple of π and $dx/dy = 0$ at every odd integral multiple of $\pi/2$. Therefore the hypocycloid has horizontal tangents at $(a,\,0)$ $(t = 0)$ and at $(-a,\,0)$ $(t = \pi)$, vertical tangents at $(0,\,a)$ $(t = \pi/2)$ and $(0,\,-a)$ $(t = 3\pi/2)$. Next,

$$\frac{d^2y}{dx^2} = \frac{1}{dx/dt} \cdot \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-\sec^2 t}{-3a\cos^2 t\,\sin t} = \frac{1}{3a\cos^4 t\,\sin t}.$$

Thus d^2y/dx^2 has the same sign as $a\sin t$, and therefore the graph of the hypocycloid is concave upward for $0 < t < \pi/2$ and for $\pi/2 < t < \pi$, concave downward for $\pi < t < 3\pi/2$ and $3\pi/2 < t < 2\pi$.

Part (c): A *Mathematica*-generated graph of the hypocycloid (in the case $a = 1$) is next. The command used to generate the graph was

```
ParametricPlot[ { (Cos[t])∧3, (Sin[t])∧3 }, { t, 0, 2*Pi },
     AspectRatio → Automatic, PlotRange → {{ -1.1, 1.1 }, { -1.1, 1.1 }} ];
```



C10S04.037: Extend OP the distance a to the point R at the "northeast" corner of Archimedes' rectangle. Because P has Cartesian coordinates

$$x = a\theta\cos\theta, \qquad y = a\theta\sin\theta,$$

it follows that R has coordinates

$$x = a\theta\cos\theta + a\cos\theta, \qquad y = a\theta\sin\theta + a\sin\theta.$$

Next, Q has coordinates

$$a\theta\cos\theta + a\cos\theta - a\theta\sin\theta, \qquad y = a\theta\sin\theta + a\sin\theta + a\theta\cos\theta.$$

Therefore the slope of PQ is

$$\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$$

The spiral has polar equation $r = a\theta$, thus parametric equations

$$x = a\theta\cos\theta, \qquad y = a\theta\sin\theta.$$

Therefore

$$\frac{dy}{dx} = \frac{a\sin\theta + a\theta cos\theta}{a\cos\theta - a\theta\sin\theta} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$$

Hence the line containing P and Q is tangent to the spiral at the point P.

C10S04.038: Part (a): Given: The cycloid with parametric equations $x = a(t - \sin t)$, $y = a(1 - \cos t)$ where $a > 0$. If t is not an integral multiple of 2π, then

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} = \frac{2 \sin\left(\dfrac{t}{2}\right) \cos\left(\dfrac{t}{2}\right)}{2 \sin^2\left(\dfrac{t}{2}\right)} = \cot\left(\frac{t}{2}\right).$$

Part (b): $\displaystyle \lim_{t \to 0} \left| \frac{dy}{dx} \right| = \lim_{t \to 0} \left| \frac{\cos(t/2)}{\sin(t/2)} \right| = +\infty$. So there is a vertical tangent at each cusp of the cycloid.

C10S04.039: If ψ is constant, then by Eq. (10) of the text

$$\begin{aligned}
&\frac{1}{r} \cdot \frac{dr}{d\theta} = k \qquad \text{(a constant);}\\
&\frac{1}{r}\, dr = k\, d\theta;\\
&\ln r = C + k\theta \qquad \text{(where } C \text{ is constant);}\\
&r = Ae^{k\theta} \qquad \text{(where } A = e^C\text{).}
\end{aligned}$$

C10S04.040: With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, we have

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Therefore if $f(\alpha) = 0$, then

$$\left.\frac{dy}{dx}\right|_{\theta=\alpha} = \frac{f'(\alpha)\sin\alpha}{f'(\alpha)\cos\alpha} = \tan\alpha,$$

and therefore the angle of inclination of the tangent line at the pole is indeed α.

C10S04.041: Let $y = tx$ where $t \geqq 0$. Then this line meets the loop at exactly one point in the first quadrant. For such points on the loop, we then have

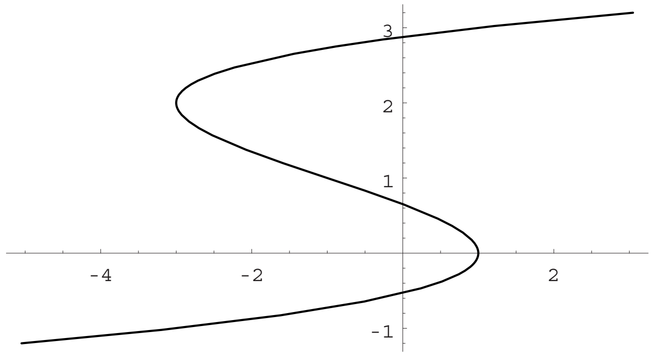
$$x^5 + t^5x^5 = 5t^2x^4; \qquad x = \frac{5t^2}{1 + t^5}, \quad y = \frac{5t^3}{1 + t^5}, \quad 0 \leqq t < +\infty.$$

C10S04.042: Suppose that the midpoint of the segment lies in the second quadrant (the other three cases are similar), so that the endpoints of the segment lie on the positive y-axis and the negative x-axis. Drop perpendiculars from the midpoint to the coordinate axes to see that the coordinates of the midpoint are $x = -a\cos\theta$ and $y = a\sin\theta$. It follows that $x^2 + y^2 = a^2$ (in all four cases), and thus the locus of the midpoint is the circle of radius a centered at the origin. The problem is more interesting if the point on the segment is one other than its midpoint, or if two points on the segment are constrained to lie on the coordinate axes while one endpoint traces out the locus.

C10S04.043: Let $f(x) = x^3 - 3x^2 + 1$. Then $f'(x) = 0$ when $x = 0$ and when $x = 2$; $f''(x) = 0$ when $x = 1$. Hence the graph of the parametric equations

$$x = t^3 - 3t^2 + 1, \qquad y = t \tag{1}$$

has vertical tangents at $(-3, 2)$ and $(1, 0)$ and an inflection point at $(-1, 1)$. There are no horizontal tangents and the only critical points occur at the points where the tangent line is vertical. The graph of the equations in (1) is next.



We generated this graph by executing the *Mathematica* command

```
ParametricPlot[ { t∧3 − 3∗t∧2 + 1, t }, { t, −1.2, 3.2 },
                AspectRatio → Automatic ];
```

C10S04.044: Let $f(x) = x^4 - 3x^3 + 5x$. Then $f'(x) = 0$ when

$$x = 1 \quad \text{and when} \quad x = \frac{5 \pm \sqrt{105}}{8}.$$

The corresponding values of $f(x)$ are

$$y = 3 \quad \text{and} \quad y = \frac{-75 \pm 105\sqrt{105}}{512}.$$

Thus the graph of the parametric equations

$$x = t^4 - 3t^3 + 5t, \qquad y = t \tag{1}$$

has vertical tangent lines at the point $(3, 1)$ and at the points with approximate coordinates

$$(-2.2479098251, -0.6558688457) \quad \text{and} \quad (1.9549410751, 1.9058688457).$$

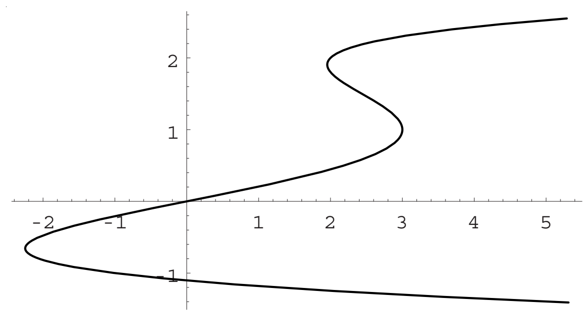
Next, $f''(x) = 0$ when $x = 0$ and when $x = 1.5$, so the graph of the equations in (1) has inflection points at

$$(0, 0) \quad \text{and} \quad \left(\frac{39}{16}, \frac{3}{2}\right) = (2.4375, 1.5).$$

There are no horizontal tangents and the only critical points are the three where the tangent line is vertical. The graph of the parametric equations in (1) is next; we generated it with the *Mathematica* command

```
ParametricPlot[ { t∧5 − 3∗t∧3 + 5∗t, t }, { t, −1.41, 2.55 },
```

```
AspectRatio → Automatic ];
```



C10S04.045: To use *Mathematica* 3.0 to help solve this problem, we let

```
g[t_] := t∧5 − 5∗t∧3 + 4
h[t_] := (Sign[g[t]])∗(Abs[g[t]])∧(1/3)
```

and

```
f[t_] := (g[t])∧(1/3)
```

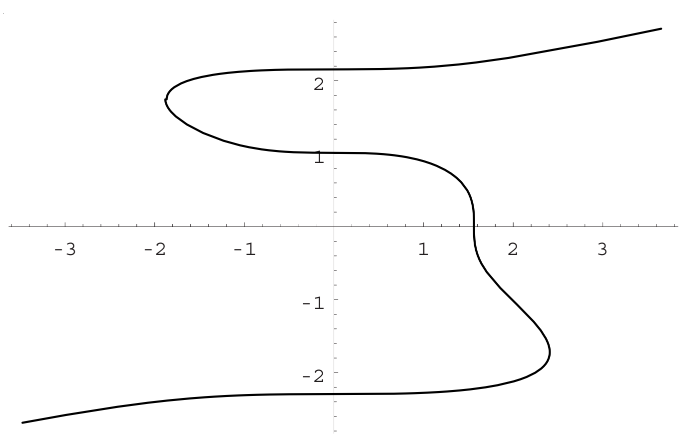
(note that $h(t) = f(t)$; we define h to avoid certain problems with cube roots of negative numbers). To see the graph of the parametric equations

$$x = (t^5 - 5t^3 + 4)^{1/3}, \qquad y = t,$$

we executed the *Mathematica* command

```
ParametricPlot[ { h[t], t }, { t, −2.7, 2.7 } ];
```

with the result shown next.



Next we found that

$$f'(t) = \frac{5t^2(t^2-3)}{3(t^5-5t^3+4)^{2/3}} \quad \text{and that} \quad f''(t) = \frac{10(t^8-9t^6+24t^3-36t)}{9(t^5-5t^3+4)^{5/3}}.$$

If follows that $f'(t) = 0$ when $t = 0$ and when $t = \pm\sqrt{3}$. Hence the graph of the equation $x^3 = y^5 - 5y^3 + 4$ has *vertical* tangents at the three points

$$\left(\left[4-6\sqrt{3}\,\right]^{1/3},\ \sqrt{3}\right) \approx (-1.855891115,\ 1.732050808),$$

$$\left(\left[4+6\sqrt{3}\,\right]^{1/3},\ -\sqrt{3}\right) \approx (2.432447355,\ 1.732050808), \qquad \text{and}$$

$$\left(4^{1/3},\ 0\right) \approx (1.587401052,\ 0).$$

The *horizontal* tangents will occur when the denominator in $f'(t) = 0$; that is, at

$$(0,\ -2.307699789), \qquad (0,\ 1), \qquad \text{and} \qquad (0,\ 2.143299604)$$

(numbers with decimal points are approximations). Finally, Newton's method yields the zeros of the numerator of $f''(t)$, and—also checking the zeros of its denominator—we find that the graph has inflection points at

$$\begin{array}{ll} (-5.150545372,\ -3.110298772), & (0,\ -2.307699789), \\ (2.037032912,\ -1.044330352), & (1.587401052,\ 0), \\ (0,\ 1), & (0,\ 2.143299604), \qquad \text{and} \\ (4.266140637,\ 2.856500901). & \end{array}$$

The first and last of these aren't shown on the preceding graph, but the graph appears to be a straight line in their vicinity, so showing more of the graph is not useful.

C10S04.046: To use *Mathematica* 3.0 to help solve this problem, we let

```
g[t_] := 5*t∧6 - 17*t∧3 + 13*t

h[t_] := (Sign[g[t]]*(Abs[g[t]])∧(1/5)
```

and

```
f[t_] := (g[t])∧(1/5)
```

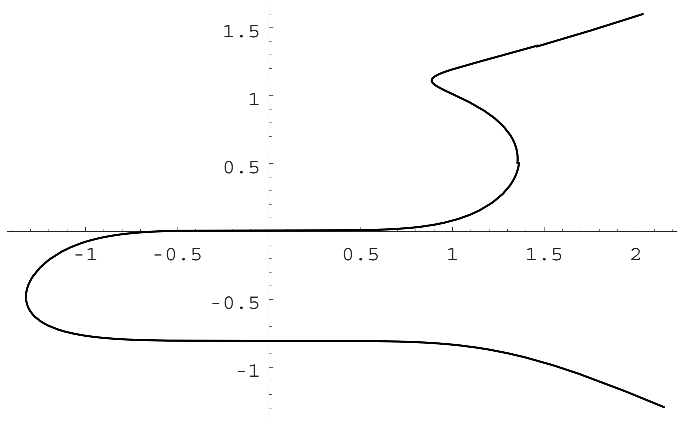
(note that $h(t) = f(t)$; we define h to avoid certain problems with odd integral roots of negative numbers). To see the graph of the parametric equations

$$x = (5t^6 - 17t^3 + 13y)^{1/5}, \qquad y = t,$$

we executed the *Mathematica* command

```
ParametricPlot[ { h[t], t }, { t, −1.3, 1.6 } ];
```

with the result shown next.



Next we found that

$$f'(t) = \frac{30t^5 - 51t^2 + 13}{5(5t^6 - 17t^3 + 13t)^{4/5}} \qquad \text{and} \qquad f''(t) = \frac{2(75t^{10} - 1530t^7 + 3315t^5 - 867t^4 - 663t^2 - 338)}{25(5t^6 - 17t^3 + 13t)^{9/5}}.$$

It follows with the aid of Newton's method that $f'(t) = 0$ when $t \approx -0.488$, when $t \approx 0.528$, and when $t \approx 1.104$. Hence the graph of the equation $x^5 = 5y^6 - 17y^3 + 13y$ has *vertical* tangents at the three points

$$(-1.338784051,\ -0.488418117), \quad (0.80783532,\ 1.103631606), \quad \text{and} \quad (1.349152308,\ 0.528310640).$$

The *horizontal* tangents will occur when the denominator in $f'(t) = 0$; that is, at

$$(0,\ -0.812678591) \quad \text{and at} \quad (0,\ 0)$$

(numbers with decimal points are approximations). Finally, Newton's method yields the zeros of the numerator of $f''(t)$ and—also checking the zeros of its denominator—we find that the graph has inflection points at

$$\begin{array}{ll} (0,\ -0.813073457), & (2.513563956,\ -1.516333702), \\ (0,\ 0), & (0.992940442,\ 1.004409592), \\ (1.185188240,\ 1.258427457), & (3.735641965,\ 2.388360391). \end{array}$$

The second and last of these don't appear on the preceding graph, but the graph appears to be a straight line in their vicinity, so showing more of the graph isn't much use.

Section 10.5

C10S05.001: The area is

$$A = \int_{-1}^{1} (2t^2+1)(3t^2)\,dt = \int_{-1}^{1} (6t^4+3t^2)\,dt = \left[\frac{6}{5}t^5 + t^3\right]_{-1}^{1} = 2\cdot\left(\frac{6}{5}+1\right) = \frac{22}{5}.$$

C10S05.002: The area is

$$A = \int_0^{\ln 2} (e^{-t})(3e^{3t})\,dt = \int_0^{\ln 2} 3e^{2t}\,dt = \left[\frac{3}{2}e^{2t}\right]_0^{\ln 2} = 6 - \frac{3}{2} = \frac{9}{2}.$$

C10S05.003: The area is

$$A = \int_0^{\pi} \sin^3 t\,dt = \int_0^{\pi} \big(\sin t - \cos^2 t\,\sin t\big)\,dt = \left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi} = \frac{4}{3}.$$

C10S05.004: The area is $A = \displaystyle\int_0^1 3e^{2t}\,dt = \left[\frac{3}{2}e^{2t}\right]_0^1 = \frac{3}{2}(e^2-1) \approx 9.5835841484.$

C10S05.005: The area is

$$A = \int_0^{\pi} e^t \sin t\,dt = \left[\frac{1}{2}e^t(\sin t - \cos t)\right]_0^{\pi} = \frac{1}{2}(e^{\pi}+1) \approx 12.0703463164.$$

See Example 5 of Section 8.3 for the evaluation of the antiderivative using integration by parts.

C10S05.006: The area is

$$A = \int_0^1 (2t+1)e^t\,dt = \Big[(2t-1)e^t\Big]_0^1 = e-(-1) = e+1 \approx 3.718281828459.$$

See Example 3 of Section 8.3 for the evaluation of the antiderivative using integration by parts.

C10S05.007: The volume is

$$\begin{aligned} V &= \int_{-1}^{1} \pi(2t^2+1)^2\cdot 3t^2\,dt = \pi\int_{-1}^{1}(12t^6+12t^4+3t^2)\,dt \\ &= \pi\left[\frac{12}{7}t^7 + \frac{12}{5}t^5 + t^3\right]_{-1}^{1} = \left(\frac{179}{35}+\frac{179}{35}\right)\pi = \frac{358}{35}\pi \approx 32.1340048567. \end{aligned}$$

C10S05.008: The volume is

$$V = \int_0^{\ln 2} \pi(e^{-2t})\cdot 3e^{3t}\,dt = \pi\int_0^{\ln 2} 3e^t\,dt = 3\pi\Big[\,e^t\,\Big]_0^{\ln 2} = 3\pi \approx 9.424777960769.$$

C10S05.009: The volume is

$$V = \int_0^\pi \pi(\sin t)^5\, dt = \pi \int_0^\pi (1 - 2\cos^2 t + \cos^4 t) \sin t\, dt$$

$$= \pi \left[-\frac{1}{5}\cos^5 t + \frac{2}{3}\cos^3 t - \cos t \right]_0^\pi = \pi \left(\frac{8}{15} + \frac{8}{15} \right) = \frac{16}{15}\pi \approx 3.351032163829.$$

C10S05.010: The volume is

$$V = \pi \int_0^\pi e^{2t} \sin t\, dt = \pi \left[\frac{1}{5} e^{2t}(2\sin t - \cos t) \right]_0^\pi$$

$$= \frac{\pi}{5} e^{2\pi} + \frac{\pi}{5} = \frac{\pi}{5}\left(e^{2\pi} + 1\right) \approx 337.087648741765.$$

See Example 5 of Section 8.3 for the technique of finding the antiderivative using integration by parts.

C10S05.011: The arc-length element is $ds = (t+4)^{1/2}\, dt$. Hence the length of the curve is

$$L = \int_5^{12} (t+4)^{1/2}\, dt = \left[\frac{2}{3}(t+4)^{3/2} \right]_5^{12} = \frac{128}{3} - 18 = \frac{74}{3} \approx 24.6666666667.$$

C10S05.012: The arc-length element is $ds = (t^2 + t^4)^{1/2}\, dt = t(t^2+1)^{1/2}\, dt$. Thus the length of the curve is

$$L = \int_0^1 t(t^2+1)^{1/2}\, dt = \left[\frac{1}{3}(t^2+1)^{3/2} \right]_0^1 = \frac{2\sqrt{2} - 1}{3} \approx 0.6094757082.$$

C10S05.013: The arc-length element is $ds = \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}\; dt = \sqrt{2}\; dt$. Therefore the length of the curve is

$$L = \int_{\pi/4}^{\pi/2} \sqrt{2}\; dt = \Big[t\sqrt{2}\, \Big]_{\pi/4}^{\pi/2} = \frac{\pi\sqrt{2}}{4} \approx 1.1107207345.$$

C10S05.014: The arc-length element is

$$ds = \sqrt{(e^t \cos t + e^t \sin t)^2 + (e^t \cos t - e^t \sin t)^2}\; dt = e^t\sqrt{2}\; dt.$$

Therefore the length of the curve is

$$L = \int_0^\pi e^t\sqrt{2}\; dt = \Big[e^t\sqrt{2}\, \Big]_0^\pi = (e^\pi - 1)\sqrt{2} \approx 31.3116678016.$$

C10S05.015: Equation (10) of the text tells us that the arc-length element in polar coordinates is

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta = \left(e^\theta + \frac{1}{4}e^\theta \right)^{1/2} d\theta = \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta.$$

Therefore the length of the curve is

$$L = \int_0^{4\pi} \frac{\sqrt{5}}{2} e^{\theta/2}\,d\theta = \left[e^{\theta/2}\sqrt{5}\, \right]_0^{4\pi} = \left(e^{2\pi} - 1\right)\sqrt{5} \approx 1195.159675159775.$$

C10S05.016: The arc-length element is $ds = \sqrt{\theta^2 + 1}\,d\theta$. Thus the length of the curve is

$$\begin{aligned} L &= \int_{2\pi}^{4\pi} \sqrt{\theta^2+1}\,d\theta = \frac{1}{2}\left[\theta\sqrt{\theta^2+1} + \ln\left(\theta + \sqrt{\theta^2+1}\,\right)\right]_{2\pi}^{4\pi} \\ &= \frac{1}{2}\left[4\pi\sqrt{16\pi^2+1} + \ln\left(4\pi + \sqrt{16\pi^2+1}\,\right) - 2\pi\sqrt{4\pi^2+1} - \ln\left(2\pi + \sqrt{4\pi^2+1}\,\right)\right] \\ &\approx 59.563021935206. \end{aligned}$$

The antiderivative was obtained using a trigonometric substitution (as in Section 8.6); alternatively, a hyperbolic substitution can be used, or simply apply integral formula 44 of the endpapers of the text.

C10S05.017: The arc-length element is $ds = \left(1 + \dfrac{1}{t}\right)^{1/2} dt$, so the surface area is

$$\begin{aligned} A = \int_1^4 2\pi \cdot 2t^{1/2} \cdot \left(1 + \frac{1}{t}\right)^{1/2} dt &= \int_1^4 4\pi(t+1)^{1/2}\,dt = \frac{8\pi}{3}\left[(t+1)^{3/2}\right]_1^4 \\ &= \frac{40\pi\sqrt{5}}{3} - \frac{16\pi\sqrt{2}}{3} = \frac{8\pi}{3}\left(5\sqrt{5} - 2\sqrt{2}\,\right) \approx 69.968820743698. \end{aligned}$$

C10S05.018: The arc-length element is

$$ds = \left[\left(4t - \frac{1}{t^2}\right)^2 + \frac{16}{t}\right]^{1/2} dt = \left(16t^2 + \frac{8}{t} + \frac{1}{t^4}\right)^{1/2} dt = \frac{(16t^6 + 8t^3 + 1)^{1/2}}{t^2}\,dt = \frac{4t^3+1}{t^2}\,dt,$$

so the surface area of revolution is

$$\begin{aligned} A &= 2\pi \int_1^2 8t^{1/2} \cdot \frac{4t^3+1}{t^2}\,dt = 2\pi \int_1^2 \frac{8(4t^3+1)}{t^{3/2}}\,dt = 2\pi \int_1^2 (32t^{3/2} + 8t^{-3/2})\,dt \\ &= 2\pi\left[\frac{64}{5}t^{5/2} - 16t^{-1/2}\right]_1^2 = 2\pi \cdot \frac{216\sqrt{2} + 16}{5} = \frac{16\pi}{5}\left(2 + 27\sqrt{2}\,\right) \approx 403.971278839858. \end{aligned}$$

C10S05.019: The arc-length element is $ds = (9t^4 + 4)^{1/2}\,dt$, but the surface area of revolution is *not*

$$\int_{-1}^1 2\pi t^3 (9t^4+4)^{1/2}\,dt.$$

The reason is that the radius of the circle of revolution is t^3, which is negative for $-1 \leqq t < 0$. But symmetry of the graph allows us to double the integral over $[0, 1]$ to find the area:

$$A = 2\int_0^1 2\pi t^3(9t^4+4)^{1/2}\,dt = \frac{2\pi}{27}\left[(9t^4+4)^{3/2}\right]_0^1 = \frac{2\pi}{27}\left(13\sqrt{13} - 8\right) \approx 9.045963922970.$$

C10S05.020: The arc-length element is $ds = \sqrt{4 + (2t+1)^2}\,dt = (4t^2 + 4t + 5)^{1/2}\,dt$, so the surface area of revolution is

$$A = \int_0^3 2\pi(2t+1)(4t^2+4t+5)^{1/2}\ dt = \frac{\pi}{3}\Big[(4t^2+4t+5)^{3/2}\Big]_0^3$$

$$= \frac{53\pi\sqrt{53}}{3} - \frac{5\pi\sqrt{5}}{3} = \frac{\pi}{3}\left(53\sqrt{53} - 5\sqrt{5}\right) \approx 392.3487776186.$$

C10S05.021: The circle with polar equation $r = 4\sin\theta,\ 0 \leqq \theta \leqq \pi,$ is to be rotated around the x-axis. The arc-length element is

$$ds = \sqrt{(4\sin\theta)^2 + (4\cos\theta)^2}\ d\theta = 4\ d\theta$$

and the radius of the circle of revolution is $y = 4\sin^2\theta,$ so the surface area of revolution is

$$A = 2\pi\int_0^\pi 16\sin^2\theta\ d\theta = 16\pi\int_0^\pi (1-\cos 2\theta)\ d\theta = 8\pi\Big[2\theta - \sin 2\theta\Big]_0^\pi = 16\pi^2 \approx 157.9136704174.$$

C10S05.022: The arc-length element is $ds = \sqrt{(e^\theta)^2 + (e^\theta)^2}\ d\theta = e^\theta\sqrt{2}\ d\theta$ and the radius of the circle of revolution is $x = r\cos\theta = e^\theta\cos\theta,$ so the surface area of revolution is

$$A = 2\pi\sqrt{2}\int_0^{\pi/2} e^{2\theta}\cos\theta\ d\theta = \frac{2\pi\sqrt{2}}{5}\Big[(2\cos\theta + \sin\theta)e^{2\theta}\Big]_0^{\pi/2} = \frac{2\pi\sqrt{2}}{5}(e^\pi - 2) \approx 37.5702490396.$$

C10S05.023: The cycloidal arch is the graph of the parametric equations $x = a(t - \sin t),\ y = a(1-\cos t),$ $0 \leqq t \leqq 2\pi,\ a > 0.$ When the region between the arch and the x-axis is rotated around the x-axis, the volume swept out is

$$V = \int_{t=0}^{2\pi} \pi y^2\ dx = \pi a^3\int_0^{2\pi}(1-\cos t)^3\ dt = \pi a^3\int_0^{2\pi}\left[1 - 3\cos t + \frac{3}{2}(1+\cos 2t) - (1-\sin^2 t)\cos t\right] dt$$

$$= \pi a^3\left[t - 3\sin t + \frac{3}{2}t + \frac{3}{4}\sin 2t - \sin t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = \frac{5}{2}\pi a^3\cdot 2\pi = 5\pi^2 a^3.$$

C10S05.024: The cycloidal arch is the graph of the parametric equations $x = a(t-\sin t),\ y = a(1-\cos t),$ $0 \leqq t \leqq 2\pi,\ a > 0.$ Suppose that it is rotated around the x-axis to generate a surface of area A. To find the arc-length element, we first compute

$$[x'(t)]^2 + [y'(t)]^2 = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 4a^2\frac{1-\cos t}{2} = 4a^2\sin^2\left(\frac{t}{2}\right).$$

Thus the arc-length element is

$$ds = 2a\left|\ \sin\left(\frac{t}{2}\right)\right|\ dt = 2a\sin\left(\frac{t}{2}\right)\ dt.$$

We may remove the absolute value symbols because $\sin(t/2) \geqq 0$ if $0 \leqq t \leqq 2\pi$. So the surface area of revolution around the x-axis is

$$A = \int_{t=0}^{2\pi} 2\pi y\ ds = 4\pi a^2 \int_0^{2\pi} (1 - \cos t) \sin\left(\frac{t}{2}\right)\ dt = 8\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right)\ dt$$

$$= 8\pi a^2 \int_0^{2\pi} \left[1 - \cos^2\left(\frac{t}{2}\right)\right] \sin\left(\frac{t}{2}\right)\ dt = 8\pi a^2 \left[-2\cos\left(\frac{t}{2}\right) + \frac{2}{3}\cos^2\left(\frac{t}{2}\right)\right]_0^{2\pi} = \frac{64}{3}\pi a^2.$$

C10S05.025: Part (a): The area of the ellipse is

$$A = 4\int_0^{\pi/2} ab\sin^2 t\ dt = 2ab\int_0^{\pi/2} (1 - \cos 2t)\ dt = 2ab\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2} = \pi ab.$$

Part (b): The volume generated when [the upper half of] the ellipse is rotated around the x-axis is

$$V = 2\int_0^{\pi/2} \pi(b^2\sin^2 t)(a\sin t)\ dt = 2\pi ab^2 \int_0^{\pi/2} (1 - \cos^2 t)\sin t\ dt = 2\pi ab^2\left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = \frac{4}{3}\pi ab^2.$$

Compare this with the solution of Problem 36 in Section 6.2.

C10S05.026: The loop in the graph of $x = t^2,\ y = t^3 - 3t$ is generated as t runs from $-\sqrt{3}$ to $\sqrt{3}$. Its area is

$$A = 2\int_0^{\sqrt{3}} -2t(t^3 - 3t)\ dt = \int_0^{\sqrt{3}} (12t^2 - 4t^4)\ dt = \left[4t^3 - \frac{4}{5}t^5\right]_0^{\sqrt{3}} = \frac{24\sqrt{3}}{5} \approx 8.3138438763.$$

C10S05.027: Using the given Cartesian parametrization, the arc-length element for the spiral is

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}\ dt = \sqrt{t^2 + 1}\ dt.$$

Therefore the arc length is

$$L = \int_0^{2\pi} \sqrt{t^2 + 1}\ dt = \frac{1}{2}\left[t\sqrt{t^2 + 1} + \ln\left(t + \sqrt{t^2 + 1}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi\sqrt{1 + 4\pi^2} + \ln\left(2\pi + \sqrt{1 + 4\pi^2}\right)\right] \approx 21.2562941482.$$

The antiderivative can be obtained with the trigonometric substitution $t = \tan\theta$ or by use of integral formula 44 of the endpapers of the textbook.

C10S05.028: The parametrization $x = b + a\cos t,\ y = a\sin t,\ 0 \leqq t \leqq 2\pi$ of the circle yields the arc-length element $ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}\ dt = a\ dt$. The radius of the circle of revolution around the y-axis is $x = b + a\cos t$, and therefore the surface area of revolution is

$$A = \int_0^{2\pi} 2\pi a(b + a\cos t)\ dt = 2\pi a\Big[bt + a\sin t\Big]_0^{2\pi} = 4\pi^2 ab.$$

C10S05.029: The area bounded by the astroid is

$$A = -4\int_{t=0}^{\pi/2} y\,dx = 4\int_0^{\pi/2} (a\sin^3 t)(3a\cos^2 t\,\sin t)\,dt = 12a^2\int_0^{\pi/2} (\sin^4 t - \sin^6 t)\,dt$$
$$= 12a^2\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}\right) = 12a^2\cdot\frac{3}{16}\pi\left(1-\frac{5}{6}\right) = \frac{3}{8}\pi a^2.$$

The first minus sign is needed because $dx < 0$. The integral was computed using integral formula 113 of the endpapers of the text.

C10S05.030: The arc-length element is

$$ds = (9a^2\cos^4 t\,\sin^2 t + 9a^2\cos^2 t\,\sin^4 t)^{1/2}\,dt$$
$$= 3a\left[(\sin^2 t\,\cos^2 t)(\cos^2 t + \sin^2 t)\right]^{1/2}\,dt = 3a\left(\frac{1}{4}\sin^2 2t\right)^{1/2}\,dt = \frac{3}{2}a\sin 2t\,dt.$$

So the total length of the astroid is

$$A = 4\int_0^{\pi/2} \frac{3}{2}a\sin 2t\,dt = \Big[-3a\cos 2t\Big]_0^{\pi/2} = 3a - (-3a) = 6a.$$

Compare this with the solution of Problem 41 in Section 6.4.

C10S05.031: The arc-length element is

$$ds = (9a^2\cos^4 t\,\sin^2 t + 9a^2\cos^2 t\,\sin^4 t)^{1/2}\,dt$$
$$= 3a\left[(\sin^2 t\,\cos^2 t)(\cos^2 t + \sin^2 t)\right]^{1/2}\,dt = 3a\left(\frac{1}{4}\sin^2 2t\right)^{1/2}\,dt = \frac{3}{2}a\sin 2t\,dt.$$

The radius of the circle of revolution is $y = a\sin^3 t$. So the surface area of revolution around the x-axis is

$$A = 2\int_{t=0}^{\pi/2} 2\pi y\,ds = 2\int_0^{\pi/2}\left(2\pi a\sin^3 t\right)\left(\frac{3}{2}a\sin 2t\right)dt$$
$$= 6\pi a^2\int_0^{\pi/2} 2\sin^4 t\,\cos t\,dt = \frac{12\pi a^2}{5}\Big[\sin^5 t\Big]_0^{\pi/2} = \frac{12}{5}\pi a^2.$$

Compare this with the solution of Problem 42 in Section 6.4.

C10S05.032: First we use Eq. (10) to compute the arc-length element. Given $r^2 = 2a^2\cos 2\theta$,

$$2r\frac{dr}{d\theta} = -4a^2\sin 2\theta;$$
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{16a^4\sin^2 2\theta}{4r^2} = \frac{4a^4\sin^2 2\theta}{2a^2\cos 2\theta} = \frac{2a^2\sin^2 2\theta}{\cos 2\theta};$$
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{2a^2\cos^2 2\theta + 2a^2\sin^2 2\theta}{\cos 2\theta} = \frac{2a^2}{\cos 2\theta};$$
$$ds = \frac{a\sqrt{2}}{\sqrt{\cos 2\theta}}\,d\theta.$$

The radius of the circle of revolution is $x = r\cos\theta = \left(a\sqrt{2\cos 2\theta}\right) \cdot \cos\theta$. Therefore the surface area of revolution around the y-axis is

$$A = 2\int_0^{\pi/4} 2\pi\left(a\sqrt{2\cos 2\theta}\right)(\cos\theta)\frac{a\sqrt{2}}{\sqrt{\cos 2\theta}}\ d\theta = 8\pi a^2 \int_0^{\pi/4} \cos\theta\ d\theta = 8\pi a^2 \Big[\sin\theta\Big]_0^{\pi/4} = 4\pi a^2\sqrt{2}.$$

The first integral here is improper; make the integrand continuous by using its right-hand limit at $\theta = \pi/4$ for its value there.

C10S05.033: The area is

$$A = 2\int_{t=0}^3 y\ dx = 2\int_0^3 \left(3t - \tfrac{1}{3}t^3\right)\left(2t\sqrt{3}\right)\ dt = \sqrt{3}\int_0^3 \left(12t^2 - \frac{4}{3}t^4\right)\ dt$$

$$= \sqrt{3}\left[4t^3 - \frac{4}{15}t^5\right]_0^3 = \frac{216\sqrt{3}}{5} \approx 74.8245948870.$$

C10S05.034: The arc-length element is

$$ds = \sqrt{12t^2 + (t^2-3)^2}\ dt = \sqrt{(t^2+3)^2}\ dt = (t^2+3)\ dt.$$

Therefore the arc length of the loop is

$$L = \int_{-3}^3 (t^2+3)\ dt = \left[\frac{1}{3}t^3 + 3t\right]_{-3}^3 = 9+9+9+9 = 36.$$

C10S05.035: The volume of revolution around the x-axis is

$$V = \int_{t=0}^3 \pi y^2\ dx = \pi\int_0^3 \left(3t - \frac{1}{3}t^3\right)^2 \cdot 2t\sqrt{3}\ dt = 2\pi\sqrt{3}\int_0^3 \left(\frac{1}{9}t^7 - 2t^5 + 9t^3\right)\ dt$$

$$= 2\pi\sqrt{3}\left[\frac{1}{72}t^8 - \frac{1}{3}t^6 + \frac{9}{4}t^4\right]_0^3 = \frac{243\pi\sqrt{3}}{4} \approx 330.5649341317.$$

C10S05.036: We saw in the solution of Problem 34 that the arc-length element is $ds = (t^2+3)\,dt$. Therefore the surface area of revolution around the x-axis is

$$A = \int_{t=0}^3 2\pi y\ ds = \int_0^3 2\pi\left(3t - \frac{1}{3}t^3\right)(t^2+3)\,dt = 2\pi\int_0^3 \left(9t + 2t^3 - \frac{1}{3}t^5\right)\,dt$$

$$= 2\pi\left[\frac{9}{2}t^2 + \frac{1}{2}t^4 - \frac{1}{18}t^6\right]_0^3 = 81\pi \approx 254.4690049408.$$

C10S05.037: Part (a): The parametrization found for the first-quadrant loop of the folium in Section 10.4 was

$$x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}, \quad 0 \leqq t < +\infty.$$

We first need to compute the arc-length element.

$$[x'(t)]^2 = \frac{9(2t^3-1)^2}{(1+t^3)^4} \qquad \text{and} \qquad [y'(t)]^2 = \frac{9(t^4-2t)^2}{(1+t^3)^4};$$

$$ds = \frac{3\sqrt{t^8+4t^6-4t^5-4t^3+4t^2+1}}{(t^3+1)^2}\ dt.$$

Part (b): We will find the length of the loop by integrating ds from $t=0$ to $t=1$ (to avoid an improper integral) and doubling the result. The length is thus

$$L = 2\int_0^1 \frac{3\sqrt{t^8+4t^6-4t^5-4t^3+4t^2+1}}{(t^3+1)^2}\ dt.$$

We used *Mathematica* 3.0 and the command

```
6*NIntegrate[ (Sqrt[ t∧8 + 4*t∧6 - 4*t∧5 - 4*t∧3 + 4*t∧2 + 1 ])/(t∧3 + 1)∧2,
            { t, 0, 1 }, MaxRecursion -> 18, WorkingPrecision -> 28 ]
```

to find that $L \approx 4.917488721682$.

C10S05.038: We use the parametrization $x = a(t - \sin t)$, $y = a(1-\cos t)$, $0 \leqq t \leqq 2\pi$, $a > 0$. The arc-length element is

$$ds = a\sqrt{2(1-\cos t)}\ dt = 2a\sqrt{\frac{1-\cos t}{2}}\ dt = 2a\left|\sin\frac{t}{2}\right|\ dt = 2a\sin\frac{t}{2}\ dt.$$

Therefore the surface area of revolution around the y-axis is

$$A = 2\pi\int_0^{2\pi} a\left(t - 2\sin\frac{t}{2}\,\cos\frac{t}{2}\right)\left(2a\sin\frac{t}{2}\right)\ dt.$$

Let $t = 2u$, so that $dt = 2\ du$. Then

$$A = 2\pi\int_0^{\pi} 2a^2\cdot 2\cdot(2u - 2\sin u\,\cos u)(\sin u)\ du = 16\pi a^2\int_0^{\pi}(u\sin u - \sin^2 u\,\cos u)\ du$$

$$= 16\pi a^2\left[\sin u - u\cos u - \frac{1}{3}\sin^3 u\right]_0^{\pi} = 16\pi^2 a^2.$$

See the solution of Problem 3 in Section 8.3 for the integration by parts to antidifferentiate $u\sin u$. Or if you prefer, imitate a computer algebra program: Assume that the antiderivative has the form $Au\sin u + Bu\cos u + C\sin u + D\cos u$. Differentiate and solve for the four coefficients A, B, C, and D.

C10S05.039: We use the parametrization $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \leqq t \leqq 2\pi$, $a > 0$. By the method of nested cylindrical shells, the volume of revolution around the y-axis is

$$V = \int_{t=0}^{2\pi} 2\pi xy\ dx = \int_0^{2\pi} 2\pi a(t-\sin t)\cdot a(1-\cos t)\cdot a(1-\cos t)\ dt$$

$$= 2\pi a^3\int_0^{2\pi}(t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\,\cos t - \sin t\,\cos^2 t)\ dt$$

$$= 2\pi a^3\int_0^{2\pi}\left(\frac{3}{2}t - 2t\cos t + \frac{1}{2}t\cos 2t - \sin t + 2\sin t\,\cos t - \sin t\,\cos^2 t\right)\ dt$$

$$= 2\pi a^3 \left[\frac{3}{4}t^2 - 2\cos t - 2t\sin t + \frac{1}{8}\cos 2t + \frac{1}{4}t\sin 2t + \cos t + \sin^2 t + \frac{1}{3}\cos^3 t \right]_0^{2\pi} = 6\pi^3 a^3.$$

See the solution of Problem 5 in Section 8.3 for the way integration by parts can be used to find the two more troublesome antiderivatives here.

C10S05.040: Part (a): The point T has coordinates $x = a\cos t$, $y = a\sin t$; let P have coordinates $(x,\, y)$. The slope of OT is $\tan t$ and hence the slope of TP is $-\cot t$. Note also that TP has length at. Therefore

$$\frac{y - a\sin t}{x - a\cos t} = -\cot t = -\frac{\cos t}{\sin t} \qquad \text{and}$$
$$(x - a\cos t)^2 + (y - a\sin t)^2 = a^2t^2.$$

Thus

$$y\sin t - a\sin^2 t = -x\cos t + a\cos^2 t; \qquad \text{that is,} \qquad y\sin t + x\cos t = a.$$

Therefore

$$\begin{aligned}
(x - a\cos t)^2 &= a^2t^2 - (y - a\sin t)^2 = a^2t^2 - \left(\frac{a - x\cos t}{\sin t} - a\sin t \right)^2 \\
&= a^2t^2 - \left(\frac{a - x\cos t - a\sin^2 t}{\sin t} \right)^2 = a^2t^2 - \frac{(a - x\cos t - a + a\cos^2 t)^2}{\sin^2 t} \\
&= a^2t^2 - \frac{1}{\sin^2 t}\left[(x - a\cos t)(-\cos t) + a - a \right]^2 = a^2t^2 - \frac{\cos^2 t}{\sin^2 t}(x - a\cos t)^2.
\end{aligned}$$

It now follows that

$$\begin{aligned}
(1 + \cot^2 t)(x - a\cos t)^2 &= a^2t^2; \\
(x - a\cos t)^2 &= a^2t^2\sin^2 t; \\
x &= a\cos t \pm at\sin t.
\end{aligned}$$

Next,

$$\begin{aligned}
(y - a\sin t)^2 &= a^2t^2 - (x - a\cos t)^2 = a^2t^2 - \left(\frac{a - y\sin t}{\cos t} - a\cos t \right)^2 \\
&= a^2t^2 - \left(\frac{a - y\sin t - a\cos^2 t}{\cos t} \right)^2 = a^2t^2 - \left(\frac{a\sin^2 t - y\sin t}{\cos t} \right)^2 \\
&= a^2t^2 - (a\sin t - y)^2\tan^2 t.
\end{aligned}$$

Therefore

$$\begin{aligned}
(1 + \tan^2 t)(y - a\sin t)^2 &= a^2t^2; \\
(y - a\sin t)^2 &= a^2t^2\cos^2 t; \\
y &= a\sin t \pm at\cos t.
\end{aligned}$$

A moment's reflection about the behavior of $P(x, y)$ for t small positive now makes it evident that the correct choice of signs is

$$x = a(\cos t + t\sin t), \qquad y = a(\sin t - t\cos t).$$

Part (b): After all that algebra, the arc-length element ds almost miraculously simplifies to $at\ dt$. Therefore the length of the involute from $t = 0$ to $t = \pi$ is

$$L = \int_0^{\pi} at\ dt = \left[\frac{a}{2}t^2\right]_0^{\pi} = \frac{\pi^2 a}{2} \approx (4.9348)a.$$

C10S05.041: We will compute the area of the part of the region above the x-axis, then double the result. On the left we see a quarter-circle of radius πa, with area

$$A_1 = \frac{1}{4}\pi(\pi a)^2 = \frac{1}{4}\pi^3 a^2.$$

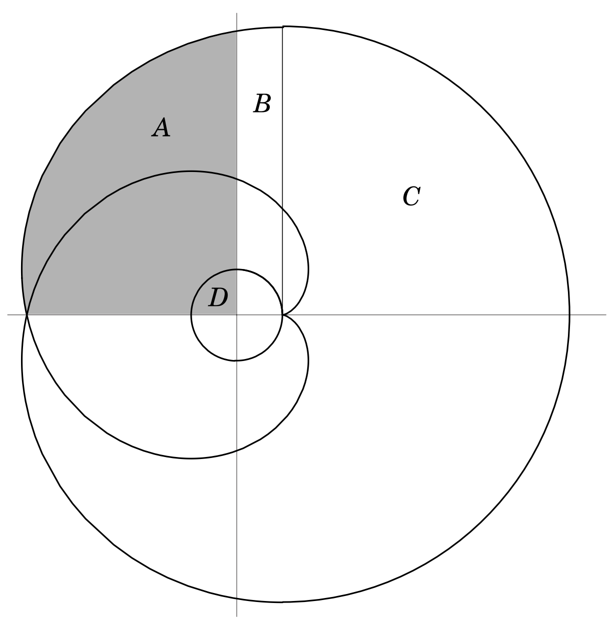
On the right, the area between the involute and the x-axis can be found with an integral:

$$A_2 = \int_0^{\pi} [-y(t)\cdot x'(t)]\ dt = \frac{a^2}{12}\Big[3t^2\sin 2t - 3\sin 2t + 6t\cos 2t + 2t^3\Big]_0^{\pi} = \frac{\pi a^2}{6}(\pi^2 + 3).$$

But we must subtract the area of the part of the water tank above the x-axis, the area of a semicircle of radius a: $A_3 = \frac{1}{2}\pi a^2$. So the total area of the region that the cow can graze is

$$A = 2(A_1 + A_2 - A_3) = \frac{5}{6}\pi^3 a^2.$$

C10S05.042: You can see in the following figure that there is a problem at the extreme left. Each involute moves away from the y-axis very briefly, then moves back toward it.



We can avoid this problem by finding the area that the cow can graze in the third quadrant by integrating *not* $y\ dx$, but instead $x\ dy$. To find the limits of integration on the parameter t, we need to know the value

t_1 of t at which the involute crosses the negative x-axis (where the two involutes cross) and the value t_2 of t at which the outer involute crosses the positive y-axis. Newton's method yields

$$t_1 \approx 4.4934094579 \qquad \text{and} \qquad t_2 \approx 6.121250466898.$$

So the area of the shaded region is

$$\int_{t=t_1}^{t_2} x(t) \cdot y'(t)\, dt \approx (23.106)a^2.$$

The area of the region bounded below by the x-axis, on the left by the y-axis, on the right by the line $x = a$, and above by the outer involute is

$$B = \int_{t=t_1}^{2\pi} [-y(t) \cdot x'(t)]\, dt \approx (6.256)a^2.$$

The area of the quarter-circle bounded below by the x-axis, on the left by the line $x = a$, and on the right and above by the circular arc of radius $2\pi a$ is

$$C = \frac{1}{4}\pi(2\pi a)^2 = \pi^3 a^2 \approx (31.006)a^2.$$

We can obtain the total area that the cow can graze by doubling the sum of the areas A, B, and C, but then we need to subtract the area occupied by the water tank, the area $D = \pi a^2 \approx (3.142)a^2$ of a circle of radius a. So the area the cow can graze is

$$2(A + B + C) - D \approx (117.596)a^2.$$

C10S05.043: Given $r(\theta) = 3\sin 3\theta$, remember that roses with *odd* coefficients are swept out *twice* in the interval $0 \leqq \theta \leqq 2\pi$. Therefore we should integrate

$$ds = \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta = \sqrt{45 + 36\cos 6\theta}\, d\theta$$

from 0 to π to obtain the total length of the rose:

$$\int_{\theta=0}^{\pi} 1\, ds = \int_0^{\pi} \sqrt{45 + 36\cos 6\theta}\, d\theta \approx 20.047339830833.$$

The *Mathematica* 3.0 command we used in Problem 43—we used appropriately modified versions of it for Problems 44 through 55—was

```
NIntegrate[ Sqrt[ 45 + 36*Cos[6*t] ], { t, 0, Pi },
                MaxRecursion -> 18, WorkingPrecision -> 28 ]
```

C10S05.044: Two integrals are required. The surface area is

$$\int_{\theta=0}^{\pi/3} 2\pi x\, ds - \int_{\theta=\pi/3}^{\pi/2} 2\pi x\, ds.$$

The minus sign is needed because $x(\theta) = r(\theta)\cos\theta$ is negative if $\pi/3 \leqq \theta \leqq \pi/2$. The total surface area is approximately 64.912021806645.

C10S05.045: Given $r(\theta) = 2\cos 2\theta$, remember than a rose with an even coefficient n of θ has $2n$ "petals," and is swept out as θ ranges from 0 to 2π. The arc length element in this case is $ds = \sqrt{10 - 6\cos 4\theta}\ d\theta$, and the length of the graph is

$$\int_0^{2\pi} \sqrt{10 - 6\cos 4\theta}\ d\theta \approx 19.376896441095$$

C10S05.046: When the rose of Problem 45 is rotated around the x-axis, the entire surface is generated twice. To obtain each part of the surface once, we will rotate the part of the rose from $\theta = 0$ to $\theta = \pi/4$ and, separately, the part from $\theta = \pi/4$ to $\pi/2$. We will set up an integral for each surface area, add the results, and double the sum. With $x(\theta) = r(\theta)\sin\theta$ and the arc length element ds of Problem 45, we get the integrals

$$\int_{\theta=0}^{\pi/4} 2\pi y\, ds \approx 5.46827 \qquad \text{and} \qquad \int_{\theta=\pi/4}^{\pi/2} (-2\pi y)\, ds \approx 16.1232,$$

for a total area of approximately 43.1829346047.

C10S05.047: Given: $r(\theta) = 5 + 9\cos\theta$, the arc length element is $ds = \sqrt{106 + 90\cos\theta}\ d\theta$, and so the total length of the limaçon is

$$\int_0^{2\pi} \sqrt{106 + 90\cos\theta}\ d\theta \approx 61.003581373850.$$

C10S05.048: The limaçon of Problem 47 is to be rotated around the x-axis. To find the surface area generated, we need to know where $r(\theta) = 0$. The solution is

$$\theta_1 = \cos^{-1}\left(-\frac{5}{9}\right) \approx 2.159827297.$$

So the surface area is

$$\int_{\theta=0}^{\theta_1} 2\pi y\ ds - \int_{\theta=\theta_1}^{\pi} 2\pi y\ ds.$$

The minus sign is needed because $y < 0$ on the part of the limaçon from $\theta = \theta_1$ to π. And we stop at $\theta = \pi$ because the same surface is swept out a second time for $\pi \leqq \theta \leqq 2\pi$. The resulting total surface area is approximately 860.260874010443.

C10S05.049: Given: $r(\theta) = \cos(7\theta/3)$. To sweep out all seven "petals" of this quasi-rose, you need to let θ vary from 0 to 3π. The length of the graph is

$$\int_0^{3\pi} \sqrt{\tfrac{1}{9}\big(29 - 20\cos(14\theta/3)\big)}\ d\theta \approx 16.342833373939.$$

C10S05.050: The length of the graph of this curve is $\displaystyle\int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t}\ dt \approx 9.429431296944.$

C10S05.051: Part (a): When the curve of Problem 50 is rotated around the x-axis, the surface generated is swept out twice. We will rotate the part of the curve in the first quadrant around the x-axis and double the result to get the total surface area

$$2\int_{t=0}^{\pi/2} 2\pi y\ ds \approx 16.057027566602.$$

Part (b): To find the volume of revolution around the x-axis, we evaluate

$$2\int_0^{\pi/2} \pi[y(t)]^2 \cdot x'(t)\ dt = 2\int_0^{\pi/2} 4\pi(\sin^2 t\ \cos t - \sin^4 t\ \cos t)\ dt$$

$$= 2\left[4\pi\left(\frac{1}{3}\sin^3 t - \frac{1}{5}\sin^5 t\right)\right]_0^{\pi/2} = \frac{16}{15}\pi \approx 3.351032163829.$$

C10S05.052: Now the curve of Problems 50 and 51 is to be rotated around the y-axis. We will use the same part of the curve (the part in the first quadrant) and double the answer.

Part (a): The surface area generated is

$$2\int_0^{\pi/2} 2\pi x(t)\sqrt{\cos^2 t + 4\cos^2 2t}\ dt \approx 17.720537653947.$$

Part (b): Using the method of cylindrical shells, the volume enclosed by that surface is

$$2\int_0^{\pi/2} 2\pi x(t)y(t)x'(t)\ dt = 4\pi\int_0^{\pi/2} 2\sin^2 t\cos^2 t\ dt$$

$$= 4\pi\int_0^{\pi/2} \frac{1}{2}(1-\cos^2 2t)\ dt = 2\pi\int_0^{\pi/2}\left(1 - \frac{1+\cos 4t}{2}\right)\ dt$$

$$= 2\pi\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right)\ dt = 2\pi\left[\frac{1}{2}t - \frac{1}{8}\sin 4t\right]_0^{\pi/2} = \frac{1}{2}\pi^2.$$

C10S05.053: The arc-length element is $ds = \sqrt{25\cos^2 5t + 9\sin^2 3t}\ dt$, and the entire Lissajous curve is obtained by letting t range from 0 to 2π. Hence the length of the graph is

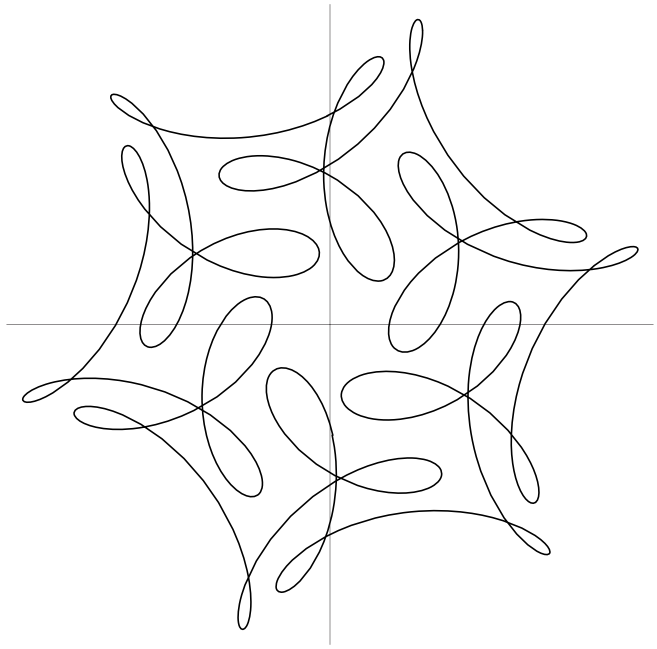
$$\int_0^{2\pi} \sqrt{25\cos^2 5t + 9\sin^2 3t}\ dt \approx 24.602961618540.$$

C10S05.054: The length of the graph is $\displaystyle\int_0^{2\pi} \sqrt{464 - 320\cos 3t}\ dt \approx 130.742666991511.$

C10S05.055: The length of the graph is

$$\int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt \approx 39.403578712896.$$

The graph is next.



Section 10.5

C10S05.001: The area is

$$A = \int_{-1}^{1} (2t^2+1)(3t^2)\,dt = \int_{-1}^{1} (6t^4+3t^2)\,dt = \left[\frac{6}{5}t^5 + t^3\right]_{-1}^{1} = 2\cdot\left(\frac{6}{5}+1\right) = \frac{22}{5}.$$

C10S05.002: The area is

$$A = \int_0^{\ln 2} (e^{-t})(3e^{3t})\,dt = \int_0^{\ln 2} 3e^{2t}\;dt = \left[\frac{3}{2}e^{2t}\right]_0^{\ln 2} = 6 - \frac{3}{2} = \frac{9}{2}.$$

C10S05.003: The area is

$$A = \int_0^{\pi} \sin^3 t\;dt = \int_0^{\pi} \big(\sin t - \cos^2 t\,\sin t\big)\,dt = \left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi} = \frac{4}{3}.$$

C10S05.004: The area is $A = \displaystyle\int_0^1 3e^{2t}\;dt = \left[\frac{3}{2}e^{2t}\right]_0^1 = \frac{3}{2}(e^2-1) \approx 9.5835841484.$

C10S05.005: The area is

$$A = \int_0^{\pi} e^t \sin t\;dt = \left[\frac{1}{2}e^t(\sin t - \cos t)\right]_0^{\pi} = \frac{1}{2}(e^{\pi}+1) \approx 12.0703463164.$$

See Example 5 of Section 8.3 for the evaluation of the antiderivative using integration by parts.

C10S05.006: The area is

$$A = \int_0^1 (2t+1)e^t\;dt = \Big[(2t-1)e^t\Big]_0^1 = e-(-1) = e+1 \approx 3.718281828459.$$

See Example 3 of Section 8.3 for the evaluation of the antiderivative using integration by parts.

C10S05.007: The volume is

$$V = \int_{-1}^{1} \pi(2t^2+1)^2\cdot 3t^2\;dt = \pi\int_{-1}^{1}(12t^6+12t^4+3t^2)\;dt$$

$$= \pi\left[\frac{12}{7}t^7 + \frac{12}{5}t^5 + t^3\right]_{-1}^{1} = \left(\frac{179}{35}+\frac{179}{35}\right)\pi = \frac{358}{35}\pi \approx 32.1340048567.$$

C10S05.008: The volume is

$$V = \int_0^{\ln 2} \pi(e^{-2t})\cdot 3e^{3t}\;dt = \pi\int_0^{\ln 2} 3e^t\;dt = 3\pi\Big[\,e^t\,\Big]_0^{\ln 2} = 3\pi \approx 9.424777960769.$$

C10S05.009: The volume is

$$V = \int_0^{\pi} \pi(\sin t)^5 \; dt = \pi \int_0^{\pi} (1 - 2\cos^2 t + \cos^4 t) \sin t \; dt$$

$$= \pi \left[-\frac{1}{5}\cos^5 t + \frac{2}{3}\cos^3 t - \cos t \right]_0^{\pi} = \pi \left(\frac{8}{15} + \frac{8}{15} \right) = \frac{16}{15}\pi \approx 3.351032163829.$$

C10S05.010: The volume is

$$V = \pi \int_0^{\pi} e^{2t} \sin t \; dt = \pi \left[\frac{1}{5} e^{2t}(2\sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{\pi}{5} e^{2\pi} + \frac{\pi}{5} = \frac{\pi}{5}\left(e^{2\pi} + 1\right) \approx 337.087648741765.$$

See Example 5 of Section 8.3 for the technique of finding the antiderivative using integration by parts.

C10S05.011: The arc-length element is $ds = (t+4)^{1/2} \; dt$. Hence the length of the curve is

$$L = \int_5^{12} (t+4)^{1/2} \; dt = \left[\frac{2}{3}(t+4)^{3/2} \right]_5^{12} = \frac{128}{3} - 18 = \frac{74}{3} \approx 24.6666666667.$$

C10S05.012: The arc-length element is $ds = (t^2 + t^4)^{1/2} \; dt = t(t^2+1)^{1/2} \; dt$. Thus the length of the curve is

$$L = \int_0^1 t(t^2+1)^{1/2} \; dt = \left[\frac{1}{3}(t^2+1)^{3/2} \right]_0^1 = \frac{2\sqrt{2} - 1}{3} \approx 0.6094757082.$$

C10S05.013: The arc-length element is $ds = \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} \; dt = \sqrt{2} \; dt$. Therefore the length of the curve is

$$L = \int_{\pi/4}^{\pi/2} \sqrt{2} \; dt = \left[t\sqrt{2} \right]_{\pi/4}^{\pi/2} = \frac{\pi\sqrt{2}}{4} \approx 1.1107207345.$$

C10S05.014: The arc-length element is

$$ds = \sqrt{(e^t \cos t + e^t \sin t)^2 + (e^t \cos t - e^t \sin t)^2} \; dt = e^t \sqrt{2} \; dt.$$

Therefore the length of the curve is

$$L = \int_0^{\pi} e^t \sqrt{2} \; dt = \left[e^t \sqrt{2} \right]_0^{\pi} = (e^{\pi} - 1)\sqrt{2} \approx 31.3116678016.$$

C10S05.015: Equation (10) of the text tells us that the arc-length element in polar coordinates is

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \; d\theta = \left(e^{\theta} + \frac{1}{4} e^{\theta} \right)^{1/2} \; d\theta = \frac{\sqrt{5}}{2} e^{\theta/2} \; d\theta.$$

Therefore the length of the curve is

$$L = \int_0^{4\pi} \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta = \left[e^{\theta/2}\sqrt{5} \right]_0^{4\pi} = \left(e^{2\pi} - 1\right)\sqrt{5} \approx 1195.159675159775.$$

C10S05.016: The arc-length element is $ds = \sqrt{\theta^2 + 1}\, d\theta$. Thus the length of the curve is

$$\begin{aligned} L &= \int_{2\pi}^{4\pi} \sqrt{\theta^2 + 1}\, d\theta = \frac{1}{2}\left[\theta\sqrt{\theta^2+1} + \ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_{2\pi}^{4\pi} \\ &= \frac{1}{2}\left[4\pi\sqrt{16\pi^2+1} + \ln\left(4\pi + \sqrt{16\pi^2+1}\right) - 2\pi\sqrt{4\pi^2+1} - \ln\left(2\pi + \sqrt{4\pi^2+1}\right)\right] \\ &\approx 59.563021935206. \end{aligned}$$

The antiderivative was obtained using a trigonometric substitution (as in Section 8.6); alternatively, a hyperbolic substitution can be used, or simply apply integral formula 44 of the endpapers of the text.

C10S05.017: The arc-length element is $ds = \left(1 + \dfrac{1}{t}\right)^{1/2} dt$, so the surface area is

$$\begin{aligned} A &= \int_1^4 2\pi \cdot 2t^{1/2} \cdot \left(1 + \frac{1}{t}\right)^{1/2} dt = \int_1^4 4\pi(t+1)^{1/2}\, dt = \frac{8\pi}{3}\left[(t+1)^{3/2}\right]_1^4 \\ &= \frac{40\pi\sqrt{5}}{3} - \frac{16\pi\sqrt{2}}{3} = \frac{8\pi}{3}\left(5\sqrt{5} - 2\sqrt{2}\right) \approx 69.968820743698. \end{aligned}$$

C10S05.018: The arc-length element is

$$ds = \left[\left(4t - \frac{1}{t^2}\right)^2 + \frac{16}{t}\right]^{1/2} dt = \left(16t^2 + \frac{8}{t} + \frac{1}{t^4}\right)^{1/2} dt = \frac{(16t^6 + 8t^3 + 1)^{1/2}}{t^2}\, dt = \frac{4t^3+1}{t^2}\, dt,$$

so the surface area of revolution is

$$\begin{aligned} A &= 2\pi \int_1^2 8t^{1/2} \cdot \frac{4t^3+1}{t^2}\, dt = 2\pi \int_1^2 \frac{8(4t^3+1)}{t^{3/2}}\, dt = 2\pi \int_1^2 (32t^{3/2} + 8t^{-3/2})\, dt \\ &= 2\pi \left[\frac{64}{5} t^{5/2} - 16t^{-1/2}\right]_1^2 = 2\pi \cdot \frac{216\sqrt{2} + 16}{5} = \frac{16\pi}{5}\left(2 + 27\sqrt{2}\right) \approx 403.971278839858. \end{aligned}$$

C10S05.019: The arc-length element is $ds = (9t^4 + 4)^{1/2}\, dt$, but the surface area of revolution is *not*

$$\int_{-1}^1 2\pi t^3 (9t^4 + 4)^{1/2}\, dt.$$

The reason is that the radius of the circle of revolution is t^3, which is negative for $-1 \leqq t < 0$. But symmetry of the graph allows us to double the integral over $[0, 1]$ to find the area:

$$A = 2\int_0^1 2\pi t^3 (9t^4+4)^{1/2}\, dt = \frac{2\pi}{27}\left[(9t^4+4)^{3/2}\right]_0^1 = \frac{2\pi}{27}\left(13\sqrt{13} - 8\right) \approx 9.045963922970.$$

C10S05.020: The arc-length element is $ds = \sqrt{4 + (2t+1)^2}\, dt = (4t^2 + 4t + 5)^{1/2}\, dt$, so the surface area of revolution is

$$A = \int_0^3 2\pi(2t+1)(4t^2+4t+5)^{1/2}\,dt = \frac{\pi}{3}\Big[(4t^2+4t+5)^{3/2}\Big]_0^3$$

$$= \frac{53\pi\sqrt{53}}{3} - \frac{5\pi\sqrt{5}}{3} = \frac{\pi}{3}\left(53\sqrt{53} - 5\sqrt{5}\right) \approx 392.3487776186.$$

C10S05.021: The circle with polar equation $r = 4\sin\theta,\ 0 \leqq \theta \leqq \pi,$ is to be rotated around the x-axis. The arc-length element is

$$ds = \sqrt{(4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta = 4\,d\theta$$

and the radius of the circle of revolution is $y = 4\sin^2\theta,$ so the surface area of revolution is

$$A = 2\pi\int_0^\pi 16\sin^2\theta\,d\theta = 16\pi\int_0^\pi (1-\cos 2\theta)\,d\theta = 8\pi\Big[2\theta - \sin 2\theta\Big]_0^\pi = 16\pi^2 \approx 157.9136704174.$$

C10S05.022: The arc-length element is $ds = \sqrt{(e^\theta)^2 + (e^\theta)^2}\,d\theta = e^\theta\sqrt{2}\,d\theta$ and the radius of the circle of revolution is $x = r\cos\theta = e^\theta\cos\theta,$ so the surface area of revolution is

$$A = 2\pi\sqrt{2}\int_0^{\pi/2} e^{2\theta}\cos\theta\,d\theta = \frac{2\pi\sqrt{2}}{5}\Big[(2\cos\theta + \sin\theta)e^{2\theta}\Big]_0^{\pi/2} = \frac{2\pi\sqrt{2}}{5}(e^\pi - 2) \approx 37.5702490396.$$

C10S05.023: The cycloidal arch is the graph of the parametric equations $x = a(t-\sin t),\ y = a(1-\cos t),$ $0 \leqq t \leqq 2\pi,\ a > 0.$ When the region between the arch and the x-axis is rotated around the x-axis, the volume swept out is

$$V = \int_{t=0}^{2\pi}\pi y^2\,dx = \pi a^3\int_0^{2\pi}(1-\cos t)^3\,dt = \pi a^3\int_0^{2\pi}\left[1 - 3\cos t + \frac{3}{2}(1+\cos 2t) - (1-\sin^2 t)\cos t\right]dt$$

$$= \pi a^3\left[t - 3\sin t + \frac{3}{2}t + \frac{3}{4}\sin 2t - \sin t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = \frac{5}{2}\pi a^3\cdot 2\pi = 5\pi^2 a^3.$$

C10S05.024: The cycloidal arch is the graph of the parametric equations $x = a(t-\sin t),\ y = a(1-\cos t),$ $0 \leqq t \leqq 2\pi,\ a > 0.$ Suppose that it is rotated around the x-axis to generate a surface of area A. To find the arc-length element, we first compute

$$[x'(t)]^2 + [y'(t)]^2 = a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) = 4a^2\frac{1-\cos t}{2} = 4a^2\sin^2\left(\frac{t}{2}\right).$$

Thus the arc-length element is

$$ds = 2a\left|\sin\left(\frac{t}{2}\right)\right|\,dt = 2a\sin\left(\frac{t}{2}\right)\,dt.$$

We may remove the absolute value symbols because $\sin(t/2) \geqq 0$ if $0 \leqq t \leqq 2\pi.$ So the surface area of revolution around the x-axis is

$$A = \int_{t=0}^{2\pi} 2\pi y \, ds = 4\pi a^2 \int_0^{2\pi} (1 - \cos t) \sin\left(\frac{t}{2}\right) dt = 8\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{t}{2}\right) dt$$

$$= 8\pi a^2 \int_0^{2\pi} \left[1 - \cos^2\left(\frac{t}{2}\right)\right] \sin\left(\frac{t}{2}\right) dt = 8\pi a^2 \left[-2\cos\left(\frac{t}{2}\right) + \frac{2}{3}\cos^2\left(\frac{t}{2}\right)\right]_0^{2\pi} = \frac{64}{3}\pi a^2.$$

C10S05.025: Part (a): The area of the ellipse is

$$A = 4\int_0^{\pi/2} ab\sin^2 t \, dt = 2ab\int_0^{\pi/2} (1 - \cos 2t) \, dt = 2ab\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2} = \pi ab.$$

Part (b): The volume generated when [the upper half of] the ellipse is rotated around the x-axis is

$$V = 2\int_0^{\pi/2} \pi(b^2\sin^2 t)(a\sin t) \, dt = 2\pi ab^2 \int_0^{\pi/2} (1 - \cos^2 t)\sin t \, dt = 2\pi ab^2\left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = \frac{4}{3}\pi ab^2.$$

Compare this with the solution of Problem 36 in Section 6.2.

C10S05.026: The loop in the graph of $x = t^2$, $y = t^3 - 3t$ is generated as t runs from $-\sqrt{3}$ to $\sqrt{3}$. Its area is

$$A = 2\int_0^{\sqrt{3}} -2t(t^3 - 3t) \, dt = \int_0^{\sqrt{3}} (12t^2 - 4t^4) \, dt = \left[4t^3 - \frac{4}{5}t^5\right]_0^{\sqrt{3}} = \frac{24\sqrt{3}}{5} \approx 8.3138438763.$$

C10S05.027: Using the given Cartesian parametrization, the arc-length element for the spiral is

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \, dt = \sqrt{t^2 + 1} \, dt.$$

Therefore the arc length is

$$L = \int_0^{2\pi} \sqrt{t^2 + 1} \, dt = \frac{1}{2}\left[t\sqrt{t^2 + 1} + \ln\left(t + \sqrt{t^2 + 1}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi\sqrt{1 + 4\pi^2} + \ln\left(2\pi + \sqrt{1 + 4\pi^2}\right)\right] \approx 21.2562941482.$$

The antiderivative can be obtained with the trigonometric substitution $t = \tan\theta$ or by use of integral formula 44 of the endpapers of the textbook.

C10S05.028: The parametrization $x = b + a\cos t$, $y = a\sin t$, $0 \leqq t \leqq 2\pi$ of the circle yields the arc-length element $ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2} \, dt = a \, dt$. The radius of the circle of revolution around the y-axis is $x = b + a\cos t$, and therefore the surface area of revolution is

$$A = \int_0^{2\pi} 2\pi a(b + a\cos t) \, dt = 2\pi a\Big[bt + a\sin t\Big]_0^{2\pi} = 4\pi^2 ab.$$

C10S05.029: The area bounded by the astroid is

$$A = -4\int_{t=0}^{\pi/2} y\,dx = 4\int_0^{\pi/2} (a\sin^3 t)(3a\cos^2 t\,\sin t)\,dt = 12a^2\int_0^{\pi/2} (\sin^4 t - \sin^6 t)\,dt$$
$$= 12a^2\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}\right) = 12a^2\cdot\frac{3}{16}\pi\left(1-\frac{5}{6}\right) = \frac{3}{8}\pi a^2.$$

The first minus sign is needed because $dx < 0$. The integral was computed using integral formula 113 of the endpapers of the text.

C10S05.030: The arc-length element is

$$ds = (9a^2\cos^4 t\,\sin^2 t + 9a^2\cos^2 t\,\sin^4 t)^{1/2}\,dt$$
$$= 3a\left[(\sin^2 t\,\cos^2 t)(\cos^2 t + \sin^2 t)\right]^{1/2}\,dt = 3a\left(\frac{1}{4}\sin^2 2t\right)^{1/2}\,dt = \frac{3}{2}a\sin 2t\,dt.$$

So the total length of the astroid is

$$A = 4\int_0^{\pi/2} \frac{3}{2}a\sin 2t\,dt = \bigg[-3a\cos 2t\bigg]_0^{\pi/2} = 3a - (-3a) = 6a.$$

Compare this with the solution of Problem 41 in Section 6.4.

C10S05.031: The arc-length element is

$$ds = (9a^2\cos^4 t\,\sin^2 t + 9a^2\cos^2 t\,\sin^4 t)^{1/2}\,dt$$
$$= 3a\left[(\sin^2 t\,\cos^2 t)(\cos^2 t + \sin^2 t)\right]^{1/2}\,dt = 3a\left(\frac{1}{4}\sin^2 2t\right)^{1/2}\,dt = \frac{3}{2}a\sin 2t\,dt.$$

The radius of the circle of revolution is $y = a\sin^3 t$. So the surface area of revolution around the x-axis is

$$A = 2\int_{t=0}^{\pi/2} 2\pi y\,ds = 2\int_0^{\pi/2} \left(2\pi a\sin^3 t\right)\left(\frac{3}{2}a\sin 2t\right)\,dt$$
$$= 6\pi a^2\int_0^{\pi/2} 2\sin^4 t\,\cos t\,dt = \frac{12\pi a^2}{5}\bigg[\sin^5 t\bigg]_0^{\pi/2} = \frac{12}{5}\pi a^2.$$

Compare this with the solution of Problem 42 in Section 6.4.

C10S05.032: First we use Eq. (10) to compute the arc-length element. Given $r^2 = 2a^2\cos 2\theta$,

$$2r\frac{dr}{d\theta} = -4a^2\sin 2\theta;$$
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{16a^4\sin^2 2\theta}{4r^2} = \frac{4a^4\sin^2 2\theta}{2a^2\cos 2\theta} = \frac{2a^2\sin^2 2\theta}{\cos 2\theta};$$
$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{2a^2\cos^2 2\theta + 2a^2\sin^2 2\theta}{\cos 2\theta} = \frac{2a^2}{\cos 2\theta};$$
$$ds = \frac{a\sqrt{2}}{\sqrt{\cos 2\theta}}\,d\theta.$$

The radius of the circle of revolution is $x = r\cos\theta = \left(a\sqrt{2\cos 2\theta}\,\right)\cdot\cos\theta$. Therefore the surface area of revolution around the y-axis is

$$A = 2\int_0^{\pi/4} 2\pi\left(a\sqrt{2\cos 2\theta}\,\right)(\cos\theta)\frac{a\sqrt{2}}{\sqrt{\cos 2\theta}}\ d\theta = 8\pi a^2\int_0^{\pi/4}\cos\theta\ d\theta = 8\pi a^2\Big[\sin\theta\Big]_0^{\pi/4} = 4\pi a^2\sqrt{2}\,.$$

The first integral here is improper; make the integrand continuous by using its right-hand limit at $\theta = \pi/4$ for its value there.

C10S05.033: The area is

$$A = 2\int_{t=0}^3 y\ dx = 2\int_0^3\left(3t - \tfrac{1}{3}t^3\right)\left(2t\sqrt{3}\,\right)\ dt = \sqrt{3}\int_0^3\left(12t^2 - \frac{4}{3}t^4\right)\ dt$$

$$= \sqrt{3}\left[4t^3 - \frac{4}{15}t^5\right]_0^3 = \frac{216\sqrt{3}}{5} \approx 74.8245948870.$$

C10S05.034: The arc-length element is

$$ds = \sqrt{12t^2 + (t^2-3)^2}\ dt = \sqrt{(t^2+3)^2}\ dt = (t^2+3)\ dt.$$

Therefore the arc length of the loop is

$$L = \int_{-3}^3 (t^2+3)\ dt = \left[\frac{1}{3}t^3 + 3t\right]_{-3}^3 = 9+9+9+9 = 36.$$

C10S05.035: The volume of revolution around the x-axis is

$$V = \int_{t=0}^3 \pi y^2\ dx = \pi\int_0^3\left(3t - \frac{1}{3}t^3\right)^2\cdot 2t\sqrt{3}\ dt = 2\pi\sqrt{3}\int_0^3\left(\frac{1}{9}t^7 - 2t^5 + 9t^3\right)\ dt$$

$$= 2\pi\sqrt{3}\left[\frac{1}{72}t^8 - \frac{1}{3}t^6 + \frac{9}{4}t^4\right]_0^3 = \frac{243\pi\sqrt{3}}{4} \approx 330.5649341317.$$

C10S05.036: We saw in the solution of Problem 34 that the arc-length element is $ds = (t^2+3)\,dt$. Therefore the surface area of revolution around the x-axis is

$$A = \int_{t=0}^3 2\pi y\ ds = \int_0^3 2\pi\left(3t - \frac{1}{3}t^3\right)(t^2+3)\,dt = 2\pi\int_0^3\left(9t + 2t^3 - \frac{1}{3}t^5\right)\ dt$$

$$= 2\pi\left[\frac{9}{2}t^2 + \frac{1}{2}t^4 - \frac{1}{18}t^6\right]_0^3 = 81\pi \approx 254.4690049408.$$

C10S05.037: Part (a): The parametrization found for the first-quadrant loop of the folium in Section 10.4 was

$$x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}, \quad 0 \leqq t < +\infty.$$

We first need to compute the arc-length element.

$$[x'(t)]^2 = \frac{9(2t^3-1)^2}{(1+t^3)^4} \qquad \text{and} \qquad [y'(t)]^2 = \frac{9(t^4-2t)^2}{(1+t^3)^4};$$

$$ds = \frac{3\sqrt{t^8+4t^6-4t^5-4t^3+4t^2+1}}{(t^3+1)^2}\,dt.$$

Part (b): We will find the length of the loop by integrating ds from $t=0$ to $t=1$ (to avoid an improper integral) and doubling the result. The length is thus

$$L = 2\int_0^1 \frac{3\sqrt{t^8+4t^6-4t^5-4t^3+4t^2+1}}{(t^3+1)^2}\,dt.$$

We used *Mathematica* 3.0 and the command

```
6*NIntegrate[ (Sqrt[ t∧8 + 4*t∧6 - 4*t∧5 - 4*t∧3 + 4*t∧2 + 1 ])/(t∧3 + 1)∧2,
        { t, 0, 1 }, MaxRecursion -> 18, WorkingPrecision -> 28 ]
```

to find that $L \approx 4.917488721682$.

C10S05.038: We use the parametrization $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \leqq t \leqq 2\pi$, $a > 0$. The arc-length element is

$$ds = a\sqrt{2(1-\cos t)}\,dt = 2a\sqrt{\frac{1-\cos t}{2}}\,dt = 2a\left|\sin\frac{t}{2}\right|\,dt = 2a\sin\frac{t}{2}\,dt.$$

Therefore the surface area of revolution around the y-axis is

$$A = 2\pi\int_0^{2\pi} a\left(t - 2\sin\frac{t}{2}\,\cos\frac{t}{2}\right)\left(2a\sin\frac{t}{2}\right)dt.$$

Let $t = 2u$, so that $dt = 2\,du$. Then

$$A = 2\pi\int_0^{\pi} 2a^2\cdot 2\cdot(2u - 2\sin u\,\cos u)(\sin u)\,du = 16\pi a^2\int_0^{\pi}(u\sin u - \sin^2 u\,\cos u)\,du$$

$$= 16\pi a^2\left[\sin u - u\cos u - \frac{1}{3}\sin^3 u\right]_0^{\pi} = 16\pi^2 a^2.$$

See the solution of Problem 3 in Section 8.3 for the integration by parts to antidifferentiate $u\sin u$. Or if you prefer, imitate a computer algebra program: Assume that the antiderivative has the form $Au\sin u + Bu\cos u + C\sin u + D\cos u$. Differentiate and solve for the four coefficients A, B, C, and D.

C10S05.039: We use the parametrization $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \leqq t \leqq 2\pi$, $a > 0$. By the method of nested cylindrical shells, the volume of revolution around the y-axis is

$$V = \int_{t=0}^{2\pi} 2\pi xy\,dx = \int_0^{2\pi} 2\pi a(t-\sin t)\cdot a(1-\cos t)\cdot a(1-\cos t)\,dt$$

$$= 2\pi a^3\int_0^{2\pi}(t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\,\cos t - \sin t\,\cos^2 t)\,dt$$

$$= 2\pi a^3\int_0^{2\pi}\left(\frac{3}{2}t - 2t\cos t + \frac{1}{2}t\cos 2t - \sin t + 2\sin t\,\cos t - \sin t\,\cos^2 t\right)dt$$

$$= 2\pi a^3 \left[\frac{3}{4}t^2 - 2\cos t - 2t\sin t + \frac{1}{8}\cos 2t + \frac{1}{4}t\sin 2t + \cos t + \sin^2 t + \frac{1}{3}\cos^3 t \right]_0^{2\pi} = 6\pi^3 a^3.$$

See the solution of Problem 5 in Section 8.3 for the way integration by parts can be used to find the two more troublesome antiderivatives here.

C10S05.040: Part (a): The point T has coordinates $x = a\cos t$, $y = a\sin t$; let P have coordinates $(x,\, y)$. The slope of OT is $\tan t$ and hence the slope of TP is $-\cot t$. Note also that TP has length at. Therefore

$$\frac{y - a\sin t}{x - a\cos t} = -\cot t = -\frac{\cos t}{\sin t} \quad \text{and}$$
$$(x - a\cos t)^2 + (y - a\sin t)^2 = a^2t^2.$$

Thus

$$y\sin t - a\sin^2 t = -x\cos t + a\cos^2 t; \qquad \text{that is,} \qquad y\sin t + x\cos t = a.$$

Therefore

$$\begin{aligned}(x - a\cos t)^2 = a^2t^2 - (y - a\sin t)^2 &= a^2t^2 - \left(\frac{a - x\cos t}{\sin t} - a\sin t\right)^2 \\ &= a^2t^2 - \left(\frac{a - x\cos t - a\sin^2 t}{\sin t}\right)^2 = a^2t^2 - \frac{(a - x\cos t - a + a\cos^2 t)^2}{\sin^2 t} \\ &= a^2t^2 - \frac{1}{\sin^2 t}\left[(x - a\cos t)(-\cos t) + a - a\right]^2 = a^2t^2 - \frac{\cos^2 t}{\sin^2 t}(x - a\cos t)^2.\end{aligned}$$

It now follows that

$$\begin{aligned}(1 + \cot^2 t)(x - a\cos t)^2 &= a^2t^2; \\ (x - a\cos t)^2 &= a^2t^2\sin^2 t; \\ x &= a\cos t \pm at\sin t.\end{aligned}$$

Next,

$$\begin{aligned}(y - a\sin t)^2 = a^2t^2 - (x - a\cos t)^2 &= a^2t^2 - \left(\frac{a - y\sin t}{\cos t} - a\cos t\right)^2 \\ &= a^2t^2 - \left(\frac{a - y\sin t - a\cos^2 t}{\cos t}\right)^2 = a^2t^2 - \left(\frac{a\sin^2 t - y\sin t}{\cos t}\right)^2 \\ &= a^2t^2 - (a\sin t - y)^2\tan^2 t.\end{aligned}$$

Therefore

$$\begin{aligned}(1 + \tan^2 t)(y - a\sin t)^2 &= a^2t^2; \\ (y - a\sin t)^2 &= a^2t^2\cos^2 t; \\ y &= a\sin t \pm at\cos t.\end{aligned}$$

A moment's reflection about the behavior of $P(x, y)$ for t small positive now makes it evident that the correct choice of signs is

$$x = a(\cos t + t \sin t), \qquad y = a(\sin t - t \cos t).$$

Part (b): After all that algebra, the arc-length element ds almost miraculously simplifies to $at\ dt$. Therefore the length of the involute from $t = 0$ to $t = \pi$ is

$$L = \int_0^{\pi} at\ dt = \left[\frac{a}{2}t^2\right]_0^{\pi} = \frac{\pi^2 a}{2} \approx (4.9348)a.$$

C10S05.041: We will compute the area of the part of the region above the x-axis, then double the result. On the left we see a quarter-circle of radius πa, with area

$$A_1 = \frac{1}{4}\pi(\pi a)^2 = \frac{1}{4}\pi^3 a^2.$$

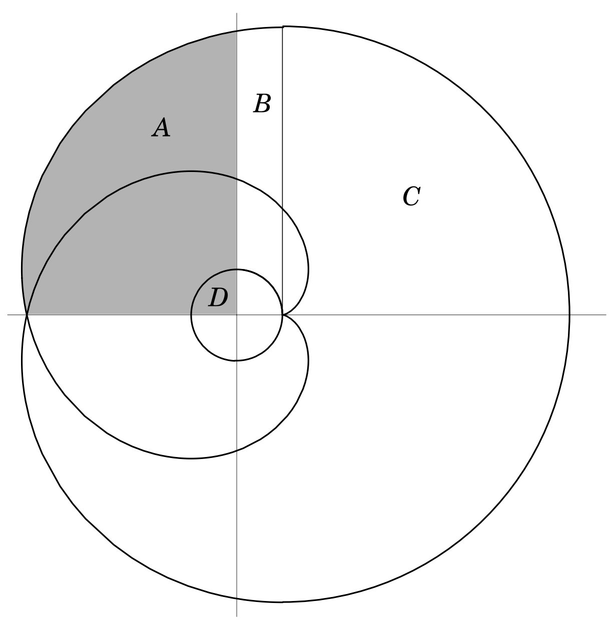
On the right, the area between the involute and the x-axis can be found with an integral:

$$A_2 = \int_0^{\pi} [-y(t) \cdot x'(t)]\ dt = \frac{a^2}{12}\Big[3t^2 \sin 2t - 3 \sin 2t + 6t \cos 2t + 2t^3\Big]_0^{\pi} = \frac{\pi a^2}{6}(\pi^2 + 3).$$

But we must subtract the area of the part of the water tank above the x-axis, the area of a semicircle of radius a: $A_3 = \frac{1}{2}\pi a^2$. So the total area of the region that the cow can graze is

$$A = 2(A_1 + A_2 - A_3) = \frac{5}{6}\pi^3 a^2.$$

C10S05.042: You can see in the following figure that there is a problem at the extreme left. Each involute moves away from the y-axis very briefly, then moves back toward it.



We can avoid this problem by finding the area that the cow can graze in the third quadrant by integrating *not* $y\ dx$, but instead $x\ dy$. To find the limits of integration on the parameter t, we need to know the value

t_1 of t at which the involute crosses the negative x-axis (where the two involutes cross) and the value t_2 of t at which the outer involute crosses the positive y-axis. Newton's method yields

$$t_1 \approx 4.4934094579 \qquad \text{and} \qquad t_2 \approx 6.121250466898.$$

So the area of the shaded region is

$$\int_{t=t_1}^{t_2} x(t) \cdot y'(t)\, dt \approx (23.106)a^2.$$

The area of the region bounded below by the x-axis, on the left by the y-axis, on the right by the line $x = a$, and above by the outer involute is

$$B = \int_{t=t_1}^{2\pi} [-y(t) \cdot x'(t)]\, dt \approx (6.256)a^2.$$

The area of the quarter-circle bounded below by the x-axis, on the left by the line $x = a$, and on the right and above by the circular arc of radius $2\pi a$ is

$$C = \frac{1}{4}\pi(2\pi a)^2 = \pi^3 a^2 \approx (31.006)a^2.$$

We can obtain the total area that the cow can graze by doubling the sum of the areas A, B, and C, but then we need to subtract the area occupied by the water tank, the area $D = \pi a^2 \approx (3.142)a^2$ of a circle of radius a. So the area the cow can graze is

$$2(A + B + C) - D \approx (117.596)a^2.$$

C10S05.043: Given $r(\theta) = 3\sin 3\theta$, remember that roses with *odd* coefficients are swept out *twice* in the interval $0 \leqq \theta \leqq 2\pi$. Therefore we should integrate

$$ds = \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta = \sqrt{45 + 36\cos 6\theta}\, d\theta$$

from 0 to π to obtain the total length of the rose:

$$\int_{\theta=0}^{\pi} 1\, ds = \int_0^{\pi} \sqrt{45 + 36\cos 6\theta}\, d\theta \approx 20.047339830833.$$

The *Mathematica* 3.0 command we used in Problem 43—we used appropriately modified versions of it for Problems 44 through 55—was

```
NIntegrate[ Sqrt[ 45 + 36*Cos[6*t] ], { t, 0, Pi },
            MaxRecursion -> 18, WorkingPrecision -> 28 ]
```

C10S05.044: Two integrals are required. The surface area is

$$\int_{\theta=0}^{\pi/3} 2\pi x\, ds - \int_{\theta=\pi/3}^{\pi/2} 2\pi x\, ds.$$

The minus sign is needed because $x(\theta) = r(\theta)\cos\theta$ is negative if $\pi/3 \leqq \theta \leqq \pi/2$. The total surface area is approximately 64.912021806645.

C10S05.045: Given $r(\theta) = 2\cos 2\theta$, remember than a rose with an even coefficient n of θ has $2n$ "petals," and is swept out as θ ranges from 0 to 2π. The arc length element in this case is $ds = \sqrt{10 - 6\cos 4\theta}\ d\theta$, and the length of the graph is

$$\int_0^{2\pi} \sqrt{10 - 6\cos 4\theta}\ d\theta \approx 19.376896441095$$

C10S05.046: When the rose of Problem 45 is rotated around the x-axis, the entire surface is generated twice. To obtain each part of the surface once, we will rotate the part of the rose from $\theta = 0$ to $\theta = \pi/4$ and, separately, the part from $\theta = \pi/4$ to $\pi/2$. We will set up an integral for each surface area, add the results, and double the sum. With $x(\theta) = r(\theta)\sin\theta$ and the arc length element ds of Problem 45, we get the integrals

$$\int_{\theta=0}^{\pi/4} 2\pi y\, ds \approx 5.46827 \qquad \text{and} \qquad \int_{\theta=\pi/4}^{\pi/2} (-2\pi y)\, ds \approx 16.1232,$$

for a total area of approximately 43.1829346047.

C10S05.047: Given: $r(\theta) = 5 + 9\cos\theta$, the arc length element is $ds = \sqrt{106 + 90\cos\theta}\ d\theta$, and so the total length of the limaçon is

$$\int_0^{2\pi} \sqrt{106 + 90\cos\theta}\ d\theta \approx 61.003581373850.$$

C10S05.048: The limaçon of Problem 47 is to be rotated around the x-axis. To find the surface area generated, we need to know where $r(\theta) = 0$. The solution is

$$\theta_1 = \cos^{-1}\left(-\frac{5}{9}\right) \approx 2.159827297.$$

So the surface area is

$$\int_{\theta=0}^{\theta_1} 2\pi y\ ds - \int_{\theta=\theta_1}^{\pi} 2\pi y\ ds.$$

The minus sign is needed because $y < 0$ on the part of the limaçon from $\theta = \theta_1$ to π. And we stop at $\theta = \pi$ because the same surface is swept out a second time for $\pi \leqq \theta \leqq 2\pi$. The resulting total surface area is approximately 860.260874010443.

C10S05.049: Given: $r(\theta) = \cos(7\theta/3)$. To sweep out all seven "petals" of this quasi-rose, you need to let θ vary from 0 to 3π. The length of the graph is

$$\int_0^{3\pi} \sqrt{\tfrac{1}{9}\big(29 - 20\cos(14\theta/3)\big)}\ d\theta \approx 16.342833373939.$$

C10S05.050: The length of the graph of this curve is $\displaystyle\int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t}\ dt \approx 9.429431296944.$

C10S05.051: Part (a): When the curve of Problem 50 is rotated around the x-axis, the surface generated is swept out twice. We will rotate the part of the curve in the first quadrant around the x-axis and double the result to get the total surface area

$$2\int_{t=0}^{\pi/2} 2\pi y\ ds \approx 16.057027566602.$$

Part (b): To find the volume of revolution around the x-axis, we evaluate

$$\begin{aligned}2\int_0^{\pi/2} \pi[y(t)]^2 \cdot x'(t)\ dt &= 2\int_0^{\pi/2} 4\pi(\sin^2 t\ \cos t - \sin^4 t\ \cos t)\ dt \\ &= 2\left[4\pi\left(\frac{1}{3}\sin^3 t - \frac{1}{5}\sin^5 t\right)\right]_0^{\pi/2} = \frac{16}{15}\pi \approx 3.351032163829.\end{aligned}$$

C10S05.052: Now the curve of Problems 50 and 51 is to be rotated around the y-axis. We will use the same part of the curve (the part in the first quadrant) and double the answer.

Part (a): The surface area generated is

$$2\int_0^{\pi/2} 2\pi x(t)\sqrt{\cos^2 t + 4\cos^2 2t}\ dt \approx 17.720537653947.$$

Part (b): Using the method of cylindrical shells, the volume enclosed by that surface is

$$\begin{aligned}2\int_0^{\pi/2} 2\pi x(t)y(t)x'(t)\ dt &= 4\pi\int_0^{\pi/2} 2\sin^2 t\cos^2 t\ dt \\ &= 4\pi\int_0^{\pi/2} \frac{1}{2}(1-\cos^2 2t)\ dt = 2\pi\int_0^{\pi/2}\left(1-\frac{1+\cos 4t}{2}\right)\ dt \\ &= 2\pi\int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 4t\right)\ dt = 2\pi\left[\frac{1}{2}t - \frac{1}{8}\sin 4t\right]_0^{\pi/2} = \frac{1}{2}\pi^2.\end{aligned}$$

C10S05.053: The arc-length element is $ds = \sqrt{25\cos^2 5t + 9\sin^2 3t}\ dt$, and the entire Lissajous curve is obtained by letting t range from 0 to 2π. Hence the length of the graph is

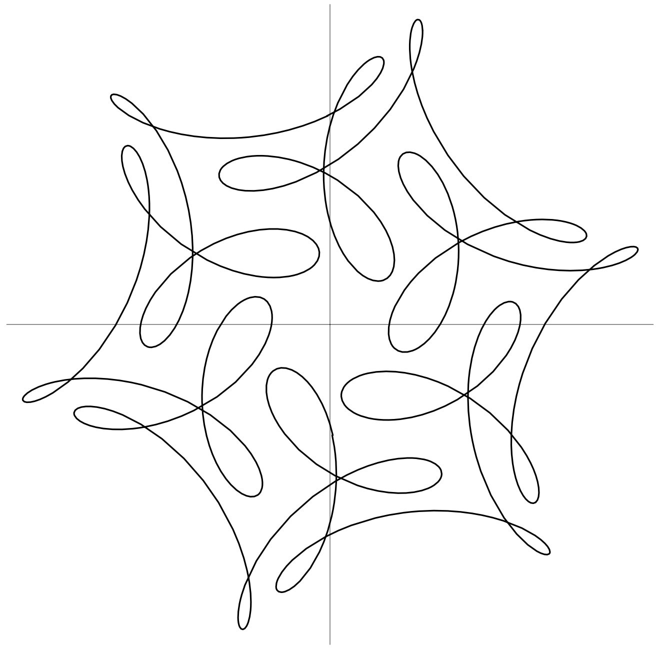
$$\int_0^{2\pi} \sqrt{25\cos^2 5t + 9\sin^2 3t}\ dt \approx 24.602961618540.$$

C10S05.054: The length of the graph is $\displaystyle\int_0^{2\pi} \sqrt{464 - 320\cos 3t}\ dt \approx 130.742666991511.$

C10S05.055: The length of the graph is

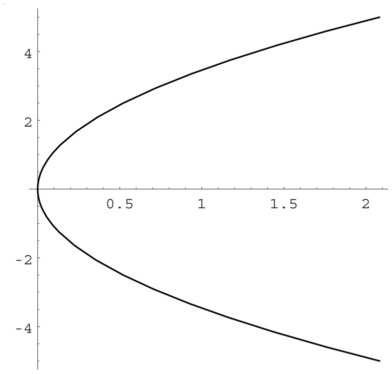
$$\int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt \approx 39.403578712896.$$

The graph is next.



Section 10.6

C10S06.001: If the vertex is at $V(0, 0)$ and the focus is at $F(3, 0)$, then the directrix must be the vertical line with equation $x = -3$. If (x, y) is a point on the parabola, then by the definition of parabola, $y^2 + (x-3)^2 = (x+3)^2$. It's easy to simplify this equation to $y^2 = 12x$. The graph is next.

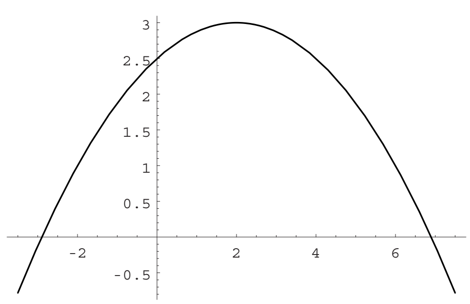


C10S06.002: If the vertex is $V(0, 0)$ and the focus is $F(0, -2)$, then the directrix must be the horizontal line $y = 2$. Using the definition of parabola, it follows that if (x, y) is a point of the parabola, then $x^2 + (y+2)^2 = (y-2)^2$. It's easy to simplify this equation to $x^2 = -8y$.

C10S06.003: If the vertex of the parabola is $V(2, 3)$ and the focus is $F(2, 1)$, then the directrix must be the horizontal line $y = 5$. Then it follows from the definition of a parabola that if (x, y) is a point of the parabola, then

$$\begin{aligned}
&(x-2)^2 + (y-1)^2 = (y-5)^2;\\
&(x-2)^2 + y^2 - 2y + 1 = y^2 - 10y + 25;\\
&(x-2)^2 = -8y + 24;\\
&(x-2)^2 = -8(y-3).
\end{aligned}$$

The graph of this parabola is next.

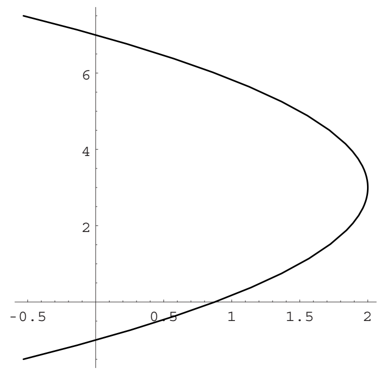


C10S06.004: If the vertex is $V(-1, -1)$ and the focus is $F(-3, -1)$, then the directrix must be the vertical line $x = 1$. If (x, y) is a point of the parabola, then by definition $(x+3)^2 + (y+1)^2 = (x-1)^2$. It's easy to simplify this equation to $(y+1)^2 = -8(x+1)$.

C10S06.005: If the vertex is $V(2, 3)$ and the focus is $F(0, 3)$, then the directrix of this parabola must be the vertical line $x = 4$. If (x, y) is a point of the parabola, then—by definition—

$$
\begin{aligned}
&x^2 + (y-3)^2 = (x-4)^2;\\
&(y-3)^2 - 8x + 16;\\
&(y-3)^2 = -8(x-2).
\end{aligned}
$$

The graph of this parabola is next.



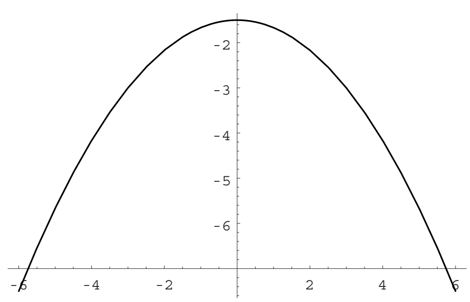
C10S06.006: With focus $F(1, 2)$ and directrix $x = -1$, the definition of parabola implies that if (x, y) is on this parabola, then

$$
\begin{aligned}
&(x-1)^2 + (y-2)^2 = (x+1)^2;\\
&x^2 - 2x + 1 + (y-2)^2 = x^2 + 2x + 1;\\
&(y-2)^2 = 4x.
\end{aligned}
$$

C10S06.007: If a parabola has focus $F(0, -3)$, directrix $y = 0$, and contains the point (x, y), then by definition

$$
x^2 + (y+3)^2 = y^2; \qquad x^2 = -6y - 9; \qquad x^2 = -6\left(y + \frac{3}{2}\right).
$$

The graph of this parabola is next.



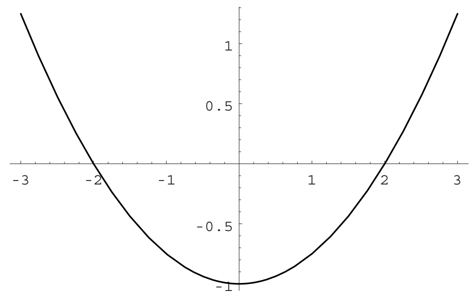
C10S06.008: With focus $F(1, -1)$ and directrix $x = 3$, the definition of parabola implies that if (x, y) is on this parabola, then

$$
(x-1)^2 + (y+1)^2 = (x-3)^2; \quad (y+1)^2 = 2x - 1 - 6x + 9; \quad (y+1)^2 = -4x + 8; \quad (y+1)^2 = -4(x-2).
$$

C10S06.009: With focus $F(0, 0)$ and directrix $y = -2$, the definition of parabola implies that if (x, y) lies on this parabola, then

$$x^2 + y^2 = (y+2)^2; \qquad x^2 = 4y + 4; \qquad x^2 = 4(y+1).$$

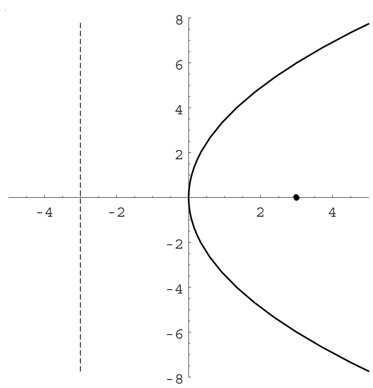
Its graph is next.



C10S06.010: With focus $F(-2, 1)$ and directrix $x = -4$, every point (x, y) of this parabola satisfies

$$(x+2)^2 + (y-1)^2 = (x+4)^2; \qquad 4x + 4 + (y-1)^2 = 8x + 16;$$
$$(y-1)^2 = 4x + 12; \qquad (y-1)^2 = 4(x+3).$$

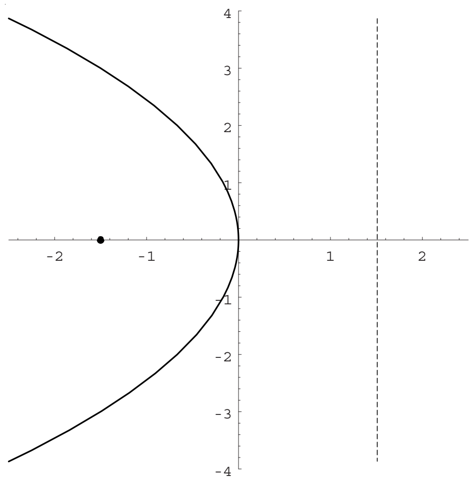
C10S06.011: The parabola with equation $y^2 = 12x$ has vertex $V(0, 0)$ and horizontal axis $y = 0$, so its focus must be at $F(c, 0)$ and its directrix must be the vertical line $x = -c$ where $c > 0$. So its equation also has the form $(x-c)^2 + y^2 = (x+c)^2$; that is, $y^2 = 4cx$. Therefore $c = 3$, the focus is $F(3, 0)$, and the directrix is the vertical line $x = -3$. The graph of this parabola is next.



C10S06.012: Clearly the parabola with equation $x^2 = -8y$ has vertex $V(0, 0)$, its axis is the y-axis, and it opens downward. So its focus is at $F(0, -c)$ and its directrix is the line $y = c$ where $c > 0$. So its equation has the form $x^2 + (y+c)^2 = (y-c)^2$; that is, $x^2 = -4cy$. Hence $c = 2$, the directrix is the line $y = 2$, and the focus is $F(0, -2)$.

C10S06.013: The parabola with equation $y^2 = -6x$ has vertex $V(0, 0)$, its axis is the x-axis, and it opens to the left. Hence it has focus $F(-c, 0)$ and directrix $x = c$ where $c > 0$. So its equation has the form $(x+c)^2 + y^2 = (x-c)^2$; that is, $y^2 = -4cx$. Therefore $c = \frac{3}{2}$, the focus is $F\left(-\frac{3}{2}, 0\right)$, and the directrix is

the line $x = \frac{3}{2}$. The graph of this parabola is next.



C10S06.014: The parabola with equation $x^2 = 7y$ clearly has vertex $V(0, 0)$, axis the y-axis, and opens upward. Hence it has focus $F(0, c)$ and directrix $y = -c$ where $c > 0$. Thus its equation has the form $x^2 + (y - c)^2 = (y + c)^2$; that is, $x^2 = 4cy$. Thus $c = \frac{7}{4}$, the focus is $F\left(0, \frac{7}{4}\right)$, and the directrix has equation $y = -\frac{7}{4}$.

C10S06.015: Given: $x^2 - 4x - 4y = 0$. We complete the square in x:

$$x^2 - 4x + 4 = 4y + 4; \qquad (x - 2)^2 = 4(y + 1). \tag{1}$$

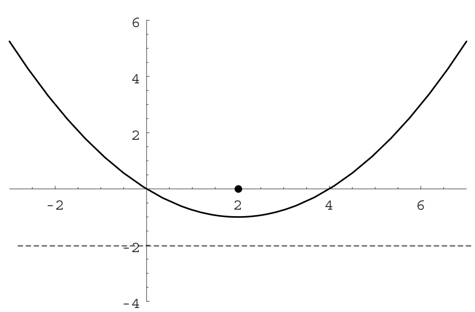
Thus this parabola has vertex $V(2, -1)$, vertical axis with equation $x = 2$, and opens upward. So its focus is at $F(2, -1 + c)$ and its directrix is $y = -1 - c$ where $c > 0$. Thus its equation has the form

$$(x - 2)^2 + (y + 1 - c)^2 = (y + 1 + c)^2;$$

$$(x - 2)^2 + (y + 1)^2 - 2c(y + 1) + c^2 = (y + 1)^2 + 2c(y + 1) + c^2;$$

$$(x - 2)^2 = 4c(y + 1).$$

By Eq. (1), $c = 1$. Therefore the focus is at $F(2, 0)$ and the directrix has equation $y = -2$. The graph of this parabola is next.



C10S06.016: Given $y^2 - 2x + 6y + 15 = 0$, we find that $(y + 3)^2 = 2(x - 3)$. So this parabola has vertex $V(3, -3)$, horizontal axis $y = -3$, and opens to the right. So its focus is $F(3 + c, -3)$ and its directrix has equation $x = 3 - c$ where $c > 0$. So its equation has the form

$$(x-3-c)^2+(y+3)^2=(x-3+c)^2;$$

$$(x-3)^2-2c(x-3)+c^2+(y+2)^2=(x-3)^2+2c(x-3)+c^2;$$

$$(y+3)^2=4c(x-3).$$

Therefore $c=\frac{1}{2}$, the focus is $F\left(\frac{7}{2},\,-3\right)$, and the directrix has equation $x=\frac{5}{2}$.

C10S06.017: Given $4x^2+4x+4y+13=0$, we complete the square in x to find that $(2x+1)^2=-4(y+3)$. Thus this parabola has vertex $V\left(-\frac{1}{2},\,-3\right)$, vertical axis $x=-\frac{1}{2}$, and opens downward. So its focus is $F\left(-\frac{1}{2},\,-3-c\right)$ and its directrix has equation $y=-3+c$ where $c>0$. So its equation has the form

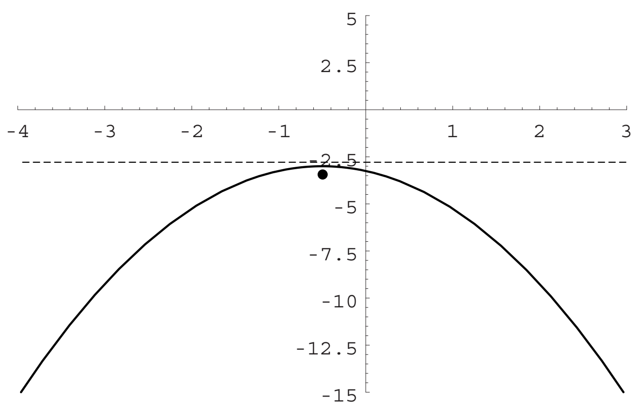
$$\left(x+\tfrac{1}{2}\right)^2+(y+3+c)^2=(y+3-c)^2;$$

$$\left(x+\tfrac{1}{2}\right)^2+(y+3)^2+2c(y+3)+c^2=(y+3)^2-2c(y+3)+c^2;$$

$$\left(x+\tfrac{1}{2}\right)^2=-4c(y+3);$$

$$(2x+1)^2=-16c(y+3).$$

Therefore $c=\frac{1}{4}$, the focus is $F\left(-\frac{1}{2},\,-\frac{13}{4}\right)$, and the directrix has equation $y=-\frac{11}{4}$. This parabola is shown next.



C10S06.018: Given $4y^2-12y+9x=0$, we complete the square in y and find that $(2y-3)^2=-9(x-1)$. So this parabola has vertex $V\left(1,\,\frac{3}{2}\right)$, horizontal axis with equation $y=\frac{3}{2}$, and opens to the left. Its focus is $F\left(1-c,\,\frac{3}{2}\right)$ and its directrix has equation $x=1+c$ where $c>0$. So its equation also takes the form

$$(x-1+c)^2+\left(y-\tfrac{3}{2}\right)^2=(x-1-c)^2;$$

$$(x-1)^2+2c(x-1)+c^2+\left(y-\tfrac{3}{2}\right)^2=(x-1)^2-2c(x-1)+c^2;$$

$$\left(y-\tfrac{3}{2}\right)^2=-4c(x-1);$$

$$(2y-3)^2=-16c(x-1).$$

Therefore $c=\frac{9}{16}$, the focus is $F\left(\frac{7}{16},\,\frac{3}{2}\right)$, and the directrix has equation $x=\frac{25}{16}$.

C10S06.019: The location of the vertices makes it clear that the center of the ellipse is at $(0,\,0)$. Therefore its equation may be written in the standard form

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1.$$

C10S06.020: We use the equation $a^2 = b^2 + c^2$ with $a = 13$ and $c = 5$ to find that $b = 12$. The major axis is horizontal, hence an equation of this ellipse is

$$\left(\frac{x}{13}\right)^2 + \left(\frac{y}{12}\right)^2 = 1.$$

C10S06.021: We use the equation $a^2 = b^2 + c^2$ with $a = 17$ and $c = 8$ to find that $b = 15$. The major axis is vertical, so an equation of this ellipse is

$$\left(\frac{x}{15}\right)^2 + \left(\frac{y}{17}\right)^2 = 1.$$

C10S06.022: Here we have $a = 6$ and $b = 4$; because the major axis is vertical, an equation of this ellipse is

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 = 1.$$

C10S06.023: Because $c = 3$ and

$$a = \frac{c}{e} = 3 \cdot \frac{4}{3} = 4,$$

we use the equation $a^2 = b^2 + c^2$ to find that $b^2 = 7$. Therefore an equation of this ellipse is

$$\frac{x^2}{16} + \frac{y^2}{7} = 1.$$

C10S06.024: Because $c = 4$ and

$$a = \frac{c}{e} = 4 \cdot \frac{3}{2} = 6,$$

we use the equation $a^2 = b^2 + c^2$ to find that $b = \sqrt{20}$. Therefore an equation of this ellipse is

$$\frac{x^2}{20} + \frac{y^2}{36} = 1.$$

C10S06.025: Because $a = 10$ and $c = ea = \frac{1}{2} \cdot 10 = 5$, it follows from the equation $a^2 = b^2 + c^2$ that $b = \sqrt{75}$. Therefore an equation of this ellipse is

$$\frac{x^2}{100} + \frac{y^2}{75} = 1.$$

C10S06.026: We have $b = 5$ and $e = \frac{1}{2}$. Thus $a = 2c$; moreover, $a^2 = b^2 + c^2$. So $4c^2 = 25 + c^2$, and it follows that $c = \frac{5}{3}\sqrt{3}$ and $a = \frac{10}{3}\sqrt{3}$. Therefore this ellipse has equation

$$\frac{x^2}{25} + \frac{3y^2}{100} = 1.$$

C10S06.027: From the information given in the problem, we see that $8 = a/e$ and $a = 2/e$. It follows that $e = \frac{1}{2}$, and so $a = 4$ and $c = 2$. Consequently $b^2 = 12$, and therefore an equation of this ellipse is

$$\frac{x^2}{16} + \frac{y^2}{12} = 1.$$

C10S06.028: First, $c = 4$ and $9 = c/e^2$; therefore $e = \frac{2}{3}$. So $a = c/e = 6$; $b^2 = a^2 - c^2 = 20$. Therefore an equation of this ellipse is

$$\frac{x^2}{20} + \frac{y^2}{36} = 1.$$

C10S06.029: Were the center at the origin, the equation would be $(x/4)^2 + (y/2)^2 = 1$. Because the center is at $C(2, 3)$, the translation principle implies that the equation is instead

$$\left(\frac{x-2}{4}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1.$$

C10S06.030: First, $a = 4$ and $e = \frac{3}{4}$. So $c = ae = 3$ and $b^2 = a^2 - c^2 = 16 - 9 = 7$; therefore this ellipse has equation

$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{7} = 1.$$

C10S06.031: The center of this ellipse is at $(1, 1)$, $c = 3$, and $a = 5$. Thus $b = 4$, and so an equation of this ellipse is

$$\left(\frac{x-1}{5}\right)^2 + \left(\frac{y-1}{4}\right)^2 = 1.$$

C10S06.032: First we note that the center is $C(-3, 2)$. It follows that $b = 3$ and $c = 2$, so that $a^2 = 13$. Therefore an equation of this ellipse is

$$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{13} = 1.$$

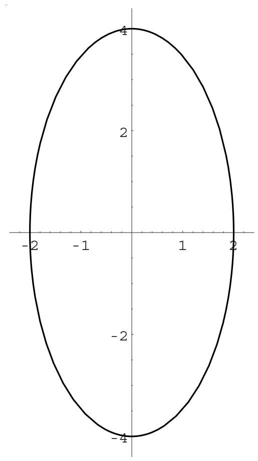
C10S06.033: The center is at $C(1, 2)$, the major axis is horizontal, and $c = 3$. Next, $a = c/e$ and $e = 1/3$, so $a = 3c = 9$. Because $b^2 = a^2 - c^2$, we see that $b = \sqrt{72}$. Thus an equation of this ellipse is

$$\frac{(x-1)^2}{81} + \frac{(y-2)^2}{72} = 1.$$

C10S06.034: In standard form, the equation of this ellipse is

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

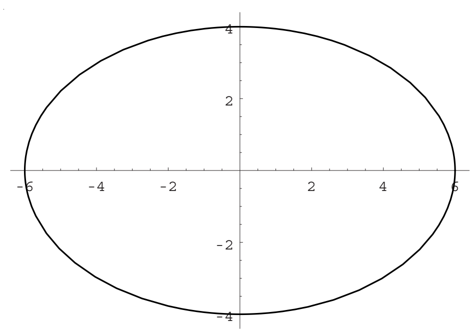
Therefore its center is $C(0,\,0)$, $a = 4$, and $b = 2$. Thus $c = 2\sqrt{3}$, so the foci are $\left(0,\,\pm 2\sqrt{3}\,\right)$. The major axis is vertical, of length 8; the minor axis has length 4. The graph is next.



C10S06.035: In standard form, the equation of this ellipse is

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = 1,$$

so its center is at $C(0,\,0)$, $a = 6$, and $b = 4$; thus $c = 2\sqrt{5}$. The foci are $\left(\pm 2\sqrt{5}\,,\,0\right)$. The major axis is horizontal, of length 12; the minor axis has length 8. The graph of this ellipse is next.



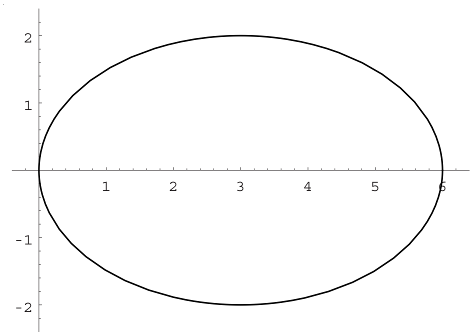
C10S06.036: Given: $4x^2 - 24x + 9y^2 = 0$. We complete the square in x and write the equation in standard form as follows:

$$x^2 - 6x + \frac{9}{4}y^2 = 0; \qquad x^2 - 6x + 9 + \frac{9}{4}y^2 = 9;$$

$$(x-3)^2 + \frac{9}{4}y^2 = 9; \qquad \left(\frac{x-3}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

Therefore the center is $C(3,\,0)$, $a = 3$, and $b = 2$. Thus $c = \sqrt{5}$. The foci are $\left(3 \pm \sqrt{5}\,,\,0\right)$, the major axis

is horizontal of length 6, and the minor axis has length 4. The graph is next.

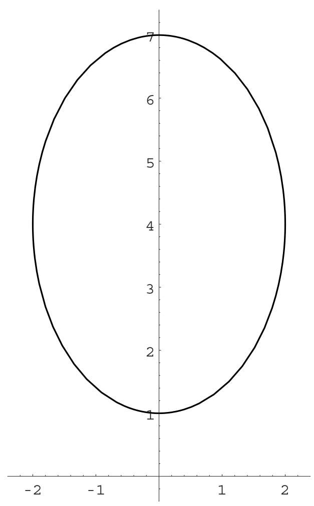


C10S06.037: We complete the square in y as follows:

$$9x^2 + 4y^2 - 32y + 28 = 0; \qquad \frac{9}{4}x^2 + y^2 - 8y + 7 = 0;$$

$$\frac{9}{4}x^2 + y^2 - 8y + 16 = 9; \qquad \left(\frac{x}{2}\right)^2 + \left(\frac{y-4}{3}\right)^2 = 1.$$

Thus this ellipse has center $C(0, 4)$, $a = 3$, and $b = 2$; thus $c = \sqrt{5}$. The foci are $\left(0, 4 \pm \sqrt{5}\,\right)$, the major axis is vertical, of length 6; the minor axis has length 4. The graph of this ellipse is next.



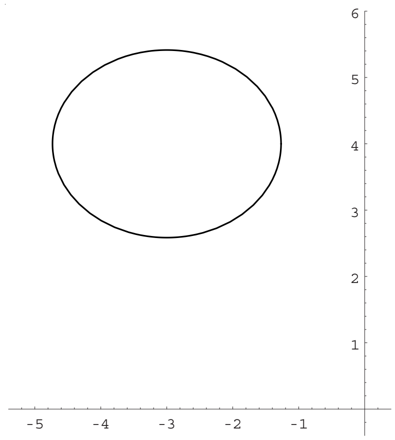
C10S06.038: We put the given equation into standard form as follows:

$$2x^2 + 3y^2 + 12x - 24y + 60 = 0; \qquad 2(x^2 + 6x) + 3(y^2 - 8y) + 60 = 0;$$

$$2(x^2 + 6x + 9) + 3(y^2 - 8y + 16) + 60 = 48 + 18 = 66; \qquad \frac{(x+3)^2}{3} + \frac{(y-4)^2}{2} = 1.$$

Thus this ellipse has center $C(-3, 4)$, $a = \sqrt{3}$, and $b = \sqrt{2}$. So $c = 1$, the foci are $(-4, 4)$ and $(-2, 4)$, the major axis is horizontal with length $2\sqrt{3}$, and the minor axis has length $2\sqrt{2}$. The graph of this ellipse is

next.



C10S06.039: The given information implies that the transverse axis is horizontal, $c = 4$, and $a = 1$. Hence $b = \sqrt{c^2 - a^2} = \sqrt{15}$. Therefore an equation of this hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{15} = 1.$$

C10S06.040: The transverse axis is vertical, $c = 3$, and $a = 2$. Therefore $b = \sqrt{c^2 - a^2} = \sqrt{5}$. Hence an equation of this hyperbola is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1.$$

C10S06.041: The given information implies that the transverse axis is horizontal, $c = 5$, and $b/a = 3/4$, so that $b = \frac{3}{4}a$. Then the equation $a^2 + b^2 = c^2$ implies that $a = 4$ and thus that $b = 3$. So an equation of this hyperbola is

$$\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 = 1.$$

C10S06.042: The given information implies that the transverse axis is horizontal, $a = 3$, and $b/a = 3/4$, so that $b = 9/4$. Hence an equation of this hyperbola is

$$\frac{x^2}{9} - \frac{16y^2}{81} = 1.$$

C10S06.043: The information given in the problem implies that the transverse axis is vertical, $a = 5$, and $a/b = 1$, so that $b = 5$ as well. Hence an equation of this hyperbola is

$$\left(\frac{y}{5}\right)^2 - \left(\frac{x}{5}\right)^2 = 1.$$

C10S06.044: By the given information, the transverse axis is horizontal, $a = 3$ and $c = ae = 5$, so that $b = 4$. Therefore this hyperbola has equation

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = 1.$$

C10S06.045: The transverse axis is vertical and $c = 6$. Hence $a = c/e = 3$, and so $b = \sqrt{27}$. Therefore an equation of this hyperbola is

$$\frac{y^2}{9} - \frac{x^2}{27} = 1.$$

C10S06.046: The transverse axis is horizontal and $a = 4$. So an equation of this hyperbola has the form

$$\frac{x^2}{16} - \frac{y^2}{b^2} = 1.$$

Because the point $(8, 3)$ satisfies this equation, it follows that $b^2 = 3$.

C10S06.047: The transverse axis is horizontal and $c = 4$. One directrix is $x = 1$, so $1 = a/e = c/e^2$. Thus $e = 2$, and so $a = 2$ and $b^2 = 12$. Thus an equation of this hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1.$$

C10S06.048: The transverse axis is vertical and $c = 9$. One directrix is $y = 4$, so $c/e^2 = 4$. Thus $e = 3/2$, so $a = c/e = 6$ and $b^2 = 45$. So an equation of this hyperbola is

$$\frac{y^2}{36} - \frac{x^2}{45} = 1.$$

For an alternative solution, use the focus $F(0, 9)$, the directrix L with equation $y = 4$, and the definition $|PF| = e \cdot |PL|$ for each point $P(x, y)$ on the hyperbola. It then follows that

$$\begin{aligned}
&\sqrt{x^2 + (y-9)^2} = e \cdot (y - 4);\\
&x^2 + y^2 - 18y + 81 = e^2(y^2 - 8y + 16);\\
&x^2 - (e^2 - 1)y^2 + (8e^2 - 18)y + (81 - 16e^2) = 0.
\end{aligned}$$

The coefficient of y in the last equation must be zero, and it follows that $e = 3/2$. The last equation becomes

$$x^2 - \frac{5}{4}y^2 = 16e^2 - 81 = -45,$$

and therefore an equation of this hyperbola is $\dfrac{y^2}{36} - \dfrac{x^2}{45} = 1$.

C10S06.049: Given: The hyperbola has center $(2, 2)$, the transverse axis is horizontal of length 6, and $e = 2$. Translate the hyperbola so that its center is at $(0, 0)$. The vertices are therefore $(-3, 0)$ and $(3, 0)$, so that $a = 3$. Then $c = ae = 6$, so that $b^2 = 27$. So the translated hyperbola has equation

$$\frac{x^2}{9} - \frac{y^2}{27} = 1.$$

Therefore an equation of the original hyperbola is $\dfrac{(x-2)^2}{9} - \dfrac{(y-2)^2}{27} = 1$.

C10S06.050: Given: The hyperbola has center $C(-1, 3)$, vertices $V_1(-4, 3)$ and $V_2(2, 3)$, and foci $F_1(-6, 3)$ and $F_2(4, 3)$. Translate the hyperbola so that its center is at the origin. The new vertices

are $(\pm 3,\ 0)$ and the new foci are $(\pm 5,\ 0)$. So the transverse axis is horizontal, $a = 3$, and $c = 5$. Therefore $b = 4$, and so the translated hyperbola has equation

$$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = 1.$$

Thus an equation of the original hyperbola is $\left(\dfrac{x+1}{3}\right)^2 - \left(\dfrac{y-3}{4}\right)^2 = 1.$

C10S06.051: Given: The hyperbola has center $C(1,\ -2)$, vertices $V_1(1,\ 1)$, and $V_2(1,\ -5)$, and asymptotes $3x - 2y = 7$ and $3x + 2y = -1$. Translate the hyperbola so that its center is at $(0,\ 0)$. The new vertices are $(0,\ \pm 3)$ and the new asymptotes have equations

$$3(x+1) - 2(y-2) = 7 \quad \text{and} \quad 3(x+1) + 2(y-2) = -1;$$

$$3x - 2y = 0 \quad \text{and} \quad 3x + 2y = 0.$$

Thus their equations are $y = \pm\frac{3}{2}x$. Therefore the translated parabola has $a = 3$ and $a/b = 3/2$, so that $b = 2$. Thus—because its transverse axis is vertical—it has equation

$$\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = 1.$$

Therefore the original hyperbola has equation $\left(\dfrac{y+2}{3}\right)^2 - \left(\dfrac{x-1}{2}\right)^2 = 1.$

C10S06.052: Given: One focus of the hyperbola is $F(8,\ -1)$ and its asymptotes have equations $3x - 4y = 13$ and $3x + 4y = 5$. If we translate this hyperbola so that its center is at $(0,\ 0)$, then its asymptotes will have the equations

$$3(x-u) - 4(y-v) = 13 \quad \text{and} \quad 3(x-u) + 4(y-v) = 5;$$

$$3x - 4y = 13 + 3u - 4v = 0 \quad \text{and} \quad 3x + 4y = 5 + 3u + 4v = 0.$$

Therefore $3u - 4v = -13$ and $3u + 4v = -5$. It follows that $u = -3$ and $v = 1$. So the given focus will be translated to the point $(5,\ 0)$. Thus the other focus is at $(-5,\ 0)$, $c = 5$, and the asymptotes are $y = \pm 3x/4 = \pm bx/a$. Thus $b = 3a/4$ and so

$$a^2 + \frac{9}{16}a^2 = 25; \qquad \frac{25}{16}a^2 = 25; \qquad a = 4.$$

Consequently $b = 3$, and an equation of the translated hyperbola is

$$\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 = 1.$$

So the original hyperbola has equation $\left(\dfrac{x-3}{4}\right)^2 - \left(\dfrac{y+1}{3}\right)^2 = 1.$

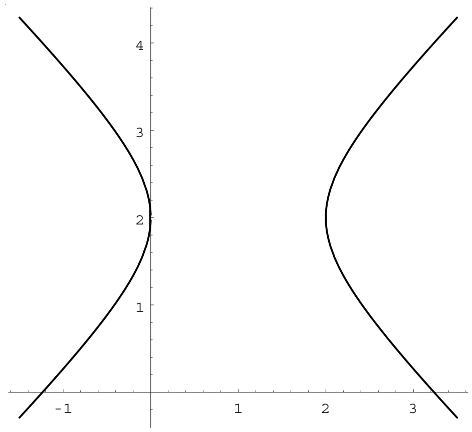
C10S06.053: Given $x^2 - y^2 - 2x + 4y = 4$, we first complete the square in the two variables:

$$x^2 - 2x - (y^2 - 4y) = 4;$$

$$x^2 - 2x + 1 - (y^2 - 4y + 4) + 4 - 1 = 4;$$

$$(x-1)^2 - (y-2)^2 = 1.$$

Thus this hyperbola has center $C(1, 2)$. Also $a = b = 1$, so $c = \sqrt{2}$. So its foci are $\left(1 \pm \sqrt{2}, 2\right)$. If its center were $(0, 0)$, its asymptotes would be $y = \pm x$. Therefore its actual asymptotes have equations $y - 2 = \pm(x - 1)$; that is, $y = x + 1$ and $y = -x + 3$. Its graph is next.



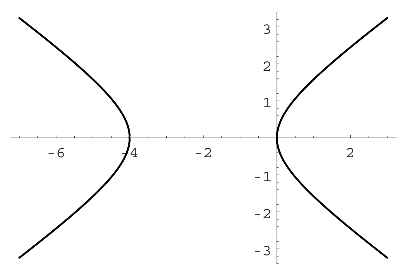
C10S06.054: The hyperbola's equation can be written in the form $(x + 2)^2 - 2y^2 = 4$, thus in the form

$$\frac{(x+2)^2}{4} - \frac{y^2}{2} = 1.$$

Thus $a = 2$, $b = \sqrt{2}$, and $c^2 = a^2 + b^2 = 6$. Therefore the center is at $(-2, 0)$, the foci are at $\left(-2 \pm \sqrt{6}, 0\right)$, and the asymptotes have the equations

$$y = \pm\frac{\sqrt{2}}{2}(x + 2).$$

The graph is next.



C10S06.055: Given the equation $y^2 - 3x^2 - 6y = 0$, complete the square:

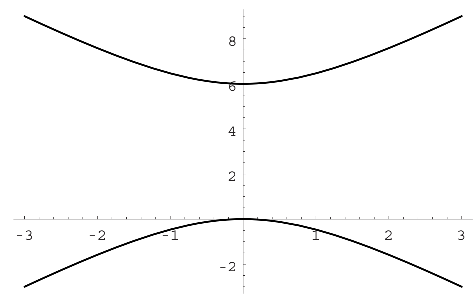
$$y^2 - 6y + 9 - 3x^2 = 9;$$

$$(y - 3)^2 - 3x^2 = 9;$$

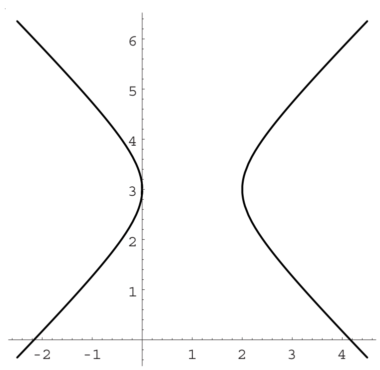
$$\frac{(y-3)^2}{9} - \frac{x^2}{3} = 1.$$

This hyperbola has center $(0, 3)$, $a = 3$, $b = \sqrt{3}$, and $c = 2\sqrt{3}$. If the center were at the origin, the hyperbola would have asymptotes with equations $y = \pm x\sqrt{3}$ and its foci would be $\left(0, \pm 2\sqrt{3}\right)$. So the given

hyperbola has asymptotes $y = 3 \pm x\sqrt{3}$ and foci $\left(0,\, 3 \pm 2\sqrt{3}\,\right)$. Its graph is next.



C10S06.056: After we complete the square, the equation of the hyperbola becomes $(x-1)^2-(y-3)^2 = 1$. Thus $a = b = 1$ and $c = \sqrt{2}$. The center is at $(1,\, 3)$ and the foci are at $\left(1 \pm \sqrt{2},\, 3\right)$. The asymptotes have equations $y = x + 2$ and $y = -x + 4$. The graph is next.



C10S06.057: First complete the square in both variables:

$$9x^2 - 4y^2 + 18x + 8y = 31;$$

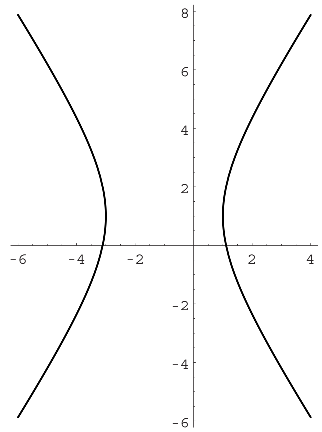
$$9(x^2 + 2x) - 4(y^2 - 2y) = 31;$$

$$9(x^2 + 2x + 1) - 4(y^2 - 2y + 1) = 31 + 9 - 4 = 36;$$

$$\left(\frac{x+1}{2}\right)^2 - \left(\frac{y-1}{3}\right)^2 = 1.$$

Thus this hyperbola has center $C(-1,\, 1)$. From its equation we also see that $a = 2$ and $b = 3$, so that $c = \sqrt{13}$. If its center were at the origin, its foci would be $\left(\pm\sqrt{13},\, 0\right)$ and its asymptotes would be $y = \pm 3x/2$. Thus its foci are at $\left(-1 \pm \sqrt{13},\, 1\right)$. Its asymptotes have equations $y - 1 = \pm 3(x+1)/2$; that

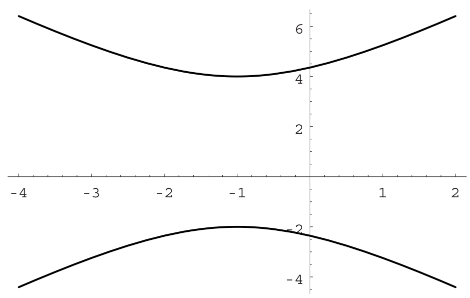
is, $2y = 3x + 5$ and $2y = -3x - 1$. The graph of this hyperbola is next.



C10S06.058: Complete the square to obtain

$$4(y-1)^2 - 9(x+1)^2 = 36; \qquad \text{that is,} \qquad \left(\frac{y-1}{3}\right)^2 - \left(\frac{x+1}{2}\right)^2 = 1.$$

From this equation we see that $a = 3$ and $b = 2$, the transverse axis is vertical, and $c = \sqrt{13}$. The center is at $(-1, 1)$, the foci are at $\left(-1, 1 \pm \sqrt{13}\right)$, and the asymptotes have equations $2y = 3x+5$ and $2y = -3x-1$. The graph is next.



C10S06.059: We read from the given equation

$$r = \frac{6}{1 + \cos\theta}$$

the information that (in the terminology of Section 10.6) $pe = 6$ and $e = 1$. Therefore the conic section is a parabola with directrix the vertical line $x = 6$; its focus is at $(0, 0)$. Conversion to Cartesian coordinates yields

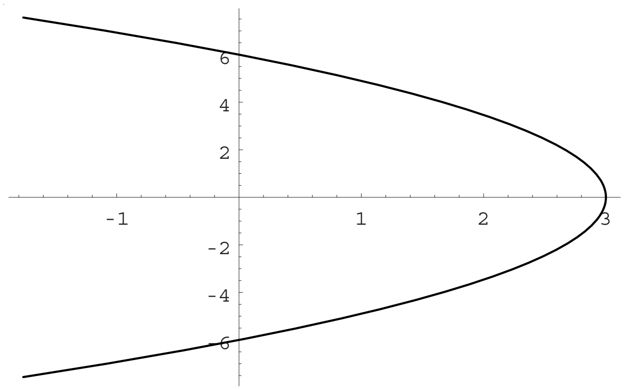
$$r + r\cos\theta = 6; \qquad x^2 + y^2 = (6-x)^2;$$

$$y^2 = 36 - 12x; \qquad x = 3 - \frac{1}{12}y^2.$$

The parabola opens to the left with vertex at $(3, 0)$; its axis is the x-axis (or the part of the x-axis for which $x \leqq 3$). To see the graph of this conic, we executed the *Mathematica* command

```
ParametricPlot[ { (6*Cos[t])/(1 + Cos[t]), (6*Sin[t])/(1 + Cos[t]) },
                                { t, −1.8, 1.8 }, PlotPoints → 47 ];
```

The result is next.



C10S06.060: We read from the given polar equation

$$r = \frac{6}{1 + 2\cos\theta}$$

the information that $pe = 6$ and $e = 2$. Hence the graph of this conic section is a hyperbola with eccentricity 2 and one directrix is the vertical line $x = 3$; the corresponding focus is at (0, 0). Conversion to Cartesian coordinates yields

$$r + 2r\cos\theta = 6; \qquad x^2 + y^2 = (6 - 2x)^2;$$

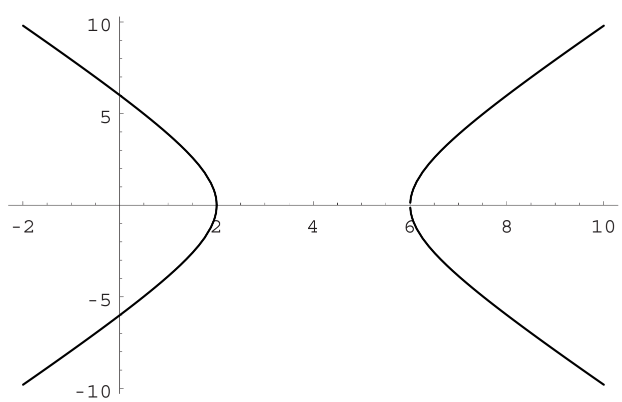
$$x^2 + y^2 = 36 - 24x + 4x^2; \qquad y^2 = 3x^2 - 24x + 36;$$

$$\frac{1}{3}y^2 = x^2 - 8x + 12; \qquad (x-4)^2 - \frac{1}{3}y^2 = 4.$$

Hence the center of the hyperbola is at (4, 0), its other focus is at (8, 0), and its vertices are at (2, 0) and (6, 0). To generate its graph, we executed the *Mathematica* command

```
Plot[ { Sqrt[ 3*x*x − 24*x + 36 ], −Sqrt[ 3*x*x − 24*x + 36 ] },
                                  { x, −2, 10 }, PlotPoints → 97 ];
```

The result is shown next.



C10S06.061: From the given polar equation

$$r = \frac{3}{1 - \cos\theta}$$

we read the information that $pe = 3$ and that $e = 1$, so that the conic section is a parabola with focus $(0, 0)$ and directrix the vertical line $x = -3$. Conversion to Cartesian coordinates yields

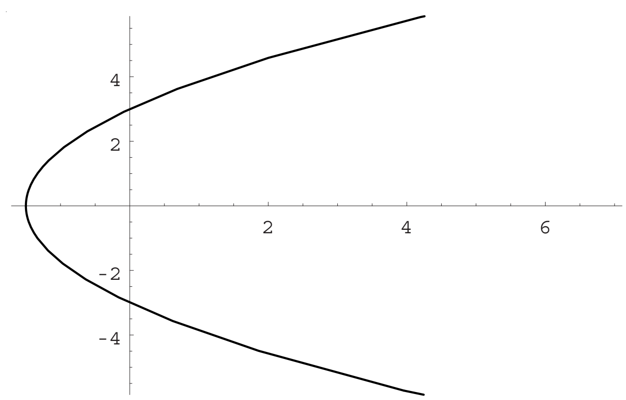
$$r - r\cos\theta = 3; \qquad r^2 = (x+3)^2;$$

$$x^2 + y^2 = x^2 + 6x + 9; \qquad x = \frac{1}{6}(y^2 - 9).$$

Hence the parabola opens to the right with vertex at $\left(-\frac{3}{2}, 0\right)$ and its axis is the x-axis. To see its graph, we executed the *Mathematica* command

```
ParametricPlot[ { (3*Cos[t])/(1 - Cos[t]), (3*Sin[t])/(1 - Cos[t]) },
                                          { t, 0.5, 2*Pi − 0.5 } ];
```

The result is next.



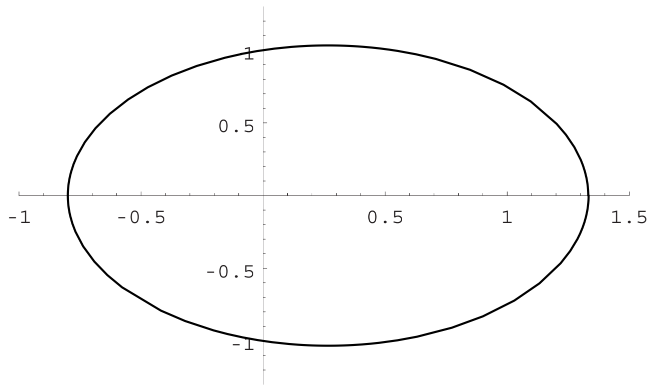
C10S06.062: From the given polar equation

$$r = \frac{8}{8 - 2\cos\theta} = \frac{1}{1 - \frac{1}{4}\cos\theta}$$

we read the information that $pe = 1$ and that $e = \frac{1}{4}$. Hence this conic section is an ellipse with one focus at $(0, 0)$ and one directrix the vertical line $x = -4$. When $\theta = 0$ we see that $r = \frac{4}{3}$; when $\theta = \pi$, $r = \frac{4}{5}$. Hence the vertices of this ellipse are at the points $\left(\frac{4}{3}, 0\right)$ and $\left(-\frac{4}{5}, 0\right)$. To see its graph, we executed the *Mathematica* command

```
ParametricPlot[ { (4*Cos[t])/(4 − Cos[t]), (4*Sin[t])/(4 − Cos[t]) },
        { t, 0, 2*Pi }, PlotRange → { { −1, 1.5 }, { −1.3, 1.3 } } ];
```

and the result is shown next.



C10S06.063: From the given polar equation

$$r = \frac{6}{2 - \sin\theta} = \frac{3}{1 - \frac{1}{2}\sin\theta}$$

we read the information that $pe = 3$ and that $e = \frac{1}{2}$. Hence $p = 6$, and therefore the conic section is an ellipse with one *horizontal* directrix $y = -6$ and one focus at $(0, 0)$. Conversion to Cartesian coordinates yields

$$2r - y = 6; \qquad 4(x^2 + y^2) = (y+6)^2;$$

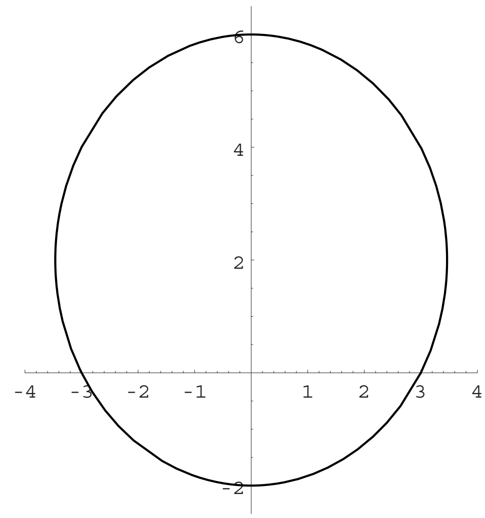
$$4x^2 + 3y^2 - 12y = 36; \qquad \frac{4}{3}x^2 + y^2 - 4y = 12;$$

$$\frac{4}{3}x^2 + y^2 - 4y + 4 = 16; \qquad \frac{4}{3}x^2 + (y-2)^2 = 16.$$

Hence this ellipse has center at $(0, 2)$ and its other focus at $(0, 4)$. When we evaluate r for $\theta = \pm\pi/2$, we find that the vertices of this ellipse are at $(0, 6)$ and $(0, -2)$. To see its graph, we executed the *Mathematica* command

```
ParametricPlot[ { (6*Cos[t])/(2 − Sin[t]), (6*Sin[t])/(2 − Sin[t]) },
            { t, 0, 2*Pi }, PlotRange → { { −4, 4 }, { −2.5, 6.5 } },
                                            AspectRatio → Automatic ];
```

and the result is shown next.



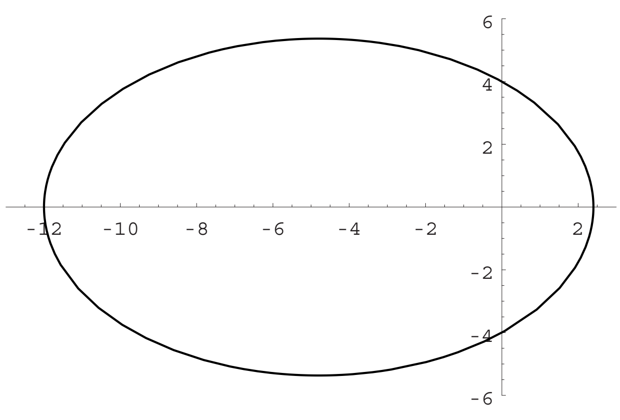
C10S06.064: From the given polar equation

$$r = \frac{12}{3 + 2\cos\theta} = \frac{4}{1 + \frac{2}{3}cos\theta}$$

we see that $pe = 4$ and that $e = \frac{2}{3}$. Hence $p = 6$, and this conic section is an ellipse with one focus at (0, 0) and one directrix with equation $x = 6$. Substitution of $\theta = 0$ and $\theta = \pi$ in its equation yields the further information that its vertices are at $\left(\frac{12}{5}, 0\right)$ and $(-12, 0)$. To see its graph, we executed the *Mathematica* command

```
ParametricPlot[ { (12*Cos[t])/(3 + 2*Cos[t]), (12*Sin[t])/(3 + 2*Cos[t]) },
                 { t, 0, 2*Pi }, PlotRange → { { −13, 3 }, { −6, 6 } } ];
```

and the result is shown next.



C10S06.065: The parabola with equation $y^2 = 4px$ has focus $F(p, 0)$. Suppose that (x, y) is a point on the parabola. Then $x = y^2/(4p)$. Our goal is to minimize $(x - p)^2 + y^2$; that is,

$$f(y) = \left(\frac{y^2}{4p} - p\right)^2 + y^2.$$

Now

$$\begin{aligned} f'(y) &= 2\left(\frac{y^2}{4p} - p\right) \cdot \frac{2y}{4p} + 2y = \frac{4y^3}{16p^2} - \frac{4py}{4p} + 2y \\ &= \frac{y^3}{4p^2} - y + 2y = \frac{y^3}{4p^2} + y = \frac{y^3 + 4p^2 y}{4p^2} = \frac{y}{4p^2}(y^2 + 4p^2). \end{aligned}$$

Therefore $f'(y) = 0$ if and only if $y = 0$; $f'(y) < 0$ if $y < 0$ and $f'(y) > 0$ if $y > 0$. So by the first derivative test, $f(y)$ has a global minimum value and it occurs where $y = 0$, so that $x = 0$ as well. Therefore the vertex $V(0,\, 0)$ of this parabola is the point of the parabola closest to its focus $F(p,\, 0)$.

C10S06.066: Suppose that the vertex of this parabola is $V(a,\, b)$. Then the parabola has equation of the form $(x - a)^2 = c(y - b)$ for some number c. Because $(2,\, 3)$ and $(4,\, 3)$ lie on the parabola, we get the simultaneous equations

$$\begin{aligned} &(2 - a)^2 = c(3 - b) \quad \text{and} \quad (4 - a)^2 = c(3 - b); \\ &(2 - a)^2 = (4 - a)^2; \\ &4 - 4a = 16 - 8a; \\ &4a = 12. \end{aligned}$$

Thus $a = 3$; note also that $c(3 - b) = 1$. Next, the point $(6,\, -5)$ is also on the parabola, so

$$(6 - 3)^2 = c(-5 - b) \quad \text{and thus} \quad 9 = -c(b + 5).$$

Therefore $-c(b + 5) = 9$ and $-c(b - 3) = 1$. Eliminate c to find that $b = 4$, and thus that $c = -1$. Therefore an equation of this parabola is $(x - 3)^2 = -(y - 4)$.

C10S06.067: Given: The point $Q(x_0,\, y_0)$ on the graph of the parabola with equation $y^2 = 4px$ $(p \neq 0)$. Using implicit differentiation,

$$2y\,\frac{dy}{dx} = 4p,$$

so the slope of the line tangent to the graph at Q is $4p/(2y_0)$. Thus it has equation

$$\begin{aligned} &y - y_0 = \frac{4p}{2y_0}(x - x_0); \\ &2y_0 y - 2y_0^2 = 4px - 4px_0; \\ &y_0 y - y_0^2 = 2px - 2px_0; \\ &y_0 y - 4px_0 = 2px - 2px_0; \\ &2px - y_0 y + 2px_0 = 0. \end{aligned}$$

In particular, when $y = 0$ we see that $x = -x_0$, so the tangent line meets the x-axis at the point $(-x_0,\, 0)$.

C10S06.068: With the origin at the focus $S(0,\, 0)$, the vertex of the parabola is at $V(a,\, 0)$ and thus the directrix has equation $x = 2a$ $(a < 0)$. The coordinates of C are $(100,\, 100)$ and the [horizontal] distance from C to the directrix is $100\sqrt{2}$, the same as its distance from the focus. Hence $2a = 100 - 100\sqrt{2}$, so

$a = 50\left(1 - \sqrt{2}\,\right)$. Therefore the closest approach of the comet to the sun is $50\left(\sqrt{2}\, - 1\right) \approx 20.7106781187$ million miles.

C10S06.069: Set up coordinates so that the parabola has vertex $V(-p,\,0)$. Then the equation of the comet's orbit is $y^2 = 4p(x+p)$. The line $y = x$ meets the orbit of the comet at the point $(a,\,b)$, which is $100\sqrt{2}$ million miles from the origin (which is also where both the sun and the focus of the parabola are located). Therefore

$$a^2 = 4p(a+p) \qquad \text{and} \qquad \sqrt{a^2+a^2} = \left(100\sqrt{2}\,\right)(10^6) = 10^8\sqrt{2}\,.$$

It follows that $a = 10^8$. Next, $a^2 = 4p(a+p)$. We apply the quadratic formula to find without difficulty that $p = \frac{1}{2}\left(\sqrt{2}\, - 1\right)(10^8)$. Now solve the equation of the orbit for x:

$$x = \frac{1}{4p}y^2 - p.$$

The area A_3 swept out by the line from the sun to the comet in three days is then

$$A_3 = \frac{1}{2}100^2 - \int_{2p}^{100}\left(\frac{1}{4p}y^2 - p\right)dy.$$

It now follows that

$$A_3 = 5000 - \frac{1}{12p}(10^6 - 8p^3) + 100p - 2p^2 \approx 2475.469.$$

The area of the "quarter-parabola" is

$$A_Q = \int_0^{2p}\left(p - \frac{1}{4p}y^2\right)dy = \frac{4}{3}p^2 \approx 571.9096.$$

So the comet will reach its point of closest approach in roughly 0.693 more days; that is, in about 16 h 38 min.

C10S06.070: We begin with Eqs. (7) and (8):

$$x = (v_0\cos\alpha)t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + (v_0\sin\alpha)t.$$

Then $t = \dfrac{x}{v_0\cos\alpha}$, so

$$\begin{aligned}
y &= -\frac{1}{2}g\cdot\frac{x^2}{(v_0\cos\alpha)^2} + \frac{v_0 x\sin\alpha}{v_0\cos\alpha} = -\frac{gx^2}{2(v_0\cos\alpha)^2} + \frac{x\sin\alpha}{\cos\alpha}\\
&= -\frac{g}{2(v_0\cos\alpha)^2}\left(x^2 - \frac{2x(v_0\cos\alpha)^2\sin\alpha}{g\cos\alpha}\right) = -\frac{g}{2(v_0\cos\alpha)^2}\left(x^2 - \frac{2xv_0^2\sin\alpha\,\cos\alpha}{g}\right)\\
&= -\frac{g}{2(v_0\cos\alpha)^2}\left(x^2 - \frac{2xv_0^2\sin\alpha\,\cos\alpha}{g} + \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2}\right) + \frac{gv_0^4\sin^2\alpha\cos^2\alpha}{2g^2v_0^2\cos^2\alpha}\\
&= -\frac{g}{2(v_0\cos\alpha)^2}\left(x - \frac{v_0\sin\alpha\cos\alpha}{g}\right)^2 + \frac{v_0^2\sin^2\alpha}{2g}.
\end{aligned}$$

Therefore the trajectory has equation

$$y - \frac{v_0^2 \sin^2 \alpha}{2g} = -\frac{g}{2(v_0 \cos \alpha)^2} \left(x - \frac{v_0 \sin \alpha \, \cos \alpha}{g} \right)^2.$$

It is now evident that the maximum height reached by the projectile will be

$$M = \frac{v_0^2 \sin^2 \alpha}{2g}.$$

This maximum will occur when x is *half* the range of the projectile (because of the symmetry of the parabola around its axis), and thus the projectile will have range

$$R = \frac{2v_0^2 \sin \alpha \, \cos \alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$

C10S06.071: With v_0 held constant, the range

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

of the projectile will be maximized when $\sin 2\alpha = 1$; that is, when $\alpha = 45°$. Thus the maximum range will be $R_{\max} = v_0^2/g$.

C10S06.072: With $v_0 = 50$ (m/s), $g = 9.8$ (m/s^2), and $\alpha = \pi/4$, the range will be

$$R = R_{\max} = \frac{2500}{9.8} \approx 255.102 \quad \text{(meters)}.$$

The maximum height reached by the projectile will be

$$M = \frac{v_0^2 \sin^2 \alpha}{2g} = \frac{2500}{(4)(9.8)} \approx 63.776 \quad \text{(meters)}.$$

C10S06.073: The range will be 125 meters when

$$\begin{aligned} \frac{2500 \sin 2\alpha}{9.8} &= 125; \\ \sin 2\alpha &= \frac{(125)(9.8)}{2500} = 0.49; \\ 2\alpha &= \arcsin(0.49). \end{aligned}$$

Therefore $\alpha \approx 14°40'13''$ and $\alpha \approx 75°19'47''$ will both produce a range of 125 meters.

C10S06.074: To find the time aloft, we solve $v_0 \sin \alpha = \frac{1}{2} gt$ for

$$t = \frac{2v_0 \sin \alpha}{g}.$$

Part (a): With $\alpha = \pi/6$ and $v_0 = 50$, the range will be

$$R = \frac{2500\sqrt{3}}{(2)(9.8)} \approx 220.925 \quad \text{(meters)}$$

and the time aloft will be

$$t = \frac{(2)(50)}{(2)(9.8)} = \frac{250}{49} \approx 5.102 \quad \text{(seconds)}.$$

Part (b): With $\alpha = \pi/3$ and $v_0 = 50$, the range will be

$$R = \frac{2500\sqrt{3}}{(2)(9.8)} \approx 220.925 \quad \text{(meters)},$$

exactly the same as in Part (a), but the time aloft will be instead

$$t = \frac{(2)(50)\sqrt{3}}{(2)(9.8)} = \frac{250\sqrt{3}}{49} \approx 8.837 \quad \text{(seconds)}.$$

C10S06.075: Given $\sqrt{x} + \sqrt{y} = \sqrt{a}$, square twice to eliminate the radicals:

$$x + 2\sqrt{xy} + y = a;$$

$$2\sqrt{xy} = a - x - y;$$

$$4xy = (a - x - y)^2 = a^2 - 2a(x + y) + x^2 + 2xy + y^2;$$

$$2xy + 2a(x + y) = x^2 + y^2 + a^2.$$

Now convert to polar coordinates:

$$r^2 + a^2 = 2r^2 \sin\theta \, \cos\theta + 2ar(\sin\theta + \cos\theta).$$

Now rotate the graph 45° (the reason is that if you graph the original equation, it resembles a parabola with axis the line $y = x$):

$$r^2 + a^2 = 2r^2 \sin\left(\theta + \frac{\pi}{4}\right) \cos\left(\theta + \frac{\pi}{4}\right) + 2ar\left[\sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right)\right];$$

$$r^2 + a^2 = 2r^2 \cdot \frac{1}{2}(\cos^2\theta - \sin^2\theta) + 2ar\sqrt{2}\,\cos\theta.$$

Finally, return to Cartesian coordinates:

$$x^2 + y^2 + a^2 = x^2 - y^2 + 2ax\sqrt{2};$$

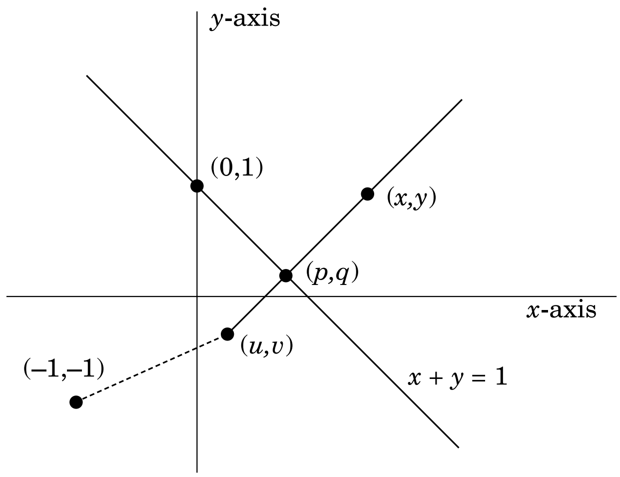
$$2ax\sqrt{2} = 2y^2 + a^2;$$

$$x = \frac{\sqrt{2}}{2a}y^2 + \frac{a\sqrt{2}}{4};$$

$$x - \frac{a\sqrt{2}}{4} = \frac{\sqrt{2}}{2a}y^2.$$

Therefore the graph of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is a parabola.

C10S06.076: Consider the following figure:



Let L be the line $x + y = 1$, let $(u,\ v)$ be a point equidistant from the point $(-1,\ -1)$ and the line L, and let M be the line through $(u,\ v)$ and perpendicular to the line L. The distance from $(u,\ v)$ to the line L is the distance from $(u,\ v)$ to the point of intersection of the lines L and M; call this point of intersection $(p,\ q)$. The slope of L is -1, so the slope of M is $+1$.

First write p and q in terms of u and v: Because the slope of M is 1, $q - v = p - u$, and because the point $(p,\ q)$ lies on the line L, $p + q = 1$. So

$$\begin{gathered} 1 - p - v = p - u; \\ p = \tfrac{1}{2}(1 + u - v) \qquad \text{and} \qquad q = \tfrac{1}{2}(1 - u + v). \end{gathered}$$

Now equate the distance between $(u,\ v)$ and $(-1,\ -1)$ with the distance between $(u,\ v)$ and $(p,\ q)$:

$$\begin{aligned} (u+1)^2 + (v+1)^2 &= (u-p)^2 + (v-q)^2 \\ &= \left(u + \frac{v-u-1}{2}\right)^2 + \left(v + \frac{u-v-1}{2}\right)^2. \end{aligned}$$

Now, in order to write the equation of the curve in terms of x and y, replace $(u,\ v)$ with $(x,\ y)$ and expand:

$$\begin{gathered} 4(x+1)^2 + 4(y+1)^2 = (2x + y - x - 1)^2 + (2y + x - y - 1)^2; \\ 4x^2 + 8x + 4 + 4y^2 + 8y + 4 = 2\left(x^2 + 2xy + y^2 - 2x - 2y + 1\right); \\ 2x^2 + 12x + 2y^2 + 12y - 4xy + 6 = 0; \\ x^2 - 2xy + y^2 + 6x + 6y + 3 = 0. \end{gathered}$$

The coefficient of x is 6, so $D = 6$.

C10S06.077: Part (a): In the usual notation, we have $e = 0.999925$ and $a - c = 0.13$ (AU). Now

$$b^2 = a^2 - c^2 = (a+c)(a-c) \qquad \text{and} \qquad a = \frac{c}{e}.$$

It follows that

$$\frac{c}{e} - c = 0.13, \qquad \text{and thus} \qquad c = (0.13)\frac{999925}{75} \approx 1733.203333.$$

Thus $a = c/e \approx 1733.246664$ and so $b \approx 12.25577415$. The maximum distance between Kahoutek and the sun is therefore $2a - 0.13 \approx 3466.363328$ (AU)—about 322 *billion* miles, about 20 light-days.

Part (b): In the case of Comet Hyakutake, we have $e = 0.999643856$ and $a - c = 0.2300232$. Thus

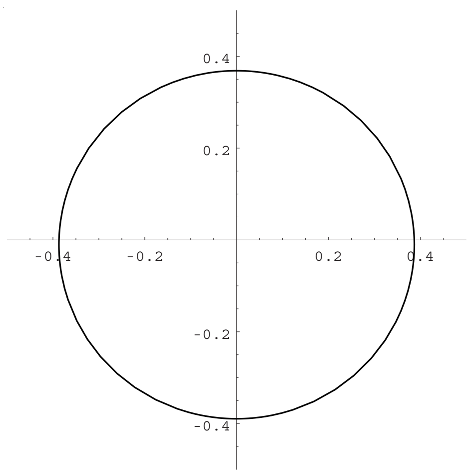
$$\frac{c}{e} - c = 0.2300232; \qquad \text{hence} \qquad c\left(\frac{1}{e} - 1\right) = 0.2300232.$$

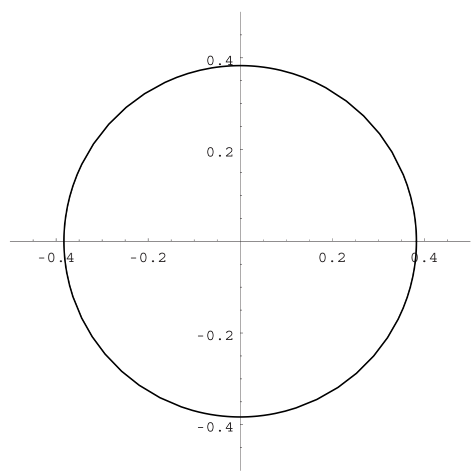
Thus $c \approx 645.64130974$. But $a = c/e$, so $a \approx 645.87133294$. So the greatest distance between Hyakutake and the sun is $2a - 0.2300232 \approx 1291.51264269$ (AU). This is about 120 billion miles, about 7.45 light-days.

C10S06.078: In the usual notation, we have $2a = 0.467 + 0.307 = 0.774$. So $a = 0.387$ and $e = 0.206$. Therefore $c = ae = 0.079722$ and

$$b = \sqrt{a^2 - c^2} \approx 0.378699621;$$

we'll use $b = 0.3787$. Therefore the ellipse has major axis 0.774, minor axis 0.7574; in terms of percentages, a is about 2.2% greater than b. Is this a nearly circular orbit? Decide for yourself: Compare the circle (on the right) below, with diameter 0.766, with the ellipse (on the left) below with the shape of the orbit of the planet.





C10S06.079: Assume that the focus on the positive y-axis is $F(0,\, c)$ and that the directrix is the line L with equation $y = c/e^2$ where $0 < e < 1$. Suppose that $P(x,\, y)$ is a point of the ellipse. Then the equation $|PF| = e \cdot |PL|$ yields

$$\sqrt{x^2 + (y-c)^2} = e \cdot \left(y - \frac{c}{e^2}\right);$$

$$x^2 + y^2 - 2cy + c^2 = e^2\left(y - \frac{c}{e^2}\right)^2;$$

$$x^2 + y^2 - 2cy + c^2 = e^2y^2 - 2cy + \frac{c^2}{e^2};$$

$$x^2 + (1-e^2)y^2 = \frac{c^2}{e^2} - c^2 = c^2\left(\frac{1}{e^2} - 1\right) = \frac{c^2}{e^2}(1-e^2).$$

Now substitute $a = c/e$:

$$x^2 + (1 - e^2)y^2 = a^2(1 - e^2);$$

$$\frac{x^2}{a^2(1 - e^2)} + \frac{y^2}{a^2} = 1.$$

Let $b^2 = a^2(1 - e^2)$ where $b > 0$. This is possible because $0 < e < 1$. Then

$$b^2 = a^2 - a^2e^2 = a^2 - c^2, \quad \text{so that} \quad a^2 + b^2 = c^2.$$

The equation of the ellipse is therefore

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1;$$

note also that $0 < b < a$ and that the directrix has equation $y = \dfrac{c}{e^2} = \dfrac{a}{e}$.

C10S06.080: Implicit differentiation of the equation of the ellipse yields

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0,$$

and therefore the line tangent to its graph at $(x_0,\, y_0)$ has slope

$$m = -\frac{2x_0}{a^2} \cdot \frac{b^2}{2y_0} = -\frac{b^2x_0}{a^2y_0}$$

and thus equation

$$y - y_0 = -\frac{b^2x_0}{a^2y_0}(x - x_0).$$

But

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1, \quad \text{so} \quad b^2x_0^2 + a^2y_0^2 = a^2b^2.$$

Thus the equation of the tangent line may be written in the form

$$a^2y_0y - a^2y_0^2 = b^2x_0^2 - b^2x_0x;$$

$$a^2y_0y + b^2x_0x = a^2b^2;$$

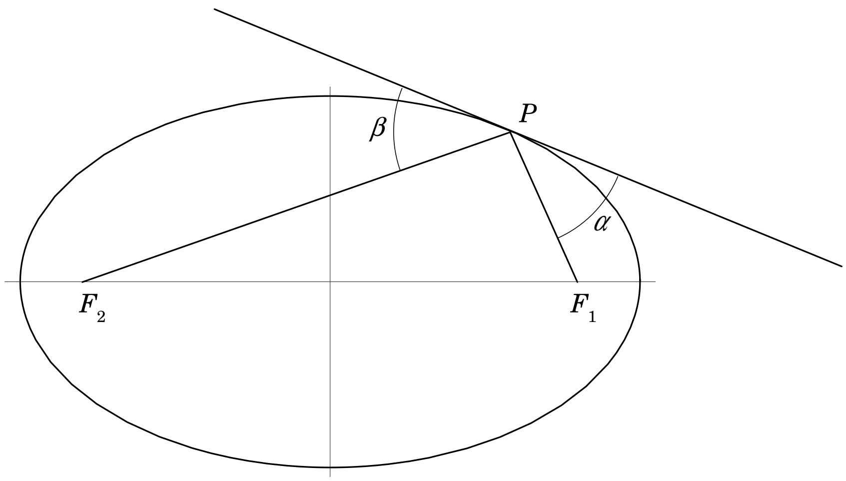
$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1.$$

The second of these last three equations will figure prominently in the solution of Problem 25.

C10S06.081: We ignore the *Suggestion* given in the statement of Problem 81. We recommend that you visit

`http://www.augsburg.edu/depts/math/MATtours/ellipses.1.09.0.html`

for an elegant two-line proof of the reflection property due to Zalman P. Usiskin. (This site was available on January 7, 2000 and has been in existence for several years; it should still be there when you read this.) Before we discovered Usiskin's proof, we constructed an algebraic proof and here it is.



See the preceding figure. Let $P(x_0,\, y_0)$ be the point of tangency. Suppose that the ellipse has equation $(x/a)^2 + (y/b)^2 = 1$ where $0 < b < a$. We saw in the solution of Problem 24 that the slope of the tangent line is

$$-\frac{b^2 x_0}{a^2 y_0}. \tag{1}$$

Let $F_1(c,\, 0)$ and $F_2(-c,\, 0)$ be the foci of the ellipse, let m_1 be the slope of F_1P, let m_2 be the slope of F_2P, let m be the slope of the tangent line, let α be the angle between F_1P and the tangent, and let β be the angle between F_2P and the tangent. Let θ_1 be the angle of inclination of F_1P, let θ_2 be the angle of inclination of F_2P, and let ϕ be the angle between the tangent line and the horizontal, so that $\alpha+\phi+\theta_1 = \pi$ and $\phi+\theta_2 = \beta$. Also note that $\tan\phi = 1/m$. Finally note that because $(x_0,\, y_0)$ lies on the ellipse, $b^2x_0^2 + a^2y_0^2 = a^2b^2$. Then

$$a^4y_0^2 - b^4x_0^2 = a^2b^2y_0^2 - a^2b^2x_0^2 + a^2b^2x_0^2 + a^4y_0^2 - b^4x_0^2 - a^2b^2y_0;$$

$$(a^2y_0 + b^2x_0)(a^2y_0 - b^2x_0) = a^2b^2y_0^2 - a^2b^2x_0^2 + (a^2 - b^2)(b^2x_0^2 + a^2y_0^2);$$

$$a^4y_0^2 - b^4x_0^2 = a^2b^2y_0^2 - a^2b^2x_0^2 + a^2b^2(a^2 - b^2);$$

$$a^4y_0^2 - b^4x_0^2 = a^2b^2(y_0^2 - x_0^2 + c^2);$$

$$2x_0y_0\frac{(a^2y_0 + b^2x_0)(a^2y_0 - b^2x_0)}{b^2x_0} = 2a^2y_0(y_0^2 - x_0^2 + c^2);$$

$$\frac{2x_0y_0}{x_0^2 - c^2} \cdot \frac{a^4y_0^2 - b^4x_0^2}{b^4x_0^2} = \frac{2a^2y_0}{b^2x_0} \cdot \frac{y_0^2 - x_0^2 + c^2}{x_0^2 - c^2};$$

$$\left(\frac{y_0}{x_0 - c} + \frac{y_0}{x_0 + c}\right) \cdot \left(\frac{a^4y_0^2}{b^4x_0^2} - 1\right) = \frac{2a^2y_0}{b^2x_0}\left(\frac{y_0^2}{x_0^2 - c^2} - 1\right);$$

$$(m_1 + m_2)(m^2 - 1) = 2m(m_1m_2 - 1);$$

$$(m_1 + m_2)\frac{m^2 - 1}{m^2} = \frac{2}{m}(m_1m_2 - 1);$$

$$(m_1 + m_2)\left(\frac{1}{m^2} - 1\right) = \frac{2}{m}(1 - m_1 m_2);$$

$$(m_1 + m_2) \cdot \frac{1}{m^2} - 2(1 - m_1 m_2) \cdot \frac{1}{m} - (m_1 + m_2) = 0;$$

$$(\tan\theta_1 + \tan\theta_2)\tan^2\phi - 2(1 - \tan\theta_1\ \tan\theta_2)\tan\phi - (\tan\theta_1 + \tan\theta_2) = 0;$$

$$\tan\theta_1 + \tan\theta_2 + 2\tan\phi - 2\tan\theta_1\ \tan\theta_2\ \tan\phi - (\tan\theta_1 + \tan\theta_2)\tan^2\phi = 0;$$

$$\begin{aligned} &\tan\theta_1 + \tan\phi - \tan\theta_1\ \tan\theta_2\ \tan\theta - \tan\theta_2\ \tan^2\phi \\ &\qquad\qquad = -\tan\theta_2 - \tan\phi + \tan\theta_1\ \tan\theta_2\ \tan\phi + \tan\theta_1\ \tan^2\phi; \end{aligned}$$

$$\frac{\tan\theta_1 + \tan\phi}{-1 + \tan\theta_1\ \tan\phi} = \frac{\tan\theta_2 + \tan\phi}{1 - \tan\theta_2\ \tan\phi};$$

$$-\tan(\theta_1 + \phi) = \tan(\theta_2 + \phi);$$

$$\tan(\pi - \theta_1 - \phi) = \tan(\phi + \theta_2);$$

$$\tan\alpha = \tan\beta.$$

Therefore $\alpha = \beta$. ◀

C10S06.082: Given: $0 < c < a$, the fixed points $F_1(-c, 0)$ and $F_2(c, 0)$, and the point $P(x, y)$, assume that $|PF_1| + |PF_2| = 2a$. Then

$$\begin{aligned} &\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a; \\ &\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}; \\ &(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2; \\ &4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx; \\ &a^2\left[(x-c)^2 + y^2\right] = a^4 - 2a^2cx + c^2x^2; \\ &a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2; \\ &a^2x^2 + a^2c^2 + a^2y^2 = a^4 + c^2x^2; \\ &(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2); \\ &b^2x^2 + a^2y^2 = a^2b^2 \qquad \text{where} \quad b^2 = a^2 - c^2; \\ &\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \end{aligned}$$

C10S06.083: Solution (a): It's clear that the center of this ellipse is at $(1, 0)$. So the ellipse has an equation of the form

$$\frac{(x-1)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Substitution of $(x, y) = (3, 0)$ in this equation yields

$$\frac{4}{a^2} = 1, \qquad \text{so that} \qquad a^2 = 4.$$

Thus we may assume that $a = 2$. Substitution of $(x,\, y) = (0,\, 2)$ then yields

$$\frac{1}{4} + \frac{4}{b^2} = 1, \qquad \text{so that} \qquad b^2 = \frac{16}{3}.$$

Thus an equation of the ellipse through the four given points is

$$\frac{(x-1)^2}{4} + \frac{3y^2}{16} = 1.$$

Solution (b) (in case it is *not* clear where the center of the ellipse is): The *Mathematica* command

```
Solve[ { ((-1 - u)/a)∧2 + ((0 - v)/b)∧2 == 1,
         ((3 - u)/a)∧2 + ((0 - v)/b)∧2 == 1,
         ((0 - u)/a)∧2 + ((2 - v)/b)∧2 == 1,
         ((0 - u)/a)∧2 + ((-2 - v)/b)∧2 == 1 }, { a, b, u, v } ]
```

returns the solutions $u = 1$, $v = 0$, $a = \pm 2$, $b = \pm 4/\sqrt{3}$ *and no others.*

C10S06.084: Because $2a = 10$, if $P(x,\, y)$ is a point of the ellipse then—by the last paragraph of the subsection on applications of the ellipse in Section 10.6—

$$\begin{aligned}
&\sqrt{(x+3)^2 + (y-3)^2} + \sqrt{(x-3)^2 + (y+3)^2} = 10;\\
&x^2 + 6x + 9 + y^2 - 6y + 9 = 100 - 20\sqrt{(x-3)^2 + (y+3)^2} + x^2 - 6x + 9 + y^2 + 6y + 9;\\
&20\sqrt{(x-3)^2 + (y+3)^2} = 100 - 12x + 12y;\\
&5\sqrt{(x-3)^2 + (y+3)^2} = 25 - 3x + 3y;\\
&25(x^2 - 6x + 9 + y^2 + 6y + 9) = 625 - 150x + 150y + 9x^2 - 18xy + 9x^2;\\
&25x^2 - 150x + 225 + 25y^2 + 150y + 225 = 625 - 150x + 150y + 9x^2 - 18xy + 9y^2;\\
&16x^2 + 18xy + 16y^2 = 175.
\end{aligned}$$

C10S06.085: Given (with a change in notation):

$$\frac{x^2}{15-q} - \frac{y^2}{q-6} = 1. \tag{1}$$

Part (a): If $6 < q < 15$, then $15 - q > 0$ and $q - 6 > 0$. So the graph of Eq. (1) is a hyperbola with horizontal transverse axis and center $C(0,\, 0)$. Also $a^2 = 15 - q$ and $b^2 = q - 6$, so that $a^2 + b^2 = 9 = c^2$. Thus $c = 3$ and so the hyperbola has foci at $(\pm 3,\, 0)$.

Part (b): $q < 6$. Then $15 - q > 0$ and $q - 6 < 0$, so the graph of Eq. (1) is an ellipse.

Part (c): $q > 15$. In this case $15 - q < 0$ and $q - 6 > 0$, so Eq. (1) takes the form

$$\frac{x^2}{q-15} + \frac{y^2}{q-6} = -1.$$

Both denominators are positive, so there are no points on the graph.

C10S06.086: Given: The hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the point $P(x_0, y_0)$ on its graph. Note that neither a nor b is zero. If $y_0 \neq 0$, then implicit differentiation yields

$$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{b^2 x}{a^2 y},$$

so that the slope of the line tangent to the graph of the hyperbola at the point P is

$$m = \frac{b^2 x_0}{a^2 y_0}.$$

(Note: This formula will be important in the solution of Problem 87.) Hence the line tangent to the hyperbola at P has equation

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0); \qquad \text{that is,} \qquad a^2 y_0 y - a^2 y_0^2 = b^2 x_0 x - b^2 x_0^2. \tag{1}$$

But

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1, \qquad \text{and so } b^2 x_0^2 - a^2 y_0^2 = a^2 b^2.$$

(Note: The second of these formulas is important in the solution of Problem 87.) Substitution in the second formula in Eq. (1) now yields $b^2 x_0 x - a^2 y_0 y = a^2 b^2$, and therefore the tangent line has equation

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

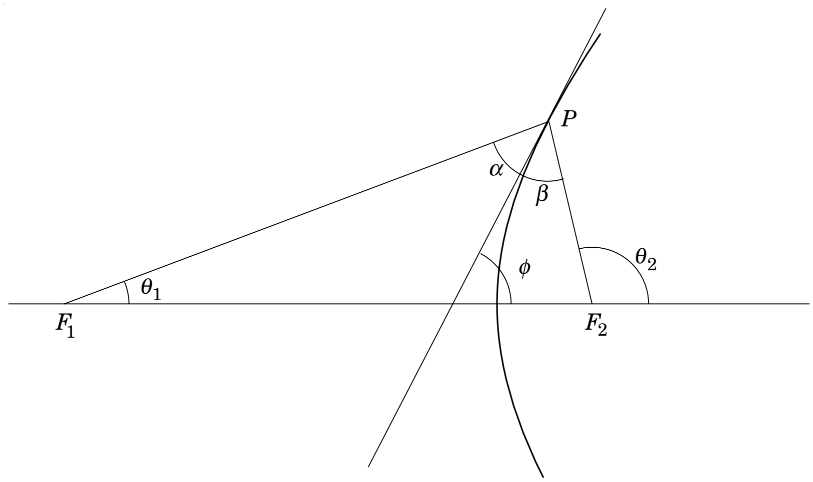
If $y_0 = 0$, work instead with dx/dy to obtain the same result by the same method.

C10S06.087: See the following figure. It shows the right branch of a hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(where $a > 0$ and $b > 0$), with foci $F_1(-c, 0)$ and $F_2(c, 0)$ (where $c > 0$). Let L be the line tangent to the hyperbola at the point $P(p, q)$ where $p > 0$ and $q \neq 0$. Let α be the angle between L and F_1P and let β be the angle between L and F_2P. Let θ_1 be the angle of inclination of F_1P, θ_2 the angle of inclination of F_2P,

m the slope of L, and ϕ the angle of inclination of L. The goal is to prove that $\alpha = \beta$.



Here's what we have to work with. First, $m = \tan\phi$; we also found in the solution of Problem 86 that

$$m = \frac{b^2 p}{a^2 q} = \tan\phi.$$

We know also (Section 10.6) that $a^2 + b^2 = c^2$. Because $(p,\, q)$ lies on the hyperbola, it also follows that $b^2p^2 - a^2q^2 = a^2b^2$. Let m_1 be the slope of F_1P and let m_2 be the slope of F_2P. Then

$$m_1 = \tan\theta_1 = \frac{q}{p+c} \qquad \text{and} \qquad m_2 = \tan\theta_2 = \frac{q}{p-c}.$$

Also,

$$\theta_1 + \alpha + \pi - \phi = \pi \qquad \text{and} \qquad \phi + \beta + \pi - \theta_2 = \pi,$$

so that $\alpha = \phi - \theta_1$ and $\beta = \theta_2 - \phi$.

We are ready to begin. The following proof was developed interactively with *Mathematica* 3.0. We can show that $\alpha = \beta$ if we can show that $\tan\alpha = \tan\beta$. This would follow if $\tan(\phi - \theta_1) = \tan(\theta_2 - \phi)$, which follows from

$$\frac{\tan\phi - \tan\theta_1}{1 + \tan\phi\,\tan\theta_1} = \frac{\tan\theta_2 - \tan\phi}{1 + \tan\theta_2\,\tan\phi}.$$

This equation follows from

$$\frac{m - m_1}{1 + mm_1} = \frac{m_2 - m}{1 + mm_2}; \qquad \text{that is,} \qquad \frac{m - m_1}{1 + mm_1} - \frac{m_2 - m}{1 + mm_2} = 0.$$

We entered the left-hand side of the last equation and applied various *Mathematica* commands to it with the following results. First we used `Together`:

$$\frac{2m - m_1 + m^2m_1 - m_2 + m^2m_2 - 2mm_1m_2}{(1 + mm_1)(1 + mm_2)}.$$

Then `Numerator`:

$$2m - m_1 + m^2m_1 - m_2 + m^2m_2 - 2mm_1m_2.$$

Then we entered % `/. m -> b*b*p/(a*a*q)`. This asks *Mathematica* to evaluate the previous expression (%) "subject to" the replacement of m with $b^2p/(a^2q)$, and we obtained

$$-m_1 - m_2 + \frac{b^4 m_1 p^2}{a^4 q^2} + \frac{b^4 m_2 p^2}{a^4 q^2} + \frac{2b^2 p}{a^2 q} - \frac{2b^2 m_1 m_2 p}{a^2 q}.$$

Similarly, we replaced m_1 with $q/(p+c)$ and m_2 with $q/(p-c)$ and thereby obtained

$$\frac{2b^2 p}{a^2 q} + \frac{b^4 p^2}{a^4 (p-c) q} + \frac{b^4 p^2}{a^4 (p+c) q} - \frac{q}{p-c} - \frac{q}{p+c} - \frac{2b^2 pq}{a^2 (p-c)(p+c)}.$$

Another application of `Together` followed by `Numerator` yielded

$$-2(a^2b^2c^2p - a^2b^2p^3 - b^4p^3 + a^4pq^2 + a^2b^2pq^2).$$

The command %`/(-2*p) // Cancel` produced

$$a^2b^2c^2 - a^2b^2p^2 - b^4p^2 + a^4q^2 + a^2b^2q^2,$$

and then % `/. c^2 -> a^2 + b^2` yielded

$$a^2b^2(a^2+b^2) - a^2b^2p^2 - b^4p^2 + a^4q^2 + a^2b^2q^2.$$

We then asked for replacement of b^2p^2 with $a^2b^2 + a^2q^2$ and obtained

$$a^2b^2(a^2+b^2) - b^4p^2 + a^4q^2 + a^2b^2q^2 - a^2(a^2b^2 + a^2q^2).$$

The command `Factor[%]` returned

$$b^2(a^2b^2 - b^2p^2 + a^2q^2).$$

We then cancelled b^2 to obtain $a^2b^2 - b^2p^2 + a^2q^2$, which we have already seen is zero. This establishes the desired conclusion: $\alpha = \beta$. ◀

C10S06.088: We begin with $0 < a < c$, $b = \sqrt{c^2 - a^2}$, the two fixed points $F_1(-c, 0)$ and $F_2(c, 0)$, and the point $P(x, y)$ satisfying the equation $|PF_1| - |PF_2| = 2a$. Thus

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a;$$

$$\sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2};$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2;$$

$$cx = a^2 + a\sqrt{(x-c)^2 + y^2};$$

$$a\sqrt{(x-c)^2 + y^2} = cx - a^2;$$

$$a^2(x^2 - 2cx + c^2 + y^2) = c^2x^2 - 2a^2cx + a^4;$$

$$(a^2 - c^2)x^2 + a^2c^2 + a^2y^2 = a^4;$$

$$-b^2x^2 + a^2y^2 = a^2(a^2 - c^2) = -a^2b^2;$$

$$b^2x^2 - a^2y^2 = a^2b^2;$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{1}$$

We have here the implicit assumption that $x > 0$, so that $x \geqq a$. So we have obtained the *right* branch of the hyperbola. But the equation $|PF_2| - |PF_1| = 2a$ yields the *left* branch, so the locus of $P(x, y)$ is indeed the entire hyperbola with the equation that appears in (1).

C10S06.089: First, $a^2 = \frac{9}{2} + \frac{9}{2} = 9$, so that $a = 3$ and thus $2a = 6$. Therefore Problem 24 implies that this hyperbola has equation

$$\begin{aligned}
&\sqrt{(x-5)^2 + (y-5)^2} + 6 = \sqrt{(x+5)^2 + (y+5)^2}\,;\\
&x^2 - 10x + y^2 - 10y + 50 + 12\sqrt{(x-5)^2 + (y-5)^2} + 36 = x^2 + 10x + y^2 + 10y + 50;\\
&12\sqrt{(x-5)^2 + (y-5)^2} = 20x + 20y - 36;\\
&3\sqrt{(x-5)^2 + (y-5)^2} = 5x + 5y - 9;\\
&9(x^2 + y^2 - 10x - 10y + 50) = 25(x^2 + 2xy + y^2) - 90x - 90y + 81;\\
&16x^2 + 50xy + 16y^2 = 369.
\end{aligned}$$

C10S06.090: Suppose that the plane is at $P(x, y)$, that A is at $(-50, 0)$, and that B is at $(50, 0)$. Let T_A and T_B denote the times for the signals from A and B (respectively) to reach the plane. Then $T_A - T_B = 400$. But

$$|PA| = 980T_A \quad \text{and} \quad |PB| = 980T_B,$$

so $|PA|/980 = 400 + |PB|/980$, and hence $|PA| - |PB| = (980)(400) = 392000$ (ft). Moreover, $b^2 = c^2 - a^2$, so (still in feet)

$$\begin{aligned}
a &= 196000,\\
c &= 264000, \quad \text{and}\\
b &= (4000)\sqrt{1955} \approx 176861.53.
\end{aligned}$$

In miles, $a \approx 37.1212$, $b \approx 33.4965$, and $c = 50$. The hyperbola on which the plane must lie has approximate equation

$$\frac{x^2}{1377.984} - \frac{y^2}{1122.016} = 1.$$

Now the plane also lies on the line $y = 50$, so when this value is substituted into the equation of the hyperbola we find that

$$x^2 \approx 4448.317, \quad \text{so that} \quad x \approx 66.6957.$$

In our coordinate system, the plane is located approximately at the point $(66.6957, 50)$ (now we stay exclusively in miles). Thus the plane is 16.6957 miles *east* of B and 50 miles north of B; alternatively, it is about 52.7138 miles from B in the direction $18°27'54''$ east of north.

C10S06.091: Suppose that the plane is at $P(x, y)$, that A is at $(-50, 0)$, and that B is at $(50, 0)$. Let $D = |AP|$ and $E = |BP|$, in feet. Then

$$\frac{D}{980} + \frac{E}{980} = 600 \quad \text{and} \quad \frac{D}{980} = \frac{E}{980} + 400.$$

Find D and E, observe in the process that $D = 5E$, and note that $P(x, y)$ satisfies both the equations $D = |AP|$ and $E = |BP|$. You should find that (in feet) $x \approx 218272.73$.

C10S06.092: We assume that the parabola opens to the right, and thus that its equation is

$$r = \frac{p}{1 + e\cos\theta}.$$

With units in millions of miles, we have

$$150 = \frac{p}{1 - \cos(\pi/4)},$$

and therefore

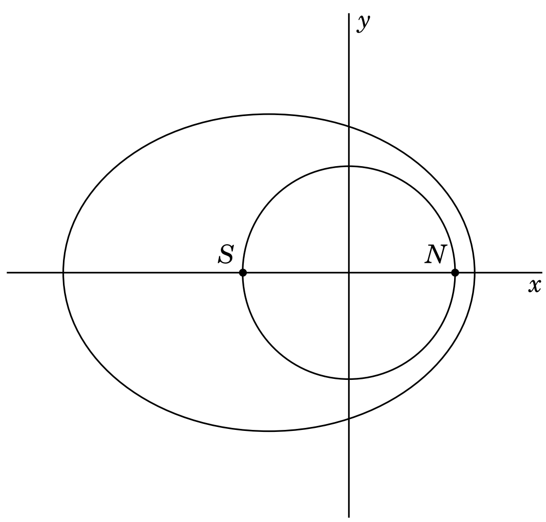
$$p = 150 \cdot \left(1 - \frac{1}{2}\sqrt{2}\right) = 75 \cdot \left(2 - \sqrt{2}\right).$$

The comet is closest to the sun at the vertex of its parabolic orbit, when $\theta = \pi$, and the closest approach is thus $p/2$, approximately 21.967 million miles. It is plausible, but stretches the interpretation of the problem slightly, to interpret the given data to mean that

$$150 = \frac{p}{1 - \cos(3\pi/4)},$$

and the minimum distance is then $p/2 \approx 128.033$ million miles—but we consider the first answer to be correct.

C10S06.093: The following figure indicates the earth (as the small circle) with north pole marked N and south pole marked S; the larger curve indicates the elliptical orbit of the satellite.



From the figure we read the information

$$\frac{pe}{1 + e} = 4500 \quad \text{and} \quad \frac{pe}{1 - e} = 9000.$$

Therefore $pe = 4500(1+e) = 9000(1-e)$, and it follows that $e = \frac{1}{3}$ and that $p = 18000$. The polar equation of the orbit of the satellite is then

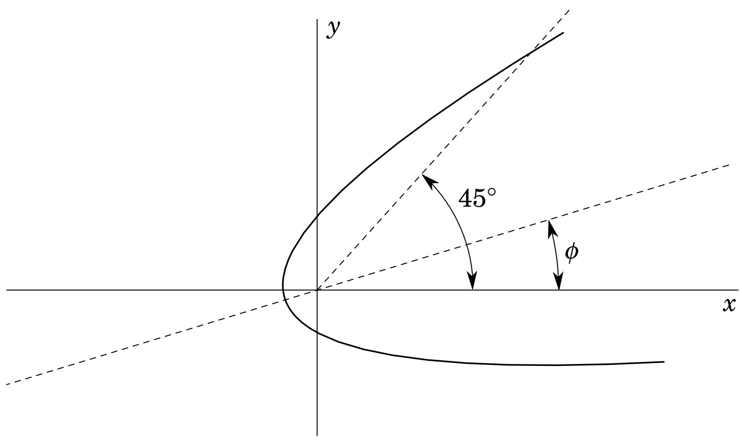
$$r = \frac{6000}{1 + \frac{1}{3}\cos\theta}.$$

The satellite crosses the equatorial plane when $\theta = \pi/2$, which yields $r = 6000$. So the height of the satellite above the surface of the earth then is $h = 6000 - 4000 = 2000$ (mi).

C10S06.094: As in Example 13 of the text, we assume that the orbit of the comet is sufficiently well approximated by a parabola near the sun that we may assume that its orbit is a parabola with eccentricity $e = 1$. Thus the orbit is approximated by the graph of the polar equation

$$r = \frac{p}{1 - \cos(\theta - \phi)},$$

where ϕ is the angle measured from the polar axis to the axis of the parabola, as shown in the following figure.



We now use the data given in the problem to find p and ϕ. First,

$$\frac{5}{2} = \frac{p}{1 - \cos\left([\pi/4] - \phi\right)} \qquad \text{and} \qquad 1 = \frac{p}{1 - \cos\left([\pi/2] - \phi\right)}.$$

By solving each equation for p and equating the results, we get

$$f(\phi) = 3 - \left(\frac{5}{2}\sqrt{2}\right)(\cos\phi + \sin\phi) + 2\sin\phi = 0.$$

To find ϕ, we apply Newton's method and obtain the following results:

$$\phi \approx 1.088587522 \qquad (\text{about} \quad 62^\circ\ 22'\ 17'') \qquad \text{and}$$

$$\phi \approx -0.2691202248 \qquad (\text{about} \quad -15^\circ\ 25'\ 10'').$$

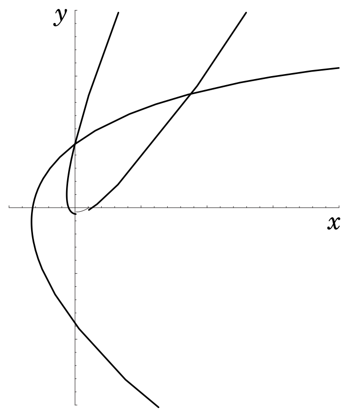
Now $p = r \cdot [1 - \cos(\theta - \phi)]$. So when ϕ takes on the first of these two values and $\theta = \pi/2$, we have $p \approx 0.1140272202$, and the closest approach of the comet is

$$0.0570136101 \quad (\text{AU; about } 5{,}302{,}270 \text{ mi}).$$

When ϕ takes on the other value, we find that $p \approx 1.265883432$, and in this case the closest approach of the comet is

$$0.6329417159 \quad (\text{AU; about } 58{,}863{,}580 \text{ mi}).$$

Are both these situations possible? Indeed they are—the following figure illustrates both cases, and is drawn to scale.



C10S06.095: Here is one solution; it may not be the simplest, and it probably isn't the most elegant—but it works. Assume that $a > b > 0$ and locate coordinate axes so that the Cartesian equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then substitute $x = r\cos\theta$ and $y = r\sin\theta$ to convert this equation to polar coordinates. It turns out that

$$\frac{1}{2}r^2 = \frac{a^2b^2}{a^2 + b^2 + (b^2 - a^2)\cos 2\theta}.$$

The area of the ellipse is

$$A = 4\int_0^{\pi/2} \frac{1}{2}r^2\, d\theta.$$

The substitution $\phi = 2\theta$ then yields

$$A = 2\int_{\phi=0}^{\pi} \frac{a^2b^2}{(a^2 + b^2) + (b^2 - a^2)\cos\phi}\, d\phi.$$

Then the substitution (see the discussion following Miscellaneous Problem 134 of Chapter 8)

$$u = \tan\frac{\phi}{2}: \qquad \phi = 2\arctan u,$$

$$d\phi = \frac{2}{1+u^2}\, du, \qquad \cos\phi = \frac{1-u^2}{1+u^2}$$

and the observation that $u = 0$ when $\phi = 0$ and that $u \to +\infty$ as $\phi \to \pi^-$ leads to the improper (but convergent!) integral

$$A = 2\int_0^\infty \frac{a^2b^2}{(a^2+b^2)+(b^2-a^2)\cdot\dfrac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$

$$= 2\int_0^\infty \frac{2a^2b^2}{(a^2+b^2)+(b^2-a^2)(1-u^2)}\,du = 4a^2b^2\int_0^\infty \frac{1}{2b^2+2a^2u^2}\,du$$

$$= 2a^2\int_0^\infty \frac{1}{1+\left(\dfrac{au}{b}\right)^2}\,du = 2a^2\cdot\left[\frac{b}{a}\arctan\left(\frac{au}{b}\right)\right]_0^\infty = 2ab\cdot\frac{\pi}{2} = \pi ab.$$

Here's an alternative solution (C.H.E.) that uses the standard polar form of the equation of an ellipse as presented in the final subsection of Section 10.6.

◇◇

We begin with the ellipse with semiaxes a and $b < a$, eccentricity $e = c/a$, and polar equation

$$r = \frac{pe}{1-e\cos\theta}.$$

Then Eq. (29) gives $pe = a(1-e^2)$, so the area of the ellipse is

$$A = 2\int_{\theta=0}^{\pi}\frac{1}{2}r^2\,d\theta = a^2(1-e^2)^2 I \quad\text{where}\quad I = \int_0^\pi \frac{1}{(1-e\cos\theta)^2}\,d\theta.$$

The substitution $u = \tan(\theta/2)$ mentioned in the previous solution now gives (after a bit of simplification)

$$I = \int_0^\infty \frac{2(1+u^2)}{\left[(1-e)+(1+e)u^2\right]^2}\,du = \int_0^\infty \frac{2(1+u^2)}{(B+Cu^2)^2}\,du$$

where $B = 1-e$ and $C = 1+e$. A simple partial-fractions expansion yields

$$I = \frac{2(C-B)}{C}I_2 + \frac{2}{C}I_1 \quad\text{where}\quad I_n = \int_0^\infty \frac{1}{(B+Cu^2)^n}\,du.$$

Then integration of I_1 by parts with

$$p = \frac{1}{B+Cu^2} \quad\text{and}\quad dq = du$$

gives

$$I_1 = \left[\frac{u}{B+Cu^2}\right]_0^\infty + \int_0^\infty \frac{2Cu^2}{(B+Cu^2)^2}\,du \qquad \text{(the first term vanishes);}$$

$$I_1 = 2\int_0^\infty \frac{B+Cu^2}{(B+Cu^2)^2}\,du - \int_0^\infty \frac{2B}{(B+Cu^2)^2}\,du = 2I_1 - 2BI_2.$$

It follows that $I_2 = \dfrac{I_1}{2B}$, so

$$I = \frac{2(C-B)}{C} \cdot \frac{I}{2B} + \frac{2}{C} I_1 = \left(\frac{1}{B} + \frac{1}{C}\right) I_1 = \left(\frac{1}{B} + \frac{1}{C}\right) \int_0^\infty \frac{1}{B + Cu^2}\, du.$$

With $k = \sqrt{\dfrac{B}{C}}$ we get

$$\begin{aligned} I &= \frac{1}{C}\left(\frac{1}{B} + \frac{1}{C}\right) \int_0^\infty \frac{1}{k^2 + u^2}\, du = \frac{1}{C}\left(\frac{1}{B} + \frac{1}{C}\right) \cdot \left[\frac{1}{k} \tan^{-1} \frac{u}{k}\right]_0^\infty \\ &= \frac{1}{C}\left(\frac{1}{B} + \frac{1}{C}\right) \cdot \frac{\pi}{2} \cdot \sqrt{\frac{C}{B}} = \frac{1}{\sqrt{BC}} \cdot \left(\frac{1}{B} + \frac{1}{C}\right) \cdot \frac{\pi}{2} \\ &= \frac{1}{\sqrt{1-e^2}} \cdot \left(\frac{1}{1-e} + \frac{1}{1+e}\right) \cdot \frac{\pi}{2} = \frac{\pi}{(1-e^2)^{3/2}}. \end{aligned}$$

Finally, we find that the area of the ellipse is

$$\begin{aligned} A &= a^2(1-e^2)^2 I = a^2(1-e^2)^2 \cdot \frac{\pi}{(1-e^2)^{3/2}} \\ &= \pi a^2 \sqrt{1-e^2} = \pi a^2 \sqrt{1 - \left(\frac{c}{a}\right)^2} = \pi a \cdot \sqrt{a^2 - c^2} = \pi a b, \end{aligned}$$

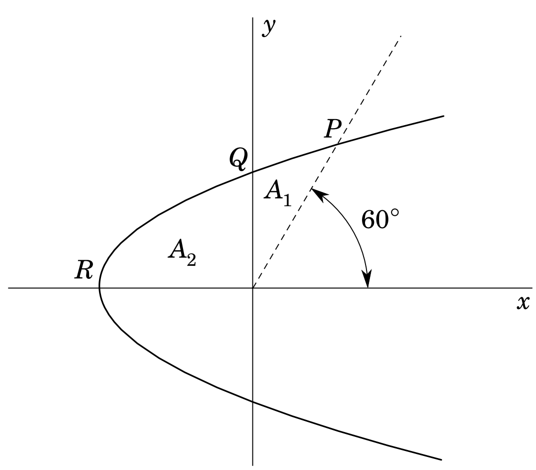
as desired.

◊◊◊

See also Carl E. Linderhold's book *Mathematics Made Difficult* (New York: World Publishing, 1971), pp. 76–77, for a two-page proof that 2 is a prime number (it involves the phrase "maximal ideal").

C10S06.096: See the next figure for the meanings of the various symbols. Note that A_1 is the area swept out by the radius vector to the comet as it moves from P to Q and that A_2 is the area swept out in moving from Q to R. Because

$$\frac{T}{A_2} = \frac{15}{A_1}, \quad \text{we see that} \quad T = \frac{15A_2}{A_1}.$$



Next,

$$A_1 = \frac{1}{2} \int_{\pi/3}^{\pi/2} \frac{1}{(1-\cos\theta)^2}\, d\theta.$$

The substitution $u = \tan(\theta/2)$ (see the solution of Problem 95) transforms this integral into

$$A_1 = \frac{1}{2} \int_{1/\sqrt{3}}^{1} \frac{1+u^2}{4u^4}\, du = \frac{1}{6}\left(3\sqrt{3} - 2\right).$$

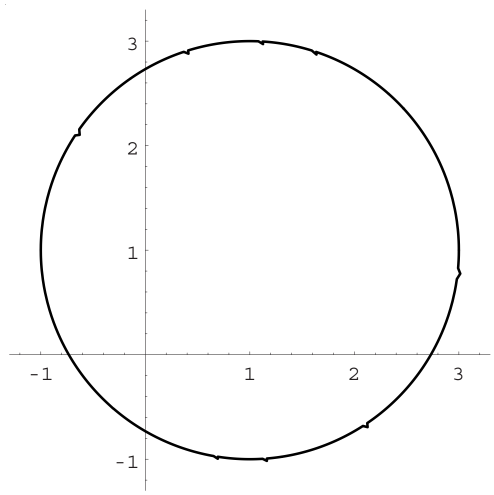
The same substitution yields

$$A_2 = \frac{1}{2} \int_{\pi/2}^{\pi} \frac{1}{(1-\cos\theta)^2}\, d\theta = \frac{1}{3}.$$

It now follows that $T = \dfrac{30}{23}\left(3\sqrt{3} + 2\right) \approx 9.3863$ days.

Chapter 10 Miscellaneous Problems

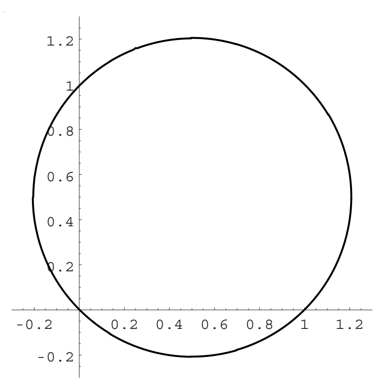
C10S0M.001: Completing the square yields $(x-1)^2 + (y-1)^2 = 4$, so this conic section is a circle with center $C(1, 1)$ and radius 2. Its graph is next.



C10S0M.002: Completing the square in both variables yields

$$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2}; \quad \text{that is,} \quad \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

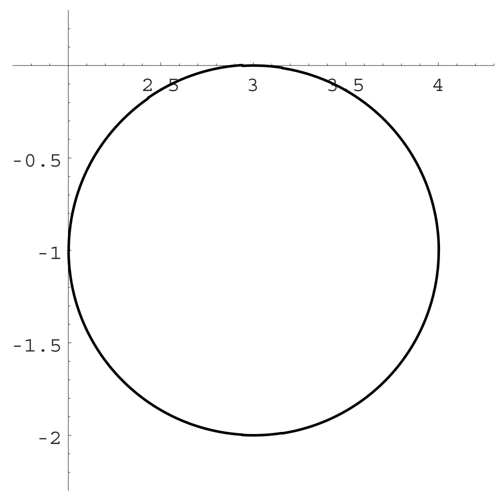
Therefore this conic section is the circle with center $C\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$. Its graph is next.



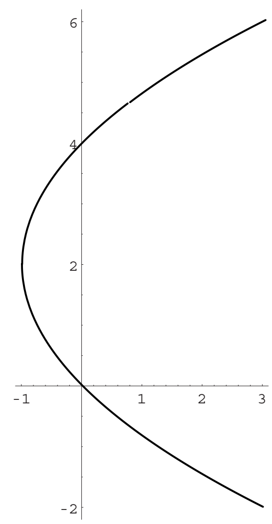
C10S0M.003: Completing the square in both variables yields

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 1; \quad \text{that is,} \quad (x-3)^2 + (y+1)^2 = 1.$$

Therefore this conic section is the circle with center $C(3, -1)$ and radius 1. Its graph is next.



C10S0M.004: Given: $y^2 = 4(x + y)$. Then $y^2 - 4y + 4 = 4x + 4$; it follows that $(y - 2)^2 = 4(x + 1)$. So the graph of this conic section is a parabola. It has directrix $x = -2$, axis $y = 2$, vertex at $(-1, 2)$, focus at $(0, 2)$, and it opens to the right. Its graph is next.

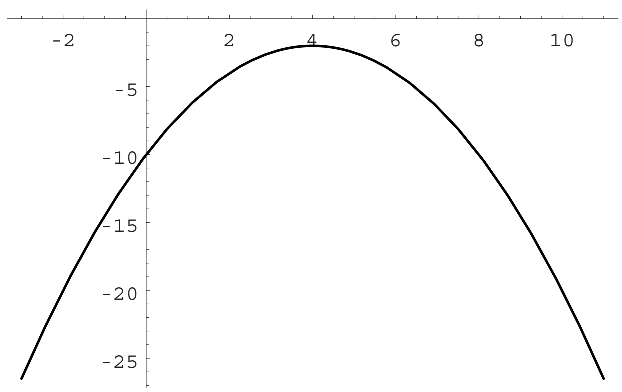


C10S0M.005: Completing the square in x yields

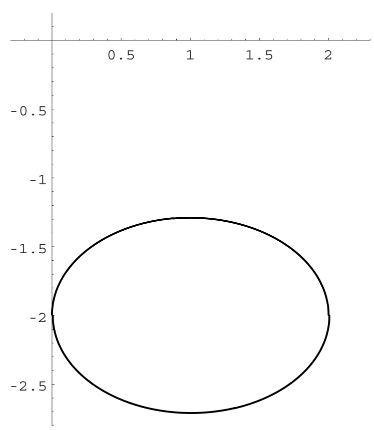
$$x^2 - 8x + 16 = -2y - 4; \quad \text{that is,} \quad (x - 4)^2 = -2(y + 2).$$

Thus this conic section is a parabola with vertex $V(4, -2)$, vertical axis with equation $x = 4$, focus at

$\left(4, -\frac{5}{2}\right)$, and opening downward. Its graph is next.



C10S0M.006: The equation can be written in the form $(x-1)^2 + 2(y+2)^2 = 1$, and therefore this conic section is an ellipse with center $(1, -2)$, major axis horizontal of length 2, and minor axis of length $\sqrt{2}$. It has vertices at $(2, -2)$, $\left(1, -2 \pm \frac{1}{2}\sqrt{2}\right)$, and $(2, -2)$; its foci are at $\left(1 \pm \frac{1}{2}\sqrt{2}, -2\right)$. Its graph is next.

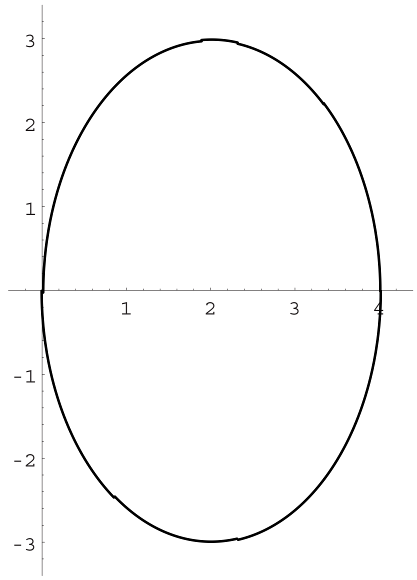


C10S0M.007: Given: $9x^2 + 4y^2 = 36x$. Complete the square in x as follows:

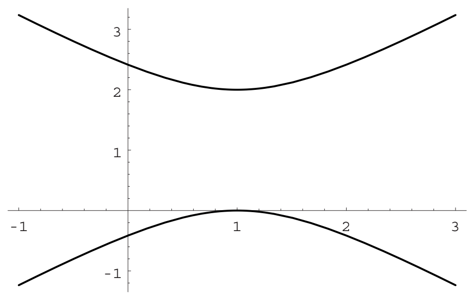
$$9(x^2 - 4x) + 4y^2 = 0; \qquad 9(x^2 - 4x + 4) + 4y^2 = 36; \qquad \left(\frac{x-2}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$$

Hence this conic section is an ellipse with center $C(2, 0)$, vertical major axis of length 6, minor axis of length

4, foci at $\left(2, \pm\sqrt{5}\right)$, and vertices at (0, 0), (4, 0), (2, 3), and (2, −3). Its graph is next.



C10S0M.008: The given equation can be written in the form $(y-1)^2 - (x-1)^2 = 1$. Therefore this conic section is a hyperbola with center (1, 1), foci at the points $\left(1,\ 1-\sqrt{2}\right)$ and $\left(1,\ 1+\sqrt{2}\right)$, vertices at (1, 2) and (1, 0), vertical transverse axis of length 2, eccentricity $e = \sqrt{2}$, directrices $y = 1 \pm \frac{1}{2}\sqrt{2}$, and asymptotes $y = x$ and $y = x + 2$. Its graph is next.



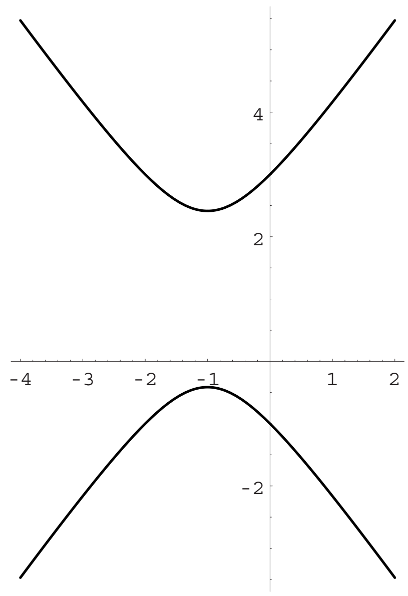
C10S0M.009: Given $y^2 - 2x^2 = 4x + 2y + 3$, complete the square in both variables as follows:

$$y^2 - 2y - 2x^2 - 4x = 3; \qquad y^2 - 2y - 2(x^2 + 2x) = 3;$$

$$y^2 - 2y + 1 - 2(x^2 + 2x + 1) = 2; \qquad \frac{(y-1)^2}{2} - (x+1)^2 = 1.$$

This conic section is a hyperbola with center $C(-1, 1)$, vertical transverse axis of length $2\sqrt{2}$, $a = \sqrt{2}$,

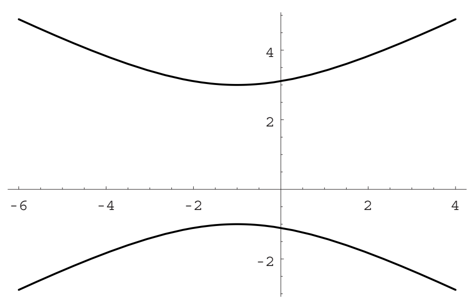
$b = 1$, $c = \sqrt{3}$, foci $F\left(-1,\, 1 \pm \sqrt{3}\right)$, and vertices $V\left(-1,\, 1 \pm \sqrt{2}\right)$. Its graph is next.



C10S0M.010: The given equation can be written in the form

$$\left(\frac{y-1}{2}\right)^2 - \left(\frac{x+1}{3}\right)^2 = 1.$$

Therefore this conic section is a hyperbola with center $C(-1,\, 1)$. Because $c = \sqrt{13}$, the foci are at the points $\left(-1,\, 1 \pm \sqrt{13}\right)$. The vertices are at $(-1,\, 3)$ and $(-1,\, -1)$. The transverse is vertical, of length 4, the eccentricity is $\frac{1}{2}\sqrt{13}$, the directrices are $y = 1 \pm \frac{4}{13}\sqrt{13}$, and the asymptotes have equations $3y = 2x + 5$ and $3y = -2x + 1$. The graph is next.



C10S0M.011: Complete the square in each variable:

$$x^2 + 2y^2 = 4x + 4y - 12; \qquad x^2 - 4x + 2y^2 - 4y = -12;$$

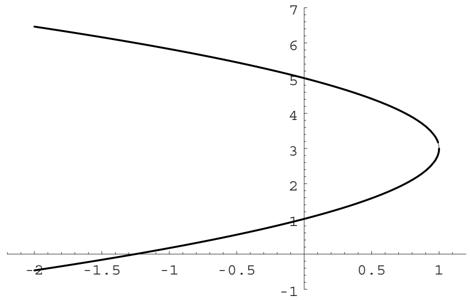
$$x^2 - 4x + 4 + 2(y^2 - 2y + 1) = -6; \qquad (x-2)^2 + 2(y-1)^2 < 0.$$

Thus there are no points on the graph of the given equation.

C10S0M.012: First complete the square:

$$y^2 - 6y + 4x + 5 = 0; \qquad y^2 - 6y + 9 + 4x = 4; \qquad (y-3)^2 = -4(x-1).$$

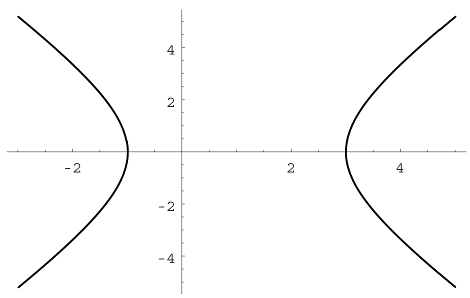
Hence this conic section is a parabola opening to the left with vertex at (1, 3) and focus at (0, 3). Its graph is next.



C10S0M.013: The given equation can be written in the form

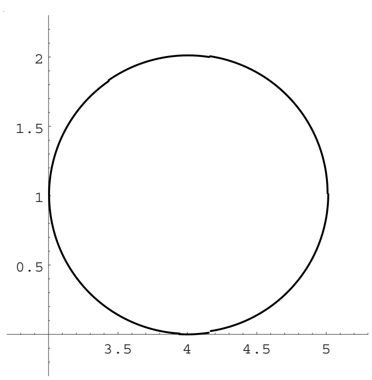
$$\frac{(x-1)^2}{4} - \frac{y^2}{9} = 1,$$

so this conic section is a hyperbola with center at (1, 0), horizontal transverse axis of length 4, foci at $\left(1 \pm \sqrt{13}, 0\right)$, and vertices at (3, 0) and (−1, 0). Its graph is next.



C10S0M.014: If $(x^2-4)(y^2-1) = 0$, then either $x^2 = 4$ or $y^2 = 1$. Therefore the graph consists of the two horizontal lines $y = \pm 1$ together with the two vertical lines $x = \pm 2$. The given equation is not the equation of a conic section.

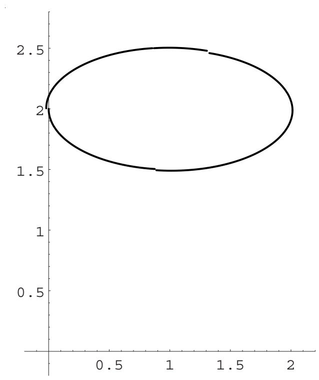
C10S0M.015: The equation can be written in the form $(x-4)^2 + (y-1)^2 = 1$; this conic section is the circle with center (4, 1) and radius 1. Its graph is next.



C10S0M.016: The given equation can be written in the form

$$(x-1)^2 + \frac{(y-2)^2}{\frac{1}{4}} = 1,$$

so this conic section is an ellipse with center (1, 2), horizontal major axis of length 2, and vertical minor axis of length 1. It has vertices at (0, 2), $\left(1, \frac{5}{2}\right)$, (2, 2), and $\left(1, \frac{3}{2}\right)$ and its foci are located at $\left(1 \pm \frac{1}{2}\sqrt{3}, 2\right)$. Its graph is next.



C10S0M.017: The given equation can be written in the form

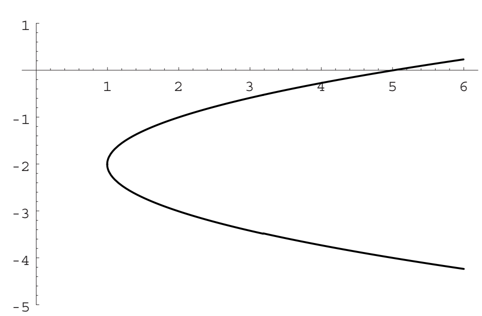
$$\left[(x-2)^2 + (y-2)^2\right] \cdot (x+y)^2 = 0,$$

and therefore either $(x-2)^2 + (y-2)^2 = 0$ or $(x+y)^2 = 0$. In the former case the only way for the sum of two squares to be zero is if each is zero, so only $(x, y) = (2, 2)$ satisfies the equation. In the latter case $(x+y)^2 = 0$ implies that $y = -x$. So the graph consists of the line $y = -x$ together with the isolated point (2, 2). It is not the graph of a conic section.

C10S0M.018: The given equation can be written in the form

$$x - 1 = (y+2)^2,$$

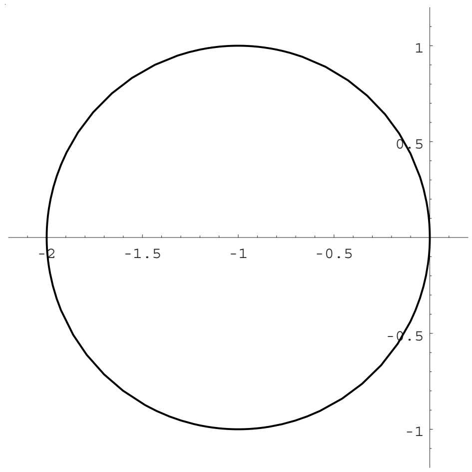
and this conic section is a parabola opening to the right with vertex at (1, −2) and focus at $\left(\frac{5}{4}, -2\right)$. Its graph is next.



C10S0M.019: Convert to Cartesian coordinates, then complete the square:

$$r = -2\cos\theta; \qquad r^2 = -2r\cos\theta;$$
$$x^2 + y^2 = -2x; \qquad x^2 + 2x + 1 + y^2 = 1;$$
$$(x+1)^2 + y^2 = 1.$$

This conic section is the circle with center $C(-1,\,0)$ and radius 1. Its graph is next.



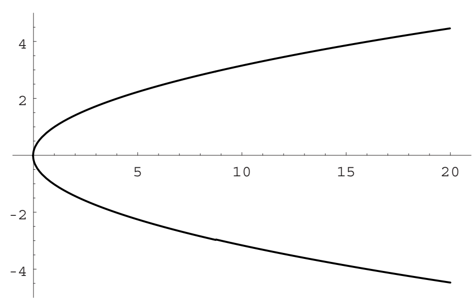
C10S0M.020: Multiply both sides of the given equation by r to obtain $r\cos\theta + r\sin\theta = 0$; in Cartesian coordinates, that's $x + y = 0$. Therefore the graph is that of the straight line $y = -x$ with slope -1 and passing through the origin.

C10S0M.021: Given

$$r = \frac{1}{\sin\theta \,+\, \cos\theta},$$

multiply each side of this equation by the denominator to obtain $r\sin\theta \,-\, r\cos\theta = 1$. In Cartesian coordinates, that's $y = x + 1$. Hence the graph is the straight line through the point $(0,\,1)$ with slope 1. One does obtain the entire graph because the denominator can take both positive and negative values arbitrarily close to zero. We omit the graph to save space.

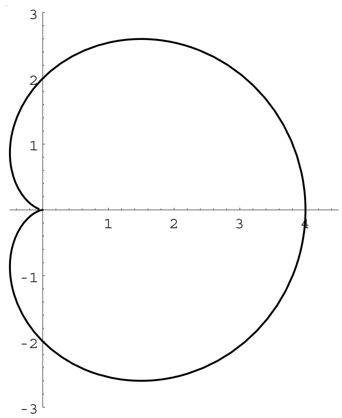
C10S0M.022: Given $r\sin^2\theta = \cos\theta$, multiply each side of this equation by r, then convert to Cartesian coordinates: $y^2 = x$. This conic section is a parabola with vertex at the origin, axis the x-axis, opening to the right, with focus $\left(\frac{1}{4},\,0\right)$ and directrix the vertical line with equation $x = -\frac{1}{4}$. The graph is next.



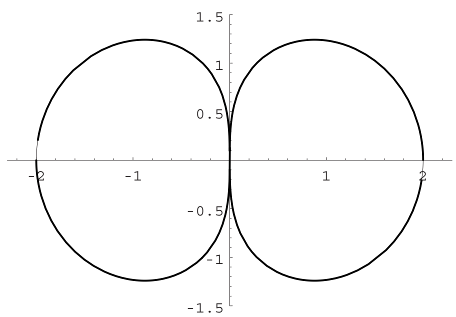
C10S0M.023: Given $r = 3\csc\theta$, rewrite this as $r\sin\theta = 3$, then convert to Cartesian coordinates: $y = 3$. The graph is the horizontal line passing through the point $(0,\,3)$. All of the line is the graph because $\csc\theta$ takes on arbitrarily large values. We omit the graph to save space.

C10S0M.024: The graph of $r = 2(\cos\theta - 1) = -2 + 2\cos\theta$ is a lovely cardioid, but it isn't a conic section.

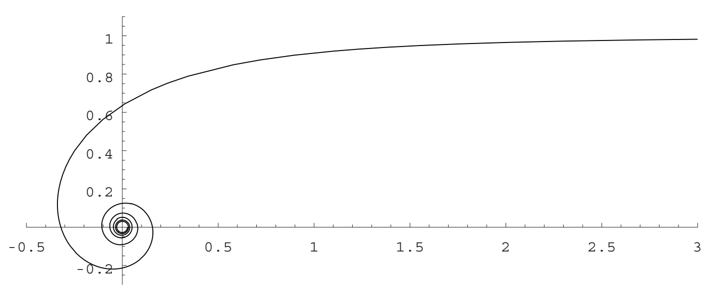
Its graph is next.



C10S0M.025: The graph of the polar equation $r^2 = 4\cos\theta$ is a pair of tangent ovals (not a conic section). It's shown next.

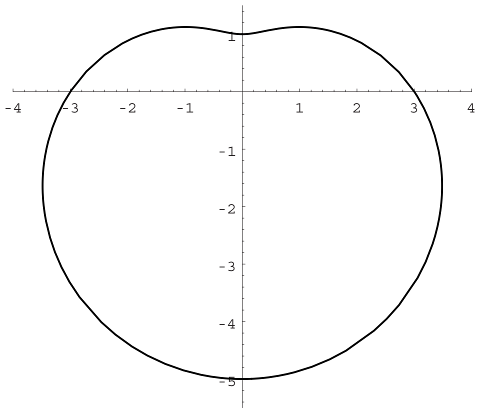


C10S0M.026: Given: $r\theta = 1$; that is, $r = 1/\theta$. This is a spiral; the part of the graph for $\theta > 0$ is shown next. The part of the graph for $\theta < 0$ is the reflection of that graph around the y-axis. The spiral is not a conic section.



C10S0M.027: The graph of the polar equation $r = 3 - 2\sin\theta$ is a limaçon (from the French word for

shell-snail); it is not a conic section. The graph is next.



C10S0M.028: Convert to Cartesian coordinates:

$$r = \frac{1}{1+\cos\theta}; \qquad r + r\cos\theta = 1; \qquad x^2 + y^2 = (1-x)^2; \qquad y^2 = -2x + 1.$$

Thus the graph is a conic section; it is the parabola with focus at $(0, 0)$, directrix $x = 1$, and vertex $\left(0, \frac{1}{2}\right)$. It opens to the left and its axis is the x-axis.

C10S0M.029: Given $r = \dfrac{4}{2+\cos\theta}$, convert to Cartesian coordinates:

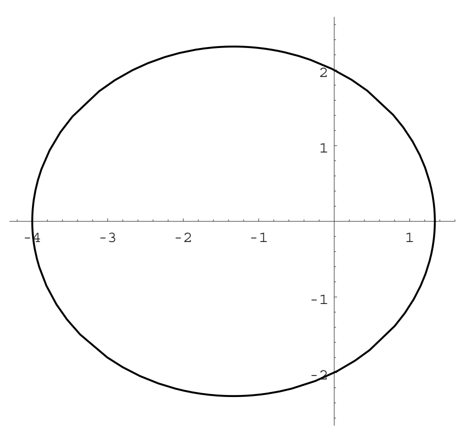
$$2r + r\cos\theta = 4; \qquad 2r = 4 - x;$$

$$4(x^2+y^2) = x^2 - 8x + 16; \qquad 3x^2 + 4y^2 + 8x = 16;$$

$$3\left(x^2 + \frac{8}{3}x\right) + 4y^2 = 16; \qquad 3\left(x^2 + \frac{8}{3}x + \frac{16}{9}\right) + 4y^2 = 16 + \frac{16}{3} = \frac{64}{3};$$

$$3\left(x + \frac{4}{3}\right)^2 + 4y^2 = \frac{64}{3}; \qquad \frac{9}{64}\left(x + \frac{4}{3}\right)^2 + \frac{3}{16}y^2 = 1.$$

Thus the graph is a conic section; it is the ellipse with center $C\left(-\frac{4}{3}, 0\right)$, $a = \frac{8}{3}$, and $b = \frac{4}{3}\sqrt{3}$. Its major axis is horizontal and its eccentricity is $e = \frac{1}{2}$. Its vertices are located at $(-4, 0)$, $\left(0, \pm\frac{4}{3}\sqrt{3}\right)$, and $\left(\frac{4}{3}, 0\right)$; its foci are at $\left(-\frac{8}{3}, 0\right)$ and $(0, 0)$. Its graph is next.

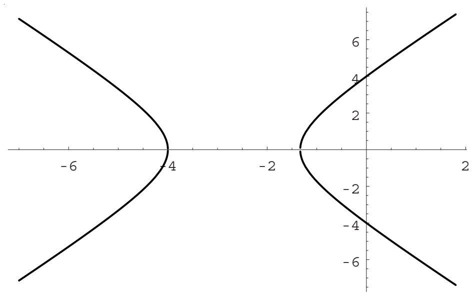


C10S0M.030: Given $r = \dfrac{4}{1 - 2\cos\theta}$, convert to Cartesian coordinates:

$$r - 2r\cos\theta = 4; \qquad r = 2x + 4;$$

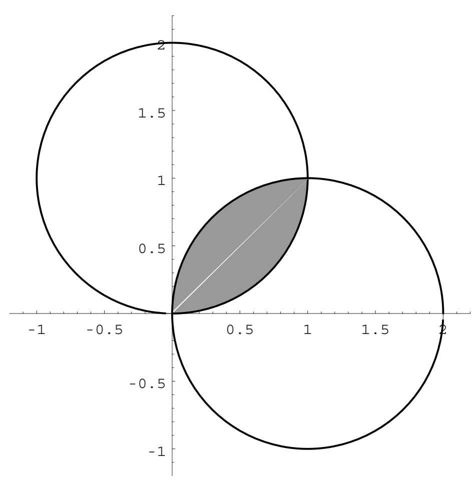
$$x^2 + y^2 = 4x^2 + 16x + 16; \qquad y^2 = 3x^2 + 16x + 16.$$

This is a conic section; it is the equation of the hyperbola with one focus at $(0, 0)$ and one directrix the vertical line $x = -2$. Its center is $\left(-\frac{8}{3}, 0\right)$, the other focus is at $\left(-\frac{16}{3}, 0\right)$, the other directrix has equation $x = -\frac{10}{3}$, and the vertices are at $\left(-\frac{4}{3}, 0\right)$ and $(-4, 0)$. It has a horizontal transverse axis of length $\frac{8}{3}$. Its graph is next.



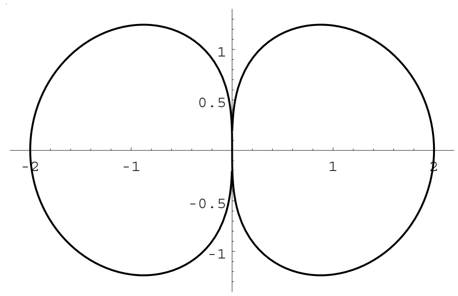
C10S0M.031: The region within both circles is shown shaded in the next figure. To find its area A, we integrate from $\theta = 0$ to $\theta = \pi/4$ and double the result:

$$A = \int_0^{\pi/4} (2\sin\theta)^2 \, d\theta = \Big[2\theta - \sin 2\theta\Big]_0^{\pi/4} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2} \approx 0.570796326795.$$



C10S0M.032: The graph of the polar equation $r^2 = 4\cos\theta$ is shown next. To find the area A that it encloses, we double the area of the oval on the right:

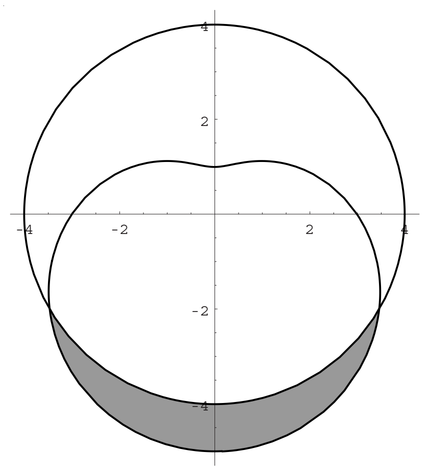
$$A = \int_{-\pi/2}^{\pi/2} 4\cos\theta \, d\theta = \Big[4\sin\theta\Big]_{-\pi/2}^{\pi/2} = 4 - (-4) = 8.$$



C10S0M.033: The circle and the limaçon are shown next. They cross where $\theta = \alpha = 7\pi/6$ and where $\theta = \beta = 11\pi/6$. To find the area A within the limaçon but outside the circle, we evaluate

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[(3 - 2\sin\theta)^2 - 16\right]\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} (4\sin^2\theta - 12\sin\theta - 7)\,d\theta = \frac{1}{2}\Big[12\cos\theta - \sin 2\theta - 5\theta\Big]_{\alpha}^{\beta}$$

$$= \frac{1}{2}\left(\frac{13\sqrt{3}}{2} - \frac{55\pi}{6}\right) - \frac{1}{2}\left(-\frac{13\sqrt{3}}{2} - \frac{35\pi}{6}\right) = \frac{39\sqrt{3} - 10\pi}{6} \approx 6.022342493215.$$



C10S0M.034: The circle and the lemniscate are shown next. They cross at the points where $\sin\theta = 0$ and where $\cos\theta = \sin\theta$. Thus we obtain the two solutions $r = 0$ and $r = \sqrt{2}$, $\theta = \pi/4$. The area of the small region within the lemniscate and outside the circle in the first quadrant is

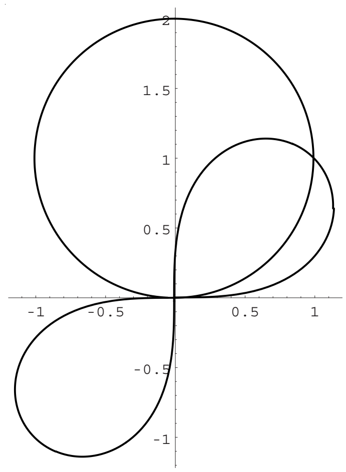
$$A_1 = \frac{1}{2}\int_0^{\pi/4} \left[(2\sin 2\theta) - (2\sin\theta)^2\right]\,d\theta = \frac{1}{2}\Big[-\cos 2\theta - 2\theta + \sin 2\theta\Big]_0^{\pi/4}$$

$$= \frac{1}{2}\left(-\frac{\pi}{2} + 1 + 1\right) = \frac{4 - \pi}{4} \approx 0.214601837.$$

The area of the part of the lemniscate in the third quadrant is

$$A_2 = \int_0^{\pi/2} \sin 2\theta = \left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$

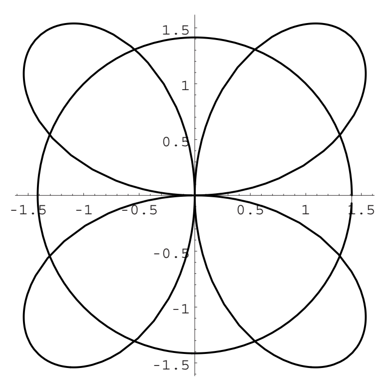
Therefore the total area outside the circle but within the lemniscate is $A_1 + A_2 = \frac{1}{4}(8 - \pi) \approx 1.214601837$.



C10S0M.035: The circle and the four-leaved rose cross where θ is an odd integral multiple of $\pi/8$. To find the area A within the rose and outside the circle, we multiple the area in the first quadrant by 4. Thus with $\alpha = \pi/8$ and $\beta = 3\pi/8$, we have

$$A = 2\int_{\alpha}^{\beta} (-2 + 4\sin^2 2\theta)\ d\theta = \int_{\alpha}^{\beta} (-4\cos 4\theta)\ d\theta = \Big[-\sin 4\theta \Big]_{\alpha}^{\beta} = 1 - (-1) = 2.$$

The circle and the rose are shown next.

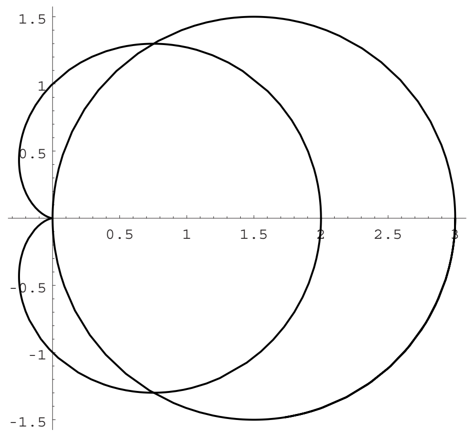


C10S0M.036: The circle and the cardioid meet at the pole and where $\theta = \pm\pi/3$. The area outside the cardioid but within the circle is, by symmetry,

$$A = \int_0^{\pi/3} \left[9\cos^2\theta - (1 + \cos\theta)^2\right] d\theta = \int_0^{\pi/3} (3 - 2\cos\theta + 4\cos 2\theta)\ d\theta$$

$$= \Big[3\theta - 2\sin\theta + 2\sin 2\theta\Big]_0^{\pi/3} = \pi - 0 = \pi \approx 3.1415926535897932385.$$

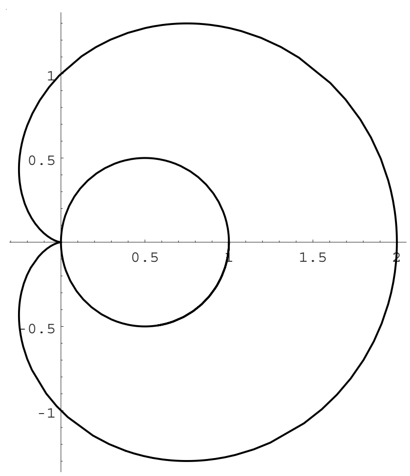
The circle and the cardioid are shown next.



C10S0M.037: The circle and the cardioid are shown next. They meet only at the pole. To find the area A inside the cardioid but outside the circle, we simply find the area within the cardioid and subtract the area of a circle of radius $\frac{1}{2}$.

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\,d\theta - \frac{\pi}{4} = -\frac{\pi}{4} + \int_0^{2\pi} 2\cos^4\frac{\theta}{2}\,d\theta$$

$$= -\frac{\pi}{4} + \left[\frac{1}{8}(6\theta + 8\sin\theta + \sin 2\theta)\right]_0^{2\pi} = -\frac{\pi}{4} + \frac{3\pi}{2} = \frac{5\pi}{4} \approx 3.926990816987.$$



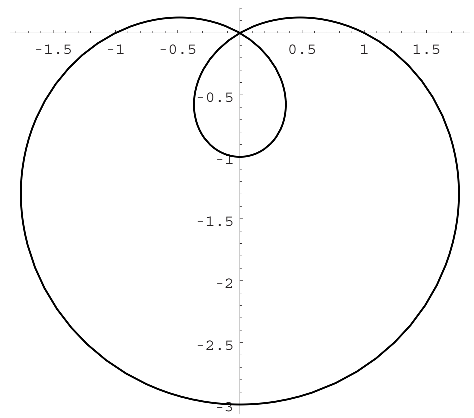
C10S0M.038: The graph of the limaçon with polar equation $r = 1 - 2\sin\theta$ is shown following this solution. The curve passes through the pole when $\theta = \alpha = \pi/6$ and again when $\theta = \beta = 5\pi/6$. The area within the smaller loop is

$$A_1 = \frac{1}{2}\int_\alpha^\beta (1-2\sin\theta)^2\,d\theta = \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_\alpha^\beta = \frac{2\pi - 3\sqrt{3}}{2} \approx 0.543516442236.$$

The area within the entire limaçon is

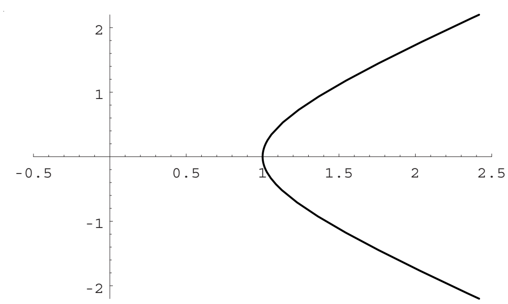
$$A_2 = \frac{1}{2}\int_{5\pi/6}^{13\pi/6} (1-2\sin\theta)^2\,d\theta = \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_{5\pi/6}^{13\pi/6} = \frac{4\pi + 3\sqrt{3}}{2}.$$

Therefore the area between the two loops is $A = A_2 - A_1 = \pi + 3\sqrt{3} \approx 8.337745076296$.

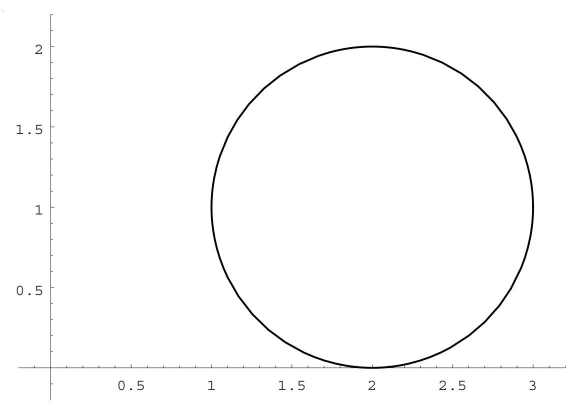


C10S0M.039: Elimination of the parameter yields the equation $y = x + 2$ of the straight line through $(0,\ 2)$ with slope 1. The graph consists of all points of this line because both $2t^3 - 1$ and $2t^3 + 1$ take on arbitrarily large positive and negative values.

C10S0M.040: Because $\cosh^2 t - \sinh^2 t = 1$, elimination of the parameter yields the equation $x^2 - y^2 = 1$ of a hyperbola with center at $(0,\ 0)$ and vertices at $(\pm 1,\ 0)$. But only the right branch of this parameter is the graph of the parametric equations, because $x = \cosh t \geqq 1$ for all t. That graph is next.



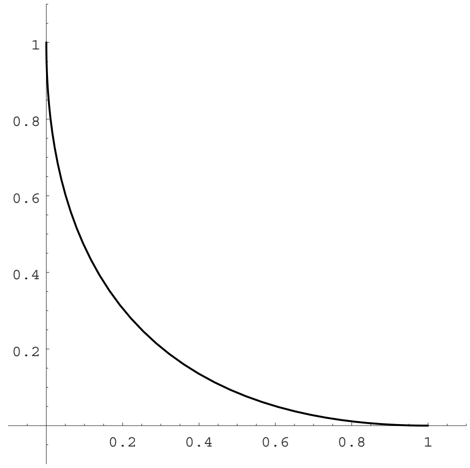
C10S0M.041: Because $(x - 2)^2 + (y - 1)^2 = \cos^2 t + \sin^2 t = 1$, the graph is the circle with center $C(2,\ 1)$ and radius 1. The entire circle is obtained as the graph of the parametric equations because $\cos t$ and $\sin t$ take on all values between -1 and $+1$ as t ranges from 0 to 2π. The graph is next.



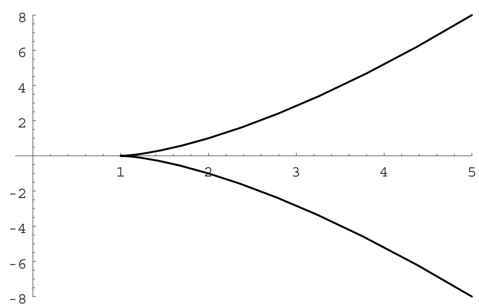
C10S0M.042: Note that neither x nor y is ever negative. Thus

$$\sqrt{x}\ + \sqrt{y}\ = \cos^2 t + \sin^2 t = 1, \qquad \text{so that} \qquad y = \left(1 - \sqrt{x}\,\right)^2.$$

Also, because $x \leqq 1$ and $y \leqq 1$ for all t, the graph terminates at the two points (1, 0) and (0, 1), as indicated in the following figure. This graph is part of a parabola; see the solution of Problem 75 of Section 10.6 for the reason.



C10S0M.043: Because $x - 1 = t^2 = y^{2/3}$, we can write the equation in the form $y^2 = (x-1)^3$. This curve is called by some a "semicubical parabola" even though it's not a parabola. Its graph is next.



C10S0M.044: By the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3t}{2}.$$

When $t = 1$, we have $x = 1$, $y = 1$, and $dy/dx = \frac{3}{2}$. So an equation of the tangent line is $y - 1 = \frac{3}{2}(x-1)$; that is, $2y + 1 = 3x$.

C10S0M.045: By the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-4\sin t} = -\frac{4}{3}\tan t.$$

When $t = \pi/4$, we have $x = \frac{3}{2}\sqrt{2}\,$, $y = 2\sqrt{2}\,$, and $dy/dx = -\frac{4}{3}$. Hence an equation of the tangent line is

$$y - 2\sqrt{2}\ = -\frac{4}{3}\left(x - \frac{3}{2}\sqrt{2}\right); \qquad \text{that is,} \qquad y = -\frac{4}{3}\left(x - 3\sqrt{2}\,\right).$$

C10S0M.046: By the chain rule,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^{t}} = -e^{-2t}.$$

When $t = 0$, we have $x = 1$, $y = 1$, and $dy/dx = -1$. Therefore an equation of the tangent line is $y - 1 = (-1)(x - 1)$; that is, $x + y = 2$.

C10S0M.047: Given the polar equation $r = \theta$, we have $x(\theta) = \theta\cos\theta$ and $y(\theta) = \theta\sin\theta$. Therefore

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\theta\cos\theta + \sin\theta}{\cos\theta - \theta\sin\theta}.$$

Thus when $\theta = \pi/2$, we have $x = 0$, $y = \pi/2$, and $dy/dx = -2/\pi$. So an equation of the tangent line is

$$y = -\frac{2}{\pi}x + \frac{\pi}{2}; \qquad \text{that is,} \qquad 4x + 2\pi y = \pi^2.$$

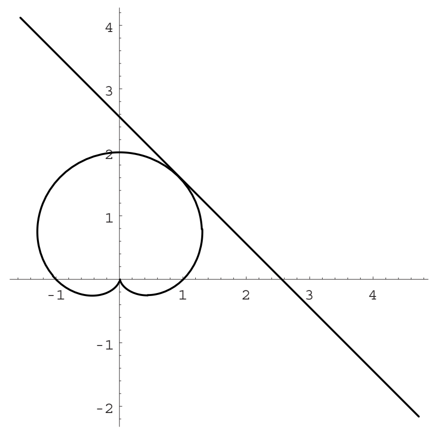
C10S0M.048: Given the cardioid with the polar equation $r = 1 + \sin\theta$, we have $x(\theta) = (1 + \sin\theta)\cos\theta$ and $y(\theta) = (1 + \sin\theta)\sin\theta$. Therefore, by the chain rule,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\,\cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}.$$

Thus when $\theta = \pi/3$, we have

$$x = \frac{1}{2}\left(1 + \frac{1}{2}\sqrt{3}\right), \quad y = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\sqrt{3}\right), \quad \text{and} \quad \frac{dy}{dx} = -1.$$

Therefore an equation of the tangent line is $y = \frac{1}{4}\left(5 + 3\sqrt{3} - 4x\right)$. See the following figure.



C10S0M.049: The area is

$$\int_{t=-1}^{2} y\,dx = \left[\frac{2}{3}(9t + t^3)\right]_{-1}^{2} = \frac{52}{3} - \left(-\frac{20}{3}\right) = 24.$$

C10S0M.050: The area is $A = \displaystyle\int_{t=0}^{10} y\,dx = \Big[\,t\,\Big]_{0}^{10} = 10.$

C10S0M.051: The area is $A = \displaystyle\int_{t=0}^{\pi/2} y\,dx = \int_0^{\pi/2} 12\cos^2 t\,dt = \Big[6t+3\sin 2t\Big]_0^{\pi/2} = 3\pi \approx 9.424777960769.$

C10S0M.052: The area is

$$A = \int_0^1 \sinh^2 t\,dt = \frac{1}{4}\int_0^1 (e^{2t} - 2 + e^{-2t})\,dt = \frac{1}{4}\left[\frac{1}{2}e^{2t} - 2t - \frac{1}{2}e^{-2t}\right]_0^1 = \frac{e^4 - 4e^2 - 1}{8e^2} \approx 0.4067151020.$$

C10S0M.053: The arc length is

$$L = \int_0^1 t(4+9t^2)^{1/2}\,dt = \left[\frac{1}{27}(4+9t^2)^{3/2}\right]_0^1 = \frac{13\sqrt{13}}{27} - \frac{8}{27} = \frac{13\sqrt{13}\ - 8}{27} \approx 1.439709873372.$$

C10S0M.054: Here, $dx/dt = -\tan t$ and $dy/dt = 1$. So

$$ds = \sqrt{1 + \tan^2 t}\ dt = \sec t\,dt$$

(not $-\sec t$, because $\sec t > 0$ if $0 \leqq t \leqq \pi/4$). So the arc length is

$$L = \int_0^{\pi/4} \sec t\,dt = \Big[\ln(\sec t + \tan t)\Big]_0^{\pi/4} = \ln\left(1 + \sqrt{2}\right) \approx 0.881373587020.$$

C10S0M.055: In this problem we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4 + \left(3t^2 - \frac{1}{3t^2}\right)^2 = 9t^4 + 2 + \frac{1}{9t^4} = \left(3t^2 + \frac{1}{3t^2}\right)^2.$$

Therefore the arc length is

$$\int_1^2 \left(3t^2 + \frac{1}{3t^2}\right)\,dt = \left[t^3 - \frac{1}{3t}\right]_1^2 = \frac{47}{6} - \frac{2}{3} = \frac{43}{6} \approx 7.166666666667.$$

C10S0M.056: Given the polar equation $r(\theta) = \sin\theta$, we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2\theta + \cos^2\theta = 1,$$

so that $ds = 1\,d\theta$. Therefore the arc length is $L = \displaystyle\int_0^{\pi} 1\,d\theta = \pi.$

C10S0M.057: Given: the polar equation $r = \sin^2(\theta/3)$, $0 \leqq \theta \leqq \pi$. Then

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^4\frac{\theta}{3} + \frac{4}{9}\sin^2\frac{\theta}{3}\,\cos^2\frac{\theta}{3},$$

and therefore

$$\begin{aligned}\sqrt{r^2 + (dr/d\theta)^2} &= \left(\sin\frac{\theta}{3}\right)\left(\sin^2\frac{\theta}{3} + \frac{4}{9}\cos^2\frac{\theta}{3}\right)^{1/2} = \frac{1}{3}\left(\sin\frac{\theta}{3}\right)\left(9\sin^2\frac{\theta}{3} + 4\cos^2\frac{\theta}{3}\right)^{1/2}\\ &= \frac{1}{3}\left(\sin\frac{\theta}{3}\right)\left(9\sin^2\frac{\theta}{3} + 9\cos^2\frac{\theta}{3} - 5\cos^2\frac{\theta}{3}\right)^{1/2} = \frac{1}{3}\left(\sin\frac{\theta}{3}\right)\left(9 - 5\cos^2\frac{\theta}{3}\right)^{1/2}.\end{aligned}$$

So the length of the graph is

$$L = \int_0^{\pi} \frac{1}{3}\left(\sin\frac{\theta}{3}\right)\left(9 - 5\cos^2\frac{\theta}{3}\right)^{1/2} d\theta.$$

Let

$$u = \cos\frac{\theta}{3}; \qquad \text{then} \qquad du = -\frac{1}{3}\sin\frac{\theta}{3}\, d\theta.$$

This substitution yields

$$L = \int_{u=1}^{1/2} -(9-5u^2)^{1/2}\, du = \int_{1/2}^{1} \sqrt{9-5u^2}\, du = \sqrt{5}\int_{1/2}^{1}\left(\frac{9}{5} - u^2\right)^{1/2} du.$$

Then integral formula 54 of the endpapers, with $a = 3/\sqrt{5} = \frac{3}{5}\sqrt{5}$, yields

$$\begin{aligned}
L &= \sqrt{5}\left[\frac{u}{2}\left(\frac{9}{5} - u^2\right)^{1/2} + \frac{9}{10}\arcsin\frac{u\sqrt{5}}{3}\right]_{1/2}^{1} \\
&= \sqrt{5}\left[\frac{1}{2}\left(\frac{9}{5} - 1\right)^{1/2} + \frac{9}{10}\arcsin\frac{\sqrt{5}}{3} - \frac{1}{4}\left(\frac{9}{5} - \frac{1}{4}\right)^{1/2} - \frac{9}{10}\arcsin\frac{\sqrt{5}}{6}\right] \\
&= \frac{1}{2}\sqrt{9-5} + \frac{9\sqrt{5}}{10}\arcsin\frac{\sqrt{5}}{3} - \frac{1}{4}\sqrt{9 - \frac{5}{4}} - \frac{9\sqrt{5}}{10}\arcsin\frac{\sqrt{5}}{6} \\
&= 1 + \frac{9\sqrt{5}}{10}\arcsin\frac{\sqrt{5}}{3} - \frac{\sqrt{31}}{8} - \frac{9\sqrt{5}}{10}\arcsin\frac{\sqrt{5}}{6} \approx 1.2281021668591117.
\end{aligned}$$

C10S0M.058: First, $dx/dt = 2t$ and $dy/dt = 3$. Hence $ds = \sqrt{9+4t^2}\, dt$. So the surface area is

$$A = \int_{t=0}^{2} 2\pi y\, ds = \int_0^2 6\pi t\sqrt{9+4t^2}\, dt = \left[\frac{\pi}{2}(9+4t^2)^{3/2}\right]_0^2 = 49\pi \approx 153.9380400259.$$

C10S0M.059: First,

$$\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/2} = \sqrt{\frac{(1+t^5)^2}{t^6}} = \frac{1+t^5}{t^3}.$$

Therefore the surface area of revolution is

$$\begin{aligned}
A = \int_{t=1}^{4} 2\pi y\, ds = \frac{\pi}{3}\int_1^4 \left(2t^5 + 5 + 3t^{-5}\right) dt &= \pi\left[\frac{1}{9}t^6 + \frac{5}{3}t - \frac{1}{4}t^{-4}\right]_1^4 \\
&= \frac{4255735\pi}{9216} - \frac{55\pi}{36} = \frac{471295\pi}{1024} \approx 1445.914950853127.
\end{aligned}$$

C10S0M.060: Given the polar equation $r = \cos\theta$, we have $x(\theta) = \cos^2\theta$ and $y(\theta) = \sin\theta\,\cos\theta$. Therefore

$$\sqrt{[x'(\theta)]^2 + [y'(\theta)]^2} = \sqrt{4\cos^2\theta\,\sin^2\theta + (\cos^2\theta - \sin^2\theta)^2} = 1.$$

Consequently the surface area of revolution is

$$A = \int_{\theta=0}^{\pi/2} 2\pi y\; ds = \Big[-\pi \cos^2\theta \Big]_0^{\pi/2} = 0 - (-\pi) = \pi.$$

C10S0M.061: Given the polar equation $r = \exp(\theta/2)$, we have

$$x(\theta) = \exp(\theta/2)\,\cos\theta \quad \text{ and } \quad y(\theta) = \exp(\theta/2)\,\sin\theta.$$

Therefore

$$[x'(\theta)]^2 + [y'(\theta)]^2 = \left(\frac{1}{2}\exp(\theta/2)\,\cos\theta - \exp(\theta/2)\,\sin\theta\right)^2 + \left(\frac{1}{2}\exp(\theta/2)\,\sin\theta + \exp(\theta/2)\,\cos\theta\right)^2$$

$$= \frac{1}{4}\exp(\theta)\,\cos^2\theta - \exp(\theta)\,\cos\theta\,\sin\theta + \exp(\theta)\,\sin^2\theta + \exp(\theta)\,\cos^2\theta + \exp(\theta)\,\cos\theta\,\sin\theta + \frac{1}{4}\exp(\theta)\,\sin^2\theta$$

$$= \frac{5}{4}\exp(\theta)\,\cos^2\theta + \frac{5}{4}\exp(\theta)\,\sin^2\theta = \frac{5}{4}\exp(\theta).$$

Therefore the surface area of revolution is

$$A = \int_0^{\pi} 2\pi y(\theta)\cdot\frac{\sqrt{5}}{2}\exp(\theta/2)\; d\theta = \pi\sqrt{5}\int_0^{\pi} e^{\theta}\sin\theta\; d\theta = \frac{\pi\sqrt{5}}{2}\Big[(\sin\theta - \cos\theta)e^{\theta}\Big]_0^{\pi}$$

$$= \frac{\pi e^{\pi}\sqrt{5}}{2} - \left(-\frac{\pi\sqrt{5}}{2}\right) = \frac{\pi(1+e^{\pi})\sqrt{5}}{2} \approx 84.791946612137.$$

See Example 5 in Section 8.3 for how to find the antiderivative using integration by parts.

C10S0M.062: Given $x(t) = e^t\cos t$ and $y(t) = e^t\sin t$, we have

$$\begin{aligned}[x'(t)]^2 + [y'(t)]^2 &= (e^t\cos t - e^t\sin t)^2 + (e^t\cos t + e^t\sin t)^2\\ &= e^{2t}\cos^2 t - 2e^{2t}\sin t\,\cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\,\cos t + e^{2t}\sin^2 t\\ &= 2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t = 2e^{2t}.\end{aligned}$$

Therefore $ds = e^t\sqrt{2}\; dt$. So the surface area of revolution is

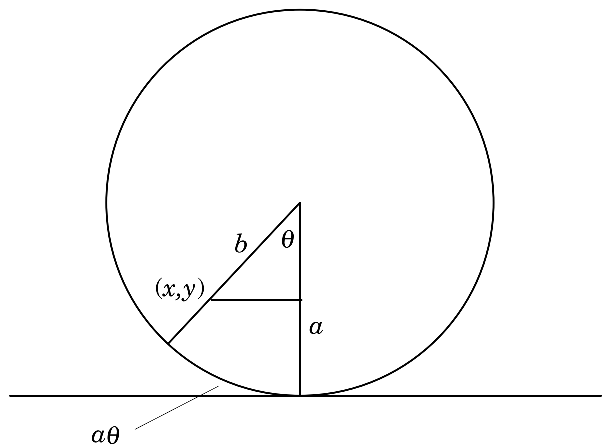
$$A = \int_{t=0}^{\pi/2} 2\pi y(t)\; ds = 2\pi\sqrt{2}\int_0^{\pi/2} e^{2t}\sin t\; dt = \frac{2\pi\sqrt{2}}{5}\Big[(2\sin t - \cos t)e^{2t}\Big]_0^{\pi/2}$$

$$= \frac{4\pi e^{\pi}\sqrt{2}}{5} - \left(-\frac{2\pi\sqrt{2}}{5}\right) = \frac{2\pi(1+2e^{\pi})\sqrt{2}}{5} \approx 84.026263955537.$$

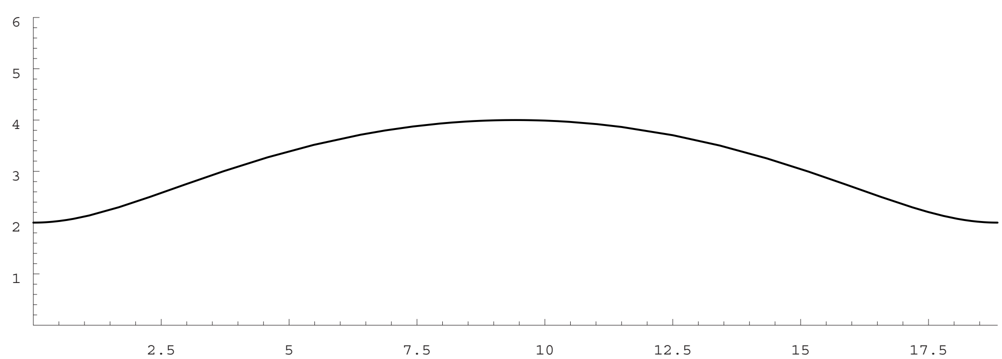
See Example 5 of Section 8.3 for the technique of finding the antiderivative using integration by parts.

C10S0M.063: Suppose that the circle rolls to the right through a central angle θ. Then its center is at the point $(a\theta,\, a)$. So the point $(x,\, y)$ that generates the trochoid is located where $x = a\theta - b\sin\theta$ and

$y = a - b\cos\theta$. This is easy to see from the small right triangle in the rolling circle shown next.

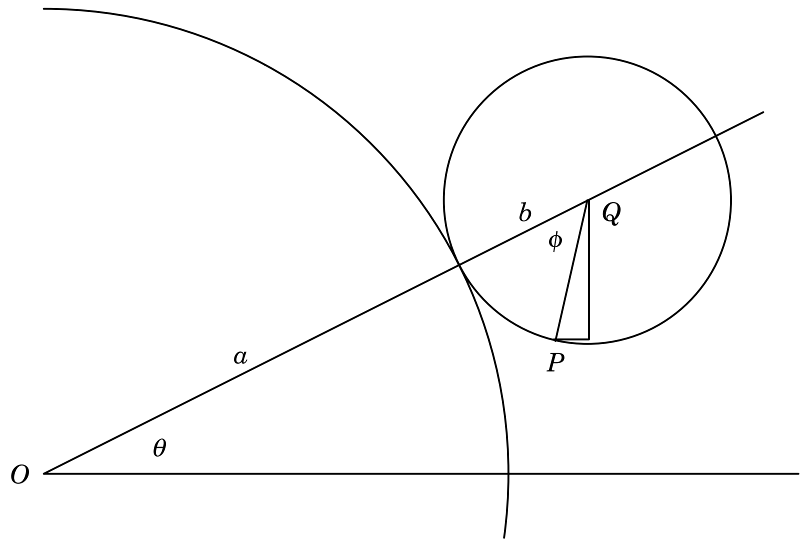


One cycle of a trochoid with $a = 3$ and $b = 1$ is shown next.

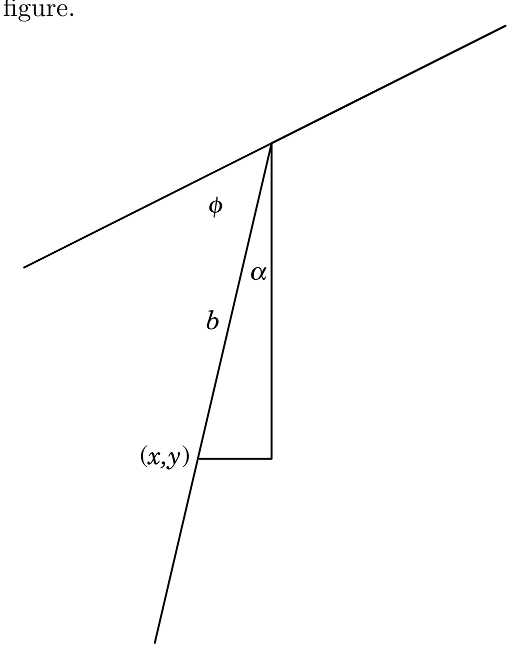


C10S0M.064: The circle of radius a has its center at the origin O; the smaller circle of radius b rolls around the circumference of the larger circle, beginning with its center on the positive x-axis. Let P be the point on the circumference of the smaller circle initially at location $(a,\, 0)$. Let $(x,\, y)$ be the locus of P. Suppose that the extended radius from the larger circle through the center of the small circle make the angle θ with

the positive x-axis, as in the following figure.



Let the angle that the radius QP of the small circle makes with the radius of the large circle be ϕ. Then the coordinates of Q are $x = (a+b)\cos\theta,\ y = (a+b)\sin\theta$. Let α be the angle between this radius and the vertical, as in the following figure.



From this figure we see that

$$\theta + \phi + \alpha = \frac{\pi}{2} \qquad \text{and} \qquad b\phi = a\theta, \quad \text{so that} \quad \phi = \frac{a}{b}\theta.$$

Hence

$$\alpha = \frac{\pi}{2} - \theta - \phi = \frac{\pi}{2} - \theta - \frac{a}{b}\theta = \frac{\pi}{2} - \frac{a+b}{b}\theta.$$

Thus the coordinates of P are

$$x = (a+b)\cos\theta - b\sin\alpha = (a+b)\cos\theta - b\sin\left(\frac{\pi}{2} - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right) \quad \text{and}$$

$$y = (a+b)\sin\theta - b\cos\alpha = (a+b)\sin\theta - b\cos\left(\frac{\pi}{2} - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

C10S0M.065: If $b = a$ in the parametric equations of the epicycloid of Problem 64, then its equations take the form

$$x = 2a\cos\theta - a\cos 2\theta,$$

$$y = 2a\sin\theta - a\sin 2\theta.$$

Shift this epicycloid a units to the left. Its equations will then be

$$x = 2a\cos\theta - a\cos 2\theta - a,$$

$$y = 2a\sin\theta - a\sin 2\theta.$$

Then substitution yields

$$\begin{aligned}
r^2 = x^2 + y^2 &= a^2(2\cos\theta - \cos^2\theta + \sin^2\theta - 1)^2 + a^2(2\sin\theta - 2\sin\theta\,\cos\theta)^2 \\
&= a^2(2\cos\theta - 2\cos^2\theta)^2 + a^2(2\sin\theta - 2\sin\theta\,\cos\theta)^2 \\
&= a^2(4\cos^2\theta - 8\cos^3\theta + 4\cos^4\theta + 4\sin^2\theta - 8\sin^2\theta\,\cos\theta + 4\sin^2\theta\,\cos^2\theta) \\
&= a^2\big[(4\cos^2\theta)(1 - 2\cos\theta + cos^2\theta) + (4\sin^2\theta)(1 - 2\cos\theta + \cos^2\theta)\big] \\
&= 4a^2(\cos^2\theta + \sin^2\theta)(1-\cos\theta)^2 = 4a^2(1-\cos\theta)^2.
\end{aligned}$$

Thus the translated epicycloid has polar equation $r = 2a(1 - \cos\theta)$, and therefore it is a cycloid.

C10S0M.066: See Fig. 10.3.15 of the text. With $r(t) = a\sqrt{2\cos 2\theta}\,$, we have parametric equations

$$x(\theta) = a(2\cos 2\theta)^{1/2}\cos\theta, \quad y(\theta) = a(2\cos 2\theta)^{1/2}\sin\theta$$

for the part of the lemniscate that lies in the first quadrant. Then, as in the solution of Problem 32 in Section 10.5, we find that $ds = a\sqrt{2\sec 2\theta}\ d\theta$. So the area generated by rotation of that quarter of the lemniscate around the x-axis is

$$\int_0^{\pi/4} 2\pi y(\theta)\cdot a\sqrt{2\sec 2\theta}\ d\theta = 4\pi a^2\int_0^{\pi/4}\sin\theta\ d\theta = 4\pi a^2\Big[-\cos\theta\Big]_0^{\pi/4} = 2\pi a^2\left(2 - \sqrt{2}\,\right).$$

Therefore the surface area generated by rotating the entire lemniscate around the x-axis is double that: $A = 4\pi a^2\left(2 - \sqrt{2}\,\right) \approx (7.361209476085)a^2$.

C10S0M.067: Using the method of cylindrical shells, we find the volume generated to be

$$V = \int_{\theta=0}^{2\pi} 2\pi x(\theta)y(\theta)\ dx = 2\pi a^3\int_0^{2\pi}(1-\cos\theta)^2(\theta - \sin\theta)\ d\theta$$

$$= \frac{1}{12}\pi a^3 \left[18\theta^2 - 18\cos\theta - 9\cos 2\theta + 2\cos 3\theta - 48\theta\sin\theta + 6\theta\sin 2\theta \right]_0^{2\pi}$$

$$= \frac{1}{12}\pi a^3 (72\pi^2 - 25 + 25) = 6\pi^3 a^3 \approx (186.037660081799)a^3.$$

The solution of Problem 3 in Section 8.3 illustrates how to use integration by parts for the antiderivatives of $\theta\cos\theta$ and $\theta\cos 2\theta$.

C10S0M.068: We begin with the parametric equations

$$x(t) = (a-b)\cos t + b\cos\left(\frac{a-b}{b}t\right),$$

$$y(t) = (a-b)\sin t - b\sin\left(\frac{a-b}{b}t\right).$$

Then

$$\big[x'(t)\big]^2 + \big[y'(t)\big]^2 = \left[(a-b)\cos t - (a-b)\cos\left(\frac{a-b}{b}t\right)\right]^2 + \left[(b-a)\sin t - (a-b)\sin\left(\frac{a-b}{b}t\right)\right]^2$$

$$= 4(a-b)^2\sin^2\left(\frac{at}{2b}\right),$$

so that the arc-length element is

$$ds = 2(a-b)\sin\left(\frac{at}{2b}\right)\,dt.$$

Therefore the arc length of one arch of the hypocycloid is

$$L = \int_{t=0}^{2\pi b/a} ds = \left[\frac{4(b-a)b}{a}\cos\left(\frac{at}{2b}\right)\right]_0^{2\pi b/a} = \frac{4(a-b)b}{a} - \left(-\frac{4(a-b)b}{a}\right) = \frac{8(a-b)b}{a}.$$

C10S0M.069: If the point C were on the x-axis, then a Cartesian equation for the circle would be

$$(x-p)^2 + y^2 = p^2; \quad \text{that is,} \quad x^2 - 2px + y^2 = 0.$$

Its polar equation is therefore $r^2 = 2px = 2pr\cos\theta$; that is, $r = 2p\cos\theta$. Because the radius to C makes the angle α with the x-axis, the actual equation we seek is therefore $r = 2p\cos(\theta - \alpha)$.

C10S0M.070: Let $P(x, y)$ be a point of the parabola with focus $(0, 0)$ and directrix $x + y - 4 = 0$. By Miscellaneous Problem 93 of Chapter 3, the distance from P to the directrix is

$$\frac{|x+y-4|}{\sqrt{2}}$$

and its distance from the focus is $\sqrt{x^2+y^2}$. Set them equal and square both sides of the equation to obtain

$$x^2 + y^2 = \frac{(x+y-4)^2}{2};$$

$$2x^2 + 2y^2 = x^2 - 2xy + y^2 - 8x - 8y + 16;$$

$$x^2 - 2xy + y^2 + 8x + 8y = 16.$$

C10S0M.071: Assume that $a > b > 0$. Parametrize the ellipse thus:

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \leqq t \leqq 2\pi.$$

The diameter with endpoints (x, y) and $(-x, -y)$ has length

$$2\sqrt{x^2 + y^2} = 2\sqrt{a^2\cos^2 t + b^2\sin^2 t}$$

where, without loss of generality, $0 \leqq t \leqq \pi/2$. Thus our goal is to maximize and minimize the function $f(t) = a^2\cos^2 t + b^2\sin^2 t$ on that domain. Now

$$f'(t) = -2a^2\sin t\,\cos t + 2b^2\sin t\,\cos t = 2(b^2 - a^2)\sin t\,\cos t.$$

Because $f'(t) < 0$ if $0 < t < \pi/2$ and f is continuous on its domain, the maximum value of f occurs when $t = 0$ and its minimum when $t = \pi/2$. Therefore the diameter of this ellipse of maximum length is its major axis, of length $2a$, and its diameter of minimum length is its minor axis, of length $2b$.

C10S0M.072: Implicit differentiation of the equation of the ellipse with respect to x yields

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0;$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\cdot\frac{b^2}{2y} = -\frac{b^2x}{a^2y}$$

if $y \neq 0$. Therefore $dy/dx = 0$ when $x = 0$ and $y = \pm b$. So the line tangent to the ellipse is horizontal at the two vertices $(0, \pm b)$. Now work with dx/dy to show that the line tangent to the ellipse is vertical at the two vertices $(\pm a, 0)$. Therefore the ellipse is normal to the coordinate axes where it crosses them.

C10S0M.073: The parabola passes through $(0, 0)$, $(b/2, h)$ (its vertex), and $(b, 0)$. Hence its equation has the form

$$y - h = c\left(x - \frac{b}{2}\right)^2.$$

Because $(0, 0)$ satisfies this equation, we find that

$$-h = c\left(-\frac{b}{2}\right)^2, \qquad \text{so that} \quad c = -\frac{4h}{b^2}.$$

Therefore the desired equation of the parabola is $y = \dfrac{4hx(b - x)}{b^2}$.

C10S0M.074: We assume that $a > b$; this is implied by the information that one focus is at $(c, 0)$. Assume that a, b, and c are all positive.

Part (a): We are to maximize and minimize $x^2 + y^2$ given the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

Thus we maximize and minimize

$$f(x) = x^2 + b^2 - \frac{b^2x^2}{a^2}, \qquad -a \leqq x \leqq a.$$

Then

$$f'(x) = 2x - \frac{2b^2x}{a^2};$$

$$f'(x) = 0 \quad \text{when} \quad x = \frac{b^2x}{a^2};$$

$$a^2x = b^2x, \quad \text{and so} \quad x = 0.$$

Because f is continuous on $[-a,\ a]$, it has both a global maximum value and a global minimum value there. It follows immediately that the points of the ellipse farthest from its center are $(\pm a,\ 0)$ and those closest to its center are $(0,\ \pm b)$.

Part (b): Now we want to maximize and minimize the expression $(x-c)^2+y^2$ given the condition in Eq. (1). Thus we maximize and minimize the function

$$g(x) = (x-c)^2 + b^2 - \frac{b^2x^2}{a^2}, \quad -a \leqq x \leqq a.$$

Now

$$g'(x) = 2(x-c) - \frac{2b^2x}{a^2};$$

$$g'(x) = 0 \quad \text{when} \quad x - c = \frac{b^2x}{a^2};$$

$$a^2x - a^2c = b^2x;$$

$$(a^2 - b^2)x = a^2c;$$

$$c^2x = a^2c;$$

$$x = \frac{a^2}{c}.$$

But $a/c > 1$, so that $a^2/c > a$, and therefore the critical point a^2/c is not in the domain of g. But g is continuous on its domain $[-a,\ a]$, and therefore its maximum must occur at one endpoint and its minimum at the other. Because $f(a) = (a-c)^2 < (a+c)^2 = f(-a)$, the point of the ellipse closest to its focus is $(a,\ 0)$ and the point farthest from its focus is $(-a,\ 0)$.

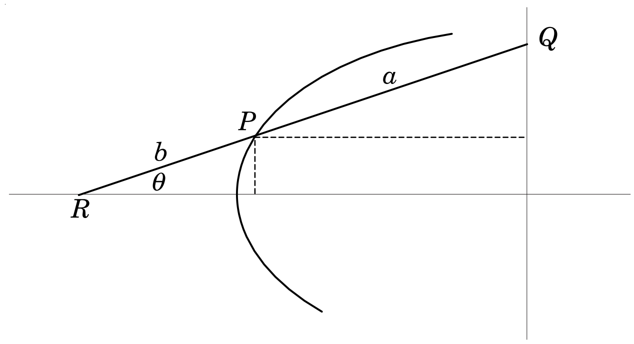
C10S0M.075: Let θ be the angle that the segment QR makes with the x-axis (see the figure that follows this solution). Then the coordinates of the point $P(x,\ y)$ satisfy the equations

$$x = -a\cos\theta \quad \text{and} \quad y = b\sin\theta$$

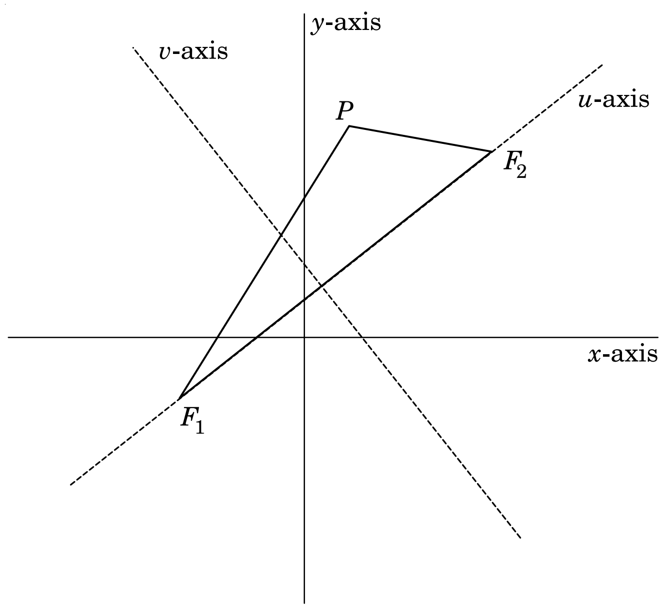
(the figure shows only the case in which P is in the second quadrant; you should check the other three cases for yourself). It now follows that

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2\theta + \sin^2\theta = 1,$$

and therefore the locus of P is an ellipse. All points of the this ellipse are obtained as θ varies from 0 to 2π.



C10S0M.076: Please refer to the following figure. We introduce a uv-coordinate system; in this rest of this solution, all coordinates will be uv-coordinates.



Choose the u- and v-axes so that $F_1 = F_1(-c,\ 0)$ and $F_2 = F_2(c,\ 0)$ where $c > 0$. Suppose that $P = P(u,\ v)$. Then $|PF_1| = 2a + |PF_2|$, and therefore

$$\sqrt{(u+c)^2 + v^2} = 2a + \sqrt{(u-c)^2 + v^2}\,.$$

Consequently

$$(u+c)^2 + v^2 = 4a^2 + 4a\sqrt{(u-c)^2 + v^2} + (u-c)^2 + v^2;$$

$$4uc - 4a^2 = 4a\sqrt{(u-c)^2 + v^2}\,;$$

$$uc - a^2 = a\sqrt{(u-c)^2 + v^2}\,;$$

$$u^2c^2 - 2a^2uc + a^4 = a^2u^2 - 2a^2uc + a^2c^2 + a^2v^2;$$

$$u^2c^2 - a^2u^2 - a^2v^2 = a^2c^2 - a^4;$$

$$u^2(c^2 - a^2) - a^2v^2 = a^2(c^2 - a^2).$$

Now $|F_1F_2| > 2a$, so $c > a$. Thus $c^2 - a^2 = b^2$ for some $b > 0$. Hence

$$b^2u^2 - a^2v^2 = a^2b^2; \quad \text{that is,}$$

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1.$$

Therefore the locus of $P(u, v)$ is a hyperbola with vertices $(\pm a, 0)$ and foci $(\pm c, 0)$ (because $c^2 = a^2 + b^2$), and so the hyperbola has foci F_1 and F_2. Finally, if a circle with radius r_2 is centered at F_2 and another with radius r_1 is centered at F_1, with r_2 and r_1 satisfying the equation $r_2 = 2a + r_1$, then the two circles will intersect at a point on the hyperbola. You may thereby construct by straightedge-and-compass methods as many points lying on the hyperbola as you please.

C10S0M.077: Please refer to the figure that follows this solution. Suppose that

$$Q_1 = Q_1\left(\frac{a^2}{4p}, a\right) \quad \text{and that} \quad Q_2 = Q_2\left(\frac{b^2}{4p}, b\right).$$

Then the slope of Q_1Q_2 will be

$$m = \frac{4p(b-a)}{b^2 - a^2} = \frac{4p}{b+a}.$$

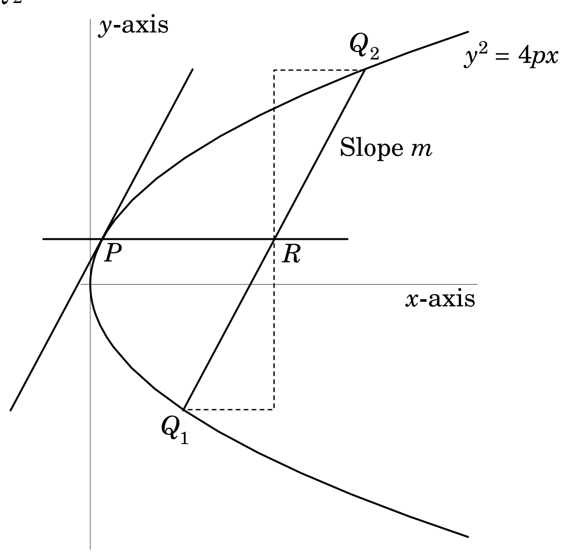
Implicit differentiation of the parabola's equation with respect to x yields

$$2y\frac{dy}{dx} = 4p, \quad \text{so that} \quad \frac{dy}{dx} = \frac{2p}{y}.$$

So we can find the y-coordinate of P (and thus of R) by solving

$$\frac{2p}{y} = \frac{4p}{b+a}: \qquad y = \frac{a+b}{2}.$$

It now follows that the two right triangles with sides Q_1R and RQ_2 are congruent, and therefore R is the midpoint of the segment Q_1Q_2.



C10S0M.078: Let $P = P(x, y)$; we have $F_1(-a, 0)$, $F_2(a, 0)$, and $|PF_1| \cdot |PF_2| = a^2$. Then

$$\sqrt{(x+a)^2+y^2}\,\sqrt{(x-a)^2+y^2} = a^2;$$

$$(x^2+2ax+a^2+y^2)(x^2-2ax+a^2+y^2) = a^4;$$

$$x^4-2ax^3+a^2x^2+x^2y^2+2ax^3-4a^2x^2+2a^3x+2axy^2$$
$$+a^2x^2-2a^3x+a^4+a^2y^2+x^2y^2-2axy^2+a^2y^2+y^4 = a^4;$$

$$x^4-2a^2x^2+2a^2y^2+2x^2y^2+y^4 = 0;$$

$$x^4+2x^2y^2+y^4 = 2a^2(x^2-y^2);$$

$$(x^2+y^2)^2 = 2a^2(x^2-y^2);$$

$$r^4 = 2a^2(r^2\cos^2\theta - r^2\sin^2\theta) = 2a^2r^2\cos 2\theta;$$

$$r^2 = 2a^2\cos 2\theta.$$

This is an equation of a lemniscate. With $a = \sqrt{2}\,$, it is the one shown in the text in Fig. 9.3.15.

C10S0M.079: First we put the given equation in standard form:

$$3x^2-y^2+12x+9=0; \qquad 3x^2+12x-y^2=-9;$$

$$3(x^2+4x+4)-y^2=3; \qquad 3(x+2)^2-y^2=3;$$

$$(x+2)^2-\frac{y^2}{3}=1.$$

With the usual meaning of the notation, this is a hyperbola with $a^2=1$ and $b^2=3$. By Eq. (20) of Section 10.6, $b^2=a^2(e^2-1)$, we see that $3=e^2-1$, so that $e^2=4$. Therefore this hyperbola has eccentricity $e=2$.

C10S0M.080: When $r=0$, $\sec\theta = 2\cos\theta$, so $\theta = \pm\pi/4$. The loop of the strophoid is obtained when θ ranges between these two values, so the area of the loop is

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec^2\theta - 4\cos\theta\,\sec\theta + 4\cos^2\theta)\,d\theta = \int_0^{\pi/4}(\sec^2\theta - 4 + 4\cos^2\theta)\,d\theta$$

$$= \Big[\tan\theta - 2\theta + \sin 2\theta\Big]_0^{\pi/4} = \frac{4-\pi}{2} \approx 0.4292036732.$$

C10S0M.081: First we convert the equation of the folium to polar coordinates:

$$x^3+y^3=3xy; \qquad r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\sin\theta\,\cos\theta;$$

$$r\cos^3\theta + r\sin^3\theta = 3\sin\theta\,\cos\theta; \qquad r = \frac{3\sin\theta\,\cos\theta}{\sin^3\theta+\cos^3\theta};$$

$$r = \frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}.$$

To obtain the area of the loop of the folium, we evaluate

$$A = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = 9\int_0^{\pi/4} \frac{\sec^2\theta\,\tan^2\theta}{(1+\tan^3\theta)^2}\,d\theta.$$

The substitution $u = \tan\theta,\ du = \sec^2\theta\,d\theta$ transforms this integral into

$$A = 3\int_0^1 \frac{3u^2}{(1+u^3)^2}\,du = 3\left[-\frac{1}{1+u^3}\right]_0^1 = 3\left(1-\frac{1}{2}\right) = \frac{3}{2}.$$

C10S0M.082: In this problem the loop has polar equation

$$r = \frac{5\cos^2\theta\,\sin^2\theta}{\cos^5\theta+\sin^5\theta} = \frac{5\sec\theta\,\tan^2\theta}{1+\tan^5\theta} \qquad \text{for} \qquad 0 \leqq \theta \leqq \pi/2.$$

Therefore the area it bounds is

$$A = \frac{25}{2}\int_0^{\pi/2} \frac{\sec^2\theta\,\tan^4\theta}{(1+\tan^5\theta)^2}\,d\theta = 25\int_0^{\pi/4} \frac{\sec^2\theta\,\tan^4\theta}{(1+\tan^5\theta)^2}\,d\theta.$$

The substitution $u = \tan\theta,\ du = \sec^2\theta\,d\theta$ transforms this integral into

$$A = 5\int_0^1 \frac{5u^4}{(1+u^5)^2}\,du = 5\left[-\frac{1}{1+u^5}\right]_0^1 = 5\left(1-\frac{1}{2}\right) = \frac{5}{2}.$$

C10S0M.083: The equation of the conic can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F \;=\; 0. \tag{1}$$

We may assume that $A = 1$. Because $(5,\,0)$ lies on the conic, $25 + 5D + F = 0$. Because $(-5,\,0)$ lies on the conic, $25 - 5D + F = 0$. Therefore $D = 0$ and $F = 25$. Because $(0,\,4)$ lies on the conic, $16C + 4E + F = 0$. Because $(0,\,-4)$ lies on the conic, $16C - 4E + F = 0$. So $E = 0$ and $F = -16C$. Therefore $16C = 25$, so that $C = \frac{25}{16}$. The equation of the conic is therefore

$$x^2 + Bxy + \frac{25}{16}y^2 - 25 = 0: \qquad 16x^2 + 16Bxy + 25y^2 = 400. \tag{2}$$

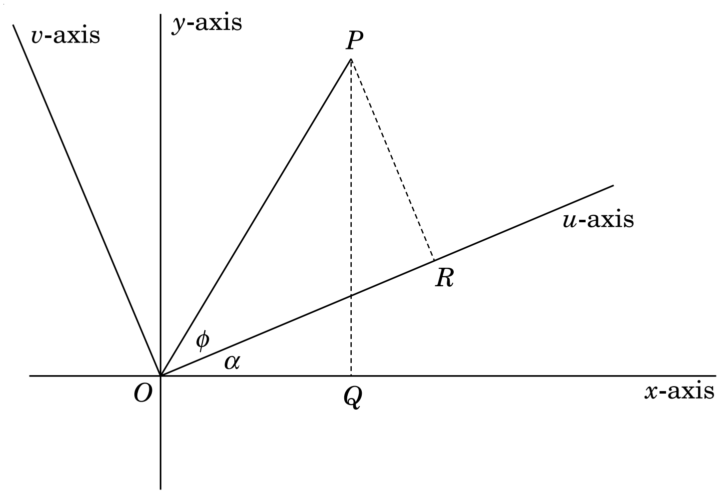
If you have studied Eq. (1), you probably learned about its *discriminant* $B^2 - 4AC$. It is known that if the discriminant is positive, then the conic is a hyperbola; if zero, a parabola; if negative, a parabola. Degenerate cases may occur, as they do in this problem. We see from Eq. (2) that if $B < \frac{5}{2}$, then the conic is an ellipse and if $B > \frac{5}{2}$, then the conic is a hyperbola. If $B = \frac{5}{2}$ then the second equation in (2) becomes

$$16x^2 + 40xy + 25y^2 = 400; \qquad (4x+5y)^2 = 400; \qquad 4x + 5y = \pm 20.$$

This is a *degenerate* parabola: two parallel lines.

If you have not studied Eq. (1) and its discriminant, you can proceed as follows. First, no [nondegenerate] parabola can contain the four given points. They are the vertices of a rhombus, and if three lay on a parabola, the fourth would be "within" the parabola. It is also clear that the four points can lie on an ellipse: Take $B = 0$ in Eq. (2). It is also clear that the four given points satisfy the equation $16x^2+100xy+25y^2 = 400$. We

claim that this is an equation of a hyperbola. To show this, we will set up a rotated rectangular uv-coordinate system in which the “mixed” term $100xy$ disappears. Please refer to the next figure.



First consider the point P, with xy-coordinates $(x,\, y)$ and uv-coordinates $(u,\, v)$. The rectangular uv-coordinate system is obtained from the xy-coordinate system by a rotation through the angle α shown in the figure. Note that

$$x = OQ = OP\cos(\alpha + \phi) \quad \text{and} \quad y = PQ = OP\sin(\alpha + \phi). \tag{3}$$

Moreover,

$$u = OR = OP\cos\phi \quad \text{and} \quad v = PR = OP\sin\phi. \tag{4}$$

Substitution of the equations in (4) into those in (3) yields

$$x = OP(\cos\alpha\,\cos\phi - \sin\alpha\,\sin\phi) = u\cos\alpha - v\sin\alpha \tag{5}$$

and

$$y = OP(\sin\alpha\,\cos\phi + \cos\alpha\,\sin\phi) = u\sin\alpha + v\cos\alpha. \tag{6}$$

We then entered the expression

```
expr = (16*x*x + 100*x*y + 25*y*y - 400)
```

in *Mathematica* 3.0. Then we made the substitutions for x and y given in Eqs. (5) and (6), expanded it with the command `Expand`, then asked for the coefficient of uv with the command `Coefficient[expr, u*v]`. *Mathematica* returned

$$100\cos^2\alpha + 18\cos\alpha\,\sin\alpha - 100\sin^2\alpha$$

We simplified this and entered the result in the form

```
expr2 = (100*Cos[2*a] + 9*Sin[2*a]),
```

using a in place of α. Then we entered `Solve[expr2 == 0, a]`, and *Mathematica* returned two angles:

$$\alpha = \frac{1}{2}\arccos\left(\frac{-9}{\sqrt{10081}}\right), \quad \alpha = -\frac{1}{2}\arccos\left(\frac{9}{\sqrt{10081}}\right).$$

We set α equal to the second of these and entered the command `expr = Expand[expr]`. This caused *Mathematica* to replace α with its numerical value throughout `expr`. The result is too long to reproduce here. We then asked for

```
Coefficient[expr, u*v]
```

and *Mathematica* returned another long expression; when we asked *Mathematica* to `Simplify` the result, the answer was 0. So we have successfully eliminated the coefficient of uv. Next we asked for

```
Coefficient[expr, u*u]
```

and the result, after `Simplify`, became

$$\frac{1}{2}\left(41 - \sqrt{10081}\right).$$

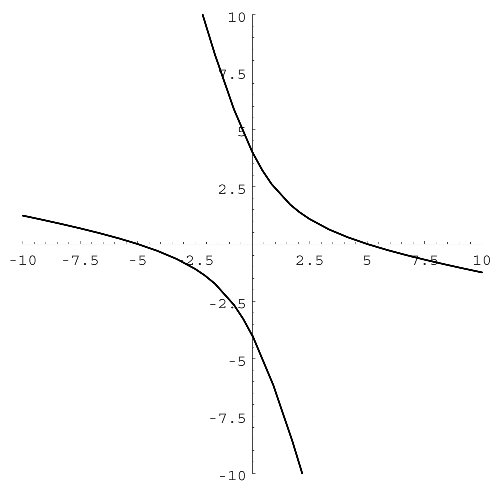
Similarly, the coefficient of v^2 turned out to be

$$\frac{1}{2}\left(41 + \sqrt{10081}\right).$$

Because the coefficient of u^2 is approximately -29.702091589893 and the coefficient of v^2 is approximately 70.702091589893, and because the equation $16x^2 + 100xy + 15y^2 = 400$ in the uv-coordinate system is

$$\frac{1}{2}\left(41 - \sqrt{10081}\right)u^2 + \frac{1}{2}\left(41 + \sqrt{10081}\right)v^2 = 400,$$

this conic section is a hyperbola. Its graph is next.



If the graph of $16x^2 + 16Bxy + 25y^2 = 400$ is normal to the y-axis at the point $(0, 4)$, then $dy/dx = 0$ there. By implicit differentiation,

$$32x + 16By + 16Bx\,\frac{dy}{dx} + 50y\,\frac{dy}{dx} = 0,$$

and when we substitute the data $x = 0$, $y = 4$, $dy/dx = 0$, we find that $64B = 0$, so that $B = 0$. In this case the graph is the ellipse with equation

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

Section 11.2

C11S02.001: The most obvious pattern is that $a_n = n^2$ for $n \geqq 1$.

C11S02.002: The most obvious pattern is that $a_n = 5n - 3$ for $n \geqq 1$.

C11S02.003: The most obvious pattern is that $a_n = \dfrac{1}{3^n}$ for $n \geqq 1$.

C11S02.004: The most obvious pattern is that $a_n = \dfrac{(-1)^{n-1}}{2^{n-1}} = \left(-\dfrac{1}{2}\right)^{n-1}$ for $n \geqq 1$.

C11S02.005: The most obvious pattern is that $a_n = \dfrac{1}{3n-1}$ for $n \geqq 1$.

C11S02.006: The most obvious pattern is that $a_n = \dfrac{1}{n^2+1}$ for $n \geqq 1$.

C11S02.007: Perhaps the most obvious pattern is that $a_n = 1 + (-1)^n$ for $n \geqq 1$.

C11S02.008: One pattern is that $a_n = \frac{15}{2} - \frac{5}{2} \cdot (-1)^n$. Another is that

$$a_n = 5 \cdot \left(1 + \frac{1 - (-1)^n}{2}\right) \qquad \text{for} \quad n \geqq 1.$$

C11S02.009: $\displaystyle \lim_{n\to\infty} \frac{2n}{5n-3} = \lim_{n\to\infty} \frac{2}{5 - \dfrac{3}{n}} = \frac{2}{5-0} = \frac{2}{5}.$

C11S02.010: $\displaystyle \lim_{n\to\infty} \frac{1-n^2}{2+3n^2} = \lim_{n\to\infty} \frac{\dfrac{1}{n^2} - 1}{\dfrac{2}{n^2} + 3} = \frac{0-1}{0+3} = -\frac{1}{3}.$

C11S02.011: $\displaystyle \lim_{n\to\infty} \frac{n^2 - n + 7}{2n^3 + n^2} = \lim_{n\to\infty} \frac{\dfrac{1}{n} - \dfrac{1}{n^2} + \dfrac{7}{n^3}}{2 + \dfrac{1}{n}} = \frac{0+0+0}{2+0} = 0.$

C11S02.012: This sequence diverges because

$$a_n = \frac{n^3}{10n^2+1} > \frac{n^3}{10n^2+10n^2} = \frac{n}{20} \to +\infty$$

as $n \to +\infty$.

C11S02.013: Example 9 tells us that if $|r| < 1$, then $r^n \to 0$ as $n \to +\infty$. Take $r = \frac{9}{10}$ to deduce that

$$\lim_{n\to\infty} \left[1 + \left(\tfrac{9}{10}\right)^n\right] = 1 + 0 = 1.$$

C11S02.014: Example 9 tells us that if $|r| < 1$, then $r^n \to 0$ as $n \to +\infty$. Take $r = -\frac{1}{2}$ to deduce that

$$\lim_{n\to\infty} \left[2 - \left(-\tfrac{1}{2}\right)^n\right] = 2 - 0 = 2.$$

C11S02.015: Given: $a_n = 1 + (-1)^n$ for $n \geqq 1$. If n is odd then $a_n = 1 + (-1) = 0$; if n is even then $a_n = 1 + 1 = 2$. Therefore the sequence $\{a_n\}$ diverges. To prove this, we appeal to the definition of limit of a sequence given in Section 11.2. Suppose that $\{a_n\}$ converges to the number L. Let $\epsilon = \frac{1}{2}$ and suppose that N is a positive integer.

Case 1: $L \geqq 1$. Then choose $n \geqq N$ such that n is odd. Then $a_n = 0$, so

$$|a_n - L| = |0 - L| = L \geqq 1 > \epsilon.$$

Case 2: $L < 1$. Then choose $n \geqq N$ such that n is even. Then $a_n = 2$, so

$$|a_n - L| = |2 - L| = 2 - L > 1 > \epsilon.$$

No matter what the value of L, it cannot be made to fit the definition of the limit of the sequence $\{a_n\}$. Therefore the sequence $\{a_n\} = \{1 + (-1)^n\}$ has no limit. (We can't even say that it approaches $+\infty$ or $-\infty$; it does not.)

C11S02.016: Because $1 + (-1)^n = 0$ if n is odd and 2 if n is even,

$$0 \leqq \frac{1 + (-1)^n}{\sqrt{n}} \leqq \frac{2}{\sqrt{n}}$$

for all $n \geqq 1$. Therefore, by the squeeze law for sequences, $\displaystyle\lim_{n\to\infty} \frac{1 + (-1)^n}{\sqrt{n}} = 0$.

C11S02.017: We use l'Hôpital's rule for sequences (Eq. (9) of Section 11.2):

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1 + (-1)^n\sqrt{n}}{\left(\frac{3}{2}\right)^n} = \lim_{x\to\infty} \frac{1 \pm x^{1/2}}{\left(\frac{3}{2}\right)^x} = \pm \lim_{x\to\infty} \frac{1}{2x^{1/2}\left(\frac{3}{2}\right)^x \ln\left(\frac{3}{2}\right)} = 0.$$

C11S02.018: We use the squeeze law for sequences (Theorem 3 of Section 11.2): $-1 \leqq \sin n \leqq 1$ for all integers $n \geqq 0$, and therefore

$$-\frac{1}{3^n} \leqq \frac{\sin n}{3^n} \leqq \frac{1}{3^n}$$

for all integers $n \geqq 1$. By the result in Example 9 of Section 11.2, $1/3^n \to 0$ as $n \to +\infty$. Therefore

$$\lim_{n\to\infty} \frac{\sin n}{3^n} = 0.$$

C11S02.019: First we need a lemma.

Lemma: If $r > 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^r} = 0$.

Proof: Suppose that $r > 0$. Given $\epsilon > 0$, let $N = 1 + [\![1/\epsilon^{1/r}]\!]$. Then N is a positive integer, and if $n > N$, then $n > 1/\epsilon^{1/r}$, so that $n^r > 1/\epsilon$. Therefore

$$\left| \frac{1}{n^r} - 0 \right| < \epsilon.$$

Thus, by definition, $\displaystyle\lim_{n\to\infty} \frac{1}{n^r} = 0$. ◀

Next, we use the squeeze law for sequences (Theorem 3 of Section 11.2): $-1 \leqq \sin n \leqq 1$ for all integers $n \geqq 1$, and therefore

$$0 \leqq \frac{\sin^2 n}{\sqrt{n}} \leqq \frac{1}{\sqrt{n}}$$

for all integers $n \geqq 1$. But by the preceding lemma, $1/\sqrt{n} \to 0$ as $n \to +\infty$. Therefore

$$\lim_{n\to\infty} \frac{\sin^2 n}{\sqrt{n}} = 0.$$

C11S02.020: First, $1 \leqq 2 + \cos n \leqq 3$ for every integer $n \geqq 1$. Thus

$$\frac{1}{n} \leqq \frac{2+\cos n}{n} \leqq \frac{3}{n}$$

for every integer $n \geqq 1$. So by the squeeze law for sequences, $(2 + \cos n)/n \to 0$ as $n \to +\infty$. Therefore, by the substitution law for sequences (Theorem 2 of Section 11.2),

$$\lim_{n\to\infty} \sqrt{\frac{2+\cos n}{n}} = 0.$$

C11S02.021: If n is a positive integer, then $\sin \pi n = 0$. Therefore $a_n = 0$ for every integer $n \geqq 0$. So $\{a_n\} \to 0$ as $n \to +\infty$.

C11S02.022: First we need a lemma.

Lemma: If the sequence $\{a_n\}$ converges, then it is bounded.

Proof: Suppose that the sequence $\{a_n\}$ converges to L. Then, given $\epsilon = 1$, there exists a positive integer N such that, if $n \geqq N$, then $|a_n - L| < \epsilon = 1$. That is, $L - 1 < a_n < L + 1$ if $n \geqq N$. Let M be the maximum of the numbers $|a_1|$, $|a_2|$, ..., $|a_N|$, $|L - 1|$, and $|L + 1|$. Then $-M \leqq a_k \leqq M$ for every integer $k \geqq 1$. Therefore the sequence $\{a_n\}$ is bounded. ◀

Thus to prove that the sequence $\{n \cos \pi n\}$ does not converge, it is sufficient to show that it is unbounded. But $n \cos \pi n = n$ if n is an odd integer. So there is no number M such that $|a_n| \leqq M$ for all n. Therefore the sequence $\{n \cos \pi n\}$ does not converge.

C11S02.023: Suppose that $a > 0$. Then $f(x) = a^x$ is continuous, so $a^x \to 1 = a^0$ as $x \to 0$. But $-\dfrac{\sin n}{n} \to 0$ as $n \to +\infty$. Therefore $\displaystyle\lim_{n\to\infty} \pi^{-(\sin n)/n} = 1$.

C11S02.024: If n is odd then $\cos n\pi = -1$; if n is even then $\cos n\pi = 1$. Therefore a_n takes on the alternating values 2 and $\frac{1}{2}$ as $n \to \infty$. Therefore, by a proof similar to the one given in the solution of Problem 15, the sequence $\{a_n\}$ has no limit as $n \to +\infty$.

C11S02.025: We use l'Hôpital's rule for sequences (Eq. (9)):

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0.$$

C11S02.026: We use l'Hôpital's rule for sequences (Eq. (9)):

$$\lim_{n\to\infty} \frac{\ln 2n}{\ln 3n} = \lim_{x\to\infty} \frac{\ln 2x}{\ln 3x} = \lim_{x\to\infty} \frac{x}{x} = 1.$$

C11S02.027: We use l'Hôpital's rule for sequences (Eq. (9)):

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

C11S02.028: Let $x = 1/n$. Then $\displaystyle\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$

C11S02.029: Because $-\pi/2 < \tan^{-1} x < \pi/2$ for all x,

$$-\frac{\pi}{2n} < \frac{\tan^{-1} n}{n} < \frac{\pi}{2n}$$

for all integers $n \geqq 1$. Therefore, by the squeeze law for limits, $\displaystyle\lim_{n\to\infty} \frac{\tan^{-1} n}{n} = 0.$

C11S02.030: We use l'Hôpital's rule for sequences (Eq. (9)):

$$\lim_{n\to\infty} \frac{n^3}{\exp(n/10)} = \lim_{x\to\infty} \frac{x^3}{\exp(x/10)} = \lim_{x\to\infty} \frac{30x^2}{\exp(x/10)} = \lim_{x\to\infty} \frac{600x}{\exp(x/10)} = \lim_{x\to\infty} \frac{6000}{\exp(x/10)} = 0.$$

C11S02.031: We use the squeeze law for limits of sequences:

$$0 < \frac{2^n + 1}{e^n} < \frac{2^n + 2^n}{e^n} = 2\left(\frac{2}{e}\right)^n.$$

By the result in Example 9, $(2/e)^n \to 0$ as $n \to +\infty$. Therefore $\displaystyle\lim_{n\to\infty} \frac{2^n+1}{e^n} = 0.$

C11S02.032: If $a_n = \dfrac{\sinh n}{\cosh n}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n\to\infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = \frac{1-0}{1+0} = 1.$$

C11S02.033: By Eq. (3) in Section 7.3,

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

C11S02.034: If $a_n = (2n+5)^{1/n}$, then—using l'Hôpital's rule for sequences—

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \exp\left(\frac{\ln(2x+5)}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{2}{2x+5}\right) = e^0 = 1.$$

C11S02.035: We use l'Hôpital's rule for sequences:

$$\lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^n = \lim_{n\to\infty} \exp\left(x \ln \frac{x-1}{x+1}\right) = \exp\left(\lim_{x\to\infty} \frac{\ln(x-1) - \ln(x+1)}{\dfrac{1}{x}}\right)$$

$$= \exp\left(\lim_{x\to\infty} \frac{\dfrac{1}{x-1} - \dfrac{1}{x+1}}{-\dfrac{1}{x^2}}\right) = \exp\left(\lim_{x\to\infty} \frac{-2x^2}{x^2-1}\right) = e^{-2}.$$

C11S02.036: By the result in Example 7, $(0.001)^{1/n} \to 1$ as $n \to +\infty$. Therefore

$$\lim_{n\to\infty} (0.001)^{-1/n} = \frac{1}{\displaystyle\lim_{n\to\infty} (0.001)^{1/n}} = \frac{1}{1} = 1.$$

C11S02.037: Let $f(x) = 2^x$ and note that f is continuous on the set of all real numbers. Thus

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f\left(\frac{x+1}{x}\right) = f\left(\lim_{x\to\infty} \frac{x+1}{x}\right) = f(1) = 2^1 = 2.$$

It is the continuity of f at $x = 1$ that makes the second equality valid.

C11S02.038: We use l'Hôpital's rule for sequences:

$$\lim_{n\to\infty} \left(2 - \frac{1}{n^2}\right)^n = \lim_{x\to\infty} \exp\left(x \ln \frac{x^2-2}{x^2}\right) = \exp\left(\lim_{x\to\infty} \frac{\ln(x^2-2) - 2\ln x}{\dfrac{1}{x}}\right)$$

$$= \exp\left(\lim_{x\to\infty} \frac{\dfrac{2x}{x^2-1} - \dfrac{2}{x}}{-\dfrac{1}{x^2}}\right) = \exp\left(\lim_{x\to\infty} \frac{-4x^2}{x^3-2x}\right) = e^0 = 1.$$

C11S02.039: By Example 11, $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$. Thus by Example 7,

$$\lim_{n\to\infty} \left(\frac{2}{n}\right)^{3/n} = \lim_{n\to\infty} \frac{8^{1/n}}{\left(n^{1/n}\right)^3} = \frac{1}{1^3} = 1.$$

C11S02.040: First,

$$\lim_{n\to\infty} (n^2+1)^{1/n} = \lim_{x\to\infty} \exp\left(\frac{\ln(x^2+1)}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{2x}{x^2+1}\right) = e^0 = 1. \qquad (1)$$

Therefore—by an argument similar to that in the solution of Problem 15, but more subtle—

$$\lim_{n\to\infty} (-1)^n (n^2+1)^{1/n} \quad \text{does not exist.}$$

For a proof, let $\epsilon = \frac{1}{2}$ and choose the integer N so large that if $n > N$, then

$$\left|(n^2+1)^{1/n} - 1\right| < \epsilon.$$

This is possible by the result in (1). But then $a_n = (-1)^n(n^2+1)^{1/n}$ lies in the interval $(1-\epsilon,\ 1+\epsilon) = \left(\frac{1}{2},\ \frac{3}{2}\right)$ whenever n is even, but a_n lies in the interval $(-1-\epsilon,\ -1+\epsilon) = \left(-\frac{3}{2},\ -\frac{1}{2}\right)$ if n is odd. Because these intervals have no points in common, no interval of the form $(L-\epsilon,\ L+\epsilon)$ can contain all a_n for all $n > K$ regardless of how large K might be. Therefore no number L can be the limit of the sequence $\{a_n\}$.

C11S02.041: Given: $a_n = \left(\dfrac{2-n^2}{3+n^2}\right)^n$. First note that

$$\lim_{n\to\infty} \frac{2-n^2}{3+n^2} = \lim_{n\to\infty} \frac{\dfrac{2}{n^2} - 1}{\dfrac{3}{n^2} + 1} = \frac{0-1}{0+1} = -1.$$

Moreover,

$$\lim_{n\to\infty}\left(\frac{n^2-2}{n^2+3}\right)^n = \lim_{x\to\infty}\exp\left(\frac{\ln(x^2-2)-\ln(x^2+3)}{\dfrac{1}{x}}\right) = \exp\left(\lim_{x\to\infty}\frac{\dfrac{2x}{x^2-2}-\dfrac{2x}{x^2+3}}{-\dfrac{1}{x^2}}\right)$$

$$= \exp\left(\lim_{x\to\infty}\frac{-x^2(2x^3+6x-2x^3+4x)}{(x^2-2)(x^2+3)}\right) = \exp\left(\lim_{x\to\infty}\frac{-10x^3}{(x^2-2)(x^2+3)}\right) = e^0 = 1.$$

Therefore, as $n \to +\infty$ through *even* values, $a_n \to 1$, whereas as $n \to +\infty$ through *odd* values, $a_n \to -1$. So we can now show that the sequence $\{a_n\}$ has no limit as $n \to \infty$.

Let $\epsilon = \frac{1}{10}$. Choose N_1 so large that if $n > N_1$ and n is even, then $|a_n - 1| < \epsilon$. Choose N_2 so large that if $n > N_2$ and n is odd, then $|a_n - (-1)| < \epsilon$. Let N be the maximum of N_1 and N_2. Then if $n > N$,

$$|a_n - 1| < \epsilon \quad \text{if } n \text{ is even;}$$

$$|a_n - (-1)| < \epsilon \quad \text{if } n \text{ is odd.}$$

Put another way, if $n > N$ and n is even, then a_n lies in the interval $(0.9, 1.1)$. If $n > N$ and n is odd, then a_n lies in the interval $(-1.1, -0.9)$. It follows that no interval of length 0.2 can contain a_n for all $n > K$, no matter how large K might be. Because every real number L is the midpoint of such an interval, this means that no real number L can be the limit of the sequence $\{a_n\}$. Therefore

$$\lim_{n\to\infty} a_n \quad \text{does not exist.}$$

C11S02.042: Given: $a_n = \dfrac{\left(\frac{2}{3}\right)^n}{1-n^{1/n}}$. First note that—by l'Hôpital's rule—

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty}\exp(\ln x^{1/x}) = \exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right) = \exp\left(\lim_{x\to\infty}\frac{1}{x}\right) = e^0 = 1.$$

Also, if $y = x^{1/x}$, then

$$\ln y = \frac{1}{x}\ln x; \qquad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\ln x;$$

$$\frac{dy}{dx} = \frac{y}{x^2}(1-\ln x); \qquad \frac{dy}{dx} = \frac{x^{1/x}}{x^2}(1-\ln x).$$

Moreover,

$$\lim_{x\to\infty} x^2\left(\frac{2}{3}\right)^x = \lim_{x\to\infty}\frac{x^2}{\left(\frac{3}{2}\right)^x} = \lim_{x\to\infty}\frac{2x}{\left(\frac{3}{2}\right)^x\ln\frac{3}{2}} = \lim_{x\to\infty}\frac{2}{\left(\frac{3}{2}\right)^x\left(\ln\frac{3}{2}\right)^2} = 0.$$

Therefore (using Theorem 4 and again using l'Hôpital's rule)

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x}{1-x^{1/x}} = \lim_{x\to\infty}\frac{x^2\left(\frac{2}{3}\right)^x\ln\frac{2}{3}}{[(\ln x)-1]\,x^{1/x}} = 0.$$

C11S02.043: Let $f(n) = \dfrac{n-2}{n+13}$. Then the *Mathematica* command

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Table[ { n, N[f[10∧n]] }, { n, 1, 7 } ]
```

yielded the response

$$\{\{1, 0.347826\},\ \{2, 0.867257\},\ \{3, 0.985192\},\ \{4, 0.998502\},\ \{5, 0.99985\},\ \{6, 0.999985\},\ \{7, 0.999999\}\}.$$

Indeed,

$$\lim_{n\to\infty} \frac{n-2}{n+13} = \lim_{n\to\infty} \frac{1-\dfrac{2}{n}}{1+\dfrac{13}{n}} = \frac{1-0}{1+0} = 1.$$

C11S02.044: Let $f(n) = \dfrac{2n+3}{5n-17}$. Results of our experiment:

n	$f(n)$
10	0.69697
100	0.42029
1000	0.40197
10000	0.40020
100000	0.40002
1000000	0.40000

Indeed, by Theorem 4 and l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{2n+3}{5n-17} = \lim_{x\to\infty} \frac{2x+3}{5x-17} = \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5}.$$

C11S02.045: Let $f(n) = a_n = \sqrt{\dfrac{4n^2+7}{n^2+3n}}$. Results of our experiment:

n	$f(n)$
10	1.76940
100	1.97083
1000	1.99701
10000	1.99970
100000	1.99997
1000000	2.00000

By Theorem 4 and l'Hôpital's rule (used twice),

$$\lim_{n\to\infty} \frac{4n^2+7}{n^2+3n} = \lim_{x\to\infty} \frac{4x^2+7}{x^2+3x} = \lim_{x\to\infty} \frac{8x}{2x+3} = \lim_{x\to\infty} \frac{8}{2} = 4.$$

Therefore, by Theorem 2, $\displaystyle\lim_{n\to\infty} a_n = \sqrt{4} = 2$.

C11S02.046: Let $f(n) = a_n = \left(\dfrac{n^3 - 5}{8n^3 + 7n}\right)^{1/3}$. Results of our experiment:

n	$f(n)$
10	0.497718
100	0.499985
1000	0.500000
10000	0.500000
100000	0.500000

Moreover,

$$\lim_{n\to\infty} \frac{n^3 - 5}{8n^3 + 7n} = \lim_{n\to\infty} \frac{1 - \dfrac{5}{n^3}}{8 + \dfrac{7}{n^2}} = \frac{1-0}{8+0} = \frac{1}{8}.$$

We may apply Theorem 2 because $g(x) = x^{1/3}$ is continuous at $x = \dfrac{1}{8}$. Thus $\displaystyle\lim_{n\to\infty} a_n = g\left(\frac{1}{8}\right) = \frac{1}{2}$.

C11S02.047: Let $f(n) = a_n = \exp\left(-1/\sqrt{n}\,\right)$. Results of an experiment:

n	$f(n)$
10	0.728893
100	0.904837
1000	0.968872
10000	0.990050
100000	0.996843
1000000	0.999000
10000000	0.999684
100000000	0.999900
1000000000	0.999968

Because $-1/\sqrt{n} \to 0$ as $n \to +\infty$ and $g(x) = e^x$ is continuous at $x = 0$, Theorem 2 implies that

$$\lim_{n\to\infty} a_n = g(0) = 1.$$

C11S02.048: Let $f(n) = a_n = n \sin \dfrac{2}{n}$. The results of an experiment:

n	$f(n)$
10	1.98669
100	1.99987
1000	2.00000
10000	2.00000

Let $x = 1/n$. Then $x \to 0^+$ as $n \to +\infty$. So

$$\lim_{n\to\infty} a_n = \lim_{x\to 0^+} \frac{\sin 2x}{x} = 2\left(\lim_{x\to 0^+} \frac{\sin 2x}{2x}\right) = 2 \cdot 1 = 2.$$

C11S02.049: Let $f(n) = a_n = 4\tan^{-1}\dfrac{n-1}{n+1}$. The results of an experiment:

n	$f(n)$
10	2.74292
100	3.10159
1000	3.13759
10000	3.14119
100000	3.14155
1000000	3.14159
10000000	3.14159

The limit appears to be π. Because

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty}\left(1 - \frac{2}{n+1}\right) = 1 - 0 = 1$$

and $g(x) = 4\tan^{-1} x$ is continuous at $x = 1$, Theorem 2 implies that

$$\lim_{n\to\infty} a_n = g(1) = 4\tan^{-1}(1) = 4 \cdot \frac{\pi}{4} = \pi.$$

C11S02.050: Let $f(n) = a_n = 3\sin^{-1}\sqrt{\dfrac{3n-1}{4n+1}}$. The results of an experiment:

n	$f(n)$
10	2.99751
100	3.12652
1000	3.14008
10000	3.14144
100000	3.14158
1000000	3.14159
10000000	3.14159

The limit seems to be π. Because

$$\lim_{n\to\infty} \frac{3n-1}{4n+1} = \lim_{n\to\infty} \frac{3 - \dfrac{1}{n}}{4 + \dfrac{1}{n}} = \frac{3-0}{4+0} = \frac{3}{4}$$

and because $g(x) = 3\arcsin\sqrt{x}$ is continuous at $x = \frac{3}{4}$, Theorem 2 implies that

$$\lim_{n\to\infty} a_n = g\left(\frac{3}{4}\right) = 3\arcsin\sqrt{\frac{3}{4}} = 3\arcsin\frac{\sqrt{3}}{2} = 3\cdot\frac{\pi}{3} = \pi.$$

C11S02.051: Proof: Suppose that

$$\lim_{n\to\infty} a_n = A \neq 0.$$

Without loss of generality we may suppose that $A > 0$. Let $\epsilon = A/3$ and choose N so large that if $n > N$, then $|a_n - A| < \epsilon$. Then if n is even, $(-1)^n a_n = a_n$; in this case $|(-1)^n a_n - A| < \epsilon$ if $n > N$. If n is odd, then $(-1)^n a_n = -a_n$; in this case $|(-1)^n a_n - (-A)| < \epsilon$ if $n > N$. In other words, $(-1)^n a_n$ lies in the interval $I = (A - \epsilon,\ A + \epsilon)$ if n is even, whereas $(-1)^n a_n$ lies in the interval $J = (-A - \epsilon,\ -A + \epsilon)$ if n is odd. This means that no open interval of length 2ϵ can contain every number $(-1)^n a_n$ for which $n > K$, no matter how large the value of K. (Note that no such interval can contains points of both I and J because the distance between their closest endpoints is 4ϵ.) Because every real number is the midpoint of an open interval of length 2ϵ, it now follows that no real number can be the limit of the sequence $\{(-1)^n a_n\}$. This concludes the proof. ◀

C11S02.052: Proof: To say that

$$\lim_{n\to\infty} a_n = +\infty \tag{1}$$

means that, for every interval of the form $(c,\ +\infty)$, there exists a positive integer N such that if $n \geqq N$, then a_n lies in the interval $(c,\ +\infty)$. If $\{a_n\}$ is an unbounded increasing sequence, then, no matter how large the number c, $a_k > c$ for some integer k. But then $a_n > c$ for all $n \geqq k$, so that a_n lies in the interval $(c,\ +\infty)$ for all $n \geqq k$. This is what Eq. (1) means.

C11S02.053: Given: $A > 0$, $x_1 \neq 0$,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) \quad \text{if} \quad n \geqq 1, \qquad \text{and} \qquad L = \lim_{n\to\infty} x_n.$$

Then

$$\lim_{n\to\infty} x_{n+1} = L \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{A}{x_n}\right) = \frac{1}{2}\left(L + \frac{A}{L}\right).$$

It follows that

$$L = \frac{1}{2}\left(L + \frac{A}{L}\right); \qquad 2L = \frac{L^2 + A}{L};$$

$$2L^2 = L^2 + A; \qquad L^2 = A.$$

Therefore $L = \pm\sqrt{A}$.

C11S02.054: Given: $A > 0$, $x_1 \neq 0$,

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{A}{x_n^2}\right) \quad \text{if} \quad n \geqq 1, \qquad \text{and} \qquad L = \lim_{n\to\infty} x_n.$$

Then

$$\lim_{n\to\infty} x_{n+1} = L \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{3}\left(2x_n + \frac{A}{x_n^2}\right) = \frac{1}{3}\left(2L + \frac{A}{L^2}\right).$$

It follows that

$$L = \frac{1}{3}\left(2L + \frac{A}{L^2}\right); \qquad 3L = \frac{2L^3 + A}{L^2};$$

$$3L^3 = 2L^3 + A; \qquad L^3 = A.$$

Therefore $L = A^{1/3}$.

C11S02.055: Part (a): Note first that $F_1 = 1$ and $F_2 = 1$. If $n \geqq 3$, then F_{n-1} is the total number of pairs present in the preceding month and F_{n-2} is the total number of productive pairs. Therefore $F_n = F_{n-1} + F_{n-2}$; that is, $F_{n+1} = F_n + F_{n-1}$ for $n \geqq 2$. So $\{F_n\}$ is the Fibonacci sequence of Example 2.

Part (b): Note first that $G_1 = G_2 = G_3 = 1$. If $n \geqq 4$, then G_{n-1} is the total number of pairs present in the preceding month and G_{n-3} is the total number of productive pairs. Therefore $G_n = G_{n-1} + G_{n-3}$; that is, $G_{n+1} = G_n + G_{n-2}$. The *Mathematica* commands

```
g[1] = 1; g[2] = 1; g[3] = 1;
g[n_] := g[n] = g[n - 1] + g[n - 3]
```

serve as one way to enter the formula for the recursively defined function g. Then the command

```
Table[ {n, g[n]}, {n, 4, 25} ]
```

produces the output

$$\{\{4,2\},\ \{5,3\},\ \{6,4\},\ \{7,6\},\ \{8,9\},\ \{9,13\},\ \{10,19\},\ \{11,28\},\ \{12,41\},$$
$$\{13,60\},\ \{14,88\},\ \{15,129\},\ \{16,189\},\ \{17,277\},\ \{18,406\},\ \{19,595\},$$
$$\{20,872\},\ \{21,1278\},\ \{22,1873\},\ \{23,2745\},\ \{24,4023\},\ \{25,5896\}\}$$

C11S02.056: Given the Fibonacci sequence $\{F_n\}$ of Example 2, note that

$$F_{n+1} = F_n + F_{n-1} \quad \text{implies that} \quad \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

Thus if

$$a_n = \frac{F_n}{F_{n-1}}, \qquad \text{then} \qquad a_{n+1} = \frac{F_{n+1}}{F_n} = 1 + \frac{1}{a_n}.$$

Assuming that τ exists, let $n \to +\infty$ in the last equation to obtain

$$\tau = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1 + \frac{1}{a_n}\right) = 1 + \frac{1}{\tau}.$$

Therefore

$$\tau^2 = \tau + 1; \qquad \tau^2 - \tau - 1 = 0; \qquad \tau = \frac{1 \pm \sqrt{5}}{2}.$$

Because $a_n > 0$ for all $n \geqq 1$, τ cannot be negative. Therefore $\tau = \frac{1}{2}(1 + \sqrt{5}\,)$.

C11S02.057: Part (a): Clearly $a_1 < 4$. Suppose that $a_k < 4$ for some integer $k \geqq 1$. Then

$$a_{k+1} = \frac{1}{2}(a_k + 4) < \frac{1}{2}(4 + 4) = 4.$$

Therefore, by induction, $a_n < 4$ for every integer $n \geqq 1$. Next, $a_2 = 3$, so that $a_1 < a_2$. Suppose that $a_k < a_{k+1}$ for some integer $k \geqq 1$. Then

$$a_k + 4 < a_{k+1} + 4; \qquad \frac{1}{2}(a_k + 4) < \frac{1}{2}(a_{k+1} + 4); \qquad a_{k+1} < a_{k+2}.$$

Therefore, by induction, $a_n < a_{n+1}$ for every integer $n \geqq 1$.

Part (b): Part (a) establishes that $\{a_n\}$ is a bounded increasing sequence. Therefore the bounded monotonic sequence property of Section 11.2 implies that the sequence $\{a_n\}$ converges. Let L denote its limit. Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 4) = \frac{1}{2}(L + 4).$$

It now follows immediately that $L = 4$.

C11S02.058: Given the positive real number r, let

$$L = \frac{1 + \sqrt{1 + 4r}}{2}.$$

Define the sequence $\{a_n\}$ recursively as follows: $a_1 = \sqrt{r}\,$, and for each integer $n \geqq 1$, $a_{n+1} = \sqrt{r + a_n}\,$. We plan to show first that $\{a_n\}$ is a bounded sequence, then that $\{a_n\}$ is an increasing sequence. Only then will we attempt to evaluate its limit, because our method for doing so depends on knowing that the sequence $\{a_n\}$ converges.

First,

$$a_1 = \sqrt{r}\, = \frac{\sqrt{4r}}{2} < \frac{1 + \sqrt{1 + 4r}}{2} = L.$$

Suppose that $a_k < L$ for some integer $k \geqq 1$. Then

$$a_k < \frac{1 + \sqrt{1 + 4r}}{2}; \qquad a_k + r < L + r;$$

$$4(a_k + r) < 2 + 2\sqrt{1 + 4r}\, + 4r; \qquad 4(a_k + r) < \left(1 + \sqrt{1 + 4r}\,\right)^2;$$

$$a_k + r < L^2; \qquad a_{k+1} = \sqrt{a_k + r} < L.$$

Therefore, by induction, $a_n < L$ for all $n \geqq 1$.

Next, $0 < \sqrt{r}\,$, so that $r < r + \sqrt{r}$. Thus $\sqrt{r} < \sqrt{r + \sqrt{r}}\,$. That is, $a_1 < a_2$. Suppose that $a_k < a_{k+1}$ for some integer $k \geqq 1$. Then

$$r + a_k < r + a_{k+1}; \qquad \sqrt{r + a_k} < \sqrt{r + a_{k+1}}\,; \qquad a_{k+1} < a_{k+2}.$$

Therefore, by induction, $a_n < a_{n+1}$ for all $n \geqq 1$.

Now that we know that the sequence $\{a_n\}$ converges, we may denote its limit by M. Then

$$M = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{r + a_n} = \sqrt{r + M}.$$

It now follows that $M^2 - M - r = 0$, and thus that

$$M = \frac{1 \pm \sqrt{1+4r}}{2}.$$

Because $M > 0$, we conclude that $\displaystyle\lim_{n\to\infty} a_n = \frac{1+\sqrt{1+4r}}{2} = L$. And in conclusion if, as in Problem 58, we take $r = 2$ we find that

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} = L = 2.$$

C11S02.059: Take $r = 20$ in the solution of Problem 58 to show that

$$\sqrt{20 + \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}} = 5.$$

C11S02.060: Take $r = 90$ in the solution of Problem 58 to show that

$$\sqrt{90 + \sqrt{90 + \sqrt{90 + \sqrt{90 + \cdots}}}} = 10.$$

C11S02.061: Suppose that $\{a_n\}$ is a bounded monotonic sequence; without loss of generality, suppose that it is an increasing sequence. Because the set of values of a_n is a nonempty set of real numbers with an upper bound, it has a least upper bound λ. We claim that λ is the limit of the sequence $\{a_n\}$.

Let $\epsilon > 0$ be given. Then there exists a positive integer N such that

$$\lambda - \epsilon < a_N \leqq \lambda.$$

For if not, then $\lambda - \epsilon$ would be an upper bound for $\{a_n\}$ smaller than λ, its least upper bound. But because $\{a_n\}$ is an increasing sequence with upper bound λ, it now follows that if $n > N$, then $a_N \leqq a_n \leqq \lambda$. Thus if $n > N$, then $|a_n - \lambda| < \epsilon$. Therefore, by definition, λ is the limit of the sequence $\{a_n\}$.

C11S02.062: Suppose that S is a nonempty set of real numbers with upper bound b. For each positive integer n, let a_n be the least integral multiple of $1/10^n$ that is an upper bound for S.

We first claim that $\{a_n\}$ is a decreasing sequence. For suppose that $a_n < a_{n+1}$. Then

$$a_n = \frac{j}{10^n} < \frac{k}{10^{n+1}} = a_{n+1}$$

for some integers j and k. This implies that

$$\frac{10j}{10^{n+1}} < \frac{k}{10^{n+1}}$$

so that $k/10^{n+1}$, chosen to be the least integral multiple of $1/10^{n+1}$ that is an upper bound for S, is larger than $10j/10^{n+1}$, a smaller integral multiple of $1/10^{n+1}$ but also an upper bound for S. This is impossible. Therefore $\{a_n\}$ is a decreasing sequence.

Next we claim that $\{a_n\}$ is bounded. Because it is decreasing, a_1 is an upper bound. Any element of S is a lower bound. This shows that $\{a_n\}$ is bounded.

Therefore $\{a_n\}$ converges; let A be its limit. We now claim that A is an upper bound for S. If not, choose s in S such that $A < s$. Let $\epsilon = s - A$. Choose N so large that $A < a_N < A + \epsilon = s$. Then a_N is not an upper bound for S. This is impossible. Hence A is an upper bound for S.

Finally we claim that A is the least upper bound of S. Suppose that B is an upper bound for S and that $B < A$. Then there exist integers k and n such that $k/10^n$ lies between B and A. This implies that $a_n < A$, which is impossible because $\{a_n\}$ is a decreasing sequence with limit—and thus lower bound—A. Therefore A is the least upper bound of S.

C11S02.063: For each integer $n \geqq$, let a_n be the largest integral multiple of $1/10^n$ such that $a_n^2 \leqq 2$. (For example, $a_1 = 1.4,\ a_2 = 1.41,$ and $a_3 = 1.414$.)

Part (a): First note that the numbers 1 and $\frac{3}{2}$ are multiples of $1/10^n$ (for each $n \geqq 1$) with $1^2 < 2$ and $\left(\frac{3}{2}\right)^2 > 2$. It follows that $1 \leqq a_n \leqq \frac{3}{2}$ for each integer $n \geqq 1$, and therefore the sequence $\{a_n\}$ is bounded. Next, a_n as an integral multiple of $1/10^n$ is also an integral multiple of $1/10^{n+1}$ whose square does not exceed 2. But a_{n+1} is the *largest* multiple of $1/10^{n+1}$ whose square does not exceed 2; it follows that $a_n \leqq a_{n+1}$, and thus the sequence $\{a_n\}$ is also an increasing sequence.

Part (b): Because $\{a_n\}$ is a bounded increasing sequence, it has a limit A. Then the limit laws give

$$A^2 = \left(\lim_{n\to\infty} a_n\right)^2 = \lim_{n\to\infty} (a_n)^2 \leqq \lim_{n\to\infty} 2 = 2,$$

so we see that $A^2 \leqq 2$.

Part (c): Assume that $A^2 < 2$. Then $2 - A^2 > 0$. Choose the integer k so large that $4/10^k \leqq 2 - A^2$. Then

$$\begin{aligned}\left(a_k + \frac{1}{10^k}\right)^2 &= a_k^2 + \frac{2a_k}{10^k} + \frac{1}{10^{2k}} \\ &< a_k^2 + \frac{4}{10^k} \qquad \left(\text{because } \ a_k < \frac{3}{2} \ \text{ and } \ \frac{1}{10^{2k}} < \frac{1}{10^k}\right) \\ &\leqq A^2 + (2 - A^2) = 2.\end{aligned}$$

Thus the assumption that $A^2 < 2$ implies that $(a_k + 1/10^k)^2 < 2$, which contradicts the fact that a_k is, by definition, the largest integral multiple of $1/10^k$ whose square does not exceed 2. It therefore follows that A^2 is *not* less than 2; that is, that $A^2 \geqq 2$.

Part (d): It follows immediately from the results in parts (c) and (d) that $A^2 = 2$.

C11S02.064: The first few terms of the sequence $\{a_n\}$ are

2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30,
31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55,
56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79,

80, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 101.

Challenge 1: Prove that a_n is never the square of an integer. Challenge 2: Prove that every positive integer not the square of an integer is a value of a_n. Challenge 3: Construct a similar sequence whose values are the positive integers that are not cubes of integers.

Section 11.3

C11S03.001: The series is geometric with first term 1 and ratio $\frac{1}{3}$. Therefore it converges to

$$\frac{1}{1 - \dfrac{1}{3}} = \frac{3}{2}.$$

C11S03.002: The series is geometric with first term 1 and ratio $1/e$. Therefore it converges to

$$\frac{1}{1 - \dfrac{1}{e}} = \frac{e}{e-1} \approx 1.581976706869.$$

C11S03.003: This series diverges by the nth-term test. Alternatively, you can show by induction that

$$S_k = \sum_{n=1}^{k} (2n-1) = k^2,$$

so this series diverges because $\lim_{k\to\infty} S_k = +\infty$.

C11S03.004: By the result in Example 7 of Section 11.2,

$$\lim_{n\to\infty} 2^{1/n} = 1.$$

Therefore the given series diverges by the nth-term test.

C11S03.005: This series is geometric but its ratio is -2 and $|-2| > 1$. Therefore the given series diverges. Alternatively, it diverges by the nth-term test for divergence.

C11S03.006: The given series is geometric with first term 1 and ratio $-\frac{1}{4}$. Therefore it converges to

$$\frac{1}{1 - \left(-\dfrac{1}{4}\right)} = \frac{1}{1 + \dfrac{1}{4}} = \frac{4}{5}.$$

C11S03.007: The given series is geometric with first term 4 and ratio $\frac{1}{3}$. Therefore it converges to

$$\frac{4}{1 - \dfrac{1}{3}} = 6.$$

C11S03.008: The given series is geometric with first term $\frac{1}{3}$ and ratio $\frac{2}{3}$. Thus it converges; its sum is

$$\frac{\dfrac{1}{3}}{1 - \dfrac{2}{3}} = 1.$$

C11S03.009: This series is geometric with first term 1 and ratio $r = 1.01$. Because $|r| > 1$, the series diverges. Alternatively, you can apply the nth-term test for divergence.

C11S03.010: By Example 11 in Section 11.2,

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Therefore the given series diverges by the nth-term test.

C11S03.011: The given series diverges by the nth-term test:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1, \qquad \text{and therefore} \qquad \lim_{n\to\infty} \frac{(-1)^n n}{n+1} \neq 0.$$

C11S03.012: The given series is geometric with first term and ratio both $e/10$, so it converges to

$$\frac{\dfrac{e}{10}}{1-\dfrac{e}{10}} = \frac{e}{10-e} \approx 0.3733022570253914872053 1323.$$

C11S03.013: The given series is geometric with first term 1 and ratio $r = -3/e$. It diverges because $|r| > 1$.

C11S03.014: The given series is the difference of two convergent geometric series, so its sum is

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\dfrac{3}{4}} - \frac{1}{1-\dfrac{1}{2}} = 4 - 2 = 2.$$

C11S03.015: The given series is geometric with first term 1 and ratio $1/\sqrt{2}$. Therefore its sum is

$$\frac{1}{1-\dfrac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}\,\left(\sqrt{2}+1\right)}{2-1} = 2+\sqrt{2} \approx 3.414213562373.$$

C11S03.016: Note that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tag{1}$$

converges (it is geometric with ratio $\frac{1}{2}$). If

$$\sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{1}{2^n}\right) \tag{2}$$

also converged, then the sum of the series in (1) and (2) would converge. Their sum is

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{2}{n} - \frac{1}{2^n}\right) = \sum_{n=1}^{\infty} \frac{2}{n}. \tag{3}$$

But if the series in (3) converged, so would

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{2}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

and this would contradict Theorem 4 of Section 11.3. Therefore the series in (2) diverges. See also Problem 62 of this section.

C11S03.017: Because the limit of the nth term is

$$\lim_{n\to\infty} \frac{n}{10n+17} = \lim_{n\to\infty} \frac{1}{10+\dfrac{17}{n}} = \frac{1}{10} \neq 0,$$

the given series diverges.

C11S03.018: By l'Hôpital's rule (used twice), the limit of the nth term is

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln(n+1)} = \lim_{x\to\infty} \frac{x^{1/2}}{\ln(x+1)} = \lim_{x\to\infty} \frac{x+1}{2x^{1/2}} = \lim_{x\to\infty} x^{1/2} = +\infty.$$

Therefore the given series diverges by the nth-term test.

C11S03.019: The given series is the difference of two convergent geometric series, so its sum is

$$\sum_{n=1}^{\infty} \left(5^{-n} - 7^{-n}\right) = \sum_{n=1}^{\infty} \frac{1}{5^n} - \sum_{n=1}^{\infty} \frac{1}{7^n} = \frac{\dfrac{1}{5}}{1-\dfrac{1}{5}} - \frac{\dfrac{1}{7}}{1-\dfrac{1}{7}} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

C11S03.020: By Example 9 in Section 11.2, $\left(\frac{9}{10}\right)^n \to 0$ as $n \to +\infty$. So the given series diverges by the nth-term test, because

$$\lim_{n\to\infty} \frac{1}{1+\left(\dfrac{9}{10}\right)^n} = \frac{1}{1+0} = 1 \neq 0.$$

C11S03.021: The series is geometric with first term e/π and ratio $r = e/\pi$. Because $|r| < 1$, the series converges to

$$\frac{\dfrac{e}{\pi}}{1-\dfrac{e}{\pi}} = \frac{e}{\pi - e} \approx 6.421479600999.$$

C11S03.022: The given series is geometric with ratio $r = \pi/e$. But its first term is nonzero and $|r| > 1$, so the series diverges.

C11S03.023: The given series is geometric with ratio $r = \frac{100}{99}$. But its first term is nonzero and $|r| > 1$, so the series diverges.

C11S03.024: The given series is geometric with first term 1 and ratio $r = \frac{99}{100}$, so it converges to

$$\frac{1}{1-\dfrac{99}{100}} = 100.$$

C11S03.025: The given series is the sum of three convergent geometric series, and its sum is

$$\sum_{n=0}^{\infty} \frac{1+2^n+3^n}{5^n} = \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

$$= \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{2}{5}} + \frac{1}{1-\frac{3}{5}} = \frac{5}{4} + \frac{5}{3} + \frac{5}{2} = \frac{65}{12} \approx 5.41666666667.$$

C11S03.026: The given series diverges because it is the sum of the two convergent geometric series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \tag{1}$$

and the divergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{5}{3}\right)^n. \tag{2}$$

The sum of the two series in (1) converges by part (1) of Theorem 2, hence the sum of that series and the series in (2) diverges by an argument similar to that used in the solution of Problem 16 or by the general argument given in the solution of Problem 62.

C11S03.027: We use both parts of Theorem 2:

$$\sum_{n=0}^{\infty} \frac{7\cdot 5^n + 3\cdot 11^n}{13^n} = 7\cdot\left[\sum_{n=0}^{\infty}\left(\frac{5}{13}\right)^n\right] + 3\cdot\left[\sum_{n=0}^{\infty}\left(\frac{11}{13}\right)^n\right]$$

$$= 7\cdot\frac{1}{1-\frac{5}{13}} + 3\cdot\frac{1}{1-\frac{11}{13}} = \frac{91}{8} + \frac{39}{2} = \frac{247}{8} = 30.875.$$

C11S03.028: The given series diverges by the nth-term test, because (by Example 7 in Section 11.2) $\lim_{n\to\infty} 2^{1/n} = 1 \neq 0$.

C11S03.029: The given series converges because it is the difference of two convergent geometric series. Its sum is

$$\sum_{n=1}^{\infty}\left[\left(\frac{7}{11}\right)^n - \left(\frac{3}{5}\right)^n\right] = \sum_{n=1}^{\infty}\left(\frac{7}{11}\right)^n - \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n = \frac{\frac{7}{11}}{1-\frac{7}{11}} - \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{7}{4} - \frac{3}{2} = \frac{1}{4}.$$

C11S03.030: The given series diverges by the nth-term test for divergence, because

$$\lim_{n\to\infty} \frac{2n}{\sqrt{4n^2+3}} = \lim_{n\to\infty}\left(\frac{4n^2}{4n^2+3}\right)^{1/2} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{3}{4n^2}}\right)^{1/2} = \left(\frac{1}{1+0}\right)^{1/2} = 1 \neq 0.$$

C11S03.031: The given series diverges by the nth-term test for divergence, because

$$\lim_{n\to\infty} \frac{n^2-1}{3n^2-1} = \lim_{n\to\infty} \frac{1-\dfrac{1}{n^2}}{3-\dfrac{1}{n^2}} = \frac{1-0}{3-0} = \frac{1}{3} \neq 0.$$

C11S03.032: The given series is geometric with first term $\sin 1$ and ratio $r = \sin 1 \approx 0.841470984808$. Therefore, because $|r| < 1$, the series converges. Its sum is

$$\frac{\sin 1}{1-\sin 1} \approx 5.307993516444.$$

C11S03.033: The given series is geometric with nonzero first term and ratio $r = \tan 1 \approx 1.557407724655$. Because $|r| > 1$, this series diverges.

C11S03.034: The given series is geometric with nonzero first term and ratio

$$r = \arcsin 1 = \frac{\pi}{2} \approx 1.570796326795.$$

Because $|r| > 1$, this series diverges.

C11S03.035: This is a geometric series with first term $\pi/4$ and ratio $r = \pi/4 \approx 0.785398163397$. Because $|r| < 1$, it converges; its sum is

$$\frac{\dfrac{\pi}{4}}{1-\dfrac{\pi}{4}} = \frac{\pi}{4-\pi} \approx 3.659792366325.$$

C11S03.036: Because

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \approx 1.570796326795,$$

the given series diverges by the nth-term test for divergence.

C11S03.037: A figure similar to Fig. 11.3.4 of the text shows that if n is an integer and $n \geqq 2$, then

$$\frac{1}{n \ln n} \geqq \int_n^{n+1} \frac{1}{x \ln x}\,dx.$$

Therefore the kth partial sum S_k of the given series satisfies the equalities and inequalities

$$S_k = \sum_{n=2}^{k} \frac{1}{x \ln x} \geqq \int_2^{k+1} \frac{1}{x \ln x}\,dx = \Big[\ln(\ln x)\Big]_2^{k+1} = \ln(\ln(k+1)) - \ln(\ln 2) = \ln\left(\frac{\ln(k+1)}{\ln 2}\right).$$

Hence the given series diverges because $\{S_k\} \to +\infty$ as $k \to +\infty$.

C11S03.038: The methods of Example 7 yield the following results.

Part (a):

$$0.666\,666\,666\,\cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots = \frac{\dfrac{6}{10}}{1-\dfrac{1}{10}} = \frac{6}{10-1} = \frac{2}{3}.$$

Part (b):

$$0.111\,111\,111\,\cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = \frac{\dfrac{1}{10}}{1 - \dfrac{1}{10}} = \frac{1}{10-1} = \frac{1}{9}.$$

Part (c):

$$0.249\,999\,999\,\cdots = \frac{2}{10} + \frac{4}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots = \frac{1}{5} + \frac{1}{25} + \frac{\dfrac{9}{1000}}{1 - \dfrac{1}{10}} = \frac{6}{25} + \frac{9}{1000-100} = \frac{1}{4}.$$

Part (d):

$$0.999\,999\,999\,\cdots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{\dfrac{9}{10}}{1 - \dfrac{1}{10}} = \frac{9}{10-1} = 1.$$

C11S03.039: Here we have

$$0.47\,47\,47\,47\,\cdots = \frac{47}{100} + \frac{47}{10000} + \frac{47}{1000000} + \cdots = \frac{\dfrac{47}{100}}{1 - \dfrac{1}{100}} = \frac{47}{99}.$$

C11S03.040: $0.25\,25\,25\,\cdots = \dfrac{25}{100} + \dfrac{25}{10000} + \dfrac{25}{1000000} + \cdots = \dfrac{\dfrac{25}{100}}{1 - \dfrac{1}{100}} = \dfrac{25}{99}.$

C11S03.041: $0.123\,123\,123\,\cdots = \dfrac{123}{1000} + \dfrac{123}{1000000} + \dfrac{123}{1000000000} + \cdots = \dfrac{\dfrac{123}{1000}}{1 - \dfrac{1}{1000}} = \dfrac{123}{999} = \dfrac{41}{333}.$

C11S03.042: $0.3377\,3377\,3377\,\cdots = \dfrac{3377}{10^4} + \dfrac{3377}{10^8} + \dfrac{3377}{10^{12}} + \cdots = \dfrac{\dfrac{3377}{10000}}{1 - \dfrac{1}{10000}} = \dfrac{3377}{9999} = \dfrac{307}{909}.$

C11S03.043: As in Example 7,

$$3.14159\,14159\,14159\,\cdots = 3 + \frac{14159}{10^5} + \frac{14150}{10^{10}} + \frac{14159}{10^{15}} + \cdots = 3 + \frac{\dfrac{14159}{100000}}{1 - \dfrac{1}{100000}}$$

$$= 3 + \frac{14159}{99999} = \frac{299997 + 14159}{99999} = \frac{314156}{99999}.$$

C11S03.044: The series is geometric with ratio $2x$. Thus it will converge when $|2x| < 1$; that is, when $-\frac{1}{2} < x < \frac{1}{2}$. For such x, we have

$$\sum_{n=1}^{\infty} (2x)^n = \frac{2x}{1-2x}.$$

C11S03.045: The series is geometric with ratio $x/3$. Thus it will converge when

$$\left|\frac{x}{3}\right| < 1; \quad \text{that is, when} \quad -3 < x < 3.$$

For such x, we have

$$\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n = \frac{\dfrac{x}{3}}{1-\dfrac{x}{3}} = \frac{x}{3-x}.$$

C11S03.046: The given series is geometric with ratio $x-1$. Hence it will converge when $|x-1| < 1$; that is, when $0 < x < 2$. For such x, we have

$$\sum_{n=1}^{\infty} (x-1)^n = \frac{x-1}{1-(x-1)} = \frac{x-1}{2-x}.$$

C11S03.047: The given series is geometric with ratio $(x-2)/3$. Hence it will converge when

$$\left|\frac{x-2}{3}\right| < 1; \quad \text{that is, when} \quad -1 < x < 5.$$

For such x, we have

$$\sum_{n=1}^{\infty} \left(\frac{x-2}{3}\right)^n = \frac{\dfrac{x-2}{3}}{1-\dfrac{x-2}{3}} = \frac{x-2}{3-(x-2)} = \frac{x-2}{5-x}.$$

C11S03.048: This series is geometric with ratio $x^2/(x^2+1)$. Therefore it will converge when

$$-1 < \frac{x^2}{x^2+1} < 1; \quad \text{that is, for all real numbers } x.$$

Finally,

$$\sum_{n=1}^{\infty} \left(\frac{x^2}{x^2+1}\right)^n = \frac{\dfrac{x^2}{x^2+1}}{1-\dfrac{x^2}{x^2+1}} = \frac{x^2}{x^2+1-x^2} = x^2.$$

C11S03.049: This series is geometric with ratio $5x^2/(x^2+16)$. Hence it will converge when

$$\begin{aligned} \frac{5x^2}{x^2+16} < 1: \quad & 5x^2 < x^2+16; \\ & 4x^2 < 16; \\ & x^2 < 4; \end{aligned}$$

$$-2 < x < 2.$$

For such x we have

$$\sum_{n=1}^{\infty}\left(\frac{5x^2}{x^2+16}\right)^n = \frac{\dfrac{5x^2}{x^2+16}}{1-\dfrac{5x^2}{x^2+16}} = \frac{5x^2}{x^2+16-5x^2} = \frac{5x^2}{16-4x^2}.$$

C11S03.050: The method of partial fractions yields

$$\frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} + \frac{-1}{2n+1}\right).$$

Therefore the kth partial sum of the given series is

$$S_k = \sum_{n=1}^{k}\frac{1}{4n^2-1} = \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{2k+1}\right) = \frac{1}{2}\left(1-\frac{1}{2k+1}\right).$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \lim_{k\to\infty} S_k = \frac{1}{2}.$$

C11S03.051: The method of partial fractions yields

$$\frac{1}{9n^2+3n-2} = \frac{1}{3}\left(\frac{1}{3n-1} + \frac{-1}{3n+2}\right).$$

Therefore the kth partial sum of the given series is

$$S_k = \sum_{n=1}^{k}\frac{1}{9n^2+3n-2} = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}+\frac{1}{5}-\frac{1}{8}+\frac{1}{8}-\frac{1}{11}+\cdots-\frac{1}{3k+2}\right) = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{3k+2}\right).$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{9n^2+3n-2} = \lim_{k\to\infty} S_k = \frac{1}{6}.$$

C11S03.052: Because $\ln\dfrac{n+1}{n} = \ln(n+1)-\ln n$, the kth partial sum of the given series is

$$S_k = \sum_{n=1}^{k}\ln\frac{n+1}{n} = \ln 2-\ln 1+\ln 3-\ln 2+\ln 4-\ln 3+\cdots+\ln(k+1)-\ln k = \ln(k+1).$$

Therefore the given series diverges because $\displaystyle\lim_{k\to\infty} S_k = +\infty$.

C11S03.053: The method of partial fractions yields

$$\frac{1}{16n^2-8n-3} = \frac{1}{4}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right).$$

Thus the kth partial sum of the given series is

$$S_k = \sum_{n=1}^{k} \frac{1}{16n^2 - 8n - 3} = \frac{1}{4}\left(1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \cdots - \frac{1}{4k+1}\right) = \frac{1}{4}\left(1 - \frac{1}{4k+1}\right).$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 3} = \lim_{k\to\infty} S_k = \frac{1}{4}.$$

C11S03.054: The method of partial fractions yields

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Therefore the kth partial sum of the given series is

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \frac{1}{n(n+2)} \\ &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{k-1} - \frac{1}{k+1} + \frac{1}{k} - \frac{1}{k+2}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right). \end{aligned}$$

Therefore the sum of the given series is $\displaystyle\lim_{k\to\infty} S_k = \frac{3}{4}$.

C11S03.055: The method of partial fractions yields

$$\frac{1}{n^2 - 1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

So the kth partial sum of the given series is

$$\begin{aligned} S_k &= \sum_{n=2}^{k} \frac{1}{n^2 - 1} \\ &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{k-2} - \frac{1}{k} + \frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1}\right). \end{aligned}$$

Thus the sum of the given series is $\displaystyle\lim_{k\to\infty} S_k = \frac{3}{4}$.

C11S03.056: In *Mathematica* 3.0 the command

```
Apart[ (2*n + 1)/(n*n*(n + 1)∧2) ]
```

produces the partial fraction decomposition

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

Therefore the kth partial sum of the given series is

$$S_k = \sum_{n=1}^{k} \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{9} + \frac{1}{9} - \frac{1}{16} + \cdots + \frac{1}{k^2} - \frac{1}{(k+1)^2} = 1 - \frac{1}{(k+1)^2}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{k\to\infty} S_k = 1.$$

C11S03.057: In *Derive* 2.56 application of the command `Expand` to the nth term of the given series yields the partial fraction decomposition

$$\frac{6n^2+2n-1}{n(n+1)(4n^2-1)} = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2n-1} - \frac{1}{2n+1}.$$

So the kth partial sum of the given series is

$$\begin{aligned}
S_k &= \sum_{n=1}^{k} \frac{6n^2+2n-1}{n(n+1)(4n^2-1)} \\
&= \left(1 - \frac{1}{2} + 1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9}\right) \\
&= \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{9} - \frac{1}{11}\right) + \left(\frac{1}{6} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{2k-1} - \frac{1}{2k+1}\right) \\
&= 1 - \frac{1}{k+1} + 1 - \frac{1}{2k+1}.
\end{aligned}$$

Therefore the sum of the given series is $\lim_{k\to\infty} S_k = 2$.

C11S03.058: In *Derive* 2.56 application of the command `Expand` to the nth term of the given series yields the partial fraction decomposition

$$\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}.$$

Thus the kth partial sum of the given series is

$$\begin{aligned}
S_k = \sum_{n=1}^{k} \frac{2}{n(n+1)(n+2)} &= \frac{1}{1} - \frac{2}{2} + \frac{1}{3} \\
&+ \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\
&+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\
&+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}\\
&+\frac{1}{6}-\frac{2}{7}+\frac{1}{8}\\
&+\cdots\\
&+\frac{1}{k-3}-\frac{2}{k-2}+\frac{1}{k-1}\\
&+\frac{1}{k-2}-\frac{2}{k-1}+\frac{1}{k}\\
&+\frac{1}{k-1}-\frac{2}{k}+\frac{1}{k+1}\\
&+\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\,.
\end{aligned}
$$

Inspect the diagonals that run from southwest to northeast. The fractions with denominator 3 cancel one another, as do those with denominators $4,\ 5,\ 6,\ \ldots,\ k-3,\ k-2,\ k-1,$ and k. Thus

$$
S_k = \frac{1}{1}-\frac{2}{2}+\frac{1}{2}+\frac{1}{k+1}-\frac{2}{k+1}+\frac{1}{k+2}=\frac{1}{2}-\frac{1}{k+1}+\frac{1}{k+2}.
$$

Therefore the sum of the given series is $\displaystyle\lim_{k\to\infty} S_k = \frac{1}{2}$.

C11S03.059: In *Mathematica* 3.0 the command

```
Apart[ 6/(n*(n + 1)*(n + 2)*(n + 3)) ]
```

yields the partial fraction decomposition

$$
\frac{6}{n(n+1)(n+2)(n+3)} = \frac{1}{n}-\frac{3}{n+1}+\frac{3}{n+2}-\frac{1}{n+3}.
$$

Therefore the kth partial sum of the given series is

$$
\begin{aligned}
S_k = \sum_{n=1}^{k}\frac{6}{n(n+1)(n+2)(n+3)} &= 1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}\\
&+\frac{1}{2}-\frac{3}{3}+\frac{3}{4}-\frac{1}{5}\\
&+\frac{1}{3}-\frac{3}{4}+\frac{3}{5}-\frac{1}{6}\\
&+\frac{1}{4}-\frac{3}{5}+\frac{3}{6}-\frac{1}{7}\\
&+\frac{1}{5}-\frac{3}{6}+\frac{3}{7}-\frac{1}{8}\\
&+\frac{1}{6}-\frac{3}{7}+\frac{3}{8}-\frac{1}{9}\\
&+\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{k-4}-\frac{3}{k-3}+\frac{3}{k-2}-\frac{1}{k-1}\\
&+\frac{1}{k-3}-\frac{3}{k-2}+\frac{3}{k-1}-\frac{1}{k}\\
&+\frac{1}{k-2}-\frac{3}{k-1}+\frac{3}{k}-\frac{1}{k+1}\\
&+\frac{1}{k-1}-\frac{3}{k}+\frac{3}{k+1}-\frac{1}{k+2}\\
&+\frac{1}{k}-\frac{3}{k+1}+\frac{3}{k+2}-\frac{1}{k+3}
\end{aligned}
$$

Examine the diagonals that run from southwest to northeast. The four fractions with denominator 4 all cancel one another, as do those with denominators $5,\ 6,\ \ldots,\ k-1,$ and k. Thus

$$
S_k = 1-\frac{2}{2}+\frac{1}{3}-\frac{1}{k+1}+\frac{2}{k+2}-\frac{1}{k+3}=\frac{1}{3}-\frac{1}{k+1}+\frac{2}{k+2}-\frac{1}{k+3}.
$$

Therefore the sum of the given series is $\displaystyle\lim_{k\to\infty} S_k = \frac{1}{3}$.

C11S03.060: In *Maple* V version 5.1, the sequence of commands

```
f := 6*n/(n∧4 - 5*n∧2 + 4);

convert(f,parfrac,n);
```

yields the partial fraction decomposition

$$
\frac{6n}{n^4-5n^2+4}=\frac{1}{n-2}-\frac{1}{n-1}-\frac{1}{n+1}+\frac{1}{n+2}.
$$

Thus the kth partial sum of the given series is

$$
\begin{aligned}
S_k=\sum_{n=3}^{k}\frac{6n}{n^4-5n^2+4}&=\frac{1}{1}-\frac{1}{2}-\frac{1}{4}+\frac{1}{5}\\
&+\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}\\
&+\frac{1}{3}-\frac{1}{4}-\frac{1}{6}+\frac{1}{7}\\
&+\frac{1}{4}-\frac{1}{5}-\frac{1}{7}+\frac{1}{8}\\
&+\frac{1}{5}-\frac{1}{6}-\frac{1}{8}+\frac{1}{9}\\
&+\frac{1}{6}-\frac{1}{7}-\frac{1}{9}+\frac{1}{10}\\
&+\cdots\\
&+\frac{1}{k-7}-\frac{1}{k-6}-\frac{1}{k-4}+\frac{1}{k-3}
\end{aligned}
$$

$$+\frac{1}{k-6}-\frac{1}{k-5}-\frac{1}{k-3}+\frac{1}{k-2}$$

$$+\frac{1}{k-5}-\frac{1}{k-4}-\frac{1}{k-2}+\frac{1}{k-1}$$

$$+\frac{1}{k-4}-\frac{1}{k-3}-\frac{1}{k-1}+\frac{1}{k}$$

$$+\frac{1}{k-3}-\frac{1}{k-2}-\frac{1}{k}+\frac{1}{k+1}$$

$$+\frac{1}{k-2}-\frac{1}{k-1}-\frac{1}{k+1}+\frac{1}{k+2}\,.$$

The fractions with denominator 5 cancel, as do those with denominators 6, 7, 8, ..., $k-3$, and $k-2$. After a few other cancellations we find that

$$S_k = 1-\frac{1}{4}-\frac{1}{k-1}+\frac{1}{k+2}.$$

Thus the sum of the given series is $\displaystyle\lim_{k\to\infty} S_k = \frac{3}{4}$.

C11S03.061: By part 2 of Theorem 2, if $c \neq 0$ and $\sum ca_n$ converges, then

$$\sum \frac{1}{c}\cdot ca_n = \sum a_n$$

converges. Therefore if $c \neq 0$ and $\sum a_n$ diverges, then $\sum ca_n$ diverges.

C11S03.062: Note first that $\sum(-1)a_n$ converges. If $\sum(a_n+b_n)$ also converges, then their sum $\sum b_n$ converges. Therefore if $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n+b_n)$ diverges.

C11S03.063: Let

$$S_n=\sum_{i=1}^{n} a_i \quad\text{and}\quad T_n=\sum_{i=1}^{n} b_i,$$

let k be a fixed positive integer, and suppose that $a_j = b_j$ for every integer $j \geqq k$. If $n = k+1$, then

$$S_n - T_n = (S_k+a_n)-(T_k+b_n) = (S_k+a_n)-(T_k+a_n) = S_k - T_k.$$

Assume that $S_n - T_n = S_k - T_k$ for some integer $n \geqq k+1$. Then

$$S_{n+1}-T_{n+1} = (S_n+a_{n+1})-(T_n+b_{n+1}) = (S_n+a_{n+1})-(T_n+a_{n+1}) = S_n - T_n = S_k - T_k.$$

Therefore, by induction, $S_n - T_n = S_k - T_k$ for every integer $n \geqq k+1$.

C11S03.064: Figure 11.3.5 of the text makes it clear that the total distance the bouncing ball travels is

$$D = a+2ra+2r^2a+2r^3a+\cdots = -a+2a(1+r+r^2+r^3+\cdots) = -a+\frac{2a}{1-r} = a\cdot\frac{1+r}{1-r}.$$

C11S03.065: The total time the ball spends bouncing is

$$\begin{aligned}
T &= \sqrt{2a/g} + 2\sqrt{2ar/g} + 2\sqrt{2ar^2/g} + 2\sqrt{2ar^3/g} + \cdots \\
&= -\sqrt{2a/g} + 2\sqrt{2a/g}\left(1 + r^{1/2} + r + r^{3/2} + \cdots\right) = -\sqrt{2a/g} + 2\sqrt{2a/g}\left(\frac{1}{1 - r^{1/2}}\right) \\
&= \sqrt{2a/g}\left(-1 + \frac{2}{1 - r^{1/2}}\right) = \sqrt{2a/g}\left(\frac{-1 + r^{1/2} + 2}{1 - r^{1/2}}\right) = \sqrt{2a/g}\left(\frac{1 + r^{1/2}}{1 - r^{1/2}}\right).
\end{aligned}$$

If we take $r = 0.64$, $a = 4$, and $g = 32$, we find the total bounce time to be

$$T = \sqrt{8/32}\left(\frac{1 + 0.8}{1 - 0.8}\right) = \frac{1}{2} \cdot \frac{1.8}{0.2} = 4.5 \quad \text{(seconds)}.$$

C11S03.066: The total spending will be (in billions of dollars)

$$1 + 0.9 + (0.9)^2 + (0.9)^3 + \cdots + (0.9)^n + \cdots = \frac{1}{1 - 0.9} = 10.$$

C11S03.067: Let $r = 0.95$. Then $M_1 = r\,M_0$, $M_2 = rM_1 = r^2M_0$, and so on; in the general case, $M_n = r^nM_0$. Because $-1 < r < 1$, it now follows that

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} r^nM_0 = 0.$$

C11S03.068: Suppose that Mary tosses first. With H for "heads" and T for "tails," here are her winning patterns and their respective probabilities:

$$\begin{array}{ll}
H & \dfrac{1}{2} \\[2ex]
T\;T\;H & \dfrac{1}{2^3} = \dfrac{1}{8} \\[2ex]
T\;T\;T\;T\;H & \dfrac{1}{2^5} = \dfrac{1}{32} \\[2ex]
T\;T\;T\;T\;T\;T\;H & \dfrac{1}{2^7} = \dfrac{1}{128} \\[2ex]
\vdots & \vdots
\end{array}$$

Evidently Mary's probability of winning the game is the sum of the probabilities in the right-hand column. This sum is a geometric series with first term 0.5 and ratio 0.25, and therefore its sum is $\frac{2}{3}$. The probability that Paul wins can be computed in much the same way, or simply compute $1 - \frac{2}{3} = \frac{1}{3}$.

C11S03.069: Peter's probability of winning is the sum of:

The probability that he wins in the first round;

The probability that everyone tosses tails in the first round and Peter wins in the second round;

The probability that everyone tosses tails in the first two rounds and Peter wins in the third round;

Et cetera, et cetera, et cetera.

Thus his probability of winning is

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{10}} + \cdots = \frac{\dfrac{1}{2}}{1 - \dfrac{1}{8}} = \frac{4}{7}.$$

Similarly, the probability that Paul wins is

$$\frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \frac{1}{2^{11}} + \cdots = \frac{\dfrac{1}{4}}{1 - \dfrac{1}{8}} = \frac{2}{7}$$

and the probability that Mary wins is

$$\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \cdots = \frac{\dfrac{1}{8}}{1 - \dfrac{1}{8}} = \frac{1}{7}$$

Note that the three probabilities have sum 1, as they should.

C11S03.070: Let X denote 1, 2, 3, 4, and 5. Peter (who goes first) has these winning patterns with these probabilities:

$$\begin{array}{ll}
6 & \dfrac{1}{6} \\[2ex]
X\ X\ X\ 6 & \dfrac{5^3}{6^4} \\[2ex]
X\ X\ X\ X\ X\ X\ 6 & \dfrac{5^6}{6^7} \\[2ex]
X\ X\ X\ X\ X\ X\ X\ X\ X\ 6 & \dfrac{5^9}{6^{10}} \\[2ex]
\vdots & \vdots
\end{array}$$

Thus Peter wins with probability

$$\frac{1}{6} + \frac{5^3}{6^4} + \frac{5^6}{6^7} + \frac{5^9}{6^{10}} + \cdots = \frac{\dfrac{1}{6}}{1 - \dfrac{5^3}{6^3}} = \frac{1}{6} \cdot \frac{1}{1 - \dfrac{125}{216}} = \frac{36}{91} \approx 0.395604395604.$$

Paul (who goes second) has these winning patterns with these probabilities:

$$\begin{array}{ll}
X\ 6 & \dfrac{5}{6^2} \\[2ex]
X\ X\ X\ X\ 6 & \dfrac{5^4}{6^5}
\end{array}$$

$$X\ X\ X\ X\ X\ X\ X\ 6 \qquad \frac{5^7}{6^8}$$

$$X\ X\ X\ X\ X\ X\ X\ X\ X\ X\ 6 \qquad \frac{5^{10}}{6^{11}}$$

$$\vdots \qquad \vdots$$

Thus Paul wins with probability

$$\frac{5}{6^2} + \frac{5^4}{6^5} + \frac{5^7}{6^8} + \frac{5^{10}}{6^{11}} + \cdots = \frac{\dfrac{5}{36}}{1 - \dfrac{5^3}{6^3}} = \frac{5}{36} \cdot \frac{1}{1 - \dfrac{125}{216}} = \frac{30}{91} \approx 0.329670329670.$$

Mary (who goes third) has these winning patterns with these probabilities:

$$X\ X\ 6 \qquad \frac{5^2}{6^3}$$

$$X\ X\ X\ X\ X\ 6 \qquad \frac{5^5}{6^6}$$

$$X\ X\ X\ X\ X\ X\ X\ X\ 6 \qquad \frac{5^8}{6^9}$$

$$X\ X\ X\ X\ X\ X\ X\ X\ X\ X\ X\ 6 \qquad \frac{5^{11}}{6^{12}}$$

$$\vdots \qquad \vdots$$

Thus Mary wins with probability

$$\frac{5^2}{6^3} + \frac{5^5}{6^6} + \frac{5^8}{6^9} + \frac{5^{11}}{6^{12}} + \cdots = \frac{\dfrac{25}{216}}{1 - \dfrac{5^3}{6^3}} = \frac{25}{216} \cdot \frac{1}{1 - \dfrac{125}{216}} = \frac{25}{91} \approx 0.274725274725.$$

The sum of the three probabilities is $\dfrac{36 + 30 + 25}{91} = 1,$ exactly as it should be.

C11S03.071: The amount of light transmitted is

$$\frac{I}{2^4} + \frac{I}{2^6} + \frac{I}{2^7} + \frac{I}{2^{10}} + \cdots = I \cdot \frac{\dfrac{1}{16}}{1 - \dfrac{1}{4}} = \frac{I}{12},$$

$\dfrac{1}{12}$ of the incident light.

C11S03.072: The series for x does not converge, so x is not a number; computations with x have no meaning.

Section 11.4

C11S04.001: Because $f^{(n)}(x) = (-1)^n e^{-x}$, we see that $f^{(n)}(0) = (-1)^n$ if $n \geqq 0$. Thus

$$P_5(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \quad \text{and}$$

$$R_5(x) = \frac{x^6}{6!}e^{-z} \qquad \text{for some } z \text{ between 0 and } x.$$

C11S04.002: Given $f(x) = \sin x$ and $n = 4$, we have

$$\begin{aligned}
&f'(x) = \cos x && f'(0) = 1\\
&f''(x) = -\sin x && f''(0) = 0\\
&f^{(3)}(x) = -\cos x && f^{(3)}(0) = -1\\
&f^{(4)}(x) = \sin x && f^{(4)}(0) = 0\\
&f^{(5)}(x) = \cos x
\end{aligned}$$

Therefore

$$P_4(x) = x - \frac{x^3}{3!} \qquad \text{and} \qquad R_4(x) = \frac{x^5}{5!}\cos z$$

for some number z between 0 and x.

C11S04.003: Given $f(x) = \cos x$ and $n = 4$, we have

$$\begin{aligned}
&f'(x) = -\sin x && f'(0) = 0\\
&f''(x) = -\cos x && f''(0) = -1\\
&f^{(3)}(x) = \sin x && f^{(3)}(0) = 0\\
&f^{(4)}(x) = \cos x && f^{(4)}(0) = 1\\
&f^{(5)}(x) = -\sin x
\end{aligned}$$

Therefore

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \qquad \text{and} \qquad R_4(x) = -\frac{x^5}{5!}\sin z$$

for some number z between 0 and x.

C11S04.004: Given $f(x) = (1-x)^{-1}$ and $n = 4$, we compute

$$\begin{aligned}
&f'(x) = (1-x)^{-2} && f'(0) = 1\\
&f''(x) = 2(1-x)^{-3} && f''(0) = 2\\
&f^{(3)}(x) = 6(1-x)^{-4} && f^{(3)}(0) = 6
\end{aligned}$$

$$f^{(4)}(x) = 24(1-x)^{-5} \qquad f^{(4)}(0) = 24$$

$$f^{(5)}(x) = 120(1-x)^{-6}$$

Therefore

$$P_4(x) = 1 + x + x^2 + x^3 + x^4 \qquad \text{and} \qquad R_4(x) = \frac{x^5}{(1-z)^6}$$

for some number z between 0 and x.

C11S04.005: Given $f(x) = (1+x)^{1/2}$ and $n = 3$, we compute

$$f'(x) = \frac{1}{2(1+x)^{1/2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1+x)^{3/2}} \qquad f''(0) = -\frac{1}{4}$$

$$f^{(3)}(x) = \frac{3}{8(1+x)^{5/2}} \qquad f^{(3)}(0) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16(1+x)^{7/2}}$$

Therefore

$$P_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \qquad \text{and} \qquad R_3(x) = -\frac{5x^4}{128(1+z)^{7/2}}$$

for some number z between 0 and x.

C11S04.006: Given $f(x) = \ln(1+x)$ and $n = 4$, we find that

$$f'(x) = (1+x)^{-1} \qquad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2} \qquad f''(0) = -1$$

$$f^{(3)}(x) = 2(1+x)^{-3} \qquad f^{(3)}(0) = 2$$

$$f^{(4)}(x) = -6(1+x)^{-4} \qquad f^{(4)}(0) = -6$$

$$f^{(4)}(x) = 24(1+x)^{-5}$$

Therefore

$$P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \qquad \text{and} \qquad R_4(x) = \frac{x^5}{5(1+z)^5}$$

for some number z between 0 and x.

C11S04.007: Given $f(x) = \tan x$ and $n = 3$, we find that

$$f'(x) = \sec^2 x \qquad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \, \tan x \qquad f''(0) = 0$$

$$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \qquad\qquad f^{(3)}(0) = 2$$

$$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$$

Therefore

$$P_3(x) = x + \frac{x^3}{3} \qquad \text{and} \qquad R_3(x) = \frac{x^4}{4!}(16\sec^4 z \tan z + 8\sec^2 z \tan^3 z)$$

for some number z between 0 and x.

C11S04.008: Given $f(x) = \arctan x$ and $n = 2$, we compute

$$f'(x) = \frac{1}{1+x^2} \qquad\qquad f'(0) = 1$$

$$f''(x) = -\frac{2x}{(1+x^2)^2} \qquad\qquad f''(0) = 0$$

$$f^{(3)}(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$

Therefore

$$P_2(x) = x \qquad \text{and} \qquad R_2(x) = \frac{x^3(6z^2 - 2)}{3!(1+z^2)^3}$$

for some number z between 0 and x.

C11S04.009: Given $f(x) = \arcsin x$ and $n = 2$, we compute

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \qquad\qquad f'(0) = 1$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}} \qquad\qquad f''(0) = 0$$

$$f^{(3)}(x) = \frac{1+2x^2}{(1-x^2)^{5/2}}$$

Therefore

$$P_2(x) = x \qquad \text{and} \qquad R_2(x) = \frac{x^3(1+2z^2)}{3!(1-z^2)^{5/2}}$$

for some number z between 0 and x.

C11S04.010: Given $f(x) = x^3 - 3x^2 + 5x - 7$ and $n = 4$, we compute

$$f'(x) = 3x^2 - 6x + 5 \qquad\qquad f'(0) = 5$$

$$f''(x) = 6x - 6 \qquad\qquad f''(0) = -6$$

$$f^{(3)}(x) \equiv 6 \qquad\qquad f^{(3)}(0) = 6$$

$$f^{(4)}(x) \equiv 0 \qquad\qquad f^{(4)}(0) = 0$$

$$f^{(5)}(x) \equiv 0$$

Therefore

$$P_4(x) \;=\; -7 + 5x - 3x^2 + x^3 = f(x) \qquad \text{and} \qquad R_4(x) \;\equiv\; 0.$$

C11S04.011: Because $f^{(n)}(x) = e^x$ for all $n \geqq 0$, we have $f^{(n)}(1) = e$ for such n. Therefore

$$e^x = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e^z}{120}(x-1)^5$$

for some z between 1 and x.

C11S04.012: Given: $f(x) = \cos x$, $a = \pi/4$, and $n = 3$, we compute

$$\begin{aligned} f'(x) &= -\sin x & f'(a) &= -\frac{\sqrt{2}}{2} \\ f''(x) &= -\cos x & f''(a) &= -\frac{\sqrt{2}}{2} \\ f^{(3)}(x) &= \sin x & f^{(3)}(a) &= \frac{\sqrt{2}}{2} \\ f^{(4)}(x) &= \cos x \end{aligned}$$

Therefore

$$\cos x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\cos z}{24}\left(x - \frac{\pi}{4}\right)^4$$

for some number z between $\pi/4$ and x.

C11S04.013: Given: $f(x) = \sin x$, $a = \pi/6$, and $n = 3$. We compute

$$\begin{aligned} f'(x) &= \cos x & f'(a) &= \frac{\sqrt{3}}{2} \\ f''(x) &= -\sin x & f''(a) &= -\frac{1}{2} \\ f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= -\frac{\sqrt{3}}{2} \\ f^{(4)}(x) &= \sin x \end{aligned}$$

Therefore

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sin z}{24}\left(x - \frac{\pi}{6}\right)^4$$

for some number z between $\pi/6$ and x.

C11S04.014: Given $f(x) = x^{1/2}$, $a = 100$, and $n = 3$, we compute

$$f'(x) = \frac{1}{2x^{1/2}} \qquad f'(a) = \frac{1}{20}$$

$$f''(x) = -\frac{1}{4x^{3/2}} \qquad f''(a) = -\frac{1}{4000}$$

$$f^{(3)}(x) = \frac{3}{8x^{5/2}} \qquad f^{(3)}(a) = \frac{3}{800000}$$

$$f^{(4)}(x) = -\frac{15}{16x^{7/2}}$$

Therefore

$$\sqrt{x} = 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2 + \frac{1}{1600000}(x-100)^3 - \frac{5}{128z^{7/2}}(x-100)^4$$

for some number z between 100 and x. The Taylor polynomial of degree 3 shown here can be used to approximate square roots of numbers near 100 with some accuracy. For example,

$$\sqrt{101} \approx 10 + \frac{1}{20} - \frac{1}{8000} + \frac{1}{1600000} = \frac{16078901}{1600000} \approx 10.049875625.$$

Ths error in this approximation is less than 4×10^{-9}. To obtain an accurate upper bound for the error in such an approximation, use the remainder term.

C11S04.015: Given $f(x) = (x-4)^{-2}$, $a = 5$, and $n = 5$, we compute

$$f'(x) = -2(x-4)^{-3} \qquad f'(a) = -2$$

$$f''(x) = 6(x-4)^{-4} \qquad f''(a) = 6$$

$$f^{(3)}(x) = -24(x-4)^{-5} \qquad f^{(3)}(a) = -24$$

$$f^{(4)}(x) = 120(x-4)^{-6} \qquad f^{(4)}(a) = 120$$

$$f^{(5)}(x) = -720(x-4)^{-7} \qquad f^{(5)}(a) = -720$$

$$f^{(6)}(x) = 5040(x-4)^{-8}$$

Therefore

$$\frac{1}{(x-4)^2} = 1 - 2(x-5) + 3(x-5)^2 - 4(x-5)^3 + 5(x-4)^4 - 6(x-5)^5 + \frac{(x-5)^6}{720} \cdot \frac{5040}{(z-4)^8}$$

for some number z between 5 and x.

C11S04.016: Given $f(x) = \tan x$, $a = \pi/4$, and $n = 4$, we compute

$$f(x) = \tan x \qquad f(a) = 1$$

$$f'(x) = \sec^2 x \qquad f'(a) = 2$$

$$f''(x) = 2\sec^2 x \tan x \qquad f''(a) = 4$$

$$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \qquad f^{(3)}(a) = 16$$

$$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x \qquad f^{(4)}(a) = 80$$

$$f^{(5)}(x) = 16\sec^6 x + 88\sec^4 x\,\tan^2 x + 16\sec^2 x\,\tan^4 x$$

Therefore

$$\begin{aligned}\tan x \;=\; 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 \\ + \frac{1}{120}\left(x - \frac{\pi}{4}\right)^5 \cdot (16\sec^6 z + 88\sec^4 z\,\tan^2 z + 16\sec^2 z\,\tan^4 z)\end{aligned}$$

for some number z between $\pi/4$ and x.

C11S04.017: Given $f(x) = \cos x$, $a = \pi$, and $n = 4$, we compute

$$\begin{aligned}
f(x) &= \cos x & f(a) &= -1 \\
f'(x) &= -\sin x & f'(a) &= 0 \\
f''(x) &= -\cos x & f''(a) &= 1 \\
f^{(3)}(x) &= \sin x & f^{(3)}(a) &= 0 \\
f^{(4)}(x) &= \cos x & f^{(4)}(a) &= -1 \\
f^{(5)}(x) &= -\sin x
\end{aligned}$$

Therefore

$$\cos x = -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} - \frac{\sin z}{120}(x-\pi)^5$$

for some number z between π and x.

C11S04.018: Given $f(x) = \sin x$, $a = \pi/2$, and $n = 4$, we compute

$$\begin{aligned}
f(x) &= \sin x & f(a) &= 1 \\
f'(x) &= \cos x & f'(a) &= 0 \\
f''(x) &= -\sin x & f''(a) &= -1 \\
f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= 0 \\
f^{(4)}(x) &= \sin x & f^{(4)}(a) &= 1 \\
f^{(5)}(x) &= \cos x
\end{aligned}$$

Therefore

$$\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 + \frac{\cos z}{120}\left(x - \frac{\pi}{2}\right)^5$$

for some number z between $\pi/2$ and x.

C11S04.019: Given $f(x) = x^{3/2}$, $a = 1$, and $n = 4$, we compute

$$
\begin{aligned}
f(x) &= x^{3/2} & f(a) &= 1 \\
f'(x) &= \frac{3}{2}x^{1/2} & f'(a) &= \frac{3}{2} \\
f''(x) &= \frac{3}{4}x^{-1/2} & f''(a) &= \frac{3}{4} \\
f^{(3)}(x) &= -\frac{3}{8}x^{-3/2} & f^{(3)}(a) &= -\frac{3}{8} \\
f^{(4)}(x) &= \frac{9}{16}x^{-5/2} & f^{(4)}(a) &= \frac{9}{16} \\
f^{(5)}(x) &= -\frac{45}{32}x^{-7/2}
\end{aligned}
$$

Therefore

$$
x^{3/2} = 1 + \frac{3}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{3}{128}(x-1)^4 - \frac{(x-1)^5}{120} \cdot \frac{45}{32z^{7/2}}
$$

for some number z between 1 and x.

C11S04.020: Given $f(x) = (1-x)^{-1/2}$, $a = 0$, and $n = 4$, we compute

$$
\begin{aligned}
f(x) &= (1-x)^{-1/2} & f(a) &= 1 \\
f'(x) &= \frac{1}{2}(1-x)^{-3/2} & f'(a) &= \frac{1}{2} \\
f''(x) &= \frac{3}{4}(1-x)^{-5/2} & f''(a) &= \frac{3}{4} \\
f^{(3)}(x) &= \frac{15}{8}(1-x)^{-7/2} & f^{(3)}(a) &= \frac{15}{8} \\
f^{(4)}(x) &= \frac{105}{16}(1-x)^{-9/2} & f^{(4)}(a) &= \frac{105}{16} \\
f^{(5)}(x) &= \frac{945}{32}(1-x)^{-11/2}
\end{aligned}
$$

Therefore

$$
\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \frac{x^5}{120} \cdot \frac{945}{32(1-z)^{11/2}}
$$

for some number z between 0 and x.

C11S04.021: Substitution of $-x$ for x in the series in Eq. (19) yields

$$
e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.
$$

This representation is valid for all x.

C11S04.022: Substitution of $2x$ for x in the series in Eq. (19) yields

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \frac{32x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

This representation is valid for all x.

C11S04.023: Substitution of $-3x$ for x in the series in Eq. (19) yields

$$e^{-3x} = 1 - 3x + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{n!}.$$

This representation is valid for all x.

C11S04.024: Substitution of x^3 for x in the series in Eq. (19) yields

$$\exp\left(x^3\right) = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}.$$

This representation is valid for all x.

C11S04.025: Substitution of $2x$ for x in the series in Eq. (22) yields

$$\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}.$$

This representation is valid for all x.

C11S04.026: Substitution of $x/2$ for x in the series in Eq. (22) yields

$$\sin \frac{x}{2} = \frac{x}{2} - \frac{x^3}{3! \cdot 8} + \frac{x^5}{5! \cdot 32} - \frac{x^7}{7! \cdot 128} + \frac{x^9}{9! \cdot 512} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)! \cdot 2^{2n+1}}.$$

This representation is valid for all x.

C11S04.027: Substitution of x^2 for x in the series in Eq. (22) yields

$$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}.$$

This representation is valid for all x.

C11S04.028: Substitution of $2x$ for x in the series in Eq. (21) yields

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \frac{2x^{10}}{14175} - \frac{2x^{12}}{467775} + \cdots.$$

This representation is valid for all x.

C11S04.029: Given $f(x) = \ln(1+x)$ and $a = 0$, we compute:

$$f(x) = \ln(1+x) \qquad\qquad f(a) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(a) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(a) = -1$$

$$f^{(3)}(x) = \frac{2}{(1+x)^3} \qquad f^{(3)}(a) = 2$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4} \qquad f^{(4)}(a) = -6$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5} \qquad f^{(5)}(a) = 24$$

$$f^{(6)}(x) = -\frac{120}{(1+x)^6} \qquad f^{(6)}(a) = -120$$

Evidently $f^{(n)}(a) = (-1)^{n+1}(n-1)!$ if $n \geqq 1$. (For a proof, use proof by induction. We omit the proof to save space.) Therefore the Taylor series for $f(x)$ at $a = 0$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!x^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

This representation of $f(x) = \ln(1+x)$ is valid if $-1 < x \leqq 1$.

C11S04.030: Given $f(x) = 1/(1-x)$ and $a = 0$, we compute:

$$f(x) = \frac{1}{1-x} \qquad f(a) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(a) = 1$$

$$f''(x) = \frac{2}{(1-x)^3} \qquad f''(a) = 2$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4} \qquad f^{(3)}(a) = 6$$

$$f^{(4)}(x) = \frac{24}{(1-x)^5} \qquad f^{(4)}(a) = 24$$

$$f^{(5)}(x) = \frac{120}{(1-x)^6} \qquad f^{(5)}(a) = 120$$

$$f^{(6)}(x) = \frac{720}{(1-x)^7} \qquad f^{(6)}(a) = 720$$

Evidently $f^{(n)}(a) = n!$ if $n \geqq 0$. Therefore the Taylor series for $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{n!x^n}{n!} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots .$$

This representation of $f(x) = 1/(1-x)$ is valid if $-1 < x < 1$.

C11S04.031: If $f(x) = e^{-x}$, then $f^{(n)}(x) = (-1)^n e^{-x}$ for all $n \geqq 0$. With $a = 0$, this implies that $f^{(n)}(a) = (-1)^n$ for all $n \geqq 0$. Therefore the Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots .$$

This representation of $f(x) = e^{-x}$ is valid for all x.

C11S04.032: Given $f(x) = \sin x$ and $a = \pi/2$, we compute:

$$\begin{aligned}
f(x) &= \sin x & f(a) &= 1\\
f'(x) &= \cos x & f'(a) &= 0\\
f''(x) &= -\sin x & f''(a) &= -1\\
f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= 0\\
f^{(4)}(x) &= \sin x & f^{(4)}(a) &= 1\\
f^{(5)}(x) &= \cos x & f^{(5)}(a) &= 0\\
f^{(6)}(x) &= -\sin x & f^{(6)}(a) &= -1
\end{aligned}$$

Evidently $f^{(n)}(a) = 0$ if n is odd, whereas $f^{(n)}(a) = (-1)^{n/2}$ if n is even. After simplifications we find the Taylor series for $f(x)$ at a to be

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots .$$

This representation of $f(x) = \sin x$ is valid for all x.

C11S04.033: Given $f(x) = \ln x$ and $a = 1$, we compute:

$$\begin{aligned}
f(x) &= \ln x & f(a) &= 0\\
f'(x) &= \frac{1}{x} & f'(a) &= 1\\
f''(x) &= -\frac{1}{x^2} & f''(a) &= -1\\
f^{(3)}(x) &= \frac{2}{x^3} & f^{(3)}(a) &= 2\\
f^{(4)}(x) &= -\frac{6}{x^4} & f^{(4)}(a) &= -6\\
f^{(5)}(x) &= \frac{24}{x^5} & f^{(5)}(a) &= 24\\
f^{(6)}(x) &= -\frac{120}{x^6} & f^{(6)}(a) &= -120
\end{aligned}$$

We have here convincing evidence that if $n \geqq 1$, then $f^{(n)}(a) = (-1)^{n+1}(n-1)!$. (To prove this rigorously, use proof by induction; we omit any proof to save space.) Therefore the Taylor series for $f(x) = \ln x$ at $a = 1$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!(x-1)^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \cdots .$$

This representation of $f(x) = \ln x$ is valid if $0 < x \leqq 2$.

C11S04.034: Given $f(x) = e^{2x}$ and $a = 0$, we compute:

$$\begin{aligned}
f(x) &= e^{2x} & f(a) &= 1 \\
f'(x) &= 2e^{2x} & f'(a) &= 2 \\
f''(x) &= 4e^{2x} & f''(a) &= 4 \\
f^{(3)}(x) &= 8e^{2x} & f^{(3)}(a) &= 8 \\
f^{(4)}(x) &= 16e^{2x} & f^{(4)}(a) &= 16 \\
f^{(5)}(x) &= 32e^{2x} & f^{(5)}(a) &= 32 \\
f^{(6)}(x) &= 64e^{2x} & f^{(6)}(a) &= 64
\end{aligned}$$

It's clear that $f^{(n)}(a) = 2^n$ if $n \geqq 0$. Therefore the Taylor series for $f(x) = e^{2x}$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \frac{4x^5}{15} + \frac{4x^6}{45} + \frac{8x^7}{315} + \frac{2x^7}{315} + \cdots .$$

C11S04.035: Given $f(x) = \cos x$ and $a = \pi/4$, we compute:

$$\begin{aligned}
f(x) &= \cos x & f(a) &= \frac{\sqrt{2}}{2} \\
f'(x) &= -\sin x & f'(a) &= -\frac{\sqrt{2}}{2} \\
f''(x) &= -\cos x & f''(a) &= -\frac{\sqrt{2}}{2} \\
f^{(3)}(x) &= \sin x & f^{(3)}(a) &= \frac{\sqrt{2}}{2} \\
f^{(4)}(x) &= \cos x & f^{(4)}(a) &= \frac{\sqrt{2}}{2} \\
f^{(5)}(x) &= -\sin x & f^{(5)}(a) &= -\frac{\sqrt{2}}{2}
\end{aligned}$$

$$f^{(6)}(x) = -\cos x \qquad\qquad f^{(6)}(a) = -\frac{\sqrt{2}}{2}$$

It should be clear that

$$f^{(n)}(a) = \frac{\sqrt{2}}{2} \qquad \text{if } n \text{ is of the form } 4k \text{ or } 4k+3, \text{ whereas}$$

$$f^{(n)}(a) = -\frac{\sqrt{2}}{2} \qquad \text{if } n \text{ is of the form } 4k+1 \text{ or } 4k+2.$$

Therefore the Taylor series for $f(x) = \cos x$ at $a = \pi/4$ is

$$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2!\cdot 2}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{3!\cdot 2}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{4!\cdot 2}\left(x - \frac{\pi}{4}\right)^4 - \cdots.$$

This representation of $f(x) = \cos x$ is valid for all x.

C11S04.036: Given $f(x) = 1/(1-x)^2$ and $a = 0$, we compute:

$$\begin{aligned}
f(x) &= (1-x)^{-2} & f(a) &= 1\\
f'(x) &= 2(1-x)^{-3} & f'(a) &= 2\\
f''(x) &= 6(1-x)^{-4} & f''(a) &= 6\\
f^{(3)}(x) &= 24(1-x)^{-5} & f^{(3)}(a) &= 24\\
f^{(4)}(x) &= 120(1-x)^{-6} & f^{(4)}(a) &= 120\\
f^{(5)}(x) &= 720(1-x)^{-7} & f^{(5)}(a) &= 720\\
f^{(6)}(x) &= 5040(1-x)^{-8} & f^{(6)}(a) &= 5040
\end{aligned}$$

It should be clear that $f^{(n)}(a) = (n+1)!$ for $n \geqq 0$. (Prove this by induction; we omit the proof to save space.) Therefore the Taylor series for $f(x)$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{(n+1)!x^n}{n!} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + \cdots.$$

This representation of $f(x)$ is valid for $-1 < x < 1$.

C11S04.037: Given $f(x) = \dfrac{1}{x}$ and $a = 1$, we compute:

$$\begin{aligned}
f(x) &= \frac{1}{x} & f(a) &= 1\\
f'(x) &= -\frac{1}{x^2} & f'(a) &= -1\\
f''(x) &= \frac{2}{x^3} & f''(a) &= 2\\
f^{(3)}(x) &= -\frac{6}{x^4} & f^{(3)}(a) &= -6
\end{aligned}$$

$$f^{(4)}(x) = \frac{24}{x^5} \qquad f^{(4)}(a) = 24$$

$$f^{(5)}(x) = -\frac{120}{x^6} \qquad f^{(5)}(a) = -120$$

$$f^{(6)}(x) = \frac{720}{x^7} \qquad f^{(6)}(a) = 720$$

Clearly $f^{(n)}(a) = (-1)^n \cdot n!$ for $n \geqq 0$. Therefore the Taylor series for $f(x)$ at $a = 1$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-1)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-4)^4 - (x-1)^5 + (x-1)^6 - (x-1)^7 + \cdots.$$

This representation of $f(x)$ is valid for $0 < x < 2$.

C11S04.038: Given $f(x) = \cos x$ and $a = \pi/2$, we compute:

$$f(x) = \cos x \qquad f(a) = 0$$

$$f'(x) = -\sin x \qquad f'(a) = -1$$

$$f''(x) = -\cos x \qquad f''(a) = 0$$

$$f^{(3)}(x) = \sin x \qquad f^{(3)}(a) = 1$$

$$f^{(4)}(x) = \cos x \qquad f^{(4)}(a) = 0$$

$$f^{(5)}(x) = -\sin x \qquad f^{(5)}(a) = -1$$

$$f^{(6)}(x) = -\cos x \qquad f^{(6)}(a) = 0$$

Therefore the Taylor series for $f(x)$ at $x = a$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \left(x - \frac{\pi}{2}\right)^{2n-1} = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x - \frac{\pi}{2}\right)^7 - \cdots.$$

This representation of $f(x) = \cos x$ is valid for all x.

C11S04.039: Given $f(x) = \sin x$ and $a = \pi/4$, we compute:

$$f(x) = \sin x \qquad f(a) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \cos x \qquad f'(a) = \frac{\sqrt{2}}{2}$$

$$f''(x) = -\sin x \qquad f''(a) = -\frac{\sqrt{2}}{2}$$

$$f^{(3)}(x) = -\cos x \qquad f^{(3)}(a) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \sin x \qquad\qquad f^{(4)}(a) = \frac{\sqrt{2}}{2}$$

$$f^{(5)}(x) = \cos x \qquad\qquad f^{(5)}(a) = \frac{\sqrt{2}}{2}$$

$$f^{(6)}(x) = -\sin x \qquad\qquad f^{(6)}(a) = -\frac{\sqrt{2}}{2}$$

Therefore the Taylor series for $f(x) = \sin x$ at $a = \pi/4$ is

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2! \cdot 2}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{3! \cdot 2}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{4! \cdot 2}\left(x - \frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{5! \cdot 2}\left(x - \frac{\pi}{4}\right)^5 - \cdots .$$

This representation of $f(x) = \sin x$ is valid for all x.

C11S04.040: Given $f(x) = \sqrt{1+x}$ and $a = 0$, we compute:

$$f(x) = (1+x)^{1/2} \qquad\qquad f(a) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad\qquad f'(a) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \qquad\qquad f''(a) = -\frac{1}{4}$$

$$f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} \qquad\qquad f^{(3)}(a) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2} \qquad\qquad f^{(4)}(a) = -\frac{15}{16}$$

$$f^{(5)}(x) = \frac{105}{32}(1+x)^{-9/2} \qquad\qquad f^{(5)}(a) = \frac{105}{32}$$

$$f^{(6)}(x) = -\frac{945}{64}(1+x)^{-11/2} \qquad\qquad f^{(6)}(a) = -\frac{945}{64}$$

If $n \geqq 1$, the coefficient of x^n in the Taylor series for $f(x)$ is therefore

$$\frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{n! \cdot 2^n} = \frac{(-1)^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot 2 \cdot 4 \cdot 6 \cdots (2n-2)}{n! \cdot 2^n \cdot (n-1)! \cdot 2^{n-1}} = \frac{(-1)^{n+1} \cdot (2n-2)!}{n! \cdot (n-1)! \cdot 2^{2n-1}}.$$

Therefore the Taylor series for $f(x)$ at $x = a$ is

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (2n-2)! x^n}{n! \cdot (n-1)! \cdot 2^{2n-1}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \frac{33x^7}{2048} - \frac{429x^8}{32768} + \cdots .$$

This representation of $f(x) = \sqrt{1+x}$ is valid for $-1 < x < 1$. Numerical evidence suggests that it is not valid if $x = \pm 1$.

C11S04.041: Given $f(x) = \sin x$ and $a = 0$, we compute:

$$f(x) = \sin x \qquad\qquad f(a) = 0$$

$$
\begin{aligned}
f'(x) &= \cos x & f'(a) &= 1 \\
f''(x) &= -\sin x & f''(a) &= 0 \\
f^{(3)}(x) &= -\cos x & f^{(3)}(a) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(a) &= 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(a) &= 1 \\
f^{(6)}(x) &= -\sin x & f^{(6)}(a) &= 0
\end{aligned}
$$

Therefore Taylor's formula for $f(x)$ at $a = 0$ is

$$
f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + (-1)^{n+1} \frac{\sin z}{(2n+3)!} x^{2n+3} \tag{1}
$$

for some number z between 0 and x. Because $|\cos z\,| \leqq 1$ for all z, it follows from Eq. (18) of the text that the remainder term in Eq. (1) approaches zero as $n \to \infty$. Therefore the Taylor series of $f(x) = \sin x$ at $a = 0$ is

$$
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,
$$

and this representation is valid for all x.

C11S04.042: Assuming that termwise differentiation of these series is legitimate (it is), we have

$$
\begin{aligned}
D_x \cos x &= D_x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots \right) \\
&= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots = -\sin x
\end{aligned}
$$

and

$$
\begin{aligned}
D_x \sin x &= D_x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots \right) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots = \cos x.
\end{aligned}
$$

C11S04.043: Given $f(x) = \cosh x$, $g(x) = \sinh x$, and $a = 0$, we compute:

$$
\begin{aligned}
f(x) &= \cosh x & f(a) &= 1 \\
f'(x) &= \sinh x & f'(a) &= 0 \\
f''(x) &= \cosh x & f''(a) &= 1 \\
f^{(3)}(x) &= \sinh x & f^{(3)}(a) &= 0 \\
f^{(4)}(x) &= \cosh x & f^{(4)}(a) &= 1 \\
f^{(5)}(x) &= \sinh x & f^{(5)}(a) &= 0
\end{aligned}
$$

$$f^{(6)}(x) = \cosh x \qquad\qquad f^{(6)}(a) = 1$$

Evidently $f^{(n)}(a) = 1$ if n is even and $f^{(n)}(a) = 0$ if n is odd. Therefore the Maclaurin series for $f(x) = \cosh x$ is

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots . \tag{1}$$

The remainder term in Taylor's formula is

$$\frac{\sinh z}{(2n+1)!} x^{2n+1}$$

where z is between 0 and x. The remainder term approaches zero as $n \to +\infty$ by Eq. (18) of the text. Therefore the series in Eq. (1) converges to $f(x) = \cosh x$ for all x. Similarly,

$$\begin{aligned}
g(x) &= \sinh x & \qquad g(a) &= 0 \\
g'(x) &= \cosh x & \qquad g'(a) &= 1 \\
g''(x) &= \sinh x & \qquad g''(a) &= 0 \\
g^{(3)}(x) &= \cosh x & \qquad g^{(3)}(a) &= 1 \\
g^{(4)}(x) &= \sinh x & \qquad g^{(4)}(a) &= 0 \\
g^{(5)}(x) &= \cosh x & \qquad g^{(5)}(a) &= 1 \\
g^{(6)}(x) &= \sinh x & \qquad g^{(6)}(a) &= 0
\end{aligned}$$

It is clear that $g^{(n)}(a) = 0$ if n is even, whereas $g^{(n)} = 1$ if n is odd. Therefore the Maclaurin series for $g(x) = \sinh x$ is

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots . \tag{2}$$

This series converges to $g(x) = \sinh x$ for all x by an argument very similar to that given for the hyperbolic cosine series.

Next, substitution of ix for x yields

$$\cosh ix = 1 + \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} + \frac{(ix)^6}{6!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x.$$

Similarly, $\sinh ix = \sin x$. This is one way to describe the relationship of the hyperbolic functions to the circular functions. A more prosaic response to the concluding question in Problem 43 would be that if the signs in the Maclaurin series for the cosine function are changed so that they are all plus signs, you get the Maclaurin series for the hyperbolic cosine function; the same relation hold for the sine and hyperbolic sine series.

C11S04.044: First,

$$\begin{aligned}\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right)\\ &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) + \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots .\end{aligned}$$

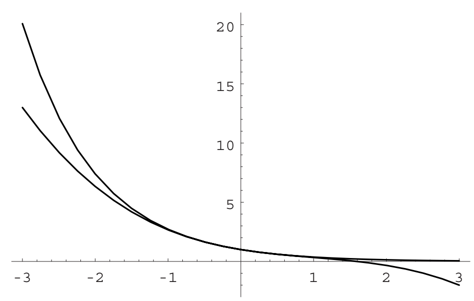
Similarly,

$$\begin{aligned}\sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)\\ &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) - \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots .\end{aligned}$$

C11S04.045: Given $f(x) = e^{-x}$, its plot together with that of

$$P_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

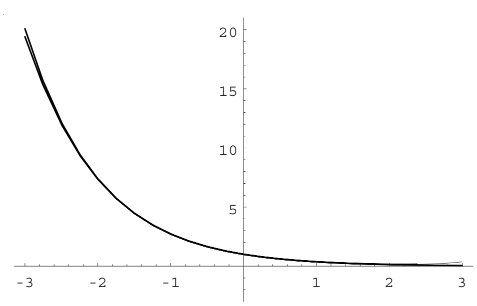
are shown next.



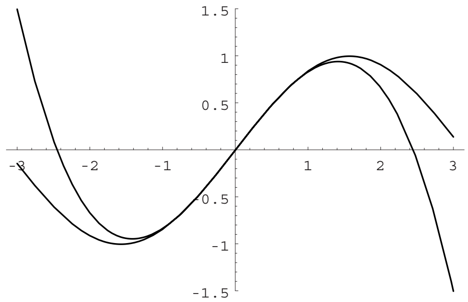
The graphs of $f(x) = e^{-x}$ and

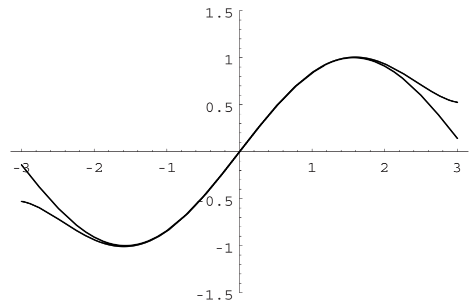
$$P_6(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!}$$

are shown together next.



C11S04.046: The graphs of $f(x) = \sin x$ and $P_3(x) = x - \frac{1}{6}x^3$ are shown together next, on the left; the graphs of f and $P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ are shown together on the right.

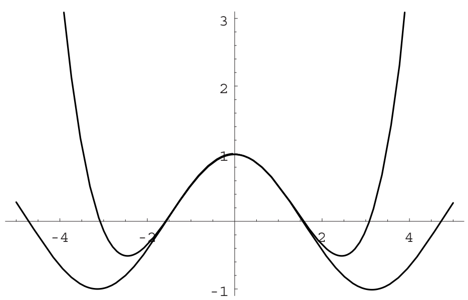


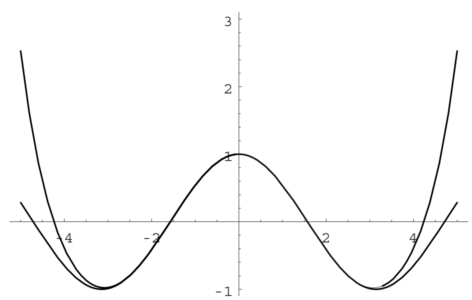


C11S04.047: Given $f(x) = \cos x$, two of its Taylor polynomials are

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{and} \quad P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

The graphs of f and P_4 are shown next, on the left; the graph of f and P_8 are on the right.

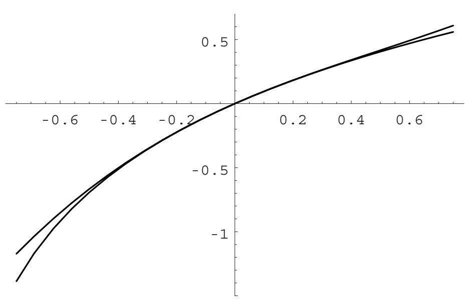


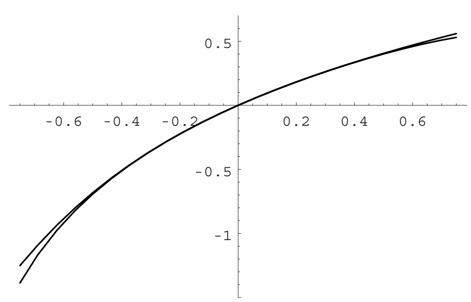


C11S04.048: Given $f(x) = \ln(1 + x)$, two of its Taylor polynomials are

$$P_2(x) = 1 - \frac{x^2}{2} \quad \text{and} \quad P_4(x) = 1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The graphs of f and P_2 are shown together next, on the left; the graphs of f and P_4 are on the right.

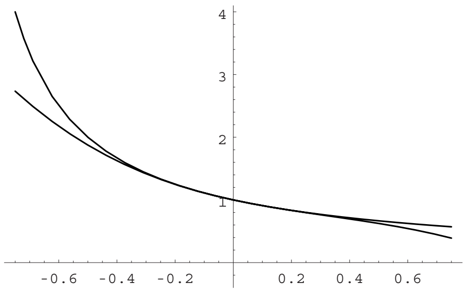


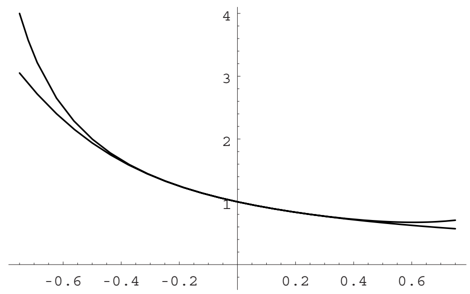


C11S04.049: Given $f(x) = \dfrac{1}{1+x}$, two of its Taylor polynomials are

$$P_3(x) = 1 - x + x^2 - x^3 \quad \text{and} \quad P_4(x) = 1 - x + x^2 - x^3 + x^4.$$

The graphs of f and P_3 are shown together next, on the left; the graphs of f and P_4 are on the right.

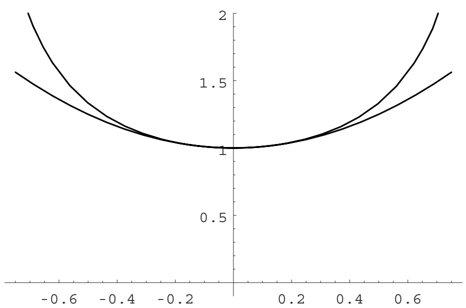


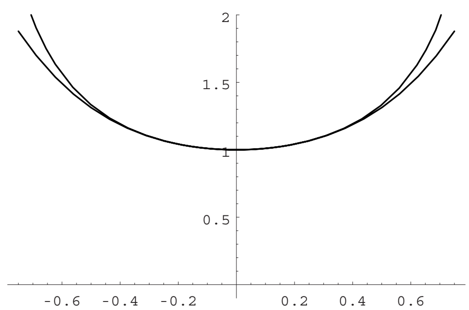


C11S04.050: Given $f(x) = \dfrac{1}{1-x^2}$, two of its Taylor polynomials are

$$P_3(x) = 1 + x^2 \quad \text{and} \quad P_6(x) = 1 + x^2 + x^4 + x^6.$$

The graphs of f and P_3 are shown together next, on the left; the graphs of f and P_6 are on the right.

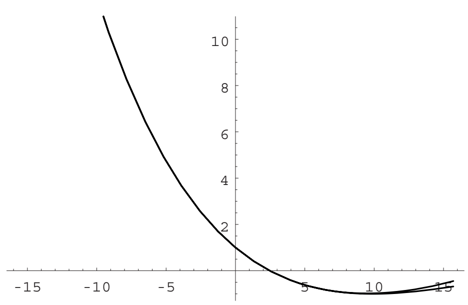




C11S04.051: The graph of the Taylor polynomial

$$P_4(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!}$$

of $f(x)$ and the graph of $g(x)$ are shown together, next.



C11S04.052: Given: $\alpha = \tan^{-1}(1/5)$.

Part (a): $\tan 2\alpha = \dfrac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{25}} = \dfrac{5}{12}.$

Part (b): $\tan 4\alpha = \dfrac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{25}{144}} = \dfrac{120}{119}.$

Part (c): $\tan\left(\dfrac{\pi}{4} - 4\alpha\right) = \dfrac{1 - \frac{120}{119}}{1 + \frac{120}{119}} = -\dfrac{1}{239}.$

Part (d): $\tan\left(\dfrac{\pi}{4} - 4\alpha\right) = -\dfrac{1}{239}$; $\quad \dfrac{\pi}{4} - 4\alpha = -\arctan\dfrac{1}{239}$; $\quad 4\arctan\dfrac{1}{5} - \arctan\dfrac{1}{239} = \dfrac{\pi}{4}$.

C11S04.053: We begin with the formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Let $A = \arctan x$ and $B = \arctan y$. Thus

$$\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}, \qquad \text{so that} \qquad \arctan x + \arctan y = \arctan\frac{x+y}{1-xy}$$

(if $xy < 1$). Thus

$$\arctan\frac{1}{2} + \arctan\frac{1}{5} = \arctan\frac{\frac{7}{10}}{\frac{9}{10}} = \arctan\frac{7}{9}.$$

Therefore

$$\arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8} = \arctan\frac{7}{9} + \arctan\frac{1}{8} = \arctan\frac{\frac{65}{72}}{\frac{65}{72}} = \arctan 1 = \frac{\pi}{4}.$$

C11S04.054: The first six terms of the series in (27) give $a = \arctan\frac{1}{5} \approx 0.1973955598$ with ten-place accuracy. The first two terms of that series give $b = \arctan\frac{1}{239} \approx 0.0041840760$ with ten-place accuracy. Then $16a - 4b \approx 3.141592653$ is in error in only the last digit as an approximation to $\pi \approx 3.141592654$.

If we use instead the approximation

$$\arctan x \ = \ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots - \frac{x^{47}}{47} + \frac{x^{49}}{49},$$

then substitution of $x = \dfrac{1}{5}$ yields (to the number of digits shown)

$$a \approx 0.19739555984988075837004976519479029349010164238671$$

and substitution of $x = \dfrac{1}{239}$ yields (again, to the number of digits shown)

$$b \approx 0.00418407600207472386453821495928545274104806530763.$$

Then

$$16a - 4b \approx 3.14159265358979323846264338327950288487743410695687;$$

—compare this with

$$\pi \approx 3.14159265358979323846264338327950288419716939937510582$$

(also accurate to the number of digits shown here). The error in this last approximation is less than 7×10^{-37}.

C11S04.055: Prove that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for every real number x.

Proof: Suppose that x is a real number. Choose the integer k so large that $k > |2x|$. Let $L = |x|^k/k!$. Suppose that $n = k + 1$. Then

$$\frac{|x|^n}{n!} = \frac{|x|^{k+1}}{(k+1)!} = \frac{|x|^k}{k!} \cdot \frac{|x|}{k+1} < \frac{L}{2} = \frac{L}{2^{n-k}}$$

because $|2x| < k < k + 1$ and $n - k = 1$. Next, assume that

$$\frac{|x|^m}{m!} < \frac{L}{2^{m-k}}$$

for some integer $m > k$. Then

$$\frac{|x|^{m+1}}{(m+1)!} = \frac{|x|^m}{m!} \cdot \frac{|x|}{m+1} < \frac{L}{2^{m-k}} \cdot \frac{1}{2} = \frac{L}{2^{m+1-k}}$$

because $|2x| < k < m$. Therefore, by induction,

$$\frac{|x|^n}{n!} < \frac{L}{2^{n-k}}$$

for every integer $n > k$. Now let $n \to +\infty$ to conclude that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

C11S04.056: Suppose that $0 < x \leqq 1$. Then because

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \frac{(-1)^{n+1}t^{n+1}}{1+t},$$

we have

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \frac{(-1)^{n+1}t^{n+1}}{1+t}\right) dt;$$

$$\Big[\ln(1+t)\Big]_0^x = \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots + (-1)^n \frac{t^{n+1}}{n+1}\right]_0^x + R_n \qquad (1)$$

where

$$R_n = \int_0^x \frac{(-1)^{n+1}t^{n+1}}{1+t}\,dt.$$

Now

$$|R_n| \leqq \int_0^x t^{n+1}\,dt = \frac{x^{n+2}}{n+2},$$

and therefore $R_n \to 0$ as $n \to +\infty$. Thus upon evaluation of Eq. (1), we find that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + R_n,$$

but because $R_n \to 0$ as $n \to +\infty$, we may now conclude that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

if $0 \leqq x \leqq 1$.

C11S04.057: By Theorem 4 of Section 11.3, S is not a number. Thus attempts to do "arithmetic" with S are meaningless and may lead to all sorts of absurd results.

C11S04.058: Replacement of x with $-x$ in the result in Problem 56 yields

$$\begin{aligned}\ln(1-x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-x)^n}{n} = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{n} + \cdots \\ &= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.\end{aligned}$$

Therefore

$$\begin{aligned}\ln\frac{1+x}{1-x} &= \ln(1+x) \ - \ \ln(1-x) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right) \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots\right) = \sum_{n \text{ odd}} \frac{2x^n}{n}\end{aligned}$$

if $0 \leqq x \leqq 1$.

C11S04.059: Results: With $x = 1$ in the Maclaurin series in Problem 56, we find that

$$a = \sum_{n=1}^{50} \frac{(-1)^{n+1}}{n} \approx 0.68324716057591818842565811649.$$

With $x = \frac{1}{3}$ in the second series in Problem 58, we find that

$$b = \sum_{\substack{n=1 \\ n \text{ odd}}}^{49} \frac{2}{n \cdot 3^n} \approx 0.69314718055994530941723210107.$$

Because $|a - \ln 2| \approx 0.009900019984$, whereas $|b - \ln 2| \approx 2.039 \times 10^{-26}$, it is clear that the second series of Problem 58 is far superior to the series of Problem 56 for the accurate approximation of $\ln 2$.

Section 11.5

C11S05.001: $\displaystyle\int_0^\infty \frac{x}{x^2+1}\,dx = \left[\frac{1}{2}\ln(x^2+1)\right]_0^\infty = +\infty.$ Therefore $\displaystyle\sum_{n=1}^\infty \frac{n}{n^2+1}$ diverges.

C11S05.002: $\displaystyle\int_0^\infty x\exp(-x^2)\,dx = \left[-\frac{1}{2}\exp(-x^2)\right]_0^\infty = \frac{1}{2} < +\infty.$ Therefore

$$\sum_{n=1}^\infty \frac{n}{\exp(n^2)}$$

converges. The *Mathematica* 3.0 command

```
NSum[ n/Exp[n*n], {n, 1, Infinity}, WorkingPrecision → 32 ]
```

yields the result that the sum of this series is approximately 0.4048813985713107.

C11S05.003: $\displaystyle\int_0^\infty (x+1)^{-1/2}\,dx = \left[2(x+1)^{1/2}\right]_0^\infty = +\infty.$ Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n+1}}$ diverges.

C11S05.004: $\displaystyle\int_0^\infty (x+1)^{-4/3}\,dx = \left[-\frac{3}{(x+1)^{1/3}}\right]_0^\infty = 3 < +\infty.$ Therefore

$$\sum_{n=1}^\infty \frac{1}{(n+1)^{4/3}}$$

converges. The *Mathematica* 3.0 command

```
NSum[ 1/(n + 1)∧(4/3), {n, 1, Infinity}, WorkingPrecision → 32 ]
```

reveals that the sum of the given series is approximately 2.6009377504588624.

C11S05.005: $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx = \left[\arctan x\right]_0^\infty = \frac{\pi}{2} < +\infty.$ Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges.

This is a special case of series 6.1.32 in Eldon R. Hansen, *A Table of Series and Products*, Prentice-Hall Inc. (Englewood Cliffs, N.J.), 1975. According to Hansen, its sum is

$$-\frac{1}{2} + \frac{\pi}{2}\coth\pi \approx 1.07667404746858117413405079475 0.$$

Mathematica 3.0 reports that the sum of the first 1,000,000 terms of this series is approximately 1.07667.

C11S05.006: The method of partial fractions yields

$$\int_1^\infty \frac{1}{x(x+1)}\,dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$
$$= \left[\ln\frac{x}{x+1}\right]_1^\infty = \left(\lim_{x\to\infty} \ln\frac{x}{x+1}\right) - \ln\frac{1}{2} = \ln 1 - \ln\frac{1}{2} = \ln 2 < +\infty.$$

Therefore the given series converges. In Example 3 of Section 11.3 we found that its sum is 1.

C11S05.007: $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \Big[\ln(\ln x)\Big]_2^\infty = +\infty$. Therefore $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln n}$ diverges.

C11S05.008: $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^\infty = +\infty$. Therefore $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n}$ diverges.

C11S05.009: $\displaystyle\int_0^\infty 2^{-x}\,dx = \left[-\frac{1}{2^x\ln 2}\right]_0^\infty = \frac{1}{\ln 2} < +\infty$. Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{2^n}$ converges (to 1).

C11S05.010: Let $u = x$ and $dv = e^{-x}\,dx$. Then $du = dx$ and we may choose $v = -e^{-x}$. Thus

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

Therefore

$$\int_0^\infty xe^{-x}\,dx = \Big[-(x+1)e^{-x}\Big]_0^\infty = 1 < +\infty,$$

and thus $\displaystyle\sum_{n=1}^\infty \frac{n}{e^n}$ converges. To find its sum, note that

$$\sum_{n=1}^\infty \frac{n}{e^n} = \frac{1}{e}\sum_{n=1}^\infty n\left(\frac{1}{e}\right)^{n-1} = \frac{1}{e}f(x)$$

where $x = 1/e$ and

$$f(x) = \sum_{n=1}^\infty nx^{n-1}.$$

Now $f(x) = g'(x)$ where

$$g(x) = \sum_{n=1}^\infty x^n = \frac{x}{1-x}.$$

It follows that

$$f(x) = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

Therefore

$$\sum_{n=1}^\infty \frac{n}{e^n} = \frac{1}{e}f\left(\frac{1}{e}\right) = \frac{e}{(e-1)^2} \approx 0.920673594207792319.$$

Alternatively, the *Mathematica* 3.0 command

```
Sum[ n/Exp[n], {n, 1, Infinity} ]
```

almost immediately produces the same exact value of the sum.

C11S05.011: For each positive integer n, let

$$I_n = \int x^n e^{-x}\,dx.$$

Let $u = x^n$ and $dv = e^{-x}\,dx$. Then $du = nx^{n-1}\,dx$; choose $v = -e^{-x}$. Then

$$I_n = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx = -x^n e^{-x} + nI_{n-1}.$$

Therefore

$$\int x^2 e^{-x}\,dx = I_2 = -x^2 e^{-x} + 2I_1 = -x^2 e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\,dx\right) = -(x^2 + 2x + 2)e^{-x} + C.$$

Hence

$$\int_0^\infty x^2 e^{-x}\,dx = -\Big[(x^2 + 2x + 2)e^{-x}\Big]_0^\infty = 2 < +\infty,$$

and so $\displaystyle\sum_{n=1}^\infty \frac{n^2}{e^n}$ converges. It can be shown that its sum is

$$\frac{e(e+1)}{(e-1)^3} \approx 1.992294767125.$$

C11S05.012: $\displaystyle\int_1^\infty \frac{1}{17x - 13}\,dx = \left[\frac{1}{17}\ln(17x - 13)\right]_1^\infty = +\infty$. Therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{17n - 13}$ diverges.

C11S05.013: Choose $u = \ln x$ and $dv = \dfrac{1}{x^2}\,dx$. Then $du = \dfrac{1}{x}\,dx$; choose $v = -\dfrac{1}{x}$. Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Thus

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \left[-\frac{1 + \ln x}{x}\right]_1^\infty = \frac{1+0}{1} - \lim_{x\to\infty}\frac{1 + \ln x}{x} = 1 - \lim_{x\to\infty}\frac{1}{x} = 1 - 0 = 1 < +\infty$$

(we used l'Hôpital's rule to find the limit). Therefore $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2}$ converges.

C11S05.014: Because

$$\int_1^\infty \frac{x+1}{x^2}\,dx = \int_1^\infty \left(\frac{1}{x} + \frac{1}{x^2}\right)dx = \left[-\frac{1}{x} + \ln x\right]_1^\infty = +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{n+1}{n^2}$ diverges.

C11S05.015: Because

$$\int_0^\infty \frac{x}{x^4 + 1}\,dx = \left[\frac{1}{2}\arctan\left(x^2\right)\right]_0^\infty = \frac{\pi}{4} < +\infty,$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

With the aid of *Mathematica* 3.0 and Theorem 2, we find that the sum of this series is approximately 0.694173022150715 (only the last digit shown here is in doubt; it may round to 6 instead of 5).

C11S05.016: Because

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^3+x}\,dx &= \int_1^{\infty} \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \\ &= \left[(\ln x) - \frac{1}{2}\ln\left(x^2+1\right)\right]_1^{\infty} = \frac{1}{2}\left[\ln \frac{x^2}{x^2+1}\right]_1^{\infty} = \frac{1}{2}\ln 2 < +\infty,\end{aligned}$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

Use of a *Mathematica* command similar to that in the solutions of Problems 2 and 4 reveals that the sum of the given series is approximately 0.671865985524009838. *Mathematica* 2.2 cannot find the exact sum, but a *Mathematica* 3.0 command similar to the one in the solution of Problem 10 returns the exact value

$$\text{EulerGamma} + \frac{\text{PolyGamma}(0,\ 1-i)\ +\ \text{PolyGamma}(0,\ 1+i)}{2}.$$

Here, `EulerGamma` is Euler's constant $\gamma \approx 0.577216$ and `PolyGamma[n,z]` returns the nth derivative of the digamma function $\psi(z)$. (We mention all this to give you a reference point at which to begin further research if you wish.)

C11S05.017: Because

$$\int_1^{\infty} \frac{2x+5}{x^2+5x+17}\,dx = \Big[\ln(x^2+5x+17)\Big]_1^{\infty} = +\infty,$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{n^2+5n+17}$ diverges.

C11S05.018: Integration by parts (as in Example 1 of Section 8.3) yields

$$\int_1^{\infty} \ln\left(\frac{x+1}{x}\right) dx = \Big[(x+1)\ln(x+1) - x\ln x\Big]_1^{\infty} = \left[\ln \frac{(x+1)^{x+1}}{x^x}\right]_1^{\infty}.$$

Now

$$\frac{(x+1)^{x+1}}{x^x} = (x+1)\left(\frac{x+1}{x}\right)^x = (x+1)\left(1+\frac{1}{x}\right)^x.$$

Because

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e, \quad \text{it follows that} \quad \lim_{x\to\infty}\frac{(x+1)^{x+1}}{x^x} = +\infty.$$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges. See also Problem 52 of Section 11.3.

C11S05.019: Choose $u = \ln(1 + x^{-2})$ and $dv = dx$. Then

$$du = \frac{-2x^{-3}}{1 + x^{-2}}\,dx = -\frac{2}{x^3 + x}\,dx;$$

choose $v = x$. Thus

$$\begin{aligned}\int_1^\infty \ln\left(1 + \frac{1}{x^2}\right)\,dx &= \Big[x\ln(1 + x^{-2})\Big]_1^\infty + 2\int_1^\infty \frac{1}{1 + x^2}\,dx \\ &= \lim_{x\to\infty} \frac{\ln(1 + x^{-2})}{1/x} - \ln 2 + \Big[2\arctan x\Big]_1^\infty \\ &= \left[\lim_{z\to 0^+} \frac{\ln(1 + z^2)}{z}\right]_1^\infty - \ln 2 + \pi - \frac{\pi}{2} = \frac{\pi}{2} - \ln 2 < +\infty\end{aligned}$$

(use l'Hôpital's rule to evaluate the last limit). Therefore $\displaystyle\sum_{n=1}^\infty \ln\left(1 + \frac{1}{n^2}\right)$ converges.

C11S05.020: Because

$$\int_1^\infty \frac{2^{1/x}}{x^2}\,dx = \left[-\frac{2^{1/x}}{\ln 2}\right]_1^\infty = \frac{2}{\ln 2} - \frac{1}{\ln 2} = \frac{1}{\ln 2} < +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{2^{1/n}}{n^2}$ converges. *Mathematica* 3.0 reports that its sum is approximately 2.8069937050197894.

C11S05.21: Because

$$\int_1^\infty \frac{x}{4x^2 + 5}\,dx = \left[\frac{1}{8}\ln(4x^2 + 5)\right]_1^\infty = +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{n}{4n^2 + 5}$ diverges.

C11S05.022: Because

$$\int_1^\infty \frac{x}{(4x^2 + 5)^{3/2}}\,dx = \left[-\frac{1}{4\sqrt{4x^2 + 5}}\right]_1^\infty = \frac{1}{12} < +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{n}{(4n^2 + 5)^{3/2}}$ converges. Its sum (via *Mathematica*) is about 0.1030899515824624.

C11S05.023: Because

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx = \int_2^\infty \frac{(\ln x)^{-1/2}}{x}\,dx = \Big[2(\ln x)^{1/2}\Big]_2^\infty = +\infty,$$

the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln n}}$ diverges.

C11S05.024: Because

$$\int_2^\infty \frac{1}{x}(\ln x)^{-3}\,dx = \left[-\frac{1}{2(\ln x)^2}\right]_2^\infty = \frac{1}{2(\ln 2)^2} < +\infty,$$

the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges. Its sum (via *Mathematica*) is approximately 2.06588653888413525.

C11S05.025: The substitution $u = 2x$ and integral formula 17 of the endpapers of the text yields

$$\int \frac{1}{4x^2+9}\,dx = \frac{1}{2}\int \frac{1}{u^2+9}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right) + C.$$

Therefore

$$\int_1^\infty \frac{1}{4x^2+9}\,dx = \left[\frac{1}{6}\arctan\left(\frac{2x}{3}\right)\right]_1^\infty = \frac{1}{6}\cdot\frac{\pi}{2} - \frac{1}{6}\arctan\left(\frac{2}{3}\right) < +\infty,$$

and thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+9}$ converges.

This series is a special case of Eq. (6.1.32) of Eldon R. Hansen's *A Table of Series and Products*, Prentice-Hall, Inc. (Englewood Cliffs, N.J.), 1975. *Mathematica* 3.0 summed this series in a fraction of a second and obtained the same answer as Hansen, viz.,

$$-\frac{1}{18} + \frac{\pi}{12}\coth\left(\frac{3\pi}{2}\right) \approx 0.20628608982235128529.$$

C11S05.026: First,

$$\int_1^\infty \frac{x+1}{x+100}\,dx = \int_1^\infty \left(1 - \frac{99}{x+100}\right)dx = \Big[x - 99\ln(x+100)\Big]_1^\infty = +\infty$$

because, by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{x}{99\ln(x+100)} = \lim_{x\to\infty} \frac{x+100}{99} = +\infty.$$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n+100}$ diverges.

C11S05.027: Because

$$\int_1^\infty \frac{x}{(x^2+1)^2}\,dx = \left[-\frac{1}{2(x^2+1)}\right]_1^\infty = \frac{1}{4} < +\infty,$$

the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1}$ converges.

The *Mathematica* command

```
NSum[ n/(n∧4 + 2*n∧2 + 1), { n, 1, Infinity }, WorkingPrecision -> 28 ] // Timing
```

when executed on a Power Macintosh 7600/120 yielded the approximate sum 0.39711677137965943 in 3.45 seconds.

C11S05.028: Because

$$\int_1^\infty (x+1)^{-3}\,dx = \left[-\frac{1}{2}(x+1)^{-2}\right]_1^\infty = \frac{1}{8} < +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{1}{(n+1)^3}$ converges.

The sum of this series is $\zeta(3) - 1$, where the *Riemann zeta function* $\zeta(k)$ is defined for integers $k \geqq 2$ by the formula

$$\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}.$$

It is known that $\zeta(3) \approx 1.2020569031595942854$, so the sum of the series in this problem is approximately 0.2020569031595942854.

C11S05.029: Because

$$\int_1^\infty \frac{\arctan x}{x^2+1}\,dx = \left[\frac{1}{2}(\arctan x)^2\right]_1^\infty = \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3}{32}\pi^2 < +\infty,$$

the series $\displaystyle\sum_{n=1}^\infty \frac{\arctan n}{n^2+1}$ converges.

C11S05.030: Because

$$\int_3^\infty \frac{1}{x(\ln x)\,[\ln(\ln x)]}\,dx = \Big[\ln(\ln(\ln x))\Big]_3^\infty = +\infty,$$

the series $\displaystyle\sum_{n=3}^\infty \frac{1}{n(\ln n)\,[\ln(\ln n)]}$ diverges.

C11S05.031: The integral test cannot be applied because this is not a positive-term series. In Section 11.7 you will see how to prove that it converges, and in Problem 61 there you will see that its sum is $-\ln 2$.

C11S05.032: This is not a positive-term series; $\sin n$ is negative for infinitely many positive integral values of n. (Exercise: Prove this assertion.) After you study absolute convergence in Section 11.7, you will see how to apply the integral test to prove that this series converges. The *Mathematica* 3.0 command

```
N[ Sum[ N[ Exp[-n]*Sin[n], 100 ], {n, 1, 100} ], 100 ]          (1)
```

asks the program to compute the first hundred terms of the terms while carrying 100-digit accuracy, add them, and report the answer (also to 100-digit accuracy, although now only the first 98 can be trusted). Because $|\sin x| \leqq 1$ for all x, the number

$$\sum_{n=101}^\infty e^{-n} = \frac{1}{(e-1)e^{100}} \approx 2.165 \times 10^{-44}$$

overestimates the error in approximating the sum of the entire series with the partial sum in (1). Hence we can rely on more than 40 decimal digits of accuracy in the response to the command in (1): The sum of the series in Problem 32 is approximately 0.4195697895124155513.

C11S05.033: The terms of this series are not monotonically decreasing. For specific examples, if we let $a_n = (2 + \sin n)/(n^2)$, then

$$a_5 \approx 0.0416430290 < 0.0477940139 \approx a_6, \qquad a_{11} \approx 0.0082645437 < 0.0101626881 \approx a_{12},$$

$$a_{18} \approx 0.0038549776 < 0.0059553385 \approx a_{19}, \qquad a_{24} \approx 0.0019000376 < 0.0029882372 \approx a_{25},$$

$$a_{30} \approx 0.0011244093 < 0.0016607309 \approx a_{31}, \qquad a_{37} \approx 0.0009908414 < 0.0015902829 \approx a_{38}.$$

In Section 11.6 you will see how to use the *comparison test* to prove that this series converges.

C11S05.034: The terms of this series are not monotonically decreasing. For specific examples, let $a_n = [(\sin n)/n]^4$. Then

$$a_3 \approx 0.0000048963 < 0.0012814163 \approx a_4, \qquad a_7 \approx 0.0000775950 < 0.0002339130 \approx a_8,$$

$$a_{10} \approx 0.0000087592 < 0.0000682987 \approx a_{11}, \qquad a_{16} \approx 0.0000001048 < 0.0000102286 \approx a_{17},$$

$$a_{19} \approx 0.0000000039 < 0.0000043417 \approx a_{20}, \qquad a_{28} \approx 0.0000000088 < 0.0000002742 \approx a_{29}.$$

After you study the comparison test in Section 11.6, you will know how to prove that this series converges. Granted that it does converge, we can combine the integral test remainder estimate and the comparison test to obtain a fairly accurate approximation to its sum. Because $0 \leqq \sin^4 x \leqq 1$ for all x, we see that

$$\sum_{n=1}^{1000} \left(\frac{\sin n}{n}\right)^4 < \sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^4 < \left[\sum_{n=1}^{1000} \left(\frac{\sin n}{n}\right)^4\right] + \int_{1000}^{\infty} \frac{1}{x^4}\,dx = \left[\sum_{n=1}^{1000} \left(\frac{\sin n}{n}\right)^4\right] + \frac{1}{3000000000}.$$

The leftmost member of the preceding inequality can be accurately approximated by means of the *Mathematica* 3.0 command

```
N[ Sum[ N[ ((Sin[n])/n)∧4, {n, 1, 1000} ], 100 ]
```

and thus we find that the sum of the original series is approximately 0.547197551.

C11S05.035: There is some implication that we are to use the integral test to solve this problem. Hence we consider only the case in which $p > 0$. And if $p \neq 1$, then

$$\int_1^{\infty} p^{-x}\,dx = \left[-\frac{p^{-x}}{\ln p}\right]_{x=1}^{\infty} = \frac{1}{p \ln p} - \lim_{x\to\infty} \frac{1}{p^x \ln p}. \tag{1}$$

If $0 < p < 1$, then the limit in (1) is $+\infty$. If $p > 1$ then the limit in (1) is 0. If $p = 1$ then the series diverges because

$$\int_1^{\infty} \frac{1}{p^x}\,dx = \int_1^{\infty} 1\,dx = +\infty.$$

Answer: The series diverges if $0 < p \leqq 1$ and converges if $p > 1$.

C11S05.036: Clearly the given series diverges if $p \leqq 0$, so we assume that $p > 0$. If $p \neq 1$ then

$$\int_1^{\infty} \frac{x}{(x^2+1)^p}\,dx = \left[\frac{(x^2+1)^{1-p}}{2(1-p)}\right]_1^{\infty} = \left[-\frac{1}{2(p-1)(x^2+1)^{p-1}}\right]_1^{\infty}.$$

This improper integral diverges if $0 < p < 1$ and converges if $p > 1$. If $p = 1$ then

$$\int_1^\infty \frac{x}{(x^2+1)^p}\,dx = \left[\frac{1}{2}\ln(x^2+1)\right]_1^\infty = +\infty,$$

and therefore $\displaystyle\sum_{n=1}^\infty \frac{n}{(n^2+1)^p}$ diverges if $p \leqq 1$ and converges if $p > 1$.

C11S05.037: If $p = 1$ then

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_2^\infty \frac{1}{x\ln x}\,dx = \Big[\ln(\ln x)\Big]_2^\infty = +\infty.$$

Otherwise,

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_2^\infty \frac{(\ln x)^{-p}}{x}\,dx = \left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^\infty = \left[-\frac{1}{(p-1)(\ln x)^{p-1}}\right]_2^\infty.$$

So this improper integral diverges if $p < 1$. If $p > 1$ then it converges to

$$\frac{1}{(p-1)(\ln 2)^{p-1}} < +\infty.$$

Therefore the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ diverges if $p \leqq 1$ and converges if $p > 1$.

C11S05.038: If $p = 1$ then

$$\int_3^\infty \frac{1}{x(\ln x)\,[\ln(\ln x)]^p}\,dx = \int_3^\infty \frac{1}{x(\ln x)\,[\ln(\ln x)]}\,dx = \Big[\ln(\ln(\ln x))\Big]_3^\infty = +\infty.$$

If $p < 0$ then the series clearly diverges. If $p = 0$ then it is the series of Problem 7, which we have already shown to be divergent by the integral test. Thus we suppose that $p > 0$ and $p \neq 1$. Then

$$\int_3^\infty \frac{1}{x(\ln x)\,[\ln(\ln x)]^p}\,dx = \left[\frac{[\ln(\ln x)]^{1-p}}{1-p}\right]_3^\infty = \left[-\frac{1}{(p-1)\,[\ln(\ln x)]^{p-1}}\right]_3^\infty.$$

Therefore the improper integral diverges if $0 < p < 1$ and converges if $p > 1$. So the given series

$$\sum_{n=3}^\infty \frac{1}{n(\ln n)\,[\ln(\ln n)]^p}$$

diverges if $p \leqq 1$ and converges if $p > 1$. When $p = 1$ it is a very slowly divergent series; the *Mathematica* 3.0 command

```
NSum[ 1/(n*(Log[n])*(Log[Log[n]])), {n, 3, 1000000000}, WorkingPrecision -> 28 ]
```

yielded the partial sum 5.77285617911296384. When $p = 2$ it converges to a surprisingly large value; the *Mathematica* command

```
NSum[ 1/(n*(Log[n])*(Log[Log[n]])^2), {n, 3, Infinity}, WorkingPrecision -> 28 ]
```

returned the approximate sum 38.4067680928217863.

C11S05.039: We require $R_n < 0.0001$. This will hold provided that

$$\int_n^\infty \frac{1}{x^2}\,dx < 0.0001$$

because R_n cannot exceed the integral. So we require

$$\left[-\frac{1}{x}\right]_n^\infty < 0.0001;$$

that is, that $n > 10000$.

C11S05.040: We require $R_n < 0.00005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^2}\,dx < 0.00005$$

because R_n cannot exceed the integral. So we require

$$\left[-\frac{1}{x}\right]_n^\infty < 0.00005;$$

that is, that $n > 20000$.

C11S05.041: We require $R_n < 0.00005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^3}\,dx < 0.00005$$

because R_n cannot exceed the integral. So we require

$$\left[-\frac{1}{2x^2}\right]_n^\infty < 0.00005;$$

$$\frac{1}{2n^2} < 0.00005;$$

$$2n^2 > 20000;$$

thus we require that $n > 100$.

C11S05.042: We require $R_n < 2 \times 10^{-11}$. This will hold provided that

$$\int_n^\infty \frac{1}{x^6}\,dx < 2 \times 10^{-11}$$

because R_n cannot exceed the integral by Theorem 2. So we require

$$\left[-\frac{1}{5x^5}\right]_n^\infty < 2 \times 10^{-11},$$

and it follows that $n^5 > 10^{10}$, and thus that $n > 100$. The exact value of the sum is

$$\sum_{n=1}^\infty \frac{1}{n^6} = \zeta(6) = \frac{\pi^6}{945} \approx 1.017343061984.$$

C11S05.043: We require $R_n < 0.005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^{3/2}}\,dx < 0.005;$$

that is, provided that

$$\left[-\frac{2}{x^{1/2}}\right]_n^\infty < 0.005,$$

so that $n^{1/2} > 400$, and thus $n > N = 160000$. *Mathematica* 3.0 reports that

$$S_N = \sum_{n=1}^{N} \frac{1}{n^{3/2}} \approx 2.607375356498 \quad \text{and that} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \approx 2.612375348685.$$

Note that $S - S_N \approx 0.004999992187 < 0.005$.

C11S05.044: We require $R_n < 0.0005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2} < 0.0005,$$

so that $n > 31.6228$. Choose $N = 32$. Then *Mathematica* 3.0 reports that

$$S_N = \sum_{n=1}^{N} \frac{1}{n^3} \approx 1.201583642358 \quad \text{and that} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202056903160.$$

Note that $S - S_N \approx 0.000473260802 < 0.0005$.

The exact value of the sum of the given series is denoted by $\zeta(3)$, where ζ is the *Riemann zeta function*; it is discussed briefly in the Project that follows Section 11.5. It is known that

$$\zeta(3) \approx 1.202056903159594285399738161511449990764986292340498881792,$$

correct to the number of digits shown here.

C11S05.045: We require $R_n < 0.000005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^5}\,dx = \frac{1}{4n^4} < 0.000005,$$

so that $n > 14.9535$. Choose $N = 15$. Then *Mathematica* 3.0 reports that

$$S_N = \sum_{n=1}^{N} \frac{1}{n^5} \approx 1.036923438841 \quad \text{and that} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.036927755143.$$

Note that $S - S_N \approx 0.000004316302 < 0.000005$.

C11S05.046: We require $R_n < 0.00000005$. This will hold provided that

$$\int_n^\infty \frac{1}{x^7}\,dx = \frac{1}{6n^6} < 0.00000005,$$

so that $n > 12.2221$. Choose $N = 13$. Then *Mathematica* 3.0 reports that

$$S_N = \sum_{n=1}^{N} \frac{1}{n^7} \approx 1.008349250111 \quad \text{and that} \quad S = \sum_{n=1}^{\infty} \frac{1}{n^7} \approx 1.008349277382.$$

Note that $S - S_N \approx 0.00000002727 < 0.00000005$. The exact value of the sum of the given series is denoted by $\zeta(7)$; see the concluding remarks in the solution of Problem 44.

C11S05.047: If $p = 1$, then

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \int_1^\infty \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^\infty = +\infty,$$

so in this case the given series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ diverges. Otherwise (with the aid of *Mathematica* 3.0 for the antiderivative)

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \left[\frac{x^{1-p}\ln x - x^{1-p}}{(1-p)^2}\right]_1^\infty = \left[\frac{-1+\ln x}{(p-1)^2x^{p-1}}\right]_1^\infty.$$

Thus if $p < 1$ the given series diverges, whereas if $p > 1$ it converges. Answer: $p > 1$.

C11S05.048: We may assume that $p > 0$. If $p = e$ then

$$\int_1^\infty \frac{1}{p^{\ln x}}\,dx = \int_1^\infty \frac{1}{x}\,dx = \Big[\ln x\Big]_1^\infty = +\infty,$$

so in this case the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p^{\ln n}}$ diverges. Otherwise

$$\int_1^\infty \frac{1}{p^{\ln x}}\,dx = \left[\frac{x}{(1-\ln p)p^{\ln x}}\right]_1^\infty.$$

If $p > e$, then $p^{\ln x} > e^{\ln x} = x$, and thus

$$\left(\lim_{x\to\infty} \frac{x}{(1-\ln p)p^{\ln x}}\right) - \frac{1}{1-\ln p} = 0 + \frac{1}{-1+\ln p} < +\infty.$$

If $0 < p < e$, then $p^{\ln x} < e^{\ln x} = x$, and in this case

$$\lim_{x\to\infty} \frac{x}{(1-\ln p)p^{\ln x}} = +\infty.$$

Therefore the given series converges exactly when $p > e$.

C11S05.049: From the proof of Theorem 1 (the integral test), we see that if

$$a_n = \frac{1}{n}, \quad f(x) = \frac{1}{x}, \quad \text{and} \quad S_n = \sum_{k=1}^{n} a_n$$

for each integer $n \geqq 1$, then

$$S_n \geqq \int_1^{n+1} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{n+1} = \ln(n+1)$$

and

$$S_n - a_1 \leqq \int_1^n \frac{1}{x}\,dx = \Big[\ln x\Big]_1^n = \ln n.$$

Therefore

$$\ln n < \ln(n+1) \leqq S_n \leqq 1 + \ln n;$$

put another way,

$$\ln n \leqq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leqq 1 + \ln n$$

for every integer $n \geqq 1$. So if a computer adds a million terms of the harmonic series every second, the partial sum S_n will first reach 50 when $n \leqq e^{50} \leqq e \cdot n$. This means that n must satisfy the inequalities

$$1 + \llbracket e^{49} \rrbracket \leqq n \leqq \llbracket e^{50} \rrbracket\,;$$

that is,

$$1907346572495099690526 \leqq n \leqq 5184705528587072464087.$$

Divide the smaller of these bounds by one million (additions the computer carries out each second), then by 3600 to convert to hours, by 24 and then by 365.242199 to convert to years, and finally by 100 to convert to an answer: It will require over 604414 centuries. For a more precise answer, if $N = 2911002088526872100231$, then *Mathematica* 3.0 reports that

$$\sum_{n=1}^{N-1} \frac{1}{n} \approx 49.9999999999999999999999713 \qquad \text{and}$$

$$\sum_{n=1}^{N} \frac{1}{n} \approx 50.0000000000000000000000057.$$

After converting to centuries as before, we finally get the "right" answer: It will require a little over 922460 centuries.

C11S05.050: Part (a): Subtraction of $\ln n$ from each member of the inequality in Problem 49 yields

$$0 \leqq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n \leqq 1.$$

Part (b): First,

$$\frac{1}{n+1} \leqq \int_n^{n+1} \frac{1}{x}\,dx = \Big[\ln x\Big]_n^{n+1} = \ln(n+1) - \ln n;$$

that is,

$$\ln(n+1) - \ln n - \frac{1}{n+1} \geqq 0$$

for every positive integer n. Therefore

$$c_n - c_{n+1} = -\ln n - \frac{1}{n+1} + \ln(n+1) \geqq 0$$

for such n, and therefore the sequence $\{c_n\}$ is a decreasing sequence. Because it is bounded, it converges by the bounded monotonic sequence property discussed in Section 11.2 of the text. Its limit, *Euler's constant,* is denoted by γ, so that

$$\gamma = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right) \approx 0.5772156649015328606065120900824.$$

C11S05.051: Suppose that f is continuous and $f(x) > 0$ for all $x \geqq 1$. For each positive integer n, let

$$b_n = \int_1^n f(x)\,dx.$$

Part (a): Note that the sequence $\{b_n\}$ is increasing. Suppose that it is bounded, so that

$$B = \lim_{n\to\infty} b_n$$

exists. The definition of the value of an improper integral then implies that

$$\int_1^\infty f(x)\,dx = \lim_{\alpha\to\infty} \int_1^\alpha f(x)\,dx. \tag{1}$$

Therefore, by Theorem 4 in Section 11.2,

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} b_n = B.$$

Part (b): If the increasing sequence $\{b_n\}$ is not bounded, then by Problem 52 of Section 11.2,

$$\lim_{n\to\infty} b_n = +\infty.$$

Then Eq. (1) implies that

$$\int_1^\infty f(x)\,dx = +\infty$$

because $\displaystyle\int_1^\alpha f(x)\,dx$ is an increasing function of α.

Section 11.6

C11S06.001: The series

$$\sum_{n-1}^{\infty} \frac{1}{n^2+n+1} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges because it is a p-series with $p = 2 > 1$. Therefore the dominated series also converges.

C11S06.002: The series

$$\sum_{n=1}^{\infty} \frac{n^3+1}{n^4+2}$$

diverges by limit-comparison with the harmonic series, demonstrated by the computation

$$\frac{\dfrac{n^3+1}{n^4+2}}{\dfrac{1}{n}} = \frac{n^4+n}{n^4+2} = \frac{1+\dfrac{1}{n^3}}{1+\dfrac{2}{n^4}} \to \frac{1+0}{1+0} = 1$$

as $n \to +\infty$.

C11S06.003: The series

$$\sum_{n=1}^{\infty} \frac{1}{n+n^{1/2}}$$

diverges by limit-comparison with the harmonic series, demonstrated by the computation

$$\frac{\dfrac{1}{n+n^{1/2}}}{\dfrac{1}{n}} = \frac{n}{n+n^{1/2}} = \frac{1}{1+\dfrac{1}{n^{1/2}}} \to \frac{1}{1+0} = 1$$

as $n \to +\infty$.

C11S06.004: The series

$$\sum_{n=1}^{\infty} \frac{1}{n+n^{3/2}} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$$

and the latter converges because it is a p-series with $p = \frac{3}{2} > 1$. Therefore the dominated series also converges. The *Mathematica* 3.0 command

```
NSum[ 1/(n + n∧(3/2)), {n, 1, Infinity}, WorkingPrecision → 29 ]
```

yields the information that

$$\sum_{n=1}^{\infty} \frac{1}{n+n^{3/2}} \approx 1.68400947026785195.$$

C11S06.005: The series

$$\sum_{n=1}^{\infty} \frac{1}{1+3^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{3^n},$$

and the latter converges because it is a geometric series with ratio $\frac{1}{3} < 1$. Therefore the dominated series also converges.

C11S06.006: The series

$$\sum_{n=1}^{\infty} \frac{10n^2}{n^4+1} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{10}{n^2},$$

and the latter converges because it is a constant multiple of the p-series with $p = 2 > 1$. Therefore the dominated series also converges. A *Mathematica* 3.0 command similar to the one in the solution of Problem 4 yields the approximation

$$\sum_{n=1}^{\infty} \frac{10n^2}{n^4+1} \approx 11.2852792472431008541.$$

C11S06.007: The series

$$\sum_{n=2}^{\infty} \frac{10n^2}{n^3-1}$$

diverges by limit-comparison with the harmonic series, demonstrated by the computation

$$\frac{\dfrac{10n^2}{n^3-1}}{\dfrac{1}{n}} = \frac{10n^3}{n^3-1} = \frac{10}{1-\dfrac{1}{n^3}} \to \frac{10}{1-0} = 10$$

as $n \to +\infty$.

C11S06.008: First note that if $n \geqq 1$, then $\dfrac{n^2-n}{n^4+2} \leqq \dfrac{n^2}{n^4} = \dfrac{1}{n^2}$. Therefore the series

$$\sum_{n=1}^{\infty} \frac{n^2-n}{n^4+2} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and the latter series converges because it is a p-series with $p = 2 > 1$. Therefore the dominated series also converges. A *Mathematica* 3.0 command similar to those in the solutions of Problems 4 and 6 yields the approximate sum 0.4266730151703227152.

C11S06.009: First note that if $n \geqq 1$, then $\dfrac{1}{\sqrt{37n^3+3}} \leqq \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$. Therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{37n^3+3}} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$$

and the latter series converges because it is a p-series with $p = \frac{3}{2} > 1$. Therefore the dominated series also converges.

C11S06.010: The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

diverges by limit-comparison with the harmonic series, demonstrated by the computation

$$\frac{\dfrac{1}{\sqrt{n^2+1}}}{\dfrac{1}{n}} = \frac{n}{(n^2+1)^{1/2}} = \frac{1}{\left(1+\dfrac{1}{n^2}\right)^{1/2}} \to \frac{1}{\sqrt{1+0}} = 1$$

as $n \to +\infty$. The divergence of the given series, like that of the harmonic series, is quite slow. *Mathematica* 3.0 reports that the sum of its first million terms is only about 14.01049385339896.

C11S06.011: Because $\dfrac{\sqrt{n}}{n^2+n} \leqq \dfrac{n^{1/2}}{n^2} = \dfrac{1}{n^{3/2}}$, the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$$

which converges because it is a p-series with $p = \frac{3}{2} > 1$. Therefore the dominated series also converges.

C11S06.012: The series

$$\sum_{n=1}^{\infty} \frac{1}{3+5^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{5^n},$$

and the latter series converges because it is geometric with ratio $\frac{1}{5} < 1$. Thus the dominated series also converges. Its sum, elicited by a *Mathematica* 3.0 command similar to those used in earlier solutions, is approximately 0.170518822699190828424247791489.

C11S06.013: First we need a lemma: $\ln x < x$ if $x > 0$.

Proof: Let $f(x) = x - \ln x$. Then

$$f'(x) = 1 - \frac{1}{x}.$$

Because $f'(x) < 0$ if $0 < x < 1$, $f'(1) = 0$, and $f'(x) > 0$ if $1 < x$, the graph of $y = f(x)$ has a global minimum value at $x = 1$. Its minimum is $f(1) = 1 - \ln 1 = 1 > 0$, and $f(x) \geqq f(1)$ if $x > 0$. Therefore $f(x) > 0$ for all $x > 0$; that is, $\ln x < x$ if $x > 0$. ◀

Therefore the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \quad \text{dominates} \quad \sum_{n=2}^{\infty} \frac{1}{n}.$$

The latter series diverges because it is "eventually the same" as the harmonic series, and therefore the dominating series also diverges.

C11S06.014: Note that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

This result can be derived by a single application of l'Hôpital's rule; it is also a consequence of other earlier results and problems. Therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{n - \ln n}$$

diverges by limit comparison with the harmonic series, demonstrated by the computation

$$\lim_{n\to\infty} \frac{\dfrac{1}{n - \ln n}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{n - \ln n} = \lim_{n\to\infty} \frac{1}{1 - \dfrac{\ln n}{n}} = \frac{1}{1-0} = 1.$$

As one would expect—as a consequence of the limit-comparison test—the divergence of the given series is quite slow. The sum of its first thousand terms is only about 8.76261.

C11S06.015: Because $0 \leqq \sin^2 n \leqq 1$ for every integer $n \geqq 1$, the series

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2+1} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The latter series converges because it is a p-series with $p = 2 > 1$. Therefore the dominated series also converges.

C11S06.016: Because $0 \leqq \cos^2 n \leqq 1$ for every integer $n \geqq 1$, the series

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{3^n}.$$

The latter series converges because it is a geometric series with ratio $\frac{1}{3} < 1$. Therefore the dominated series also converges by the comparison test.

C11S06.017: First we need a lemma: If n is a positive integer, then $n < 2^n$.

Proof: The lemma is true for $n = 1$ because $1 < 2$, so that $1 < 2^1$. Suppose that $k < 2^k$ for some integer $k \geqq 1$. Then

$$2^{k+1} = 2 \cdot 2^k \geqq 2 \cdot k = k + k \geqq k + 1.$$

Thus whenever the lemma holds for the integer $k \geqq 1$, it also hold for $k+1$. Therefore, by induction, $n < 2^n$ for every integer $n \geqq 1$. ◀

Next, note that as a consequence of the lemma,

$$\frac{n+2^n}{n+3^n} \leqq \frac{2^n+2^n}{3^n} = \frac{2 \cdot 2^n}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n.$$

Therefore the series

$$\sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} 2 \cdot \left(\frac{2}{3}\right)^n.$$

The latter series converges because it is geometric with ratio $\frac{2}{3} < 1$. Therefore the dominated series also converges by the comparison test.

C11S06.018: The given series

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{3^n},$$

which converges because it is geometric with ratio $\frac{1}{3} < 1$. Therefore the dominated series converges by the comparison test. Its sum is between

$$S_{100} = \sum_{n=1}^{100} \frac{1}{2^n + 3^n} \quad \text{and} \quad S_{100} + T \quad \text{where} \quad T = \sum_{n=101}^{\infty} \frac{1}{3^n} = \frac{2}{3^{100}}.$$

Because $0 < T < 4 \times 10^{48}$, all 29 digits of the *Mathematica*-generated approximation

$$S_{100} \approx 0.32135438719750624899165047695$$

are accurate as an approximation to the sum S of the given series.

C11S06.019: Because $\dfrac{1}{n^2 \ln n} \leqq \dfrac{1}{n^2}$ if $n \geqq 3$, the given series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \quad \text{is eventually dominated by} \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

The latter series converges because it is eventually the same as the p-series with $p = 2 > 1$. Therefore the dominated series converges by the comparison test. See the discussion of "eventual domination" following the proof of Theorem 1 (the comparison test) in Section 11.6 of the text.

C11S06.020: Because $\dfrac{1}{n^{1+\sqrt{n}}} \leqq \dfrac{1}{n^2}$ if $n \geqq 1$, the given series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\sqrt{n}}} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The latter series converges because it is the p-series with $p = 2 > 1$. Therefore the dominated series converges by the comparison test; *Mathematica* 3.0 reports that its sum is approximately 1.2619486400097.

C11S06.021: First, a lemma: There is a positive integer K such that $\ln n \leqq \sqrt{n}$ if n is a positive integer and $n \geqq K$.

Proof: We use l'Hôpital's rule:

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0.$$

Therefore there exists a positive integer K such that $\ln n \leqq \sqrt{n}$ if $n \geqq K$. ◀

Alternatively, let $f(x) = x^{1/2} - \ln x$. Apply methods of calculus to show that $f'(x) < 0$ if $0 < x < 4$, $f'(4) = 0$, and $f'(x) > 0$ if $x > 4$. It follows that $f(x) \geqq f(4) = 2 - \ln 2 > 0$ for all $x > 0$, and hence $x^{1/2} > \ln x$ for all $x > 0$. Thus the integer K of the preceding proof may be chosen to be 1. But relying only on the lemma, we now conclude that

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \quad \text{is eventually dominated by} \quad \sum_{n=1}^{\infty} \frac{n^{1/2}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

The last series converges because it is the p-series with $p = \frac{3}{2} > 1$. Therefore the dominated series converges by the comparison test.

C11S06.022: The series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n}$ diverges by limit-comparison with the harmonic series, shown by the computation

$$\lim_{n\to\infty} \frac{\dfrac{\arctan n}{n}}{\dfrac{1}{n}} = \lim_{n\to\infty} \arctan n = \frac{\pi}{2}.$$

C11S06.023: Because $0 \leqq \sin^2(1/n) \leqq 1$ for every positive integer n, the given series

$$\sum_{n=1}^{\infty} \frac{\sin^2(1/n)}{n^2} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The latter series converges because it is the p-series with $p = 2 > 1$. Therefore the dominated series also converges by the comparison test.

C11S06.024: The given series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges by limit-comparison with the harmonic series, demonstrated by the computation

$$\lim_{n\to\infty} \frac{\dfrac{e^{1/n}}{n}}{\dfrac{1}{n}} = \lim_{n\to\infty} e^{1/n} = \lim_{u\to 0^+} e^u = e^0 = 1.$$

C11S06.025: We showed in the solution of Problem 13 that $\ln n \leqq n$ for every positive integer n. We showed in the solution of Problem 17 that $n \leqq 2^n$ for every positive integer n. Therefore

$$\sum_{n=1}^{\infty} \frac{\ln n}{e^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{2^n}{e^n} = \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n.$$

The last series converges because it is geometric with ratio $2/e < 1$. Therefore the dominated series converges by the comparison test.

C11S06.026: The series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+3n}$ diverges by limit-comparison with the harmonic series, shown by the computation

$$\frac{\dfrac{n^2+2}{n^3+3n}}{\dfrac{1}{n}} = \frac{n^3+2n}{n^3+3n} = \frac{1+\dfrac{2}{n^2}}{1+\dfrac{3}{n^2}} \to \frac{1+0}{1+0} = 1$$

as $n \to +\infty$.

C11S06.027: The given series

$$\sum_{n=1}^{\infty} \frac{n^{3/2}}{n^2+4} \quad \text{diverges by limit-comparison with} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$

shown by the computation

$$\lim_{n\to\infty} \frac{\dfrac{n^{3/2}}{n^2+4}}{\dfrac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = \lim_{n\to\infty} \frac{1}{1+\dfrac{4}{n^2}} = \frac{1}{1+0} = 1;$$

note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges because it is a p-series with $p = \frac{1}{2} \leqq 1$.

C11S06.028: The given series

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{2^n},$$

which converges because it is a geometric series with ratio $\frac{1}{2} < 1$. Therefore the dominated series converges by the comparison test. To find its sum, observe that

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{x^n}{n} = f(x)$$

where

$$x = \frac{1}{2} \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}.$$

(We will see in Section 11.8 the conditions under which this last "term-by-term" differentiation of a series in powers of x is valid. It is valid in this case provided that $-1 < x < 1$. We evaluated the sum of the last series using the fact that it is geometric with ratio x.) We now see by antidifferentiation that

$$f(x) = C - \ln(1-x); \quad 0 = f(0) = C - \ln 1 = C, \quad \text{and hence} \quad f(x) = -\ln(1-x)$$

if $-1 < x < 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = f\left(\frac{1}{2}\right) = -\ln\left(\frac{1}{2}\right) = \ln 2. \tag{1}$$

The series in (1) converges quite rapidly in the sense that you don't need to add a huge number of terms to get good approximations to its sum. For example,

$$\sum_{n=1}^{100} \frac{1}{n \cdot 2^n} \approx 0.6931471805599453094172321214581688, \tag{2}$$

which agrees with the exact decimal expansion of $\ln 2$ in the first 31 digits to the right of the decimal. By contrast, the better known series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,$$

which also converges to $\ln 2$ (see Problem 61 in Section 11.7), does so much more slowly. For example,

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} \approx 0.692647,$$

and only the first two digits to the right of the decimal are correct even though we summed ten times as many terms as in (2).

C11S06.029: First note that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \tag{1}$$

diverges because it is a p-series with $p = \frac{1}{2} \leqq 1$. Therefore the given series

$$\sum_{n=1}^{\infty} \frac{3}{4 + \sqrt{n}}$$

diverges by limit-comparison with the series in (1), as shown by the computation

$$\lim_{n\to\infty} \frac{\dfrac{3}{4 + n^{1/2}}}{\dfrac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{3n^{1/2}}{4 + n^{1/2}} = \lim_{n\to\infty} \frac{3}{\dfrac{4}{n^{1/2}} + 1} = \frac{3}{0+1} = 3.$$

C11S06.030: First observe that

$$\frac{n^2+1}{e^n(n+1)^2} = \frac{n^2+1}{e^n(n^2+2n+1)} \leqq \frac{n^2+1}{e^n(n^2+1)} = \frac{1}{e^n}$$

for each integer $n \geqq 1$. Therefore the given series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{e^n(n+1)^2} \quad \text{converges by comparison with} \quad \sum_{n=1}^{\infty} \frac{1}{e^n};$$

the latter series converges because it is geometric with ratio $1/e < 1$. The original series can be summed exactly using the *Mathematica* 3.0 command

```
Sum[ (n∧2 + 1)/((Exp[n])∗(n + 1)∧2), {n, 1, Infinity} ]
```

—the result (after simplifications) is

$$\frac{1}{e-1} - 2e + 2e\ln(e-1) + 2e\text{Li}_2\left(\frac{1}{e}\right) \approx 0.31057878433676.$$

Here, $\text{Li}_n(z)$ is the *polylogarithm function*, written `PolyLog[n, z]` in *Mathematica* and defined by

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

This result suggests that the sum of the original series in Problem 30 cannot be expressed exactly in terms of elementary functions. By contrast, the sum of the related series

$$\sum_{n=1}^{\infty} \frac{n^2+1}{e^n(n+1)}$$

is elementary and can be found by hand without great difficulty by using techniques of Section 11.9. You might later enjoy verifying our result, which was

$$\frac{1}{(e-1)^2} - 2 + 2e[1 - \ln(e-1)] \approx 0.8323135130814966353 2.$$

C11S06.031: First note that

$$\frac{2n^2-1}{n^2 \cdot 3^n} \leqq \frac{2n^2}{n^2 \cdot 3^n} = \frac{2}{3^n}$$

for each positive integer n. Therefore the given series

$$\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2 \cdot 3^n} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{2}{3^n}.$$

The latter series converges because it is geometric with ratio $\frac{1}{3} < 1$. Therefore the series of Problem 31 converges by the comparison test.

C11S06.032: First note that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \tag{1}$$

converges because it is a p-series with $p = \frac{4}{3} > 1$. Therefore the given series

$$\sum_{n=1}^{\infty} \frac{1}{(2n^4+1)^{1/3}}$$

converges by limit-comparison with the series in (1), shown by the following computation:

$$\lim_{n\to\infty} \frac{\dfrac{1}{(2n^4+1)^{1/3}}}{\dfrac{1}{n^{4/3}}} = \lim_{n\to\infty} \frac{n^{4/3}}{(2n^4+1)^{1/3}} = \lim_{n\to\infty} \frac{1}{\left(2 + \dfrac{1}{n^4}\right)^{1/3}} = \frac{1}{(2+0)^{1/3}} = \frac{1}{2^{1/3}}.$$

The sum of the original series is reported by *Mathematica* 3.0 to be approximately 2.754012386799936.

C11S06.033: Because $1 \leqq 2 + \sin n \leqq 3$ for each integer $n \geqq 1$, the given series

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{3}{n^2}.$$

The latter series converges because it is a constant multiple of the p-series with $p = 2 > 1$. Therefore the dominated series converges as well by the comparison test.

C11S06.034: In the solution of Problem 13 we showed that $\ln x < x$ for all $x > 0$. Therefore the given series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The latter series converges because it is the p-series with $p = 2 > 1$. Therefore the dominated series converges by the comparison test. *Mathematica* 3.0 can sum the original series exactly (in some sense); the command

```
Sum[ (Log[n])/(n∧3), {n, 2, Infinity} ]
```

elicits the response $-\zeta'(3)$, approximately 0.19812624288564; the *Riemann zeta function* $\zeta(z)$ is discussed briefly in the project that follows Section 11.5.

C11S06.035: The given series

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$$

diverges by limit-comparison with the harmonic series, demonstrated by the following computation:

$$\lim_{n\to\infty} \frac{\dfrac{(n+1)^n}{n^{n+1}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

C11S06.036: First note that $0 \leqq \sin^4 n \leqq 1$ for each integer $n \geqq 1$. Therefore

$$\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^4 = \sum_{n=1}^{\infty} \frac{\sin^4 n}{n^4} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

The latter series converges because it is the p-series with $p = 4 > 1$. Therefore the dominated series converges by the comparison test. The dominated series has sum approximately 0.5471975512. Compare this series and this result with those in the solution of Problem 34 of Section 11.5.

C11S06.037: The sum of the first ten terms of the given series is

$$S_{10} = \sum_{n=1}^{10} \frac{1}{n^2+1} = \frac{1662222227}{1693047850} \approx 0.981792822335.$$

The error in using S_{10} to approximate the sum S of the infinite series is

$$S - S_{10} = R_{10} \leqq \int_{10}^{\infty} \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.099668652491.$$

Because $S \approx 1.076674047469$, the true value of the error is approximately 0.09488123.

C11S06.038: The sum of the first ten terms of the given series is

$$S_{10} = \sum_{n=1}^{10} \frac{1}{3^n+1} = \frac{76943801855199427217}{190429124708983981100} \approx 0.404054799773.$$

The error in using S_{10} to approximate the sum S of the infinite series is

$$S - S_{10} = R_{10} \leqq \sum_{n=11}^{\infty} \frac{1}{3^n} = \frac{2}{3^{10}} \approx 0.000033870176.$$

Because $S \approx 0.40406326728086 1808$, the true value of the error is approximately 0.00000847.

C11S06.039: The sum of the first ten terms of the given series is

$$\sum_{n=1}^{10} \frac{\cos^2 n}{n^2} \approx 0.528869678057.$$

The error in using S_{10} to approximate the sum S of the infinite series is

$$S - S_{10} = R_{10} \leqq \int_{10}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{10}^{\infty} = \frac{1}{10} = 0.1.$$

Because $S \approx 0.574137740053$, the true value of the error is approximately 0.04526806.

C11S06.040: The sum of the first ten terms of the given series is

$$S_{10} = \sum_{n=2}^{11} \frac{1}{(n+1)(\ln n)^2} \approx 1.224893289245.$$

The error in using S_{10} to approximate the sum S of the infinite series is

$$S - S_{10} = R_{10} \leqq \int_{11}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \left[-\frac{1}{\ln x}\right]_{11}^{\infty} = \frac{1}{\ln 11} \approx 0.417032391424.$$

Because $S \approx 1.625972613903$, the true value of the error is approximately 0.40107933.

C11S06.041: The sum of the series is

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx 0.686503342339,$$

and $S - 0.005 \approx 0.681503342339$. Because

$$\sum_{n=1}^{9} \frac{1}{n^3+1} \approx 0.680981 < S - 0.005 < 0.681980 = \sum_{n=1}^{10} \frac{1}{n^3+1},$$

the smallest positive integer n such that $R_n < 0.005$ is $n = 10$. Without advance knowledge of the sum of the given series, you can obtain a conservative overestimate of n in the following way. We know that

$$R_n \leqq \int_n^{\infty} \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_n^{\infty} = \frac{1}{2n^2}.$$

So it will be sufficient if

$$\frac{1}{2n^2} < 0.005; \qquad 2n^2 > 200; \qquad n > 10;$$

that is, if $n = 11$. More accuracy, and a smaller value of n, might be obtained had we used instead the better estimate

$$R_n \leqq \int_n^{\infty} \frac{1}{x^3+1}\,dx,$$

and if $n = 10$ the value of this integral is approximately 0.004998001249, but one must question whether the extra work in evaluating the antiderivative and solving the resulting inequality would be worth the trouble.

C11S06.042: The sum of the series is

$$S = \sum_{n=1}^{\infty} \frac{n}{(n+1)\cdot 2^n} \;=\; 2(1-\ln 2) \;\approx\; 0.613705638880, \tag{1}$$

and $S - 0.005 \approx 0.608705638880$. Because

$$\sum_{n=1}^{7} \frac{n}{(n+1)\cdot 2^n} \approx 0.606687 < S - 0.005 < 0.610159 \approx \sum_{n=1}^{8} \frac{n}{(n+1)\cdot 2^n},$$

the smallest positive integer n such that $R_n < 0.005$ is $n = 8$. Without advance knowledge of the sum of the given series, you can obtain a conservative overestimate of n in the following way. We know that

$$R_n \leqq \int_n^{\infty} \frac{1}{2^x}\,dx = \left[-\frac{2^{-x}}{\ln 2}\right]_n^{\infty} = \frac{1}{2^n \ln 2}.$$

So it will be sufficient if

$$\frac{1}{2^n \ln 2} < 0.005; \qquad 2^n > \frac{200}{\ln 2} \approx 289; \qquad n = 9.$$

We might have found a smaller value of n by instead solving

$$\int_n^{\infty} \frac{x}{(x+1)\cdot 2^x}\,dx < 0.005,$$

but the antiderivative we need appears to be a nonelementary function. (Techniques of Section 11.9 can be used to find the exact sum in Eq. (1).)

C11S06.043: The sum of the series is

$$S = \sum_{n=1}^{\infty} \frac{\cos^4 n}{n^4} \approx 0.100714442927,$$

and $S - 0.005 \approx 0.095714442927$. Because

$$\sum_{n=1}^{2} \frac{\cos^4 n}{n^4} \approx 0.087095 < S - 0.005 < 0.098954 \approx \sum_{n=1}^{3} \frac{\cos^4 n}{n^4},$$

the smallest positive integer n such that $R_n < 0.005$ is $n = 3$. Without advance knowledge of the sum of the given series, you can obtain a conservative overestimate of n in the following way. We know that

$$R_n \leqq \int_n^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3}.$$

So it will be sufficient if

$$\frac{1}{3n^3} < 0.005; \qquad 3n^3 > 200; \qquad n^3 > 67;$$

that is, $n = 5$. There are ways to lower this estimate but they are highly technical and probably not worth the extra trouble.

C11S06.044: The sum of the series is

$$S = \sum_{n=1}^{\infty} \frac{1}{n^{2+(1/n)}} \approx 1.4759745320,$$

and $S - 0.005 \approx 1.4709745320$. Because

$$\sum_{n=1}^{196} \frac{1}{n^{2+(1/n)}} \approx 1.47095965 < S - 0.005 < 1.47098473 \approx \sum_{n=1}^{197} \frac{1}{n^{2+(1/n)}},$$

the smallest positive integer n such that $R_n < 0.005$ is $n = 197$. Without advance knowledge of the sum of the given series, you can obtain a conservative overestimate of n in the following way. We note that $n^{2+(1/n)} > n^2$ for each positive integer n, and thus

$$R_n \leqq \int_n^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_n^{\infty} = \frac{1}{n}.$$

So it will be sufficient if

$$\frac{1}{n} < 0.005; \qquad n > 200; \qquad n = 201.$$

C11S06.045: We suppose that $\sum a_n$ is a convergent positive-term series. Apply the mean value theorem to $f(t) = (\sin t)/t$ on the interval $[0, x]$ to show that $\sin x < x$ for all $x > 0$. Moreover, the converse of Theorem 3 in Section 11.3 implies that $a_n \to 0$ as $n \to +\infty$. Thus there exists a positive integer K such that if $n \geqq K$, then $a_n < \pi$. Therefore

$$0 < \sin(a_n) < a_n \qquad \text{if} \qquad n \geqq K.$$

Consequently $\sum a_n$ eventually dominates the eventually positive-term series $\sum \sin(a_n)$. Therefore the latter series converges because the values of its terms for $1 \leqq n < K$ cannot affect its convergence or divergence.

C11S06.046: Part (a): By l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{n^{1/8}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/8}} = \lim_{x\to\infty} \frac{8x^{7/8}}{x} = \lim_{x\to\infty} \frac{8}{x^{1/8}} = 0.$$

Therefore there exists a positive integer K such that $\ln n < n^{1/8}$ for all $n \geqq K$. (If you're curious, you can use Newton's method to discover that $K = 2149100652958$ is the least integer that "works.".)

Part (b): We know from part (a) that $\dfrac{1}{n} < \dfrac{1}{(\ln n)^8}$ for all $n \geqq K$ (the same K of part (a)). Therefore

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^8} \quad \text{eventually dominates} \quad \sum_{n=2}^{\infty} \frac{1}{n},$$

and the latter series diverges because it is eventually the same as the harmonic series.

C11S06.047: If $\sum a_n$ is a convergent positive-term series, then we may assume that $n \geqq 1$, and hence

$$0 < \frac{a_n}{n} \leqq a_n \quad \text{for all} \quad n.$$

Therefore $\sum a_n$ dominates the positive-term series $\sum (a_n/n)$, so by the comparison test the latter series converges as well.

C11S06.048: Because $\{c_n\} \to 0$, there exists a positive integer K such that $c_n \leqq 1$ for all $n \geqq K$. Hence $0 \leqq a_n c_n \leqq a_n$ if $n \geqq K$. Therefore $\sum a_n$ eventually dominates $\sum a_n c_n$. Thus by the comparison test, the latter series converges.

C11S06.049: Convergence of $\sum b_n$ implies that $\{b_n\} \to 0$ (the converse of Theorem 3 of Section 10.3). Therefore $\sum a_n b_n$ converges by Problem 48.

C11S06.050: First, Eq. (7) in Section 5.3 tells us that for each positive integer n,

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.$$

A proof by induction is quite easy to construct. We omit it to save space. Using this result,

$$\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + 4 + \cdots + n} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}. \tag{1}$$

This series is dominated by double the convergent p-series with $p = 2$, and therefore the series in (1) converges. Moreover,

$$\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1},$$

so the kth partial sum of the series in (1) is

$$\begin{aligned} S_k &= \sum_{n-1}^{k} \frac{2}{n(n+1)} \\ &= \frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \frac{2}{4} - \frac{2}{5} + \cdots + \frac{2}{k-1} - \frac{2}{k} + \frac{2}{k} - \frac{2}{k+1} = 2 - \frac{2}{k+1}. \end{aligned}$$

Therefore the sum of the series in (1) is $\displaystyle\lim_{k\to\infty} S_k = 2$.

C11S06.051: By Problem 50 in Section 10.5, if n is a positive integer then

$$0 \leqq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n \ \leqq\ 1.$$

Therefore

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \ \leqq\ 1 + \ln n$$

for every positive integer n. Hence

$$\sum_{n=1}^{\infty} \frac{1}{1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}} \qquad \text{dominates} \qquad \sum_{n=1}^{\infty} \frac{1}{1 + \ln n}.$$

But we showed in the solution of Problem 13 of this section that $\ln n < n$ for all $n \geqq 1$. So the last series dominates

$$\sum_{n=1}^{\infty} \frac{1}{1+n},$$

which diverges because it is eventually the same as the harmonic series. Therefore the series of Problem 51 diverges.

C11S06.052: Suppose that $\sum a_n$ and $\sum b_n$ are positive-term series.

Part (a): Suppose that $\sum b_n$ converges and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Then there exists a positive integer K such that $a_n \leqq b_n$ for all $n \geqq K$. Thus $\sum b_n$ eventually dominates $\sum a_n$. Therefore $\sum a_n$ converges.

Part (b): Suppose that $\sum b_n$ diverges and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty.$$

Then there exists a positive integer K such that $a_n \geqq b_n$ for all $n \geqq K$. Thus $\sum a_n$ eventually dominates $\sum b_n$, and therefore $\sum a_n$ diverges.

Section 11.7

C11S07.001: The sequence $\{1/n^2\}$ is monotonically decreasing with limit zero. So the given series meets both criteria of the alternating series test and therefore converges. It is known that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \approx 0.822467033424;$$

this can be derived from results in Problem 68 of Section 11.8.

C11S07.002: The sequence $\left\{1/\sqrt{n^2+1}\,\right\}$ is monotonically decreasing with limit zero, so the given series meets both criteria of the alternating series test. Therefore the series converges. Its sum is approximately 0.440917473865.

C11S07.003: Because

$$\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} \neq 0, \qquad \lim_{n\to\infty} \frac{(-1)^n n}{3n+2} \quad \text{does not exist.}$$

Therefore the given series diverges by the nth-term test for divergence.

C11S07.004: The sequence $\left\{n/(3n^2+2)\right\}$ is monotonically decreasing with limit zero, so the given series meets both criteria of the alternating series test. Therefore this series converges. The *Mathematica* 3.0 command

```
Sum[ (n*(-1)∧(n)/(3*n*n + 2), {n, 1, Infinity} ] // Timing        (1)
```

returns the exact value of the sum; it is

$$\frac{1}{12}\left[\psi\left(\frac{1}{2} - \frac{i}{\sqrt{6}}\right) - \psi\left(1 - \frac{i}{\sqrt{6}}\right) + \psi\left(\frac{1}{2} + \frac{i}{\sqrt{6}}\right) - \psi\left(1 + \frac{i}{\sqrt{6}}\right)\right] \approx -0.11608887384310614138585 6.$$

The `Timing` command in (1) returns a computational time of about 2.167 seconds on a PowerMacIntosh 7600/120, by modern standards a relatively slow machine. The *digamma function* $\psi(z)$ is defined to be the *logarithmic derivative* of the gamma function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)};$$

in the language of *Mathematica* 3.0, it is `PolyGamma[0,z]`, where `PolyGamma[n,z]` $= \psi^{(n)}(z)$ is the nth derivative of the digamma function. As usual, this additional information is provided for the benefit of readers who are interested in additional reading and research on these topics.

C11S07.005: Because

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+2}} = \lim_{n\to\infty} \frac{1}{\left(1 + \dfrac{2}{n^2}\right)^{1/2}} = 1 \neq 0, \qquad \lim_{n\to\infty} \frac{(-1)^{n+1} n}{\sqrt{n^2+2}} \quad \text{does not exist.}$$

Therefore the given series diverges by the nth-term test for divergence.

C11S07.006: Let

$$f(x) = \frac{x^2}{(x^5+5)^{1/2}} \quad \text{for} \quad x \geqq 1.$$

Then

$$f'(x) = \frac{x(20-x^5)}{2(x^5+5)^{3/2}},$$

and therefore f is decreasing if $x \geqq 2$. Therefore the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{\sqrt{n^5+5}} = \sum_{n=1}^{\infty} (-1)^{n+1}a_n$$

satisfies the inequalities $a_1 < a_2 > a_3 > a_4 > a_5 > \cdots$; that is, after the first term, its terms are monotonically decreasing with limit

$$\lim_{n\to\infty} \frac{n^2}{(n^5+5)^{1/2}} = \lim_{n\to\infty} \frac{1}{\left(n + \dfrac{5}{n^4}\right)^{1/2}} = 0.$$

Therefore both criteria of the alternating series test are (effectively) met and thus the given series converges. Its sum is approximately 0.0577598154958.

C11S07.007: We showed in the solution of Problem 13 of Section 11.6 that $n > \ln n$ for every integer $n \geqq 1$. Also, by l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{n}{\ln n} = +\infty,$$

so the given series diverges by the nth-term test for divergence.

C11S07.008: Let

$$f(x) = \frac{\ln x}{x^{1/2}}; \quad \text{then} \quad f'(x) = \frac{2-\ln x}{2x^{3/2}}.$$

Hence f is decreasing if $\ln x > 2$; that is, if $x > e^2 \approx 7.389$. Let $a_n = (\ln n)/\sqrt{n}$. Then even though the sequence $\{a_n\}$ is monotonically increasing for $1 \leqq n \leqq 7$, it is monotonically decreasing thereafter. By l'Hôpital's rule,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0.$$

Therefore after the first seven terms, the terms of this sequence meet both criteria of the alternating series test. Altering the first seven terms of a series cannot change the fact of its convergence or divergence, only its sum; therefore the given series converges.

The *Mathematica* 3.0 command

```
Sum[ ((−1)∧n)∗(Log[n])/Sqrt[n], {n, 1, Infinity} ]
```

yields the exact value of its sum:

$$\left(\sqrt{2}\,\ln 2\right)\zeta\left(\frac{1}{2}\right)+\left(1-\sqrt{2}\,\right)\zeta'\left(\frac{1}{2}\right)\;\approx\;0.1932888316392827389654090859\,14;$$

the *Riemann zeta function* $\zeta(z)$ is discussed briefly in the project following Section 11.5 of the textbook.

C11S07.009: First we claim that if n is a positive integer, then

$$\frac{n}{2^n}\;\geqq\;\frac{n+1}{2^{n+1}}. \tag{1}$$

This assertion is true if $n = 1$ because

$$\frac{1}{2}\;\geqq\;\frac{2}{4}, \qquad \text{and thus} \qquad \frac{1}{2^1}\;\geqq\;\frac{2}{2^2}.$$

Suppose that the inequality in (1) holds for some integer $k \geqq 1$. Then

$$\frac{k}{2^k}\;\geqq\;\frac{k+1}{2^{k+1}}; \qquad \frac{k}{2^k}+\frac{1}{2^k}\;\geqq\;\frac{k+1}{2^{k+1}}+\frac{2}{2^{k+1}};$$
$$\frac{k+1}{2^k}\;\geqq\;\frac{k+3}{2^{k+1}}; \qquad \frac{k+1}{2^{k+1}}\;>\;\frac{k+2}{2^{k+2}}.$$

Therefore, by induction, the inequality in (1) holds for every integer $n \geqq 1$; indeed, strict inequality holds if $n \geqq 2$. Therefore if $a_n = n/2^n$ for $n \geqq 1$, then the sequence $\{a_n\}$ is monotonically decreasing. Its limit is zero by l'Hôpital's rule:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty}\frac{x}{2^x} = \lim_{x\to\infty}\frac{1}{2^x\ln 2} = 0.$$

Therefore the given series satisfies both criteria of the alternating series test and thus it converges. To find its sum, note that

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^n} = f\left(\frac{1}{2}\right) \qquad \text{where} \qquad f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}nx^n = x\sum_{n=1}^{\infty}(-1)^n nx^{n-1} = xg(x)$$

where $g(x) = h'(x)$ if we let

$$h(x)=\sum_{n=1}^{\infty}(-1)^n x^n = -x+x^2-x^3+x^4-x^5+\cdots = -\frac{x}{1+x}.$$

Thus

$$g(x)=h'(x)=-\frac{1}{(1+x)^2}, \qquad \text{so that} \qquad f(x)=-\frac{x}{(1+x)^2}.$$

It can be shown that all these computations are valid if $-1 < x < 1$, and therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^n} = f\left(\frac{1}{2}\right) = -\frac{2}{9}\approx -0.222222222222.$$

C11S07.010: The ratio test yields

$$\lim_{n\to\infty}\left|\frac{(n+1)\left(-\frac{2}{3}\right)^{n+2}}{n\left(-\frac{2}{3}\right)^{n+1}}\right| = \lim_{n\to\infty}\frac{2(n+1)}{3n} = \frac{2}{3} < 1.$$

Therefore the alternating series

$$\sum_{n=1}^{\infty} n\cdot\left(-\frac{2}{3}\right)^{n+1}$$

converges absolutely, and thus it converges. To find its sum, note that it is

$$f\left(-\frac{2}{3}\right) \qquad \text{where} \qquad f(x) = \sum_{n=1}^{\infty} nx^{n+1} = x^2 g(x)$$

where

$$g(x) = \sum_{n=1}^{\infty} nx^{n-1}.$$

But $g(x) = h'(x)$ where

$$h(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

It now follows that

$$g(x) = \frac{1}{(x-1)^2} \qquad \text{and that} \qquad f(x) = \frac{x^2}{(x-1)^2}.$$

These computations can be shown valid provided that $-1 < x < 1$. So the sum of the original series is $f\left(-\frac{2}{3}\right) = \frac{4}{25} = 0.16$.

C11S07.011: Given the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n n}{\sqrt{2^n+1}}, \tag{1}$$

first observe that, by l'Hôpital's rule,

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{(2^x+1)^{1/2}} &= \lim_{x\to\infty}\frac{2(2^x+1)^{1/2}}{2^x\ln 2} = \lim_{x\to\infty}\frac{2(2^x+1)^{1/2}}{(2^{2x})^{1/2}\ln 2}\\ &= \lim_{x\to\infty}\frac{2}{\ln 2}\left(\frac{2^x+1}{2^{2x}}\right)^{1/2} = \lim_{x\to\infty}\frac{2}{\ln 2}\left(\frac{1}{2^x}+\frac{1}{2^{2x}}\right)^{1/2} = 0.\end{aligned}$$

Next,

$$\lim_{n\to\infty}\frac{2^{n+1}+1}{2^n+1} = \lim_{n\to\infty}\frac{2+2^{-n}}{1+2^{-n}} = \frac{2+0}{1+0} = 2$$

and

$$\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = 1^2 = 1.$$

Therefore there exists a positive integer K such that, if $n \geqq K$, then

$$\frac{2^{n+1}+1}{2^n+1} > \frac{3}{2} > \frac{(n+1)^2}{n^2}.$$

For such n, it follows that

$$\frac{n^2}{2^n+1} > \frac{(n+1)^2}{2^{n+1}+1}; \quad \text{thus}$$

$$\frac{n}{\sqrt{2^n+1}} > \frac{n+1}{\sqrt{2^{n+1}+1}}.$$

This shows that the terms of the series in (1) are monotonically decreasing for $n \geqq K$, and so both criteria of the alternating series test are met for $n \geqq K$. Altering the terms for $n < K$ cannot change the convergence or divergence of a series, so the series in (1) converges. Its sum is approximately -0.178243455603. (By the way, the least value of K that "works" in this proof is $K = 5$, although the terms of the series begin to decrease in magnitude after $n = 3$.)

C11S07.012: This series diverges by the nth-term test for divergence because $(n\pi/10)^{n+1} \to +\infty$ as $n \to +\infty$.

C11S07.013: The values of $\sin(n\pi/2)$ for $n = 1, 2, 3, \ldots$ are $1, 0, -1, 0, 1, 0, -1, 0, 1 \cdots$. So we rewrite the given series in the form

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^{2/3}}$$

to present it as an alternating series in the strict sense of the definition. Because the sequence $\{1/(2n-1)^{2/3}\}$ clearly meets the criteria of the alternating series test, this series converges. Its sum is approximately 0.711944418056.

C11S07.014: This series converges by the alternating series test, because the numerators take the values $-1, 1, -1, 1, -1, \ldots$ as $n = 1, 2, 3, 4, 5, \ldots$, and the denominators are monotonically increasing positive numbers with limit $+\infty$.

A *Mathematica* command similar to those used previously in the solutions for this chapter yields the exact value of the sum of the given series; it is

$$-1 + \frac{\sqrt{2}}{2}\zeta\left(\frac{3}{2}\right) \approx -0.76514702462540794536726875 86.$$

C11S07.015: Because

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \lim_{u\to 0^+} \sin u = 0$$

and because $\sin u$ decreases monotonically through positive values as $u \to 0^+$, this series converges by the alternating series test. Its sum is approximately -0.550796848134.

C11S07.016: Because

$$\lim_{n\to\infty} n\sin\left(\frac{\pi}{n}\right) = \lim_{u\to 0^+} \frac{\sin \pi u}{u} = \pi\left(\lim_{u\to 0^+} \frac{\sin \pi u}{\pi u}\right) = \pi \cdot 1 = \pi \neq 0,$$

the given series diverges by the nth-term test for divergence.

C11S07.017: By Example 7 of Section 11.2, $2^{1/n} \to 1$ as $n \to +\infty$. So the given series diverges by the nth-term test for divergence.

C11S07.018: Lemma: If $a > 1$ and k is a positive integer, then

$$\lim_{n\to\infty} \frac{a^n}{n^k} = +\infty. \tag{1}$$

Proof: If $k = 1$, then l'Hôpital's rule yields

$$\lim_{n\to\infty} \frac{a^n}{n^k} = \lim_{x\to\infty} \frac{a^x}{x} = \lim_{x\to\infty} \frac{a^x \ln a}{1} = +\infty$$

because $\ln a > 0$. So the result in Eq. (1) holds when $k = 1$. Assume that Eq. (1) holds for some integer $k \geqq 1$. Then by l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{a^n}{n^{k+1}} = \lim_{x\to\infty} \frac{a^x}{x^{k+1}} = \lim_{x\to\infty} \frac{a^x \ln a}{(k+1)x^k} = \left(\frac{\ln a}{k+1}\right)\left(\lim_{x\to\infty} \frac{a^x}{x^k}\right) = +\infty$$

because $(\ln a)/(k+1) > 0$. Thus whenever Eq. (1) holds for some positive integer $k \geqq 1$, it also holds for $k+1$. Therefore, by induction, Eq. (1) holds for every integer $k \geqq 1$. ◀

Therefore

$$\lim_{n\to\infty} \frac{(1.01)^{n+1}}{n^4} = (1.01)\left(\lim_{n\to\infty} \frac{(1.01)^n}{n^4}\right) = +\infty.$$

Consequently the given series diverges by the nth-term test for divergence.

The point of the Lemma is that every exponential function with base $a > 1$ eventually outruns any polynomial function, no matter how high its degree. In the case of this particular series, its 1000th term is approximately 2.11687×10^{-8}, rather close to zero, but its millionth term exceeds 2.38839×10^{4297}.

C11S07.019: By the result in Example 11 of Section 11.2 of the text, $n^{1/n} \to 1$ as $n \to +\infty$. Therefore the given series diverges by the nth-term test for divergence.

C11S07.020: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)!(2n)!}{n!(2n+2)!} = \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{1}{2(2n+1)} = 0 < 1.$$

So the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{(2n)!}$$

converges absolutely and therefore converges. Its sum is approximately 0.42443638350202229593. The convergence of this series is particularly fast; this 20-place accuracy was obtained by adding only the first 14 terms of the series.

The exact value of the sum of the series (obtained with the usual *Mathematica* command) is

$$\frac{\sqrt{\pi}}{2ie^{1/4}} \operatorname{erf}\left(\frac{i}{2}\right)$$

where the *error function* is defined by

$$\operatorname{erf}(z) \;=\; \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-t^2\right)\,dt.$$

See Eq. (15) of Section 8.8 and the surrounding discussion for more information about the error function.

C11S07.021: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2} < 1,$$

so the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$$

converges absolutely. Because it is geometric with ratio $r = -\frac{1}{2}$ and first term $\frac{1}{2}$, its sum is $\frac{1}{3}$.

C11S07.022: The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad \text{is dominated by} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and the latter converges because it is the p-series with $p = 2 > 1$. Therefore the first series converges by the comparison test. It converges absolutely because it is a positive-term series. For additional discussion of this series, see the solution to Problem 5 in Section 11.5 and the solution to Problem 37 in Section 11.6. The usual *Mathematica* 3.0 command returns the exact value of the sum of this series; it is

$$\frac{\pi\cosh\pi \,-\, \sinh\pi}{2\sinh\pi} \;\approx\; 1.0766740474685811741340507947 5.$$

C11S07.023: If

$$f(x) = \frac{\ln x}{x}, \qquad \text{then} \qquad f'(x) = \frac{1-\ln x}{x^2},$$

so the sequence $\{(\ln n)/n\}$ is monotonically decreasing if $n \geqq 3$. By l'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Therefore the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

converges by the alternating series test. Because

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^{\infty} = +\infty,$$

the given series converges conditionally rather than absolutely. Its sum is approximately 0.159868903742.

C11S07.024: The given series

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \quad \text{is eventually dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and the latter series converges because it is the p-series with $p = 2 > 1$. Therefore the given series converges by the comparison test. It converges absolutely because it is a positive-term series. Its sum is approximately 1.291285997062663540407282590 (with the aid of the usual `NSum` command in *Mathematica* 3.0). We examine this series from a completely different perspective in Problem 65 of Section 11.8.

C11S07.025: The series

$$\sum_{n=1}^{\infty} \left(\frac{10}{n}\right)^n$$

converges absolutely by the root test, because

$$\rho = \lim_{n\to\infty} \left[\left(\frac{10}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{10}{n} = 0 < 1.$$

Its sum is approximately 186.724948614024. In spite of the relatively large sum, this series converges extremely rapidly; for example, the sum of its first 25 terms is approximately 186.724948614005. The sum is large because the first ten terms of the series are each at least 10; the largest is the fourth term, 39.0625. But the 25th term is less than 1.126×10^{-10}.

C11S07.026: Given the infinite series

$$\sum_{n=1}^{\infty} \frac{3^n}{n!n},$$

the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{3^{n+1}n!n}{3^n(n+1)!(n+1)} = \lim_{n\to\infty} \frac{3n}{(n+1)^2} = 0.$$

Therefore the given series converges absolutely. *Mathematica* 3.0 reports that the exact value of its sum is

$$-\gamma - \ln 3 + \mathrm{Ei}\,(3) \approx 8.258004617055774006006578692211837127301,$$

where γ is Euler's constant and $\mathrm{Ei}\,(z)$ is the [principal value of the] *exponential integral function*

$$\mathrm{Ei}\,(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t}\,dt.$$

C11S07.027: Given the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{n!10^{n+1}}{(n+1)!10^n} = \lim_{n\to\infty} \frac{10}{n+1} = 0.$$

Therefore this series converges absolutely. It is the result of substitution of -10 for x in the Maclaurin series for $f(x) = e^x$ (see Eq. (19) in Section 11.4); therefore its sum is $e^{-10} \approx 0.0000453999297625$.

C11S07.028: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{n^n}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)!\,n^n}{n!\,(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}.$$

Because $\rho < 1$, the given series converges absolutely. Its sum, using the *Mathematica* 3.0 command `NSum`, is approximately 0.6558316008674916.

C11S07.029: The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n+1}\right)^n$ diverges by the nth-term test for divergence because

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \neq 0.$$

The evaluation of the limit is made easy by Eq. (3) in Section 7.3.

C11S07.030: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,n^2}{(2n)!}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)^2(2n)!}{n!\,n^2(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)^3}{n^2(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^3+2n^2} = \lim_{n\to\infty} \frac{\dfrac{1}{n} + \dfrac{2}{n^2} + \dfrac{1}{n^3}}{4 + \dfrac{2}{n}} = \frac{0+0+0}{4+0} = 0.$$

Therefore the given series converges absolutely. The *Mathematica* 3.0 `Sum` command yields the exact value of its sum:

$$\frac{1}{32}\left[14 + 13e^{1/4}\sqrt{\pi}\,\operatorname{erf}\left(\frac{1}{2}\right)\right] \approx 0.9187409358813278427240872318129884.$$

C11S07.031: Given the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$, the root test yields

$$\rho = \lim_{n\to\infty} \left[\left(\frac{\ln n}{n}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

(with the aid of l'Hôpital's rule). Because $\rho < 1$, the given series converges absolutely. Its sum is approximately 0.187967875056.

C11S07.032: Because

$$\lim_{n\to\infty} \frac{2^{3n}}{7^n} = \lim_{n\to\infty} \frac{8^n}{7^n} = \lim_{n\to\infty} \left(\frac{8}{7}\right)^n = +\infty,$$

the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{3n}}{7^n}$ diverges by the nth-term test for divergence.

C11S07.033: First note that

$$\sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

as $n \to +\infty$. Moreover, the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ is monotonically decreasing. Proof: Suppose that n is a positive integer. Then

$$\begin{aligned} n^2 + 2n + 1 &> n^2 + 2n; & n+1 &> \sqrt{n^2 + 2n} \\ 2n + 2 &> 2\sqrt{n^2 + 2n}\,; & 4(n+1) &> n + 2\sqrt{n^2 + 2n} + n + 2; \\ 2\sqrt{n+1} &> \sqrt{n} + \sqrt{n+2}\,; & \sqrt{n+1} - \sqrt{n} &> \sqrt{n+2} - \sqrt{n+1}\,. \end{aligned}$$

Therefore the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$ converges by the alternating series test.

Its sum is approximately 0.760209625219. It converges conditionally, not absolutely. The reason is that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \qquad \text{dominates} \qquad \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n+1}}\,,$$

which diverges because it is a constant multiple of a series eventually the same as the p-series with $p = \frac{1}{2} \leqq 1$.

C11S07.034: Given the series $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{4}\right)^n$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)\cdot\left(\frac{3}{4}\right)^{n+1}}{n\cdot\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty} \frac{3(n+1)}{4n} = \frac{3}{4}.$$

Because $\rho < 1$, the series converges absolutely. To find its sum, note that

$$\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{4}\right)^n = f\left(\frac{3}{4}\right) \qquad \text{where} \qquad f(x) = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1}.$$

The last series is

$$g(x) = \sum_{n=1}^{\infty} nx^{n-1} = G'(x) \qquad \text{where} \qquad G(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

(These computations all can be shown valid provided that $-1 < x < 1$.) Thus

$$g(x) = G'(x) = \frac{1}{(x-1)^2}, \qquad \text{so that} \qquad f(x) = \frac{x}{(x-1)^2}.$$

Therefore the sum of the original series is $f\left(\dfrac{3}{4}\right) = 12$.

C11S07.035: Because

$$\lim_{n\to\infty} \left(\ln\frac{1}{n}\right)^n = \lim_{n\to\infty} (-\ln n)^n$$

does not exist, the series $\displaystyle\sum_{n=1}^{\infty} \left(\ln\frac{1}{n}\right)^n$ diverges by the nth-term test for divergence.

C11S07.036: Given the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{[(n+1)!]^2(2n)!}{(n!)^2(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{2(2n+1)}$$

$$= \lim_{n\to\infty} \frac{n+1}{4n+2} = \lim_{x\to\infty} \frac{x+1}{4x+2} = \lim_{x\to\infty} \frac{1}{4} = \frac{1}{4}$$

by l'Hôpital's rule. Because $\rho < 1$, the series converges absolutely.

The *Mathematica* 3.0 command

```
Sum[ ((n!)∧2)/((2∗n)!), {n, 1, Infinity} ]
```

returns the exact value of its sum:

$$\frac{9 + 2\pi\sqrt{3}}{27} \approx 0.7363998587187150779097 95.$$

The remarkable simplicity of this result suggests that

$$\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!} \tag{1}$$

is an elementary function. Indeed, the *Mathematica* 3.0 `Sum` command, followed by the powerful command `FullSimplify`, reveals that the sum of the power series in (1) is

$$\frac{4}{4-x} + \frac{4\sqrt{x}}{(4-x)^{3/2}} \arcsin\left(\frac{\sqrt{x}}{2}\right).$$

C11S07.037: First, for each positive integer n,

$$\frac{3^n}{n(2^n+1)} \geqq \frac{3^n}{n(2^n+2^n)} = \frac{3^n}{2n \cdot 2^n} = \frac{1}{2n} \cdot \left(\frac{3}{2}\right)^n.$$

Next, using l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\left(\frac{3}{2}\right)^x}{2x} = \lim_{x\to\infty} \frac{\left(\frac{3}{2}\right)^x \ln\left(\frac{3}{2}\right)}{2} = +\infty$$

because $\ln\left(\frac{3}{2}\right) > 0$ and because, if $a > 1$, then $a^x \to +\infty$ as $x \to +\infty$. Therefore the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{n(2^n+1)}$$

diverges by the nth-term test for divergence.

C11S07.038: Claim: $f(x) = \dfrac{\arctan x}{x}$ is decreasing for $x \geqq 2$. Proof:

$$f'(x) = \frac{x - (1+x^2)\arctan x}{x^2(1+x^2)}, \quad \text{so}$$

$$f'(x) < \frac{x - (1+x^2)}{x^2(1+x^2)} \quad \text{if} \quad x \geqq 2; \quad \text{thus}$$

$$f'(x) < \frac{2x - (1+x^2)}{x^2(1+x^2)} = -\frac{(x-1)^2}{x^2(x^2+1)}$$

if $x \geqq 2$. Moreover,

$$0 \leqq \arctan x \leqq \frac{\pi}{2} \quad \text{if} \quad x > 0, \quad \text{and therefore} \quad 0 \leqq \frac{\arctan x}{x} \leqq \frac{\pi}{2x} \quad \text{if} \quad x > 0.$$

Consequently

$$\lim_{x\to\infty} \frac{\arctan x}{x} = 0$$

by the squeeze law for limits (Section 2.3).

Therefore $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{n+1}\arctan n}{n}$ converges by the alternating series test. Its sum is about 0.465712303526. But this series is conditionally convergent, not absolutely convergent. The reason: $\arctan 1 = \pi/4 > 1/2$, and hence

$$\sum_{n=1}^{\infty} \frac{\arctan n}{n} \quad \text{dominates} \quad \sum_{n=1}^{\infty} \frac{1}{2n},$$

and the latter series diverges because it is a nonzero multiple of the harmonic series.

C11S07.0:39 Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n-1)}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)!\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{n!\cdot 1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)} = \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}.$$

Because $\rho < 1$, the series in question converges absolutely. Its sum is approximately 0.586781998767.

C11S07.040: Given the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 4\cdot 7\cdots(3n-2)}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)\cdot 1\cdot 4\cdot 7\cdots(3n-2)}{1\cdot 3\cdot 5\cdots(2n-1)\cdot 1\cdot 4\cdot 7\cdots(3n-2)\cdot(3n+1)} = \lim_{n\to\infty} \frac{2n+1}{3n+1} = \frac{2}{3}.$$

Because $\rho < 1$, the original series converges absolutely. The `Sum` command in *Mathematica* 3.0 yields the exact value of the sum of this series; it is

$$\texttt{HypergeometricPFQ}\left[\left\{1,\, \frac{3}{2}\right\},\, \left\{\frac{4}{3}\right\},\, -\frac{2}{3}\right] \;=\; {}_2\mathrm{F}_1\left(1,\, \frac{3}{2};\, \frac{4}{3};\, -\frac{2}{3}\right) \;\approx\; 0.5644219964461680148.$$

Note carefully the punctuation in the arguments of the *generalized hypergeometric function*

$${}_p\mathrm{F}_q(a_1,\, a_2,\, \ldots,\, a_p;\; b_1,\, b_2,\, \ldots,\, b_q;\; z) \;=\; \sum_{k=0}^{\infty} \frac{\displaystyle\prod_{i=1}^{p} (a_i)_k}{\displaystyle k!\prod_{i=1}^{q} (b_i)_k}\, z^k$$

where $(c)_k \;=\; \displaystyle\prod_{j=1}^{k} (a+j-1)$.

The hypergeometric function finds its way into extremely diverse branches of advanced mathematics. For example,

$$\cos z = {}_0\mathrm{F}_1\left(\,;\ \frac{1}{2};\ -\frac{1}{4}z^2\right);$$

$$\ln(z+1) = z \cdot {}_2\mathrm{F}_1\,(1,\ 1;\ 2;\ -z)\,;$$

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} \cdot {}_1\mathrm{F}_1\left(\frac{1}{2};\ \frac{3}{2};\ -z^2\right);$$

$$\pi = 4 - \frac{8}{9} \cdot {}_3\mathrm{F}_2\left(\frac{1}{2},\ \frac{1}{2},\ 1;\ \frac{5}{2},\ \frac{5}{2};\ -1\right).$$

(To interpret the right-hand side of the first formula, you need to know that the "empty product" of no numbers is defined to have the value 1, just as $2^0 = 0! = 1$.)

C11S07.041: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{3^n(n!)^2}$, the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+3)!3^n(n!)^2}{(n+2)!3^{n+1}\left[(n+1)!\right]^2} = \lim_{n\to\infty} \frac{n+3}{3(n+1)^2} = \lim_{n\to\infty} \frac{\dfrac{1}{n} + \dfrac{3}{n^2}}{3\left(1+\dfrac{1}{n}\right)^2} = \frac{0+0}{3 \cdot 1} = 0.$$

Because $\rho < 1$, the given series converges absolutely. Its sum can be computed exactly, as follows. Note first that

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{3^n(n!)^2} = \sum_{n=1}^{\infty} \frac{(n+2)(n+1)}{3^n \cdot (n!)} = f\left(\frac{1}{3}\right)$$

where

$$f(x) = \sum_{n=1}^{\infty} \frac{(n+2)(n+1)x^n}{n!}.$$

But $f(x) = g'(x)$ where

$$g(x) = \sum_{n=1}^{\infty} \frac{(n+2)x^{n+1}}{n!},$$

and $g(x) = h'(x)$ where

$$h(x) = \sum_{n=1}^{\infty} \frac{x^{n+2}}{n!} = x^2 \sum_{n=1}^{\infty} \frac{x^n}{n!} = x^2(e^x - 1).$$

But then, $f(x) = h''(x) = (x^2 + 4x + 2)e^x - 2$, so the sum of the series in this problem is

$$f\left(\frac{1}{3}\right) = \frac{31e^{1/3} - 18}{9} \approx 2.807109464185.$$

This is confirmed by *Mathematica* 3.0, which in response to the command

```
NSum[ ((n + 2)*(n + 1))/((3∧n)*(n!)), { n, 1, Infinity } ]
```

returns the approximate sum 2.80711.

C11S07.042: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^n}{3^{n^2}}$, the root test yields

$$\rho = \lim_{n\to\infty} \left[\frac{n^n}{3^{n^2}}\right]^{1/n} = \lim_{n\to\infty} \frac{n}{3^n} = \lim_{x\to\infty} \frac{x}{3^x} = \lim_{x\to\infty} \frac{1}{3^x \ln 3} = 0.$$

Because $\rho < 1$, the original series converges absolutely. Moreover, its convergence is extremely rapid in spite of the very rapidly increasing factor in the numerator of each term: The sum of its first 20 terms and the sum of its first 40 terms agree to the first 60 decimal places; its sum is approximately 0.2853164160576381077.

C11S07.043: The sum of the first five terms of the given series is

$$S_5 = \sum_{n=1}^{5} \frac{(-1)^{n+1}}{n^3} = \frac{195353}{216000} \approx 0.904412037037.$$

The sixth term of the series is

$$-\frac{1}{216} \approx -0.004629629629.$$

Thus S_5 approximates the sum S of the series with error less than 0.005. Indeed, we can conclude that $S_6 \approx 0.899782 < S < 0.904412 \approx S_5$. To two decimal places, $S \approx 0.90$. *Mathematica* 3.0 reports that $S \approx 0.901542677370$.

C11S07.044: The sum of the first eight terms of the given series is

$$S_8 = \sum_{n=1}^{8} \frac{(-1)^{n+1}}{3^n} = \frac{1640}{6561} \approx 0.249961896052.$$

The ninth term of the series is

$$\frac{1}{19683} \approx 0.000050805263.$$

Thus S_8 approximates the sum S of the series with error less than 0.000051. Indeed, we can conclude that $S_8 \approx 0.249962 < S < 0.250012 \approx S_9$. (Here we round *down* lower bounds and round *up* upper bounds.) To four decimal places, $S \approx 0.2500$. *Mathematica* 3.0 reports (using the command `NSum`) that $S = 0.250000000000$. Indeed, the series is geometric with first term $\frac{1}{3}$ and ratio $-\frac{1}{3}$, so its sum is exactly $\frac{1}{4}$. *Mathematica* 3.0 can find the exact sums of a wide variety series, including geometric series (try `Sum` instead of `NSum`); the command

```
Sum[ ((-1)∧(n+1))/(3∧n), { n, 1, Infinity } ]
```

elicits the response $\dfrac{1}{4}$.

C11S07.045: The sum of the first six terms of the given series is

$$S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n!} = \frac{91}{144} \approx 0.631944444444.$$

The seventh term of the series is

$$\frac{1}{5040} \approx 0.000198412698.$$

Thus S_6 approximates the sum S of the series with error less than 0.0002. Indeed, we can conclude that $S_6 \approx 0.631945 < S < 0.632142 \approx S_7$ (here we round *down* lower bounds and round *up* upper bounds). To three places, $S \approx 0.632$. *Mathematica* 3.0 reports that $S \approx 0.632120558829$. Using Eq. (19) in Section 11.4, we see that the exact value of the sum is

$$S = 1 - \frac{1}{e}.$$

We have in this problem another example of a series that *Mathematica* 3.0 can sum exactly (using the command `Sum` instead of `NSum`); the command

```
Sum[ ((-1)∧(n+1))/(n!), { n, 1, Infinity } ]
```

produces the exact answer in the form $-\dfrac{1-e}{e}$.

C11S07.046: The sum of the first seven terms of the given series is

$$S_7 = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^n} = \frac{376274084904457}{480290277600000} \approx 0.783430567832.$$

The eighth term of the series is

$$-\frac{1}{16777216} \approx -0.0000000596.$$

Thus S_7 approximates the sum S of the series with error less than 0.00000006. In fact, we may conclude that $S_8 \approx 0.78343051 < S < S_7 \approx 0.78343056$. Thus to six decimal places, $S \approx 0.783431$. *Mathematica* 3.0 reports that $S \approx 0.7834305107121344$.

C11S07.047: The sum of the first 12 terms of the series is

$$S_{12} = \sum_{n=1}^{12} \frac{(-1)^{n+1}}{n} = \frac{18107}{27720} \approx 0.653210678211.$$

The 13th term of the series is

$$\frac{1}{13} \approx 0.076923076923.$$

Thus S_{12} approximates the sum S of the series with error less than 0.08. Indeed, we may conclude that $S_{12} \approx 0.653211 < S < 0.730133 \approx S_{13}$ (we round *down* lower bounds and round *up* upper bounds). Thus to one decimal place, $S \approx 0.7$. This is a series that *Mathematica* 3.0 can sum exactly; the command

```
Sum[ ((-1)∧(n+1))/n, { n, 1, Infinity } ]
```

produces the response $\ln 2$.

C11S07.048: The sum of the first 15 terms of the series is

$$S_{15} = \sum_{n=1}^{15} \frac{(-1)^{n+1}}{n^2} = \frac{107074439839}{129859329600} \approx 0.824541757368.$$

The 16th term of the series is

$$-\frac{1}{256} = -0.00390625.$$

Thus S_{15} approximates the sum S of the series with error less than 0.004. In fact, we may conclude that $S_{16} \approx 0.820636 < S < 0.824541 \approx S_{15}$. (We round lower bounds *down* and upper bounds *up*.) Thus to two places, $S \approx 0.82$. This is another series than *Mathematica* 3.0 can sum exactly: The command

```
Sum[ ((-1)∧(n+1))/(n∧2), { n, 1, Infinity } ]
```

produces the response

$$\frac{\pi^2}{12}$$

—approximately 0.822467033424. *Mathematica* 3.0 can also sum many telescoping series exactly; it reports that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(exactly). It was unable to sum exactly the telescoping (!) series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right),$$

which we first saw in Johnson and Kiokemeister's freshman calculus book in 1967.

C11S07.049: The condition

$$\frac{1}{n^4} < 0.0005 \quad \text{leads to} \quad n > 6.69,$$

so the sum of the terms through $n = 6$ will provide three-place accuracy. The sum of the first six terms of the series is

$$\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^4} = \frac{4090037}{4320000} \approx 0.946767824074,$$

so to three places, the sum of the infinite series is 0.947. The exact value of the sum of this series is

$$\frac{7\pi^4}{720} \approx 0.947032829497.$$

C11S07.050: The condition

$$\frac{1}{n^5} < 0.00005 \quad \text{leads to} \quad n > 7.25,$$

so the sum of the terms through $n = 7$ will provide four-place accuracy. The sum of the first seven terms of the series is

$$\sum_{n=1}^{7} \frac{(-1)^{n+1}}{n^5} = \frac{12705011703799}{13069123200000} \approx 0.972139562033,$$

so, to four places, the sum of the infinite series is 0.9721. The exact value of the sum of this series is not known to be expressible in terms of elementary functions, but the *Mathematica* 3.0 command

```
Sum[ ((-1)∧(n+1))/(n∧5), { n, 1, Infinity } ]
```

yields the result

$$\frac{15}{16}\zeta(5) \approx 0.97211977044690930593565514355.$$

You can prove this for yourself, by hand, beginning with the definition of

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}.$$

Recent releases of computer algebra programs such as *Maple* and *Mathematica* are familiar with the Riemann zeta function $\zeta(z)$ and can evaluate in closed form the sums of many related (or seemingly related) series. If you have *Mathematica* 3.0 or later available, try the command

```
Sum[ ((−1)∧(n+1))/((2∗n − 1)∧5), { n, 1, Infinity } ]
```

or, if *Maple* V version 5.1 is handy,

```
sum( ((−1)∧(n+1))/((2∗n − 1)∧5), n=1..infinity);
```

Also experiment with sums such as

$$1 - \frac{1}{3^k} + \frac{1}{5^k} - \frac{1}{7^k} + \frac{1}{9^k} - \cdots$$

where k is a fixed positive integer. (The case $k = 2$ is especially interesting!)

C11S07.051: The condition

$$\frac{1}{n! \cdot 2^n} < 0.00005 \quad \text{leads to} \quad 5 < n < 6,$$

so the sum of the terms through $n = 5$ will provide four-place accuracy. The sum of the first six terms of the series is

$$\sum_{n=0}^{5} \frac{(-1)^n}{n! \cdot 2^n} = \frac{2329}{3840} \approx 0.606510416667,$$

so to four places, the sum of the infinite series is 0.6065. The exact value of the sum of this series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot 2^n} = e^{-1/2} \approx 0.606530659713.$$

C11S07.052: The condition

$$\frac{1}{(2n)!} < 0.000005 \quad \text{leads to} \quad 4 < n < 5,$$

so the sum of the terms through $n = 4$ will provide five-place accuracy. The sum of the first five terms of the series is

$$\sum_{n=0}^{4} \frac{(-1)^n}{(2n)!} = \frac{4357}{8064} \approx 0.540302579365,$$

so to five places, the sum of the infinite series is 0.54030. The exact value of the sum of the infinite series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1) \approx 0.540302305868.$$

C11S07.053: The condition

$$\frac{1}{(2n+1)!} \cdot \left(\frac{\pi}{3}\right)^{2n+1} < 0.000005 \quad \text{leads to} \quad 4 < n < 5,$$

so the sum of the terms through $n = 4$ will provide five-place accuracy. The sum of the first five terms of the series is

$$\sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)!} \cdot \left(\frac{\pi}{3}\right)^{2n+1} \approx 0.866025445100,$$

so to five places, the sum of the infinite series is 0.86603. The exact value of the sum of the infinite series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left(\frac{\pi}{3}\right)^{2n+1} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \approx 0.866025403784.$$

C11S07.054: The condition

$$\frac{1}{n \cdot 10^n} < 0.00000005 \quad \text{leads to} \quad 6 < n < 7,$$

so the sum of the first six terms of the series will provide seven-place accuracy. The sum of the first six terms is

$$\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n \cdot 10^n} = \frac{571861}{6000000} \approx 0.095310166667,$$

so to seven places, the sum of the infinite series is 0.0953102. The exact value of the sum of this series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 10^n} = \ln(1.1) \approx 0.095310179804.$$

C11S07.055: Because

$$0 < a_n \leqq \frac{1}{n} \quad \text{for all} \quad n \geqq 1,$$

$a_n \to 0$ as $n \to +\infty$ by the squeeze law for limits (Section 2.3 of the text). The alternating series test does not apply because the sequence $\{a_n\}$ is not monotonically decreasing. The series $\sum a_n$ diverges because its $2n$th partial sum S_{2n} satisfies the inequality

$$S_{2n} > 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2n},$$

and the last expression is half the nth partial sum of the harmonic series. Similar remarks hold for S_{2n+1}, and hence $S_n \to +\infty$ as $n \to +\infty$. Therefore $\sum a_n$ diverges.

C11S07.056: Because

$$0 < a_n \leqq \frac{1}{\sqrt{n}} \quad \text{for all} \quad n \geqq 1,$$

$a_n \to 0$ as $n \to +\infty$ by the squeeze law for limits (Section 2.3 of the text). The alternating series test does not apply because the sequence $\{a_n\}$ is not monotonically decreasing. The series $\sum a_n$ diverges because its $2n$th partial sum S_{2n} satisfies the inequality

$$S_{2n} > 1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \cdots + \frac{1}{\sqrt{2n-1}} > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2n},$$

and the last expression is half the nth partial sum of the harmonic series. Similar remarks hold for S_{2n+1}, and hence $S_n \to +\infty$ as $n \to +\infty$. Therefore $\sum a_n$ diverges.

C11S07.057: Let

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n}} \quad \text{for} \quad n \geqq 1.$$

Then $\sum a_n$ and $\sum b_n$ converge by the alternating series test. But

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges because it is the harmonic series.

C11S07.058: This is merely the contrapositive of Theorem 3, so its proof is the same.

C11S07.059: Let $b = |a|$. Then the ratio test applied to $\sum (a^n/n!)$ yields

$$\rho = \lim_{n \to \infty} \frac{n!\, b^{n+1}}{(n+1)!\, b^n} = \lim_{n \to \infty} \frac{b}{n+1} = 0 < 1.$$

Therefore the series

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \tag{1}$$

converges for every real number a. Thus by the nth-term test for divergence, it follows that

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0$$

for every real number a. The sum of the series in (1) is e^a.

C11S07.060: Part (a): Given: $-1 < r < 1$ and $\displaystyle\sum_{n=0}^{\infty} nr^n$, the ratio test yields

$$\rho = \lim_{n \to \infty} \frac{(n+1)|r|^{n+1}}{n|r|^n} = \lim_{n \to \infty} \frac{n+1}{n}|r| = |r| < 1.$$

Therefore the series in question converges. For later use in part (b), note that

$$\lim_{n\to\infty} nr^n = 0 \tag{1}$$

by the nth-term test for divergence.

Part (b): Let S denote the sum of the series in part (a). Then

$$\begin{aligned}
(1-r)S &= \sum_{n=0}^{\infty} n(1-r)r^n = \lim_{k\to\infty} \sum_{n=0}^{k} (nr^n - nr^{n+1}) \\
&= \lim_{k\to\infty} (r - r^2 + 2r^2 - 2r^3 + 3r^3 - 3r^4 + \cdots + (k-1)r^{k-1} - (k-1)r^k + kr^k - kr^{k+1}) \\
&= \lim_{k\to\infty} (1 + r + r^2 + r^3 + \cdots + r^k - kr^{k+1} - 1) \\
&= \lim_{k\to\infty} \left(\frac{1-r^{k+1}}{1-r} - kr^{k+1} - 1 \right) = \frac{1}{1-r} - 1 = \frac{r}{1-r}.
\end{aligned}$$

(Note that $kr^{k+1} \to 0$ as $k \to +\infty$ by the concluding remark in part (a).) Therefore

$$\sum_{n=0}^{\infty} nr^n = S = \frac{r}{(1-r)^2}.$$

C11S07.061: We are given

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{and} \qquad S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

Part (a): Note first that

$$S_2 = 1 - \frac{1}{2} \qquad \text{and} \qquad H_2 - H_1 = 1 + \frac{1}{2} - 1,$$

so $S_{2n} = H_{2n} - H_n$ if $n = 1$. Assume that $S_{2m} = H_{2m} - H_m$ for some integer $m \geqq 1$. Then

$$\begin{aligned}
S_{2(m+1)} = S_{2m} + \frac{1}{2m+1} - \frac{1}{2m+2} &= H_{2m} - H_m + \frac{1}{2m+1} - \frac{1}{2m+2} \\
= H_{2m} + \frac{1}{2m+1} + \frac{1}{2m+2} - H_m - \frac{2}{2m+2} &= H_{2(m+1)} - H_{m+1}.
\end{aligned}$$

Therefore, by induction, $S_{2n} = H_{2n} - H_n$ for every positive integer n.

Part (b): Let $m = 2n$. Then

$$\lim_{n\to\infty} (H_{2n} - \ln 2n) = \lim_{m\to\infty} (H_m - \ln m) = \gamma$$

by Problem 50 in Section 11.5.

Part (c): By the results in parts (a) and (b),

$$\begin{aligned}
&\lim_{n\to\infty} (H_{2n} - \ln 2n - H_n + \ln n) = 0; \\
&\lim_{n\to\infty} (S_{2n} - \ln 2 - \ln n + \ln n) = 0; \\
&\lim_{n\to\infty} S_{2n} = \ln 2.
\end{aligned}$$

Thus the "even" partial sums of the alternating harmonic series converge to $\ln 2$. But the alternating harmonic series converges by the alternating series test. Therefore the sequence of *all* of its partial sums converges to $\ln 2$; that is,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2.$$

But see the solution of Problem 28 in Section 11.6 for a better way to approximate $\ln 2$.

C11S07.062: Part (a): We may suppose that none of the terms of the series is zero. If $a_n > 0$, then

$$a_n^+ = \frac{a_n + |a_n|}{2} = \frac{a_n + a_n}{2} = a_n$$

and if $a_n < 0$, then

$$a_n^- = \frac{a_n - |a_n|}{2} = \frac{a_n + a_n}{2} = a_n.$$

Thus $\sum a_n^+$ is the series of positive terms of $\sum a_n$ and $\sum a_n^-$ is the series of negative terms of $\sum a_n$, arranged in each case in the same order in which they appear in $\sum a_n$.

Part (b): If both $\sum a_n^+$ and $\sum a_n^-$ converged, then both would converge absolutely. This would imply that $\sum a_n$ also converges absolutely, but it does not. If $\sum a_n^+$ converged and $\sum a_n^-$ diverged, then their sum would diverge by Problem 62 in Section 11.3, but their sum is $\sum a_n$, which converges. Therefore $\sum a_n^+$ and $\sum a_n^-$ both diverge.

Without loss of generality, suppose that the real number r is nonnegative. Sum enough terms of $\sum a_n^+$ to exceed r, but don't use more terms than are necessary. This is possible because $\sum a_n^+ = +\infty$. Then add enough terms of $\sum a_n^-$ so that the resulting sum is less than r, but use no more terms than are necessary. This is possible because $\sum a_n^- = -\infty$. Repeat this process using only terms not used in the previous step. Because $a_n \to 0$ as $n \to +\infty$, the partial sums of the resulting series will converge to r. Thus some rearrangement of the conditionally convergent series $\sum a_n$ converges to r.

C11S07.063: The answer consists of the first twelve terms of the following series:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} + \frac{1}{21} - \frac{1}{12} + \frac{1}{23} + \frac{1}{25} - \frac{1}{14}$$
$$+ \frac{1}{27} - \frac{1}{16} + \frac{1}{29} + \frac{1}{31} - \frac{1}{18} + \frac{1}{33} + \frac{1}{35} - \frac{1}{20} + \frac{1}{37} + \frac{1}{39} - \frac{1}{22} + \frac{1}{41} + \frac{1}{43} - \frac{1}{24} + \frac{1}{45} + \frac{1}{47} - \frac{1}{26} + \cdots.$$

The 12th partial sum of the series shown here is

$$\frac{353201}{360360} \approx 0.9801337551337551$$

and the 13th is

$$\frac{6364777}{6126120} \approx 1.0389572845455198,$$

so the convergence to the sum 1 is quite slow (as might be expected when dealing with variations of the harmonic series). To generate and view many more partial sums, enter the following commands in *Mathematica* 3.0 (or modify them to use in another computer algebra program):

```
u = Table[ 1/(2*n - 1), { n, 1, 2 + 1000 } ]
```

$$\left\{1,\ \frac{1}{3},\ \frac{1}{5},\ \frac{1}{7},\ \ldots,\ \frac{1}{2003}\right\}$$

(Of course, the ellipsis is ours, not *Mathematica's*. And you may replace 1000 in the first two commands with as large a positive integer as you and your computer will tolerate.)

```
v = Table[ 1/(2*n), { n, 1, 2 + 1000 } ]
```

$$\left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{6},\ \frac{1}{8},\ \ldots,\ \frac{1}{2004}\right\}$$

```
x = 0;   i = 0;   j = 0;
```

(Here x denotes the running sum of the first k terms of the series; i and j are merely subscripts to be used in the arrays u and v, respectively.)

```
While[ i < 1000, {
     While[ x <= 1, { i = i + 1, x = x + u[[i]], Print[ { i, u[[i]], N[x,40] } ] } ],
     While[ x >= 1, { j = j + 1, x = x - v[[j]], Print[ { j, v[[j]], N[x,40] } ] } ] } ]
```

If you execute these commands, be prepared for 1543 lines of output, concluding with

$$\left\{1001,\ \frac{1}{2001},\ 1.000352986739167522306758169577325155187\right\}$$

$$\left\{\ 542,\ \frac{1}{1084},\ 0.9994304775140752138424894448507423267\,8\ \right\}$$

There is evidence that the series is converging to 1 but still stronger evidence that the convergence is painfully slow.

C11S07.064: Part (a): The method has already been explained in the solution of Problem 62. Here is the beginning of the rearranged series that converges to -2:

$$\begin{aligned}
&-\sum_{n=1}^{31}\frac{1}{2n}+\sum_{n=1}^{1}\frac{1}{2n-1}-\sum_{n=32}^{227}\frac{1}{2n}+\sum_{n=2}^{2}\frac{1}{2n-1}\\
&-\sum_{n=228}^{440}\frac{1}{2n}+\sum_{n=3}^{3}\frac{1}{2n-1}-\sum_{n=441}^{658}\frac{1}{2n}+\sum_{n=4}^{4}\frac{1}{2n-1}\\
&-\sum_{n=659}^{876}\frac{1}{2n}+\sum_{n=5}^{5}\frac{1}{2n-1}-\sum_{n=877}^{1094}\frac{1}{2n}+\sum_{n=6}^{6}\frac{1}{2n-1}\\
&-\sum_{n=1095}^{1312}\frac{1}{2n}+\sum_{n=7}^{7}\frac{1}{2n-1}-\sum_{n=1313}^{1530}\frac{1}{2n}.
\end{aligned}$$

The partial sum at this point is approximately -2.00015.

Part (b): Add enough positive terms to just exceed 3. Then add just enough negative terms for the partial sum to drop below 2. Then add enough positive terms to just exceed 4. Then add just enough negative

terms for the partial sum to drop below 3. Then exceed 5, then drop below 4, and so on. The rearranged series begins in this way:

$$\sum_{n=1}^{57} \frac{1}{2n-1} - \sum_{n=1}^{4} \frac{1}{2n} + \sum_{n=58}^{3361} \frac{1}{2n-1} - \sum_{n=5}^{33} \frac{1}{2n}$$
$$+ \sum_{n=3362}^{184479} \frac{1}{2n-1} - \sum_{n=34}^{248} \frac{1}{2n} + \sum_{n=184480}^{10111149} \frac{1}{2n-1} - \sum_{n=249}^{1836} \frac{1}{2n}.$$

At this point the partial sum of the rearrangement is approximately 4.999913358948.

C11S07.065: The sum of the first 50 terms of the rearrangement is $S_{50} \approx -0.00601599$. Also,

$$S_{500} \approx -0.000622656, \qquad S_{5000} \approx -0.0000624766,$$
$$S_{50000} \approx -0.0000062497656, \qquad S_{500000} \approx -0.00000062499766,$$
$$S_{5000000} \approx -0.000000062499977, \quad \text{and} \quad S_{50000000} \approx -0.0000000062499998.$$

We have strong circumstantial evidence here that the sum of the series is 0. (It is.)

Section 11.8

C11S08.001: Given the series $\displaystyle\sum_{n=1}^{\infty} nx^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n|x|^n} = |x|,$$

so the series converges if $-1 < x < 1$. It clearly diverges at both endpoints of this interval, so its interval of convergence is $(-1,\,1)$. To find its sum, note that

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = xf'(x)$$

where

$$f(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, \qquad \text{so that} \qquad f'(x) = \frac{1}{(1-x)^2}.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ if $-1 < x < 1$.

C11S08.002: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^{1/2}|x|^{n+1}}{(n+1)^{1/2}|x|^n} = |x|,$$

so the series converges if $-1 < x < 1$. It clearly diverges if $x = 1$ (it dominates the harmonic series) but converges if $x = -1$ by the alternating series test. Thus its interval of convergence is $[-1,\,1)$. The *Mathematica* 3.0 command

```
Sum[ (x∧n)/Sqrt[n], {n, 1, Infinity} ]
```

returns the sum of this series in the closed, but redundant, form

$$\texttt{PolyLog}\left[\frac{1}{2},\ \texttt{x}\right]$$

where `PolyLog[n, z]` is the polylogarithm function

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

C11S08.003: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)2^n|x|^{n+1}}{n2^{n+1}|x|^n} = \frac{|x|}{2},$$

so the series converges if $-2 < x < 2$. It diverges at each endpoint of this interval by the nth-term test for divergence, so its interval of convergence is $(-2,\,2)$.

C11S08.004: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^{1/2}5^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^{1/2}5^n|x|^{n+1}}{(n+1)^{1/2}5^{n+1}|x|^n} = \frac{|x|}{5},$$

so the series converges if $-5 < x < 5$. It also converges if $x = 5$ by the alternating series test, but diverges if $x = -5$ by domination of the harmonic series. Thus its interval of convergence is $(-5, 5]$.

C11S08.005: Given the series $\displaystyle\sum_{n=1}^{\infty} n!x^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{n!|x|^n} = \lim_{n\to\infty} n|x|.$$

This limit is zero if $x = 0$ but is $+\infty$ otherwise. Therefore the series converges only at the real number $x = 0$. Thus its interval of convergence is $[0, 0]$. If you prefer the strict interpretation of the word "interval," the interval $[a, b]$ is defined only if $a < b$ according to Appendix A. If so, we must say that this series has no interval of convergence and that it converges only if $x = 0$.

C11S08.006: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^n|x|^{n+1}}{(n+1)^{n+1}|x|^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{|x|}{n+1} = \frac{1}{e} \cdot \left(\lim_{n\to\infty} \frac{|x|}{n+1}\right) = 0$$

for all real x. So the series converges for all x.

C11S08.007: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{n^3}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^3 3^{n+1}|x|^{n+1}}{(n+1)^3 3^n|x|^n} = 3|x|,$$

so the series converges if $-\frac{1}{3} < x < \frac{1}{3}$. When $x = \frac{1}{3}$ it is the p-series with $p = 3 > 1$, and thus it converges. When $x = -\frac{1}{3}$ the series converges by the alternating series test. Therefore its interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

C11S08.008: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n x^n}{\sqrt{2n+1}}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(2n+1)^{1/2}4^{n+1}|x|^{n+1}}{(2n+3)^{1/2}4^n|x|^n} = 4|x|,$$

so the series converges if $-\frac{1}{4} < x < \frac{1}{4}$. When $x = \frac{1}{4}$, the series converges by the alternating series test. When $x = -\frac{1}{4}$, the series diverges by limit-comparison with the p-series for which $p = \frac{1}{2}$.

C11S08.009: Given the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{1/2}(2x)^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)^{1/2}2^{n+1}|x|^{n+1}}{n^{1/2}2^n|x|^n} = 2|x|,$$

so the series converges if $-\frac{1}{2} < x < \frac{1}{2}$. It diverges at each endpoint of this interval by the nth-term test for divergence, and therefore its interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

C11S08.010: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2x^n}{3n-1}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)^2(3n-1)|x|^{n+1}}{n^2(3n+1)|x|^n} = |x|,$$

so this series converges if $-1 < x < 1$. At each endpoint of this interval it diverges by the nth-term test for divergence, so its interval of convergence is $(-1,\,1)$. The `Sum` command in *Mathematica* 3.0 returns the sum of the series in the form

$$\frac{4x - x^2}{9(x-1)^2} + \frac{x}{18}\,{}_2\mathrm{F}_1\!\left(1,\ \frac{1}{2};\ \frac{5}{3};\ x\right)$$

where ${}_2\mathrm{F}_1$ is the hypergeometric function discussed in one of the solutions in Section 10.7.

C11S08.011: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n nx^n}{2^n(n+1)^3}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)^4 2^n|x|^{n+1}}{n(n+2)^3 2^{n+1}|x|^n} = \frac{|x|}{2},$$

so this series converges if $-2 < x < 2$. If $x = 2$ it becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)^3},$$

which converges by the alternating series test. If $x = -2$ it becomes

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^3},$$

which converges because it is dominated by the p-series with $p = 2 > 1$. Therefore its interval of convergence is $[-2,\,2]$.

C11S08.012: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}x^n}{10^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)^{10}10^n|x|^{n+1}}{n^{10}10^{n+1}|x|^n} = \frac{|x|}{10},$$

so this series converges if $-10 < x < 10$. At each endpoint of this interval it diverges by the nth-term test for divergence. Therefore its interval of convergence is $(-10,\,10)$. We were astounded to find—with the aid of the `Sum` command in *Mathematica* 3.0—that the sum of this series (on that interval) is a rational function; viz.,

$$-\frac{10x}{(x-10)^{11}} \cdot (x^9 + 10130x^8 + 4784000x^7 + 455192000x^6 + 13103540000x^5$$
$$+ 131035400000x^4 + 455192000000x^3 + 478400000000x^2 + 101300000000x + 1000000000).$$

C11S08.013: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)x^n}{3^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{[\ln(n+1)]\,3^n|x|^{n+1}}{[\ln n]\,3^{n+1}|x|^n} = \frac{|x|}{3},$$

so the series converges if $-3 < x < 3$. At the two endpoints of this interval it diverges by the nth-term test for divergence. Hence its interval of convergence is $(-3, 3)$. Note:

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

by l'Hôpital's rule.

C11S08.014: Given the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 4^n x^n}{n \ln n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n\,[\ln n]\,4^{n+1}|x|^{n+1}}{(n+1)\,[\ln(n+1)]\,4^n|x|^n} = 4|x|,$$

so this series converges if $-\frac{1}{4} < x < \frac{1}{4}$. If $x = \frac{1}{4}$ then this series converges by the alternating series test. If $x = -\frac{1}{4}$ then it diverges by the integral test—see the solution of Problem 7 of Section 11.5. Thus its interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right]$. Use l'Hôpital's rule as in the solution of Problem 13 to show that

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = 1.$$

C11S08.015: Given the series $\displaystyle\sum_{n=0}^{\infty} (5x-3)^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{|5x-3|^{n+1}}{|5x-3|^n} = |5x-3|,$$

and we solve $|5x-3| < 1$ as follows:

$$-1 < 5x - 3 < 1; \qquad 2 < 5x < 4; \qquad \frac{2}{5} < x < \frac{4}{5}.$$

So this series converges on the interval $I = \left(\frac{2}{5}, \frac{4}{5}\right)$. It diverges at each endpoint by the nth-term test for divergence, so I is its interval of convergence. On this interval it is a geometric series with ratio in $(-1, 1)$, so its sum is

$$\sum_{n=0}^{\infty} (5x-3)^n = \frac{1}{1-(5x-3)} = \frac{1}{4-5x}$$

provided that x is in I.

C11S08.016: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^4+16}$, the ratio test yields

$$\lim_{n\to\infty} \frac{\left[n^4+16\right]|2x-1|^{n+1}}{\left[(n+1)^4+16\right]|2x-1|^n} = |2x-1|.$$

Note:

$$\lim_{n\to\infty} \frac{n^4+16}{(n+1)^4+16} = \lim_{n\to\infty} \frac{1+\dfrac{16}{n^4}}{\left(\dfrac{n+1}{n}\right)^4 + \dfrac{16}{n^4}} = \frac{1+0}{1+0} = 1.$$

Next we solve $|2x-1| < 1$:

$$-1 < 2x - 1 < 1; \qquad 0 < 2x < 2; \qquad 0 < x < 1.$$

So the given series converges if $0 < x < 1$. If $x = 0$ it converges by the alternating series test. If $x = 1$ it converges because it is dominated by the p-series with $p = 4 > 1$. Therefore its interval of convergence is $[0, 1]$. *Mathematica* 3.0 can sum this series in closed form with its `Sum` command, but the resulting expression is a nested sum of about 113 terms involving the imaginary number i and the Lerch transcendent function

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}$$

(in which any term in the sum for which $k + a = 0$ is excluded). Therefore we have not included the *Mathematica* output here. Out of curiosity, we tested *Mathematica* with some much simpler variants of the original series in Problem 16; here are the results:

```
Sum[ (x∧n)/(n∧4), {n, 1, Infinity} ]
```

$$\mathrm{Li}_4(x).$$

```
Sum[ (x∧n)/(n∧2 + 1), {n, 1, Infinity} ]
```

$$\frac{1-i}{4}\big[-1 - i + (1+i)\cdot {}_2\mathrm{F}_1(i,\, 1;\; 1+i;\; x) + x \cdot {}_2\mathrm{F}_1(1,\, 1-i;\; 2-i;\; x)\big].$$

The polylogarithm and hypergeometric functions are discussed earlier in these solutions.

C11S08.017: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{n^2}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^2 2^{n+1}|x-3|^{n+1}}{(n+1)^2 2^n |x-3|^n} = 2|x-3|,$$

so this series converges if $2|x - 3| < 1$:

$$|x-3| < \frac{1}{2}; \qquad -\frac{1}{2} < x - 3 < \frac{1}{2}; \qquad \frac{5}{2} < x < \frac{7}{2}.$$

If $x = \frac{5}{2}$, the series converges by the alternating series test. If $x = \frac{7}{2}$, it converges because it is the p-series with $p = 2 > 1$. Thus its interval of convergence is $\left[\frac{5}{2},\, \frac{7}{2}\right]$.

C11S08.018: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)!n^n|x|^{n+1}}{n!(n+1)^{n+1}|x|^n} = |x| \cdot \left[\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n\right] = \frac{|x|}{e}.$$

We are given divergence at the endpoints, but this can be derived using the result in Miscellaneous Problem 61 of Chapter 11. Either way, the interval of convergence of this series is $(-e, e)$.

C11S08.019: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!}x^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{n!(2n+2)!|x|^{n+1}}{(n+1)!(2n)!|x|^n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)|x|}{n+1} = \lim_{n\to\infty} 2(2n+1)|x|.$$

The last limit is $+\infty$ if $x \neq 0$ but zero if $x = 0$. Therefore this series converges only at the single point $x = 0$.

C11S08.020: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} x^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{n! \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)|x|^{n+1}}{(n+1)! \cdot 1 \cdot 3 \cdot 5 \cdots (2n+1)|x|^n} = \lim_{n\to\infty} \frac{2n+3}{n+1}|x| = 2|x|,$$

so this series converges if $-\frac{1}{2} < x < \frac{1}{2}$. We are given divergence at both endpoints, so its interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. To *derive* divergence at the endpoints, note that

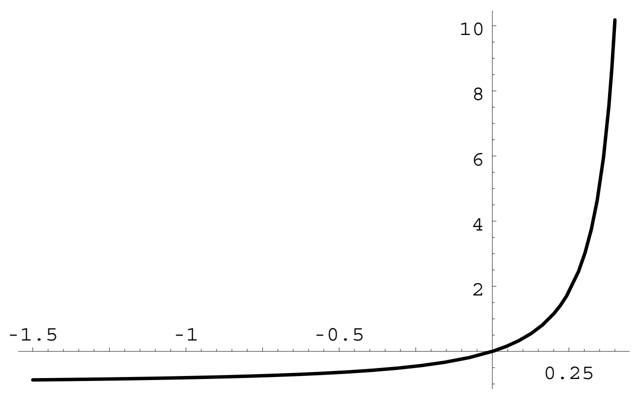
$$\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n! \cdot 2^n} = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdots \frac{2n+1}{2n} > 1,$$

so that the power series fails to converge if $x = \pm\frac{1}{2}$ by the nth-term test for divergence.

The usual *Mathematica* 3.0 command yields a simple algebraic function for the sum of this series on its interval of convergence; it is

$$f(x) = \frac{1}{(1-2x)^{3/2}} - 1,$$

and its graph is next.



C11S08.021: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(x+1)^n}{3^n}$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)^3 3^n |x+1|^{n+1}}{n^3 3^{n+1}|x+1|^n} = \frac{|x+1|}{3},$$

so the given series converges if

$$-1 < \frac{x+1}{3} < 1; \qquad -3 < x+1 < 3; \qquad -4 < x < 2.$$

At the endpoints of this interval, the series diverges by the nth-term test for divergence. Thus its interval of convergence is $(-4, 2)$. To find its sum in closed form, let

$$f(x) = \sum_{n=1}^{\infty} \frac{n^3(x+1)^n}{3^n}.$$

Then

$$f(x) = (x+1)\sum_{n=1}^{\infty} \frac{n^3(x+1)^{n-1}}{3^n} = (x+1)g'(x)$$

where

$$g(x) = \sum_{n=1}^{\infty} \frac{n^2(x+1)^n}{3^n} = (x+1)\sum_{n=1}^{\infty} \frac{n^2(x+1)^{n-1}}{3^n}.$$

But $g(x) = (x+1)h'(x)$ where

$$h(x) = \sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^n} = (x+1)\sum_{n=1}^{\infty} \frac{n(x+1)^{n-1}}{3^n} = (x+1)k'(x)$$

where

$$k(x) = \sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n} = \frac{x+1}{2-x}$$

if $-4 < x < 2$ because the last series is geometric and convergent for such x. It now follows that

$$k'(x) = \frac{3}{(x-2)^2}; \qquad h(x) = \frac{3(x+1)}{(x-2)^2};$$

$$h'(x) = -\frac{3(x+4)}{(x-2)^3}; \qquad g(x) = -\frac{3(x+1)(x+4)}{(x-2)^3};$$

$$g'(x) = \frac{3(x^2+14x+22)}{(x-2)^4}; \qquad f(x) = \frac{3(x+1)(x^2+14x+22)}{(x-2)^4}.$$

C11S08.022: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n^2}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^2|x-2|^{n+1}}{(n+1)^2|x-2|^n} = |x-2|,$$

and thus the series converges if $-1 < x-2 < 1$; that is, if $1 < x < 3$. It also converges if $x = 3$ by the alternating series test and converges if $x = 1$ because then it is the p-series with $p = 2 > 1$. Therefore its interval of convergence is $[1, 3]$.

C11S08.023: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(3-x)^n}{n^3}$, the ratio test yields

$$\lim_{n\to\infty} \frac{n^3|3-x|^{n+1}}{(n+1)^3|3-x|^n} = |3-x| = |x-3|,$$

so this series converges if $-1 < x-3 < 1$; that is, if $2 < x < 4$. It also converges at $x = 2$ because it is the p-series with $p = 3 > 1$ and converges at $x = 4$ by the alternating series test. Therefore its interval of convergence is $[2, 4]$.

C11S08.024: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}10^n}{n!}(x-10)^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{n!10^{n+1}|x-10|^{n+1}}{(n+1)!10^n|x-10|^n} = \lim_{n\to\infty} \frac{10|x-10|}{n+1} = 0$$

for all real numbers x. Therefore this series converges on the set $(-\infty, +\infty)$ of all real numbers. You can use the series in Eq. (19) of Section 11.4—of which this series is a special case—to write its sum in closed form. (Watch out for the "missing" term corresponding to $n = 0$.) You should find that its sum is $1 - \exp(100 - 10x)$.

C11S08.025: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}(x-5)^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{(n+1)!2^n|x-5|^{n+1}}{n!2^{n+1}|x-5|^n} = \lim_{n\to\infty} \frac{n+1}{2}|x-5| = +\infty$$

unless $x = 5$, in which case the limit is zero. So this series converges only at the single point $x = 5$; its radius of convergence is zero.

C11S08.026: Given the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\cdot 10^n}(x-2)^n$, the ratio test yields

$$\lim_{n\to\infty} \frac{n\cdot 10^n\cdot|x-2|^{n+1}}{(n+1)\cdot 10^{n+1}\cdot|x-2|^n} = \frac{|x-2|}{10},$$

and therefore this series converges if $-10 < x-2 < 10$; that is, if $-8 < x < 12$. If $x = 12$ it converges by the alternating series test; if $x = -8$ it diverges because it is then the harmonic series. To write its sum in closed form, let

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\cdot 10^n}(x-2)^n; \qquad \text{then} \qquad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{10^n}(x-2)^{n-1}.$$

The series for $f'(x)$ is geometric, with sum

$$\frac{\frac{1}{10}}{1+\frac{x-2}{10}} = \frac{1}{10+x-2} = \frac{1}{x+8}.$$

Therefore $f(x) = C + \ln(x+8)$; $f(2) = 0 = C + \ln 10$, and thus

$$f(x) = -\ln 10 + \ln(x+8) = \ln\left(\frac{x+8}{10}\right).$$

C11S08.027: Given the series $\displaystyle\sum_{n=0}^{\infty} x^{(2^n)}$, the ratio test yields

$$\lim_{n\to\infty} \frac{|x|^{(2^{n+1})}}{|x|^{(2^n)}} = \lim_{n\to\infty} |x|^{(2^n)}.$$

This limit is zero if $-1 < x < 1$, is 1 if $x = \pm 1$, and is $+\infty$ if $|x| > 1$. The series diverges if $x = \pm 1$ by the nth-term test for divergence, and hence its interval of convergence is $(-1, 1)$.

C11S08.028: Given the series

$$\sum_{n=0}^{\infty}\left(\frac{x^2+1}{5}\right)^n,$$

note that it is geometric with first term 1 and ratio $(x^2+1)/5$. Hence it converges if

$$-1 < \frac{x^2+1}{5} < 1; \qquad -5 < x^2+1 < 5;$$
$$0 \leqq x^2 < 4; \qquad -2 < x < 2.$$

Its interval of convergence is $(-2,\,2)$ and its sum there is $\dfrac{5}{4-x^2}$.

C11S08.029: Given the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$, the ratio test yields

$$\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot|x|^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)\cdot|x|^n} = \lim_{n\to\infty}\frac{|x|}{2n+1} = 0$$

for all x. Hence the interval of convergence of this series is $(-\infty,\,+\infty)$.

C11S08.030: Given the series $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}x^n$, the ratio test yields

$$\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)\cdot 2\cdot 5\cdot 8\cdots(3n-1)\cdot|x|^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2\cdot 5\cdot 8\cdots(3n-1)\cdot(3n+2)\cdot|x|^n} = \lim_{n\to\infty}\frac{2n+1}{3n+2}|x| = \frac{2|x|}{3}.$$

Therefore this series converges if $|x| < \frac{3}{2}$ and diverges if $|x| > \frac{3}{2}$. If $x = \frac{3}{2}$, then the given series has nth term

$$a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot 3^n}{2\cdot 5\cdot 8\cdots(3n-1)\cdot 2^n},$$

and

$$\frac{a_{n+1}}{a_n} = \frac{3(2n+1)}{2(3n+2)} = \frac{6n+3}{6n+4} = 1-\frac{1}{6n+4} > 1-\frac{1}{n+1} = \frac{n}{n+1}$$

for every integer $n \geqq 1$. Thus for such n we have

$$a_{n+1} > \frac{n}{n+1}\cdot a_n > \frac{n}{n+1}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdots\frac{1}{2}\cdot a_1;$$
$$a_{n+1} > \frac{1}{2(n+1)}\cdot\frac{3}{4} = \frac{3}{8(n+1)}.$$

Therefore $a_n > \dfrac{3}{8n}$ for all $n \geqq 1$. Hence

$$\sum_{n=1}^{\infty} a_n \quad \text{dominates} \quad \frac{3}{8}\sum_{n=1}^{\infty}\frac{1}{n};$$

thus the series of this problem diverges if $x = \frac{3}{2}$. For the case $x = -\frac{3}{2}$, more care is needed. Substitute $x = +\frac{3}{2}$. Then the nth term of the original series is

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 3^n}{2 \cdot 5 \cdot 8 \cdots (3n-1) \cdot 2^n} = \frac{\dfrac{1}{2} \cdot \dfrac{3}{2} \cdot \dfrac{5}{2} \cdots \dfrac{2n-1}{2}}{\dfrac{2}{3} \cdot \dfrac{5}{3} \cdot \dfrac{8}{3} \cdots \dfrac{3n-1}{3}} = \frac{\left(1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)\left(3-\frac{1}{2}\right)\cdots\left(n-\frac{1}{2}\right)}{\left(1-\frac{1}{3}\right)\left(2-\frac{2}{3}\right)\left(3-\frac{1}{3}\right)\cdots\left(n-\frac{1}{3}\right)}.$$

Therefore

$$\ln a_n = \sum_{k=1}^{n} \ln\left(\frac{k-\frac{1}{2}}{k-\frac{1}{3}}\right) = \sum_{k=1}^{n} \ln\left(\frac{k-\frac{1}{3}-\frac{1}{6}}{k-\frac{1}{3}}\right) = \sum_{k=1}^{n} \ln\left(1 - \frac{1}{6k-2}\right).$$

The linear approximation to $f(x) = \ln x$ at $(1,\,0)$ is $L(x) = x - 1$. Because the graph of f is concave downward everywhere, $\ln x < x - 1$ if $0 < x < 1$. Substitute

$$x = 1 - \frac{1}{6k-2}$$

in the inequality $\ln x < x - 1$ to conclude that

$$\ln\left(1 - \frac{1}{6k-2}\right) < -\frac{1}{6k-2}$$

for every positive integer k. Therefore

$$\ln a_n < -\sum_{k=1}^{n} \frac{1}{6k-2}$$

for every positive integer n. The last series diverges to $-\infty$ by limit-comparison with the harmonic series. It follows that

$$\lim_{n\to\infty} \ln a_n = -\infty, \qquad \text{and thus that} \qquad \lim_{n\to\infty} a_n = 0.$$

Therefore when we substitute $x = -\frac{3}{2}$ in the original power series of this problem, the resulting numerical series satisfies the criteria of the alternating series test and consequently converges. Thus the interval of convergence of the given power series is $\left[-\frac{3}{2},\, \frac{3}{2}\right)$.

The reader familiar with convergence tests will recognize that the last argument of this solution derives from a *failure* of Raabe's test; we owe great thanks to Ed Azoff for suggestions that led to this argument. The sum of this series (on its interval of convergence) can be expressed in closed form with the aid of *Mathematica*; it is

$$\frac{x}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{3}\right)} \cdot {}_2\mathrm{F}_1\left(1,\, \frac{3}{2};\, \frac{5}{3};\, \frac{2x}{3}\right).$$

See earlier solutions in Chapter 11 for information about the hypergeometric function ${}_2\mathrm{F}_1$; see the subsection on *Special Functions* in Section 8.8, and several end-of-section problems, for more about the gamma function.

C11S08.031: The function is the sum of a geometric series with first term and ratio x, and hence

$$f(x) = \frac{x}{1-x} = x + x^2 + x^3 + x^4 + x^5 + \cdots.$$

This series has radius of convergence 1 and interval of convergence $(-1,\,1)$.

C11S08.032: We write $f(x)$ in the form of the sum of a geometric series:

$$f(x) = \frac{1}{10+x} = \frac{\frac{1}{10}}{1+\frac{1}{10}x} = \frac{1}{10} - \frac{x}{10^2} + \frac{x^2}{10^3} - \frac{x^3}{10^4} + \frac{x^4}{10^5} - \cdots .$$

The radius of convergence of this series is 10 and its interval of convergence is $(-10, 10)$.

C11S08.033: Substitute $-3x$ for x in the Maclaurin series for e^x in Eq. (2), then multiply by x^2 to obtain

$$\begin{aligned} f(x) = x^2e^{-3x} &= x^2\left(1 - \frac{3x}{1!} + \frac{9x^2}{2!} - \frac{27x^3}{3!} + \frac{81x^4}{4!} - \frac{243x^5}{5!} + \cdots\right) \\ &= x^2 - \frac{3x^3}{1!} + \frac{3^2x^4}{2!} - \frac{3^3x^5}{3!} + \frac{3^4x^6}{4!} - \frac{3^5x^7}{5!} + \cdots . \end{aligned}$$

The ratio test gives radius of convergence $+\infty$, so the interval of convergence of this series is $(-\infty, +\infty)$.

C11S08.034: Write $f(x)$ in the form of the sum of a geometric series:

$$f(x) = \frac{x}{9-x^2} = \frac{\frac{1}{9}x}{1-\left(\frac{1}{3}x\right)^2} = \frac{x}{3^2} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \frac{x^7}{3^8} + \frac{x^9}{3^{10}} + \cdots .$$

This series has radius of convergence 3 and interval of convergence $(-3, 3)$.

C11S08.035: Substitute x^2 for x in the Maclaurin series in (4) to obtain

$$f(x) = \sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots .$$

The ratio test yields radius of convergence $+\infty$, so the interval of convergence of this series is $(-\infty, +\infty)$.

C11S08.036: Substitution in the Maclaurin series in Eq. (3) yields

$$\begin{aligned} f(x) = \cos^2 2x &= \frac{1}{2}(1+\cos 4x) \\ &= \frac{1}{2}\left(1 + 1 - \frac{4^2x^2}{2!} + \frac{4^4x^4}{4!} - \frac{4^6x^6}{6!} + \frac{4^8x^8}{8!} - \frac{4^{10}x^{10}}{10!} + \cdots\right) \\ &= 1 - \frac{2^3x^2}{2!} + \frac{2^7x^4}{4!} - \frac{2^{11}x^6}{6!} + \frac{2^{15}x^8}{8!} - \frac{2^{19}x^{10}}{10!} + \cdots . \end{aligned}$$

The ratio test yields radius of convergence $+\infty$, so the interval of convergence of this series is $(-\infty, +\infty)$.

C11S08.037: Substitution of $\alpha = \frac{1}{3}$ in the binomial series in Eq. (14) yields

$$\begin{aligned} (1+x)^{1/3} &= 1 + \frac{1}{3}x - \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{x^3}{3!} - \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdot\frac{x^4}{4!} + \cdots \\ &= 1 + \frac{1}{3}x - \frac{2}{3^2}\cdot\frac{x^2}{2!} + \frac{2\cdot 5}{3^3}\cdot\frac{x^3}{3!} - \frac{2\cdot 5\cdot 8}{3^4}\cdot\frac{x^4}{4!} + \cdots . \end{aligned}$$

Next, replacement of x with $-x$ yields

$$f(x) = (1-x)^{1/3} = 1 - \frac{1}{3}x - \frac{2}{3^2}\cdot\frac{x^2}{2!} - \frac{2\cdot 5}{3^3}\cdot\frac{x^3}{3!} - \frac{2\cdot 5\cdot 8}{3^4}\cdot\frac{x^4}{4!} - \frac{2\cdot 5\cdot 8\cdot 11}{3^5}\cdot\frac{x^5}{5!} - \cdots .$$

The radius of convergence of this series is 1.

C11S08.038: Substitution of $\alpha = \frac{3}{2}$ in the binomial series in Eq. (14) yields

$$(1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{3\cdot 1}{2^2}\cdot\frac{x^2}{2!} - \frac{3\cdot 1\cdot 1}{2^3}\cdot\frac{x^3}{3!} + \frac{3\cdot 1\cdot 1\cdot 3}{2^4}\cdot\frac{x^4}{4!} - \frac{3\cdot 1\cdot 1\cdot 3\cdot 5}{2^5}\cdot\frac{x^5}{5!} + \cdots.$$

Then replacement of x with x^2 yields

$$f(x) = (1+x^2)^{3/2} = 1 + \frac{3}{2}x^2 + \frac{3}{2^2}\cdot\frac{x^4}{2!} - \frac{3\cdot 1}{2^3}\cdot\frac{x^6}{3!} + \frac{3\cdot 1\cdot 3}{2^4}\cdot\frac{x^8}{4!} - \frac{3\cdot 1\cdot 3\cdot 5}{2^5}\cdot\frac{x^{10}}{5!} + \cdots.$$

The radius of convergence of this series is 1.

C11S08.039: Substitution of $\alpha = -3$ in the binomial series in Eq. (14) yields

$$f(x) = (1+x)^{-3} = 1 - 3x + 3\cdot 4\cdot\frac{x^2}{2!} - 3\cdot 4\cdot 5\cdot\frac{x^3}{3!} + 3\cdot 4\cdot 5\cdot 6\cdot\frac{x^4}{4!} - \cdots.$$

The radius of convergence of this series is 1.

C11S08.040: Rewrite $f(x)$ in such a way that the binomial series in Eq. (14) can be used:

$$f(x) = (9+x^3)^{-1/2} = \frac{1}{3}\left(1 + \frac{x^3}{9}\right)^{-1/2}.$$

According to Eq. (14),

$$(1+z)^{-1/2} = 1 - \frac{1}{2}z + \frac{1\cdot 3}{2^2}\cdot\frac{z^2}{2!} - \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{z^3}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{z^4}{4!} - \cdots.$$

Multiply by $\frac{1}{3}$ and substitute $x^3/9$ for z to obtain

$$f(x) = \frac{1}{\sqrt{9+x^3}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}x^3 + \frac{1\cdot 3}{2^2\cdot 3^5}\cdot\frac{x^6}{2!} - \frac{1\cdot 3\cdot 5}{2^3\cdot 3^7}\cdot\frac{x^9}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 3^9}\cdot\frac{x^{12}}{4!} - \cdots.$$

This series converges when $-1 < \dfrac{x^3}{9} < 1$, so its radius of convergence is $9^{1/3}$.

C11S08.041: Let $g(x) = \ln(1+x)$. Then

$$g'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots, \quad -1 < x < 1.$$

So

$$g(x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

by Theorem 3. Also $0 = g(0) = \ln 1 = C$, so that

$$g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots.$$

Therefore

$$f(x) = \frac{g(x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \cdots.$$

The ratio test tells us that the radius of convergence is 1; the interval of convergence is $(-1, 1]$.

C11S08.042: Let $g(x) = \arctan x$. Then

$$g'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - \cdots, \quad -1 < x < 1.$$

Thus

$$g(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots.$$

Also $0 = g(0) + C$. So

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots,$$

and thus

$$x - \arctan x = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \frac{x^{11}}{11} - \cdots.$$

So

$$f(x) = \frac{x - \arctan x}{x^3} = \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \frac{x^6}{9} + \frac{x^8}{11} - \cdots.$$

The ratio test indicates radius of convergence 1; the interval of convergence is $[-1, 1]$.

C11S08.043: Termwise integration yields

$$\begin{aligned} f(x) &= \int_0^x \sin t^3 \, dt = \int_0^x \left(t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \frac{t^{21}}{7!} + \cdots \right) dt \\ &= \left[\frac{t^4}{4} - \frac{t^{10}}{3!10} + \frac{t^{16}}{5!16} - \frac{t^{22}}{7!22} + \cdots \right]_0^x = \frac{x^4}{4} - \frac{x^{10}}{3!10} + \frac{x^{16}}{5!16} - \frac{x^{22}}{7!22} + \cdots. \end{aligned}$$

This representation is valid for all real x.

C11S08.044: Termwise integration yields

$$\begin{aligned} f(x) &= \int_0^x \frac{\sin t}{t} \, dt = \int_0^x \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots \right) dt \\ &= \left[t - \frac{t^3}{3!3} + \frac{t^5}{5!5} - \frac{t^7}{7!7} + \cdots \right]_0^x = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \frac{x^7}{7!7} + \cdots. \end{aligned}$$

This representation is valid for all real x.

C11S08.045: Termwise integration yields

$$\begin{aligned} f(x) &= \int_0^x \exp\left(-t^3\right) \, dt = \int_0^x \left(1 - t^3 + \frac{t^6}{2!} - \frac{t^9}{3!} + \frac{t^{12}}{4!} - \cdots \right) dt \\ &= \left[t - \frac{t^4}{4} + \frac{t^7}{2!7} - \frac{t^{10}}{3!10} + \frac{t^{13}}{4!13} - \cdots \right]_0^x = x - \frac{x^4}{4} + \frac{x^7}{2!7} - \frac{x^{10}}{3!10} + \frac{x^{13}}{4!13} - \cdots. \end{aligned}$$

This representation is valid for all real x.

C11S08.046: One result in the solution of Problem 42 yields

$$f(x) = \int_0^x \frac{\arctan t}{t}\,dt = \int_0^x \left(1 - \frac{t^2}{3} + \frac{t^4}{5} - \frac{t^6}{7} + \frac{t^8}{9} - \cdots\right) dt$$

$$= \left[t - \frac{t^3}{3^2} + \frac{t^5}{5^2} - \frac{t^7}{7^2} + \frac{t^9}{9^2} - \cdots\right]_0^x = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \frac{x^9}{9^2} - \cdots.$$

This representation is valid for $-1 \leqq x \leqq 1$. The sum of the last series in the case $x = 1$ is known as *Catalan's constant*, which is connected with estimates in the theory of combinatorial functions.

C11S08.047: First,

$$1 - \exp\left(-t^2\right) = 1 - \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) = t^2 - \frac{t^4}{2!} + \frac{t^6}{3!} - \frac{t^8}{4!} + \frac{t^{10}}{5!} - \cdots.$$

Then termwise integration yields

$$f(x) = \int_0^x \frac{1 - \exp\left(-t^2\right)}{t^2}\,dt = \int_0^x \left(1 - \frac{t^2}{2!} + \frac{t^4}{3!} - \frac{t^6}{4!} + \frac{t^8}{5!} - \cdots\right) dt$$

$$= \left[t - \frac{t^3}{2!\cdot 3} + \frac{t^5}{3!\cdot 5} - \frac{t^7}{4!\cdot 7} + \frac{t^9}{5!\cdot 9} - \cdots\right]_0^x = x - \frac{x^3}{2!\cdot 3} + \frac{x^5}{3!\cdot 5} - \frac{x^7}{4!\cdot 7} + \frac{x^9}{5!\cdot 9} - \cdots.$$

This representation is valid for all real x.

C11S08.048: Termwise integration yields

$$\tanh^{-1} x = \int_0^x \frac{1}{1 - t^2}\,dt = \int_0^x \left(1 + t^2 + t^4 + t^6 + t^8 + \cdots\right) dt$$

$$= \left[t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \frac{t^9}{9} + \cdots\right]_0^x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots.$$

This representation is valid if $-1 < x < 1$.

C11S08.049: We begin with

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1 - x}, \quad -1 < x < 1.$$

Then termwise differentiation yields

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1 - x)^2}; \quad \text{thus}$$

$$xf'(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1 - x)^2}, \quad -1 < x < 1.$$

C11S08.050: Termwise differentiation of the second series in the solution of Problem 49 yields

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}. \quad \text{Hence}$$

$$x^2 f''(x) = \sum_{n=2}^{\infty} n(n-1)x^{n} = \frac{2x^2}{(1-x)^3}, \quad -1 < x < 1.$$

C11S08.051: We found in the solution of Problem 49 that if

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1,$$

then

$$xf'(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad -1 < x < 1.$$

Therefore

$$D_x\left[xf'(x)\right] = \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3},$$

and hence

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \quad -1 < x < 1.$$

C11S08.052: We saw in the solution of Problem 49 that

$$G(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad -1 < x < 1.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = G\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{4}} = 2.$$

We saw in the solution of Problem 51 that

$$H(x) = \sum_{n=1}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}, \quad -1 < x < 1.$$

Thus

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = H\left(\frac{1}{3}\right) = \frac{\frac{1}{3}+\frac{1}{9}}{\left(\frac{2}{3}\right)^3} = \frac{3}{2}.$$

In connection with our earlier discussions of the polylogarithm function, you can verify that both the series of Problem 52 are special cases of the series

$$\sum_{n=1}^{\infty} \frac{n^p}{q^n} \;=\; \mathrm{Li}_{(-p)}\left(\frac{1}{q}\right).$$

C11S08.053: If

$$\begin{aligned} y = e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots, \quad \text{then} \\ \frac{dy}{dx} &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!} + \frac{(n+1)x^n}{(n+1)!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \cdots = e^x = y. \end{aligned}$$

C11S08.054: If

$$\begin{aligned} y = \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots, \quad \text{then} \\ u = \frac{dy}{dx} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots = \cos x, \\ \frac{d^2y}{dx^2} &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots = -\sin x, \quad \text{and} \\ \frac{d^2u}{dx^2} = \frac{d^3y}{dx^3} &= -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = -\cos x. \end{aligned}$$

Therefore both the sine and cosine functions satisfy the differential equation $\dfrac{d^2y}{dx^2} + y \;=\; 0.$

C11S08.055: If

$$\begin{aligned} y = \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \cdots, \quad \text{then} \\ \frac{dy}{dx} &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \cdots = \cosh x \quad \text{and} \\ \frac{d^2y}{dx^2} &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sinh x. \end{aligned}$$

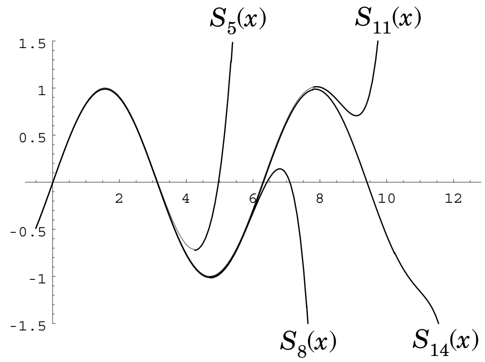
Therefore both the hyperbolic sine and hyperbolic cosine functions satisfy the differential equation

$$\frac{d^2y}{dx^2} \;-\; y \;=\; 0.$$

C11S08.056: We let

$$S_k(x) = \sum_{n=1}^{k} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$$

for $k \geqq 1$ and plotted $y = S_k(x)$ for $k = 5,\ 8,\ 11,$ and 14 simultaneously using *Mathematica* 3.0. These graphs are next.



All four graphs cross first cross the positive x-axis at a point between 3.1 and 3.2. To locate this point more accurately, we applied Newton's method to the equation $S_k(x) = 0$ for all four values of k, using the initial guess $x_0 = 3$. Here are the results:

	$k = 5$	$k = 8$	$k = 11$	$k = 14$
x_1	3.1490725899	3.1425458981	3.1425465431	3.1425465431
x_2	3.1486900652	3.1415918805	3.1415926533	3.1415926533
x_3	3.1486900715	3.1415918808	3.1415926536	3.1415926536
x_4	3.1486900716	3.1415918808	3.1415926536	3.1415926536

There can be little doubt that, to ten places, $\pi = 3.1415926536$.

C11S08.057: From Example 7 we have

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}; \qquad \text{we are also given} \qquad J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n!(n+1)!}.$$

We apply the ratio test to the series for $J_1(x)$ with the following result:

$$\lim_{n\to\infty} \frac{2^{2n+1} n!(n+1)!|x|^{2n+3}}{2^{2n+3}(n+1)!(n+2)!|x|^{2n+1}} = \lim_{n\to\infty} \frac{x^2}{4(n+1)(n+2)} = 0$$

for all x. Therefore this series converges for all x. Next,

$$\begin{aligned} J_0'(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+1}(n+1)!n!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!n!} = -J_1(x). \end{aligned}$$

C11S08.058: First,

$$xJ_0(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n}(n!)^2}.$$

Therefore term-by-term integration yields

$$\int xJ_0(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)2^{2n}(n!)^2} + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1}(n+1)!n!} + C$$
$$= x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(n+1)!n!} + C = xJ_1(x) + C.$$

C11S08.059: We begin with

$$y(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}. \quad \text{Then:}$$
$$y'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{2^{2n+2}[(n+1)!]^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2^{2n+1}n!(n+1)!};$$
$$y''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)x^{2n}}{2^{2n+1}n!(n+1)!};$$
$$x^2y''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)x^{2n+2}}{2^{2n+1}n!(n+1)!};$$
$$xy'(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+2}}{2^{2n+1}n!(n+1)!};$$
$$x^2y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}.$$

Note that the coefficient n in Bessel's equation is zero. Therefore

$$x^2y''(x) + xy'(x) + x^2y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}\left[-\frac{2n+1}{2(n+1)} - \frac{1}{2(n+1)} + 1\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}\left[-\frac{2n+2}{2n+2} + 1\right] \equiv 0.$$

C11S08.060: We begin with

$$y(x) = J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}n!(n+1)!}. \qquad \text{Then:}$$
$$x^2y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{2^{2n+1}n!(n+1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n+1}}{2^{2n-1}(n-1)!n!};$$
$$y'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)x^{2n}}{2^{2n+1}n!(n+1)!};$$
$$xy'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)x^{2n+1}}{2^{2n+1}n!(n+1)!};$$

$$y''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)(2n)x^{2n-1}}{2^{2n+1}n!(n+1)!};$$

$$x^2y''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)(2n)x^{2n+1}}{2^{2n+1}n!(n+1)!},$$

Take $n = 1$ in Bessel's equation. Then

$$\begin{aligned} x^2y''(x) + xy'(x) + (x^2-1)y(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)(2n)x^{2n+1}}{2^{2n+1}n!(n+1)!} + \frac{x}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)x^{2n+1}}{2^{2n+1}n!(n+1)!} \\ &\quad - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}n!(n+1)!} - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}n!(n+1)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n-1}(n-1)!n!} \left[\frac{(2n+1)(2n)}{4n(n+1)} + \frac{2n+1}{4n(n+1)} - 1 - \frac{1}{4n(n+1)} \right]. \end{aligned}$$

The last factor (in brackets) is

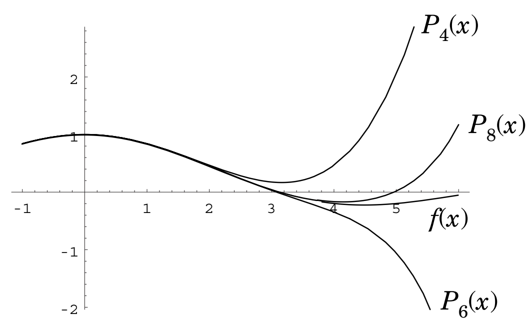
$$\frac{2n+1}{2(n+1)} + \frac{2n+1}{4n(n+1)} - 1 - \frac{1}{4n(n+1)} = \frac{2n+1}{2(n+1)} + \frac{1}{2(n+1)} - 1 = \frac{2n+2}{2n+2} - 1 \equiv 0.$$

This establishes that $y(x) = J_1(x)$ satisfies Bessel's equation of order 1.

C11S08.061: The Taylor series of f centered at $a = 0$ is

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots.$$

This representation is valid for all real x. We plotted the Taylor polynomials for $f(x)$ with center $a = 0$ of degree 4, 6, and 8 and the graph of $y = f(x)$ simultaneously, with the following result.



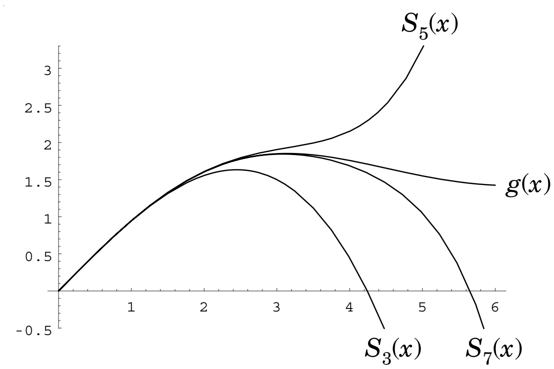
C11S08.062: Using the solution of Problem 61, we have

$$g(x) = \left[t - \frac{t^3}{3!3} + \frac{t^5}{5!5} - \frac{t^7}{7!7} + \cdots \right]_0^x = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \frac{x^7}{7!7} + \cdots.$$

The ratio test yields

$$\lim_{n\to\infty} \frac{(2n+1)!(2n+1)|x|^{2n+3}}{(2n+3)!(2n+3)|x|^{2n+1}} = \lim_{n\to\infty} \frac{(2n+1)x^2}{(2n+3)^2(2n+2)} = 0$$

for all real x. Hence this series converges for all x. The following figure shows the graph of $y = g(x)$ and the graphs of its Taylor polynomials $S_k(x)$ of degree k for $k = 3,\ 5,$ and 7.



C11S08.063: Equation 20 is

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \quad -1 < x < 1.$$

Thus

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}, \quad -1 < x < 1.$$

Consequently,

$$\pi = 6 \cdot \frac{\pi}{6} = 6 \cdot \arctan \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2n} = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n}.$$

Mathematica 3.0 reports that

$$\frac{6}{\sqrt{3}} \sum_{n=0}^{10} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n} \approx 3.1415933045030815,$$

$$\frac{6}{\sqrt{3}} \sum_{n=0}^{20} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n} \approx 3.1415926535956350,$$

$$\frac{6}{\sqrt{3}} \sum_{n=0}^{30} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n} \approx 3.1415926535897932, \quad \text{and}$$

$$\frac{6}{\sqrt{3}} \sum_{n=0}^{40} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n} \approx 3.1415926535897932.$$

C11S08.064: We begin by replacing $\sin xt$ with its Maclaurin series:

$$\begin{aligned}
\int_0^{\infty} e^{-t} \sin xt \, dt &= \int_0^{\infty} e^{-t} \left(xt - \frac{x^3 t^3}{3!} + \frac{x^5 t^5}{5!} - \frac{x^7 t^7}{7!} + \cdots \right) dt \\
&= x \int_0^{\infty} te^{-t} \, dt - \frac{x^3}{3!} \int_0^{\infty} t^3 e^{-t} \, dt + \frac{x^5}{5!} \int_0^{\infty} t^5 e^{-t} \, dt - \frac{x^7}{7!} \int_0^{\infty} t^7 e^{-t} \, dt + \cdots \\
&= x - x^3 + x^5 - x^7 + \cdots = \frac{x}{1+x^2}, \quad -1 < x < 1.
\end{aligned}$$

The next-to-last equality follows from the definition of the gamma function in Eq. (7) of Section 8.8 and the results in Problems 47 and 48 there, which imply that

$$\int_0^\infty t^n e^{-t}\,dt = n!$$

if n is a nonnegative integer.

C11S08.065: Part (a): From the Maclaurin series for the natural exponential function in Eq. (2) of this section, we derive the fact that

$$e^{-t} = \frac{1}{e^t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n$$

for all real t. Substitute $t = x \ln x = \ln(x^x)$ to obtain

$$\frac{1}{x^x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x \ln x)^n$$

if $x > 0$. Part (b): The formula in Problem 53 of Section 8.8 states that if m and n are fixed positive integers, then

$$\int_0^1 x^m (\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}. \tag{1}$$

Moreover, term-by-term integration is valid in the case of the result in part (a), so

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 (x \ln x)^n\,dx.$$

Thus Eq. (1) here yields

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

C11S08.066: Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad -r < x < r$$

where $r > 0$. Then

$$a_0 = f(0) = \frac{f^{(0)}(0)}{0!}.$$

Assume that

$$a_k = \frac{f^{(k)}(0)}{k!} \quad \text{and that} \quad f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k}$$

for some integer $k \geqq 0$. Then

$$f^{(k+1)}(x) = \sum_{n=k+1}^{\infty} n(n-1)\cdots(n-k)a_n x^{n-k-1}.$$

Moreover, substitution of $x = 0$ zeros out every term in the last series except for its first, for which $n = k+1$, and therefore

$$f^{(k+1)}(0) = (k+1)!a_{k+1}, \quad \text{so that} \quad a_{k+1} = \frac{f^{(k+1)}(0)}{(k+1)!}.$$

Therefore, by induction, $a_n = f^{(n)}(0)/n!$ for all $n \geqq 0$. Thus the only power series in powers of x that represents $f(x)$ at and near $x = 0$ is its Maclaurin series.

C11S08.067: Part (a):

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n + \cdots. \end{aligned}$$

$$\begin{aligned} f'(x) = \alpha &+ \frac{\alpha(\alpha-1)}{1!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{3!}x^3 + \cdots \\ &+ \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{(n-1)!}x^{n-1} + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{n!}x^n + \cdots. \end{aligned}$$

$$\begin{aligned} xf'(x) = \alpha x &+ \frac{\alpha(\alpha-1)}{1!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{3!}x^4 + \cdots \\ &+ \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{(n-1)!}x^n + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{n!}x^{n+1} + \cdots. \end{aligned}$$

Therefore

$$\begin{aligned} (1+x)f'(x) &= f'(x) + xf'(x) \\ &= \alpha + (\alpha^2 - \alpha + \alpha)x + \frac{1}{2!}\left[\alpha(\alpha-1)(\alpha-2+2)\right]x^2 + \frac{1}{3!}\left[\alpha(\alpha-1)(\alpha-2)(\alpha-3+3)\right]x^3 + \cdots \\ &\qquad + \frac{1}{n!}\left[\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(\alpha-n+n)\right]x^n + \cdots \\ &= \alpha + \alpha^2 x + \frac{\alpha^2(\alpha-1)}{2!}x^2 + \frac{\alpha^2(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \\ &\qquad + \frac{\alpha^2(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n + \cdots = \alpha f(x). \end{aligned}$$

Part (b): From the result $(1+x)f'(x) = \alpha f(x)$ in part (a), we derive

$$\frac{f'(x)}{f(x)} = \frac{\alpha}{1+x}; \qquad \ln\left(f(x)\right) = C + \alpha\ln(1+x);$$

$$f(x) = K(1+x)^{\alpha}; \qquad 1 = f(0) = K\cdot 1^{\alpha}: \quad K = 1.$$

Therefore $f(x) = (1+x)^\alpha,\ -1 < x < 1$.

C11S08.068: Part(a): $\displaystyle \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}(\arcsin x)^2\right]_0^1 = \frac{1}{2}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{8}.$

Part (b): One of the two results in Problem 58 of Section 8.3 is that

$$\int_0^{\pi/2} (\sin x)^{2n+1}\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}$$

for every positive integer n. Thus the substitution $x = \sin u$ yields

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \frac{(\sin u)^{2n+1}}{\cos u}\cos u\,du = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}.$$

Part (c): The series in Example 12 is

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}.$$

Therefore

$$\begin{aligned}
\frac{\pi^2}{8} &= \int_0^1 \left[\frac{x}{\sqrt{1-x^2}} + \sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{(2n-1)}{2n}\cdot\frac{1}{2n+1}\cdot\frac{x^{2n+1}}{\sqrt{1-x^2}}\right]dx \\
&= \Big[-(1-x^2)^{1/2}\Big]_0^1 + \left[\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)\cdot(2n+1)}\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx\right] \\
&= 1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)\cdot(2n+1)}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1} \\
&= 1 + \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\cdots.
\end{aligned}$$

Part (d): Therefore

$$S = \sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4}S.$$

Therefore $\dfrac{3}{4}S = \dfrac{\pi^2}{8}$, and thus $S = \dfrac{\pi^2}{6}$. That is,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

C11S08.069: Assume that the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for some $x = x_0 \neq 0$. Then $\{a_n x_0^n\} \to 0$ as $n \to +\infty$. Thus there exists a positive integer K such that, if $n \geqq K$ and $|x| < |x_0|$, then

$$|a_n x_0^n| \leqq 1; \qquad |a_n x^n| \leqq \left| \frac{x^n}{x_0^n} \right|; \qquad |a_n x^n| \leqq \left| \frac{x}{x_0} \right|^n.$$

This implies that the series

$$\sum_{n=0}^{\infty} |a_n x^n| \quad \text{is eventually dominated by} \quad \sum_{n=0}^{\infty} \left| \frac{x}{x_0} \right|^n,$$

a convergent geometric series. Therefore $\sum a_n x^n$ converges absolutely if $|x| < |x_0|$.

C11S08.070: Part (a): Suppose that the power series $\sum a_n x^n$ converges for some, but not all, nonzero values of x. Let S be the set of real numbers for which the series converges absolutely. If S is not bounded above, then $\sum a_n x^n$ converges absolutely for all x by the result in Problem 69. Therefore S has an upper bound. Moreover, S is nonempty because it contains zero. Therefore S has a least upper bound.

Part (b): Let λ be the least upper bound of S. If $|x| < \lambda$, then there exists x_0 in S such that $|x| < x_0 < \lambda$. Therefore, invoking the result in Problem 69 again, $\sum a_n x^n$ converges absolutely. On the other hand, if $|x| > \lambda$ but $\sum a_n x^n$ converges, then x is an element of S. This contradicts the definition of least upper bound, and therefore $\sum a_n x^n$ diverges if $|x| > \lambda$. This proves Theorem 1 immediately with $R = \lambda$ in part (3) of the theorem.

Section 11.9

C11S09.001: To estimate $65^{1/3}$ using the binomial series, first write

$$65^{1/3} = (4^3+1)^{1/3} = 4\left(1+\frac{1}{64}\right)^{1/3}.$$

According to Eq. (14) in Section 11.8, the binomial series is

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots;$$

it has radius of convergence $R = 1$. Therefore

$$4(1+x)^{1/3} = 4+\frac{4}{3}x-\frac{4\cdot 2}{3^2}\cdot\frac{x^2}{2!}+\frac{4\cdot 2\cdot 5}{3^3}\cdot\frac{x^3}{3!}-\frac{4\cdot 2\cdot 5\cdot 8}{3^4}\cdot\frac{x^4}{4!}+\cdots.$$

With $x = \frac{1}{64}$, this series is alternating after the first two terms. For three digits correct to the right of the decimal in our approximation, we note that if $a = \frac{1}{64}$, then

$$\frac{4\cdot 2}{3^2}\cdot\frac{a^2}{2!} < 0.00011.$$

Therefore

$$65^{1/3} \approx 4+\frac{4}{3}\cdot\frac{1}{64} \approx 4.02083333 \approx 4.021.$$

For more accuracy, the sum of the first seven terms of the series is approximately 4.02072575858904; compare this with $65^{1/3} \approx 4.02072575858906$.

C11S09.002: To estimate $630^{1/4}$ using the binomial series, first write

$$630^{1/4} = 5\cdot\left(\frac{630}{625}\right)^{1/4} = 5\cdot\left(1+\frac{1}{125}\right)^{1/4}.$$

Using Eq. (14) in Section 10.8, we have

$$(1+x)^{1/4} = 1+\frac{1}{4}x-\frac{3}{4^2}\cdot\frac{x^2}{2!}+\frac{3\cdot 7}{4^3}\cdot\frac{x^3}{3!}-\frac{3\cdot 7\cdot 11}{4^4}\cdot\frac{x^4}{4!}+\cdots.$$

With $x = \frac{1}{125}$, this series is alternating after the first two terms, and

$$5\cdot\frac{3}{4^2}\cdot\frac{x^2}{2} < 0.00003,$$

and thus

$$630^{1/4} \approx 5+\frac{5}{4}\cdot\frac{1}{125} = 5.010.$$

Adding the next term of the series changes the approximation only slightly to 5.00997. The true value of $630^{1/4}$ is approximately 5.009971392346.

C11S09.003: The Maclaurin series for the sine function yields

$$\sin(0.5) = \frac{1}{2}-\frac{1}{3!\cdot 2^3}+\frac{1}{5!\cdot 2^5}-\frac{1}{7!\cdot 2^7}-\cdots.$$

The alternating series remainder estimate (Theorem 2 in Section 11.7) tells us that because

$$\frac{1}{5! \cdot 2^5} \approx 0.000260417 < 0.0003,$$

the error in approximating $\sin(0.5)$ using only the first two terms of this series will be no greater than 0.0003. Thus

$$\sin(0.5) \approx \frac{1}{2} - \frac{1}{3! \cdot 2^3} \approx 0.47916667.$$

Therefore, to three places, $\sin(0.5) \approx 0.479$. As a separate check, the sum of the first three terms of the series is approximately 0.47942708, the sum of its first six terms is approximately 0.479425538604, and this agrees with the true value of $\sin(0.5)$ to the number of digits shown.

C11S09.004: Replacement of x with $-\frac{1}{5}$ in the Maclaurin series for e^x (Eq. (2) in Section 11.8) yields

$$e^{-1/5} = 1 - \frac{1}{5} + \frac{1}{2! \cdot 5^2} - \frac{1}{3! \cdot 5^3} + \frac{1}{4! \cdot 5^4} - \cdots . \tag{1}$$

Because

$$\frac{1}{4! \cdot 5^4} \approx 0.00006667 < 0.00007,$$

the sum of the first four terms of the series in (1) will yield three-place accuracy. Compare that sum, approximately $0.81866667 \approx 0.819$, with $e^{-1/5} \approx 0.81873075$.

C11S09.005: The Maclaurin series of the inverse tangent function is given in Eq. (20) of Section 11.8; it is

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots .$$

When $x = \frac{1}{2}$, the sum of the first four terms of this series is approximately 0.4634672619. With five terms we get 0.4636842758 and with six terms we get 0.4636398868. To three places, $\arctan(0.5) \approx 0.464$.

C11S09.006: By Eq. (19) in Section 11.8,

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots .$$

If $x = \frac{1}{10}$, then this series is an alternating series; thus because

$$\frac{1}{4} \cdot \frac{1}{10^4} = 0.000025 < 0.00003,$$

the sum of the first three terms of the series should give three-place accuracy. And

$$\frac{1}{10} - \frac{1}{2 \cdot 100} + \frac{1}{3 \cdot 1000} \approx 0.09533333,$$

so that $\ln(1.1) \approx 0.0953$. Indeed, the sum of the first four terms of the series is approximately 0.09530833; moreover, $\ln(1.1) \approx 0.0953101798$.

C11S09.007: When $x = \pi/10$ is substituted in the Maclaurin series for the sine function (Eq. (4) in Section 11.8), we obtain

$$\sin\left(\frac{\pi}{10}\right) = \frac{\pi}{10} - \frac{\pi^3}{3! \cdot 10^3} + \frac{\pi^5}{5! \cdot 10^5} - \frac{\pi^7}{7! \cdot 10^7} + \cdots .$$

Because

$$\frac{\pi^5}{5! \cdot 10^5} \approx 0.000025501641 < 0.00003,$$

the sum of the first two terms of the series should yield three-place accuracy. Thus because

$$\sin\left(\frac{\pi}{10}\right) \approx \frac{\pi}{10} - \frac{\pi^3}{3! \cdot 10^3} \approx 0.30899155,$$

we may conclude that $\sin(\pi/10) \approx 0.309$ to three places. In fact, the sum of the first three terms of the series is approximately 0.30901699 and $\sin(\pi/10) \approx 0.30901699$ to the number of digits shown.

C11S09.008: When $x = \pi/20$ is substituted in the Maclaurin series for the cosine function (Eq. (3) in Section 11.8), the result is

$$\cos\left(\frac{\pi}{20}\right) = 1 - \frac{\pi^2}{2! \cdot 20^2} + \frac{\pi^4}{4! \cdot 20^4} - \frac{\pi^6}{6! \cdot 20^6} + \cdots .$$

Because

$$\frac{\pi^4}{4! \cdot 20^4} \approx 0.000025366951 < 0.00003,$$

the sum of the first two terms of the Maclaurin series should yield three-place accuracy. And

$$\cos\left(\frac{\pi}{20}\right) \approx 1 - \frac{\pi^2}{2! \cdot 20^2} \approx 0.98766299,$$

so to three places, $\cos(\pi/20) \approx 0.988$. To check, the sum of the first three terms of the Maclaurin series is approximately 0.9876883614 and $\cos(\pi/20) \approx 0.9876883406$ to the number of digits shown.

C11S09.009: First we convert $10°$ into $\pi/18$ radians, then use the Maclaurin series for the sine function (Eq. (4) of Section 11.8):

$$\sin 10° = \sin\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{\pi^3}{3! \cdot 18^3} + \frac{\pi^5}{5! \cdot 18^5} - \frac{\pi^7}{7! \cdot 18^7} + \cdots .$$

Because

$$\frac{\pi^3}{3! \cdot 18^3} \approx 0.000886096,$$

the first term of the Maclaurin series alone may not give three-place accuracy (it doesn't). But

$$\frac{\pi^5}{5! \cdot 18^5} \approx 0.0000013496016,$$

so we will certainly obtain three-place accuracy by adding the first two terms of the series:

$$\sin\left(\frac{\pi}{18}\right) \approx \frac{\pi}{18} - \frac{\pi^3}{3! \cdot 18^3} \approx 0.17364683.$$

Thus, to three places, $\sin 10° \approx 0.174$. To check, the sum of the first three terms of the series is approximately 0.1736481786 and the true value of $\sin 10°$ is approximately 0.1736481777 (to the number of digits shown).

C11S09.010: First we convert $5°$ to $\pi/36$ radians. Then the Maclaurin series for the cosine function (see Eq. (3) in Section 11.8) yields

$$\cos 5^\circ = \cos\left(\frac{\pi}{36}\right) = 1 - \frac{\pi^2}{2!\cdot 36^2} + \frac{\pi^4}{4!\cdot 36^4} - \frac{\pi^6}{6!\cdot 36^6} + \cdots.$$

Now

$$\frac{\pi^2}{2!\cdot 36^2} \approx 0.0038077177,$$

so the first term alone of the series will not give three-place accuracy, but

$$\frac{\pi^4}{4!\cdot 36^4} \approx 0.0000024164524,$$

so the sum of the first two terms will be quite enough:

$$\cos\left(\frac{\pi}{36}\right) \approx 1 - \frac{\pi^2}{2!\cdot 36^2} \approx 0.99619228,$$

and therefore—to three places—we have $\cos 5^\circ \approx 0.996$. To be certain, we found that the sum of the first three terms is approximately 0.9961946987 and that the true value of $\cos 5^\circ$ is 0.9961946981 to the number of digits shown.

C11S09.011: Four-place accuracy demands that the error not exceed 0.00005. Here we have

$$\begin{aligned} I &= \int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) dx \\ &= 1 - \frac{1}{3!3} + \frac{1}{5!5} - \frac{1}{7!7} + \frac{1}{9!9} - \frac{1}{11!11} + \cdots. \end{aligned}$$

The alternating series remainder estimate (Theorem 2 of Section 11.7), when applied, yields

$$\frac{1}{7!7} \approx 0.0000566893 \quad \text{(not good enough) and} \quad \frac{1}{9!9} \approx 0.000000008748 \quad \text{(great accuracy)},$$

so that

$$I \approx 1 - \frac{1}{3!3} + \frac{1}{5!5} - \frac{1}{7!7} \approx 0.946082766;$$

to four places, $I \approx 0.9461$. To check, the sum of the first five terms of the series is approximately 0.9460830726 and the true value of the integral is 0.9460830703671830, correct to the number of digits shown here.

C11S09.012: Here the Maclaurin series of the sine function (Eq. (4) in Section 11.8) yields

$$\begin{aligned} J &= \int_0^1 \frac{\sin x}{x^{1/2}}\,dx = \int_0^1 \left(x^{1/2} - \frac{x^{5/2}}{3!} + \frac{x^{9/2}}{5!} - \frac{x^{13/2}}{7!} + \frac{x^{17/2}}{9!} - \cdots\right) dx \\ &= \frac{2}{3} - \frac{2}{3!\cdot 7} + \frac{2}{5!\cdot 11} - \frac{2}{7!\cdot 15} + \frac{2}{9!\cdot 19} - \cdots. \end{aligned}$$

Here,

$$\frac{2}{7!\cdot 15} \approx 0.000026455026 \quad \text{(not good enough)}$$

but

$$\frac{2}{9! \cdot 19} \approx 0.00000029 \qquad \text{(four-place accuracy assured).}$$

Therefore

$$J \approx \frac{2}{3} - \frac{2}{3! \cdot 7} + \frac{2}{5! \cdot 11} - \frac{2}{7! \cdot 15} \approx 0.6205363155;$$

to four places, $J \approx 0.6205$. To check this result, the sum of the first five terms of the series is approximately 0.620536605613 and the approximate value of the integral is 0.6205366034467622 (correct to the number of digits shown here). The generalized hypergeometric function appears anew as the exact value of the integral; it is

$$\frac{2}{3} \cdot {}_1\mathrm{F}_2\left(\frac{3}{4};\ \frac{3}{2},\ \frac{7}{4};\ -\frac{1}{4}\right).$$

C11S09.013: The Maclaurin series for the inverse tangent function (Eq. (20) in Section 11.8) leads to

$$\begin{aligned} K &= \int_0^{1/2} \frac{\arctan x}{x}\,dx = \int_0^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots\right) dx \\ &= \frac{1}{2} - \frac{1}{2^3 \cdot 3^2} + \frac{1}{2^5 \cdot 5^2} - \frac{1}{2^7 \cdot 7^2} + \frac{1}{2^9 \cdot 9^2} - \cdots . \end{aligned}$$

Now

$$\frac{1}{2^9 \cdot 9^2} \approx 0.000024112654 < 0.00003,$$

so by the alternating series error estimate, the sum of the first four terms of the numerical series should yield four-place accuracy. Thus

$$K \approx \frac{1}{2} - \frac{1}{2^3 \cdot 3^2} + \frac{1}{2^5 \cdot 5^2} - \frac{1}{2^7 \cdot 7^2} \approx 0.4872016723;$$

that is, to four places $K \approx 0.4872$. To check this result, the sum of the first five terms of the series is approximately 0.487225784990 and the approximate value of the integral is 0.487222358295 (to the number of digits shown).

C11S09.014: The Maclaurin series for the sine function yields

$$\begin{aligned} L &= \int_0^1 \sin x^2\,dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots\right) dx \\ &= \frac{1}{3} - \frac{1}{3! \cdot 7} + \frac{1}{5! \cdot 11} - \frac{1}{7! \cdot 15} + \frac{1}{9! \cdot 19} - \cdots . \end{aligned}$$

Using the alternating series remainder estimate, we find that

$$\frac{1}{7! \cdot 15} \approx 0.000013227513,$$

and thus for four-place accuracy we need only add the first three terms of the series:

$$L \approx \frac{1}{3} - \frac{1}{3! \cdot 7} + \frac{1}{5! \cdot 11} \approx 0.310281385281;$$

to four places, $L \approx 0.3103$. To verify this result, the sum of the first four terms of the series is approximately 0.310268157768 and the value of the integral is 0.3102683017233811 (accurate to the number of digits shown). The *Mathematica* 3.0 command

```
Integrate[ Sin[ x∧2 ], { x, 0, 1 } ]
```

immediately returns the exact value of the integral

$$\left(\sqrt{\frac{\pi}{2}}\right) \cdot \texttt{FresnelS}\left(\sqrt{\frac{2}{\pi}}\right)$$

in terms of the *Fresnel integral* in less than 4 seconds on an old computer. This transcendental function is defined by

$$S(z) = \texttt{FresnelS}[z] = \int_0^z \sin\left(\frac{\pi t^2}{2}\right)\,dt.$$

C11S09.015: The Maclaurin series for $\ln(1+x)$, in Eq. (19) of Section 11.8, yields

$$I = \int_0^{1/10} \frac{\ln(1+x)}{x}\,dx = \int_0^{1/10} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \cdots\right)dx$$

$$= \left[x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \frac{x^6}{36} + \cdots\right]_0^{1/10} = \frac{1}{10} - \frac{1}{4 \cdot 10^2} + \frac{1}{9 \cdot 10^3} - \frac{1}{16 \cdot 10^4} + \frac{1}{25 \cdot 10^5} - \cdots.$$

Because

$$\frac{1}{16 \cdot 10^4} = 0.00000625,$$

the alternating series remainder estimate tells us that the sum of the first three terms of the numerical series should give four-place accuracy. Because

$$I \approx \frac{1}{10} - \frac{1}{4 \cdot 10^2} + \frac{1}{9 \cdot 10^3} \approx 0.097611111,$$

the four-place approximation we seek is $I \approx 0.0976$. To check this result, the sum of the first four terms of the series is approximately 0.097604861111 and the value of the integral, to the number of digits shown here, is 0.0976052352293216.

C11S09.016: The binomial series (Example 8 in Section 11.8) yields

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{1 \cdot 3x^2}{2! \cdot 2^2} - \frac{1 \cdot 3 \cdot 5x^3}{3! \cdot 2^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^4}{4! \cdot 2^4} - \cdots.$$

Therefore

$$\begin{aligned} J &= \int_0^{1/2} \frac{1}{\sqrt{1+x^4}}\,dx = \int_0^{1/2} \left(1 - \frac{x^4}{2} + \frac{1\cdot 3x^8}{2!\cdot 2^2} - \frac{1\cdot 3\cdot 5x^{12}}{3!\cdot 2^3} + \cdots\right) dx \\ &= \left[x - \frac{x^5}{2\cdot 5} + \frac{1\cdot 3x^9}{2!\cdot 2^2\cdot 9} - \frac{1\cdot 3\cdot 5x^{13}}{3!\cdot 2^3\cdot 13} + \frac{1\cdot 3\cdot 5\cdot 7x^{17}}{4!\cdot 2^4\cdot 17} - \cdots\right]_0^{1/2} \\ &= \frac{1}{2} - \frac{1}{2^6\cdot 5} + \frac{1\cdot 3}{2!\cdot 2^{11}\cdot 9} - \frac{1\cdot 3\cdot 5}{3!\cdot 2^{16}\cdot 13} + \frac{1\cdot 3\cdot 5\cdot 7}{4!\cdot 2^{21}\cdot 17} - \cdots. \end{aligned}$$

We note that

$$\frac{1\cdot 3\cdot 5}{3!\cdot 2^{16}\cdot 13} \approx 0.0000029343825 < 0.000003,$$

so the alternating series remainder estimate assures us of four-place accuracy if we sum the first three terms of the numerical series:

$$J \approx \frac{1}{2} - \frac{1}{2^6\cdot 5} + \frac{1\cdot 3}{2!\cdot 2^{11}\cdot 9} \approx 0.496956380208.$$

Thus, to four places, $J \approx 0.4970$. To check this result, the sum of the first four terms of the series is approximately 0.496953445826 and the value of the integral, accurate to the number of digits shown here, is 0.4969535632094542968468. Solution by integration is known to require nonelementary functions; *Mathematics* 3.0 uses the elliptic integral of the first kind.

C11S09.017: The Maclaurin series for the natural exponential function—Eq. (2) in Section 11.8—yields

$$\begin{aligned} J &= \int_0^{1/2} \frac{1-e^{-x}}{x}\,dx = \int_0^{1/2} \left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \cdots\right) dx \\ &= \left[x - \frac{x^2}{2!2} + \frac{x^3}{3!3} - \frac{x^4}{4!4} + \frac{x^5}{5!5} - \cdots\right]_0^{1/2} = \frac{1}{2} - \frac{1}{2!\cdot 2\cdot 2^2} + \frac{1}{3!\cdot 3\cdot 2^3} - \frac{1}{4!\cdot 4\cdot 2^4} + \cdots. \end{aligned}$$

Because

$$\frac{1}{6!\cdot 6\cdot 2^6} \approx 0.00000361690,$$

the alternating series remainder estimate assures us that the sum of the first five terms of the series, which is approximately 0.443845486111, will give us four-place accuracy: $J \approx 0.4438$. The value of the integral, accurate to the number of digits shown here, is 0.4438420791177484.

C11S09.018: The binomial series yields

$$(1+x^3)^{1/2} = 1 + \frac{x^3}{2} - \frac{x^6}{2!\cdot 2^2} + \frac{3x^9}{3!\cdot 2^3} - \frac{3\cdot 5x^{12}}{4!\cdot 2^4} + \cdots.$$

Then term-by-term integration yields

$$\int_0^{1/2} \sqrt{1+x^3}\,dx = \frac{1}{2} + \frac{1}{4\cdot 2^5} - \frac{1}{2!\cdot 7\cdot 2^9} + \frac{3}{3!\cdot 10\cdot 2^{13}} - \frac{3\cdot 5}{4!\cdot 13\cdot 2^{17}} + \cdots.$$

We find that

$$\frac{1}{2!\cdot 7\cdot 2^9} \approx 0.00014, \qquad \text{whereas} \qquad \frac{3}{3!\cdot 10\cdot 2^{13}} \approx 0.0000061.$$

The sum of the first three terms of the numerical series then yields the approximation

$$\int_0^{1/2} \sqrt{1+x^3}\ dx \approx 0.507672991071;$$

to four places, the value of the integral is 0.5077. To the number of digits shown here, its actual value is 0.50767875196789934967776718. Direct evaluation of the integral is known to require the use of nonelementary functions; *Mathematica* 3.0 uses the elliptic function of the first kind.

C11S09.019: The Maclaurin series for the natural exponential function—Eq. (2) in Section 11.8—yields

$$\exp(-x^2) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots.$$

Then termwise integration gives

$$\int_0^1 \exp(-x^2)\ dx = 1 - \frac{1}{3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} + \frac{1}{4!\cdot 9} - \frac{1}{5!\cdot 11} + \cdots.$$

Next, the sum of the first six terms of the numerical series is approximately 0.746729196729, the sum of the first seven terms is approximately 0.746836034336, and the sum of the first eight terms is approximately 0.746822806823. To four places, the value of the integral is 0.7468. The actual value of the integral, to the number of digits shown here, is 0.7468241328124270.

C11S09.020: The Maclaurin series for the cosine function yields

$$\frac{1-\cos x}{x^2} = \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \cdots.$$

Then term-by-term integration gives the series

$$\int_0^1 \frac{1-\cos x}{x^2}\ dx = \frac{1}{2!} - \frac{1}{4!\cdot 3} + \frac{1}{6!\cdot 5} - \frac{1}{8!\cdot 7} + \frac{1}{10!\cdot 9} - \cdots.$$

The sum of the first three terms of the last series is approximately 0.486388888889 and the sum of the first four terms is approximately 0.486385345805. Thus to four places, the value of the integral is 0.4864. Its actual value, to the number of digits shown here, is 0.48638537623532273234228992. To evaluate the definite integral directly, *Mathematica* 3.0 uses the *sine integral* function

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\ dt$$

and the Bessel function of the first kind of order $-\frac{1}{2}$.

C11S09.021: The binomial series (Example 8 in Section 11.8) yields

$$(1+x^2)^{1/3} = 1 + \frac{x^2}{3} - \frac{2x^4}{2!\cdot 3^2} + \frac{2\cdot 5x^6}{3!\cdot 3^3} - \frac{2\cdot 5\cdot 8x^8}{4!\cdot 3^4} + \frac{2\cdot 5\cdot 8\cdot 11x^{10}}{5!\cdot 3^5} - \cdots.$$

Then term-by-term integration yields

$$\begin{aligned} I &= \int_0^{1/2} (1+x^2)^{1/3}\ dx = \left[x + \frac{x^3}{3\cdot 3} - \frac{2x^5}{2!\cdot 3^2\cdot 5} + \frac{2\cdot 5x^7}{3!\cdot 3^3\cdot 7} - \frac{2\cdot 5\cdot 8x^9}{4!\cdot 3^4\cdot 9} + \frac{2\cdot 5\cdot 8\cdot 11x^{11}}{5!\cdot 3^5\cdot 11} - \cdots\right]_0^{1/2} \\ &= \frac{1}{2} + \frac{1}{3\cdot 3\cdot 2^3} - \frac{2}{2!\cdot 3^2\cdot 5\cdot 2^5} + \frac{2\cdot 5}{3!\cdot 3^3\cdot 7\cdot 2^7} - \frac{2\cdot 5\cdot 8}{4!\cdot 3^4\cdot 9\cdot 2^9} + \frac{2\cdot 5\cdot 8\cdot 11}{5!\cdot 3^5\cdot 11\cdot 2^{11}} - \cdots. \end{aligned}$$

The sum of the first five terms of the numerical series is approximately 0.513254407130 and the sum of its first six terms is approximately 0.513255746722. So to four places, $I \approx 0.5133$. The actual value of the integral, accurate to the number of digits shown here, is 0.5132555590033423.

C11S09.022: First, the binomial series yields

$$(1+x^3)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{3}{2!\cdot 2^2}x^6 - \frac{3\cdot 5}{3!\cdot 2^3}x^9 + \frac{3\cdot 5\cdot 7}{4!\cdot 2^4}x^{12} - \cdots.$$

Then term-by-term integration gives

$$\begin{aligned} J = \int_0^{1/2} \frac{x}{\sqrt{1+x^3}}\,dx &= \left[\frac{1}{2}x^2 - \frac{1}{10}x^5 + \frac{3}{64}x^8 - \frac{5}{176}x^{11} + \frac{5}{256}x^{14} - \cdots\right]_0^{1/2} \\ &= \frac{1}{2\cdot 2^2} - \frac{1}{10\cdot 2^5} + \frac{3}{64\cdot 2^8} - \frac{5}{176\cdot 2^{11}} + \frac{5}{256\cdot 2^{14}} - \cdots. \end{aligned}$$

.

Now

$$\frac{5}{176\cdot 2^{11}} \approx 0.00001387, \qquad \text{whereas} \qquad \frac{5}{256\cdot 2^{14}} \approx 0.00000119,$$

so the sum of the first four terms of the numerical series will give four-place accuracy. That sum is approximately 0.122044233842, and thus $J \approx 0.1220$. To check this result, the first five terms of the series sum to approximately 0.122045425935. The actual value of the integral, accurate to the number of digits shown here, is 0.1220453252641941480696898.

C11S09.023: The Maclaurin series for the natural exponential function (Eq. (2) in Section 11.8) yields

$$\lim_{x\to 0} \frac{1+x-e^x}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left(-\frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \cdots\right) = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x}{6} - \frac{x^2}{24} - \cdots\right) = -\frac{1}{2}.$$

C11S09.024: Here we could first use the product law for limits to dispense with the factor $\cos x$, but we choose the hard way:

$$\lim_{x\to 0} \frac{x-\sin x}{x^3\cos x} = \lim_{x\to 0} \frac{\dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \dfrac{x^7}{7!} - \cdots}{x^3 - \dfrac{x^5}{2!} + \dfrac{x^7}{4!} - \cdots} = \lim_{x\to 0} \frac{\dfrac{1}{3!} - \dfrac{x^2}{5!} + \dfrac{x^4}{7!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots} = \frac{1}{6}.$$

C11S09.025: The series in Eqs. (2) and (3) of Section 11.8 yield

$$\lim_{x\to 0} \frac{1-\cos x}{x(e^x-1)} = \lim_{x\to 0} \frac{\dfrac{x^2}{2!} - \dfrac{x^4}{4!} + \dfrac{x^6}{6!} - \cdots}{x^2 + \dfrac{x^3}{2!} + \dfrac{x^4}{3!} + \cdots} = \lim_{x\to 0} \frac{\dfrac{1}{2!} - \dfrac{x^2}{4!} + \dfrac{x^4}{6!} - \cdots}{1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \cdots} = \frac{1}{2}.$$

C11S09.026: The series in Eqs. (2) and (20) of Section 11.8 yield

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \arctan x} = \lim_{x\to 0} \frac{\dfrac{2x^3}{3!} + \dfrac{2x^5}{5!} + \dfrac{2x^7}{7!} + \cdots}{\dfrac{x^3}{3} - \dfrac{x^5}{5} + \dfrac{x^7}{7} - \cdots} = \lim_{x\to 0} \frac{\dfrac{2}{3!} + \dfrac{2x^2}{5!} + \dfrac{2x^4}{7!} + \cdots}{\dfrac{1}{3} - \dfrac{x^2}{5} + \dfrac{x^4}{7} - \cdots} = 1.$$

C11S09.027: The Maclaurin series for the sine function in Eq. (4) of Section 11.8 yields

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{(\sin x)-x}{x\sin x}$$

$$= \lim_{x\to 0}\frac{-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots}{x^2-\dfrac{x^4}{3!}+\dfrac{x^6}{5!}-\cdots} = \lim_{x\to 0}\frac{-\dfrac{x}{3!}+\dfrac{x^3}{5!}-\dfrac{x^5}{7!}+\cdots}{1-\dfrac{x^2}{3!}+\dfrac{x^4}{5!}-\cdots} = \frac{0}{1} = 0.$$

C11S09.028: By Example 5 of Section 11.9,

$$\ln x^2 = 2\ln x = 2(x-1)-\frac{2}{2}(x-1)^2+\frac{2}{3}(x-1)^3-\frac{2}{4}(x-4)^4+\cdots.$$

Therefore

$$\lim_{x\to 1}\frac{\ln x^2}{x-1} = \lim_{x\to 1}\left[2-(x-1)+\frac{2}{3}(x-1)^2-\frac{1}{2}(x-3)^3+\cdots\right] = 2.$$

C11S09.029: The Taylor series with center $\pi/2$ for the sine function is

$$\sin x = 1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\frac{1}{6!}\left(x-\frac{\pi}{4}\right)^6+\cdots.$$

We convert 80° into $x = 4\pi/9$ and substitute:

$$\sin 80^\circ = 1-\frac{1}{2!}\left(\frac{\pi}{18}\right)^2+\frac{1}{4!}\left(\frac{\pi}{18}\right)^4-\frac{1}{6!}\left(\frac{\pi}{18}\right)^6+\cdots. \qquad (1)$$

For four-place accuracy, we need

$$\frac{1}{n!}\left(\frac{\pi}{18}\right)^n < 0.00005,$$

and the smallest even positive integer for which this inequality holds is $n = 4$. Thus only the first two terms of the series in (1) are needed to show that, to four places, $\sin 80^\circ \approx 0.9848$. The sum of the first two terms is approximately 0.984769129011, the sum of the first three terms is approximately 0.984807792249, and $\sin 80^\circ \approx 0.984807753012$ (all digits given here are correct).

C11S09.030: The Taylor series with center $\pi/4$ for the cosine function is

$$\cos x = \frac{\sqrt{2}}{2}\left[1-\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4-\frac{1}{5!}\left(x-\frac{\pi}{4}\right)^5-\cdots\right].$$

Note the $+ - - + + - - + + - - + \cdots$ pattern of the signs. After we convert 35° to radians and substitute, we find that additional care with signs is required because

$$x-\frac{\pi}{4} = \frac{7\pi}{36}-\frac{\pi}{4} = -\frac{\pi}{18}$$

is negative and some of the exponents in the Taylor series are odd. We obtain

$$\cos 35^\circ = \frac{\sqrt{2}}{2}\left[1+\frac{\pi}{18}-\frac{1}{2!}\left(\frac{\pi}{18}\right)^2-\frac{1}{3!}\left(\frac{\pi}{18}\right)^3+\frac{1}{4!}\left(\frac{\pi}{18}\right)^4+\frac{1}{5!}\left(\frac{\pi}{18}\right)^5-\frac{1}{6!}\left(\frac{\pi}{18}\right)^6-\frac{1}{7!}\left(\frac{\pi}{18}\right)^7+\cdots\right].$$

The sign pattern is now $+\ +\ -\ -\ +\ +\ -\ -\ +\ +\ -\ -\ \cdots$. Normally one cannot combine terms in an infinite series without affecting its convergence or its sum. But the series in question is absolutely convergent; therefore we may combine the first and second terms, the third and fourth terms, and so on, to obtain a regrouped series that passes the alternating series convergence test. The sum of its first two terms is approximately 0.819123779375, the sum of its first three terms is approximately 0.819152072726, and the sum of its first four terms is approximately 0.819152044274. Hence to four places, $\cos 35^\circ \approx 0.8192$. (The actual value is approximately 0.81915204428899 with all digits shown correct or correctly rounded.)

C11S09.031: The Taylor series with center $\pi/4$ for the cosine function is

$$\cos x \;=\; \frac{\sqrt{2}}{2}\left[1-\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4-\frac{1}{5!}\left(x-\frac{\pi}{4}\right)^5-\cdots\right].$$

We convert 47° to radians and substitute to find that

$$\cos 47^\circ \;=\; \frac{\sqrt{2}}{2}\left[1-\frac{\pi}{90}-\frac{1}{2!}\left(\frac{\pi}{90}\right)^2+\frac{1}{3!}\left(\frac{\pi}{90}\right)^3+\frac{1}{4!}\left(\frac{\pi}{90}\right)^4-\frac{1}{5!}\left(\frac{\pi}{90}\right)^5-\cdots\right].$$

This series is absolutely convergent, so rearrangement will not change its convergence or its sum. We make it into an alternating series by grouping terms 2 and 3, terms 4 and 5, and so on. We seek six-place accuracy here. If $x=\pi/90$, then

$$\frac{x^3}{3!}+\frac{x^4}{4!}\approx 0.000007 \quad \text{and that} \quad \frac{x^5}{5!}+\frac{x^6}{6!}\approx 0.00000000043,$$

so the first five terms of the [ungrouped] series—those through exponent 4—yield the required six-place accuracy: $\cos 47^\circ \approx 0.681998$. The actual value of $\cos 47^\circ$ is approximately 0.6819983600624985 (the digits shown here are all correct or correctly rounded).

C11S09.032: The Taylor series with center $\pi/3$ for the sine function is

$$\sin x \;=\; \frac{\sqrt{3}}{2}+\frac{1}{2}\left(x-\frac{\pi}{3}\right)-\frac{\sqrt{3}}{2!\cdot 2}\left(x-\frac{\pi}{3}\right)^2-\frac{1}{3!\cdot 2}\left(x-\frac{\pi}{3}\right)^3$$
$$+\frac{\sqrt{3}}{4!\cdot 2}\left(x-\frac{\pi}{3}\right)^4+\frac{1}{5!\cdot 2}\left(x-\frac{\pi}{3}\right)^5-\frac{\sqrt{3}}{6!\cdot 2}\left(x-\frac{\pi}{3}\right)^6-\cdots,$$

and 58° converts into $x=29\pi/90$ radians, and hence

$$\sin 58^\circ \;=\; \frac{\sqrt{3}}{2}+\frac{1}{2}\left(\frac{\pi}{90}\right)-\frac{\sqrt{3}}{2!\cdot 2}\left(\frac{\pi}{90}\right)^2-\frac{1}{3!\cdot 2}\left(\frac{\pi}{90}\right)^3+\frac{\sqrt{3}}{4!\cdot 2}\left(\frac{\pi}{90}\right)^4+\frac{1}{5!\cdot 2}\left(\frac{\pi}{90}\right)^5-\frac{\sqrt{3}}{6!\cdot 2}\left(\frac{\pi}{90}\right)^6-\cdots.$$

We group terms in pairs, much as in the solution of Problem 30. For six-place accuracy, we find that

$$\frac{1}{3!\cdot 2}\left(\frac{\pi}{90}\right)^3+\frac{\sqrt{3}}{4!\cdot 2}\left(\frac{\pi}{90}\right)^4\approx 0.0000036,$$

which is not sufficient for the needed accuracy, but that

$$\frac{1}{5!\cdot 2}\left(\frac{\pi}{90}\right)^5+\frac{\sqrt{3}}{6!\cdot 2}\left(\frac{\pi}{90}\right)^6\approx 0.0000000002,$$

more than enough. We sum the series through the terms up to degree 4 to find that $\sin 58^\circ \approx 0.848048$. In fact, $\sin 58^\circ \approx 0.848048096156425970386$ (accurate to the number of digits shown here).

C11S09.033: Note that $e^{0.1} < 1.2 = \frac{6}{5}$, and if $|x| \leqq 0.1$, then the Taylor series remainder estimate yields

$$\left|\frac{e^z}{120}x^5\right| \leqq \frac{6}{600}\left(\frac{1}{10}\right)^5 = 10^{-7} < 0.5 \times 10^{-6},$$

so six-place accuracy is assured.

C11S09.034: The Taylor series remainder estimate yields

$$\left|\frac{\cos z}{720}x^6\right| \leqq \frac{10^{-6}}{720} \approx 1.4 \times 10^{-9} < 0.5 \times 10^{-8},$$

so eight-place accuracy is assured (too conservative, as usual; you actually get ten-place accuracy).

C11S09.035: The Taylor series remainder estimate is difficult to work with; we use instead the cruder alternating series remainder estimate:

$$\frac{(0.1)^5}{5} < 0.5 \times 10^{-5},$$

so five-place accuracy is assured.

C11S09.036: The alternating series remainder estimate gives

$$\frac{3}{16} \cdot (0.1)^3 = 1.875 \times 10^{-4} < 0.5 \times 10^{-3},$$

so three-place accuracy is assured.

C11S09.037: Clearly $|e^z| < \frac{5}{3}$ if $|z| < 0.5$. Hence the Taylor series remainder estimate yields

$$\left|\frac{e^z}{120}x^5\right| \leqq \frac{5}{3 \cdot 120}\left(\frac{1}{2}\right)^5 \approx 0.434 \times 10^{-3},$$

so two-place accuracy will be obtained if $|x| \leqq 0.5$. In particular,

$$e^{1/3} \approx 1 + \frac{1}{3} + \frac{1}{18} + \frac{1}{486} + \frac{1}{1944} \approx 1.39.$$

In fact, to the number of digits shown here, $e^{1/3} \approx 1.395612425086$.

C11S09.038: We let

$$f(x) = \left|x - \frac{x^3}{6} - \sin x\right|$$

and used *Mathematica* 3.0 to find the largest value of x for which $f(x) < 10^{-6}$. That value of x is approximately $a = 0.164396337603$, so the approximation will give five-place accuracy on the interval $[-a, a]$.

C11S09.039: The Taylor series remainder estimate is

$$|R_3(x)| = \frac{\sqrt{2}}{2} \cdot \frac{\cos z}{4!}\left(x - \frac{\pi}{4}\right)^4.$$

Part (a): If $40^\circ \leqq x^\circ \leqq 50^\circ$, then

$$\frac{2\pi}{9} \leqq x \leqq \frac{5\pi}{18} \quad \text{and} \quad \cos z \leqq \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

so

$$|R_3(x)| \leqq \frac{\sqrt{2}}{2} \cdot \frac{1}{24} \cdot \left(\frac{\pi}{36}\right)^4 \cdot \frac{\sqrt{3}}{2} \approx 0.0000014797688 < 0.000002,$$

thereby giving five-place accuracy. Part (b): If $44° \leqq x° \leqq 46°$, then

$$\frac{44\pi}{180} \leqq x \leqq \frac{46\pi}{180},$$

so that

$$|R_3(x)| \leqq \frac{\sqrt{2}}{2} \cdot \frac{1}{24} \cdot \left(\frac{\pi}{180}\right)^4 \cdot \frac{\sqrt{3}}{2} \approx 0.0000000023676302 < 0.000000003,$$

thereby giving eight-place accuracy.

C11S09.040: If $\frac{1}{6}\pi \leqq x \leqq \frac{1}{3}\pi$, then

$$|R_4(x)| = \frac{\sqrt{2}}{2} \cdot \frac{\sin z}{5!} \cdot \left|x - \frac{\pi}{4}\right|^5 \leqq \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{5!} \cdot \left(\frac{\pi}{12}\right)^5 \approx 0.0000062759,$$

not quite enough for five-place accuracy, but

$$|R_5(x)| = \frac{\sqrt{2}}{2} \cdot \frac{\cos z}{6!} \cdot \left|x - \frac{\pi}{4}\right|^6 \leqq \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{6!} \cdot \left(\frac{\pi}{12}\right)^6 \approx 0.000000273839 < 0.0000003,$$

definitely good enough. Hence we should use the approximation

$$\cos x \approx P_5(x) = \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 - \frac{1}{5!}\left(x - \frac{\pi}{4}\right)^5\right].$$

To use *Mathematica* 3.0 to demonstrate the accuracy of this approximation, recall that % refers to "last output." So execute the commands

```
Series[ Cos[x], { x, Pi/4, 5 } ] // Normal
```

and

```
Plot[ Abs[Cos[x] - %], { x, Pi/6, Pi/3 },
        PlotRange → { -0.000001, 0.000001 } ];
```

You should find that every point on the graph lies between 0 and 5×10^{-7}.

C11S09.041: The volume of revolution around the x-axis is

$$\begin{aligned}
V &= 2\int_0^{\pi} \pi \frac{\sin^2 x}{x^2}\,dx = 2\pi \int_0^{\pi} \frac{1 - \cos 2x}{2x^2}\,dx \\
&= \pi \int_0^{\pi} \frac{1}{x^2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \frac{(2x)^8}{8!} + \cdots\right) dx = \pi \int_0^{\pi} \left(\frac{2^2}{2!} - \frac{2^4 x^2}{4!} + \frac{2^6 x^4}{6!} - \frac{2^8 x^6}{8!} + \cdots\right) dx \\
&= \pi \left[\frac{2^2 x}{2!} - \frac{2^4 x^3}{4! \cdot 3} + \frac{2^6 x^5}{6! \cdot 5} - \frac{2^8 x^7}{8! \cdot 7} + \cdots\right]_0^{\pi} = \frac{(2\pi)^2}{2!} - \frac{(2\pi)^4}{4! \cdot 3} + \frac{(2\pi)^6}{6! \cdot 5} - \frac{(2\pi)^8}{8! \cdot 7} + \cdots.
\end{aligned}$$

This series converges rapidly after the first 10 or 15 terms. For example, the sum of the first seven terms is about 8.927353886225. The *Mathematica* 3.0 command

```
NSum[ ((-1)∧(k+1))∗((2∗Pi)∧(2∗k))/(((2∗k)!)∗(2∗k-1)),
    { k, 1, Infinity }, WorkingPrecision → 28 ]
```

returns the approximate sum 8.910509146510103807178167 8. The *Mathematica* 3.0 command

```
2∗Integrate[ Pi∗(Sin[x]/x)∧2, {x, 0, Pi} ]
```

produces the exact value of the volume:

$$
\begin{aligned}
V &= -1 + \text{HypergeometricPFQ}\left[\left\{-\frac{1}{2}\right\},\ \left\{\frac{1}{2}, \frac{1}{2}\right\},\ -\pi^2\right] \\
&\approx 8.9105091465101038071781677928811594135107930070735323609643.
\end{aligned}
$$

The *Mathematica* function `HypergeometricPFQ` is the generalized hypergeometric function ${}_p\mathrm{F}_q$. Space prohibits further discussion; we've mentioned this only to give you a reference in case you're interested in further details.

C11S09.042: The area is

$$
\begin{aligned}
A &= 2\int_0^{2\pi} \frac{1-\cos x}{x^2}\,dx = 2\int_0^{2\pi}\left(\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \cdots\right)dx \\
&= 2\left[\frac{x}{2!} - \frac{x^3}{4!\cdot 3} + \frac{x^5}{6!\cdot 5} - \frac{x^7}{8!\cdot 7} + \frac{x^9}{10!\cdot 9} - \cdots\right]_0^{2\pi} \\
&= \frac{2^2\pi}{2!} - \frac{2^4\pi^3}{4!\cdot 3} + \frac{2^6\pi^5}{6!\cdot 5} - \frac{2^8\pi^7}{8!\cdot 7} + \frac{2^{10}\pi^9}{10!\cdot 9} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n}\pi^{2n-1}}{(2n)!\cdot(2n-1)}.
\end{aligned}
$$

This series converges rather slowly at first. But it passes the alternating series convergence test and its 27th term is less than 4×10^{-31}, so the sum of its first 26 terms, which is approximately 2.8363031522652569, is the area (reliable to the number of digits shown here). *Mathematica* 3.0 reports that the exact value of the area is $2\,\mathrm{Si}(2\pi)$ where $\mathrm{Si}(z)$ is the sine integral function (see the solution of Problem 20).

C11S09.043: The volume is

$$
\begin{aligned}
V &= \int_0^{2\pi} 2\pi x\,\frac{1-\cos x}{x^2}\,dx = 2\pi\int_0^{2\pi}\left(\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \frac{x^9}{10!} - \cdots\right)dx \\
&= 2\pi\left[\frac{x^2}{2!\cdot 2} - \frac{x^4}{4!\cdot 4} + \frac{x^6}{6!\cdot 6} - \frac{x^8}{8!\cdot 8} + \frac{x^{10}}{10!\cdot 10} - \cdots\right]_0^{2\pi} \\
&= \frac{(2\pi)^3}{2!\cdot 2} - \frac{(2\pi)^5}{4!\cdot 4} + \frac{(2\pi)^7}{6!\cdot 6} - \frac{(2\pi)^9}{8!\cdot 8} + \frac{(2\pi)^{11}}{10!\cdot 10} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n+1}}{(2n)!\cdot 2n}.
\end{aligned}
$$

This alternating series converges slowly at first—its ninth term is approximately 0.0127—but its 21st term is less than 4×10^{-19}, so the sum of its first 20 terms is a very accurate estimate of its value. That partial sum is approximately 15.3162279832536178, and all the digits shown here are accurate.

The *Mathematica* 3.0 command

```
Integrate[ 2*Pi*x*(1 - Cos[x])/(x*x), {x, 0, 2*Pi} ]
```

produces the exact value of the volume:

$$V = 2\pi\big[\text{EulerGamma} - \text{CosIntegral}(2\pi) + \ln(2\pi)\big]$$
$$\approx 15.316227983253617819314890707059693673252358556029999057582 7.$$

Here, `EulerGamma` is Euler's constant $\gamma \approx 0.577216$, which first appears in the textbook in Problem 50 of Section 11.5; `CosIntegral` is defined to be

$$\texttt{CosIntegral}(x) = \gamma + \ln x + \int_0^x \frac{(\cos t) - 1}{t}\,dt.$$

Again, the reference is provided only for your convenience if you care to pursue further study of this topic.

C11S09.044: The volume is

$$V = 2\pi \int_0^{2\pi} \left(\frac{1-\cos x}{x^2}\right)^2 dx.$$

Now

$$\left(\frac{1-\cos x}{x^2}\right)^2 = \frac{1 - 2\cos x + \cos^2 x}{x^4} = \frac{2 - 4\cos x + 1 + \cos 2x}{2x^4} = \frac{3 - 4\cos x + \cos 2x}{2x^4}$$
$$= \frac{1}{2x^4}\left(3 - 4 + \frac{4x^2}{2!} - \frac{4x^4}{4!} + \frac{4x^6}{6!} - \frac{4x^8}{8!} + \cdots + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots\right)$$
$$= \frac{2^3-2}{4!} - \frac{2^5-2}{6!}x^2 + \frac{2^7-2}{8!}x^4 - \frac{2^9-2}{10!}x^{10} + \cdots.$$

Therefore

$$V = 2\pi \int_0^{2\pi} \left(\frac{2^3-2}{4!} - \frac{2^5-2}{6!}x^2 + \frac{2^7-2}{8!}x^4 - \frac{2^9-2}{10!}x^6 + \cdots\right) dx$$
$$= 2\pi \left[\frac{2^3-2}{4!}x - \frac{2^5-2}{6!\cdot 3}x^3 + \frac{2^7-2}{8!\cdot 5}x^5 - \frac{2^9-2}{10!\cdot 7}x^7 + \cdots\right]_0^{2\pi}$$
$$= \frac{2^3-2}{4!}(2\pi)^2 - \frac{2^5-2}{6!\cdot 3}(2\pi)^4 + \frac{2^7-2}{8!\cdot 5}(2\pi)^6 - \frac{2^9-2}{10!\cdot 7}(2\pi)^8 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2^{2n+1}-2)(2\pi)^{2n}}{(2n+2)!(2n-1)}.$$

The 31st term of this series is less than 10^{-22}, so by the alternating series remainder estimate, the sum of its first 30 terms will give an accurate estimate of the volume: It is $V \approx 3.2801806101868747$ (all digits shown are correct or correctly rounded). As an independent check of this result, the *Mathematica* 3.0 command

```
NIntegrate[ 2*Pi*((1 - Cos[x])/(x∧2))∧2, { x, 0, 2*Pi }, WorkingPrecision → 28 ]
```

returns the value 3.2801806101868746547, accurate to the number of digits shown; the exact value is also available from *Mathematica* and is

$$-\frac{2}{3}\big[\pi\,\text{Si}(2\pi) + \text{Si}(4\pi)\big]$$

where $\operatorname{Si}(z)$ is the sine integral function (see the solution of Problem 20).

C11S09.045: The long division is shown next.

$$\begin{array}{r l}
 & 1 + x + x^2 + x^3 + \cdots \\ \cline{2-2}
1 - x \,\big) & 1 \\
 & \underline{1 - x} \\
 & \quad\;\; x \\
 & \quad\;\; \underline{x - x^2} \\
 & \qquad\quad x^2 \\
 & \qquad\quad \underline{x^2 - x^3} \\
 & \qquad\qquad\;\; x^3 \\
 & \qquad\qquad\;\; \cdots
\end{array}$$

C11S09.046: The long division is shown next.

$$\begin{array}{r l}
 & x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \cdots \\ \cline{2-2}
1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \Big) & x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \\
 & x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} - \dfrac{x^7}{6!} + \cdots \\ \cline{2-2}
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{30} + \dfrac{x^7}{840} - \cdots \\
 & \qquad \dfrac{x^3}{3} - \dfrac{x^5}{6} + \dfrac{x^7}{72} - \cdots \\ \cline{2-2}
 & \qquad\qquad \dfrac{2x^5}{15} - \dfrac{4x^7}{315} + \cdots \\
 & \qquad\qquad \dfrac{2x^5}{15} - \dfrac{x^7}{15} + \cdots \\ \cline{2-2}
 & \qquad\qquad\qquad \dfrac{17x^7}{315} - \cdots
\end{array}$$

C11S09.047: The equation (actually, the *identity*)

$$(1 - x)(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots) \;=\; 1$$

leads to

$$a_0 + (a_1 - a_0)x + (a_2 - a_1)x^2 + (a_3 - a_2)x^3 + (a_4 - a_3)x^4 + \cdots \;=\; 1.$$

It now follows that $a_0 = 1$ and that $a_{n+1} = a_n$ if $n \geqq 0$, and therefore $a_n = 1$ for every integer $n \geqq 0$. Consequently

$$\frac{1}{1 - x} \;=\; 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots, \quad -1 < x < 1.$$

C11S09.048: The equation

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)^2 = 1 + x$$

leads to

$$a_0^2 + 2a_0a_1x + (a_1^2 + 2a_0a_2)x^2 + (2a_1a_2 + 2a_0a_3)x^3$$
$$+ (a_2^2 + 2a_1a_3 + 2a_0a_4)x^4 + (2a_2a_3 + 2a_1a_4 + 2a_0a_5)x^5 + \cdots = 1 + x.$$

It now follows that

$$a_0^2 = 1: \qquad a_0 = 1.$$
$$2a_0a_1 = 1: \qquad a_1 = \frac{1}{2}.$$
$$a_1^2 + 2a_0a_2 = 0: \qquad a_2 = -\frac{a_1^2}{2a_0} = -\frac{1}{8}.$$
$$2a_1a_2 + 2a_0a_3 = 0: \qquad a_3 = -\frac{a_1a_2}{a_0} = \frac{1}{16}.$$
$$a_2^2 + 2a_1a_3 + 2a_0a_4 = 0: \qquad a_4 = -\frac{a_2^2 + 2a_1a_3}{2a_0} = -\frac{5}{128}.$$

C11S09.049: The method of Example 3 uses the identity

$$\sec x \cos x = 1$$

and begins by assuming the existence of coefficients $\{a_i\}$ such that

$$\sec x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots.$$

Thus

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) = 1,$$

so that

$$a_0 + a_1x + \left(a_2 - \frac{1}{2}a_0\right)x^2 + \left(a_3 - \frac{1}{2}a_1\right)x^3 + \left(a_4 - \frac{1}{2}a_2 + \frac{1}{24}a_0\right)x^4 + \cdots = 1.$$

It now follows that

$$a_0 = 1; \qquad a_1 = 0;$$
$$a_2 = \frac{1}{2}a_0 = \frac{1}{2}; \qquad a_3 = \frac{1}{2}a_1 = 0;$$
$$a_4 = \frac{1}{2}a_2 - \frac{1}{24}a_0 = \frac{5}{24}.$$

Therefore

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \frac{50521}{3628800}x^{10} + \frac{540553}{95800320}x^{12} + \frac{199360981}{87178291200}x^{14} + \cdots.$$

We had *Mathematica* 3.0 compute a few extra terms in case you did as well and want to check your work. (We used the command

```
Series[ Sec[x], { x, 0, 20 } ] // Normal
```

but to save space we show here only the first eight terms of the resulting Taylor polynomial.)

C11S09.050: The series for $f(x) = \ln(1-x)$ can be found by the method of Example 10 of Section 11.8. We multiply $f(x)$ by the geometric series representation of $1/(1-x)$ and find that, if $-1 < x < 1$, then

$$\begin{aligned}
\frac{1}{1-x} \cdot \ln(1-x) &= (1 + x + x^2 + x^3 + x^4 + \cdots)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right) \\
&= -(1 + x + x^2 + x^3 + x^4 + \cdots)\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) \\
&= -\left[x + \left(1 + \frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^4 + \cdots\right] = -\sum_{n=1}^{\infty} H_n x^n
\end{aligned}$$

where

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

is the nth partial sum of the harmonic series.

C11S09.051: Example 10 in Section 11.8 shows how to derive the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots, \qquad -1 < x < 1.$$

Hence

$$\begin{aligned}
1 + x = \exp(\ln(1+x)) &= \sum_{n=0}^{\infty} a_n \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)^n \\
&= a_0 + a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) + a_2\left(x^2 - x^3 + \left[\frac{1}{4} + \frac{2}{3}\right]x^4 - \cdots\right) \\
&\qquad + a_3\left(x^3 + \left[-\frac{1}{2} - 1\right]x^4 + \cdots\right) + \cdots \\
&= a_0 + a_1 x + \left(a_2 - \frac{1}{2}a_1\right)x^2 + \left(a_3 - a_2 + \frac{1}{3}a_1\right)x^3 + \cdots.
\end{aligned}$$

Therefore

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}, \qquad \text{and} \qquad a_3 = a_2 - \frac{1}{3}a_1 = \frac{1}{6}.$$

C11S09.052: Assume that there exist coefficients $\{a_i\}$ such that

$$\frac{x}{\sin x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

on some open interval containing the origin (there do). Then

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) = x,$$

and therefore

$$a_0 x + a_1 x^2 + \left(a_2 - \frac{1}{6}a_0\right)x^3 + \left(a_3 - \frac{1}{6}a_1\right)x^4 + \left(a_4 - \frac{1}{6}a_2 + \frac{1}{120}a_0\right)x^5 + \cdots = x.$$

It now follows that

$$\begin{aligned} a_0 &= 1, & a_1 &= 0, \\ a_2 &= \frac{1}{6}a_0 = \frac{1}{6}, & a_3 &= \frac{1}{6}a_1 = 0, \quad \text{and} \\ a_4 &= \frac{1}{6}a_2 - \frac{1}{120}a_0 = \frac{1}{36} - \frac{1}{120} = \frac{7}{360}. \end{aligned}$$

Therefore

$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \frac{127x^8}{604800} + \frac{73x^{10}}{3421440} + \frac{1414477x^{12}}{653837184000} + \cdots.$$

We had *Mathematica* 3.0 compute a few extra coefficients in case you did as well and want to check your work.

C11S09.053: Long division of the finite power series $1 + x + x^2$ into the finite power series $2 + x$ proceeds as shown next.

$$\begin{array}{r|l} & 2 - x - x^2 + \cdots \\ \hline 1 + x + x^2 & 2 + x \\ & 2 + 2x + 2x^2 \\ \cline{2-2} & \quad -x - 2x^2 \\ & \quad -x - x^2 - x^3 \\ \cline{2-2} & \qquad -x^2 + x^3 \\ & \qquad -x^2 - x^3 - x^4 \\ \cline{2-2} & \qquad\quad 2x^3 + x^4 \\ & \qquad\quad \cdots \end{array}$$

The next dividend is the original dividend with exponents increased by 3, so the next three terms in the numerator can be obtained by multiplying the first three by x^3. Thus we obtain the series representation

$$\frac{2 + x}{1 + x + x^2} = 2 - x - x^2 + 2x^3 - x^4 - x^5 + 2x^6 - x^7 - x^8 + \cdots.$$

This series is the sum of three geometric series each with ratio x^3, so it converges on the interval $(-1, 1)$. Summing the three geometric series separately, we obtain

$$\frac{2}{1 - x^3} - \frac{x}{1 - x^3} - \frac{x^2}{1 - x^3} = \frac{2 - x - x^2}{1 - x^3} = \frac{(1 - x)(2 + x)}{(1 - x)(1 + x + x^2)},$$

thus verifying our computations.

C11S09.054: Using the series in Problem 53, we find that

$$\int_0^{1/2} \frac{x+2}{x^2+x+1}\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{2x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} + \frac{2x^7}{7} - \cdots\right]_0^{1/2}$$

$$= \frac{2}{2} - \frac{1}{2^2\cdot 2} - \frac{1}{2^3\cdot 3} + \frac{2}{2^4\cdot 4} - \frac{1}{2^5\cdot 5} - \frac{1}{2^6\cdot 6} + \frac{2}{2^7\cdot 7} - \cdots .$$

The sum of the first nine terms of the last series is approximately 0.857256. The computer algebra system *Mathematica* 3.0 reports that

$$\int_0^{1/2} \frac{x+2}{x^2+x+1}\,dx = \left[\sqrt{3}\,\arctan\frac{2x+1}{\sqrt{3}} + \frac{1}{2}\ln(x^2+x+1)\right]_0^{1/2}$$

$$= \sqrt{3}\,\arctan\left(\frac{2}{\sqrt{3}}\right) + \frac{1}{2}\ln\left(\frac{7}{4}\right) - \frac{\pi}{2\sqrt{3}} \approx 0.857400371269052481.$$

C11S09.055: The first two steps in the long division of $1 + x^2 + x^4$ into 1 give quotient $1 - x^2$ and remainder (and new dividend) x^6. So the process will repeat with exponents increased by 6, and thus

$$\frac{1}{1+x^2+x^4} = 1 - x^2 + x^6 - x^8 + x^{12} - x^{14} + x^{18} - x^{20} + \cdots .$$

Therefore

$$\int_0^{1/2} \frac{1}{1+x^2+x^4}\,dx = \left[x - \frac{x^3}{3} + \frac{x^7}{7} - \frac{x^9}{9} + \frac{x^{13}}{13} - \frac{x^{15}}{15}\cdots\right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{2^3\cdot 3} + \frac{1}{2^7\cdot 7} - \frac{1}{2^9\cdot 9} + \frac{1}{2^{13}\cdot 13} - \frac{1}{2^{15}\cdot 15} + \cdots .$$

The sum of the first five terms of the last series is approximately 0.459239824988 and the sum of the first six terms is approximately 0.459239825000. The *Mathematica* 3.0 command

```
NIntegrate[ 1/(1 + x∧2 + x∧4), { x, 0, 1/2 }, WorkingPrecision → 28 ]
```

returns 0.4592398249998759. The computer algebra system *Derive* 2.56 yields

$$\int_0^{1/2} \frac{1}{1+x^2+x^4}\,dx$$

$$= \left[\frac{\sqrt{3}}{6}\left\{\arctan\left(\frac{\sqrt{3}}{3}(2x+1)\right) + \arctan\left(\frac{\sqrt{3}}{3}(2x-1)\right)\right\} + \frac{1}{4}\ln\left(\frac{x^2+x+1}{x^2-x+1}\right)\right]_0^{1/2}$$

$$= \frac{\sqrt{3}}{6}\arctan\left(\frac{2\sqrt{3}}{3}\right) + \frac{1}{4}\ln\left(\frac{7}{3}\right) \approx 0.45923982499987591403.$$

C11S09.056: The first two steps in the long division of $1 + x^4 + x^8$ into 1 give quotient $1 - x^4$ and remainder (and new dividend) x^{12}. So the process will repeat with exponents increased by 12, and thus

$$\frac{1}{1+x^4+x^8} = 1 - x^4 + x^{12} - x^{16} + x^{24} - x^{28} + x^{36} - x^{40} + \cdots .$$

Therefore

$$\int_0^{1/2} \frac{1}{1+x^4+x^8}\,dx = \left[x - \frac{x^5}{5} + \frac{x^{13}}{13} - \frac{x^{17}}{17} + \frac{x^{25}}{25} - \frac{x^{29}}{29} + \cdots \right]_0^{1/2}$$
$$= \frac{1}{2} - \frac{1}{2^5 \cdot 5} + \frac{1}{2^{13} \cdot 13} - \frac{1}{2^{17} \cdot 17} + \frac{1}{2^{25} \cdot 25} - \frac{1}{2^{29} \cdot 29} + \cdots .$$

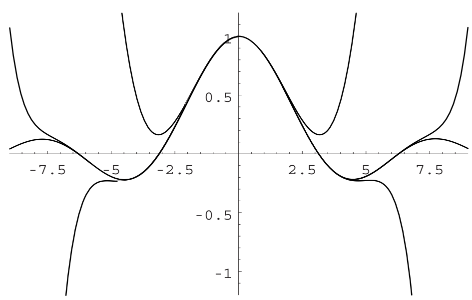
The sum of the first four terms of the last series is approximately 0.493758942364 and the sum of the first five terms is the same to 12 digits. The *Mathematica* 3.0 `NIntegrate` command yields the approximate value 0.4937589423641742. The computer algebra system *Derive* 2.56 reports that

$$\int_0^{1/2} \frac{1}{1+x^4+x^8}\,dx$$
$$= \left[\frac{\sqrt{3}}{6} \left\{ \arctan\left(\frac{\sqrt{3}}{3}(2x+1) \right) + \arctan\left(\frac{\sqrt{3}}{3}(2x-1) \right) \right\} + \frac{\sqrt{3}}{12} \ln\left(\frac{x^2 + x\sqrt{3}\,+1}{x^2 - x\sqrt{3}\,+1} \right) \right]_0^{1/2}$$
$$= \frac{\sqrt{3}}{6} \arctan\left(\frac{2\sqrt{3}}{3} \right) - \frac{\sqrt{3}}{6} \ln 2 - \frac{\sqrt{3}}{12} \ln\left(\frac{37 - 20\sqrt{3}}{52} \right) \approx 0.49375894236417420129656 58.$$

C11S09.057: See the solution of Problem 61 in Section 11.8. We plotted the Taylor polynomials with center zero for $f(x)$,

$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^{2k-2}}{(2k-1)!},$$

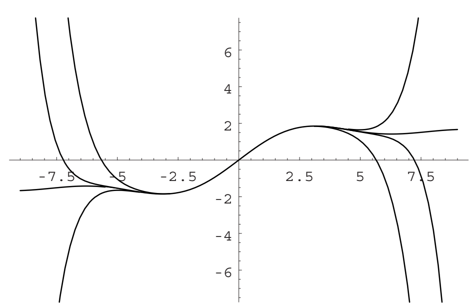
for $n = 3,\ 6,$ and 9. Their graphs, together with the graph of f, are shown next.



C11S09.058: See the solution to Problem 62 of Section 11.8. The Taylor series with center zero for $f(x)$ is

$$f(x) = \int_0^x \frac{\sin t}{t}\,dt = x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \frac{x^9}{3265920} - \frac{x^{11}}{439084800} + \cdots .$$

We plotted the graph of f and its Taylor polynomials of degrees 7, 9, and 11; the results are shown next.



C11S09.059: The four Maclaurin series we need are in Eq. (4) of Section 11.8, Eq. (7) of Section 11.9, Eq. (21) of Section 11.8, and Eq. (20) of Section 11.8. They are

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\,, \\
\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots\,, \\
\arcsin x &= x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots\,, \qquad \text{and} \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,.
\end{aligned}$$

Therefore

$$\frac{\sin x - \tan x}{\arcsin x - \arctan x} = \frac{-\frac{1}{2}x^3 - \frac{1}{8}x^5 - \cdots}{\frac{1}{2}x^3 + \frac{1}{8}x^5 + \cdots} = \frac{-\frac{1}{2} - \frac{1}{8}x^2 - \cdots}{\frac{1}{2} + \frac{1}{8}x^2 + \cdots} \to \frac{-\frac{1}{2} - 0 - 0 - \cdots}{\frac{1}{2} + 0 + 0 + \cdots} = -1$$

as $x \to 0$.

Use of l'Hôpital's rule to solve this problem—even with the aid of a computer algebra program—is troublesome. Using *Mathematica* 3.0, we first defined

$$f(x) \;=\; \sin x \;-\; \tan x \qquad \text{and} \qquad g(x) \;=\; \arcsin x \;-\; \arctan x.$$

Then the command

```
Limit[ f[x]/g[x], x → 0 ]
```

elicited the disappointing response "Indeterminate." But when we computed the quotient of the derivatives using

```
f′[x]/g′[x]
```

we obtained the expected fraction

$$\frac{\cos x \;-\; \sec^2 x}{\dfrac{1}{\sqrt{1 - x^2}} - \dfrac{1}{1 + x^2}},$$

and *Mathematica* reported that the limit of this fraction, as $x \to 0$, was -1. To avoid *Mathematica's* invocation of l'Hôpital's rule (with the intent of obtaining a fraction that was *not* indeterminate), we had *Mathematica* find the quotient of the derivatives of the last numerator and denominator; the resulting numerator was

$$-2x\sqrt{1 - x^2}\,(\cos x - \sec^2 x) + \frac{x(1 + x^2)(\cos x - \sec^2 x)}{\sqrt{1 - x^2}} + (1 + x^2)\sqrt{1 - x^2}\,(\sin x + 2\sec^2 x \tan x)$$

and the corresponding denominator was

$$-2x - \frac{x}{\sqrt{1 - x^2}}.$$

The quotient is still indeterminate, but repeating the process—you really don't want to see the results—next (and finally) led to a form not indeterminate, whose value at $x = 0$ was (still) -1.

C11S09.060: We used *Mathematica* 3.0 to generate these Maclaurin series:

$$\sin(\tan x) = x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{55x^7}{1008} - \frac{143x^9}{3456} - \frac{968167x^{11}}{39916800} + \cdots \qquad \text{and}$$
$$\tan(\sin x) = x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{107x^7}{5040} - \frac{73x^9}{24192} + \frac{41897x^{11}}{39916800} + \cdots .$$

To find the series for $\arcsin(\arctan x)$, we substituted the Taylor polynomial with center zero and degree 11 for $\arctan x$ into the corresponding Taylor polyomial for $\arcsin x$. We used *Mathematica* 3.0 to expand the resulting expression and thereby discovered that

$$\arcsin(\arctan x) = x - \frac{x^3}{6} + \frac{13x^5}{120} - \frac{341x^7}{5040} + \frac{18649x^9}{362880} - \frac{177761x^{11}}{4435200} - \cdots .$$

Similarly, we found that

$$\arctan(\arcsin x) = x - \frac{x^3}{6} + \frac{13x^5}{120} - \frac{173x^7}{5040} + \frac{12409x^9}{362880} - \frac{123379x^{11}}{13305600} - \cdots .$$

Therefore

$$\sin(\tan x) - \tan(\sin x) = -\frac{x^7}{30} - \frac{29x^9}{756} - \frac{1913x^{11}}{75600} + \cdots$$

and

$$\arcsin(\arctan x) - \arctan(\arcsin x) = -\frac{x^7}{30} + \frac{13x^9}{756} - \frac{2329x^{11}}{75600} + \cdots .$$

Thus (after cancelling the common factor x^7) we see that

$$\lim_{x\to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)} = \lim_{x\to 0} \frac{-\dfrac{1}{30} - \dfrac{29}{756}x^2 - \dfrac{1913}{75600}x^4 + \cdots}{-\dfrac{1}{30} + \dfrac{13}{756}x^2 - \dfrac{2329}{75600}x^4 + \cdots} = 1.$$

It would not be practical to try to find this limit using l'Hôpital's rule, even with a computer. The common factor x^7 tells us that the rule would have to be applied seven times. The result of just the first application of the rule yields

$$\lim_{x\to 0} \frac{(1-x^2)^{1/2}(1+x^2)(1+\arcsin^2 x)(1-\arctan^2 x)^{1/2}[(\cos(\tan x))\sec^2 x - \cos x\, \sec^2(\sin x)]}{(1-x^2)^{1/2} + (1-x^2)^{1/2}\arcsin^2 x - (1-\arctan^2 x)^{1/2} - x^2(1-\arctan^2 x)^{1/2}}.$$

C11S09.061: Part (a): Assume that $a \geqq b > 0$. The parametrization $x = a\cos t,\ y = b\sin t$ yields arc length element

$$ds = (a^2\sin^2 t + b^2\cos^2 t)^{1/2}\ dt = [a^2\sin^2 t + a^2\cos^2 t + (b^2 - a^2)\cos^2 t]^{1/2}\ dt$$
$$= [a^2 + (b^2 - a^2)\cos^2 t]^{1/2}\ dt = a\left[1 - \frac{a^2 - b^2}{a^2}\cos^2 t\right]^{1/2}\ dt = a(1 - \epsilon^2\cos^2 t)^{1/2}\ dt$$

where

$$\epsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - (b/a)^2}$$

is the eccentricity of the ellipse; recall that $0 \leqq \epsilon < 1$ for the case of an ellipse. We multiply the length of the part of the ellipse in the first quadrant by 4 to find its total arc length is

$$p = 4a \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \cos^2 t}\ dt.$$

Part (b): The binomial formula yields

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{2^2} \cdot \frac{x^2}{2!} - \frac{3}{2^3} \cdot \frac{x^3}{3!} - \frac{3 \cdot 5}{2^4} \cdot \frac{x^4}{4!} - \frac{3 \cdot 5 \cdot 7}{2^5} \cdot \frac{x^5}{5!} - \cdots.$$

Consequently,

$$\begin{aligned} \sqrt{1 - \epsilon^2 \cos^2 t} = 1 - \frac{1}{2}\epsilon^2 \cos^2 t - \frac{1}{2^2 \cdot 2!}\epsilon^4 \cos^4 t - \frac{3}{2^3 \cdot 3!}\epsilon^6 \cos^6 t \\ - \frac{3 \cdot 5}{2^4 \cdot 4!}\epsilon^8 \cos^8 t - \frac{3 \cdot 5 \cdot 7}{2^5 \cdot 5!}\epsilon^{10} \cos^{10} t - \cdots. \end{aligned}$$

Then formula 113 in the Table of Integrals in the text yields

$$\begin{aligned} p &= 4a \int_0^{\pi/2} \sqrt{1 - \epsilon^2 \cos^2 t}\ dt \\ &= 4a \left(\Big[\, t \,\Big]_0^{\pi/2} - \frac{1}{2}\epsilon^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2^2 \cdot 2!}\epsilon^4 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2} \right. \\ &\qquad \left. - \frac{3}{2^3 \cdot 3!}\epsilon^6 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{\pi}{2} - \frac{3 \cdot 5}{2^4 \cdot 4!}\epsilon^8 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{\pi}{2} - \cdots \right) \\ &= 2\pi a \left(1 - \frac{1}{4}\epsilon^2 - \frac{3}{64}\epsilon^4 - \frac{5}{256}\epsilon^6 - \frac{175}{16384}\epsilon^8 - \cdots \right). \end{aligned}$$

C11S09.062: First we apply the binomial series to express A in terms of a and ϵ:

$$\begin{aligned} A = \frac{1}{2}(a+b) &= \frac{1}{2}\left(a + a\sqrt{1 - \epsilon^2} \right) \\ &= \frac{a}{2}\left(1 + 1 - \frac{1}{2}\epsilon^2 - \frac{1}{2^2} \cdot \frac{\epsilon^4}{2!} - \frac{3}{2^3} \cdot \frac{\epsilon^6}{3!} - \frac{3 \cdot 5}{2^4} \cdot \frac{\epsilon^8}{4!} - \cdots \right) \\ &= a\left(1 - \frac{\epsilon^2}{4} - \frac{\epsilon^4}{16} - \frac{\epsilon^6}{32} - \frac{5\epsilon^8}{256} - \cdots \right). \end{aligned}$$

Next,

$$
\begin{aligned}
R = \sqrt{\frac{1}{2}(a^2+b^2)} &= \left[\frac{1}{2}\left\{a^2 + a^2(1-\epsilon^2)\right\}\right]^{1/2} = a\left(1-\frac{\epsilon^2}{2}\right)^{1/2} \\
&= a\left(1 - \frac{\epsilon^2}{4} - \frac{1}{2^2}\cdot\frac{\epsilon^4}{2!\cdot 4} - \frac{3}{2^3}\cdot\frac{\epsilon^6}{3!\cdot 8} - \frac{3\cdot 5}{2^4}\cdot\frac{\epsilon^8}{4!\cdot 16} - \cdots\right) \\
&= a\left(1 - \frac{\epsilon^2}{4} - \frac{\epsilon^4}{32} - \frac{\epsilon^6}{128} - \frac{5\epsilon^8}{2048} - \cdots\right).
\end{aligned}
$$

Therefore

$$
\frac{1}{2}(A+R) = a\left(1 - \frac{\epsilon^2}{4} - \frac{3\epsilon^4}{64} - \frac{5\epsilon^6}{256} - \frac{180\epsilon^8}{16384} - \cdots\right).
$$

It follows that the arc length of the ellipse is

$$
p = \pi(A+R) + \frac{5\pi a\epsilon^8}{8192} + \cdots . \tag{1}
$$

If ϵ is close to zero then the perimeter of the ellipse is almost exactly

$$
\pi(A+R) = \pi\left(\frac{a+b}{2} + \sqrt{\frac{a^2+b^2}{2}}\right).
$$

If $a = 238857$ (miles, exactly) and $\epsilon = 0.0549$ (exactly), then Eq. (1) and *Mathematica* 3.0 predict that the arc length of the elliptical orbit of the Moon is approximately

$$
1499651.3094565814 \quad \text{(miles); that is,} \quad 1499651 \text{ mi } 1633 \text{ ft } 11.169 \text{ in.}
$$

Section 11.10

C11S10.001: We use series methods to solve $\dfrac{dy}{dx} = y$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0,$$

and hence

$$a_{n+1} = \frac{a_n}{n+1} \qquad \text{for} \quad n \geqq 0.$$

Thus

$$\begin{array}{rlrl} & a_1 = a_0, & & a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2}, \\ a_3 = \dfrac{a_2}{3} = \dfrac{a_0}{3 \cdot 2}, & & a_4 = \dfrac{a_3}{4} = \dfrac{a_0}{4!}, & \\ a_5 = \dfrac{a_4}{5} = \dfrac{a_0}{5!}, & & \ldots . & \end{array}$$

In general, $a_n = \dfrac{a_0}{n!}$ if $n \geqq 0$. Hence

$$y(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Finally,

$$\lim_{n\to\infty} \left| \frac{n! x^{n+1}}{(n+1)! x^n} \right| = |x| \cdot \left(\lim_{n\to\infty} \frac{1}{n+1} \right) = 0,$$

so the radius of convergence of the series we found is $+\infty$.

C11S10.002: We use series methods to solve $\dfrac{dy}{dx} = 4y$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} - 4a_n]\, x^n \;=\; 0,$$

so that

$$a_{n+1} \;=\; \frac{4a_n}{n+1} \qquad \text{if} \quad n \geqq 0.$$

Therefore

$$\begin{aligned} a_1 &= 4a_0, & a_2 &= \frac{4a_1}{2} = \frac{4^2 a_0}{2}, \\ a_3 = \frac{4a_2}{3} &= \frac{4^3 a_0}{3!}, & a_4 &= \frac{4a_3}{4} = \frac{4^4 a_0}{4!}, \\ a_5, &= \frac{4^5 a_0}{5!}, & & \dots\,. \end{aligned}$$

Thus $a_n \;=\; \dfrac{4^n}{n!}a_0$ if $n \geqq 0$. Therefore

$$y(x) \;=\; a_0 \sum_{n=0}^{\infty} \frac{4^n}{n!}x^n \;=\; a_0 \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} \;=\; a_0 e^{4x}.$$

By a computation almost identical to that in the solution of Problem 1, this series has radius of convergence $+\infty$.

C11S10.003: We use series methods to solve $2\dfrac{dy}{dx} + 3y \;=\; 0$. Assume that

$$y(x) \;=\; \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) \;=\; \sum_{n=1}^{\infty} n a_n x^{n-1} \;=\; \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} [2(n+1)a_{n+1} + 3a_n]\, x^n \;=\; 0,$$

and thus

$$a_{n+1} \;=\; -\frac{3a_n}{2(n+1)} \qquad \text{if} \quad n \geqq 0.$$

Therefore

$$a_1 = -\frac{3}{2}a_0, \qquad a_2 = -\frac{3}{2} \cdot \frac{a_1}{2} = \left(\frac{3}{2}\right)^2 \cdot \frac{a_0}{2},$$

$$a_3 = -\frac{3}{2} \cdot \frac{a_2}{3} = -\left(\frac{3}{2}\right)^3 \cdot \frac{a_0}{3!}, \qquad \dots .$$

In general,

$$a_n = (-1)^n \left(\frac{3}{2}\right)^n \cdot \frac{a_0}{n!} \qquad \text{for} \quad n \geqq 1.$$

Therefore

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n \cdot \frac{x_n}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{3x}{2}\right)^n = a_0 e^{-3x/2}.$$

By computations quite similar to those in the solution of Problem 1, this series has radius of convergence $+\infty$.

C11S10.004: We use series methods to solve $\dfrac{dy}{dx} + 2xy = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0;$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0;$$

$$a_1 + \sum_{n=1}^{\infty} \left[(n+1)a_{n+1} + 2a_{n-1}\right] x^n = 0.$$

Therefore

$$a_1 = 0 \qquad \text{and} \qquad (n+1)a_{n+1} = -2a_{n-1} \quad \text{if} \quad n \geqq 1;$$

that is,

$$a_{n+2} = -\frac{2a_n}{n+2} \quad \text{if} \quad n \geqq 0.$$

Therefore $a_3 = a_5 = a_7 = \cdots = 0$ and

$$a_2 = -\frac{2a_0}{2} = -\frac{a_0}{1!}, \qquad a_4 = -\frac{2a_2}{4} = \frac{2^2 a_0}{4 \cdot 2} = \frac{a_0}{2!},$$

$$a_6 = -\frac{2a_4}{6} = -\frac{2^3 a_0}{6 \cdot 4 \cdot 2} = -\frac{a_0}{3!}, \qquad a_8 = -\frac{2^4 a_0}{8 \cdot 6 \cdot 4 \cdot 2} = \frac{a_0}{4!};$$

in general,

$$a_{2n} = \frac{(-1)^n}{n!} a_0 \qquad \text{if} \quad n \geqq 1.$$

Therefore

$$y(x) = a_0 \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots \right) = a_0 \exp\left(-x^2\right).$$

The radius of convergence of this series is $+\infty$.

C11S10.005: We use series methods to solve $\dfrac{dy}{dx} = x^2 y$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Therefore $a_1 = 0$, $a_2 = 0$, and $(n+1)a_{n+1} = a_{n-2}$ if $n \geqq 2$; that is,

$$a_1 = a_2 = a_4 = a_5 = a_7 = a_8 = \cdots 0 \quad \text{and} \quad a_{n+3} = \frac{a_n}{n+3}$$

if $n \geqq 0$. Hence

$$a_3 = \frac{a_0}{3} = \frac{a_0}{1! \cdot 3}, \qquad a_6 = \frac{a_3}{6} = \frac{a_0}{6 \cdot 3} = \frac{a_0}{2! \cdot 3^2},$$

$$a_9 = \frac{a_6}{9} = \frac{a_0}{9 \cdot 6 \cdot 3} = \frac{a_0}{3! \cdot 3^3}, \qquad \ldots;$$

in general,

$$a_{3n} = \frac{a_0}{n! \cdot 3^n} \quad \text{if } n \geqq 1.$$

Therefore

$$y(x) = a_0 \left[1 + \frac{1}{1!} \cdot \frac{x^3}{3} + \frac{1}{2!} \left(\frac{x^3}{3} \right)^2 + \frac{1}{3!} \left(\frac{x^3}{3} \right)^3 + \cdots \right] = a_0 \exp\left(\frac{x^3}{3} \right).$$

As in previous solutions, the radius of convergence of this series is $+\infty$.

C11S10.006: We use series methods to solve $(x-2)\dfrac{dy}{dx} + y = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0;$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

When $n = 0$, we have $-2a_1 + a_0 = 0$, and thus

$$a_1 = \frac{a_0}{2}.$$

If $n \geqq 1$, then $na_n - 2(n+1)a_{n+1} + a_n = 0$, so that $2(n+1)a_{n+1} = (n+1)a_n$. Therefore

$$a_{n+1} = \frac{a_n}{2} \quad \text{if} \quad n \geqq 1.$$

Hence

$$a_2 = \frac{a_1}{2} = \frac{a_0}{2^2}, \qquad a_3 = \frac{a_2}{2} = \frac{a_0}{2^3},$$

$$a_4 = \frac{a_3}{2} = \frac{a_0}{2^4}, \qquad \dots;$$

that is,

$$a_n = \frac{a_0}{2^n} \quad \text{if} \quad n \geqq 1.$$

Therefore

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{2^n} = a_0 \cdot \frac{1}{1 - \dfrac{x}{2}} = \frac{2a_0}{2-x}$$

because the series is geometric; for the same reason, its radius of convergence is $R = 2$.

C11S10.007: We use series methods to solve $(2x-1)\dfrac{dy}{dx} + 2y = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

If $n = 0$, we find that $-a_1 + 2a_0 = 0$, and thus that $a_1 = 2a_0$. If $n \geqq 1$, then

$$2na_n - (n+1)a_{n+1} + 2a_n = 0; \qquad (n+1)a_{n+1} = 2(n+1)a_n; \qquad a_{n+1} = 2a_n.$$

Hence $a_1 = 2a_0$, $a_2 = 2^2 a_0$, $a_3 = 2^3 a_0$, etc.; in general, $a_n = 2^n a_0$ if $n \geqq 1$. Therefore

$$y(x) = a_0 \sum_{n=0}^{\infty} 2^n x^n = a_0 \sum_{n=0}^{\infty} (2x)^n = \frac{a_0}{1-2x}$$

because the series is geometric; for the same reason, its radius of convergence is $R = \frac{1}{2}$.

C11S10.008: We use series methods to solve $2(x+1)\dfrac{dy}{dx} = y$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n.$$

When $n = 0$, we have

$$2a_1 = a_0, \qquad \text{so that} \qquad a_1 = \frac{1}{2}a_0.$$

If $n \geqq 1$, then $2na_n + 2(n+1)a_{n+1} = a_n$: $2(n+1)a_{n+1} = -(2n-1)a_n$, and thus

$$a_{n+1} = -\frac{2n-1}{2n+2}a_n.$$

Consequently,

$$a_2 = -\frac{1}{4}a_1 = -\frac{1}{4\cdot 2}a_0, \qquad a_3 = -\frac{3}{6}a_2 = \frac{3\cdot 1}{6\cdot 4\cdot 2}a_0,$$

$$a_4 = -\frac{5}{8}a_3 = -\frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}a_0, \qquad \ldots .$$

Thus

$$y(x) = a_0\left(1+\frac{1}{x}x-\frac{1}{4\cdot 2}x^2+\frac{3\cdot 1}{6\cdot 4\cdot 2}x^3-\frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}x^4+\frac{7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}x^5-\cdots\right) = a_0\sqrt{1+x}\,.$$

The radius of convergence of this binomial series is $R = 1$.

C11S10.009: We use series methods to solve $(x-1)\dfrac{dy}{dx} + 2y = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

When $n = 0$, we have $-a_1 + 2a_0 = 0$, so that $a_1 = 2a_0$. If $n \geqq 1$, then

$$na_n - (n+1)a_{n+1} + 2a_n = 0: \;\; (n+1)a_{n+1} = (n+2)a_n,$$

and hence

$$a_{n+1} = \frac{n+2}{n+1}a_n \quad \text{if} \quad n \geqq 0.$$

Therefore

$$a_2 = \frac{3}{2}a_1 = 3a_0, \qquad a_3 = \frac{4}{3}a_2 = 4a_0,$$

$$a_4 = \frac{5}{4}a_3 = 5a_0, \qquad \ldots;$$

in general, $a_n = (n+1)a_0$ if $n \geqq 1$. Therefore

$$y(x) = a_0\sum_{n=0}^{\infty}(n+1)x^n.$$

Now $y(x) = F'(x)$ where

$$F(x) = a_0\sum_{n=0}^{\infty} x^{n+1} = \frac{a_0 x}{1-x}.$$

Consequently,

$$y(x) = F'(x) = \frac{a_0(1-x+x)}{(1-x)^2} = \frac{a_0}{(1-x)^2}.$$

The radius of convergence of the series for $y(x)$ is $R = 1$.

C11S10.010: We use series methods to solve $2(x-1)\dfrac{dy}{dx} = 3y$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=1}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} 3a_n x^n.$$

It is convenient, and has no effect, if we change the range of the index in the first sum from $1 \leqq n < +\infty$ to $0 \leqq n < +\infty$. Thus we find that, if $n \geqq 0$, then

$$2na_n - 2(n+1)a_{n+1} = 3a_n; \quad 2(n+1)a_{n+1} = (2n-3)a_n; \quad a_{n+1} = \frac{2n-3}{2n+2}a_n.$$

Therefore

$$\begin{aligned} a_1 &= -\frac{3}{2}a_0, & a_2 &= -\frac{1}{4}a_1 = \frac{1\cdot 3}{4\cdot 2}a_0, \\ a_3 &= \frac{1}{6}a_2 = \frac{1\cdot 1\cdot 3}{6\cdot 4\cdot 2}a_0, & a_4 &= \frac{3}{8}a_3 = \frac{3\cdot 1\cdot 1\cdot 3}{8\cdot 6\cdot 4\cdot 2}a_0, \end{aligned}$$

and so on. Thus

$$y(x) = a_0\left(1 - \frac{3}{2}x + \frac{1\cdot 3}{4\cdot 2}x^2 + \frac{1\cdot 1\cdot 3}{6\cdot 4\cdot 2}x^3 + \frac{3\cdot 1\cdot 1\cdot 3}{8\cdot 6\cdot 4\cdot 2}x^4 + \frac{5\cdot 3\cdot 1\cdot 1\cdot 3}{10\cdot 8\cdot 6\cdot 4\cdot 2}x^5 + \cdots\right) = a_0(1-x)^{3/2}.$$

The radius of convergence of this binomial series is $R = 1$.

C11S10.011: We use series methods to solve the differential equation $y'' = y$. Assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Then substitution in the given differential equation yields

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad \text{for} \quad n \geqq 0.$$

Therefore

$$a_2 = \frac{a_0}{2 \cdot 1}, \qquad a_3 = \frac{a_1}{3 \cdot 2},$$

$$a_4 = \frac{a_0}{4!}, \qquad a_5 = \frac{a_1}{5!},$$

and so on. Hence

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \right) \\ &= a_0 \cosh x + a_1 \sinh x. \end{aligned}$$

The radius of convergence of all series here is $R = +\infty$. The solution may also be expressed in the form $y(x) = c_1 e^x + c_2 e^{-x}$.

C11S10.012: We use series methods to solve the differential equation $y'' = 4y$. Assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Much as in the solution of Problem 11, substitution in the given differential equation leads to

$$a_{n+2} = \frac{4a_n}{(n+2)(n+1)} \quad \text{for} \quad n \geqq 0.$$

Consequently,

$$a_2 = \frac{4a_0}{2!}, \quad a_3 = \frac{4a_1}{3!}, \quad a_4 = \frac{4^2 a_0}{4!}, \quad a_5 = \frac{4^2 a_1}{5!},$$

and so on. Therefore

$$\begin{aligned} y(x) &= a_0 \left(1 + \frac{4x^2}{2!} + \frac{4^2 x^4}{4!} + \frac{4^3 x^6}{6!} + \cdots \right) + a_1 \left(x + \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} + \frac{4^3 x^7}{7!} + \cdots \right) \\ &= a_0 \left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots \right) + \frac{a_1}{2} \left(2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \frac{(2x)^7}{7!} + \cdots \right) \end{aligned}$$

$$= a_0 \cosh 2x + \frac{1}{2} a_1 \sinh 2x.$$

Each series here has radius of convergence $+\infty$. The solution of the given differential equation can also be expressed in the form $y(x) = c_1 e^{2x} + c_2 e^{-2x}$.

C11S10.013: We use series methods to solve the differential equation $y'' + 9y = 0$. Assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Then substitution in the given differential equation leads—as in the solution of Problem 11—to the recursion formula $(n+2)(n+1)a_{n+2} + 9a_n = 0$, and thus

$$a_{n+2} = -\frac{9}{(n+2)(n+1)} a_n \quad \text{for} \quad n \geqq 0.$$

Hence

$$\begin{aligned}
a_2 &= -\frac{9}{2!} a_0, & a_3 &= -\frac{9}{3!} a_1, \\
a_4 &= \frac{9^2}{4!} a_0, & a_5 &= \frac{9^2}{5!} a_1, \\
a_6 &= -\frac{9^3}{6!} a_0, & a_7 &= -\frac{9^3}{7!} a_1,
\end{aligned}$$

and so on. Hence

$$\begin{aligned}
y(x) &= a_0 \left(1 - \frac{9x^2}{2!} + \frac{9^2 x^4}{4!} - \frac{9^3 x^6}{6!} + \cdots \right) + a_1 \left(x - \frac{9x^3}{3!} + \frac{9^2 x^5}{5!} - \frac{9^3 x^7}{7!} + \cdots \right) \\
&= a_0 \left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots \right) + \frac{a_1}{3} \left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots \right) \\
&= a_0 \cos 3x + \frac{a_1}{3} \sin 3x = c_1 \cos 3x + c_2 \sin 3x.
\end{aligned}$$

The radius of convergence of each series here is $R = +\infty$.

C11S10.014: We use series methods to solve the differential equation $y'' + y = x$. Assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Then substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = x.$$

The case $n = 0$ yields $2a_2 + a_0 = 0$, and hence

$$a_2 = -\frac{1}{2}a_0.$$

The case $n = 1$ yields $6a_3 + a_1 = 1$, and thereby

$$a_3 = -\frac{a_1 - 1}{6}.$$

And if $n \geqq 2$, we obtain

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

The last recursion formula then yields

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}, \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1 - 1}{5!},$$

$$a_6 = -\frac{a_4}{6 \cdot 5} = -\frac{a_0}{6!}, \qquad a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{a_1 - 1}{7!},$$

and so on. Therefore

$$\begin{aligned} y(x) &= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &\qquad + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots \\ &= a_0 \cos x + a_1 \sin x + x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) \\ &= a_0 \cos x + a_1 \sin x + x - \sin x = a_0 \cos x + (a_1 - 1)\sin x + x = x + c_1 \cos x + c_2 \sin x. \end{aligned}$$

The radius of convergence of each series here is $R = +\infty$.

C11S10.015: Given the differential equation $x\dfrac{dy}{dx} + y = 0$, substitution of the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

as in earlier solutions in this section yields

$$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

It then follows that $a_0 = 0$ and that $na_n + a_n = 0$ if $n \geqq 1$. The latter equation implies that $a_n = 0$ if $n \geqq 1$. Thus we obtain only the trivial solution $y(x) \equiv 0$, which is not part of the general solution because it contains no arbitrary constant and is not independent of any other solution. Part of the reason that the series has no solution of the form in (1) is that a general solution is

$$y(x) = \frac{C}{x}.$$

This solution is undefined at $x = 0$ and, of course, has no power series expansion with center $c = 0$. Here's an experiment for you: Assume a solution of the form

$$\sum_{n=0}^{\infty} b_n (x-1)^n$$

and see what happens. Then assume a solution of the form

$$\sum_{n=-1}^{\infty} c_n x^n$$

and see what happens. You can learn more about these ideas, and their consequences, in a standard course in differential equations (make sure that the syllabus includes the topic of series solution of ordinary differential equations).

C11S10.016: Given the differential equation $2x\dfrac{dy}{dx} = y$, substitution of the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

as in earlier solutions in this section yields

$$\sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} a_n x^n.$$

Whe $n = 0$, this equation yields $a_0 = 0$. If $n \geqq 1$, it implies that $2na_n = a_n$, and hence that $a_n = 0$ for all $n \geqq 0$. So the given differential equation has no series solution of the form in (1) other than the trivial solution $y(x) \equiv 0$. A general solution of the given differential equation is $y(x) = C\sqrt{x}$. Perhaps it would be possible to discover a general solution by series methods were you to begin with the assumption of the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} b_n (x-1)^n$$

.

C11S10.017: Given the differential equation $x\dfrac{dy}{dx} + y = 0$, substitution of the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

as in earlier solutions in this section yields

$$\sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0;$$

$$\sum_{n=2}^{\infty} (n-1)a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Examination of the cases $n = 0$ and $n = 1$ yields $a_0 = a_1 = 0$. If $n \geqq 2$ we see that $(n-1)a_{n-1} + a_n = 0$, and hence that $a_n = 0$ for all $n \geqq 0$. Thus the series method using the form in (1) uncovers only the trivial solution $y(x) \equiv 0$, not a general solution of the given differential equation. Part of the reason is that a general solution of the differential equation is

$$y(x) = C \exp\left(\frac{1}{x}\right).$$

C11S10.018: Given the differential equation $x^3 \dfrac{dy}{dx} = 2y$, substitution of the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

as in earlier solutions in this section yields

$$\sum_{n=1}^{\infty} na_n x^{n+2} = \sum_{n=0}^{\infty} 2a_n x^n; \qquad \text{that is,}$$

$$\sum_{n=3}^{\infty} (n-2)a_{n-2} x^n = \sum_{n=0}^{\infty} 2a_n x^n.$$

It follows that $a_0 = a_1 = a_2 = 0$ and that

$$a_n = \frac{n-2}{2} a_{n-2} \qquad \text{if} \quad n \geqq 3.$$

Therefore $a_n = 0$ for all $n \geqq 0$, and so the series method yields only the trivial solution $y(x) \equiv 0$, not a general solution of the given differential equation. A general solution of that equation is

$$y(x) = C \exp\left(-\frac{1}{x^2}\right),$$

and this is part of the reason that the given differential equation has no solution of the form in Eq. (1).

C11S10.019: Given the initial value problem

$$y'' + 4y = 0; \qquad y(0) = 0, \quad y'(0) = 3,$$

we assume the existence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution in the given differential equation yields $(n+2)(n+1)a_{n+2} + 4a_n = 0$, from which we obtain the recurrence relation

$$a_{n+2} = -\frac{4}{(n+2)(n+1)} a_n \qquad \text{for} \quad n \geqq 0.$$

Thus we may choose a_0 and a_1 to be arbitrary constants, and find that

$$\begin{aligned} a_2 &= -\frac{4}{2!} a_0, & a_3 &= -\frac{4}{3!} a_1, \\ a_4 = -\frac{4}{4 \cdot 3} a_2 &= \frac{4^2}{4!} a_0, & a_5 &= \frac{4^2}{5!} a_1, \\ a_6 &= -\frac{4^3}{6!} a_0, & a_7 &= -\frac{4^3}{7!} a_1, \end{aligned}$$

and so on. Therefore the general solution of the given differential equation may be written in the form

$$\begin{aligned} y(x) &= a_0 \left(1 - \frac{4x^2}{2!} + \frac{4^2 x^4}{4!} - \frac{4^3 x^6}{6!} + \frac{4^4 x^8}{8!} - \cdots \right) + a_1 \left(x - \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} - \frac{4^3 x^7}{7!} + \frac{4^4 x^9}{9!} - \cdots \right) \\ &= a_0 \cos 2x + \frac{a_1}{2} \sin 2x = A \cos 2x + B \sin 2x. \end{aligned}$$

Substitution of the initial conditions yields $a_0 = y(0) = 0$ and $a_1 = y'(0) = 3$, so the particular solution of the differential equation is

$$y(x) = \frac{3}{2} \sin 2x.$$

C11S10.020: Given the initial value problem

$$y'' - 4y = 0; \qquad y(0) = 2, \quad y'(0) = 0,$$

we assume the existence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution in the given differential equation yields $(n+2)(n+1)a_{n+2} = 4a_n$, and thus

$$a_{n+2} = \frac{4}{(n+2)(n+1)}a_n, \qquad n \geqq 0.$$

Thus a_0 and a_1 may be chosen to be arbitrary constants, and

$$\begin{aligned} a_2 &= \frac{4}{2!}a_0, & a_3 &= \frac{4}{3!}a_1, \\ a_4 = \frac{4}{4\cdot 3}a_2 &= \frac{4^2}{4!}a_0, & a_5 &= \frac{4}{5\cdot 4}a_3 = \frac{4^2}{5!}a_1, \\ a_6 &= \frac{4^3}{6!}a_0, & a_7 &= \frac{4^3}{7!}a_1, \end{aligned}$$

and so on. Hence

$$\begin{aligned} y(x) &= a_0\left(1 + \frac{4x^2}{2!} + \frac{4x^4}{4!} + \frac{4^3x^6}{6!} + \cdots\right) + a_1\left(x + \frac{4x^3}{3!} + \frac{4^2x^5}{5!} + \frac{4^3x^7}{7!} + \cdots\right) \\ &= a_0\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + \cdots\right) + \frac{a_1}{2}\left(2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \frac{(2x)^7}{7!} + \cdots\right) \\ &= a_0\cosh 2x + \frac{a_1}{2}\sinh 2x = A\cosh 2x + B\sinh 2x. \end{aligned}$$

Then the initial conditions yield $A = y(0) = 2$ and $2B = y'(0) = 0$. Therefore the particular solution of the given initial value problem is

$$y(x) = 2\cosh 2x = e^{2x} + e^{-2x}.$$

C11S10.021: Given the initial value problem

$$y'' - 2y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1,$$

we assume the existence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Substitution in the given differential equation then yields

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0 \qquad \text{for} \quad n \geqq 0,$$

so that

$$a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+2)(n+1)}, \qquad n \geqq 0.$$

At this point it would be easier to use the information that $a_0 = 0$ and $a_1 = 1$ to help find the general form of the coefficient a_n, but we choose to demonstrate that it is not necessary and, instead, find the general solution of the differential equation in terms of a_0 and a_1 as yet unspecified. Using the recursion formula just derived, we find that

$$\begin{aligned}
a_2 &= \frac{2a_1 - a_0}{2 \cdot 1} = \frac{2a_1 - a_0}{2!}, \\
a_3 &= \frac{4a_2 - a_1}{3 \cdot 2} = \frac{4a_1 - 2a_0 - a_1}{3 \cdot 2} = \frac{3a_1 - 2a_0}{3!}, \\
a_4 &= \frac{6a_3 - a_2}{4 \cdot 3} = \frac{3a_1 - 2a_0 - a_1 + \frac{1}{2}a_0}{4 \cdot 3} = \frac{4a_1 - 3a_0}{4!}, \qquad \text{and} \\
a_5 &= \frac{8a_4 - a_3}{5 \cdot 4} = \frac{\frac{4}{3}a_1 - a_0 - \frac{1}{2}a_1 + \frac{1}{3}a_0}{5 \cdot 4} = \frac{8a_1 - 6a_0 - 3a_1 + 2a_0}{5!} = \frac{5a_1 - 4a_0}{5!}.
\end{aligned}$$

At this point one might conjecture that

$$a_n = \frac{na_1 - (n-1)a_0}{n!} \qquad \text{if} \quad n \geqq 2,$$

and this can be established using a proof by induction on n. That granted, it follows that

$$\begin{aligned}
y(x) &= a_0 + a_1 x + \frac{2a_1 - a_0}{2!}x^2 + \frac{3a_1 - 2a_0}{3!}x^3 + \frac{4a_1 - 3a_0}{4!}x^4 + \frac{5a_1 - 4a_0}{5!}x^5 + \cdots \\
&= a_0\left(1 - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{3x^4}{4!} - \frac{4x^5}{5!} - \cdots\right) + a_1\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right) \\
&= a_0\left(1 - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{3x^4}{4!} - \frac{4x^5}{5!} - \cdots\right) + a_1 x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right).
\end{aligned}$$

Let

$$F(x) = 1 - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{3x^4}{4!} - \frac{4x^5}{5!} - \cdots.$$

Then

$$\begin{aligned}
F'(x) &= -x - x^2 - \frac{x^3}{2!} - \frac{x^4}{3!} - \frac{x^5}{4!} - \cdots \\
&= -x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = -xe^x.
\end{aligned}$$

Therefore $F(x) = (1-x)e^x + C$. Moreover, $F(0) = 1$, so that $C = 0$. Consequently,

$$y(x) = a_0(1-x)e^x + a_1 xe^x = a_0 e^x + (a_1 - a_0)xe^x = Ae^x + Bxe^x.$$

Finally, the given initial conditions imply that $A = y(0) = 0$ and that $B = y'(0) = 1$. Therefore the particular solution of the original initial value problem is $y(x) = xe^x$.

C11S10.022: Given the initial value problem

$$y'' + y' - 2y = 0; \qquad y(0) = 1, \quad y'(0) = -2,$$

we assume the existence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

Substitution in the given differential equation then yields

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - 2a_n = 0 \qquad \text{for} \quad n \geqq 0,$$

so that

$$a_{n+2} = \frac{2a_n - (n+1)a_{n+1}}{(n+2)(n+1)}, \qquad n \geqq 0.$$

At this point it would be easier to use the information that $a_0 = 1$ and $a_1 = -2$ to help discover the general form of the coefficient a_n, but we choose to demonstrate that it is not necessary and, instead, find the general solution of the differential equation in terms of the unspecified constants a_0 and a_1. Rewriting the expressions for a_0 and a_1 with the aid of hindsight and using the recursion formula for a_n, we find that

$$a_0 = \frac{1 \cdot a_0 - 0 \cdot a_1}{0!}, \qquad a_1 = \frac{1 \cdot a_1 - 0 \cdot a_0}{1!},$$

$$a_2 = \frac{2 \cdot a_0 - 1 \cdot a_1}{2!}, \qquad a_3 = \frac{3 \cdot a_1 - 2 \cdot a_0}{3!},$$

$$a_4 = \frac{6 \cdot a_0 - 5 \cdot a_1}{4!}, \qquad a_5 = \frac{11 \cdot a_1 - 10 \cdot a_0}{5!},$$

$$a_6 = \frac{22 \cdot a_0 - 21 \cdot a_1}{6!}, \qquad a_7 = \frac{43 \cdot a_1 - 42 \cdot a_0}{7!},$$

$$a_8 = \frac{86 \cdot a_0 - 85 \cdot a_1}{8!}, \qquad a_9 = \frac{171 \cdot a_1 - 170 \cdot a_0}{9!}.$$

The problem now is to discover the pattern in the coefficients. Let c_n denote the coefficient of a_0 in the numerator in a_n and let d_n denote the coefficient of a_1 in the same numerator. The data we have accumulated may be summarized in the following table.

n	c_n	d_n
0	1	0
1	0	1
2	2	−1
3	−2	3
4	6	−5
5	−10	11
6	22	−21
7	−42	43
8	86	−85
9	−170	171

It appears that $c_{n+1} = 2 - 2c_n$ for $n \geqq 0$; clearly $c_0 = 1$, and $c_1 = 0$. The first of these equations yields

$$c_{n+1} + 2c_n = 2 = c_{n+2} + 2c_{n+1},$$

and thus we obtain the *linear second-order homogeneous difference equation*

$$c_{n+2} + c_{n+1} - 2c_n = 0$$

with *initial conditions* $c_0 = 1$, $c_1 = 0$. Note the similarity to the characteristic equation of the original differential equation; note also the familiarity of the following method of solution of this difference equation. We assume a solution of the form $c_n = r^n$ where r is a nonzero constant. Substitution in the difference equation yields

$$r^{n+2} + r^{n+1} - 2r^n = 0, \qquad \text{so that} \qquad r^2 + r - 2 = 0.$$

This quadratic equation has the two solutions $r_1 = 1$ and $r_2 = -2$. Linearity of the difference equation implies that a linear combination of solutions is a solution, and thus the difference equation has *general solution*

$$c_n = A \cdot 1^n + B \cdot (-2)^n$$

for $n \geqq 0$. Then the initial conditions $c_0 = 1$ and $c_1 = 0$ yield $A = \frac{2}{3}$ and $B = \frac{1}{3}$. Hence

$$c_n = \frac{2 + (-2)^n}{3} \qquad \text{for} \quad n \geqq 0.$$

Moreover, $c_n + d_n = 1$, and it follows that

$$d_n = 1 - c_n = \frac{1 - (-2)^n}{3}.$$

Thus we find that

$$\begin{aligned} a_n &= \frac{c_n a_0 + d_n a_1}{n!} \\ &= \frac{[2+(-2)^n]\,a_0 + [1-(-2)^n]\,a_1}{n!\cdot 3} = \frac{(2a_0+a_1)+(a_0-a_1)(-2)^n}{n!\cdot 3}, \end{aligned}$$

a result that should be routine, though perhaps not simple, to establish with a proof by induction on n. Finally, this yields the general solution

$$\begin{aligned} y(x) &= \frac{2a_0+a_1}{3}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right) + \frac{a_0-a_1}{3}\left(1-2x+\frac{(-2x)^2}{2!}+\frac{(-2x)^3}{3!}+\frac{(-2x)^4}{4!}+\cdots\right) \\ &= \frac{1}{3}(2a_0+a_1)e^x + \frac{1}{3}(a_0-a_1)e^{-2x} = Ae^x + Be^{-2x}. \end{aligned}$$

The initial conditions in the original initial value problem now imply that $a_0 = 1$ and $a_1 = -2$, and hence its particular solution is $y(x) = e^{-2x}$.

Note: We obtained the coefficients c_n and d_n by using *Mathematica* 3.0 as follows: We executed the commands

```
a[0] = a0;   a[1] = a1;   a[n_] := a[n] = (2*a[n−2] ∗ (n − 1)∗a[n−1])/(n∗(n − 1))
```

and

```
ColumnForm[ Table[ { n, Simplify[ a[n] ] }, {n, 0, 9 } ] ]
```

(If you experiment, you will find that the alternative command

```
a[n_] := (2*a[n−2] ∗ (n − 1)∗a[n−1])/(n∗(n − 1))
```

will require substantically greater execution time.)

C11S10.023: Suppose that the differential equation

$$x^2y'' + x^2y' + y = 0$$

has a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$\begin{aligned} y'(x) &= \sum_{n=1}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n \qquad \text{and} \\ y''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n. \end{aligned}$$

Substitution in the given differential equation then yields

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=1}^{\infty} nc_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^n = 0;$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} (n-1)c_{n-1}x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

It then follows that $c_0 = c_1 = 0$ and that, if $n \geqq 2$,

$$n(n-1)c_n + (n-1)c_{n-1} + c_n = 0; \quad \text{that is,} \quad c_n = -\frac{n-1}{n^2-n+1}c_{n-1}.$$

Thus

$$c_2 = -\frac{1}{3}c_1 = 0, \quad c_3 = -\frac{2}{7}c_2 = 0, \quad c_4 = -\frac{3}{13}c_3 = 0,$$

and so on: $c_n = 0$ for all $n \geqq 0$. Therefore the only solution discovered by the series method used here is the trivial solution $y(x) \equiv 0$. Not only do we not find two linearly independent solutions, there is not even one because the trivial solution is neither independent of any solution nor has it the form of a general solution.

C11S10.024: Given the differential equation (the Bessel equation of order zero)

$$xy'' + y' + xy = 0,$$

we assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^{n-1}, \qquad \text{for which}$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^{n-1}.$$

Substitution in Bessel's equation then yields

$$\sum_{n=1}^{\infty} (n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

If $n = 0$, this equation yields $a_1 = 0$. If $n \geqq 1$, we find that

$$(n+1)na_{n+1} + (n+1)a_{n+1} + a_{n-1} = 0,$$

so that

$$a_{n+2} = -\frac{a_n}{(n+2)^2} \qquad \text{if} \quad n \geqq 0.$$

Thus $a_1 = a_3 = a_5 = \cdots = 0$, and

$$a_2 = -\frac{a_0}{2^2}, \qquad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{4^2 \cdot 2^2},$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{6^2 \cdot 4^2 \cdot 2^2}, \qquad a_8 = \frac{a_0}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2};$$

in general, if $n \geqq 1$ (and even if $n = 0$), then

$$\begin{aligned}
a_{2n} &= \frac{(-1)^n a_0}{(2n)^2 \cdot (2n-2)^2 \cdots 6^2 \cdot 4^2 \cdot 2^2} \\
&= \frac{(-1)^n a_0}{2^{2n} \cdot n^2 \cdot (n-1)^2 \cdots 3^2 \cdot 2^2 \cdot 1^2} = \frac{(-1)^n a_0}{2^{2n} \cdot (n!)^2}.
\end{aligned}$$

Therefore

$$J_0(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Because the coefficients with odd subscripts are all zero, there is no second linearly independent solution produced by this variation of the infinite series method. There does exist a second linearly independent solution, but finding it requires advanced techniques; see, for example, Section 8.4 of Edwards and Penney: *Differential Equations and Boundary Value Problems: Computing and Modeling*, 2nd ed. (Upper Saddle River, N.J.: Prentice Hall, 2000).

C11S10.025: Part (a): The method of separation of variables yields

$$\begin{aligned}
\frac{1}{1+y^2}\,dy &= 1\,dx; & \arctan y &= x + C; \\
y(x) &= \tan(x+C). & 0 &= y(0) = \tan C: \\
C &= n\pi \quad (n \text{ is an integer}); & y(x) &= \tan(x+n\pi) = \tan x.
\end{aligned}$$

Part (b): If

$$\begin{aligned}
y(x) &= x + c_3x^3 + c_5x^5 + c_7x^7 + c_9x^9 + \cdots, \qquad \text{then} \\
y'(x) &= 1 + 3c_3x^2 + 5c_5x^4 + 7c_7x^6 + 9c_9x^8 + \cdots. \qquad \text{Hence} \\
1 + [y(x)]^2 &= 1 + x^2 + 2c_3x^4 + (c_3^2 + 2c_5)x^6 + (2c_3c_5 + 2c_7)x^8 \\
&\quad + (2c_3c_7 + c_5^2 + 2c_9)x^{10} + (2c_3c_9 + 2c_5c_7 + 2c_{11})x^{12} + (2c_3c_{11} + 2c_5c_9 + c_7^2 + 2c_{13})x^{14} + \cdots \\
&= y'(x) = 1 + 3c_3x^2 + 5c_5x^4 + 7c_7x^6 + 9c_9x^8 + 11c_{11}x^{10} + \cdots.
\end{aligned}$$

It follows that

$$\begin{aligned}
3c_3 &= 1: & c_3 &= \frac{1}{3}. \\
5c_5 &= 2c_3 = \frac{2}{3}: & c_5 &= \frac{2}{15}. \\
7c_7 &= c_3^2 + 2c_5 = \frac{1}{9} + \frac{4}{15} = \frac{17}{45}: & c_7 &= \frac{17}{315}. \\
9c_9 &= 2c_3c_5 + 2c_7 = \frac{62}{315}: & c_9 &= \frac{62}{2835}. \\
11c_{11} &= 2c_3c_7 + c_5^2 + 2c_9 = \frac{1382}{14175}: & c_{11} &= \frac{1382}{155925}.
\end{aligned}$$

Part (c): Continuing in this manner, we find that

$$\begin{aligned}\tan x \;=\; & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13} + \frac{929569}{638512875}x^{15} \\ & + \frac{6404582}{10854518875}x^{17} + \frac{443861162}{1856156927825}x^{19} + \frac{18888466084}{194896477400625}x^{21} \\ & + \frac{113927491862}{2900518163668125}x^{23} + \frac{58870668456604}{3698160658676859375}x^{25} + \cdots .\end{aligned}$$

Chapter 11 Miscellaneous Problems

C11S0M.001: Divide each term in numerator and denominator by n^2:

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+4} = \lim_{n\to\infty} \frac{1+\dfrac{1}{n^2}}{1+\dfrac{4}{n^2}} = \frac{1+0}{1+0} = 1.$$

C11S0M.002: Divide each term in numerator and denominator by n:

$$\lim_{n\to\infty} \frac{8n-7}{7n-8} = \lim_{n\to\infty} \frac{8-\dfrac{7}{n}}{7-\dfrac{8}{n}} = \frac{8-0}{7-0} = \frac{8}{7}.$$

C11S0M.003: In Example 9 of Section 11.2, it is shown that if $|r| < 1$, then $r^n \to 0$ as $n \to +\infty$. Therefore

$$\lim_{n\to\infty} \left[10 - (0.99)^n\right] = 10 - 0 = 10.$$

C11S0M.004: Because $\sin \pi n = 0$ for every integer n, $\displaystyle\lim_{n\to\infty} n \sin \pi n = 0$.

C11S0M.005: Because

$$0 \leqq |a_n| = \frac{|1+(-1)^n\sqrt{n}\,|}{n+1} \leqq \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} \to 0$$

as $n \to +\infty$, the sequence with the given general term converges to 0 by the squeeze law for limits. This problem is the result of a typograhical error; it was originally intended to be the somewhat more challenging problem in which

$$a_n = \frac{1+(-1)^n n^{1/n}}{n+1}$$

for $n \geqq 1$.

C11S0M.006: In Example 9 of Section 11.2 it is shown that if $|r| < 1$, then $r^n \to 0$ as $n \to +\infty$. Hence there exists a positive integer K such that $0.5 < 1 + (-0.5)^n < 1.5$ if $n \geqq K$. Thus

$$\frac{0.5}{n+1} < \frac{1+(-0.5)^n}{n+1} < \frac{1.5}{n+1}$$

if $n \geqq K$. Therefore, by the squeeze law for limits (Theorem 3 of Section 10.2),

$$\lim_{n\to\infty} \frac{1+(-0.5)^n}{n+1} = 0.$$

Therefore $\displaystyle\lim_{n\to\infty} a_n = 0$ by Theorem 2 of Section 11.2.

C11S0M.007: Because $-1 \leqq \sin 2n \leqq 1$ for every positive integer n,

$$-\frac{1}{n} \leqq \frac{\sin 2n}{n} \leqq \frac{1}{n}$$

for every positive integer n. Therefore, by the squeeze law for limits (Theorem 3 of Section 11.2),

$$\lim_{n\to\infty} \frac{\sin 2n}{n} = 0.$$

C11S0M.008: Use l'Hôpital's rule to show that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

By Theorem 4 of Section 11.2,

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

Because $f(x) = 2^{-x}$ is continuous at $x = 0$, it now follows from Theorem 2 of Section 11.2 that

$$\lim_{n\to\infty} 2^{-(\ln n)/n} = 2^0 = 1.$$

C11S0M.009: If n is an even positive integer, then $\sin(n\pi/2) = 0$. Therefore $a_n = (-1)^0 = 1$ for arbitrarily large values of n. Hence if the sequence $\{a_n\}$ has a limit, it must be 1. But if n is an odd positive integer, then $\sin(n\pi/2) = \pm 1$. Therefore $a_n = -1$ for arbitrarily large values of n. So if the sequence $\{a_n\}$ has a limit, it must be -1. Because $1 \neq -1$, the sequence $\{a_n\}$ has no limit as $n \to +\infty$.

C11S0M.010: By l'Hôpital's rule—applied three times—

$$\lim_{x\to\infty} \frac{(\ln x)^3}{x^2} = \lim_{x\to\infty} \frac{3(\ln x)^2}{2x^2} = \lim_{x\to\infty} \frac{6\ln x}{4x^2} = \lim_{x\to\infty} \frac{6}{8x^2} = 0.$$

Therefore by Theorem 4 of Section 11.2, $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^3}{n^2} = 0.$

C11S0M.011: Because $-1 \leqq \sin x \leqq 1$ for all x,

$$-\frac{1}{n} \leqq \frac{1}{n}\sin\frac{1}{n} \leqq \frac{1}{n}$$

for every positive integer n. Therefore by the squeeze law for sequences, $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sin\frac{1}{n} = 0.$

C11S0M.012: Use l'Hôpital's rule and Theorem 4 of Section 11.2, or—more simply—

$$\lim_{n\to\infty} \frac{n - e^n}{n + e^n} = \lim_{n\to\infty} \frac{\dfrac{n}{e^n} - 1}{\dfrac{n}{e^n} + 1} = \frac{0-1}{0+1} = -1.$$

(Equation (8) of Section 7.2, with $k = 1$ and x replaced with n, tells us that $n/e^n \to 0$ as $n \to +\infty$.)

C11S0M.013: Here we have

$$\lim_{n\to\infty} \frac{\sinh n}{n} = \lim_{n\to\infty} \frac{e^n - e^{-n}}{2n} = \lim_{n\to\infty} \frac{1 - e^{-2n}}{2ne^{-n}}.$$

The numerator in the last fraction is approaching 1 as $n \to +\infty$, but the denominator is approaching zero through positive values (by Eq. (8) of Section 7.2). Therefore $a_n \to +\infty$ as $n \to +\infty$. Alternatively, you may say that the limit in question does not exist.

C11S0M.014: Let $m = 2n$. Then

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{2n} = \lim_{m\to\infty}\left(1+\frac{4}{m}\right)^{m} = e^4$$

by Eq. (4) in Section 7.2: $e^x = \lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$.

C11S0M.015: Example 7 in Section 11.2 implies that $3^{1/n} \to 1$ as $n \to +\infty$. Example 11 in Section 11.2 shows that $n^{1/n} \to 1$ as $n \to +\infty$. Moreover, if n is a positive integer, then

$$n^{1/n} \leqq (2n^2+1)^{1/n} \leqq (3n^2)^{1/n} = 3^{1/n}\cdot\left(n^{1/n}\right)^2,$$

and

$$\lim_{n\to\infty} 3^{1/n}\cdot\left(n^{1/n}\right)^2 = \left(\lim_{n\to\infty} 3^{1/n}\right)\cdot\left(\lim_{n\to\infty} n^{1/n}\right)^2 = 1\cdot 1 = 1.$$

Therefore, by the squeeze law for limits,

$$\lim_{n\to\infty}(2n^2+1)^{1/n} = 1.$$

C11S0M.016: Given the infinite series $\displaystyle\sum_{n=1}^{\infty}\frac{(n^2)!}{n^n}$ with nth term a_n, the ratio test yields

$$\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n^2+2n+1)!n^n}{(n^2)!(n+1)^{n+1}} \geqq \lim_{n\to\infty}\frac{n^2+n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{n}{e} = +\infty.$$

Therefore this series diverges.

The accuracy checkers were of the opinion that this problem was the result of a typographical error, because a similar but more interesting problem would be to determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{n^n}.$$

In this case the ratio test again yields divergence:

$$\rho = \lim_{n\to\infty}\frac{[(n+1)!]^2\cdot n^n}{(n+1)^{n+1}\cdot(n!)^2} = \lim_{n\to\infty}\frac{(n+1)^2\cdot n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}(n+1)\cdot\left(\frac{n}{n+1}\right)^n = +\infty$$

because $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$ by Eq. (3) in Section 7.2 and the quotient law for limits.

C11S0M.017: By l'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1}{x} = 0,$$

so $(\ln n)/n \to 0$ as $n \to +\infty$ by Theorem 4 of Section 11.2. Also, if

$$f(x) = \frac{\ln x}{x}, \qquad \text{then} \qquad f'(x) = \frac{1-\ln x}{x^2},$$

which is negative for $x > e$, and so the sequence $\{(\ln n)/n\}$ is monotonically decreasing if $n \geqq 3$. Therefore the given series converges by the alternating series test. Its sum is approximately 0.080357603217. The *Mathematica* 3.0 command

```
Sum[ ((−1)∧(n+1))∗(Log[n])/n, {n, 2, Infinity} ]
```

almost immediately produces the exact value of the sum of the series; it is

$$(-\gamma + \ln 2) \ln 2 \approx 0.0803576032166697405766033928384159153690544520408140507626\,08$$

(Euler's constant γ is first discussed in Problem 50 of Section 10.5 of the text).

C11S0M.018: The positive-term series

$$\sum_{n=0}^{\infty} \frac{3^n}{2^n + 4^n} \quad \text{is dominated by} \quad \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n,$$

which converges because it is geometric with ratio $\frac{3}{4}$. Therefore the dominated series converges as well. *Mathematica* 3.0 reports that its sum is approximately 3.0604250945355420520954 6181 (as usual, all digits shown are correct).

C11S0M.019: This series converges because the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)!\exp(n^2)}{n!\exp\left([n+1]^2\right)} = \lim_{n\to\infty} (n+1)\exp\left(n^2 - [n+1]^2\right) = \lim_{n\to\infty} \frac{n+1}{\exp(2n+1)} = 0$$

by a result from Chapter 7 and the squeeze law for limits:

$$0 \leqq \frac{n+1}{e^{2n+1}} \leqq \frac{2n+1}{e^{2n+1}}$$

for every positive integer n. The sum of the series is approximately 1.405253880284.

C11S0M.020: Because $-1 \leqq \sin x \leqq 1$ for all x,

$$\left| \frac{1}{n^{3/2}} \sin \frac{1}{n} \right| \leqq \frac{1}{n^{3/2}}$$

for every integer $n \geqq 1$. Therefore the given series converges absolutely because

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^{3/2}} \sin \frac{1}{n} \right| \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}};$$

the latter series converges because it is a p-series with $p = \frac{3}{2} > 1$. Because the original series converges absolutely, it converges by Theorem 3 of Section 11.7. The sum of the original series is approximately 1.1739398073796145.

C11S0M.021: For every positive integer n,

$$\left| \frac{(-2)^n}{3^n + 1} \right| \leqq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Therefore the series

$$\sum_{n=0}^{\infty} \left| \frac{(-2)^n}{3^n+1} \right| \quad \text{is dominated by} \quad \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n.$$

The latter series converges because it is geometric with ratio $\frac{2}{3}$, and therefore the dominated series converges. Hence the original series of Problem 21 converges absolutely, and therefore it converges by Theorem 3 of Section 11.7. Its sum is approximately 0.230836643803.

C11S0M.022: Because

$$\lim_{n\to\infty} \frac{2}{n^2} = 0, \quad \text{it follows that} \quad \lim_{n\to\infty} 2^{-(2/n^2)} = 2^0 = 1$$

by Theorem 4 of Section 11.2. Thus $\displaystyle\sum_{n=1}^{\infty} 2^{-(2/n^2)}$ diverges by the nth-term test (Theorem 3, Section 11.3).

C11S0M.023: Three applications of l'Hôpital's rule yield

$$\lim_{x\to\infty} \frac{x}{(\ln x)^3} = \lim_{x\to\infty} \frac{x}{3(\ln x)^2} = \lim_{x\to\infty} \frac{x}{6\ln x} = \lim_{x\to\infty} \frac{x}{6} = +\infty.$$

Therefore $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{(\ln n)^3}$ diverges by the nth-term test for divergence (Theorem 3 of Section 11.3).

C11S0M.024: By Example 7 of Section 11.2,

$$\lim_{n\to\infty} \frac{1}{10^{1/n}} = \frac{1}{1} = 1.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{10^{1/n}}$ diverges by the nth-term test for divergence.

C11S0M.025: For every positive integer n,

$$0 \leqq \frac{n^{1/2}+n^{1/3}}{n^2+n^3} \leqq \frac{2n^{1/2}}{n^3} = \frac{2}{n^{5/2}}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{n^{1/2}+n^{1/3}}{n^2+n^3} \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} \frac{2}{n^{5/2}}.$$

The latter series converges because it is a constant multiple of the p-series with $p = \frac{5}{2} > 1$. Therefore the dominated series converges by the comparison test (Theorem 1 of Section 11.6). The sum of the given series is approximately 1.459973884376.

C11S0M.026: We plan to show that the conditions in the alternating series test for convergence are met by the given series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{[1+(1/n)]}}. \tag{1}$$

Claim 1: If n is a positive integer, then

$$2 \leqq \left(\frac{n+1}{n}\right)^n.$$

To show this, let

$$f(x) = \left(\frac{x+1}{x}\right)^x, \quad x \geqq 1.$$

Then

$$\begin{aligned}
&\ln f(x) = x\ln(x+1) - x\ln x;\\
&\frac{f'(x)}{f(x)} = \frac{x}{x+1} + \ln(x+1) - 1 - \ln x;\\
&f'(x) = \left(\frac{x+1}{x}\right)^x \left[\ln(x+1) - \ln x - \frac{1}{x+1}\right].
\end{aligned}$$

Because $f(1) = 2$, we can establish our claim by showing that

$$\ln(x+1) - \ln x - \frac{1}{x+1} > 0 \tag{2}$$

if $x > 1$. If $a \geqq 1$, then the line tangent to the graph of $g(x) = \ln x$ at the point $(a+1,\, g(a+1))$ has slope $1/(a+1)$, and this line is otherwise completely above the graph of g because the graph of g is concave downward everywhere. Hence the line through the point $(a,\, g(a))$ with slope $1/(a+1)$ passes below the point $(a+1,\, g(a+1))$. That is,

$$\frac{1}{a+1} + \ln a < \ln(a+1).$$

Because this inequality holds for all $a \geqq 1$, we have established the inequality in (2) and this proves our first claim.

Claim 2: If n is a positive integer, then $n < 2^{n+2}$. This certainly holds if $n = 1$ because $1 < 8 = 2^3$. Suppose that $k < 2^{k+2}$ for some integer $k \geqq 1$. Then

$$2^{k+3} = 2 \cdot 2^{k+2} > 2k \geqq k+1;$$

that is, $k+1 < 2^{(k+1)+2}$. Therefore, by induction,

$$n < 2^{n+2}$$

for every positive integer n. This establishes our second claim.

Then, for every such integer n,

$$\begin{aligned}
1 &< \frac{1}{n} \cdot 2 \cdot 2^{n+1}; &\qquad n+1 &< \frac{n+1}{n} \cdot \left(\frac{n+1}{n}\right)^n \left[\left(\frac{n+1}{n}\right)^n\right]^{n+1};\\
n+1 &< \left(\frac{n+1}{n}\right)^{n^2+2n+1}; &\qquad (n+1) \cdot n^{(n+1)^2} &< (n+1)^{n^2+2n+1};\\
n^{(n+1)^2} &< (n+1)^{n^2+2n}; &\qquad n^{n+1} &< (n+1)^{n(n+2)/(n+1)};
\end{aligned}$$

$$n^{(n+1)/n} < (n+1)^{(n+2)/(n+1)}; \qquad n^{1+(1/n)} < (n+1)^{1+1/(n+1)};$$

$$\frac{1}{n^{1+(1/n)}} > \frac{1}{(n+1)^{1+1/(n+1)}}.$$

Therefore the sequence of terms of the series in (1) is monotonically decreasing in magnitude. Moreover, for each positive integer n,

$$0 \leqq \frac{1}{n^{1+(1/n)}} \leqq \frac{1}{n},$$

so by the squeeze law for limits, the terms of the series in (1) have limit zero. Finally, because they alternate in sign, the alternating series test—Theorem 1 in Section 11.7—guarantees that the series in (1) converges. Its sum (according to *Mathematica* 3.0) is approximately 0.779511537393.

C11S0M.027: Given: The alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \arctan n}{\sqrt{n}}. \tag{1}$$

We plan to show that this series meets the criterion for convergence stated in the alternating series test (Theorem 1 of of Section 11.7). First,

$$0 \leqq \arctan n \leqq \frac{\pi}{2}$$

for every positive integer n. Thus for such n,

$$0 \leqq \frac{\arctan n}{\sqrt{n}} \leqq \frac{\pi}{2\sqrt{n}}.$$

Therefore, by the squeeze law for limits,

$$\lim_{n\to\infty} \frac{\arctan n}{\sqrt{n}} = 0.$$

Now let

$$f(x) = \frac{\arctan x}{\sqrt{x}} \quad \text{for} \quad x \geqq 1. \quad \text{Then:}$$

$$\begin{aligned} f'(x) &= \frac{1}{x} \cdot \left(\frac{x^{1/2}}{1+x^2} - \frac{\arctan x}{2x^{1/2}} \right) = \frac{1}{x^{3/2}} \left(\frac{x}{1+x^2} - \frac{\arctan x}{2} \right) \\ &= \frac{1}{2x^{3/2}} \left(\frac{2x}{1+x^2} - \arctan x \right) = \frac{2x - (1+x^2)\arctan x}{2x^{3/2}(1+x^2)}. \end{aligned}$$

Now if $x \geqq 2$, then $1 \leqq \arctan x$. Therefore

$$\begin{aligned} 1 + x^2 &\leqq (1+x^2)\arctan x; & -(1+x^2)\arctan x &\leqq -(1+x^2); \\ 2x - (1+x^2)\arctan x &\leqq 2x - (1+x^2); & 2x - (1+x^2)\arctan x &\leqq -(x-1)^2; \\ 2x - (1+x^2)\arctan x &< 0 \quad \text{if} \quad x > 1; & f'(x) &< 0 \quad \text{if} \quad x > 1. \end{aligned}$$

Consequently the sequence of terms of the series in (1) is (after the first term) monotonically decreasing in magnitude. Because they alternate in sign, the series in (1) converges by the alternating series test. Its sum is approximately 0.378868816198.

C11S0M.028: Let $x = 1/n$. Then

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{x\to 0^+} \frac{1}{x} \sin x = 1.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ diverges by the nth-term test for divergence.

C11S0M.029: We use the integral test (Theorem 1 of Section 11.5):

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)}\,dx = \Big[\ln(\ln \ln x)\Big]_3^{\infty} = +\infty,$$

and therefore $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$ diverges.

C11S0M.030: We use the integral test:

$$\int_3^{\infty} \frac{1}{x(\ln x)(\ln \ln x)^2}\,dx = \left[-\frac{1}{\ln(\ln x)}\right]_3^{\infty} = \frac{1}{\ln(\ln 3)} < +\infty.$$

Therefore the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)^2}$ converges. The *Mathematica* 3.0 command

```
NSum[ 1/(n*(Log[n])*(Log[Log[n]])∧2),
    {n, 3, Infinity}, WorkingPrecision → 29 ] // Timing
```

yielded the approximation 38.4067680928211786 to its sum in about 8 seconds on a fairly slow computer.

C11S0M.031: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}\cdot n!\cdot |x|^{n+1}}{2^n\cdot (n+1)!\cdot |x|^n} = \lim_{n\to\infty} \frac{2|x|}{n+1} = 0$$

for every real number x. Therefore the given series converges for all x; its interval of convergence is $(-\infty, +\infty)$. Its sum is e^{2x}.

C11S0M.032: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}\cdot |3x|^{n+1}}{2^{n+2}\cdot |3x|^n} = \frac{3|x|}{2},$$

so the given series converges if $|x| < \frac{2}{3}$. It diverges if $x = \pm\frac{2}{3}$ by the nth-term test for divergence. Hence its interval of convergence is $\left(-\frac{2}{3}, \frac{2}{3}\right)$. The given series is, of course, geometric, and on its interval of convergence we have

$$\sum_{n=0}^{\infty} \frac{(3x)^n}{2^{n+1}} = \frac{1}{2-3x}.$$

C11S0M.033: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{n \cdot 3^n \cdot |x-1|^{n+1}}{(n+1)\cdot 3^{n+1} \cdot |x-1|^n} = \lim_{n\to\infty} \frac{n\cdot|x-1|}{3(n+1)} = \frac{|x-1|}{3}.$$

So the given series converges if $-3 < x - 1 < 3$; that is, if $-2 < x < 4$. It diverges if $x = 4$ (because it becomes the harmonic series). It converges if $x = -2$ by the alternating series test. Thus its interval of convergence is $[-2, 4)$.

C11S0M.034: The given series is geometric with ratio $\dfrac{2x-3}{4}$. Hence it converges exactly when

$$-1 < \frac{2x-3}{4} < 1; \qquad -4 < 2x - 3 < 4;$$

$$-1 < 2x < 7; \qquad -\frac{1}{2} < x < \frac{7}{2}.$$

Thus its interval of convergence is $\left(-\frac{1}{2}, \frac{7}{2}\right)$, and on that interval its sum is $\dfrac{4}{7-2x}$.

C11S0M.035: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(4n^2-1)\cdot|x|^{n+1}}{[4(n+1)^2-1]\cdot|x|^n} = \lim_{n\to\infty} \frac{(4n^2-1)\cdot|x|}{4n^2+8n+3} = \lim_{n\to\infty} \frac{\left(4-\dfrac{1}{n^2}\right)\cdot|x|}{4+\dfrac{8}{4}+\dfrac{3}{n^2}} = |x|.$$

Thus the series converges if $-1 < x < 1$. If $x = \pm 1$ then the given series converges absolutely because it is dominated by the p-series with $p = 2 > 1$. Hence the interval of convergence of the given series is $[-1, 1]$. The *Mathematica* 3.0 command

```
Sum[ ((-1)∧n)*(x∧n)/(4*n*n - 1), {n, 1, Infinity} ]
```

quickly returns the value of the sum of this series on *part* of its interval of convergence; the response is

$$\frac{\sqrt{x} - \arctan(\sqrt{x}\,) - x\arctan(\sqrt{x}\,)}{2\sqrt{x}}.$$

This result raises some intriguing new questions concerning the behavior of the series, and particularly of its sum, for $-1 \leqq x < 0$.

C11S0M.036: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n^2+1)\cdot|2x-1|^{n+1}}{[(n+1)^2+1]\cdot|2x-1|^n} = \lim_{n\to\infty} \frac{(n^2+1)\cdot|2x-1|}{n^2+2n+2} = \lim_{n\to\infty} \frac{\left(1+\dfrac{1}{n^2}\right)\cdot|2x-1|}{1+\dfrac{2}{n}+\dfrac{2}{n^2}} = |2x-1|.$$

So the given series converges if $-1 < 2x - 1 < 1$; that is, if $0 < x < 1$. It converges absolutely at the endpoints of this interval because it is dominated by the p-series with $p = 2 > 1$. Hence its interval of convergence is $[0, 1]$. On that interval *Mathematica* 3.0 reports that the exact value of its sum is

$$\frac{1-i}{4}\,[1 + i + (1+i)\cdot {}_2\mathrm{F}_1(i,\, 1;\; 1+i;\; 2x-1)$$

$$- {}_2\mathrm{F}_1(1,\ 1-i;\ 2-i;\ 2x-1) + 2x \cdot {}_2\mathrm{F}_1(1,\ 1-i;\ 2-i;\ 2x-1)\,]$$

where ${}_2\mathrm{F}_1$ is, again, the generalized hypergeometric function.

C11S0M.037: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(n+1)! \cdot 10^n \cdot |x|^{2n+2}}{n! \cdot 10^{n+1} \cdot |x|^{2n}} = \lim_{n\to\infty} \frac{(n+1)x^2}{10} = +\infty$$

if $x \neq 0$. Therefore the given series converges only if $x = 0$.

C11S0M.038: Note first that, by l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} = \lim_{x\to\infty} \frac{x+1}{x} = 1.$$

Therefore the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(\ln n) \cdot |x|^{n+1}}{[\ln(n+1)] \cdot |x|^n} = \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\,|x| = |x|.$$

So the given series converges if $-1 < x < 1$. If $x = 1$ then it diverges because it dominates the harmonic series (we have seen many proofs that $\ln n < n$ if $n \geqq 1$). If $x = -1$ then it converges by the alternating series test. The series passes the criteria of that test because $f(x) = \ln x$ is monotonically increasing and approaches $+\infty$ as $x \to +\infty$. Therefore the given series has interval of convergence $[-1,\ 1)$.

C11S0M.039: Note that

$$\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n! \cdot 2}\, x^n = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{1}$$

So the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{(2n)! \cdot x^{2n+2}}{(2n+2)! \cdot x^{2n}} = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

for all real x. Hence the interval of convergence of the series in (1) is $(-\infty,\ +\infty)$. This series converges to $f(x) = \cosh x$ (see Example 6 in Section 11.8).

C11S0M.040: Recall that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

by Eq. (3) of Section 7.2. Thus the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{\left(1 + \dfrac{1}{n+1}\right)^{n+1} |x-1|^{n+1}}{\left(1 + \dfrac{1}{n}\right)^n |x-1|^n} = \frac{e \cdot |x-1|}{e} = |x-1|.$$

So the given series converges if $-1 < x - 1 < 1$; that is, if $0 < x < 2$. It diverges at the endpoints of this interval by the nth-term test for divergence. Thus its interval of convergence is $(0,\ 2)$.

C11S0M.041: The given series diverges for every real number x by the nth-term test for divergence.

C11S0M.042: The given series is geometric with ratio $\ln x$, so it converges exactly when $-1 < \ln x < 1$; that is, when $e^{-1} < x < e$.

C11S0M.043: The ratio test yields

$$\rho = \lim_{n\to\infty} \frac{n! \cdot e^{(n+1)x}}{(n+1)! \cdot e^{nx}} = \lim_{n\to\infty} \frac{e^x}{n+1} = 0$$

for every real number x, so the given series converges for all x. Its sum is $\exp(e^x)$.

C11S0M.044: The rational number with decimal expansion $2.71828182818 28\cdots$ is

$$2 + \frac{7}{10} + \frac{1828}{10^5} + \frac{1828}{10^9} + \frac{1828}{10^{13}} + \cdots = \frac{27}{10} + \frac{1828 \cdot 10^{-5}}{1 - 10^{-4}} = \frac{27}{10} + \frac{1828}{10^5 - 10} = \frac{27}{10} + \frac{1828}{99990} = \frac{271801}{99990}.$$

C11S0M.045: Let

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n}} \quad \text{for} \quad n \geqq 1.$$

Then $\sum a_n$ and $\sum b_n$ converge by the alternating series test, but $\sum a_n b_n$ diverges because it is the harmonic series.

C11S0M.046: This is a special case of Problem 49 in Section 11.6; the proof used there can be adapted to create a solution of this problem.

C11S0M.047: Assuming that A exists, we have

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{1+a_n}\right) = 1 + \frac{1}{1+A}$$

because $A \neq -1$. Therefore $A + A^2 = 2 + A$, and it follows that $A^2 = 2$. Because $A \geqq 0$ (the limit of a sequence of positive numbers cannot be negative), $A = \sqrt{2}$.

C11S0M.048: Part (a): First, $F_1 = 1 < 2^1$ and $F_2 = 1 < 2^2$. If $n \geqq 2$, then $F_{n+1} = F_n + F_{n-1}$, so $F_n > 0$ for all $n \geqq 1$. Moreover, if $F_{n-1} < 2^{n-1}$ and $F_n < 2^n$, then

$$F_{n+1} \leqq 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}.$$

Therefore, by induction, $F_n < 2^n$ for all $n \geqq 1$. Hence

$$\sum_{n=1}^{\infty} F_n x^n \quad \text{is dominated by} \quad \sum_{n=1}^{\infty} 2^n x^n.$$

The latter series converges absolutely for $|x| < \frac{1}{2}$ (it is geometric with ratio $2x$), so the dominated series also converges for such x.

Part (b): We use the formula $F_{n+1} - F_n - F_{n-1} = 0$ for $n \geqq 2$.

$$
\begin{aligned}
(1 - x - x^2)(F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \cdots)&\\
= F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + \cdots&\\
- F_1 x^2 - F_2 x^3 - F_3 x^4 - F_4 x^5 - \cdots&\\
- F_1 x^3 - F_2 x^4 - F_3 x^5 - \cdots&\\
\cdots&\\
= F_1 x + (F_2 - F_1)x^2 + (F_3 - F_2 - F_1)x^3&\\
+ (F_4 - F_3 - F_2)x^4&\\
+ (F_5 - F_4 - F_3)x^5 + \cdots&\\
= x + (1-1)x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots \;=\; x.&
\end{aligned}
$$

Therefore $F(x) = \dfrac{x}{1 - x - x^2}$.

Note: Application of the ratio test to the given power series yields the limit $\frac{1}{2}\left(1 + \sqrt{5}\,\right)|x|$, so the radius of convergence of the series for $F(x)$ is actually

$$
\frac{1}{\tau} = \frac{\sqrt{5}\,-1}{2} \approx 0.6180339887498948482045868.
$$

C11S0M.049: If $a_n = \dfrac{1}{n}$, then the series

$$
\sum_{n=1}^{\infty} \ln(1 + a_n) \;=\; \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)
$$

diverges because

$$
\begin{aligned}
S_k &= \sum_{n=1}^{k} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{k} [\ln(n+1) - \ln n]\\
&= \ln 2 - \ln 1 + \ln 3 - \ln 2 + \ln 4 - \ln 3 + \cdots + \ln(k+1) - \ln k = \ln(k+1),
\end{aligned}
$$

and therefore $S_k \to +\infty$ as $k \to +\infty$. Alternatively, using the integral test,

$$
J = \int_1^{\infty} \ln\left(1 + \frac{1}{x}\right) dx = \int_1^{\infty} [\ln(x+1) - \ln x]\; dx = \Big[(x+1)\ln(x+1) - x\ln x\Big]_1^{\infty} = +\infty
$$

because

$$
\lim_{x\to\infty} [(x+1)\ln(x+1) - x\ln x] \;\geqq\; \lim_{x\to\infty} [(x+1)\ln x - x\ln x] = \lim_{x\to\infty} \ln x = +\infty
$$

and, at the lower limit $x = 1$ of integration, we have $(x+1)\ln(x+1) - x\ln x = \ln 4$. Therefore, because

$$
J = \int_1^{\infty} \ln\left(1 + \frac{1}{x}\right) dx = +\infty,
$$

the infinite product $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ diverges.

C11S0M.050: We must test for convergence the infinite series

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right). \tag{1}$$

An easy integration by parts yields

$$\begin{aligned}I&=\int_1^{\infty}\ln\left(1+\frac{1}{x^2}\right)\,dx=\int_1^{\infty}\left[\ln(x^2+1)-2\ln x\right]\,dx\\&=\Big[2\arctan x-2x\ln x+x\ln(1+x^2)\Big]_1^{\infty}=\pi-\left(\frac{\pi}{2}+\ln 2\right)=\frac{\pi}{2}-\ln 2.\end{aligned}$$

To evaluate the limit of the antiderivative as $x\to+\infty$, use l'Hôpital's rule:

$$\lim_{x\to\infty}\left[x\ln(1+x^2)-2x\ln x\right]=\lim_{x\to\infty}x\ln\left(\frac{1+x^2}{x^2}\right)=\lim_{x\to\infty}\frac{\ln(1+x^2)-\ln(x^2)}{\dfrac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\dfrac{2x}{1+x^2}-\dfrac{2}{x}}{-\dfrac{1}{x^2}}=\lim_{x\to\infty}\left(2x-\frac{2x^3}{1+x^2}\right)=\lim_{x\to\infty}\frac{2x+2x^3-2x^3}{1+x^2}=\lim_{x\to\infty}\frac{2x}{1+x^2}=0.$$

Therefore

$$\lim_{x\to\infty}\left[2\arctan x-2x\ln x+x\ln(1+x^2)\right]=2\cdot\frac{\pi}{2}+0=\pi.$$

Thus by the integral test, the series in Eq. (1) converges; it now follows that the infinite product converges as well. The integral test remainder estimate for the series yields

$$\pi-(n+1)\ln\left(1+\frac{1}{(n+1)^2}\right)-2\arctan(n+1)\ \leqq\ R_n\ \leqq\ \pi-n\ln\left(1+\frac{1}{n^2}\right)-2\arctan n.$$

With $n=100$, the upper and lower estimates here differ by about 0.000099005, so that

$$0.00990082840327\ \leqq\ R_n\ \leqq\ 0.00999983334000$$

(we round *down* on the left and *up* on the right). Moreover,

$$S_{100}\approx 1.29189639611721,\qquad\text{so that}\qquad 1.301797226\leqq\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)\ \leqq\ 1.301896231.$$

We apply the natural exponential function to the last inequality and thereby conclude that

$$3.675897152\ \leqq\ \prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\ \leqq\ 3.676261103.$$

So to three places, the value of the infinite product is 3.676. The *Mathematica* 3.0 command

```
Product[ 1 + 1/(n*n), { n, 1, Infinity } ]
```

almost immediately produces the response

$$\frac{\sinh \pi}{\pi}, \qquad \text{about} \qquad 3.676077910374977720695697492028 26$$

(all the digits shown in the approximation are correct). This product also appears in a somewhat more general form in Eldon R. Hansen's *A Table of Series and Products* (Englewood Cliffs, N. J.): Prentice-Hall, Inc., 1975, Eq. (89.5.16). The related infinite product

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$$

is much easier to prove convergent and to evaluate.

C11S0M.051: The binomial series is

$$(1+x)^{1/5} = 1 + \frac{x}{5} - \frac{4}{5^2} \cdot \frac{x^2}{2!} + \frac{4 \cdot 9}{5^3} \cdot \frac{x^4}{3!} - \frac{4 \cdot 9 \cdot 14}{5^4} \cdot \frac{x^4}{4!} + \cdots .$$

Substitution of $x = \frac{1}{2}$ and summing the first five terms of this series yields 1.0839 (exactly); summing the first six terms yields 1.0843788 (exactly). So to three places, $\left(1 + \frac{1}{2}\right)^{1/5} \approx 1.084$. The true value of the expression is closer to 1.084471771198.

C11S0M.052: The Taylor series with center zero for $\ln(1+x)$ can be found in Eq. (19) of Section 11.8; it is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots .$$

Substitution of $x = \frac{1}{5}$ and summing the first three terms of this series yields 0.182666666667 (approximately); summing the first four terms yields 0.182266666667 (approximately); summing the first five terms yields 0.182330666667 (approximately). So to three places, $\ln(1.2) \approx 0.182$. The true value is closer to 0.182321556794.

C11S0M.053: We substitute $-x^2$ for x in the Maclaurin series for the natural exponential function. Thus we find that

$$\begin{aligned}
\int_0^{1/2} \exp(-x^2)\,dx &= \int_0^{1/2} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right) dx \\
&= \left[x - \frac{x^3}{3} + \frac{x^5}{2! \cdot 5} - \frac{x^7}{3! \cdot 7} + \frac{x^9}{4! \cdot 9} - \cdots\right]_0^{1/2} \\
&= \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{2! \cdot 5 \cdot 2^5} - \frac{1}{3! \cdot 7 \cdot 2^7} + \frac{1}{4! \cdot 9 \cdot 2^9} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot (2n+1) \cdot 2^{2n+1}}.
\end{aligned}$$

The sum of the first two terms and the sum of the first three terms of this series are

$$\frac{443}{960} \approx 0.461458333333 \qquad \text{and} \qquad \frac{4133}{8960} \approx 0.461272321429,$$

respectively. Thus the value of the integral to three places is approximately 0.461. A closer approximation is 0.4612810064127924 (to the number of digits shown).

C11S0M.054: The binomial series takes the form

$$(1+x^4)^{1/3} = 1 + \frac{x^4}{3} - \frac{2}{3^2}\cdot\frac{x^8}{2!} + \frac{2\cdot 5}{3^3}\cdot\frac{x^{12}}{3!} - \frac{2\cdot 5\cdot 8}{3^4}\cdot\frac{x^{16}}{4!} + \cdots.$$

Then termwise integration yields

$$\begin{aligned}\int_0^{1/2}(1+x^4)^{1/3}\,dx &= \left[x + \frac{x^5}{3\cdot 5} - \frac{2}{3^2}\cdot\frac{x^9}{2!\cdot 9} + \frac{2\cdot 5}{3^3}\cdot\frac{x^{13}}{3!\cdot 13} - \frac{2\cdot 5\cdot 8}{3^4}\cdot\frac{x^{17}}{4!\cdot 17} + \cdots\right]_0^{1/2}\\ &= \frac{1}{2} + \frac{1}{2^5\cdot 3\cdot 5} - \frac{2}{2!\cdot 2^9\cdot 3^2\cdot 9} + \frac{2\cdot 5}{3!\cdot 2^{13}\cdot 3^3\cdot 13} - \frac{2\cdot 5\cdot 8}{4!\cdot 2^{17}\cdot 3^4\cdot 17} + \cdots.\end{aligned}$$

The sum of the first two terms and the sum of the first three terms of the last series are

$$\frac{241}{480} \approx 0.502083333333 \quad\text{and}\quad \frac{104107}{207360} \approx 0.502059220679,$$

respectively. So to three places, the value of the integral is approximately 0.502. A closer approximation is 0.5020597824999187833. The *Mathematica* 3.0 command

```
Integrate[ ( 1 + x∧4)∧(1/3), {x, 0, 1/2} ]
```

elicits the warning "`Unable to check convergence`," but produces the response

$$\frac{3}{28}\cdot\left(\frac{17}{2}\right)^{1/3} + \frac{2}{7}\cdot {}_2\mathrm{F}_1\left(\frac{1}{4}, \frac{2}{3};\ \frac{5}{4};\ -\frac{1}{16}\right),$$

which is plausible because it has the same decimal expansion as the approximation given earlier. And the generalized hypergeometric function now makes its graceful but permanent exit from the pages of this manual.

C11S0M.055: The Maclaurin series for the natural exponential function yields

$$\frac{1}{x}(1-e^{-x}) = \frac{1}{x}\left(x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \frac{x^5}{6!} + \cdots.$$

Then termwise integration produces

$$\begin{aligned}\int_0^1 \frac{1-e^{-x}}{x}\,dx &= \left[x - \frac{x^2}{2!\cdot 2} + \frac{x^3}{3!\cdot 3} - \frac{x^4}{4!\cdot 4} + \frac{x^5}{5!\cdot 5} - \frac{x^6}{6!\cdot 6} + \cdots\right]_0^1\\ &= 1 - \frac{1}{2!\cdot 2} + \frac{1}{3!\cdot 3} - \frac{1}{4!\cdot 4} + \frac{1}{5!\cdot 5} - \frac{1}{6!\cdot 6} + \cdots.\end{aligned}$$

The sum of the first five terms of the last series and the sum of its first six terms are

$$\frac{5737}{7200} \approx 0.796805555556 \quad\text{and}\quad \frac{8603}{10800} \approx 0.796574074074,$$

respectively. So to three places, the value of the integral is 0.797. A more accurate approximation is 0.7965995992970531.

C11S0M.056: The *Mathematica* 3.0 command

```
Series[ Exp[x], { x, 0, 13 } ] // Normal
```

produces the Taylor polynomial $P_{13}(x)$ of degree 13 with center zero for the natural exponential function. Then the command

```
Series[ Sin[x], { x, 0, 13 } ] // Normal
```

produces the Taylor polynomial $Q_{13}(x)$ of degree 13 with center zero for the sine function. The command

```
%% /.  x → %
```

then asks *Mathematica* to substitute $Q_{13}(x)$ for x in $P_{13}(x)$. In other words, compute $P_{13}(Q_{13}(x))$. Then the `Expand` command expands the resulting expression, thereby forming a polynomial of degree 169. Its first 14 terms will be the degree 13 Taylor polynomial with center zero of $\exp(\sin x)$, and it is

$$1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} - \frac{x^6}{240} + \frac{x^7}{90} + \frac{31x^8}{5760} + \frac{x^9}{5670} - \frac{2951x^{10}}{3628800} - \frac{x^{11}}{3150} + \frac{181x^{12}}{14515200} + \frac{2417x^{13}}{48648600}.$$

To check this result, simply enter the *Mathematica* 3.0 command

```
Series[ Exp[Sin[x]], {x, 0, 13} ] // Normal
```

—the response will be almost instantaneous.

C11S0M.057: We will need both the recursion formula

$$\int_0^\infty t^{2n} \exp(-t^2)\,dt = \frac{2n-1}{2} \int_0^\infty t^{2n-2} \exp(-t^2)\,dt \quad (n \geqq 1),$$

which follows from the formula in Problem 50 of Section 8.3, and the famous formula

$$\int_0^\infty \exp(-t^2)\,dt = \frac{\sqrt{\pi}}{2},$$

which is derived in Example 5 of Section 14.4 (it is Eq. (9) there). We begin with the Maclaurin series of the cosine function.

$$\begin{aligned}
\int_0^\infty \exp(-t^2)\cos 2xt\,dt &= \int_0^\infty \exp(-t^2)\left(1 - \frac{2^2x^2t^2}{2!} + \frac{2^4x^4t^4}{4!} - \frac{2^6x^6t^6}{6!} + \frac{2^8x^8t^8}{8!} - \cdots\right)dt \\
&= \int_0^\infty \left(\exp(-t^2) - \frac{2^2x^2}{2!}t^2\exp(-t^2) + \frac{2^4x^4}{4!}t^4\exp(-t^2) - \frac{2^6x^6}{6!}t^6\exp(-t^2) + \cdots\right)dt \\
&= \int_0^\infty e^{-t^2}\,dt - \frac{2^2x^2}{2!}\int_0^\infty t^2e^{-t^2}\,dt + \frac{2^4x^4}{4!}\int_0^\infty t^4e^{-t^2}\,dt - \frac{2^6x^6}{6!}\int_0^\infty t^6e^{-t^2}\,dt + \cdots \\
&= \int_0^\infty e^{-t^2}\,dt - \frac{2^2x^2}{2!}\cdot\frac{1}{2}\int_0^\infty e^{-t^2}\,dt + \frac{2^4x^4}{4!}\cdot\frac{3}{2}\cdot\frac{1}{2}\int_0^\infty e^{-t^2}\,dt - \frac{2^6x^6}{6!}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\int_0^\infty e^{-t^2}\,dt + \cdots \\
&= \left(\int_0^\infty e^{-t^2}\,dt\right)\left(1 - \frac{2^2x^2}{2!}\cdot\frac{1}{2} + \frac{2^4x^4}{4!}\cdot\frac{3\cdot 1}{2^2} - \frac{2^6x^6}{6!}\cdot\frac{5\cdot 3\cdot 1}{2^3} + \frac{2^8x^8}{8!}\cdot\frac{7\cdot 5\cdot 3\cdot 1}{2^4} - \cdots\right).
\end{aligned}$$

The typical term in the last infinite series is

$$\frac{2^{2n}x^{2n}}{(2n)!}\cdot\frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^n} = \frac{2^{2n}x^{2n}}{(2n)!}\cdot\frac{(2n)!}{2^n\cdot(2n)(2n-2)\cdots 6\cdot 4\cdot 2} = \frac{2^{2n}x^{2n}}{n!\cdot 2^n\cdot 2^n} = \frac{(x^2)^n}{n!}.$$

Consequently,

$$\int_0^\infty e^{-t^2}\cos 2xt\,dt = \left(\int_0^\infty e^{-t^2}\,dt\right)\left(1 - \frac{x^2}{1!} + \frac{(x^2)^2}{2!} - \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} - \cdots\right)$$
$$= \left(\int_0^\infty e^{-t^2}\,dt\right)e^{-x^2} = \frac{\sqrt{\pi}}{2}\,e^{-x^2}.$$

C11S0M.058: The first equality is derived in Example 5 of Section 7.6 and appears in Eq. (30) there. Thus we have

$$\tanh^{-1} x = \int_0^x \frac{1}{1-t^2}\,dt = \int_0^x \left(1 + t^2 + t^4 + t^6 + t^8 + \cdots\right)\,dt$$
$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

provided that $|x| < 1$.

C11S0M.059: The binomial series takes the form

$$(1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{1\cdot 3}{2^2}\cdot\frac{t^4}{2!} - \frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{t^6}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{t^8}{4!} - \cdots.$$

Thus

$$\sinh^{-1} x = \int_0^x (1+t^2)^{-1/2}\,dt = x - \frac{x^3}{3} + \frac{1\cdot 3}{2^2\cdot 5}\cdot\frac{x^5}{2!} - \frac{1\cdot 3\cdot 5}{2^3\cdot 7}\cdot\frac{x^7}{3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 9}\cdot\frac{x^9}{4!} - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}\cdot\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^n\cdot n!\cdot 2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\cdot(n!)^2}\cdot\frac{x^{2n+1}}{2n+1}$$

provided that $|x| < 1$.

C11S0M.060: We begin with the assumption that there exist coefficients $\{a_n\}$ such that $\tan y = \sum a_n y^n$. The Maclaurin series for the inverse tangent function is derived in Example 11 of Section 11.8 and appears in Eq. (20) there; it is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$$

Thus

$$x = \tan(\tan^{-1} x) = a_0 + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$$
$$+ a_2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)^2$$
$$+ a_3\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)^3 + \cdots$$

$$
\begin{aligned}
&= a_0 + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)\\
&\quad + a_2\left(x^2 - \frac{2x^4}{3} + \frac{23x^6}{45} - \frac{44x^8}{105} + \frac{563x^{10}}{1575} - \cdots\right)\\
&\quad + a_3\left(x^3 - x^6 + \frac{14x^7}{15} - \frac{818x^9}{945} + \frac{141x^{11}}{175} - \cdots\right)\\
&\quad + a_4\left(x^4 - \frac{4x^6}{3} + \frac{22x^8}{15} - \frac{1436x^{10}}{945} + \frac{21757x^{12}}{14175} - \cdots\right) + \cdots .
\end{aligned}
$$

Therefore

$$x = a_0 + a_1x + a_2x^2 + \left(a_3 - \frac{a_1}{3}\right)x^3 + \left(a_4 - \frac{2a_2}{3}\right)x^4 + \cdots .$$

It now follows that $a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = \frac{1}{3}$, and $a_4 = 0$. Thus the Maclaurin series for the tangent function—if it exists (it does)—begins

$$\tan x \;=\; x + \frac{x^3}{3} + \cdots .$$

For more about this series, see Eq. (7) of Section 11.9 and the discussion that precedes and follows it; see also Problem 25 of Section 11.10.

C11S0M.061: We let *Mathematica* 3.0 do this problem. First we defined

```
mu = 1/(12*n) - 1/(360*n∧3) + 1/(1260*n∧5)
```

Then the command

```
Series[ Exp[x], { x, 0, 10 } ] // Normal
```

produced the 10th-degree Taylor polynomial with center zero for e^x:

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800}.$$

Recall that % means “last output” to *Mathematica.* Thus the next command

```
% /.  x → mu
```

tells *Mathematica* to substitute `mu` for x in the Taylor polynomial, producing the response

$$
\begin{aligned}
&1 + \left(\frac{1}{1260n^5} - \frac{1}{360n^3} + \frac{1}{12n}\right) + \frac{1}{2}\left(\frac{1}{1260n^5} - \frac{1}{360n^3} + \frac{1}{12n}\right)^2 + \frac{1}{6}\left(\frac{1}{1260n^5} - \frac{1}{360n^3} + \frac{1}{12n}\right)^3\\
&\quad + \frac{1}{24}\left(\frac{1}{1260n^5} - \frac{1}{360n^3} + \frac{1}{12n}\right)^4 + \cdots + \frac{1}{3628800}\left(\frac{1}{1260n^5} - \frac{1}{360n^3} + \frac{1}{12n}\right)^{10}.
\end{aligned}
$$

Finally, the `Expand` command resulted in almost a full page of output, including the answer:

$$\exp(\mu(n)) = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + \cdots .$$

C11S0M.062: Given: $T(n) = \displaystyle\int_0^{\pi/4} (\tan x)^n\,dx.$

Part (a): Equation (12) in Section 8.4 gives the reduction formula

$$\int (\tan x)^n \, dx = \frac{1}{n-1}(\tan x)^{n-1} - \int (\tan x)^{n-2} \, dx$$

if n is an integer and $n \geqq 2$. It follows immediately that if n is a nonnegative integer, then

$$T(n+2) = \frac{1}{n+1} - T(n).$$

Part (b): We show that $T(n) \to 0$ as $n \to +\infty$ by an argument very similar to that used in the solution of Problem 61 of Section 8.3. Indeed, it may be helpful to examine the figure that accompanies that solution. Given $\epsilon > 0$ (but quite close to zero), choose the positive integer k so large that

$$\left[\tan\left(\frac{\pi}{4} - \frac{\epsilon}{2}\right)\right]^k < \frac{2\epsilon}{\pi}.$$

This is possible because

$$0 < a = \tan\left(\frac{\pi}{4} - \frac{\epsilon}{2}\right) < 1,$$

and thus $a^n \to 0$ as $n \to +\infty$. Then the region bounded above by the graph of $y = (\tan x)^k$ and below by the x-axis over the interval $0 \leqq x \leqq \pi/4$ is contained in the union of two rectangles: One with northwest vertex at $(0,\, 2\epsilon/\pi)$ and southeast vertex at $(\pi/4,\, 0)$, the other rectangle with southwest vertex at $\left(\frac{1}{4}\pi - \frac{1}{2}\epsilon,\, 0\right)$ and northeast vertex at $(\pi/4,\, 1)$. It now follows that

$$T(n) = \int_0^{\pi/4} (\tan x)^n \, dx < \frac{\pi}{4} \cdot \frac{2\epsilon}{\pi} + 1 \cdot \frac{\epsilon}{2} = \epsilon$$

if $n \geqq k$, and this shows that $T(n) \to 0$ as $n \to +\infty$.

Part (c): $T(0) = \displaystyle\int_0^{\pi/4} 1 \, dx = \frac{\pi}{4}$ and

$$T(1) = \int_0^{\pi/4} \tan x \, dx = \Big[\ln(\sec x)\Big]_0^{\pi/4} = \ln\sqrt{2} = \frac{1}{2}\ln 2.$$

Part (d): First,

$$T(2) = 1 - T(0) = 1 - \frac{\pi}{4} = (-1)^2\left(1 - \frac{\pi}{4}\right),$$

so the formula in Part (d) holds when $n = 2$. Assume that it holds for some integer $k \geqq 2$; that is, assume that

$$T(2k) = (-1)^{k+1}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{2k-1} - \frac{\pi}{4}\right).$$

Then

$$\begin{aligned} T(2k+2) = \frac{1}{2k+1} - T(2k) &= \frac{1}{2k+1} - (-1)^{k+1}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{2k-1} - \frac{\pi}{4}\right) \\ &= \frac{1}{2k+1} + (-1)^{k+2}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{2k-1} - \frac{\pi}{4}\right) \\ &= (-1)^{k+2}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{2k-1} \mp \frac{1}{2k+1} - \frac{\pi}{4}\right). \end{aligned}$$

Therefore, by induction, the formula of Part (d) holds for every positive integer n.

Part (e): We now know that if n is a positive integer, then

$$T(2n) \;=\; (-1)^{n+1}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{2n-1}-\frac{\pi}{4}\right).$$

Therefore

$$\begin{aligned}
(-1)^{n+1}T(2n) \;&=\; 1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{2n-1}-\frac{\pi}{4};\\
\frac{\pi}{4}+(-1)^{n+1}T(2n) \;&=\; 1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{2n-1}.
\end{aligned}$$

Now let $n \to +\infty$ to conclude that

$$\frac{\pi}{4}=\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\pm\frac{1}{2n-1}\right)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots.$$

Part (f): First,

$$T(3)=\frac{1}{2}-T(1)=\frac{1}{2}-\frac{1}{2}\ln 2=\frac{1}{2}(-1)^2(1-\ln 2)=\frac{1}{2}(-1)^{1+1}(1-\ln 2).$$

Therefore

$$T(2n+1)=\frac{1}{2}(-1)^{n+1}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{n}-\ln 2\right) \tag{1}$$

if $n=1$. Assume that Eq. (1) holds for some integer $k \geqq 1$. Then

$$\begin{aligned}
T(2k+3)&=\frac{1}{2k+2}-\frac{1}{2}(-1)^{k+1}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{k}-\ln 2\right)\\
&=\frac{1}{2k+2}+\frac{1}{2}(-1)^{k+2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{k}-\ln 2\right)\\
&=\pm\frac{1}{2}(-1)^{k+2}\cdot\frac{1}{k+1}+\frac{1}{2}(-1)^{k+2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{k}-\ln 2\right)\\
&=\frac{1}{2}(-1)^{k+2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{k}\mp\frac{1}{k+1}-\ln 2\right).
\end{aligned}$$

Therefore, by induction, Eq. (1) holds for every positive integer n.

Part (g): For each positive integer n, we have

$$\frac{1}{2}(-1)^{n+1}T(2n+1)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{n}-\ln 2.$$

Therefore

$$\frac{1}{2}(-1)^{n+1}T(2n+1)+\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\pm\frac{1}{n}.$$

Now let $n \to +\infty$ to conclude that

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots.$$

C11S0M.063: Proof: Assume that e is a rational number. Then $e = p/q$ where p and q are positive integers and $q > 1$ (because e is not an integer). Thus

$$\frac{p}{q} = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + R_q$$

where

$$\begin{aligned} R_q &= \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \frac{1}{(q+4)!} + \cdots \\ &= \frac{1}{q!} \cdot \left(\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \right) \\ &< \frac{1}{q!} \cdot \left(\frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots \right) = \frac{1}{q!} \cdot \frac{\dfrac{1}{q+1}}{1 - \dfrac{1}{q+1}} = \frac{1}{q! \cdot q}. \end{aligned}$$

Thus

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} \ < \ \frac{p}{q} \ < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!} + \frac{1}{q! \cdot q}.$$

If the left member of the last inequality is multiplied by $q!$, the product is an integer; call it M. Thus when all three members of the last inequality are multiplied by $q!$, the result is

$$M \ < \ (q-1)! \cdot p \ < \ M + \frac{1}{q} \ < \ M + 1.$$

This is a contradiction because it asserts that the *integer* $(q-1)! \cdot p$ lies strictly between the *consecutive integers* M and $M + 1$. Therefore e is irrational. ◀

C11S0M.064: The partial product P_k has many cancellations, much like a telescoping series:

$$P_k = \prod_{n=2}^{k} \frac{n^2}{n^2 - 1} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdots \frac{(k-1) \cdot (k-1)}{(k-2) \cdot k} \cdot \frac{k \cdot k}{(k-1) \cdot (k+1)} = \frac{2k}{k+1}.$$

Therefore

$$\prod_{n=2}^{\infty} \frac{n^2}{n^2 - 1} = \lim_{k \to \infty} P_k = 2.$$

C11S0M.065: Suppose that $x^2 = 5$. Then

$$x^2 - 4 = 1; \quad x - 2 = \frac{1}{2+x}; \quad x = 2 + \frac{1}{2+x}.$$

Now substitute the last expression for the last x. The result is

$$x = 2 + \cfrac{1}{4 + \cfrac{1}{2+x}}.$$

Repeat: Substitute the right-hand side of the last equation for the last x. Thus

$$x = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{2 + x}}}.$$

Continue this process. It follows that $a_0 = 2$ and that $a_n = 4$ for all $n \geqq 1$.

C11S0M.066: The series

$$S = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \frac{1}{10} + \cdots \tag{1}$$

is not absolutely convergent, so we must use special care in finding its sum. If both converge, then the series in (1) has the same sum as the series

$$\left(1 + \frac{1}{2}\right) - \frac{2}{3} + \left(\frac{1}{4} + \frac{1}{5}\right) - \frac{2}{6} + \left(\frac{1}{7} + \frac{1}{8}\right) - \frac{2}{9} + \left(\frac{1}{10} + \frac{1}{11}\right) - \cdots \tag{2}$$

because the $2n$th partial sum of the series in (2) is equal to the $3n$th partial sum of the series in (1). Moreover, the terms of the series in (2) are clearly approaching zero, and their absolute values do so monotonically because

$$\frac{1}{3k-2} + \frac{1}{3k-1} - \frac{2}{3k} = \frac{9k^2 - 3k + 9k^2 - 6k - 18k^2 + 18k - 4}{(3k-2)(3k-1)(3k)} = \frac{9k-4}{(3k-2)(3k-1)(3k)} \ > \ 0$$

and

$$\frac{2}{3k} - \frac{1}{3k+1} - \frac{1}{3k+2} = \frac{18k^2 + 18k + 4 - 9k^2 - 3k - 9k^2 - 6k}{(3k)(3k+1)(3k+1)} = \frac{9k+4}{(3k)(3k+1)(3k+1)} \ > \ 0.$$

Therefore the series in (2) converges by the alternating series test, and so the series in (1) converges as well. To evaluate its sum, let

$$f(x) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \frac{x^{10}}{10} + \cdots. \tag{3}$$

Then $f(x)$ exists at least on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ by the ratio test (Theorem 1 of Section 11.8; see also Problem 70 of that section). Therefore power series manipulations using calculus are valid. We find that

$$\begin{aligned}
f'(x) &= 1 + x - 2x^2 + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + x^9 + \cdots \\
&= \left(1 + x^3 + x^6 + x^9 + \cdots\right) + \left(x + x^4 + x^7 + x^{10} + \cdots\right) - 2\left(x^2 + x^5 + x^8 + x^{11} + \cdots\right),
\end{aligned}$$

$-1 < x < 1$. These rearrangement are permissible because the first series for $f'(x)$ is absolutely convergent on $(-1, 1)$. Moreover, $f'(x)$ is the sum of three geometric series, each of which converges on $(-1, 1)$. Therefore

$$f'(x) = \frac{1}{1-x^3} + \frac{x}{1-x^3} - \frac{2x^2}{1-x^3} = \frac{1 + x - 2x^2}{1-x^3} = \frac{(1-x)(1+2x)}{(1-x)(1+x+x^2)} = \frac{2x+1}{x^2+x+1}.$$

Therefore, if $-1 < x < 1$, then $f(x) = C + \ln(x^2 + x + 1)$. Also $0 = f(0) = C$, so that

$$f(x) = \ln(x^2 + x + 1), \quad -1 < x < 1.$$

Now we invoke a theorem formulated and proved by the brilliant Norwegian mathematician Neils Henrik Abel (1802–1829). It implies that if the series obtained by substitution of $x = 1$ in Eq. (3) converges (we

have seen that this is so) *and* the function f is continuous at $x = 1$ (this is clear), then $f(1)$ is the sum of the series. Therefore

$$1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \frac{1}{10} + \cdots = f(1) = \ln 3.$$

To check this result, the *Mathematica* 3.0 command

```
Sum[ 1/(3*n - 2) + 1/(3*n - 1) - 2/(3*n), {n, 1, Infinity} ]
```

yields the sum

$$\ln 3 \approx 1.0986122886681096913952452369225257046474905578227494517$$

almost instantaneously.

Bonus: One of our students contributed the following problem. Test for convergence:

$$\sum_{n=1}^{\infty} \frac{1}{2^1 \cdot 2^{1/2} \cdot 2^{1/3} \cdots 2^{1/n}}.$$

C10S0M.Extra: Curious about the Riemann zeta function? Questions about its behavior are currently the deepest and most important unsolved problems in mathematics; some of the answers have important consequences in the theory of the distribution of prime numbers. Some of those consequences are related to a remarkable identity discovered by Leonhard Euler:

Theorem: If $s > 1$ then

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Note that the product is taken over all *primes* p.

Recall that if s is a real number and $s > 1$, then

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and that the function ζ may be extended to most other numbers, including most complex numbers, by the condition that it is required to be infinitely differentiable. We'll have no need for its values at complex numbers here; we are mostly concerned with its values when s is an integer and $s > 1$. In the text we have seen a few of the values of the zeta function; for example,

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \qquad \text{and} \qquad \zeta(8) = \frac{\pi^8}{9450}.$$

It is known that $\zeta(2n)$ is a rational multiple of π^{2n} if n is a positive integer; much less is known about $\zeta(n)$ if n is odd and $n \geqq 3$. The values of $\zeta(2n)$ continue the preceding list as follows:

$$\frac{\pi^{10}}{93555}, \quad \frac{691\pi^{12}}{638512875}, \quad \frac{2\pi^{14}}{18243225}, \quad \frac{3617\pi^{16}}{325641566250}, \quad \frac{43867\pi^{18}}{38979295480125}, \quad \text{and} \quad \frac{174611\pi^{20}}{1531329465290625}.$$

The pattern of the coefficients is related to the *Bernoulli numbers* $\{B_n\}$, the values of which may be defined as follows. Write the Taylor series with center zero for

$$g(t) \;=\; \frac{t}{e^t - 1}.$$

(Note that $g(0)$ may be defined by the usual requirement that g be continuous at $t = 0$.) The resulting series is

$$g(t) \;=\; 1 - \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \frac{t^6}{30240} - \frac{t^8}{1209600} + \frac{t^{10}}{47900160} - \cdots . \tag{1}$$

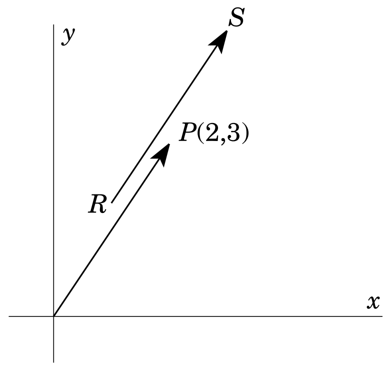
Then for n an even nonnegative integer, the nth Bernoulli number B_n may be defined to be the product of $n!$ and the coefficient of t^n in the series in (1). Finally, if n is an integer and $n \geqq 1$, then the coefficient of π^{2n} in the expression for $\zeta(2n)$ is of the form

$$\frac{2^j\,|B_{2n}|}{(2k)!}$$

where j and k are integers very closely related to n. We leave it to you to discover that simple relationship—extrapolation from the data given here will yield a valid result. Finally, if you need more numbers, the *Mathematica* commands `Zeta[ n ]` and `BernoulliB[ n ]`, or the *Maple* commands `Zeta(n)` and `bernoulli(n)`, will provide you with more values of the zeta function and more Bernoulli numbers.

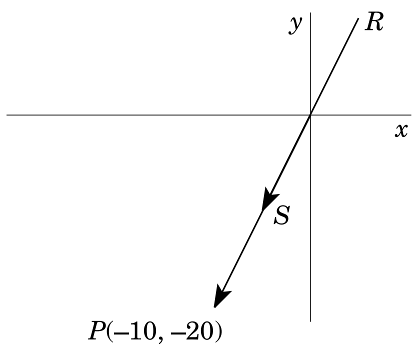
Section 12.1

C12S01.001: $\mathbf{v} = \overrightarrow{RS} = \langle\, 3-1,\ 5-2\,\rangle = \langle\, 2,\ 3\,\rangle$. The position vector of the point $P(2,\ 3)$ and $\overrightarrow{RS}$ are shown next.



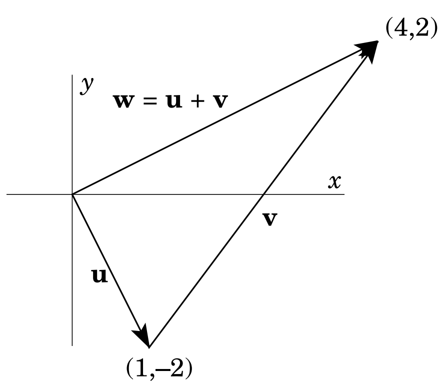
C12S01.002: $\mathbf{v} = \overrightarrow{RS} = \langle\, 1-(-2),\ 4-(-3)\,\rangle = \langle\, 3,\ 7\,\rangle$.

C12S01.003: $\mathbf{v} = \overrightarrow{RS} = \langle\, -5-5,\ -10-10\,\rangle = \langle\, -10,\ -20\,\rangle$. The position vector of the point P and $\overrightarrow{RS}$ are shown next.



C12S01.004: $\mathbf{v} = \overrightarrow{RS} = \langle\, 15-(-10),\ -25-20\,\rangle = \langle 25,\ -45\,\rangle$.

C12S01.005: $\mathbf{w} = \mathbf{u}+\mathbf{v} = \langle\, 1,\ -2\,\rangle + \langle\, 3,\ 4\,\rangle = \langle\, 1+3,\ -2+4\,\rangle = \langle\, 4,\ 2\,\rangle$. The next figure illustrates this computation in the form of the triangle law for vector addition (see Fig. 12.1.6 of the text).

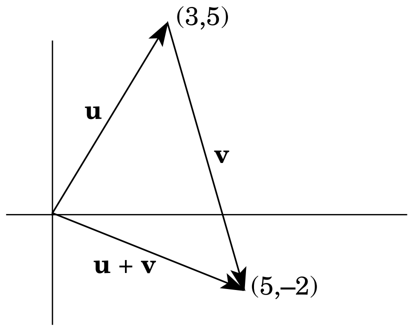


C12S01.006: $\mathbf{u}+\mathbf{v} = \langle\, 4,\ 2\,\rangle + \langle\, -2,\ 5\,\rangle = \langle\, 4-2,\ 2+5\,\rangle = \langle\, 2,\ 7\,\rangle$.

C12S01.007: Given: $\mathbf{u} = 3\mathbf{i}+5\mathbf{j}$, $\mathbf{v} = 2\mathbf{i}-7\mathbf{j}$:

$$\mathbf{u} + \mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{i} - 7\mathbf{j} = (3+2)\mathbf{i} + (5-7)\mathbf{j} = 5\mathbf{i} - 2\mathbf{j}.$$

The next figure illustrates the triangle law for vector addition using $\mathbf{u}$ and $\mathbf{v}$.



C12S01.008: $\mathbf{u} + \mathbf{v} = \langle 7 - 10,\ 5 + 0 \rangle = \langle -3,\ 5 \rangle = -3\mathbf{i} + 5\mathbf{j}$.

C12S01.009: Given: $\mathbf{a} = \langle 1,\ -2 \rangle$ and $\mathbf{b} = \langle -3,\ 2 \rangle$. Then:

$$|\mathbf{a}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}\,,$$

$$|-2\mathbf{b}| = |\langle -6,\ 4 \rangle| = \sqrt{36 + 16} = 2\sqrt{13}\,,$$

$$|\mathbf{a} - \mathbf{b}| = |\langle 1 - (-3),\ -2 - 2 \rangle| = \sqrt{16 + 16} = 4\sqrt{2}\,,$$

$$\mathbf{a} + \mathbf{b} = \langle 1 - 3,\ -2 + 2 \rangle = \langle -2,\ 0 \rangle,$$

$$3\mathbf{a} - 2\mathbf{b} = \langle 3,\ -6 \rangle - \langle -6,\ 4 \rangle = \langle 3 - (-6),\ -6 - 4 \rangle = \langle 9,\ -10 \rangle.$$

C12S01.010: Given: $\mathbf{a} = \langle 3,\ 4 \rangle$ and $\mathbf{b} = \langle -4,\ 3 \rangle$. Then:

$$|\mathbf{a}| = \sqrt{9 + 16} = \sqrt{25} = 5,$$

$$|-2\mathbf{b}| = |\langle -8,\ 6 \rangle| = \sqrt{64 + 36} = 10,$$

$$|\mathbf{a} - \mathbf{b}| = |\langle 3 - (-4),\ 4 - 3 \rangle| = \sqrt{49 + 1} = 5\sqrt{2}\,,$$

$$\mathbf{a} + \mathbf{b} = \langle 3 - 4,\ 4 + 3 \rangle = \langle -1,\ 7 \rangle,$$

$$3\mathbf{a} - 2\mathbf{b} = \langle 9,\ 12 \rangle - \langle -8,\ 6 \rangle = \langle 9 - (-8),\ 12 - 6 \rangle = \langle 17,\ 6 \rangle.$$

C12S01.011: Given: $\mathbf{a} = \langle -2,\ -2 \rangle$ and $\mathbf{b} = \langle -3,\ -4 \rangle$. Then:

$$|\mathbf{a}| = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}\,,$$

$$|-2\mathbf{b}| = |\langle 6,\ 8 \rangle| = \sqrt{36 + 64} = 10,$$

$$|\mathbf{a} - \mathbf{b}| = |\langle -2 - (-3),\ -2 - (-4) \rangle| = |\langle 1,\ 2 \rangle| = \sqrt{1 + 4} = \sqrt{5}\,,$$

$$\mathbf{a} + \mathbf{b} = \langle -2 - 3,\ -2 - 4 \rangle = \langle -5,\ -6 \rangle,$$

$$3\mathbf{a} - 2\mathbf{b} = \langle -6,\ -6 \rangle - \langle -6,\ -8 \rangle = \langle -6 - (-6),\ -6 - (-8) \rangle = \langle 0,\ 2 \rangle.$$

C12S01.012: Given: $\mathbf{a} = -2\langle\, 4,\ 7\,\rangle = \langle\, -8,\ -14\,\rangle$ and $\mathbf{b} = -3\langle\, -4,\ -2\,\rangle = \langle\, 12,\ 6\,\rangle$. Then:

$$|\mathbf{a}| = \sqrt{64+196} = \sqrt{260} = 2\sqrt{65}\,,$$

$$|-2\mathbf{b}| = |\langle\, -24,\ -12\,\rangle| = \sqrt{576+144} = \sqrt{720} = 12\sqrt{5}\,,$$

$$|\mathbf{a}-\mathbf{b}| = |\langle\, -8-12,\ -14-6\,\rangle| = |\langle\, -20,\ -20\,\rangle| = \sqrt{400+400} = 20\sqrt{2}\,,$$

$$\mathbf{a}+\mathbf{b} = \langle\, -8+12,\ -14+6\,\rangle = \langle\, 4,\ -8\,\rangle,$$

$$3\mathbf{a}-2\mathbf{b} = \langle\, -24,\ -42\,\rangle - \langle\, 24,\ 12\,\rangle = \langle\, -24-24,\ -42-12\,\rangle = \langle\, -48,\ -54\,\rangle.$$

C12S01.013: Given: $\mathbf{a} = \mathbf{i}+3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i}-5\mathbf{j}$. Then:

$$|\mathbf{a}| = |\mathbf{i}+3\mathbf{j}| = \sqrt{1+9} = \sqrt{10}\,,$$

$$|-2\mathbf{b}| = |4\mathbf{i}-10\mathbf{j}| = \sqrt{16+100} = 2\sqrt{29}\,,$$

$$|\mathbf{a}-\mathbf{b}| = |-\mathbf{i}+8\mathbf{j}| = \sqrt{1+64} = \sqrt{65}\,,$$

$$\mathbf{a}+\mathbf{b} = \mathbf{i}+3\mathbf{j}+2\mathbf{i}-5\mathbf{j} = 3\mathbf{i}-2\mathbf{j},$$

$$3\mathbf{a}-2\mathbf{b} = 3\mathbf{i}+9\mathbf{j}-4\mathbf{i}+10\mathbf{j} = -\mathbf{i}+19\mathbf{j}.$$

C12S01.014: Given: $\mathbf{a} = 2\mathbf{i}-5\mathbf{j}$ and $\mathbf{b} = \mathbf{i}-6\mathbf{j}$. Then:

$$|\mathbf{a}| = |2\mathbf{i}-5\mathbf{j}| = \sqrt{4+25} = \sqrt{29}\,,$$

$$|-2\mathbf{b}| = |-2\mathbf{i}+12\mathbf{j}| = \sqrt{4+144} = 2\sqrt{37}\,,$$

$$|\mathbf{a}-\mathbf{b}| = |\mathbf{i}+\mathbf{j}| = \sqrt{2}\,,$$

$$\mathbf{a}+\mathbf{b} = 2\mathbf{i}-5\mathbf{j}+\mathbf{i}-6\mathbf{j} = 3\mathbf{i}-11\mathbf{j},$$

$$3\mathbf{a}-2\mathbf{b} = 6\mathbf{i}-15\mathbf{j}-2\mathbf{i}+12\mathbf{j} = 4\mathbf{i}-3\mathbf{j}.$$

C12S01.015: Given: $\mathbf{a} = 4\mathbf{i}$ and $\mathbf{b} = -7\mathbf{j}$. Then:

$$|\mathbf{a}| = |4\mathbf{i}| = \sqrt{16} = 4,$$

$$|-2\mathbf{b}| = |14\mathbf{j}| = \sqrt{(14)^2} = 14,$$

$$|\mathbf{a}-\mathbf{b}| = |4\mathbf{i}+7\mathbf{j}| = \sqrt{16+49} = \sqrt{65}\,,$$

$$\mathbf{a}+\mathbf{b} = 4\mathbf{i}-7\mathbf{j},$$

$$3\mathbf{a}-2\mathbf{b} = 12\mathbf{i}+14\mathbf{j}.$$

C12S01.016: Given: $\mathbf{a} = -\mathbf{i}-\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i}+2\mathbf{j}$. Then:

$$|\mathbf{a}| = \sqrt{1+1} = \sqrt{2}\,,$$

$$|-2\mathbf{b}| = |-4\mathbf{i}-4\mathbf{j}| = \sqrt{32} = 4\sqrt{2}\,,$$

$$|\mathbf{a}-\mathbf{b}| = |-3\mathbf{i}-3\mathbf{j}| = \sqrt{18} = 3\sqrt{2},$$

$$\mathbf{a}+\mathbf{b} = -\mathbf{i}-\mathbf{j}+2\mathbf{i}+2\mathbf{j} = \mathbf{i}+\mathbf{j},$$

$$3\mathbf{a}-2\mathbf{b} = -3\mathbf{i}-3\mathbf{j}-4\mathbf{i}-4\mathbf{j} = -7\mathbf{i}-7\mathbf{j}.$$

C12S01.017: Because $|\mathbf{a}| = \sqrt{9+16} = 5$,

$$\mathbf{u} = \frac{1}{5}\mathbf{a} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \quad \text{and} \quad \mathbf{v} = -\frac{1}{5}\mathbf{a} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

C12S01.018: Because $|\mathbf{a}| = \sqrt{25+144} = 13$,

$$\mathbf{u} = \frac{1}{13}\mathbf{a} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j} \quad \text{and} \quad \mathbf{v} = -\frac{1}{13}\mathbf{a} = -\frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}.$$

C12S01.019: Because $|\mathbf{a}| = \sqrt{64+225} = \sqrt{289} = 17$,

$$\mathbf{u} = \frac{1}{17}\mathbf{a} = \frac{8}{17}\mathbf{i} + \frac{15}{17}\mathbf{j} \quad \text{and} \quad \mathbf{v} = -\frac{1}{17}\mathbf{a} = -\frac{8}{17}\mathbf{i} - \frac{15}{17}\mathbf{j}.$$

C12S01.020: Because $|\mathbf{a}| = \sqrt{49+576} = \sqrt{625} = 25$,

$$\mathbf{u} = \frac{1}{25}\mathbf{a} = \frac{7}{25}\mathbf{i} - \frac{24}{25}\mathbf{j} \quad \text{and} \quad \mathbf{v} = -\frac{1}{25}\mathbf{a} = -\frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}.$$

C12S01.021: $\mathbf{a} = \overrightarrow{PQ} = \langle 3-3,\ -2-2 \rangle = \langle 0,\ -4 \rangle = -4\mathbf{j}$.

C12S01.022: $\mathbf{a} = \overrightarrow{PQ} = \langle -3-(-3),\ 6-5 \rangle = \langle 0,\ 1 \rangle = \mathbf{j}$.

C12S01.023: $\mathbf{a} = \overrightarrow{PQ} = \langle 4-(-4),\ -7-7 \rangle = \langle 8,\ -14 \rangle = 8\mathbf{i} - 14\mathbf{j}$.

C12S01.024: $\mathbf{a} = \overrightarrow{PQ} = \langle -4-1,\ -1-(-1) \rangle = \langle -5,\ 0 \rangle = -5\mathbf{i}$.

C12S01.025: Given $\mathbf{a} = \langle 6,\ 0 \rangle$ and $\mathbf{b} = \langle 0,\ -7 \rangle$, $\mathbf{c} = \mathbf{b} - \mathbf{a} = \langle -6,\ -7 \rangle$. Then

$$|\mathbf{c}|^2 = 36 + 49 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular because of the (true) converse of the Pythagorean theorem. See Example 5.

C12S01.026: Given $\mathbf{a} = \langle 0,\ 3 \rangle$ and $\mathbf{b} = \langle 3,\ -1 \rangle$, $\mathbf{c} = \mathbf{b} - \mathbf{a} = \langle 3,\ -4 \rangle$. Then

$$|\mathbf{c}|^2 = 9 + 16 = 25 \neq 9 + 10 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular by the (true) contrapositive of the Pythagorean theorem. See Example 5.

C12S01.027: Given $\mathbf{a} = \langle 2,\ -1 \rangle$ and $\mathbf{b} = \langle 4,\ 8 \rangle$, $\mathbf{c} = \mathbf{b} - \mathbf{a} = \langle 2,\ 9 \rangle$. Then

$$|\mathbf{c}|^2 = 4 + 81 = 85 = 5 + 80 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular by the (true) converse of the Pythagorean theorem. See Example 5.

C12S01.028: Given $\mathbf{a} = \langle 8, 10 \rangle$ and $\mathbf{b} = \langle 15, -12 \rangle$, $\mathbf{c} = \mathbf{b} - \mathbf{a} = \langle 7, -22 \rangle$. Then

$$|\mathbf{c}|^2 = 49 + 484 = 533 = 164 + 369 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular by the (true) converse of the Pythagorean theorem. See Example 5.

C12S01.029: Given $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$, we have

$$3\mathbf{a} - 2\mathbf{b} = 6\mathbf{i} + 9\mathbf{j} - 6\mathbf{i} - 8\mathbf{j} = \mathbf{j} \quad \text{and}$$

$$4\mathbf{a} - 3\mathbf{b} = 8\mathbf{i} + 12\mathbf{j} - 9\mathbf{i} - 12\mathbf{j} = -\mathbf{i}.$$

Therefore $\mathbf{i} = -4\mathbf{a} + 3\mathbf{b}$ and $\mathbf{j} = 3\mathbf{a} - 2\mathbf{b}$.

C12S01.030: Given $\mathbf{a} = 5\mathbf{i} - 9\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 7\mathbf{j}$,

$$4\mathbf{a} - 5\mathbf{b} = 20\mathbf{i} - 36\mathbf{j} - 20\mathbf{i} + 35\mathbf{j} = -\mathbf{j} \quad \text{and}$$

$$7\mathbf{a} - 9\mathbf{b} = 35\mathbf{i} - 63\mathbf{j} - 36\mathbf{i} + 63\mathbf{j} = -\mathbf{i}.$$

Therefore $\mathbf{i} = -7\mathbf{a} + 9\mathbf{b}$ and $\mathbf{j} = -4\mathbf{a} + 5\mathbf{b}$.

C12S01.031: Given $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$, we have

$$\mathbf{c} = 2\mathbf{i} - 3\mathbf{j} = r\mathbf{a} + s\mathbf{b} = r\mathbf{i} + r\mathbf{j} + s\mathbf{i} - s\mathbf{j} = (r + s)\mathbf{i} + (r - s)\mathbf{j}.$$

It follows that $r + s = 2$ and that $r - s = -3$. These simultaneous equations are easily solved, and thereby we find that $\mathbf{c} = -\frac{1}{2}\mathbf{a} + \frac{5}{2}\mathbf{b}$.

C12S01.032: Given $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 8\mathbf{i} + 5\mathbf{j}$, we have

$$\mathbf{c} = 7\mathbf{i} + 9\mathbf{j} = r\mathbf{a} + s\mathbf{b} = 3r\mathbf{i} + 2r\mathbf{j} + 8s\mathbf{i} + 5s\mathbf{j} = (3r + 8s)\mathbf{i} + (2r + 5s)\mathbf{j}.$$

Thus $3r + 8s = 7$ and $2r + 5s = 9$. These simultaneous equations are easily solved for $r = 37$ and $s = -13$, and thus $\mathbf{c} = 37\mathbf{a} - 13\mathbf{b}$.

C12S01.033: Part (a): $3(5\mathbf{i} - 7\mathbf{j}) = 15\mathbf{i} - 21\mathbf{j}$. Part (b): $\dfrac{1}{3}(5\mathbf{i} - 7\mathbf{j}) = \dfrac{5}{3}\mathbf{i} - \dfrac{7}{3}\mathbf{j}$.

C12S01.034: Part (a): $-4(-3\mathbf{i} + 5\mathbf{j}) = 12\mathbf{i} - 20\mathbf{j}$. Part (b): $-\dfrac{1}{4}(-3\mathbf{i} + 5\mathbf{j}) = \dfrac{3}{4}\mathbf{i} - \dfrac{5}{4}\mathbf{j}$.

C12S01.035: Part (a): If $\mathbf{a} = 7\mathbf{i} - 3\mathbf{j}$, then $|\mathbf{a}| = \sqrt{58}$. Thus a unit vector with the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{\mathbf{a}}{\sqrt{58}}. \qquad \text{Answer:} \quad 5\mathbf{u} = \frac{5\sqrt{58}}{58}(7\mathbf{i} - 3\mathbf{j}).$$

Part (b): If $\mathbf{b} = 8\mathbf{i} + 5\mathbf{j}$, then $|\mathbf{b}| = \sqrt{89}$. Thus a unit vector with the same direction as $\mathbf{b}$ is

$$\mathbf{v} = \frac{\mathbf{b}}{\sqrt{89}}. \qquad \text{Answer:} \qquad -5\mathbf{v} = -\frac{5\sqrt{89}}{89}(8\mathbf{i} + 5\mathbf{j}).$$

C12S01.036: If $\mathbf{a} = \langle\, c,\, 2\,\rangle$ and $\mathbf{b} = \langle\, c,\, -8\,\rangle$, then let $\mathbf{w} = \mathbf{b} - \mathbf{a} = \langle\, 0,\, -10\,\rangle$. Perpendicularity of $\mathbf{a}$ and $\mathbf{b}$ is equivalent to the Pythagorean relation

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{w}|^2;$$

that is, $c^2 + 4 + c^2 + 64 = 100$, so that $c^2 = 16$. Answer: There are two solutions: $c = 4$ and $c = -4$. The corollary to Theorem 1 in Section 12.2 gives a more efficient method for solving such problems.

C12S01.037: Given $\mathbf{a} = \langle\, 2c,\, -4\,\rangle$ and $\mathbf{b} = \langle\, 3,\, c\,\rangle$, let $\mathbf{w} = \mathbf{b} - \mathbf{a} = \langle\, 3 - 2c,\, c + 4\,\rangle$. Perpendicularity of $\mathbf{a}$ and $\mathbf{b}$ is equivalent to the Pythagorean relation

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{w}|^2;$$

that is, $4c^2 + 16 + c^2 + 9 = 5c^2 - 4c + 25$, and thus the unique solution is $c = 0$.

C12S01.038: Let $\mathbf{u} = \overrightarrow{AB} = \langle\, -7,\, 4\,\rangle$, $\mathbf{v} = \overrightarrow{BC} = \langle\, 6,\, -12\,\rangle$, and $\mathbf{w} = \overrightarrow{CA} = \langle\, 1,\, 8\,\rangle$. Then

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = \langle\, -7 + 6 + 1,\, 4 - 12 + 8\,\rangle = \langle\, 0,\, 0\,\rangle = \mathbf{0}.$$

C12S01.039: With $\mathbf{a} = \langle\, a_1,\, a_2\,\rangle$, $\mathbf{b} = \langle\, b_1,\, b_2\,\rangle$, and $\mathbf{c} = \langle\, c_1,\, c_2\,\rangle$, we have

$$\begin{aligned}
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle\, a_1,\, a_2\,\rangle + (\langle\, b_1,\, b_2\,\rangle + \langle\, c_1,\, c_2\,\rangle) = \langle\, a_1,\, a_2\,\rangle + \langle b_1 + c_1,\, b_2 + c_2\,\rangle \\
&= \langle\, a_1 + (b_1 + c_1),\, a_2 + (b_2 + c_2)\,\rangle = \langle\, (a_1 + b_1) + c_1,\, (a_2 + b_2) + c_2\,\rangle \\
&= \langle\, a_1 + b_1,\, a_2 + b_2\,\rangle + \langle\, c_1,\, c_2\,\rangle = (\langle\, a_1,\, a_2\,\rangle + \langle\, b_1,\, b_2\,\rangle) + \langle\, c_1,\, c_2\,\rangle = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.
\end{aligned}$$

C12S01.040: With $\mathbf{a} = \langle\, a_1,\, a_2\,\rangle$ and scalars r and s, we have

$$\begin{aligned}
(r + s)\mathbf{a} = (r + s)\langle\, a_1,\, a_2\,\rangle &= \langle\, (r + s)a_1,\, (r + s)a_2\,\rangle = \langle\, ra_1 + sa_1,\, ra_2 + sa_2\,\rangle \\
&= \langle\, ra_1,\, ra_2\,\rangle + \langle\, sa_1,\, sa_2\,\rangle = r\langle\, a_1,\, a_2\,\rangle + s\langle\, a_1,\, a_2\,\rangle = r\mathbf{a} + s\mathbf{a}.
\end{aligned}$$

C12S01.041: With $\mathbf{a} = \langle\, a_1,\, a_2\,\rangle$ and scalars r and s, we have

$$(rs)\mathbf{a} = (rs)\langle\, a_1,\, a_2\,\rangle = \langle\, (rs)a_1,\, (rs)a_2\,\rangle = \langle\, r(sa_1),\, r(sa_2)\,\rangle = r\langle\, sa_1,\, sa_2\,\rangle = r(s\langle\, a_1,\, a_2\,\rangle) = r(s\mathbf{a}).$$

C12S01.042: With $\mathbf{a} = \langle\, a_1,\, a_2\,\rangle$, $\mathbf{b} = \langle\, b_1,\, b_2\,\rangle$, and the assumption that $\mathbf{a} + \mathbf{b} = \mathbf{a}$, we have

$$\begin{aligned}
\langle\, a_1,\, a_2\,\rangle + \langle\, b_1,\, b_2\,\rangle &= \langle\, a_1,\, a_2\,\rangle; \qquad & \langle\, a_1 + b_1,\, a_2 + b_2\,\rangle &= \langle\, a_1,\, a_2\,\rangle; \\
a_1 + b_1 = a_1 \quad \text{and} \quad a_2 + b_2 &= a_2; \qquad & b_1 = b_2 &= 0;
\end{aligned}$$

therefore $\mathbf{b} = \mathbf{0}$.

C12S01.043: See Fig. 12.1.13. The angle each cable makes with the horizontal is 30°, so

$$T_1 \cos 30^\circ = T_2 \cos 30^\circ. \qquad \text{Therefore} \qquad T_1 = T_2.$$

Also

$$T_1 \sin 30^\circ + T_2 \sin 30^\circ = 100; \qquad 2 \cdot \frac{1}{2} \cdot T_1 = 100; \qquad T_1 = T_2 = 100.$$

C12S01.044: See Fig. 12.1.14; let T_1 be the tension in the right-hand cable and T_2 the tension in the left-hand cable. Then

$$T_1 \cos 30^\circ = T_2 \cos 45^\circ \qquad \text{and} \qquad T_1 \sin 30^\circ + T_2 \sin 45^\circ = 50.$$

Thus $T_2 \sin 45^\circ = T_2 \cos 45^\circ = T_1 \cos 30^\circ$, and therefore

$$T_1 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) = 50 :$$

$$T_1 = \frac{100}{1+\sqrt{3}} = \frac{100\left(\sqrt{3}\ -1\right)}{2} = 50\left(\sqrt{3}\ -1\right) \approx 36.602540 \quad \text{(lb)}.$$

It now follows that

$$T_2 = T_1 \cdot \frac{\cos 30^\circ}{\cos 45^\circ} = T_1 \cdot \frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{2}} = \frac{\sqrt{6}}{2} \cdot T_1 = 25\left(3\sqrt{2}\ - \sqrt{6}\ \right) \approx 44.828774 \qquad \text{(lb)}.$$

C12S01.045: See Fig. 12.1.15; let T_1 be the tension in the right-hand cable and T_2 the tension in the left-hand cable. The equations that balance the horizontal and vertical forces, respectively, are

$$T_1 \cos 40^\circ = T_2 \cos 55^\circ \qquad \text{and} \qquad T_1 \sin 40^\circ + T_2 \sin 55^\circ = 125.$$

A *Mathematica* 3.0 command for solving these equations simultaneously is

```
Solve[ { t1*Cos[40*Pi/180] == t2*Cos[55*Pi/180],
           t1*Sin[40*Pi/180] + t2*Sin[55*Pi/180] == 125 }, {t1,t2} ]
```

and it yields the solution

$$T_1 = \frac{125\cos(11\pi/15)}{\cos(11\pi/36)\,\sin(2\pi/9) + \cos(2\pi/9)\,\sin(11\pi/36)} \approx 71.970925644575 \quad \text{(lb)},$$

$$T_2 = \frac{125\cos(2\pi/9)}{\cos(11\pi/36)\,\sin(2\pi/9) + \cos(2\pi/9)\,\sin(11\pi/36)} \approx 96.121326055335 \quad \text{(lb)}.$$

C12S01.046: See Fig. 12.1.16; let T_1 be the tension in the right-hand cable and T_2 the tension in the left-hand cable. Let θ_1 be the acute angle that the right-hand cable makes with the horizontal support; let θ_2 be the acute angle that the left-hand cable makes with the horizontal support. The hypotenuse of the right triangle formed by the two cables and the support has length 5, so that

$$\cos\theta_1 = \frac{3}{5} = \sin\theta_2 \qquad \text{and} \qquad \cos\theta_2 = \frac{4}{5} = \sin\theta_1.$$

Balancing the horizontal forces and balancing the vertical forces leads, respectively, to

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 \qquad \text{and} \qquad T_1 \sin\theta_1 + T_2 \sin\theta_2 = 150;$$

thus

$$\begin{aligned} \frac{3}{5}T_1 &= \frac{4}{5}T_2; &\qquad T_2 &= \frac{3}{4}T_1; \\ \frac{4}{5}T_1 + \frac{3}{4}\cdot\frac{3}{5}T_1 &= 150; &\qquad \left(\frac{4}{5}+\frac{9}{20}\right)T_1 &= 150; \\ \frac{5}{4}T_1 &= 150; &\qquad T_1 &= 120 \quad \text{(lb)}. \end{aligned}$$

Then it follows that $T_2 = \frac{3}{4}T_1 = 90$ (lb).

C12S01.047: $\mathbf{v}_a = \mathbf{v}_g - \mathbf{w} = \langle\, 500,\, 0\,\rangle - \langle\, -25\sqrt{2},\, -25\sqrt{2}\,\rangle = \langle\, 500 + 25\sqrt{2},\, 25\sqrt{2}\,\rangle$. This corresponds to a compass bearing of about $86^\circ\, 13'$ (about 3.778° north of east) with airspeed approximately 536.52 mi/h (about 467 knots).

C12S01.048: $\mathbf{v}_a = \mathbf{v}_g - \mathbf{w} = \langle\, -500,\, 0\,\rangle - \langle\, -25\sqrt{2},\, -25\sqrt{2}\,\rangle = \langle\, 25\sqrt{2} - 500,\, 25\sqrt{2}\,\rangle$. This corresponds to a compass bearing of about $274^\circ\, 21'$ (about 4.351° north of west) with airspeed approximately 466 mi/h (about 405 knots).

C12S01.049: $\mathbf{v}_a = \mathbf{v}_g - \mathbf{w} = \langle\, -250\sqrt{2},\, 250\sqrt{2}\,\rangle - \langle\, -25\sqrt{2},\, -25\sqrt{2}\,\rangle = \langle -225\sqrt{2},\, 275\sqrt{2}\,\rangle$. This corresponds to a compass bearing of about $320^\circ\, 43'$ (about 50.71° north of west) with airspeed approximately 502 mi/h (about 437 knots).

C12S01.050: Assume that $A = (a_1,\, a_2)$, that $B = (b_1,\, b_2)$, and that $C = (c_1,\, c_2)$. Then

$$\overrightarrow{AB} = \langle\, b_1 - a_1,\, b_2 - a_2\,\rangle, \quad \overrightarrow{BC} = \langle\, c_1 - b_1,\, c_2 - b_2\,\rangle, \quad \text{and} \quad \overrightarrow{CA} = \langle\, a_1 - c_1,\, a_2 - c_2\,\rangle.$$

Therefore

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \langle\, b_1 - a_1 + c_1 - b_1 + a_1 - c_1,\, b_2 - a_2 + c_2 - b_2 + a_2 - c_2\,\rangle = \langle\, 0,\, 0\,\rangle = \mathbf{0}.$$

C12S01.051: Denote the origin by O. Then $\mathbf{a} = \overrightarrow{OP}$ and $\mathbf{b} = \overrightarrow{OQ}$. Let M denote the midpoint of the line segment PQ. Then $\overrightarrow{PQ} = \mathbf{b} - \mathbf{a}$, so that

$$\overrightarrow{PM} = \frac{1}{2}\,(\mathbf{b} - \mathbf{a}).$$

Hence the position vector of M is

$$\overrightarrow{OP} + \overrightarrow{MP} = \mathbf{a} + \frac{1}{2}\,(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\,(\mathbf{a} + \mathbf{b}).$$

C12S01.052: Let $\mathbf{u} = \overrightarrow{AM}$, let $\mathbf{v} = \overrightarrow{NA}$, and let $\mathbf{w} = \overrightarrow{BC}$. Let $\mathbf{x} = \overrightarrow{MN}$. (Perhaps you should draw a figure.) Then

$$2\mathbf{u} + \mathbf{w} + 2\mathbf{v} = \mathbf{0}; \qquad \mathbf{u} + \mathbf{x} + \mathbf{v} = \mathbf{0};$$

$$\mathbf{w} = -2\mathbf{u} - 2\mathbf{v} = 2\mathbf{x}; \qquad \mathbf{x} = \frac{1}{2}\mathbf{w}.$$

Therefore $\mathbf{x}$ and $\mathbf{w}$ are parallel and $|\mathbf{x}| = \frac{1}{2}|\mathbf{w}|$. Thus the line segment joining the midpoints of two sides of an arbitrary triangle is parallel to the third side and half its length.

C12S01.053: Let A, B, C, and D be the vertices of a parallelogram, in order, counterclockwise. Then its diagonals are AC and BD. Let M be the point where the diagonals cross. Let $\mathbf{u} = \overrightarrow{AB}$ and let $\mathbf{v} = \overrightarrow{BC}$. Then

$$\overrightarrow{AC} = \mathbf{u} + \mathbf{v} \quad \text{and} \quad \overrightarrow{BD} = \mathbf{v} - \mathbf{u}.$$

Then $\overrightarrow{AM} = r(\mathbf{u} + \mathbf{v})$ and $\overrightarrow{MC} = (1 - r)(\mathbf{u} + \mathbf{v})$ for some scalar r. Similarly, $\overrightarrow{BM} = s(\mathbf{v} - \mathbf{u})$ and $\overrightarrow{MD} = (1 - s)(\mathbf{v} - \mathbf{u})$ for some scalar s. Therefore

$$\overrightarrow{AB} + \overrightarrow{BM} = \overrightarrow{AM}; \qquad \mathbf{u} + s(\mathbf{v} - \mathbf{u}) = r(\mathbf{u} + \mathbf{v});$$

$$(1 - r - s)\mathbf{u} = (r - s)\mathbf{v}; \qquad 1 - r - s = 0 = r - s$$

because $\mathbf{u}$ and $\mathbf{v}$ are not parallel. It follows that $r = s = \frac{1}{2}$, so that $|AM| = |MC|$ and $|BM| = |MD|$. Therefore the diagonals of the arbitrary parallelogram $ABCD$ bisect each other.

C12S01.054: Let A, C, E, and G be the vertices of an arbitrary quadrilateral in the plane, in order, counterclockwise. Let B be the midpoint of AC, D the midpoint of CE, F the midpoint of EG, and H the midpoint of GA. We are to prove that $BDFH$ is a parallelogram. Let $\mathbf{u} = \overrightarrow{AB}$. Then $\mathbf{u} = \overrightarrow{BC}$. Let $\mathbf{v} = \overrightarrow{CD}$. Then $\mathbf{v} = \overrightarrow{DE}$. Let $\mathbf{w} = \overrightarrow{EF}$. Then $\mathbf{w} = \overrightarrow{FG}$. Let $\mathbf{x} = \overrightarrow{GH}$. Then $\mathbf{x} = \overrightarrow{HA}$. It now follows that

$$2\mathbf{u} + 2\mathbf{v} + 2\mathbf{w} + 2\mathbf{x} = \mathbf{0}; \qquad \mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x} = \mathbf{0};$$

$$\overrightarrow{BD} = \mathbf{u} + \mathbf{v} = -(\mathbf{w} + \mathbf{x}) = \overrightarrow{FH}; \qquad |\overrightarrow{BD}| = |\overrightarrow{FH}| \quad \text{and} \quad \overrightarrow{BD} \parallel \overrightarrow{FH}.$$

According to a theorem of Euclid, if two sides of a quadrilateral are parallel and have the same length, then the quadrilateral is a parallelogram. Or, if you prefer, you can use these computations to show also that $\overrightarrow{DF} \parallel \overrightarrow{HB}$, so that $BDFH$ is a parallelogram by definition.

C12S01.055: See Fig. 12.1.18, which makes it easy to see that $a_1 = r\cos\theta$ and $a_2 = r\sin\theta$. Then

$$\mathbf{a}_\perp = \left[r\cos\left(\theta + \frac{\pi}{2}\right)\right]\mathbf{i} + \left[r\sin\left(\theta + \frac{\pi}{2}\right)\right]\mathbf{j} = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -a_2\mathbf{i} + a_1\mathbf{j}.$$

Section 12.2

C12S02.001: Given $\mathbf{a} = \langle\, 2,\ 5,\ -4\,\rangle$ and $\mathbf{b} = \langle\, 1,\ -2,\ -3,\,\rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle\, 5,\ 8,\ -11\,\rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle\, 2,\ 23,\ 0\,\rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 1 - 5 \cdot 2 + 4 \cdot 3 = 4$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle\, 1,\ 7,\ -1\,\rangle| = \sqrt{51}\,$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{4 + 25 + 16}}\langle\, 2,\ 5,\ -4\,\rangle = \dfrac{\sqrt{5}}{15}\langle\, 2,\ 5,\ -4\,\rangle = \left\langle\, \dfrac{2\sqrt{5}}{15},\ \dfrac{\sqrt{5}}{3},\ -\dfrac{4\sqrt{5}}{15}\,\right\rangle$.

C12S02.002: Given $\mathbf{a} = \langle\, -1,\ 0,\ 2\,\rangle$ and $\mathbf{b} = \langle\, 3,\ 4,\ -5\,\rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle\, 1,\ 4,\ -1\,\rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle\, -15,\ -16,\ 26\,\rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = -1 \cdot 3 + 0 \cdot 4 - 2 \cdot 5 = -13$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle\, -4,\ -4,\ 7\,\rangle| = \sqrt{16 + 16 + 49} = \sqrt{81} = 9$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{5}}\langle\, -1,\ 0,\ 2\,\rangle = \left\langle\, -\dfrac{\sqrt{5}}{5},\ 0,\ \dfrac{2\sqrt{5}}{5}\,\right\rangle$.

C12S02.003: Given $\mathbf{a} = \langle\, 1,\ 1,\ 1\,\rangle$ and $\mathbf{b} = \langle\, 0,\ 1,\ -1\,\rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle\, 2,\ 3,\ 1\,\rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle\, 3,\ -1,\ 7\,\rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 0 + 1 \cdot 1 - 1 \cdot 1 = 0$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle\, 1,\ 0,\ 2\,\rangle| = \sqrt{5}\,$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{3}}\langle\, 1,\ 1,\ 1\,\rangle = \left\langle\, \dfrac{\sqrt{3}}{3},\ \dfrac{\sqrt{3}}{3},\ \dfrac{\sqrt{3}}{3}\,\right\rangle$.

C12S02.004: Given $\mathbf{a} = \langle\, 2,\ -3,\ 5\,\rangle$ and $\mathbf{b} = \langle\, 5,\ 3,\ -7\,\rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle\, 9,\ -3,\ 3\,\rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle\, -14,\ -21,\ 43\,\rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 5 - 3 \cdot 3 - 7 \cdot 5 = -34$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle\, -3,\ -6,\ 12\,\rangle| = \sqrt{9 + 36 + 144} = 3\sqrt{21}\,$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{4 + 9 + 25}}\langle\, -2,\ -3,\ 5\,\rangle = \left\langle\, \dfrac{\sqrt{38}}{19},\ \dfrac{-3\sqrt{38}}{38},\ \dfrac{5\sqrt{38}}{38}\,\right\rangle$.

C12S02.005: Given $\mathbf{a} = \langle\, 2,\ -1,\ 0\,\rangle$ and $\mathbf{b} = \langle\, 0,\ 1,\ -3\,\rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle\, 4,\ -1,\ -3\,\rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle\, 6,\ -7,\ 12\,\rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = 2 \cdot 0 - 1 \cdot 1 + 0 \cdot (-3) = -1$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle\, 2,\ -2,\ 3\,\rangle| = \sqrt{17}\,$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{5}}\langle\, 2,\ -1,\ 0\,\rangle = \left\langle\, \dfrac{2\sqrt{5}}{5},\ -\dfrac{\sqrt{5}}{5},\ 0\,\right\rangle$.

C12S02.006: Given $\mathbf{a} = \langle 1, -2, 3 \rangle$ and $\mathbf{b} = \langle 1, 3, -2 \rangle$, we find that

(a): $2\mathbf{a} + \mathbf{b} = \langle 3, -1, 4 \rangle$,

(b): $3\mathbf{a} - 4\mathbf{b} = \langle -1, -18, 17 \rangle$,

(c): $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 1 - 2 \cdot 3 - 3 \cdot 2 = -11$,

(d): $|\mathbf{a} - \mathbf{b}| = |\langle 0, -5, 5 \rangle| = \sqrt{50} = 5\sqrt{2}$, and

(e): $\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{\sqrt{1+4+9}} \langle 1, -2, 3 \rangle = \left\langle \dfrac{\sqrt{14}}{14}, -\dfrac{\sqrt{14}}{7}, \dfrac{3\sqrt{14}}{14} \right\rangle$.

C12S02.007: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{4}{\sqrt{45}\,\sqrt{14}} = \frac{2\sqrt{70}}{105} \approx 0.15936371,$$

and therefore $\theta \approx 80.830028^\circ \approx 81^\circ$.

C12S02.008: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = -\frac{13}{\sqrt{5}\,\sqrt{50}} = -\frac{13\sqrt{10}}{50} \approx -0.82219219,$$

and therefore $\theta \approx 145.304864^\circ \approx 145^\circ$.

C12S02.009: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = 0 \quad \text{(exactly)},$$

and therefore $\theta = 90^\circ$ (exactly).

C12S02.010: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{10 - 9 - 35}{\sqrt{38}\,\sqrt{83}} = -\frac{34}{\sqrt{3154}} = -\frac{34\sqrt{3154}}{3154} \approx -0.60540788,$$

and therefore $\theta \approx 127.258199^\circ \approx 127^\circ$.

C12S02.011: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{-1}{5\sqrt{2}} = -\frac{\sqrt{2}}{10} \approx -0.14142136,$$

and therefore $\theta \approx 98.130102^\circ \approx 98^\circ$.

C12S02.012: If θ is the angle between **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = -\frac{11}{14} \approx -0.78571429,$$

and therefore $\theta \approx 141.786789^\circ \approx 142^\circ$.

C12S02.013: Refer to Problem 1;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{4\sqrt{5}}{15} \quad \text{and} \quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{2\sqrt{14}}{7}.$$

C12S02.014: Refer to Problem 2;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = -\frac{13\sqrt{5}}{5} \quad \text{and} \quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = -\frac{13\sqrt{2}}{10}.$$

C12S02.015: Refer to Problem 3;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 0 = \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.$$

C12S02.016: Refer to Problem 4;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = -\frac{17\sqrt{38}}{19} \quad \text{and} \quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = -\frac{34\sqrt{83}}{83}.$$

C12S02.017: Refer to Problem 5;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = -\frac{\sqrt{5}}{5} \quad \text{and} \quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = -\frac{\sqrt{10}}{10}.$$

C12S02.018: Refer to Problem 6;

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = -\frac{11\sqrt{14}}{14} \quad \text{and} \quad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = -\frac{11\sqrt{14}}{14}.$$

C12S02.019: An equation of the sphere is $(x-3)^2 + (y-1)^2 + (z-2)^2 = 25$; that is,

$$x^2 - 6x + y^2 - 2y + z^2 - 4z = 11.$$

C12S02.020: A equation of the sphere is $(x+2)^2 + (y-1)^2 + (z+5)^2 = 7$; that is,

$$x^2 + 4x + y^2 - 2y + z^2 + 10z + 23 = 0.$$

C12S02.021: The center of the sphere is at

$$\left(\frac{3+7}{2},\ \frac{5+3}{2},\ \frac{-3+1}{2}\right) = (5,\ 4,\ -1)$$

and its radius is

$$\frac{1}{2}\sqrt{(7-3)^2 + (3-5)^2 + (1+3)^2} = \frac{1}{2}\sqrt{16+4+16} = 3.$$

Therefore an equation of this sphere is $(x-5)^2 + (y-4)^2 + (z+1)^2 = 9$; that is,

$$x^2 - 10x + y^2 - 8y + z^2 + 2z + 33 = 0.$$

C12S02.022: The radius of the sphere is

$$\sqrt{(4-1)^2 + 5^2 + (-2)^2} = \sqrt{9+25+4} = \sqrt{38},$$

and thus an equation of the sphere is $(x-4)^2 + (y-5)^2 + (z+2)^2 = 38$; that is,

$$x^2 - 8x + y^2 - 10y + z^2 + 4z + 7 = 0.$$

C12S02.023: The sphere has radius 2, thus equation $x^2 + y^2 + (z-2)^2 = 4$; that is, $x^2 + y^2 + z^2 - 4z = 0$.

C12S02.024: The sphere has radius 4, thus equation $(x-3)^2 + (y+4)^2 + (z-3)^2 = 16$; that is,

$$x^2 - 6x + y^2 + 8y + z^2 - 6z + 18 = 0.$$

C12S02.025: We complete the square in x and y:

$$x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 = 13;$$

$$(x+2)^2 + (y-3)^2 + (z-0)^2 = 13.$$

This sphere has center at $(-2, 3, 0)$ and radius $\sqrt{13}$.

C12S02.026: We complete the square in all three variables:

$$x^2 - 8x + 16 + y^2 - 9y + \frac{81}{4} + z^2 + 10z + 25 + 40 = \frac{245}{4};$$

$$(x-4)^2 + \left(y - \frac{9}{2}\right)^2 + (z+5)^2 = \frac{85}{4}.$$

This sphere has center at $\left(4, \frac{9}{2}, -5\right)$ and radius $\dfrac{\sqrt{85}}{2}$.

C12S02.027: We complete the square in z:

$$x^2 + y^2 + z^2 - 6z + 9 = 16 + 9 = 25;$$

$$(x-0)^2 + (y-0)^2 + (z-3)^2 = 5^2.$$

This sphere has center at $(0, 0, 3)$ and radius 5.

C12S02.028: We complete the square in all three variables:

$$x^2 - \frac{7}{2}x + y^2 - \frac{9}{2}y + z^2 - \frac{11}{2}z = 0;$$

$$x^2 - \frac{7}{2}x + \frac{49}{16} + y^2 - \frac{9}{2}y + \frac{81}{16} + z^2 - \frac{11}{2}z + \frac{121}{16} = \frac{251}{16};$$

$$\left(x - \frac{7}{4}\right)^2 + \left(y - \frac{9}{4}\right)^2 + \left(z - \frac{11}{4}\right)^2 = \frac{251}{16}.$$

This sphere has center at $\left(\frac{7}{4}, \frac{9}{4}, \frac{11}{4}\right)$ and radius $\dfrac{\sqrt{251}}{4}$.

C12S02.029: The equation $z = 0$ is an equation of the xy-plane.

C12S02.030: The equation $x = 0$ is an equation of the yz-plane.

C12S02.031: The equation $z = 10$ is an equation of the plane parallel to the xy-plane and passing through the point $(0, 0, 10)$.

C12S02.032: The equation $xy = 0$ holds when either $x = 0$ or $y = 0$ (or both). So its graph is the union of the yz-plane and the xz-plane.

C12S02.033: The equation $xyz = 0$ holds when any one (or more) of the three variables is zero. Hence its graph is the union of the three coordinate planes.

C12S02.034: If $x^2 + y^2 + z^2 + 7 = 0$, then the sum of the three nonnegative numbers x^2, y^2, and z^2 is -7, which is negative. This is impossible regardless of the values of x, y, and z. Therefore there are no points on the graph of the given equation.

C12S02.035: Given the equation $x^2 + y^2 + z^2 = 0$, the sum of the three nonnegative numbers x^2, y^2, and z^2 can be zero only if none of x, y, and z is nonzero. Therefore the graph of the given equation consists of the single point $(0, 0, 0)$.

C12S02.036: The equation $x^2 + y^2 + z^2 - 2x + 1 = 0$ can be rewritten in the form $(x-1)^2 + y^2 + z^2 = 0$, and—as in the solution of Problem 35—the sum of three nonnegative real numbers can be zero only if none is nonzero. Therefore the graph of the given equation consists of the single point $(1, 0, 0)$.

C12S02.037: Complete the square in x and y:

$$\begin{aligned}
&x^2 + y^2 + z^2 - 6x + 8y + 25 = 0;\\
&x^2 - 6x + 9 + y^2 + 8y + 16 + z^2 + 25 = 25;\\
&(x-3)^2 + (y+4)^2 + z^2 = 0.
\end{aligned}$$

The sum of three nonnegative real numbers can be zero only if none is positive. So the graph of the given equation consists of the single point $(3, -4, 0)$.

C12S02.038: Given the equation $x^2 + y^2 = 0$, the argument used in the previous few solutions demonstrates that $x = 0 = y$, but z is arbitrary. Hence the graph of the given equation is the z-axis.

C12S02.039: If $\mathbf{a} = \langle 4, -2, 6 \rangle$ and $\mathbf{b} = \langle 6, -3, 9 \rangle$, then $\mathbf{b} = \frac{3}{2}\mathbf{a}$. Therefore $\mathbf{a}$ and $\mathbf{b}$ are parallel. Because they are nonzero and parallel, they are not perpendicular; alternatively, they are not perpendicular because $\mathbf{a} \cdot \mathbf{b} = 84 \neq 0$.

C12S02.040: Suppose that $\mathbf{a} = \langle 4, -2, 6 \rangle$ and $\mathbf{b} = \langle 4, 2, 2 \rangle$. If $\mathbf{b} = \lambda\mathbf{a}$ for some scalar λ, then examination of first components shows that $\lambda = 1$, whereas examination of second components shows that $\lambda = -1$. Therefore there is no such scalar, and thus $\mathbf{a}$ and $\mathbf{b}$ are not parallel. Moreover, $\mathbf{a} \cdot \mathbf{b} = 24 \neq 0$, so these two vectors are also not perpendicular.

C12S02.041: Because

$$\mathbf{b} = -9\mathbf{i} + 15\mathbf{j} - 12\mathbf{k} = -\frac{3}{4}(12\mathbf{i} - 20\mathbf{j} + 16\mathbf{k}) = -\frac{3}{4}\mathbf{a},$$

the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel. Because they are nonzero as well, they are not perpendicular; alternatively, $\mathbf{a} \cdot \mathbf{b} = -600 \neq 0$, and hence they are not perpendicular.

C12S02.042: If there existed a scalar ξ such that $\xi\mathbf{a} = \mathbf{b}$, then

$$\xi(12\mathbf{i} - 20\mathbf{j} + 17\mathbf{k}) = -9\mathbf{i} + 15\mathbf{j} + 24\mathbf{k},$$

so that $12\xi = -9$, $-20\xi = 15$, and $17\xi = 24$. The first two equations agree that $\xi = -\frac{3}{4}$, but the third implies that $\xi = \frac{24}{17}$. Hence there is no such scalar ξ, and therefore the vectors $\mathbf{a}$ and $\mathbf{b}$ are not parallel. But $\mathbf{a} \cdot \mathbf{b} = -9 \cdot 12 - 20 \cdot 15 + 24 \cdot 17 = -108 - 300 + 408 = 0$, and thus $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

C12S02.043: The distinct points P, Q, and R lie on a single straight line if and only if the vectors representing $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel. Here we have

$$3\overrightarrow{PQ} = 3\left(\langle 1, -3, 5\rangle - \langle 0, -2, 4\rangle\right) = 3\langle 1, -1, 1\rangle = \langle 3, -3, 3\rangle = \langle 4, -6, 8\rangle - \langle 1, -3, 5\rangle = \overrightarrow{QR}.$$

Therefore $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel, and so the three points P, Q, and R do lie on a single straight line.

C12S02.044: The distinct points P, Q, and R lie on a single straight line if and only if the vectors representing $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel. Here we have

$$-3\overrightarrow{PQ} = -3\left(\langle 3, 3, 3\rangle - \langle 6, 7, 8\rangle\right) = -3\langle -3, -4, -5\rangle = \langle 9, 12, 15\rangle = \langle 12, 15, 18\rangle - \langle 3, 3, 3\rangle = \overrightarrow{QR}.$$

Therefore $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel, and so the three points P, Q, and R do lie on a single straight line.

C12S02.045: The sides of triangle ABC are represented by the three vectors $\mathbf{u} = \overrightarrow{AB} = \langle -1, 1, 0\rangle$, $\mathbf{v} = \overrightarrow{BC} = \langle 0, -1, 1\rangle$, and $\mathbf{w} = \overrightarrow{AC} = \langle -1, 0, 1\rangle$. Let A denote the angle of the triangle at vertex A. Then

$$\cos A = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| \cdot |\mathbf{w}|} = \frac{(-1)(-1)}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2},$$

and therefore $A = 60^\circ$ (exactly). Similarly,

$$\cos B = \frac{-\mathbf{u} \cdot \mathbf{v}}{|-\mathbf{u}| \cdot |\mathbf{v}|} = \frac{(-1)(-1)}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2},$$

and so $B = 60^\circ$ as well. Finally,

$$\cos C = \frac{-\mathbf{v} \cdot (-\mathbf{w})}{|-\mathbf{v}| \cdot |-\mathbf{w}|} = \frac{(-1)(-1)}{\sqrt{2}\,\sqrt{2}} = \frac{1}{2},$$

and it follows that $C = 60^\circ$.

C12S02.046: The sides of triangle ABC are represented by the three vectors $\mathbf{u} = \overrightarrow{AB} = \langle 0, 2, 0\rangle$, $\mathbf{v} = \overrightarrow{BC} = \langle 0, 0, 3\rangle$, and $\mathbf{w} = \overrightarrow{AC} = \langle 0, 2, 3\rangle$. Let A denote the angle of the triangle at vertex A. Then

$$\cos A = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| \cdot |\mathbf{w}|} = \frac{4}{\sqrt{4}\,\sqrt{13}} = \frac{2\sqrt{13}}{13} \approx 0.55470020,$$

and therefore $A \approx 56.309932^\circ \approx 56^\circ$. Similarly,

$$\cos B = \frac{-\mathbf{u} \cdot \mathbf{v}}{|-\mathbf{u}| \cdot |\mathbf{v}|} = \frac{0}{\sqrt{4}\,\sqrt{9}} = 0,$$

and so $B = 90^\circ$ (exactly). Finally,

$$\cos C = \frac{-\mathbf{v} \cdot (-\mathbf{w})}{|-\mathbf{v}| \cdot |-\mathbf{w}|} = \frac{(-3)(-3)}{\sqrt{9}\,\sqrt{13}} = \frac{3\sqrt{13}}{13} \approx 0.83205029,$$

and it follows that $C \approx 33.690067° \approx 34°$.

C12S02.047: The sides of triangle ABC are represented by the three vectors $\mathbf{u} = \overrightarrow{AB} = \langle 2, -3, 2 \rangle$, $\mathbf{v} = \overrightarrow{BC} = \langle 0, 6, 3 \rangle$, and $\mathbf{w} = \overrightarrow{AC} = \langle 2, 3, 5 \rangle$. Let A denote the angle of the triangle at vertex A. Then

$$\cos A = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| \cdot |\mathbf{w}|} = \frac{4 - 9 + 10}{\sqrt{17}\,\sqrt{38}} = \frac{5\sqrt{646}}{646} \approx 0.19672237,$$

and therefore $A \approx 78.654643° \approx 79°$. Similarly,

$$\cos B = \frac{-\mathbf{u} \cdot \mathbf{v}}{|-\mathbf{u}| \cdot |\mathbf{v}|} = \frac{18 - 6}{\sqrt{17}\,\sqrt{45}} = \frac{4\sqrt{85}}{85} \approx 0.43386082,$$

and so $B \approx 64.287166° \approx 64°$. Finally,

$$\cos C = \frac{-\mathbf{v} \cdot (-\mathbf{w})}{|-\mathbf{v}| \cdot |-\mathbf{w}|} = \frac{18 + 15}{\sqrt{38}\,\sqrt{45}} = \frac{11\sqrt{190}}{190} \approx 0.79802388,$$

and it follows that $C \approx 37.058191° \approx 37°$.

C12S02.048: The sides of triangle ABC are represented by the three vectors $\mathbf{u} = \overrightarrow{AB} = \langle -1, 1, 0 \rangle$, $\mathbf{v} = \overrightarrow{BC} = \langle -1, -3, -2 \rangle$, and $\mathbf{w} = \overrightarrow{AC} = \langle -2, -2, -2 \rangle$. Let A denote the angle of the triangle at vertex A. Then

$$\cos A = \frac{\mathbf{u} \cdot \mathbf{w}}{|\mathbf{u}| \cdot |\mathbf{w}|} = \frac{2 - 2}{\sqrt{2}\,\sqrt{12}} = 0,$$

and therefore $A = 90°$ (exactly). Similarly,

$$\cos B = \frac{-\mathbf{u} \cdot \mathbf{v}}{|-\mathbf{u}| \cdot |\mathbf{v}|} = \frac{-1 + 3}{\sqrt{2}\,\sqrt{14}} = \frac{\sqrt{7}}{7} \approx 0.37796447,$$

and so $B \approx 67.792346° \approx 68°$. Finally,

$$\cos C = \frac{-\mathbf{v} \cdot (-\mathbf{w})}{|-\mathbf{v}| \cdot |-\mathbf{w}|} = \frac{2 + 6 + 4}{\sqrt{12}\,\sqrt{14}} = \frac{\sqrt{42}}{7} \approx 0.92582010,$$

and it follows that $C \approx 22.207654° \approx 22°$.

C12S02.049: If $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 5, 5 \rangle$ and its direction angles are α, β, and γ (as in Fig. 11.2.14), then

$$\cos\alpha = \frac{2}{|\mathbf{a}|} = \frac{2}{\sqrt{54}} = \frac{\sqrt{6}}{9} \approx 0.27216553,$$

so that $\alpha \approx 74.206830951736°$. Also

$$\cos\beta = \cos\gamma = \frac{5}{|\mathbf{a}|} = \frac{5}{\sqrt{54}} = \frac{5\sqrt{6}}{18} \approx 0.68041382,$$

and therefore $\beta = \gamma \approx 47.124011333364°$.

C12S02.050: If $\mathbf{a} = \overrightarrow{PQ} = \langle -1, 3, -6 \rangle$ and its direction angles are α, β, and γ (as in Fig. 12.2.14), then

$$\cos\alpha = \frac{-1}{|\mathbf{a}|} = -\frac{1}{\sqrt{46}} = -\frac{\sqrt{46}}{46} \approx -0.14744196,$$

so that $\alpha \approx 98.478713147086°$. Next,

$$\cos\beta = \frac{3}{|\mathbf{a}|} = \frac{3}{\sqrt{46}} = \frac{3\sqrt{46}}{46} \approx 0.44232587,$$

and therefore $\beta \approx 63.747624882279°$. Finally,

$$\cos\gamma = \frac{-6}{|\mathbf{a}|} = -\frac{6}{\sqrt{46}} = -\frac{6\sqrt{46}}{46} \approx -0.88465174,$$

and so $\gamma \approx 152.208694355221°$.

C12S02.051: If $\mathbf{a} = \overrightarrow{PQ} = \langle 6, 8, 10 \rangle$ and its direction angles are α, β, and γ (as in Fig. 12.2.14), then

$$\cos\alpha = \frac{6}{|\mathbf{a}|} = \frac{6}{\sqrt{200}} = \frac{3\sqrt{2}}{10} \approx 0.42426407,$$

so that $\alpha \approx 64.895909749779°$. Next,

$$\cos\beta = \frac{8}{|\mathbf{a}|} = \frac{8}{\sqrt{200}} = \frac{2\sqrt{2}}{5} \approx 0.56568543,$$

and therefore $\beta \approx 55.550098012047°$. Finally,

$$\cos\gamma = \frac{10}{|\mathbf{a}|} = \frac{10}{\sqrt{200}} = \frac{\sqrt{2}}{2} \approx 0.70710678,$$

and so $\gamma = 45°$ (exactly).

C12S02.052: If $\mathbf{a} = \overrightarrow{PQ} = \langle 5, 12, 13 \rangle$ and its direction angles are α, β, and γ (as in Fig. 12.2.14), then

$$\cos\alpha = \frac{5}{|\mathbf{a}|} = \frac{5}{\sqrt{338}} = \frac{5\sqrt{2}}{26} \approx 0.27196415,$$

so that $\alpha \approx 74.218821492928°$. Next,

$$\cos\beta = \frac{12}{|\mathbf{a}|} = \frac{12}{\sqrt{338}} = \frac{6\sqrt{2}}{13} \approx 0.65271395,$$

and therefore $\beta \approx 49.253463877931°$. Finally,

$$\cos\gamma = \frac{13}{|\mathbf{a}|} = \frac{13}{\sqrt{338}} = \frac{\sqrt{2}}{2} \approx 0.70710678,$$

and so $\gamma = 45°$ (exactly).

C12S02.053: The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = 3\mathbf{i} + \mathbf{j}$, and consequently the work done is simply $\mathbf{F} \cdot \mathbf{D} = (\mathbf{i} - \mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j}) = 3$.

C12S02.054: The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = -6\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}$, and consequently the work done is $\mathbf{F} \cdot \mathbf{D} = (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \cdot (-6\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}) = 48$.

C12S02.055: $40 \cdot (\cos 40^\circ) \cdot 1000 \cdot (0.239) \approx 7323.385$ (cal; less than 8 Cal).

C12S02.056: $200 \cdot (\sec 5^\circ) \cdot 10 \cdot \dfrac{22}{15} \cdot \dfrac{1}{550} \approx 5.353706$ (hp).

C12S02.057: Set up a coordinate system in which the lowest point of the inclined plane is at the origin in the xy-plane and its highest point is at $(x,\, h)$. Then

$$\frac{h}{x} = \tan\alpha, \qquad \text{so that} \qquad x = \frac{h}{\tan\alpha} = h\cot\alpha.$$

Therefore the displacement vector in this problem is $\mathbf{D} = \langle\, h\cot\alpha,\ h\,\rangle$. A unit vector parallel to the inclined plane is $\langle \cos\alpha,\ \sin\alpha\,\rangle$, so a unit vector in the direction of $\mathbf{N}$ is $\langle -\sin\alpha,\ \cos\alpha\,\rangle$ (by Problem 55 in Section 12.1). Therefore $\mathbf{N} = \lambda\langle -\sin\alpha,\ \cos\alpha\,\rangle$ where λ is a positive scalar. If we denote by F the magnitude $|\mathbf{F}|$ of $\mathbf{F}$, then

$$\mathbf{F} = \langle\, F\cos\alpha,\ F\sin\alpha\,\rangle.$$

The horizontal forces acting on the weight must balance, as must the vertical forces, yielding (respectively) the scalar equations

$$\lambda\sin\alpha = F\cos\alpha \qquad \text{and} \qquad \lambda\cot\alpha + F\sin\alpha = mg.$$

From the first of these equations we see that $\lambda = F\cos\alpha$, and substitution of this in the second equation yields

$$F \cdot \left(\frac{\cos^2\alpha}{\sin\alpha}\right) + F \cdot \left(\frac{\sin^2\alpha}{\sin\alpha}\right) = mg,$$

and it follows that $F = mg\sin\alpha$. Therefore $\mathbf{F} = \langle\, mg\sin\alpha\cos\alpha,\ mg\sin^2\alpha\,\rangle$. So the work done by $\mathbf{F}$ in moving the weight from the bottom to the top of the inclined plane is (if there's no friction)

$$W = \mathbf{F} \cdot \mathbf{D} = mgh\cos^2\alpha + mgh\sin^2\alpha = mgh.$$

C12S02.058: Because $|\cos\theta| \leqq 1$ for all θ, if θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot |\cos\theta| \leqq |\mathbf{a}| \cdot |\mathbf{b}|.$$

For an alternative proof that relies only on the definition of the dot product and does not use Theorem 1, suppose that

$$\mathbf{a} = \langle\, a_1,\ a_2\,\rangle \qquad \text{and} \qquad \mathbf{b} = \langle\, b_1,\ b_2\,\rangle.$$

Then

$$0 \leqq (a_1b_2 - a_2b_1)^2;$$

$$0 \leqq a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2;$$

$$a_1^2b_1^2 + 2a_1a_2b_1b_2 + a_2^2b_2^2 \leqq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2;$$

$$(a_1b_1 + a_1b_2)^2 \leqq (a_1^2 + a_2^2)(b_1^2 + b_2^2);$$

$$|a_1b_1 + a_2b_2| \leqq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2};$$

$$|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}| \cdot |\mathbf{b}|.$$

For a proof for three-dimensional vectors, begin with the inequality

$$0 \leqq (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2$$

and proceed in like manner.

C12S02.059: By the first equation in (9), we have

$$|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}).$$

Then we use the fact that $x \leqq |x|$ for each real number x (in the first line) and the Cauchy-Schwarz inequality of Problem 58 (in the second line):

$$\begin{aligned} |\mathbf{a} + \mathbf{b}|^2 = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 &\leqq |\mathbf{a}|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 \\ &\leqq |\mathbf{a}|^2 + 2|\mathbf{a}| \cdot |\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2. \end{aligned}$$

All these expression are nonnegative. Therefore, for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}|$.

C12S02.060: Following the *Suggestion*, we write

$$\mathbf{a} = (\mathbf{a} - \mathbf{b}) + \mathbf{b}, \quad \text{so that}$$

$$|\mathbf{a}| = |(\mathbf{a} - \mathbf{b}) + \mathbf{b}| \leqq |\mathbf{a} - \mathbf{b}| + |\mathbf{b}|.$$

Therefore $|\mathbf{a}| - |\mathbf{b}| \leqq |\mathbf{a} - \mathbf{b}|$ for any two vectors $\mathbf{a}$ and $\mathbf{b}$.

C12S02.061: Suppose that $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and that $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. If both are the zero vector, let $\mathbf{w} = \mathbf{i}$. Otherwise we may suppose without loss of genrality that $\mathbf{u} \neq \mathbf{0}$. Indeed, we may further suppose without loss of generality that $u_1 \neq 0$. We need to choose $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ such that

$$\mathbf{u} \cdot \mathbf{w} = u_1w_1 + u_2w_2 + u_3w_3 = 0 \quad \text{and}$$

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = 0.$$

This will obtain provided that

$$u_1v_1w_1 + u_2v_1w_2 + u_3v_1w_3 = 0 \quad \text{and}$$

$$u_1v_1w_1 + u_1v_2w_2 + u_1v_3w_3 = 0.$$

Subtraction of the first of these equations from the second yields the sufficient condition

$$(u_1v_2 - u_2v_1)w_2 + (u_1v_3 - u_3v_1)w_3 = 0.$$

Thus our goal will be accomplished if we let

$$\begin{aligned} w_2 &= u_1v_3 - u_3v_1 \quad \text{and} \\ w_3 &= -(u_1v_2 - u_2v_1) = u_2v_1 - u_1v_2. \end{aligned}$$

Now let

$$w_1 = -\frac{u_2w_2 + u_3w_3}{u_1},$$

and we have reached our goal. In particular, if $\mathbf{u} = \langle 1, 2, -3 \rangle$ and $\mathbf{v} = \langle 2, 0, 1 \rangle$, then we obtain

$$\begin{aligned} w_2 &= 1 \cdot 1 + 3 \cdot 2 = 7, \\ w_3 &= 2 \cdot 2 - 1 \cdot 0 = 4, \quad \text{and} \\ w_1 &= -(2 \cdot 7 - 3 \cdot 4) = -2, \end{aligned}$$

and thus any nonzero multiple of $\mathbf{w} = \langle -2, 7, 4 \rangle$ is a correct answer to this problem. We will see a more efficient construction of a nonzero vector perpendicular to two given nonparallel vectors in Section 12.3.

C12S02.062: The edge of the cube on the x-axis coincides with the vector $\mathbf{i} = \langle 1, 0, 0 \rangle$. The diagonal of the cube coincides with the vector $\mathbf{a} = \overrightarrow{OP} = \langle 1, 1, 1 \rangle$. So the angle θ between them satisfies

$$\cos\theta = \frac{\mathbf{i} \cdot \mathbf{a}}{|\mathbf{i}| \cdot |\mathbf{a}|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Therefore $\theta = \arccos\left(\dfrac{\sqrt{3}}{3}\right) \approx 0.95531662$ (radians; approximately $54^\circ\ 44'\ 8.197''$).

A more interesting problem is to find the angle between the long diagonal of a four-dimensional unit hypercube and one of its incident edges. Then generalize the result to dimension n.

C12S02.063: Given: The points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Let $\mathbf{a} = \overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Then $\mathbf{a}$ coincides with the line segment joining P_1 with P_2. Hence if the initial point of $\frac{1}{2}\mathbf{a}$ is placed at P_1, then its terminal point will coincide with the midpoint of the segment P_1P_2. That is, $\overrightarrow{OP_1} + \frac{1}{2}\mathbf{a}$ will be the position vector of the midpoint of P_1P_2. But

$$\overrightarrow{OP_1} + \frac{1}{2}\mathbf{a} = \langle x_1, y_1, z_1 \rangle + \frac{1}{2}\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \left\langle \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right\rangle = \overrightarrow{OM}.$$

This is enough to establish that M is the midpoint of the segment P_1P_2.

C12S02.064: Place the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ so that they have the same initial point. Next,

$$\mathbf{c} = \frac{b\mathbf{a} + a\mathbf{b}}{a+b}$$

is a linear combination of $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{c}$ lies in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Let ϕ denote the angle between the vectors $\mathbf{a}$ and $\mathbf{c}$. Then

$$\cos\phi = \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}| \cdot |\mathbf{c}|};$$

moreover,

$$\mathbf{a} \cdot \mathbf{c} = \frac{\mathbf{a} \cdot (b\mathbf{a} + a\mathbf{b})}{a+b} = \frac{a^2 b + a^2 b \cos\theta}{a+b}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Hence

$$\mathbf{a} \cdot \mathbf{c} = \frac{a^2 b}{a+b}(1 + \cos\theta) = \frac{2a^2 b}{a+b} \cos^2\left(\frac{\theta}{2}\right).$$

But $|\mathbf{a}| = a$; moreover,

$$\begin{aligned} |\mathbf{c}| &= \frac{\sqrt{(b\mathbf{a} + a\mathbf{b}) \cdot (b\mathbf{a} + a\mathbf{b})}}{a+b} = \frac{\sqrt{b^2a^2 + 2ab(\mathbf{a} \cdot \mathbf{b}) + a^2b^2}}{a+b} \\ &= \frac{\sqrt{2a^2b^2 + 2a^2b^2\cos\theta}}{a+b} = \frac{ab}{a+b}\sqrt{4\left(\frac{1+\cos\theta}{2}\right)} = \frac{2ab}{a+b}\cos\left(\frac{\theta}{2}\right). \end{aligned}$$

Therefore

$$\cos\phi = 2 \cdot \frac{a^2 b}{a+b}\left[\cos^2\left(\frac{\theta}{2}\right)\right] \cdot \frac{a+b}{2a^2 b\cos(\theta/2)} = \cos\left(\frac{\theta}{2}\right).$$

It now follows that $\phi = \frac{1}{2}\theta$. Similarly, the angle between the vectors $\mathbf{b}$ and $\mathbf{c}$ is also equal to $\frac{1}{2}\theta$. Consequently $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

Alternative proof: Both $b\mathbf{a}$ and $a\mathbf{b}$ have length $|\mathbf{a}| \cdot |\mathbf{b}|$. So they form two adjacent sides of a rhombus. Their sum is a diagonal of the rhombus, and a theorem of geometry tells us that each diagonal of a rhombus bisects the angles at its ends.

C12S02.065: Suppose that the vectors $\mathbf{a}$ and $\mathbf{b}$ in the plane are nonzero and nonparallel. We will provide a proof in the case that the first components of $\mathbf{a}$ and $\mathbf{b}$ are nonzero. Let $\mathbf{a} = \langle a_1, a_2, \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$. To find α and β such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$, solve

$$\begin{aligned} c_1 &= \alpha a_1 + \beta b_1, \\ c_2 &= \alpha a_2 + \beta b_2; \end{aligned}$$

$$\begin{aligned} a_2 c_1 &= \alpha a_1 a_2 + \beta a_2 b_1, \\ a_1 c_2 &= \alpha a_1 a_2 + \beta a_1 b_2. \end{aligned}$$

Subtract the next-to-last equation from the last:

$$a_1c_2 - a_2c_1 = \beta(a_1b_2 - a_2b_1);$$

$$\beta = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

But what if the last denominator is zero? Then $a_1b_2 = a_2b_1$, so that

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = k,$$

and thus $a_2 = ka_1$ and $b_2 = kb_1$. Therefore

$$\mathbf{a} = \langle\, a_1,\, ka_1 \,\rangle = a_1\langle\, 1,\, k \,\rangle \quad \text{and}$$

$$\mathbf{b} = \langle\, b_1,\, kb_1 \,\rangle = b_1\langle\, 1,\, k \,\rangle,$$

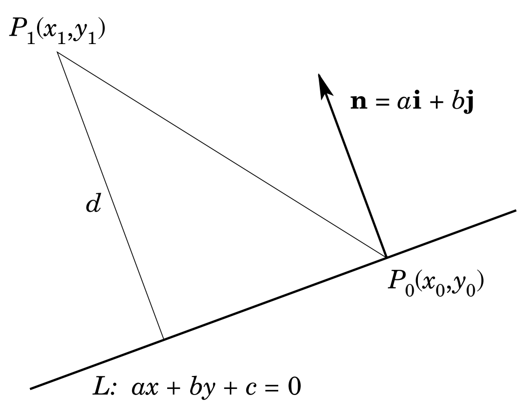
so that **a** and **b** are parallel. Hence $a_1b_2 - a_2b_1 \neq 0$.
In a similar fashion we find that

$$\alpha = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}.$$

All that remains is the verification that α and β perform as advertised:

$$\begin{aligned}
\alpha\mathbf{a} + \beta\mathbf{b} &= \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\langle\, a_1,\, a_2 \,\rangle + \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\langle\, b_1,\, b_2 \,\rangle \\
&= \frac{1}{a_1b_2 - a_2b_1}\big(\langle\, a_1b_2c_1 - a_1b_1c_2,\, a_2b_2c_1 - a_2b_1c_2 \,\rangle + \langle a_1b_1c_2 - a_2b_1c_1,\, a_1b_2c_2 - a_2b_2c_1 \,\rangle\big) \\
&= \frac{1}{a_1b_2 - a_2b_1}\langle\, (a_1b_2 - a_2b_1)c_1,\, (a_1b_2 - a_2b_1)c_2 \,\rangle = \langle\, c_1,\, c_2 \,\rangle = \mathbf{c}.
\end{aligned}$$

C12S02.066: See the following figure.



Given: d is the perpendicular distance from P_1 to L. Then

$$d = \left|\text{comp}_{\mathbf{n}}\left(\overrightarrow{P_0P_1}\right)\right| = \frac{|\,\mathbf{n}\cdot\overrightarrow{P_0P_1}\,|}{|\mathbf{n}|} = \frac{|\,ax_1 - ax_0 + by_1 - by_0\,|}{\sqrt{a^2+b^2}}.$$

Because P_0 is on L, $ax_0 + by_0 + c = 0$. Then the result in Problem 66 follows immediately.

C12S02.067: The equation

$$(x-3)^2+(y+2)^2+(z-4)^2=(x-5)^2+(y-7)^2+(z+1)^2$$

can be simplified to $2x+9y-5z=23$. This is an equation of the plane that bisects the segment AB and is perpendicular to it.

C12S02.068: First, $\overrightarrow{AP}=\langle\, x-1,\ y-3,\ z-5\,\rangle$. To express the condition that $\mathbf{n}$ and $\overrightarrow{AP}$ are perpendicular, simply write $\mathbf{n}\cdot\overrightarrow{AP}=0$. When simplified, we obtain $x-y+2z=8$. This is an equation of the plane through the point A perpendicular to the vector $\mathbf{n}$.

C12S02.069: The distance between the last two points listed in Problem 69 is

$$\sqrt{(1-0)^2+(0-1)^2+(1-1)^2}=\sqrt{2}\,,$$

and the other five distance computations also yield the same result. Hence the given points are the vertices of a regular tetrahedron. Let $\mathbf{a}$ be the position vector of $(1,\ 1,\ 0)$ and let $\mathbf{b}$ be the position vector of $(1,\ 0,\ 1)$. Then the angle θ between $\mathbf{a}$ and $\mathbf{b}$ satisfies the equation

$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}=\frac{1\cdot1+1\cdot0+0\cdot1}{\sqrt{2}\,\sqrt{2}}=\frac{1}{2}.$$

Therefore $\theta=\dfrac{\pi}{3}$.

C12S02.070: Let $\mathbf{a}$ be the vector with initial point the center of the molecule and terminal point $(0,\ 0,\ 0)$ and let $\mathbf{b}$ be the vector with initial point the center of the molecule and terminal point $(1,\ 1,\ 0)$. Double these two vectors to avoid fractions. Thus we use instead $2\mathbf{a}=\langle\,-1,\ -1,\ -1\,\rangle$ and $2\mathbf{b}=\langle\,1,\ 1,\ -1\,\rangle$. Then the bond angle α satisfies the equation

$$\cos\alpha=\frac{(2\mathbf{a})\cdot(2\mathbf{b})}{|2\mathbf{a}|\cdot|2\mathbf{b}|}=\frac{-1-1+1}{\sqrt{3}\,\sqrt{3}}=-\frac{1}{3}.$$

Therefore $\alpha\approx109.471220634491^\circ$; that is, approximately $109^\circ\ 28'\ 16.3943''$.

Section 12.3

C12S03.001: If $\mathbf{a} = \langle 5, -1, -2 \rangle$ and $\mathbf{b} = \langle -3, 2, 4 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & -2 \\ -3 & 2 & 4 \end{vmatrix} = \langle -4+4,\ 6-20,\ 10-3 \rangle = \langle 0, -14, 7 \rangle.$$

C12S03.002: If $\mathbf{a} = \langle 3, -2, 0 \rangle$ and $\mathbf{b} = \langle 0, 3, -2 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 0 & 3 & -2 \end{vmatrix} = \langle 4-0,\ 0-(-6),\ 9-0 \rangle = \langle 4, 6, 9 \rangle.$$

C12S03.003: If $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -2 & 3 & 1 \end{vmatrix} = (-1-9)\mathbf{i} + (-6-1)\mathbf{j} + (3-2)\mathbf{k} = -10\mathbf{i} - 7\mathbf{j} + \mathbf{k}.$$

C12S03.004: If $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & -2 \\ 2 & -5 & 5 \end{vmatrix} = (10-10)\mathbf{i} + (-4-20)\mathbf{j} + (-20-4)\mathbf{k} = -24\mathbf{j} - 24\mathbf{k}.$$

C12S03.005: If $\mathbf{a} = \langle 2, -3, 0 \rangle$ and $\mathbf{b} = \langle 4, 5, 0 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 4 & 5 & 0 \end{vmatrix} = \langle 0-0,\ 0-0,\ 10+12 \rangle = \langle 0, 0, 22 \rangle.$$

C12S03.006: If $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i} - 11\mathbf{j}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 2 & 0 \\ 7 & -11 & 0 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (55-14)\mathbf{k} = 41\mathbf{k}.$$

C12S03.007: If $\mathbf{a} = \langle 3, 12, 0 \rangle$ and $\mathbf{b} = \langle 0, 4, 3 \rangle$, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 12 & 0 \\ 0 & 4 & 3 \end{vmatrix} = \langle\, 36-0,\ 0-9,\ 12-0\,\rangle = \langle\, 36,\ -9,\ 12\,\rangle.$$

The magnitude of $\mathbf{a}\times\mathbf{b}$ is

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{36^2+9^2+12^2} = \sqrt{1521} = 39,$$

so the two unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ are

$$\mathbf{u} = \frac{1}{39}\langle\, 36,\ -9,\ 12\,\rangle = \left\langle \frac{12}{13},\ -\frac{3}{13},\ \frac{4}{13} \right\rangle \quad \text{and} \quad \mathbf{v} = -\frac{1}{39}\langle\, 36,\ -9,\ 12\,\rangle = \left\langle -\frac{12}{13},\ \frac{3}{13},\ -\frac{4}{13} \right\rangle.$$

C12S03.008: If $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = (10-9)\mathbf{i}+(6-5)\mathbf{j}+(3-4)\mathbf{k} = \mathbf{i}+\mathbf{j}-\mathbf{k}.$$

The magnitude of the cross product is

$$|\mathbf{i}+\mathbf{j}-\mathbf{k}| = \sqrt{1+1+1} = \sqrt{3},$$

so the two unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ are

$$\mathbf{u} = \frac{\sqrt{3}}{3}(\mathbf{i}+\mathbf{j}-\mathbf{k}) = \frac{\sqrt{3}}{3}\mathbf{i}+\frac{\sqrt{3}}{3}\mathbf{j}-\frac{\sqrt{3}}{3}\mathbf{k} \quad \text{and} \quad \mathbf{v} = -\frac{\sqrt{3}}{3}(\mathbf{i}+\mathbf{j}-\mathbf{k}) = -\frac{\sqrt{3}}{3}\mathbf{i}-\frac{\sqrt{3}}{3}\mathbf{j}+\frac{\sqrt{3}}{3}\mathbf{k}.$$

C12S03.009: The definition of the cross product in Eq. (5) yields

$$\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0-0)\mathbf{i}+(0-0)\mathbf{j}+(1-0)\mathbf{k} = \mathbf{k},$$

$$\mathbf{j}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1-0)\mathbf{i}+(0-0)\mathbf{j}+(0-0)\mathbf{k} = \mathbf{i}, \qquad \text{and}$$

$$\mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (0-0)\mathbf{i}+(1-0)\mathbf{j}+(0-0)\mathbf{k} = \mathbf{j}.$$

C12S03.010: The definition of the cross product in Eq. (5) yields

$$\mathbf{j}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-1)\mathbf{k} = -\mathbf{k},$$

$$\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (0-1)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = -\mathbf{i}, \qquad \text{and}$$

$$\mathbf{i}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0-0)\mathbf{i} + (0-1)\mathbf{j} + (0-0)\mathbf{k} = -\mathbf{j}.$$

C12S03.011: If $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}$, and $\mathbf{c} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, then

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{a}\times(\mathbf{i}-\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -\mathbf{k},$$

whereas

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \times\mathbf{c} = \mathbf{k}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -\mathbf{i}+\mathbf{j}.$$

Thus the vector product is not associative.

C12S03.012: Let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = 2\mathbf{i}$, and $\mathbf{c} = 3\mathbf{i}$. Then $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$ because both products are $\mathbf{0}$ (because $\mathbf{a}$ is parallel to $\mathbf{b}$ and $\mathbf{a}$ is parallel to $\mathbf{c}$). But $\mathbf{b} \neq \mathbf{c}$. Thus "cancellation" involving the vector product may produce invalid results.

C12S03.013: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular, note that $\mathbf{b}\times\mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$, hence $\mathbf{b}\times\mathbf{c}$ is parallel to $\mathbf{a}$. Therefore $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{0}$.

C12S03.014: By Eq. (10) (and the discussion that precedes it and follows it), the area of triangle PQR is $A = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$. But

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = -\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

Therefore the area of triangle PQR is $A = \frac{1}{2}\sqrt{3}$.

C12S03.015: By Eq. (10) (and the discussion that precedes and follows it), the area of triangle PQR is $A = \frac{1}{2}|\overrightarrow{PQ} + \overrightarrow{PR}|$. But

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 7 \\ -4 & -5 & 4 \end{vmatrix} = 39\mathbf{i} - 32\mathbf{j} - \mathbf{k}.$$

The magnitude of $39\mathbf{i} - 32\mathbf{j} - \mathbf{k}$ is

$$|39\mathbf{i} - 32\mathbf{j} - \mathbf{k}| = \sqrt{1521 + 1024 + 1} = \sqrt{2546}\,.$$

Therefore the area of triangle PQR is $A = \frac{1}{2}\sqrt{2546} \approx 25.2289516231$.

C12S03.016: Let $\mathbf{a} = \overrightarrow{OP} = \langle 1, 1, 0 \rangle$, $\mathbf{b} = \overrightarrow{OQ} = \langle 1, 0, 1 \rangle$, and $\mathbf{c} = \overrightarrow{OR} = \langle 0, 1, 1 \rangle$. Then the volume V of the parallelepiped having these vectors as adjacent edges is given by the scalar triple product in Theorem 4: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$. Here we have

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} + \mathbf{k} = \langle -1, -1, 1 \rangle.$$

Therefore $V = |\langle 1, 1, 0 \rangle \cdot \langle -1, -1, 1 \rangle| = |-1 - 1 + 0| = 2$.

C12S03.017: Let $\mathbf{a} = \overrightarrow{OP} = \langle 1, 3, -2 \rangle$, $\mathbf{b} = \overrightarrow{OQ} = \langle 2, 4, 5 \rangle$, and $\mathbf{c} = \overrightarrow{OR} = \langle -3, -2, 2 \rangle$. Part (a): By Theorem 4 and part (4) of Theorem 3, the volume of the parallelepiped with adjacent edges these three vectors is given by $V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$. Here we have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & 4 & 5 \end{vmatrix} = \langle 23, -9, -2 \rangle,$$

and therefore $V = |\langle 23, -9, -2 \rangle \cdot \langle -3, -2, 2 \rangle| = |-55| = 55$.

Part (b): By Example 7 of Section 12.3, the volume of the pyramid is $\dfrac{55}{6}$.

C12S03.018: A vector $\mathbf{n}$ normal to the plane containing $P(1, 3, -2)$, $Q(2, 4, 5)$, and $R(-3, -2, 2)$ is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 7 \\ -4 & -5 & 4 \end{vmatrix} = \langle 39, -32, -1 \rangle.$$

The magnitude of $\mathbf{n}$ is $\sqrt{2546}$, so a unit vector normal to the plane is

$$\mathbf{u} = \frac{1}{\sqrt{2546}}\mathbf{n} = \frac{\sqrt{2546}}{2546}\langle 39, -32, -1 \rangle.$$

Then the (perpendicular) distance from the origin to this plane is

$$-\mathbf{n}\cdot\overrightarrow{OP} = -\frac{\sqrt{2546}}{2546}\langle\, 39,\ -32,\ -1\,\rangle\cdot\langle\, 1,\ 3,\ -2\,\rangle = \frac{55\sqrt{2546}}{2546} \approx 1.090017548523.$$

C12S03.019: Here we have $\overrightarrow{AB} = \langle\, 2,\ 1,\ 3\,\rangle$, $\overrightarrow{AC} = \langle\, 1,\ -3,\ 0\,\rangle$, and $\overrightarrow{AD} = \langle\, 3,\ 5,\ 6\,\rangle$. By Theorem 4 and Eq. (17), the volume of the parallelepiped having these vectors as adjacent sides is the absolute value of

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -3 & 0 \\ 3 & 5 & 6 \end{vmatrix} = (-1)\cdot(6-15) - 3\cdot(12-9) = 9-9 = 0$$

(we expanded the determinant using its second row; see the discussion following Example 1 of the text). By the reasoning given in Example 8, the four given points are coplanar.

C12S03.020: Here we have $\overrightarrow{AB} = \langle\, 12,\ 11,\ 15\,\rangle$, $\overrightarrow{AC} = \langle\, 11,\ -13,\ 12\,\rangle$, and $\overrightarrow{AD} = \langle\, 13,\ 35,\ 18\,\rangle$. By Theorem 4 and Eq. (17), the volume of the parallelepiped having these vectors as adjacent sides is the absolute value of

$$\begin{vmatrix} 12 & 11 & 15 \\ 11 & -13 & 12 \\ 13 & 35 & 18 \end{vmatrix} = 12\cdot(-234-420) - 11\cdot(198-156) + 15\cdot(385+169) = -7848 - 462 + 8310 = 0$$

(we expanded the determinant using its first row; see the discussion following Example 1 of the text). By the reasoning given in Example 8, the four given points are coplanar.

C12S03.021: Here we have $\overrightarrow{AB} = \langle\, 1,\ 2,\ 3\,\rangle$, $\overrightarrow{AC} = \langle\, 2,\ 3,\ 4\,\rangle$, and $\overrightarrow{AD} = \langle\, 9,\ 12,\ 21\,\rangle$. By Theorem 4 and Eq. (17), the volume of the parallelepiped having these vectors as adjacent sides is the absolute value of

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 9 & 12 & 21 \end{vmatrix} = 1\cdot(63-48) - 2\cdot(42-36) + 3\cdot(24-27) = 15 - 12 - 9 = -6$$

(we expanded the determinant using its first row; see the discussion following Example 1 of the text). Therefore the four given points are not coplanar. The parallelepiped having the three vectors as adjacent sides has volume 6 and the tetrahedron (pyramid) with the four given points as its vertices has volume 1.

C12S03.022: Here we have $\overrightarrow{AB} = \langle\, 11,\ 12,\ 13\,\rangle$, $\overrightarrow{AC} = \langle\, 2,\ 3,\ 4\,\rangle$, and $\overrightarrow{AD} = \langle\, 9,\ 12,\ 21\,\rangle$. By Theorem 4 and Eq. (17), the volume of the parallelepiped having these vectors as adjacent sides is the absolute value of

$$\begin{vmatrix} 11 & 12 & 13 \\ 2 & 3 & 4 \\ 9 & 12 & 21 \end{vmatrix} = 11\cdot(63-48) - 12\cdot(42-36) + 13\cdot(24-27) = 165 - 72 - 39 = 54$$

(we expanded the determinant using its first row; see the discussion following Example 1 of the text). By Theorem 4, the four given points are not coplanar. The tetrahedron with those four points as vertices has volume $\frac{54}{6} = 9$.

C12S03.023: Name the vertices of the polygon counterclockwise, beginning at the origin: O, A, and B. Then

$$\mathbf{a} = \overrightarrow{OA} = \langle\, 176\cos 15^\circ,\ 176\sin 15^\circ,\ 0\,\rangle \approx \langle\, 170.002945,\ 45.552152,\ 0\,\rangle \quad \text{and}$$

$$\mathbf{b} = \overrightarrow{OB} = \langle\, 83\cos 52^\circ,\ 83\sin 52^\circ,\ 0\,\rangle \approx \langle\, 51.099902,\ 65.404893,\ 0\,\rangle.$$

Using the exact rather than the approximate values, we entered the *Mathematica* 3.0 commands

```
c = Cross[a,b]   (* to compute the vector product of a and b *)
area = (1/2)*Sqrt[c.c]   (* to compute half the magnitude of c *)
N[area,20]   (* to print a 20-place approximation to the area *)
```

The response to the last command rounds to the answer, $4395.6569291026\ \ (\text{m}^2)$.

C12S03.024: Name the vertices of the polygon counterclockwise, beginning at the origin: O, A, and B. Then

$$\mathbf{a} = \overrightarrow{OA} = \langle\, 255\cos 27^\circ,\ 255\sin 27^\circ,\ 0\,\rangle \approx \langle\, 227.206664,\ 115.767577,\ 0\,\rangle \quad \text{and}$$

$$\mathbf{b} = \overrightarrow{OB} = \mathbf{a} - \langle\, 225\cos 9^\circ,\ 225\sin 9^\circ,\ 0\,\rangle \approx \langle\, 4.976787,\ 80.569823,\ 0\,\rangle.$$

Using the exact rather than the approximate values, we entered the *Mathematica* 3.0 commands

```
c = Cross[a,b]   (* to compute the vector product of a and b *)
area = (1/2)*Sqrt[c.c]   (* to compute half the magnitude of c *)
N[area,20]   (* to print a 20-place approximation to the area *)
```

The response to the last command rounds to the answer, $8864.925026\ \ (\text{ft}^2)$.

C12S03.025: Name the vertices of the polygon counterclockwise, beginning at the origin: O, A, B, and C. Then

$$\mathbf{a} = \overrightarrow{OA} = \langle\, 220\cos 25^\circ,\ 220\sin 25^\circ,\ 0\,\rangle \approx \langle\, 227.206664,\ 115.767577,\ 0\,\rangle,$$

$$\mathbf{b} = \overrightarrow{OB} = \mathbf{a} + \langle\, -210\cos 40^\circ,\ 210\sin 40^\circ,\ 0\,\rangle \approx \langle\, 38.518380,\ 227.961416,\ 0\,\rangle, \quad \text{and}$$

$$\mathbf{c} = \overrightarrow{OC} = \langle\, -150\cos 63^\circ,\ 150\sin 63^\circ,\ 0\,\rangle \approx \langle\, -68.098575,\ 133.650979,\ 0\,\rangle.$$

Using the exact rather than the approximate values, we entered the *Mathematica* 3.0 commands

```
p = Cross[a,b]   (* to compute the vector product of a and b *)
area1 = (1/2)*Sqrt[p.p]   (* to compute half the magnitude of p *)
q = Cross[b,c]
area2 = (1/2)*Sqrt[q.q]
```

```
N[area1 + area2, 20]   (* to print a 20-place approximation to the total area *)
```

The response to the last command rounds to the answer, 31271.643253 (ft^2).

C12S03.026: Name the vertices of the polygon counterclockwise, beginning at the origin: O, A, B, C, and D. Then

$$\mathbf{a} = \overrightarrow{OA} = \langle\, 175,\, 0,\, 0 \,\rangle,$$

$$\mathbf{b} = \overrightarrow{OB} = \mathbf{a} + \langle\, 200\cos 50^\circ,\ 200\sin 50^\circ,\ 0 \,\rangle \approx \langle\, 303.557522,\ 153.208889,\ 0 \,\rangle,$$

$$\mathbf{c} = \overrightarrow{OC} = \mathbf{b} + \langle\, -150\cos 40^\circ,\ 150\sin 40^\circ,\ 0 \,\rangle \approx \langle\, 188.650855,\ 249.627030,\ 0 \,\rangle, \qquad \text{and}$$

$$\mathbf{d} = \overrightarrow{OD} = \langle\, -125\cos 70^\circ,\ 125\sin 70^\circ,\ 0 \,\rangle \approx \langle\, -42.752518,\ 117.461578,\ 0 \,\rangle.$$

Using the exact rather than the approximate values, we entered the *Mathematica* 3.0 commands

```
p = Cross[a,b]   (* to compute the vector product of a and b *)
area1 = (1/2)*Sqrt[p.p]   (* to compute half the magnitude of p *)
q = Cross[b,c]
area2 = (1/2)*Sqrt[q.q]
r = Cross[c,d]
area3 = (1/2)*Sqrt[r.r]
N[area1 + area2 + area3, 20]   (* to print a 20-place approximation to total area *)
```

The response to the last command rounds to the answer, 53258.070719 (m^2).

C12S03.027: Let $\mathbf{a} = \langle\, a_1,\, a_2,\, a_3 \,\rangle$ and $\mathbf{b} = \langle\, b_1,\, b_2,\, b_3 \,\rangle$. Then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle\, a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3\ \ a_1b_2 - a_2b_1 \,\rangle \qquad \text{and}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \langle\, b_2a_3 - b_3a_2,\ b_3a_1 - b_1a_3\ \ b_1a_2 - b_2a_1 \,\rangle$$

$$= -\langle\, a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \,\rangle = -(\mathbf{a}\times\mathbf{b}).$$

C12S03.028: Let $\mathbf{a} = \langle\, a_1,\, a_2,\, a_3 \,\rangle$, $\mathbf{b} = \langle\, b_1,\, b_2,\, b_3 \,\rangle$, and $\mathbf{c} = \langle\, c_1,\, c_2,\, c_3 \,\rangle$. Then

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot\, \mathbf{c} = \langle\, a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \,\rangle \cdot \langle\, c_1,\, c_2,\, c_3 \,\rangle$$

$$= c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1).$$

The last expression is the expansion of the determinant in Eq. (17) along its third row (see the discussion following Eq. (3) of the text), and therefore we have shown that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for all three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

C12S03.029: Part (a): Please refer to Fig. 12.3.13 of the text. Because the segment of length d is perpendicular to the segment PQ, we know from elementary geometry that the area of the triangle is $\frac{1}{2}|\overrightarrow{PQ}| \cdot d$. Therefore

$$\frac{1}{2}|\overrightarrow{AP} \times \overrightarrow{AQ}| = \frac{1}{2}|\overrightarrow{PQ}| \cdot d, \quad \text{and thus} \quad d = \frac{|\overrightarrow{AP} \times \overrightarrow{AQ}|}{|\overrightarrow{PQ}|}. \tag{1}$$

Part (b): With $A = (1, 0, 1)$, $P = (2, 3, 1)$, and $Q = (-3, 1, 4)$, we have

$$\overrightarrow{AP} = \langle 1, 3, 0 \rangle, \quad \overrightarrow{AQ} = \langle -4, 1, 3 \rangle, \quad \text{and} \quad \overrightarrow{PQ} = \langle -5, -2, 3 \rangle.$$

To use the formula in (1), we first compute

$$\overrightarrow{AP} \times \overrightarrow{AQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ -4 & 1 & 3 \end{vmatrix} = \langle 9 - 0, \ 0 - 3, \ 1 - (-12) \rangle = \langle 9, -3, 13 \rangle.$$

Then the formula in (1) implies that the distance from the given point to the given line is

$$d = \frac{|\overrightarrow{AP} \times \overrightarrow{AQ}|}{|\overrightarrow{PQ}|} = \frac{|\langle 9, -3, 13 \rangle|}{\sqrt{25 + 4 + 9}} = \frac{\sqrt{259}}{\sqrt{38}} = \frac{\sqrt{9842}}{38} \approx 2.61070670.$$

C12S03.030: Let d be the perpendicular distance from A to the plane determined by the three (non-collinear) points P, Q, and R. Then the volume of the pyramid $APQR$ is one-third the product of the area of its base and its height, but it is also one-sixth the volume of the parallelepiped determined by the three vectors $\overrightarrow{AP}$, $\overrightarrow{AQ}$, and $\overrightarrow{AR}$. The area of the (triangular) base of the pyramid is half the magnitude of the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and—putting this all together—we have

$$\frac{1}{3} \cdot d \cdot \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{6}|\overrightarrow{AP} \cdot (\overrightarrow{AQ} \times \overrightarrow{AR})|, \quad \text{and thus} \quad d = \frac{|\overrightarrow{AP} \cdot (\overrightarrow{AQ} \times \overrightarrow{AR})|}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}. \tag{1}$$

Next, to find the distance d from the point $A(1, 0, 1)$ to the plane $\mathcal{P}$ determined by the three points $P(2, 3, 1)$, $Q(3, -1, 4)$, and $R(0, 0, 2)$, we first compute

$$\overrightarrow{PQ} = \langle 1, -4, 3 \rangle, \quad \overrightarrow{PR} = \langle -2, -3, 1 \rangle, \quad \overrightarrow{AP} = \langle 1, 3, 0 \rangle, \quad \overrightarrow{AQ} = \langle 2, -1, 3 \rangle, \quad \text{and} \quad \overrightarrow{AR} = \langle -1, 0, 1 \rangle.$$

Then by Eq. (17) of the text, we have

$$\overrightarrow{AP} \cdot (\overrightarrow{AQ} \times \overrightarrow{AR}) = \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 3 \\ -1 & 0 & 1 \end{vmatrix} = 1 \cdot (-1 - 0) - 3 \cdot (2 + 3) = -1 - 15 = -16$$

(we evaluated the determinant by expansion along its first row). Next,

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ -2 & -3 & 1 \end{vmatrix} = \langle -4+9,\ -6-1,\ -3-8 \rangle = \langle 5,\ -7,\ -11 \rangle$$

so that $|\overrightarrow{PQ}\times\overrightarrow{PR}| = \sqrt{25+49+121} = \sqrt{195}$. Thus by the formula in (1),

$$d = \frac{|-16|}{\sqrt{195}} = \frac{16\sqrt{195}}{195} \approx 1.14578380.$$

C12S03.031: Let $\mathbf{u} = \overrightarrow{P_1Q_1}$, $\mathbf{v} = \overrightarrow{P_2Q_2}$, and $\mathbf{w} = \overrightarrow{P_1P_2}$. A vector $\mathbf{n}$ that is perpendicular to both lines is one perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and an obvious choice is $\mathbf{n} = \mathbf{u}\times\mathbf{v}$ ($\mathbf{n}$ will be nonzero because the two lines are not parallel). The projection of $\mathbf{w}$ in the direction of $\mathbf{n}$ will be $\pm d$ where d is the (perpendicular) distance between the two lines. Thus

$$d = \frac{|\mathbf{w}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|\overrightarrow{P_1P_2}\cdot(\overrightarrow{P_1Q_1}\times\overrightarrow{P_2Q_2})|}{|\overrightarrow{P_1Q_1}\times\overrightarrow{P_2Q_2}|}.$$

Comment: Rather than memorizing formulas such as those in Problems 29, 30, and 31, one should learn the general technique for finding the distance between two objects (points, lines, or planes) in space. First find a vector $\mathbf{n}$ normal to both objects. Then construct another vector $\mathbf{c}$ "connecting" the two objects; that is, with initial point on one and terminal point on the other. Then the projection of $\mathbf{c}$ in the direction of $\mathbf{n}$ is the (perpendicular) distance between the two objects (or its negative); that is, the distance d is given by the simple formula

$$d = \frac{|\mathbf{c}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

C12S03.032: Part (a): Because $\mathbf{I}$ is parallel to $\mathbf{a}$, there exists a scalar a_1 such that $\mathbf{a} = a_1\mathbf{I}$. Because $\mathbf{b}$ lies in the plane determined by $\mathbf{I}$ and $\mathbf{J}$, if b_1 is the projection of $\mathbf{b}$ in the direction of $\mathbf{I}$ and b_2 is the projection of $\mathbf{b}$ in the direction of $\mathbf{J}$, it follows immediately that $\mathbf{b} = b_1\mathbf{I} + b_2\mathbf{J}$. Because every vector is a linear combination of $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$, if we let c_1 be the projection of $\mathbf{c}$ in the direction of $\mathbf{I}$, c_2 be the projection of $\mathbf{c}$ in the direction of $\mathbf{J}$, and c_3 be the projection of $\mathbf{c}$ in the direction of $\mathbf{K}$, it then follows that $\mathbf{c} = c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K}$.

Part (b): Because $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ form a right-handed triple of unit vectors, we have much the same results as expressed in the equations in (11):

$$\begin{aligned} &\mathbf{I}\times\mathbf{J} = \mathbf{K}, \qquad &&\mathbf{J}\times\mathbf{K} = \mathbf{I}, \qquad &&\mathbf{K}\times\mathbf{I} = \mathbf{J}, \\ &\mathbf{J}\times\mathbf{I} = -\mathbf{K}, \qquad &&\mathbf{K}\times\mathbf{J} = -\mathbf{I}, \quad \text{and} \quad &&\mathbf{I}\times\mathbf{K} = -\mathbf{J}. \end{aligned}$$

Thus

$$\mathbf{a}\times\mathbf{b} = a_1\mathbf{I}\times(b_1\mathbf{I} + b_2\mathbf{J}) = a_1b_1(\mathbf{I}\times\mathbf{I}) + a_1b_2(\mathbf{I}\times\mathbf{J}) = a_1b_2\mathbf{K}.$$

It now follows that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = a_1b_2\mathbf{K}\times(c_1\mathbf{I}+c_2\mathbf{J}+c_3\mathbf{K}) = a_1b_2c_1(\mathbf{K}\times\mathbf{I}) + a_1b_2c_2(\mathbf{K}\times\mathbf{J}) = a_1b_2c_1\mathbf{J} - a_1b_2c_2\mathbf{I}.$$

Part (c): Note that $a_1\mathbf{I} = \mathbf{a}$ and that $b_2\mathbf{J} = \mathbf{b} - b_1\mathbf{I}$. Thus

$$a_1b_2\mathbf{J} = a_1\mathbf{b} - a_1b_1\mathbf{I} = a_1\mathbf{b} - b_1\mathbf{a}.$$

(We are taking a little care here to avoid division by a_1 because it might be zero.) Thus

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = a_1b_2c_1\mathbf{J} - a_1b_2c_2\mathbf{I} = c_1(a_1\mathbf{b} - b_1\mathbf{a}) - b_2c_2\mathbf{a} = a_1c_1\mathbf{b} - (b_1c_1 + b_2c_2)\mathbf{a} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

C12S03.033: Given the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in space, we have

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -(\mathbf{b}\times\mathbf{c})\times\mathbf{a} \quad \text{(by Eq. (12))}\\ &= -\left[(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}\right] \quad \text{(by the result in Problem 32)}\\ &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \quad \text{(this establishes Eq. (16)).}\end{aligned}$$

C12S03.034: Note that $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is perpendicular to $\mathbf{a}\times\mathbf{b}$ and the latter is perpendicular to $\mathbf{a}$ and to $\mathbf{b}$. Therefore $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ lies in the plane determined by $\mathbf{a}$ and $\mathbf{b}$. Consequently

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) \;=\; r_1\mathbf{a} + r_2\mathbf{b}$$

for some scalars r_1 and r_2. (If $\mathbf{a}$ and $\mathbf{b}$ are parallel—so that there is no plane determined by $\mathbf{a}$ and $\mathbf{b}$—then $\mathbf{a}\times\mathbf{b} = \mathbf{0}$; in this case, simply choose $r_1 = r_2 = 0$.) The other half of the proof follows by a similar argument (or by interchanging $\mathbf{a}$ with $\mathbf{c}$ and $\mathbf{b}$ with $\mathbf{d}$ in the preceding proof).

C12S03.035: Given: The triangle in the plane with vertices at $P(x_1,\,y_1,\,0)$, $Q(x_2,\,y_2,\,0)$, and $R(x_3,\,y_3,\,0)$. Let

$$\mathbf{u} = \overrightarrow{PQ} = \langle\, x_2 - x_1,\; y_2 - y_1,\; 0\,\rangle \quad\text{ and }\quad \mathbf{v} = \overrightarrow{PR} = \langle\, x_3 - x_1,\; y_3 - y_1,\; 0\,\rangle.$$

The area of the triangle is $A = \frac{1}{2}\,|\mathbf{u}\times\mathbf{v}|$. Now

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \langle\, 0,\; 0,\; (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)\,\rangle,$$

so that

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}| &= |\,x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 - x_3y_2 + x_3y_1 + x_1y_2 - x_1y_1\,|\\ &= |\,x_2y_3 - x_2y_1 - x_1y_3 - x_3y_2 + x_3y_1 + x_1y_2\,|\\ &= |\,(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\,|,\end{aligned}$$

which is the *absolute value* of the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{1}$$

(expanded using its first column—see the discussion following Eq. (3) in the text). Thus the area of triangle PQR is half the absolute value of the determinant in (1).

C12S03.036: We let *Mathematica* 3.0 work this problem. The command `Cross[u,v]` returns the vector product of $\mathbf{u}$ and $\mathbf{v}$. The command `u.v` returns the scalar (dot) product of $\mathbf{u}$ and $\mathbf{v}$. Finally, the vector $\langle x, y, z \rangle$ is expressed in *Mathematica* as the ordered list `{x,y,z}`. Here's a transcript of the *Mathematica* session.

```
a = {a1, a2, a3}; b = {b1, b2, b3};    (* semicolons suppress output *)
c = Cross[a,b]
```

$$\{-a_3b_2 + a_2b_3,\ a_3b_1 - a_1b_3,\ -a_2b_1 + a_1b_2\}$$

```
lhs = c.c    (* ``left-hand side'' of the identity in Problem 36 *)
```

$$(-a_2b_1 + a_1b_2)^2 + (a_3b_1 - a_1b_3)^2 + (-a_3b_2 + a_2b_3)^2$$

```
rhs = (a.a)*(b.b) - (a.b)∧2
```

$$-(a_1b_1 + a_2b_2 + a_3b_3)^2 - (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

```
Simplify[lhs - rhs]
```

$$0$$

Section 12.4

C12S04.001: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = t\mathbf{v} + \overrightarrow{OP}$ yields

$$\langle\, x,\ y,\ z\,\rangle = t\langle\, 1,\ 2,\ 3\,\rangle + \langle\, 0,\ 0,\ 0\,\rangle = \langle\, t,\ 2t,\ 3t\,\rangle,$$

so that the parametric equations of the line are $x = t,\ y = 2t,\ z = 3t$.

C12S04.002: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = t\mathbf{v} + \overrightarrow{OP}$ yields

$$\langle\, x,\ y,\ z\,\rangle = t\langle\, -2,\ 7,\ 3\,\rangle + \langle\, 3,\ -4,\ 5\,\rangle = \langle\, -2t+3,\ 7t-4,\ 3t+5\,\rangle,$$

and thus the parametric equations of the line are $x = -2t+3,\ y = 7t-4,\ z = 3t+5$.

C12S04.003: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = t\mathbf{v} + \overrightarrow{OP}$ yields

$$\langle\, x,\ y,\ z\,\rangle = t\langle\, 2,\ 0,\ -3\,\rangle + \langle\, 4,\ 13,\ -3\,\rangle = \langle\, 2t+4,\ 13,\ -3t-3\,\rangle,$$

and thus the parametric equations of the line are $x = 2t+4,\ y = 13,\ z = -3t-3$.

C12S04.004: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = t\mathbf{v} + \overrightarrow{OP}$ yields

$$\langle\, x,\ y,\ z\,\rangle = t\langle\, -17,\ 13,\ 31\,\rangle + \langle\, 17,\ -13,\ -31\,\rangle = \langle\, -17t+17,\ 13t-13,\ 31t-31\,\rangle,$$

and thus the parametric equations of the line are $x = -17(t-1),\ y = 13(t-1),\ z = 31(t-1)$.

C12S04.005: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP_1} + t\,\overrightarrow{P_1P_2}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 0,\ 0,\ 0\,\rangle + t\langle\, -6,\ 3,\ 5\,\rangle,$$

so the line has parametric equations $x = -6t,\ y = 3t,\ z = 5t$.

C12S04.006: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP_1} + t\,\overrightarrow{P_1P_2}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 3,\ 5,\ 7\,\rangle + t\langle\, 3,\ -13,\ 3\,\rangle = \langle\, 3+3t,\ 5-13t,\ 7+3t\,\rangle$$

so the line has parametric equations $x = 3t+3,\ y = -13t+5,\ z = 3t+7$.

C12S04.007: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP_1} + t\,\overrightarrow{P_1P_2}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 3,\ 5,\ 7\,\rangle + t\langle\, 3,\ 0,\ -3\,\rangle = \langle\, 3+3t,\ 5,\ 7-3t\,\rangle$$

so the line has parametric equations $x = 3t+3,\ y = 5,\ z = -3t+7$.

C12S04.008: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP_1} + t\,\overrightarrow{P_1P_2}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 29,\ -47,\ 13\,\rangle + t\langle\, 44,\ 100,\ -80\,\rangle = \langle\, 29+44t,\ -47+100t,\ 13-80t\,\rangle$$

so the line has parametric equations $x = 44t+29,\ y = 100t-47,\ z = -80t+13$.

C12S04.009: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP} + t\mathbf{v}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 2,\ 3,\ -4\,\rangle + t\langle\, 1,\ -1,\ -2\,\rangle = \langle\, 2+t,\ 3-t,\ -4-2t\,\rangle$$

so the line has parametric equations $x = t+2,\ y = -t+3,\ z = -2t-4$ and symmetric equations

$$x-2 \;=\; -y+3 \;=\; -\frac{z+4}{2}.$$

C12S04.010: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP} + t\overrightarrow{PQ}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 2,\ 5,\ -7\,\rangle + t\langle\, 2,\ -2,\ 15\,\rangle = \langle\, 2+2t,\ 5-2t,\ -7+15t\,\rangle$$

so the line has parametric equations $x = 2t+2,\ y = -2t+5,\ z = 15t-7$ and symmetric equations

$$\frac{x-2}{2} \;=\; -\frac{y-5}{2} \;=\; \frac{z+7}{15}.$$

C12S04.011: Solving the vector equation $\langle\, x,\ y,\ z\,\rangle = \overrightarrow{OP} + t\mathbf{k}$ yields

$$\langle\, x,\ y,\ z\,\rangle = \langle\, 1,\ 1,\ 1\,\rangle + t\langle\, 0,\ 0,\ 1\,\rangle = \langle\, 1,\ 1,\ 1+t\,\rangle$$

so the line has parametric equations $x = 1,\ y = 1,\ z = t+1$. In a strict sense, the line doesn't have symmetric equations, but its Cartesian equations are

$$x = 1, \quad y = 1.$$

The fact that z is not mentioned means that z is *arbitrary*.

C12S04.012: A line normal to the plane with Cartesian equation $x+y+z=1$ is parallel to its normal vector $\mathbf{n} = \langle\, 1,\ 1,\ 1\,\rangle$. Because the line in this problem passes through the origin, it has vector equation $\langle\, x,\ y,\ z\,\rangle = t\mathbf{n}$, parametric equations $x = t,\ y = t,\ z = t,$ and symmetric equations $x = y = z$.

C12S04.013: A line normal to the plane with Cartesian equation $2x - y + 3z = 4$ is parallel to its normal vector $\mathbf{n} = \langle\, 2,\ -1,\ 3\,\rangle$. The line of this problem also passes through the point $P(2,\ -3,\ 4)$ and thus has vector equation $\langle\, x,\ y,\ z\,\rangle = t\langle\, 2,\ -1,\ 3\,\rangle + \langle\, 2,\ -3,\ 4\,\rangle$, parametric equations $x = 2t+2,\ y = -t-3,\ z = 3t+4,$ and symmetric equations

$$\frac{x-2}{2} \;=\; -(y+3) \;=\; \frac{z-4}{3}.$$

C12S04.014: The line with parametric equations $x = 3t,\ y = 2+t,\ z = 2-t$ contains the points $Q(0,\ 2,\ 2)$ (take $t = 0$) and $R(3,\ 3,\ 1)$ (take $t = 1$). Thus this line, and any parallel line, is parallel to the vector $\overrightarrow{QR} = \langle\, 3,\ 1,\ -1\,\rangle$. Hence such a line passing through the point $P(2,\ -1,\ 5)$ has vector equation $\langle\, x,\ y,\ z\,\rangle = t\langle\, 3,\ 1,\ -1\,\rangle + \langle\, 2,\ -1,\ 5\,\rangle$, parametric equations $x = 3t+2,\ y = t-1,\ z = -t+5$, and symmetric equations

$$\frac{x-2}{3} \;=\; y+1 \;=\; -(z-5).$$

C12S04.015: Given: The lines L_1 and L_2 with symmetric equations

$$x - 2 = \frac{y+1}{2} = \frac{z-3}{3} \quad \text{and} \quad \frac{x-5}{3} = \frac{y-1}{2} = z - 4, \tag{1}$$

respectively. Points on L_1 include $P_1(2, -1, 3)$ and $Q_1(3, 1, 6)$, and thus L_1 is parallel to the vector $\overrightarrow{P_1Q_1} = \langle 1, 2, 3 \rangle$. Points on L_2 include $P_2(5, 1, 4)$ and $Q_2(8, 3, 5)$, and thus L_2 is parallel to the vector $\overrightarrow{P_2Q_2} = \langle 3, 2, 1 \rangle$. Clearly L_1 and L_2 are not parallel. To determine whether they intersect, the *Mathematica* 3.0 command for solving the equations in (1) simultaneously is

```
Solve[ {x - 2 == (y + 1)/2, x - 2 == (z - 3)/3,
        (x - 5)/3 == (y - 1)/2, (x - 5)/3 == z - 4 }, { x, y, z } ]
```

and this command yields the response $x = 2$, $y = -1$, $z = 3$. Hence the two lines meet at the single point $(2, -1, 3)$.

C12S04.016: When we solve the equations of L_1 and L_2 simultaneously, we obtain the unique solution $x = 7$, $y = 5$, $z = -3$. Therefore the lines are not parallel and meet at the single point $(7, 5, -3)$.

C12S04.017: When we try to solve the equations of L_1 and L_2 simultaneously, we find that there is no solution. Thus the lines do not intersect. Two points on L_1 are $P_1(6, 5, 7)$ and $Q_1(8, 7, 10)$, so L_1 is parallel to the vector $\mathbf{v}_1 = \overrightarrow{P_1Q_1} = \langle 2, 2, 3 \rangle$. Two points on L_2 are $P_2(7, 5, 10)$ and $Q_2(10, 8, 15)$, so L_2 is parallel to the vector $\mathbf{v}_2 = \overrightarrow{P_2Q_2} = \langle 3, 3, 5 \rangle$. The equation $\lambda \mathbf{v}_1 = \mathbf{v}_2$ has no solution, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Therefore L_1 and L_2 are skew lines.

C12S04.018: To determine if L_1 and L_2 intersect, we solve simultaneously

$$14 + 3t = 5 + 3s \quad \text{and} \quad 7 + 2t = 15 + 5s$$

and find the unique solution $t = -\frac{23}{3}$, $s = -\frac{14}{3}$. Then we solve simultaneously

$$14 + 3t = 5 + 3s \quad \text{and} \quad 21 + 5t = 10 + 7s$$

and find the unique solution $t = -5$, $s = -2$. Therefore the lines do not intersect. Points on L_1 include $P_1(14, 7, 21)$ and $Q_1(17, 9, 26)$, so a vector parallel to L_1 is $\mathbf{v}_1 = \overrightarrow{P_1Q_1} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$. Points on L_2 include $P_2(5, 15, 10)$ and $Q_2(8, 20, 17)$, so a vector parallel to L_2 is $\mathbf{v}_2 = \overrightarrow{P_2Q_2} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$. These vectors are not parallel. Therefore L_1 and L_2 are skew lines.

C12S04.019: We solve the equations

$$\frac{x-7}{6} = \frac{y+5}{4}, \quad \frac{x-7}{6} = \frac{9-z}{8}, \quad \frac{11-x}{9} = \frac{7-y}{6}, \quad \text{and} \quad \frac{7-y}{6} = \frac{z-13}{12}$$

simultaneously and find that there is no solution. So L_1 and L_2 do not intersect. Two points on L_1 are $P_1(7, -5, 9)$ and $Q_1(13, -1, 1)$, so L_1 is parallel to the vector $\mathbf{v}_1 = \overrightarrow{P_1Q_1} = 6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$. Two points of L_2 and $P_2(11, 7, 13)$ and $Q_2(2, 1, 25)$, so L_2 is parallel to the vector $\mathbf{v}_2 = \overrightarrow{P_2Q_2} = -9\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$. Because $-\frac{3}{2}\mathbf{v}_1 = \mathbf{v}_2$, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, and therefore L_1 and L_2 are parallel as well.

C12S04.020: The *Mathematica* 3.0 command

```
Solve[ { 13 + 12*t == 22 + 9*s, -7 + 20*t == 8 + 15*s,
         11 - 28*t == -10 - 21*s }, { s, t } ]
```

produces the response $s = -1 + \frac{4}{3}t$. When this substitution is made in the equations for L_2, the equations of L_1 result. Therefore L_1 and L_2 coincide.

C12S04.021: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$, and because $\overrightarrow{OP} = \mathbf{0}$ it follows that a Cartesian equation of the plane is $x + 2y + 3z = 0$.

C12S04.022: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle -2, 7, 3 \rangle \cdot \langle x, y, z \rangle = \langle -2, 7, 3 \rangle \cdot \langle 3, -4, 5 \rangle,$$

and it follows that a Cartesian equation of the plane is $2x - 7y - 3z = 19$.

C12S04.023: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle 1, 0, -1 \rangle \cdot \langle x, y, z \rangle = \langle 1, 0, -1 \rangle \cdot \langle 5, 12, 13 \rangle,$$

and thus a Cartesian equation of the plane is $x - z + 8 = 0$.

C12S04.024: A plane with normal vector $\mathbf{j}$ is parallel to the xz-plane, so its equation is of the form $y = c$ where c is a constant. The plane of Problem 24 also passes through the point $(5, 12, 13)$, so $c = 12$. Thus a Cartesian equation of this plane is $y = 12$.

C12S04.025: A plane parallel to the xz-plane has an equation of the form $y = c$ where c is a constant. The plane of Problem 25 also passes through the point $(5, 7, -6)$, so $c = 7$. Thus a Cartesian equation of this plane is $y = 7$.

C12S04.026: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle 2, 2, -1 \rangle \cdot \langle x, y, z \rangle = \langle 2, 2, -1 \rangle \cdot \langle 1, 0, -1 \rangle.$$

Thus a Cartesian equation of this plane is $2x + 2y - z = 3$.

C12S04.027: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle 7, 11, 0 \rangle \cdot \langle x, y, z \rangle = \langle 7, 11, 0 \rangle \cdot \langle 10, 4, -3 \rangle.$$

Thus a Cartesian equation of this plane is $7x + 11y = 114$.

C12S04.028: The vector equation of the plane is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle 1, -3, 2 \rangle \cdot \langle x, y, z \rangle = \langle 1, -3, 2 \rangle \cdot \langle 1, -3, 2 \rangle.$$

Thus a Cartesian equation of this plane is $x - 3y + 2z = 14$.

C12S04.029: A plane parallel to the plane with Cartesian equation $3x + 4y - z = 10$ has normal vector $\mathbf{n} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, thus a Cartesian equation of the form $3x + 4y - z = c$ for some constant c. The plane of Problem 29 also passes through the origin, so that $c = 0$. Hence it has Cartesian equation $3x + 4y - z = 0$.

C12S04.030: The plane with Cartesian equation $x + y - 2z = 0$ has normal vector $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, so the plane of Problem 30 has vector equation

$$\mathbf{n} \cdot \langle\, x,\ y,\ z\,\rangle = \mathbf{n} \cdot \langle\, 5,\ 1,\ 4\,\rangle$$

and thus Cartesian equation $x + y - 2z + 2 = 0$.

C12S04.031: The plane through the origin $O(0,\ 0,\ 0)$ and the two points $P(1,\ 1,\ 1)$ and $Q(1,\ -1,\ 3)$ has normal vector

$$\mathbf{n} \ = \ \overrightarrow{OP} \times \overrightarrow{OQ} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} \ = \ \langle\, 4,\ -2,\ -2\,\rangle.$$

Therefore this plane has a Cartesian equation of the form $4x - 2y - 2z = c$ where c is a constant. Because the plane of Problem 31 passes through the origin, $c = 0$. Therefore a Cartesian equation of this plane is $2x - y - z = 0$.

C12S04.032: The plane $\mathcal{P}$ through the three points $A(1,\ 0,\ -1)$, $B(3,\ 3,\ 2)$, and $C(4,\ 5,\ -1)$ has normal vector

$$\mathbf{n} \ = \ \overrightarrow{AB} \times \overrightarrow{AC} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ 3 & 5 & 0 \end{vmatrix} \ = \ \langle\, -15,\ 9,\ 1\,\rangle.$$

Thus it has vector equation $\mathbf{n} \cdot \langle\, x,\ y,\ z\,\rangle \ = \ \mathbf{n} \cdot \overrightarrow{OA} \ = \ \langle\, -15,\ 9,\ 1\,\rangle \cdot \langle\, 1,\ 0,\ -1\,\rangle \ = \ -16$. Therefore the plane $\mathcal{P}$ has Cartesian equation $15x - 9y - z = 16$.

C12S04.033: The plane $\mathcal{P}$ that contains $P(2,\ 4,\ 6)$ and the line L with parametric equations $x = 7 - 3t$, $y = 3 + 4t$, $z = 5 + 2t$ also contains the two points $Q(7,\ 3,\ 5)$ (set $t = 0$) and $R(4,\ 7,\ 4)$ (set $t = 1$) of L. So the vectors $\overrightarrow{PQ} = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$ amd $\overrightarrow{PR} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ are parallel to $\mathcal{P}$, and thus $\mathcal{P}$ has normal vector

$$\mathbf{n} \ = \ \overrightarrow{PQ} \times \overrightarrow{PR} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & -1 \\ 2 & 3 & 1 \end{vmatrix} \ = \ 2\mathbf{i} - 7\mathbf{j} + 17\mathbf{k}.$$

So $\mathcal{P}$ has vector equation $\mathbf{n} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{n} \cdot \overrightarrow{OP} = 78$ and thus Cartesian equation $2x - 7y + 17z = 78$.

C12S04.034: The plane $\mathcal{P}$ that contains the point $P(13,\ -7,\ 29)$ and the line L with parametric equations $x = 17 - 9t$, $y = 23 + 14t$, $z = 35 - 41t$ also contains the two points $Q(17,\ 23,\ 35)$ (set $t = 0$) and $R(8,\ 37,\ -6)$ (set $t = 1$) of L. So the vectors $\overrightarrow{PQ} = 4\mathbf{i} + 30\mathbf{j} + 6\mathbf{k}$ and $\overrightarrow{PR} = -5\mathbf{i} + 44\mathbf{j} - 35\mathbf{k}$ are parallel to $\mathcal{P}$, and thus $\mathcal{P}$ has normal vector

$$\mathbf{n} \ = \ \overrightarrow{PQ} \times \overrightarrow{PR} \ = \ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 30 & 6 \\ -5 & 44 & -35 \end{vmatrix} \ = \ -1314\mathbf{i} + 110\mathbf{j} + 326\mathbf{k}.$$

So $\mathcal{P}$ has vector equation $\mathbf{n} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{n} \cdot \overrightarrow{OP} = -8398$ and thus Cartesian equation

$$1314x - 110y - 326z = 8398; \qquad \text{that is,} \qquad 657x - 55y - 163z = 4199.$$

C12S04.035: There are several ways to solve this problem.

First solution: Two points of L are $P(7, 3, 9)$ and $Q(3, 9, 14)$, so the vector $\overrightarrow{PQ} = \langle -4, 6, 5 \rangle$ is parallel to L. The vector $\mathbf{n} = \langle 4, 1, 2 \rangle$ is normal to the plane $\mathcal{P}$. But

$$\mathbf{n} \cdot \overrightarrow{PQ} = \langle 4, 1, 2 \rangle \cdot \langle -4, 6, 5 \rangle = -16 + 6 + 10 = 0,$$

so that $\mathbf{n}$ and $\overrightarrow{PQ}$ are perpendicular. Therefore L is parallel to $\mathcal{P}$. Moreover, P does not satisfy the equation of $\mathcal{P}$ because

$$4 \cdot 7 + 1 \cdot 3 + 2 \cdot 9 = 28 + 3 + 18 = 49 \neq 17.$$

Because L is parallel to $\mathcal{P}$ and contains a point not on $\mathcal{P}$, the line and the plane cannot coincide.

Second solution: If the line and the plane both contain the point (x, y, z), then there exists a scalar t such that all four equations in Problem 35 are simultaneously true. The *Mathematica* 3.0 command

```
Solve[ { x == 7 - 4*t, y == 3 + 6*t, z == 9 + 5*t,
          4*x + y + 2*z == 17 }, { x, y, z, t } ]
```

returns $\{\,\}$, the "empty set," telling us that these four equations have no simultaneous solution. So no point of L lies in the plane $\mathcal{P}$; the line and the plane are parallel and do not coincide.

Third solution: If $x = 7 - 4t$, $y = 3 + 6t$, and $z = 9 + 5t$ are substituted in the equation $4x + y + 2z = 17$ of the plane, the result is $49 = 17$. This is impossible, so no point of L lies in the plane $\mathcal{P}$. We reach the same conclusion as in the previous two solutions.

C12S04.036: If $x = 15+7t$, $y = 10+12t$, and $z = 5-4t$ are substituted in the equation $12x-5y+6z = 50$ of the plane, the result is $160 = 50$. This is impossible, so the line and the plane have no points in common and are therefore parallel.

C12S04.037: Simultaneous solution of the four equations given in the statement of Problem 37 yields the unique solution $x = \frac{9}{2}$, $y = \frac{9}{4}$, $z = \frac{17}{4}$, and $t = \frac{3}{4}$. So the line and the plane are not parallel and meet at the single point $\left(\frac{9}{2}, \frac{9}{4}, \frac{17}{4}\right)$. The easiest way to solve these equations by hand is to substitute the three parametric equations into the equation of the plane and solve for t:

$$3(3 + 2t) + 2(6 - 5t) - 4(2 + 3t) = 1; \qquad 9 + 6t + 12 - 10t - 8 - 12t = 1;$$

$$-16t = -12; \qquad t = \frac{3}{4}.$$

Then substitution in the parametric equations yields the values of x, y, and z given here.

C12S04.038: Simultaneous solution of the four equations given in the statement of Problem 38 yields the unique solution $x = \frac{237}{20}$, $y = \frac{3}{4}$, $z = \frac{63}{10}$, $y = \frac{21}{20}$. Therefore the line and the plane are not parallel and they meet at the single point $\left(\frac{237}{20}, \frac{3}{4}, \frac{63}{10}\right)$. See the solution of Problem 37 for additional comments.

C12S04.039: The vector $\mathbf{n}_1 = \langle 1, 0, 0 \rangle$ is normal to the first plane; the vector $\mathbf{n}_2 = \langle 1, 1, 1 \rangle$ is normal to the second. If θ is the angle between the normals (this is, by definition, the angle between the planes), then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

so $\theta \approx 54.735610317245^\circ$.

C12S04.040: The vector $\mathbf{n}_1 = \langle 2, -1, 1 \rangle$ is normal to the first plane; the vector $\mathbf{n}_2 = \langle 1, 1, -1 \rangle$ is normal to the second. If θ is the angle between the normals (this is, by definition, the angle between the planes), then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{0}{1 \cdot \sqrt{3}} = 0,$$

so $\theta = 90^\circ$ (exactly).

C12S04.041: The vector $\mathbf{n}_1 = \langle 1, -1, -2 \rangle$ is normal to the first plane; the vector $\mathbf{n}_2 = \langle 1, -1, -2 \rangle$ is normal to the second. If θ is the angle between the normals (this is, by definition, the angle between the planes), then $\theta = 0$ because $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

C12S04.042: The vector $\mathbf{n}_1 = \langle 2, 1, 1 \rangle$ is normal to the first plane; the vector $\mathbf{n}_2 = \langle 3, -1, -1 \rangle$ is normal to the second. If θ is the angle between the normals (this is, by definition, the angle between the planes), then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{4}{\sqrt{66}} = \frac{2\sqrt{66}}{33},$$

so $\theta \approx 60.503791503434^\circ$.

C12S04.043: By inspection, two points that lie on both planes are $P(10, 0, -10)$ and $Q(10, 1, -11)$. Hence a vector parallel to their line of intersection L is $\mathbf{v} = \overrightarrow{PQ} = \langle 0, 1, -1 \rangle$. So a vector equation of L is

$$\langle x, y, z \rangle = \overrightarrow{OP} + t\mathbf{v} = \langle 10, 0, -10 \rangle + t\langle 0, 1, -1 \rangle = \langle 10, t, -10 - t \rangle,$$

its parametric equations are $x = 10$, $y = t$, $z = -10 - t$, and its Cartesian equations are $x = 10$, $y = -10 - z$.

C12S04.044: Let $z = 0$ and solve the equations of the planes simultaneously to find that one point on their line of intersection L is $P(2, -1, 0)$. Repeat with $z = 1$ to find that another such point is $Q(2, 0, 1)$. So L is parallel to the vector $\mathbf{v} = \overrightarrow{PQ} = \langle 0, 1, 1 \rangle$ and thus has vector equation

$$\langle x, y, z \rangle = \overrightarrow{OP} + t\mathbf{v} = \langle 2, -1, 0 \rangle + t\langle 0, 1, 1 \rangle = \langle 2, t - 1, t \rangle,$$

parametric equations $x = 2$, $y = t - 1$, $z = t$, and Cartesian equations $x = 2$, $y + 1 = z$.

C12S04.045: The planes of Problem 41 are parallel, so there is no line of intersection.

C12S04.046: Substitute $z = 0$, then solve the equations of the planes simultaneously to find that one point on their line of intersection L is $P\left(\frac{7}{5}, \frac{6}{5}, 0\right)$. Repeat with $z = 1$ to find that another such point is $Q\left(\frac{7}{5}, \frac{1}{5}, 1\right)$. Hence the vector $\mathbf{v} = \overrightarrow{PQ} = \langle 0, -1, 1 \rangle$ is parallel to L, and thus L has vector equation

$$\langle x,\, y,\, z \rangle = \overrightarrow{OP} + t\mathbf{v} = \langle \tfrac{7}{5},\, \tfrac{6}{5},\, 0 \rangle + t\langle 0,\, -1,\, 1 \rangle.$$

Thus L has parametric equations $x = \frac{7}{5},\; y = \frac{6}{5} - t,\; z = t$ and Cartesian equations $x = \frac{7}{5},\; -y + \frac{6}{5} = z$.

C12S04.047: Substitute $z = 0$, then solve the equations of the planes simultaneously to find that one point on their line of intersection L is $Q\left(\frac{7}{5},\, \frac{6}{5},\, 0\right)$. Repeat with $z = 1$ to find that another such point is $R\left(\frac{7}{5},\, \frac{1}{5},\, 1\right)$. Hence the vector $\mathbf{v} = \overrightarrow{QR} = \langle 0,\, -1,\, 1 \rangle$ is parallel to L. Therefore the parallel line through the point $P(3,\, 3,\, 1)$ has vector equation

$$\langle x,\, y,\, z \rangle = \overrightarrow{OP} + t\mathbf{v} = \langle 3,\, 3,\, 1 \rangle + t\langle 0,\, -1,\, 1 \rangle.$$

Thus L has parametric equations $x = 3,\; y = 3 - t,\; z = 1 + t$ and Cartesian equations $x = 3,\; -y + 3 = z - 1$.

C12S04.048: A plane $\mathcal{P}$ perpendicular to the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ will be parallel to both of their normals, so the cross product of these normals will be normal to $\mathcal{P}$. The first plane given in Problem 48 has normal $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and the second has normal $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{k}$, so a normal to $\mathcal{P}$ is

$$\mathbf{n} \;=\; \mathbf{n}_1 \times \mathbf{n}_2 \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 2 & 0 & 1 \end{vmatrix} \;=\; \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}.$$

Therefore a vector equation of $\mathcal{P}$ is

$$\mathbf{n} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \;=\; \mathbf{n} \cdot \overrightarrow{OP} = \mathbf{n} \cdot (3\mathbf{i} + 3\mathbf{j} + \mathbf{k}),$$

and therefore $\mathcal{P}$ has Cartesian equation $x - 5y - 2z + 14 = 0$.

C12S04.049: Because the xy-plane has equation $z = 0$, the plane with equation $3x + 2y - z = 6$ intersects the xy-plane in the line with equation $3x + 2y = 6$. So three points on the plane $\mathcal{P}$ we seek are $P(1,\, 1,\, 1)$, $Q(2,\, 0,\, 0)$, and $R(0,\, 3,\, 0)$. Thus a normal to $\mathcal{P}$ is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{vmatrix} \;=\; \langle 3,\, 2,\, 1 \rangle.$$

Therefore $\mathcal{P}$ has vector equation $\mathbf{n} \cdot \langle x,\, y,\, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\begin{aligned} \langle 3,\, 2,\, 1 \rangle \cdot \langle x,\, y,\, z \rangle \;&=\; \langle 3,\, 2,\, 1 \rangle \cdot \langle 1,\, 1,\, 1 \rangle; \\ 3x + 2y + z \;&=\; 6. \end{aligned}$$

C12S04.050: Set $z = 0$ and solve the equations of the two given planes simultaneously to find that one point on their line of intersection is $Q(1,\, 0,\, 0)$. Repeat with $z = 1$ to find that another such point is $R(1,\, 1,\, 1)$. A third point in the plane $\mathcal{P}$ whose equation we seek is $P(1,\, 3,\, -2)$, so the vectors $\overrightarrow{PQ} = \langle 0,\, -3,\, 2 \rangle$ and $\overrightarrow{PR} = \langle 0,\, -2,\, 3 \rangle$ are parallel to $\mathcal{P}$. Hence a normal to $\mathcal{P}$ is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & 2 \\ 0 & -2 & 3 \end{vmatrix} = \langle -5, 0, 0 \rangle.$$

Thus a vector equation of $\mathcal{P}$ is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle -5, 0, 0 \rangle \cdot \langle x, y, z \rangle = \langle -5, 0, 0 \rangle \cdot \langle 1, 3, -2 \rangle; \qquad -5x = -5; \qquad x = 1.$$

C12S04.051: The plane $\mathcal{P}$ whose equation we seek passes through $P(1, 0, -1)$ and $Q(2, 1, 0)$, and is thus parallel to $\overrightarrow{PQ} = \langle 1, 1, 1 \rangle$. To find two points in the line of intersection of the other two planes, set $z = 1$ and solve their equations simultaneously to find that one such point is $S(2, 2, 1)$. Repeat with $z = 5$ to find that another such point is $R(1, -1, 5)$. Hence another vector parallel to $\mathcal{P}$ is $\overrightarrow{RS} = \langle 1, 3, -4 \rangle$. If $\overrightarrow{RS}$ had turned out to be parallel to $\overrightarrow{PQ}$, then there would have been insufficient information to solve the problem, but they are not parallel. Hence a normal to $\mathcal{P}$ is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{RS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 3 & -4 \end{vmatrix} = \langle -7, 5, 2 \rangle.$$

Hence $\mathcal{P}$ has vector equation $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle -7, 5, 2 \rangle \cdot \langle x, y, z \rangle = \langle -7, 5, 2 \rangle \cdot \langle 1, 0, -1 \rangle; \qquad -7x + 5y + 2z = -9; \qquad 7x - 5y - 2z = 9.$$

C12S04.052: Solve the given equations simultaneously—perhaps using the *Mathematica* 3.0 command

```
Solve[ { x - 1 == (y + 1)/2, x - 1 == z - 2,
         x - 2 == (y - 2)/3, x - 2 == (z - 4)/2 }, { x, y, z } ]
```

—to find that the point $P(1, -1, 2)$ is the unique point that lies on both lines. By inspection, another point on the first line is $Q(2, 1, 3)$ and another point on the second line is $R(2, 2, 4)$. Hence a vector normal to the plane $\mathcal{P}$ that contains those two lines is

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} = \langle 1, -1, 1 \rangle.$$

Therefore a vector equation of $\mathcal{P}$ is $\mathbf{n} \cdot \langle x, y, z \rangle = \mathbf{n} \cdot \overrightarrow{OP}$; that is,

$$\langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = \langle 1, -1, 1 \rangle \cdot \langle 1, -1, 2 \rangle; \qquad x - y + z = 4.$$

C12S04.053: Set $z = 1$ and solve the equations of the two planes simultaneously to find that one point on their line of intersection is $P(1, 1, 1)$. Repeat with $z = 5$ to find that another such point is $Q(-5, 6, 5)$.

Substitute $t = 0$ to find that a point on the given line is $R(1, 3, 2)$; substitute $t = 1$ to find that another such point is $S(7, -2, -2)$. Then $\overrightarrow{PQ} = \langle -6, 5, 4 \rangle$ and $\overrightarrow{RS} = \langle 6, -5, -4 \rangle$, so it's clear that the two lines are parallel. Note that R does not lie on the first given plane, so the given line and the line of intersection of the two given planes do not coincide. To obtain two nonparallel vectors in the plane $\mathcal{P}$ that contains both lines, use $\overrightarrow{PQ}$ and $\overrightarrow{PR} = \langle 0, 2, 1 \rangle$. Then a normal to $\mathcal{P}$ will be

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 5 & 4 \\ 0 & 2 & 1 \end{vmatrix} = \langle -3, 6, -12 \rangle.$$

Replace $\mathbf{n}$ with the simpler $\langle 1, -2, 4 \rangle$ and write a vector equation of $\mathcal{P}$ in the form

$$\langle 1, -2, 4 \rangle \cdot \langle x, y, z \rangle = \langle 1, -2, 4 \rangle \cdot \langle 1, 1, 1 \rangle,$$

from which we read the Cartesian equation $x - 2y + 4z = 3$.

C12S04.054: See the *Comment* in the solution of Problem 31 of Section 12.3 for our strategy in solving this problem. Without loss of generality we may assume that $a \neq 0$. Then a point on the given plane is $(d/a, 0, 0)$. So a vector "connecting" the given point with the given plane is

$$\mathbf{c} = \left\langle x_0 - \frac{d}{a}, y_0, z_0 \right\rangle$$

and a vector normal to the plane is $\mathbf{n} = \langle a, b, c \rangle$. Hence the distance between them is

$$D = \frac{|\mathbf{n} \cdot \mathbf{c}|}{|\mathbf{n}|} = \frac{|ax_0 - d + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}.$$

C12S04.055: The formula in Problem 54 with $x_0 = y_0 = z_0 = 0$, $a = b = c = 1$, and $d = 10$ yields distance

$$D = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3} \approx 5.773502691896.$$

C12S04.056: The formula in Problem 54 with $x_0 = 5,\ y_0 = 12,\ z_0 = -13,\ a = 3,\ b = 4,\ c = 5,$ and $d = 12$ yields

$$D = \frac{|15 + 48 - 65 - 12|}{\sqrt{9 + 16 + 25}} = \frac{14}{5\sqrt{2}} = \frac{7\sqrt{2}}{5} \approx 1.979898987322.$$

C12S04.057: If L_1 and L_2 are skew lines, choose two points P_1 and Q_1 in L_1 and two points P_2 and Q_2 in L_2. Let $\mathbf{n} = \overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}$. Let $\mathcal{P}_1$ be the plane through P_1 with normal vector $\mathbf{n}$ and let $\mathcal{P}_2$ be the plane through P_2 with normal vector $\mathbf{n}$. Clearly $\mathcal{P}_1$ contains L_1, $\mathcal{P}_2$ contains L_2, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel.

C12S04.058: Without loss of generality we may assume that $a \neq 0$. Then the point $Q(d_2/a, 0, 0)$ is in the second plane. By the formula in Problem 54, the distance between Q and the first plane is

$$D = \frac{|a \cdot (d_2/a) + b \cdot 0 + c \cdot 0 - d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

C12S04.059: See the *Comment* in the solution of Problem 31 of Section 12.3 for our strategy in the following solution. By inspection, $P_1(1, -1, 4)$ and $Q_1(3, 0, 4)$ lie in L_1 and we are given the two points $P_2(2, 1, -3)$ and $Q_2(0, 8, 4)$ in L_2. Hence $\mathbf{v}_1 = \overrightarrow{P_1Q_1} = \langle 2, 1, 0 \rangle$ is parallel to L_1 and $\mathbf{v}_2 = \overrightarrow{P_2Q_2} = \langle -2, 7, 7 \rangle$ is parallel to L_2.

Part (a): The line L_2 has vector equation

$$\langle x, y, z \rangle = t\mathbf{v}_2 + \langle 2, 1, -3 \rangle = \langle 2 - 2t, 1 + 7t, -3 + 7t \rangle,$$

and thus symmetric equations

$$\frac{-x+2}{2} = \frac{y-1}{7} = \frac{z+3}{7}.$$

It is clear that the two lines are not parallel because $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$. The *Mathematica* 3.0 command

```
Solve[ { (2 - x)/2 == (y - 1)/7, (2 - x)/2 == (z + 3)/7,
          x - 1 == 2*y + 2, z == 4 }, { x, y, z } ]
```

for solving the equations of the two lines simultaneously returns the information that there is no solution. Therefore the two lines are skew lines.

Part (b): Let $\mathbf{c} = \overrightarrow{P_1P_2}$ be a "connector" from L_1 to L_2 and note that $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 7, -14, 16 \rangle$ is normal to both lines. Therefore the distance between the lines is

$$D = \frac{|\mathbf{n} \cdot \mathbf{c}|}{|\mathbf{n}|} = \frac{133}{\sqrt{501}} = \frac{133\sqrt{501}}{501} \approx 5.942001786397.$$

Alternative Part (b): If we follow the method required in the statement of Problem 59, we find that the first line lies in the plane with equation $7x - 14y + 16z = 85$ and the second line lies in the plane with equation $7x - 14y + 16z = -48$. Therefore, by the formula in Problem 58, the distance between them is

$$D = \frac{|85 - (-48)|}{\sqrt{49 + 196 + 256}} = \frac{133}{\sqrt{501}} = \frac{133\sqrt{501}}{501}.$$

C12S04.060: Two points on L_1 are $P_1(7, 11, 13)$ and $Q_1(9, 6, 19)$, and a vector parallel to L_1 is thus $\mathbf{v}_1 = \overrightarrow{P_1Q_1} = \langle 2, -5, 6 \rangle$. To find two points on L_2, we set $x = 4$ and solved the equations of the two planes simultaneously using *Mathematica* 3.0: The command

```
Solve[ { 3*x - 2*y + 4*z == 10, 5*x + 3*y - 2*z == 15, x == 4 }, { x, y, z } ]
```

produced the response $(x, y, z) = (4, -3, -2) = P_2$. Repeating with $x = -4$ produced a second point on L_2: $(x, y, z) = (-4, 23, 17) = Q_2$. Thus a vector parallel to L_2 is $\mathbf{v}_2 = \overrightarrow{P_2Q_2} = \langle -8, 26, 19 \rangle$. So the vector $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -251, -86, 12 \rangle$ is normal to both lines. A vector that connects L_1 to L_2 is $\mathbf{c} = \overrightarrow{P_1P_2} = \langle -3, -14, -15 \rangle$. Hence the distance between the two lines is

$$D = \frac{|\mathbf{n} \cdot \mathbf{c}|}{|\mathbf{n}|} = \frac{1777}{\sqrt{70541}} = \frac{1777\sqrt{70541}}{70541} \approx 6.690623965208.$$

Section 12.5

C12S05.001: Because $y^2 + z^2 = 1$ while x is arbitrary, the graph lies on the cylinder of radius 1 with axis the x-axis. A small part of the graph is shown in Fig. 12.5.17.

C12S05.002: Because $x^2 + y^2 = 1$ while z varies between -1 and 1, the graph lies on the part of the cylinder with radius 1 and axis the z-axis that lies between $z = -1$ and $z = 1$. A small part of the graph is shown in Fig. 12.5.18.

C12S05.003: Because $x^2 + y^2 = t^2 = z^2$, the graph lies on the cone with axis the z-axis and equation $z^2 = x^2 + y^2$. A small part of the graph is shown in Fig. 12.5.16.

C12S05.004: First note that

$$x^2 + y^2 = (\cos^2 t + \sin^2 t)\sin^2 4t = \sin^2 4t,$$

so that

$$x^2 + y^2 + z^2 = \sin^2 4t + \cos^2 4t = 1.$$

Therefore the graph lies on the sphere with radius 1 and center (0, 0, 0). A small part of the graph is shown in Fig. 12.5.15.

C12S05.005: If $\mathbf{r}(t) = 3\mathbf{i} - 2\mathbf{j}$, then $\mathbf{r}'(t) = \mathbf{0} = \mathbf{r}''(t)$, and hence $\mathbf{r}'(1) = \mathbf{0} = \mathbf{r}''(1)$.

C12S05.006: If $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j}$, then $\mathbf{r}'(t) = 2t\mathbf{i} - 3t^2\mathbf{j}$ and $\mathbf{r}''(t) = 2\mathbf{i} - 6t\mathbf{j}$. Therefore $\mathbf{r}'(2) = 4\mathbf{i} - 12\mathbf{j}$ and $\mathbf{r}''(2) = 2\mathbf{i} - 12\mathbf{j}$.

C12S05.007: If $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{-t}\mathbf{j}$, then $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} - e^{-t}\mathbf{j}$ and $\mathbf{r}''(t) = 4e^{2t}\mathbf{i} + e^{-t}\mathbf{j}$. Therefore $\mathbf{r}'(0) = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{r}''(0) = 4\mathbf{i} + \mathbf{j}$.

C12S05.008: If $\mathbf{r}(t) = \mathbf{i}\cos t + \mathbf{j}\sin t$, then $\mathbf{r}'(t) = -\mathbf{i}\sin t + \mathbf{j}\cos t$ and $\mathbf{r}''(t) = -\mathbf{i}\cos t - \mathbf{j}\sin t$. Hence

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} \quad \text{and} \quad \mathbf{r}''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}.$$

C12S05.009: If $\mathbf{r}(t) = 3\mathbf{i}\cos 2\pi t + 3\mathbf{j}\sin 2\pi t$, then

$$\mathbf{r}'(t) = -6\pi\mathbf{i}\sin 2\pi t + 6\pi\mathbf{j}\cos 2\pi t \quad \text{and} \quad \mathbf{r}''(t) = -12\pi^2\mathbf{i}\cos 2\pi t - 12\pi^2\mathbf{j}\sin 2\pi t.$$

Therefore

$$\mathbf{r}'\left(\frac{3}{4}\right) = 6\pi\mathbf{i} \quad \text{and} \quad \mathbf{r}''\left(\frac{3}{4}\right) = 12\pi^2\mathbf{j}.$$

C12S05.010: If $\mathbf{r}(t) = 5\mathbf{i}\cos t + 4\mathbf{j}\sin t$, then $\mathbf{r}'(t) = -5\mathbf{i}\sin t + 4\mathbf{j}\cos t$ and $\mathbf{r}''(t) = -5\mathbf{i}\cos t - 4\mathbf{j}\sin t$. Hence $\mathbf{r}'(\pi) = -4\mathbf{j}$ and $\mathbf{r}''(\pi) = 5\mathbf{i}$.

C12S05.011: If $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, then

$$\mathbf{v}(t) = \langle\, 1,\ 2t,\ 3t^2 \,\rangle,$$

$$v(t) = \sqrt{1 + 4t^2 + 9t^4}\,, \quad \text{and}$$

$$\mathbf{a}(t) = \langle\, 0,\ 2,\ 6t \,\rangle.$$

C12S05.012: If $\mathbf{r}(t) = \langle\, 3t^2,\ 4t^2,\ -12t^2 \,\rangle$, then

$$\mathbf{v}(t) = \langle\, 6t,\ 8t,\ -24t \,\rangle, \quad v(t) = \sqrt{36t^2 + 64t^2 + 576t^2} = |26t|, \quad \text{and} \quad \mathbf{a}(t) = \langle\, 6,\ 8,\ -24 \,\rangle.$$

C12S05.013: If $\mathbf{r}(t) = \langle\, t,\ 3e^t,\ 4e^t \,\rangle$, then

$$\mathbf{v}(t) = \langle\, 1,\ 3e^t,\ 4e^t \,\rangle, \quad v(t) = \sqrt{1 + 25e^{2t}}\,, \quad \text{and} \quad \mathbf{a}(t) = \langle\, 0,\ 3e^t,\ 4e^t \,\rangle.$$

C12S05.014: If $\mathbf{r}(t) = \langle\, e^t,\ e^{2t},\ e^{3t} \,\rangle$, then

$$\mathbf{v}(t) = \langle\, e^t,\ 2e^{2t},\ 3e^{3t} \,\rangle, \quad v(t) = \sqrt{e^{2t} + 4e^{4t} + 9e^{6t}}\,, \quad \text{and} \quad \mathbf{a}(t) = \langle\, e^t,\ 4e^{2t},\ 9e^{3t} \,\rangle.$$

C12S05.015: If $\mathbf{r}(t) = \langle\, 3\cos t,\ 3\sin t,\ -4t \,\rangle$, then

$$\mathbf{v}(t) = \langle\, -3\sin t,\ 3\cos t,\ -4 \,\rangle, \quad v(t) = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5, \quad \text{and} \quad \mathbf{a}(t) = \langle\, -3\cos t,\ -3\sin t,\ 0 \,\rangle.$$

C12S05.016: If $\mathbf{r}(t) = \langle\, 12t,\ 5\sin 2t,\ -5\cos 2t \,\rangle$, then

$$\mathbf{v}(t) = \langle\, 12,\ 10\cos 2t,\ 10\sin 2t \,\rangle, \quad v(t) = 2\sqrt{86}\,, \quad \text{and} \quad \mathbf{a}(t) = \langle\, 0,\ -20\sin 2t,\ 20\cos 2t \,\rangle.$$

C12S05.017: By Eq. (16), we have

$$\int_0^{\pi/4} \langle\, \sin t,\ 2\cos t \,\rangle\, dt = \Big[\langle\, -\cos t,\ 2\sin t \,\rangle\Big]_0^{\pi/4} = \left\langle\, -\frac{\sqrt{2}}{2},\ \sqrt{2} \,\right\rangle - \langle\, -1,\ 0 \,\rangle = \left\langle\, \frac{2-\sqrt{2}}{2},\ \sqrt{2} \,\right\rangle.$$

C12S05.018: By Eq. (16), we have

$$\int_1^e \left\langle\, \frac{1}{t},\ -1 \,\right\rangle\, dt = \Big[\langle\, \ln t,\ -t \,\rangle\Big]_1^e = \langle\, 1,\ -e \,\rangle - \langle\, 0,\ -1 \,\rangle = \langle\, 1,\ 1-e \,\rangle.$$

C12S05.019: By Eq. (16), we have

$$\int_0^2 \langle\, t^2(1+t^3)^{3/2},\ 0 \,\rangle\, dt = \left[\left\langle\, \frac{2}{15}(1+t^3)^{5/2},\ 0 \,\right\rangle\right]_0^2 = \left(\frac{162}{5} - \frac{2}{15}\right)\mathbf{i} = \frac{484}{15}\,\mathbf{i}.$$

C12S05.020: By Eq. (16), we have

$$\int_0^1 \langle\, e^t,\ -t\exp(-t^2) \,\rangle\, dt = \left[\left\langle\, e^t,\ \frac{1}{2}\exp(-t^2) \,\right\rangle\right]_0^1 = \left\langle\, e,\ \frac{1}{2e} \,\right\rangle - \left\langle\, 1,\ \frac{1}{2} \,\right\rangle = \left\langle\, e-1,\ \frac{1-e}{2e} \,\right\rangle.$$

C12S05.021: Given $\mathbf{u}(t) = \langle\, 3t,\ -1\,\rangle$ and $\mathbf{v}(t) = \langle\, 2,\ -5t\,\rangle$, Theorem 2 yields

$$D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t) = \langle\, 3t,\ -1\,\rangle\cdot\langle\, 0,\ -5\,\rangle + \langle\, 3,\ 0\,\rangle\cdot\langle 2,\ -5t\,\rangle = 5 + 6 = 11.$$

C12S05.022: Given $\mathbf{u}(t) = \langle\, t,\ t^2\,\rangle$ and $\mathbf{v}(t) = \langle\, t^2,\ -t\,\rangle$, Theorem 2 yields

$$D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t) = \langle\, t,\ t^2\,\rangle\cdot\langle\, 2t,\ -1\,\rangle + \langle\, 1,\ 2t\,\rangle\cdot\langle t^2,\ -t\,\rangle = 2t^2 - t^2 + t^2 - 2t^2 = 0.$$

C12S05.023: Given $\mathbf{u}(t) = \langle\, \cos t,\ \sin t\,\rangle$ and $\mathbf{v}(t) = \langle\, \sin t,\ -\cos t\,\rangle$, Theorem 2 yields

$$\begin{aligned} D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t) &= \langle\, \cos t,\ \sin t\,\rangle\cdot\langle\, \cos t,\ \sin t\,\rangle + \langle\, -\sin t,\ \cos t\,\rangle\cdot\langle \sin t,\ -\cos t\,\rangle \\ &= \cos^2 t + \sin^2 t - \sin^2 t - \cos^2 t = 0. \end{aligned}$$

C12S05.024: Given $\mathbf{u}(t) = \langle\, t,\ t^2,\ t^3\,\rangle$ and $\mathbf{v}(t) = \langle\, \cos 2t,\ \sin 2t,\ e^{-3t}\,\rangle$, Theorem 2 yields

$$\begin{aligned} D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] &= \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t) \\ &= \langle\, t,\ t^2,\ t^3\,\rangle\cdot\langle\, -2\sin 2t,\ 2\cos 2t,\ -3e^{-3t}\,\rangle + \langle\, 1,\ 2t,\ 3t^2\,\rangle\cdot\langle\, \cos 2t,\ \sin 2t,\ e^{-3t}\,\rangle \\ &= -2t\sin 2t + 2t^2\cos 2t - 3t^3e^{-3t} + \cos 2t + 2t\sin 2t + 3t^2e^{-3t} \\ &= (2t^2+1)\cos 2t + (3t^2 - 3t^3)e^{-3t}. \end{aligned}$$

C12S05.025: Given $\mathbf{a} = \mathbf{0} = \langle\, 0,\ 0,\ 0\,\rangle$, it follows that $\mathbf{v}(t) = \langle\, c_1,\ c_2,\ c_3\,\rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$\mathbf{k} = \langle\, 0,\ 0,\ 1\,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\ c_2,\ c_3\,\rangle$$

implies that $c_1 = c_2 = 0$ and $c_3 = 1$. Hence $\mathbf{v}(t) = \langle\, 0,\ 0,\ 1\,\rangle$, and therefore

$$\mathbf{r}(t) = \langle\, k_1,\ k_2,\ t + k_3\,\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$\mathbf{i} = \langle\, 1,\ 0,\ 0\,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\ k_2,\ k_3\,\rangle$$

leads to $k_1 = 1$ and $k_2 = k_3 = 0$. Therefore $\mathbf{r}(t) = \langle\, 1,\ 0,\ t\,\rangle$.

C12S05.026: Given $\mathbf{a} = \langle\, 2,\ 0,\ 0\,\rangle$, it follows that $\mathbf{v}(t) = \langle\, 2t + c_1,\ c_2,\ c_3\,\rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$4\mathbf{k} = \langle\, 0,\ 0,\ 4\,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\ c_2,\ c_3\,\rangle$$

implies that $c_1 = c_2 = 0$ and $c_3 = 4$. Hence $\mathbf{v}(t) = \langle\, 2t,\ 0,\ 4\,\rangle$, and therefore

$$\mathbf{r}(t) = \langle\, t^2 + k_1,\ k_2,\ 4t + k_3\,\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$3\mathbf{j} = \langle\, 0,\, 3,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\, k_2,\, k_3 \,\rangle$$

leads to $k_2 = 3$ and $k_1 = k_3 = 0$. Therefore $\mathbf{r}(t) = \langle\, t^2,\, 3,\, 4t \,\rangle$.

C12S05.027: Given $\mathbf{a} = \langle 2,\, 0,\, -4 \rangle$, it follows that $\mathbf{v}(t) = \langle\, 2t + c_1,\, c_2,\, -4t + c_3 \,\rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$10\mathbf{j} = \langle\, 0,\, 10,\, 0 \,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\, c_2,\, c_3 \,\rangle$$

implies that $c_1 = c_3 = 0$ and $c_2 = 10$. Hence $\mathbf{v}(t) = \langle\, 2t,\, 10,\, -4t \,\rangle$, and therefore

$$\mathbf{r}(t) = \langle\, t^2 + k_1,\, 10t + k_2,\, -2t^2 + k_3 \,\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$\mathbf{0} = \langle\, 0,\, 0,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\, k_2,\, k_3 \,\rangle$$

leads to $k_1 = k_2 = k_3 = 0$. Therefore $\mathbf{r}(t) = \langle\, t^2,\, 10t,\, -2t^2 \,\rangle$.

C12S05.028: Given $\mathbf{a} = \langle 1,\, -1,\, 3 \rangle$, it follows that $\mathbf{v}(t) = \langle\, t + c_1,\, -t + c_2,\, 3t + c_3 \,\rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$7\mathbf{j} = \langle\, 0,\, 7,\, 0 \,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\, c_2,\, c_3 \,\rangle$$

implies that $c_1 = c_3 = 0$ and $c_2 = 7$. Hence $\mathbf{v}(t) = \langle\, t,\, -t + 7,\, 3t \,\rangle$, and therefore

$$\mathbf{r}(t) = \left\langle\, \frac{1}{2}t^2 + k_1,\, -\frac{1}{2}t^2 + 7t + k_2,\, \frac{3}{2}t^2 + k_3 \,\right\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$5\mathbf{i} = \langle\, 5,\, 0,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\, k_2,\, k_3 \,\rangle$$

leads to $k_1 = 5$ and $k_2 = k_3 = 0$. Therefore

$$\mathbf{r}(t) = \left\langle\, \frac{1}{2}t^2 + 5,\, -\frac{1}{2}t^2 + 7t,\, \frac{3}{2}t^2 \,\right\rangle.$$

C12S05.029: Given $\mathbf{a} = \langle 0,\, 2,\, -6t \rangle$, it follows that $\mathbf{v}(t) = \langle\, c_1,\, 2t + c_2,\, -3t^2 + c_3 \,\rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$5\mathbf{k} = \langle\, 0,\, 0,\, 5 \,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\, c_2,\, c_3 \,\rangle$$

implies that $c_1 = c_2 = 0$ and $c_3 = 5$. Hence $\mathbf{v}(t) = \langle\, 0,\, 2t,\, -3t^2 + 5 \,\rangle$, and therefore

$$\mathbf{r}(t) = \langle\, k_1,\, t^2 + k_2,\, -t^3 + 5t + k_3 \,\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$2\mathbf{i} = \langle\, 2,\, 0,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\, k_2,\, k_3 \,\rangle$$

leads to $k_1 = 2$ and $k_2 = k_3 = 0$. Therefore

$$\mathbf{r}(t) = \left\langle 2,\ t^2,\ -t^3 + 5t \right\rangle.$$

C12S05.030: Given $\mathbf{a} = \langle 6t, -5, 12t^2 \rangle$, it follows that $\mathbf{v}(t) = \langle 3t^2 + c_1, -5t + c_2, 4t^3 + c_3 \rangle$ where $c_1,\ c_2,$ and c_3 are constants. Then

$$4\mathbf{j} - 5\mathbf{k} = \langle 0, 4, -5 \rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle c_1, c_2, c_3 \rangle$$

implies that $c_1 = 0,\ c_2 = 4,$ and $c_3 = -5$. Hence $\mathbf{v}(t) = \langle 3t^2, -5t + 4, 4t^3 - 5 \rangle$, and therefore

$$\mathbf{r}(t) = \left\langle t^3 + k_1, -\frac{5}{2}t^2 + 4t + k_2, t^4 - 5t + k_3 \right\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$3\mathbf{i} + 4\mathbf{j} = \langle 3, 4, 0 \rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle k_1, k_2, k_3 \rangle$$

leads to $k_1 = 3,\ k_2 = 4,$ and $k_3 = 0$. Therefore

$$\mathbf{r}(t) = \left\langle t^3 + 3,\ -\frac{5}{2}t^2 + 4t + 4,\ t^4 - 5t \right\rangle.$$

C12S05.031: Given $\mathbf{a} = \langle t, t^2, t^3 \rangle$, it follows that

$$\mathbf{v}(t) = \left\langle \frac{1}{2}t^2 + c_1, \frac{1}{3}t^3 + c_2, \frac{1}{4}t^4 + c_3 \right\rangle$$

where $c_1,\ c_2,$ and c_3 are constants. Then

$$10\mathbf{j} = \langle 0, 10, 0 \rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle c_1, c_2, c_3 \rangle$$

implies that $c_1 = c_3 = 0,$ and $c_2 = 10$. Hence

$$\mathbf{v}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{3}t^3 + 10, \frac{1}{4}t^4 \right\rangle,$$

and therefore

$$\mathbf{r}(t) = \left\langle \frac{1}{6}t^3 + k_1, \frac{1}{12}t^4 + 10t + k_2, \frac{1}{20}t^5 + k_3 \right\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$10\mathbf{i} = \langle 10, 0, 0 \rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle k_1, k_2, k_3 \rangle$$

leads to $k_1 = 10$ and $k_2 = k_3 = 0$. Therefore

$$\mathbf{r}(t) = \left\langle \frac{1}{6}t^3 + 10,\ \frac{1}{12}t^4 + 10t,\ \frac{1}{20}t^5 \right\rangle.$$

C12S05.032: Given $\mathbf{a} = \langle t, e^{-t}, 0 \rangle$, it follows that

$$\mathbf{v}(t) = \left\langle \frac{1}{2}t^2 + c_1, -e^{-t} + c_2, c_3 \right\rangle$$

where $c_1, \ c_2,$ and c_3 are constants. Then

$$5\mathbf{k} = \langle\, 0,\, 0,\, 5 \,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\, -1 + c_2,\, c_3 \,\rangle$$

implies that $c_1 = 0, \ c_2 = 1,$ and $c_3 = 5$. Hence

$$\mathbf{v}(t) = \left\langle\, \frac{1}{2}t^2,\, 1 - e^{-t},\, 5 \,\right\rangle$$

and therefore

$$\mathbf{r}(t) = \left\langle\, \frac{1}{6}t^3 + k_1,\, e^{-t} + t + k_2,\, 5t + k_3 \,\right\rangle$$

where $k_1, \ k_2,$ and k_3 are constants. Then

$$3\mathbf{i} + 4\mathbf{j} = \langle\, 3,\, 4,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\, 1 + k_2,\, k_3 \,\rangle$$

leads to $k_1 = 3, \ k_2 = 3,$ and $k_3 = 0$. Therefore

$$\mathbf{r}(t) = \left\langle\, \frac{1}{6}t^3 + 3,\ e^{-t} + t + 3,\ 5t \,\right\rangle.$$

C12S05.033: Given $\mathbf{a} = \langle\, \cos t,\, \sin t,\, 0 \,\rangle$, it follows that

$$\mathbf{v}(t) = \langle\, c_1 + \sin t,\, c_2 - \cos t,\, c_3 \,\rangle$$

where $c_1, \ c_2,$ and c_3 are constants. Then

$$-\mathbf{i} + 5\mathbf{k} = \langle\, -1,\, 0,\, 5 \,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, c_1,\, -1 + c_2,\, c_3 \,\rangle$$

implies that $c_1 = -1, \ c_2 = 1,$ and $c_3 = 5$. Hence

$$\mathbf{v}(t) = \langle\, -1 + \sin t,\, 1 - \cos t,\, 5 \,\rangle,$$

and therefore

$$\mathbf{r}(t) = \langle\, -t - \cos t + k_1,\, t - \sin t + k_2,\, 5t + k_3 \,\rangle$$

where $k_1, \ k_2,$ and k_3 are constants. Then

$$\mathbf{j} = \langle\, 0,\, 1,\, 0 \,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, -1 + k_1,\, k_2,\, k_3 \,\rangle$$

leads to $k_1 = 1, \ k_2 = 1,$ and $k_3 = 0$. Therefore

$$\mathbf{r}(t) = \langle\, 1 - t - \cos t,\ 1 + t - \sin t,\ 5t \,\rangle.$$

C12S05.034: Given $\mathbf{a} = \langle\, 9\sin 3t,\, 9\cos 3t,\, 4 \,\rangle$, it follows that

$$\mathbf{v}(t) = \langle\, -3\cos 3t + c_1,\, 3\sin 3t + c_2,\, 4t + c_3 \,\rangle$$

where $c_1, \ c_2,$ and c_3 are constants. Then

$$2\mathbf{i} - 7\mathbf{k} = \langle\, 2,\ 0,\ -7\,\rangle = \mathbf{v}(0) = \mathbf{v}_0 = \langle\, -3 + c_1,\ c_2,\ c_3\,\rangle$$

implies that $c_1 = 5,\ c_2 = 0,$ and $c_3 = -7$. Hence

$$\mathbf{v}(t) = \langle\, 5 - 3\cos 3t,\ 3\sin 3t,\ 4t - 7\,\rangle,$$

and therefore

$$\mathbf{r}(t) = \langle\, 5t - \sin 3t + k_1,\ -\cos 3t + k_2,\ 2t^2 - 7t + k_3\,\rangle$$

where $k_1,\ k_2,$ and k_3 are constants. Then

$$3\mathbf{i} + 4\mathbf{j} = \langle\, 3,\ 4,\ 0\,\rangle = \mathbf{r}(0) = \mathbf{r}_0 = \langle\, k_1,\ -1 + k_2,\ k_3\,\rangle$$

leads to $k_1 = 3,\ k_2 = 5,$ and $k_3 = 0$. Therefore

$$\mathbf{r}(t) = \langle\, 5t - \sin 3t + 3,\ 5 - \cos 3t,\ 2t^2 - 7t\,\rangle.$$

C12S05.035: The position vector of the moving point is $\mathbf{r}(t) = \langle\, 3\cos 2t,\ 3\sin 2t,\ 8t\,\rangle$. Hence its velocity, speed, and acceleration are

$$\begin{aligned}
&\mathbf{v}(t) = \langle\, -6\sin 2t,\ 6\cos 2t,\ 8\,\rangle,\\
&v(t) = |\mathbf{v}(t)| = \sqrt{36(\sin^2 2t + \cos^2 2t) + 64} = 10, \qquad \text{and}\\
&\mathbf{a}(t) = \langle\, -12\cos 2t,\ -12\sin 2t,\ 0\,\rangle,
\end{aligned}$$

respectively. Therefore

$$\mathbf{v}\left(\tfrac{7}{8}\pi\right) = \langle\, 3\sqrt{2}\,,\ 3\sqrt{2}\,,\ 8\,\rangle, \quad v\left(\tfrac{7}{8}\pi\right) = 10, \quad \text{and} \quad \mathbf{a}\left(\tfrac{7}{8}\pi\right) = \langle\, -6\sqrt{2}\,,\ 6\sqrt{2}\,,\ 0\,\rangle.$$

C12S05.036: Given $\mathbf{u}(t) = \langle\, t,\ t^2,\ t^3\,\rangle$ and $\mathbf{v}(t) = \langle\, e^t,\ \cos t,\ \sin t\,\rangle$, we have

$$\begin{aligned}
D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] &= \mathbf{u}(t)\cdot\mathbf{v}'(t) + \mathbf{u}'(t)\cdot\mathbf{v}(t)\\
&= \langle\, t,\ t^2,\ t^3\,\rangle\cdot\langle\, e^t,\ -\sin t,\ \cos t\,\rangle + \langle\, 1,\ 2t,\ 3t^2\,\rangle\cdot\langle\, e^t,\ \cos t,\ \sin t\,\rangle\\
&= (t+1)e^t + 2t^2\sin t + (t^3 + 2t)\cos t.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&D_t\,[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)\\
&\qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ t & t^2 & t^3\\ e^t & -\sin t & \cos t \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2t & 3t^2\\ e^t & \cos t & \sin t \end{vmatrix}\\
&\qquad = \langle\, t^2\cos t + t^3\sin t,\ t^3e^t - t\cos t,\ -t\sin t - t^2e^t\,\rangle
\end{aligned}$$

$$+ \langle\, 2t\sin t - 3t^2\cos t,\ 3t^2e^t - \sin t,\ \cos t - 2te^t \,\rangle$$

$$= \langle\, (t^3 + 2t)\sin t - 2t^2\cos t,\ (t^3 + 3t^2)e^t - \sin t - t\cos t,\ \cos t - t\sin t - (t^2 + 2t)e^t \,\rangle.$$

C12S05.037: Given $\mathbf{u}(t) = \langle\, 0,\, 3,\, 4t \,\rangle$ and $\mathbf{v}(t) = \langle\, 5t,\, 0,\, -4 \,\rangle$, we first compute

$$\mathbf{u}(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4t \\ 5t & 0 & -4 \end{vmatrix} = \langle\, -12,\, 20t^2,\, -15t \,\rangle.$$

Therefore $D_t\,[\mathbf{u}(t) \times \mathbf{v}(t)] = \langle\, 0,\, 40t,\, -15 \,\rangle$. Next,

$$\mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4t \\ 5 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 5t & 0 & -4 \end{vmatrix}$$

$$= \langle\, 0,\, 20t,\, -15 \,\rangle + \langle\, 0,\, 20t,\, 0 \,\rangle = \langle\, 0,\, 40t,\, -15 \,\rangle = D_t\,[\mathbf{u}(t) \times \mathbf{v}(t)]\,.$$

C12S05.038: We asked *Mathematica* 3.0 to prove part 5 of Theorem 2. First we define arbitrary 3-vectors $\mathbf{u}(t)$ and $\mathbf{v}(t)$:

```
u[t_] := { u1[t], u2[t], u3[t] }
v[t_] := { v1[t], v2[t], v3[t] }
```

Then we form their cross product:

```
c = Cross[ u[t], v[t] ]
{-u3[t]*v2[t] + u2[t]*v3[t], u3[t]*v1[t] - u1[t]*v3[t], -u2[t]*v1[t] + u1[t]*v2[t]}
```

Then we compute the derivative of the last expression with respect to t:

```
side1 = D[c,t]
  { v3[t]*u2'[t] - v2[t]*u3'[t] - u3[t]*v2'[t] + u2[t]*v3'[t],
    -v3[t]*u1'[t] + v1[t]*u3'[t] + u3[t]*v1'[t] - u1[t]*v3'[t],
    v2[t]*u1'[t] - v1[t]*u2'[t] - u2[t]*v1'[t] + u1[t]*v2'[t] }
```

Next we compute the other side of the equation:

```
side2 = Cross[u[t],v'[t]] + Cross[u'[t],v[t]]
  { v3[t]*u2'[t] - v2[t]*u3'[t] - u3[t]*v2'[t] + u2[t]*v3'[t],
    -v3[t]*u1'[t] + v1[t]*u3'[t] + u3[t]*v1'[t] - u1[t]*v3'[t],
    v2[t]*u1'[t] - v1[t]*u2'[t] - u2[t]*v1'[t] + u1[t]*v2'[t] }
```

Now we see if the two computations produce the same result:

```
Simplify[ side1 - side2 ]
{0, 0, 0}
```

and the proof is complete.

C12S05.039: Given: $|\mathbf{r}(t)| = R$, a constant. Let $\mathbf{v}(t) = \mathbf{r}'(t)$. Then $\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2$, also a constant. Hence

$$0 = D_t \left[\mathbf{r}(t) \cdot \mathbf{r}(t) \right] = 2\mathbf{r}(t) \cdot \mathbf{v}(t),$$

so that $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$. Therefore $\mathbf{v}(t)$ is perpendicular to the radius of the sphere, so the velocity vector is everywhere tangent to the sphere.

C12S05.040: If $\mathbf{v}(t)$ is the velocity vector of the moving particle, then we are given $|\mathbf{v}(t)| = c$, a constant. Then $\mathbf{v}(t) \cdot \mathbf{v}(t) = c^2$, also a constant. Hence $0 = D_t \left[\mathbf{v}(t) \cdot \mathbf{v}(t) \right] = 2\mathbf{v}(t) \cdot \mathbf{a}(t)$ where $\mathbf{a}(t)$ is the acceleration vector of the particle. But because $\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$, it follows that $\mathbf{v}$ and $\mathbf{a}$ are always perpendicular.

C12S05.041: The ball of Example 10 has position vector $\mathbf{r}(t) = \langle\, t^2,\, 80t,\, 80t - 16t^2 \,\rangle$ and velocity vector $\mathbf{v}(t) = \langle\, 2t,\, 80,\, 80 - 32t \,\rangle$. Its maximum height occurs when the z-component of its velocity vector is zero: $80 - 32t = 0$, so that $t = \frac{5}{2}$. The speed of the ball at time t is

$$v(t) = \sqrt{4t^2 + 6400 + 6400 - 5120t + 1024t^2}\,,$$

so that

$$v\left(\tfrac{5}{2}\right) = \sqrt{25 + 12800 - 12800 + 6400} = \sqrt{6425} = 5\sqrt{257} \approx 80.156097709407$$

(ft/s). Its position then is

$$\mathbf{r}\left(\tfrac{5}{2}\right) = \left\langle\, \tfrac{25}{4},\, 200,\, 100 \,\right\rangle,$$

and the z-component of this vector is its maximum height, 100 ft.

C12S05.042: Given $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$, we have

$$\begin{aligned} \mathbf{L}'(t) &= [m\mathbf{r}(t) \times \mathbf{v}'(t)] + [m\mathbf{r}'(t) \times \mathbf{v}(t)] \\ &= [m\mathbf{r}(t) \times \mathbf{a}(t)] + m\,[\mathbf{v}(t) \times \mathbf{v}(t)] = [m\mathbf{r}(t) \times \mathbf{a}(t)] + m \cdot \mathbf{0} = \boldsymbol{\tau}(t). \end{aligned}$$

C12S05.043: Because $x_0 = y_0 = 0$, the equations in (22) and (23) take the form

$$x(t) = (v_0 \cos\alpha)t, \qquad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t.$$

To find the range, find the positive value of t for which $y(t) = 0$:

$$gt = 2v_0 \sin\alpha; \qquad t = \frac{2v_0 \sin\alpha}{g};$$

thus the range is the value of $x(t)$ then; it is

$$R = x\left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{2v_0^2 \sin\alpha\, \cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$

If $\alpha = \frac{1}{4}\pi$ and $R = 5280$ (there are 5280 feet in one mile), then

$$\frac{v_0^2}{32} = 5280, \quad \text{so that} \quad v_0 = \sqrt{32 \cdot 5280} = 32\sqrt{165} \approx 411.047442517284$$

(feet per second).

C12S05.044: Because $x_0 = y_0 = 0$, the equations in (22) and (23) take the form

$$x(t) = (v_0 \cos\alpha)t, \qquad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t.$$

The maximum height of the projectile occurs when $y'(t) = 0$; that is, when

$$-gt + v_0 \sin\alpha = 0; \quad t = \frac{v_0 \sin\alpha}{g}.$$

Hence the maximum height is

$$y\left(\frac{v_0 \sin\alpha}{g}\right) = -\frac{1}{2}g \cdot \frac{v_0 \sin^2\alpha}{g^2} + \frac{v_0^2 \sin^2\alpha}{g} = \frac{v_0^2 \sin^2\alpha}{2g}. \tag{1}$$

We saw in the solution of Problem 43 that the range of the projectile is given by

$$R = \frac{v_0^2 \sin 2\alpha}{g}, \tag{2}$$

so if $\alpha = \frac{1}{3}\pi$ and $R = 5280$, we have by Eq. (2)

$$v_0^2 = \frac{5280g}{2\sin\alpha\,\cos\alpha}.$$

Therefore by Eq. (1) the maximum height of the projectile is

$$\frac{v_0^2 \sin^2\alpha}{2g} = \frac{5280g}{2\sin\alpha\,\cos\alpha} \cdot \frac{\sin^2\alpha}{2g} = \frac{1320\sin\alpha}{\cos\alpha} = 1320\tan\left(\frac{\pi}{3}\right) = 1320\sqrt{3} \approx 2286.307065990918$$

(feet).

C12S05.045: The formula for the range is derived in the solution of Problem 43.

C12S05.046: The formula for the range, derived in the solution of Problem 43, is

$$R(\alpha) = \frac{2v_0^2 \sin\alpha\,\cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$

The range will be maximized when $R'(\alpha) = 0$; that is, when

$$\frac{2v_0^2 \cos 2\alpha}{g} = 0: \quad \alpha = \frac{\pi}{4}.$$

C12S05.047: We saw in the solution of Problem 43 that the range of the projectile is

$$R = \frac{v_0^2 \sin 2\alpha}{g}.$$

To find its maximum height, first find when $y'(t) = 0$:

$$-gt = v_0 \sin\alpha, \qquad \text{so that} \qquad t = \frac{v_0 \sin\alpha}{g}.$$

Then to find the maximum height, evaluate $y(t)$ at that value of t:

$$y\left(\frac{v_0 \sin\alpha}{g}\right) = \frac{v_0^2 \sin^2\alpha}{2g}.$$

Part (a): If $v_0 = 160$ and $\alpha = \frac{1}{6}\pi$, then the range is

$$R = \frac{(160)^2 \cdot \sqrt{3}}{2 \cdot 32} = 400\sqrt{3} \approx 692.820323027551$$

(feet) and the maximum height is

$$\frac{v_0 \sin^2\alpha}{2g} = \frac{(160)^2 \cdot 1}{4 \cdot 2 \cdot 32} = 100$$

(feet). Part (b): If $v_0 = 160$ and $\alpha = \frac{1}{4}\pi$, then the range is

$$R = \frac{(160)^2 \cdot 1}{32} = 800$$

(feet) and the maximum altitude is

$$\frac{v_0^2 \sin^2\alpha}{2g} = \frac{(160)^2 \cdot 1}{2 \cdot 2 \cdot 32} = 200$$

(feet). Part (c): If $\alpha = \frac{1}{3}\pi$ and $v_0 = 160$, then the range is

$$R = \frac{(160)^2 \cdot \sqrt{3}}{2 \cdot 32} = 400\sqrt{3} \approx 692.820323027551$$

(feet) and the maximum altitude (also in feet) is

$$y_{\max} = \frac{(160)^2 \cdot 3}{4 \cdot 2 \cdot 32} = 300.$$

C12S05.048: We have seen in previous solutions that the range and maximum height of the projectile are

$$R = \frac{v_0^2 \sin 2\alpha}{g} \qquad \text{and} \qquad y_{\max} = \frac{v_0^2 \sin^2\alpha}{2g}$$

where α is the angle from the horizontal at which the projectile is fired and v_0 is its initial velocity. To clear the hill, we require $y_{\max} > 300$; we are given $v_0 = 160$ and $R = 600$. To find α, we solve

$$\frac{(160)^2 \sin 2\alpha}{32} = 600: \qquad \sin 2\alpha = \frac{600 \cdot 32}{(160)^2} = \frac{3}{4},$$

so that $2\alpha \approx 48.590378°$ or $2\alpha \approx 131.409622°$. Thus $\alpha \approx 24.295189°$ or $\alpha \approx 65.704811°$. In the first case the maximum height of the projectile will be

$$y_{\max} = \frac{v_0^2 \sin^2\alpha}{2g} = \frac{(160)^2 \sin^2\alpha}{64} \approx 67.712434$$

(feet), not enough to clear the hill. But if $\alpha \approx 65.704811°$, then the maximum height of the projectile will be

$$y_{\max} = \frac{v_0^2 \sin^2 \alpha}{2g} = \frac{(160)^2 \sin^2 \alpha}{64} \approx 332.287565$$

(feet), so the projectile will clear the hill unless the hill has an unusual shape. Answer: Angle of elevation approximately $65° \; 42' \; 17.32''$.

C12S05.049: With $x_0 = 0$ and $y_0 = 100$, the equations in (22) and (23) of the text take the form

$$x(t) = (v_0 \cos \alpha)t, \qquad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + 100.$$

With $g = 9.8$ and $\alpha = 0$, these equations become

$$x(t) = v_0 t, \qquad y(t) = 100 - \frac{1}{2}gt^2.$$

We require $x(t) = 1000$ when $y(t) = 0$. But $y(t) = 0$ when

$$t^2 = \frac{200}{9.8} = \frac{1000}{49}, \qquad \text{so that} \qquad t = \frac{10\sqrt{10}}{7}.$$

Thus

$$1000 = x\left(\frac{10\sqrt{10}}{7}\right) = \frac{10\sqrt{10}}{7} v_0,$$

and it follows that $v_0 = 70\sqrt{10} \approx 221.359436211787$ (meters per second, approximately 726.244869461242 feet per second).

C12S05.050: First we analyze the behavior of the bomb. Suppose that it is dropped at time $t = 0$. If the projectile is fired from the origin, then the equations of motion of the bomb are

$$x(t) \equiv 800, \quad y(t) = 800 - \frac{1}{2}gt^2$$

where $g = 9.8$ (m/s^2). Now $y(t) = 400$ when

$$800 - \frac{1}{2}gt^2 = 400; \qquad \frac{1}{2}gt^2 = 400;$$

$$t^2 = \frac{800}{9.8} = \frac{4000}{49}; \qquad t = T = \frac{20\sqrt{10}}{7}.$$

Now we turn our attention to the projectile, fired at time $t = 0$ from the origin. By Eqs. (22) and (23) of the text, its equations of motion are

$$x(t) = (v_0 \cos \alpha)t, \quad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t.$$

We require $x(T) = 800$ and $y(T) = 400$. Thus

$$\frac{20\sqrt{10}}{7} \cdot v_0 \cos \alpha = 800 \qquad \text{and} \qquad -\frac{1}{2}g \cdot \frac{800}{g} + \frac{20\sqrt{10}}{7} \cdot v_0 \sin \alpha = 400;$$

$$v_0 \cos\alpha = \frac{7 \cdot 800}{20\sqrt{10}} \qquad \text{and} \qquad \frac{20\sqrt{10}}{7} \cdot v_0 \sin\alpha = 400 + 400 = 800;$$

$$v_0 \cos\alpha = 28\sqrt{10} \qquad \text{and} \qquad v_0 \sin\alpha = 28\sqrt{10}\,.$$

It now follows that $\cos\alpha = \sin\alpha$, and thus $\alpha = \frac{1}{5}\pi$. Moreover,

$$v_0 = \frac{28\sqrt{10}}{\sin(\pi/4)} = 56\sqrt{5} \approx 125.219806739988$$

meters per second, approximately 410.826137598387 feet per second.

C12S05.051: First we analyze the behavior of the bomb. Suppose that it is dropped at time $t = 0$. If the projectile is fired from the origin, then the equations of motion of the bomb are

$$x(t) \equiv 800, \quad y(t) = 800 - \frac{1}{2}gt^2$$

where $g = 9.8$ (m/s^2). Now $y(t) = 400$ when

$$800 - \frac{1}{2}gt^2 = 400; \qquad \frac{1}{2}gt^2 = 400;$$

$$t^2 = \frac{800}{9.8} = \frac{4000}{49}; \qquad t = T = \frac{20\sqrt{10}}{7}.$$

Now we turn our attention to the projectile, fired at time $t = 0$ from the origin. (We will adjust for the one-second delay later in this solution.) By Eqs. (22) and (23) of the text, its equations of motion are

$$x(t) = (v_0 \cos\alpha)t, \quad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t.$$

We require $x(T-1) = 800$ and $y(T-1) = 400$. (This is how we take care of the one-second delay). Thus

$$T - 1 = \frac{20\sqrt{10} - 7}{7};$$

$$(v_0 \cos\alpha)(T-1) = 800;$$

$$-\frac{1}{2}g(T-1)^2 + (v_0 \sin\alpha)(T-1) = 400;$$

$$v_0 \cos\alpha = \frac{800}{T-1}. \tag{1}$$

Also

$$-\frac{1}{2}g(T-1) + v_0 \sin\alpha = \frac{400}{T-1};$$

$$v_0 \sin\alpha = \frac{400}{T-1} + \frac{1}{2}g(T-1). \tag{2}$$

Division of Eq. (2) by Eq. (1) then yields

$$\tan\alpha = \frac{T-1}{800}\cdot\left[\frac{400}{T-1} + \frac{1}{2}g(T-1)\right] = \frac{1}{2} + \frac{9.8}{1600}(T-1)^2$$

$$= \frac{1}{2} + \frac{98}{16000}\cdot\left(\frac{20\sqrt{10}\,-7}{7}\right)^2 = \frac{8049-280\sqrt{10}}{8000} \approx 0.895445,$$

so that $\alpha \approx 41.842705345876°$. Then, by Eq. (1),

$$v_0 = \frac{800}{(T-1)\cos\alpha} \approx 133.645951548503$$

meters per second, approximately 438.470969647319 feet per second.

C12S05.052: With the origin at the base of the cliff, the equations of motion of the projectile are

$$x(t) = (v_0\cos\alpha)T, \quad y(t) = -\frac{1}{2}gt^2 + (v_0\sin\alpha)t + 500.$$

Here we use $v_0 = 1000$ and $g = 32$. We require $x(T) = 20000$ and $y(T) = 0$ simultaneously. Thus

$$(v_0\cos\alpha)T = 20000: \quad T = \frac{20000}{v_0\cos\alpha} = \frac{20}{\cos\alpha}$$

and

$$0 = y(T) = -2 + (1000\sin\alpha)T + 500:$$

$$-16\cdot\frac{400}{\cos^2\alpha} + \frac{20000\sin\alpha}{\cos\alpha} + 500 = 0;$$

$$6400\sec^2\alpha - 20000\tan\alpha - 500 = 0;$$

$$6400 + 6400\tan^2\alpha - 20000\tan\alpha - 500 = 0;$$

$$6400\tan^2\alpha - 20000\tan\alpha + 5900 = 0;$$

$$64\tan^2\alpha - 200\tan\alpha + 59 = 0;$$

$$\tan\alpha = \frac{200 \pm \sqrt{40000-15104}}{128} = \frac{200\pm\sqrt{24896}}{128} = \frac{25\pm\sqrt{389}}{16}.$$

It now follows that $\alpha \approx 18.252934°$ and $\alpha \approx 70.314970°$ are both solutions. Answer: There are two angles; one is approximately $18°\ 15'\ 11''$ and the other is approximately $70°\ 18'\ 54''$.

C12S05.053: We give proofs for vectors with two components; these proofs generalize readily to vectors with three or more components. Let $\mathbf{u}(t) = \langle\, u_1(t),\, u_2(t)\,\rangle$ and $\mathbf{v}(t) = \langle\, v_1(t),\, v_2(t)\,\rangle$. The assumptions that

$$\lim_{t\to a}\mathbf{u}(t) \quad\text{and}\quad \lim_{t\to a}\mathbf{v}(t)$$

both exist means that there exist vectors $\langle\, p_1,\, p_2\,\rangle$ and $\langle\, q_1,\, q_2\,\rangle$ such that

$$\lim_{t\to a} u_1(t) = p_1, \quad \lim_{t\to a} u_2(t) = p_2, \quad \lim_{t\to a} v_1(t) = q_1, \quad\text{and}\quad \lim_{t\to a} v_2(t) = q_2.$$

Part (a):

$$\lim_{t\to a}\,[\mathbf{u}(t)+\mathbf{v}(t)] = \lim_{t\to a}\langle\, u_1(t)+v_1(t),\ u_2(t)+v_2(t)\,\rangle = \langle\, p_1+q_1,\ p_2+q_2\,\rangle$$

$$= \langle\, p_1,\, p_2\,\rangle + \langle\, q_1,\, q_2\,\rangle = \left[\lim_{t\to a}\mathbf{u}(t)\right] + \left[\lim_{t\to a}\mathbf{v}(t)\right].$$

Part (b):

$$\lim_{t\to a}\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \lim_{t\to a}\,[u_1(t)v_1(t)+u_2(t)v_2(t)] = p_1q_1+p_2q_2$$

$$= \langle\, p_1,\, p_2\,\rangle\cdot\langle\, q_1,\, q_2\,\rangle = \left(\lim_{t\to a}\mathbf{u}(t)\right)\cdot\left(\lim_{t\to a}\mathbf{v}(t)\right).$$

C12S05.054: We give the proof in case the vector function $\mathbf{r}(t)$ has two components; the proof generalizes readily to vectors with three or more components. Suppose that $\mathbf{r}(t) = \langle\, r_1(t),\, r_2(t)\,\rangle$. Then

$$D_t\,[\mathbf{r}(h(t))] = D_t\langle\, r_1(h(t)),\ r_2(h(t))\,\rangle$$

$$= \langle\, r_1'(h(t))\cdot h'(t),\ r_2'(h(t))\cdot h'(t)\,\rangle = h'(t)\langle\, r_1'(h(t)),\ r_2'(h(t))\,\rangle = h'(t)\mathbf{r}'(h(t)).$$

C12S05.055: If $\mathbf{v}(t)$ is the velocity vector of the moving particle, then we are given $|\mathbf{v}(t)| = C$, a constant. Then $\mathbf{v}(t)\cdot\mathbf{v}(t) = C^2$, also a constant. Hence $0 = D_t\,[\mathbf{v}(t)\cdot\mathbf{v}(t)] = 2\mathbf{v}(t)\cdot\mathbf{a}(t)$ where $\mathbf{a}(t)$ is the acceleration vector of the particle. But because $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$, it follows that $\mathbf{v}$ and $\mathbf{a}$ are always perpendicular.

C12S05.056: Let $\mathbf{r}(t)$ be the position vector of the moving point and let R denote the radius of the circle. Then $|\mathbf{r}(t)| = R$, a constant. Thus $\mathbf{r}(t)\cdot\mathbf{r}(t) = R^2$, also a constant. Differentiation of both sides of this equation (actually, an *identity*) with respect to t yields $2\mathbf{r}(t)\cdot\mathbf{v}(t) = 0$, so that $\mathbf{r}(t)\cdot\mathbf{v}(t) = 0$ for all t. Thus $\mathbf{r}$ and $\mathbf{v}$ are always perpendicular.

C12S05.057: If $\mathbf{r}(t) = \langle\, \cosh\omega t,\ \sinh\omega t\,\rangle$, then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, \omega\sinh\omega t,\ \omega\cosh\omega t\,\rangle \quad \text{and}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle\, \omega^2\cosh\omega t,\ \omega^2\sinh\omega t\,\rangle = \omega^2\mathbf{r}(t) = c\mathbf{r}(t)$$

where $c = \omega^2 > 0$. An external force that would produce this sort of motion would be a central repulsive force proportional to distance from the origin.

C12S05.058: If $\mathbf{r}(t) = \langle\, a\cos\omega t,\ b\sin\omega t\,\rangle$, then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, -a\omega\sin\omega t,\ b\omega\cos\omega t\,\rangle \quad \text{and}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle\, -a\omega^2\cos\omega t,\ -b\omega^2\sin\omega t\,\rangle = -\omega^2\langle\, a\cos\omega t,\ b\sin\omega t\,\rangle = -\omega^2\mathbf{r}(t) = c\mathbf{r}(t)$$

where $c = -\omega^2 < 0$. An external force producing this type of motion would be a central force directed toward the origin and with magnitude proportional to distance from the origin.

C12S05.059: Given the acceleration vector $\mathbf{a} = \langle 0,\, a\,\rangle$, we first find the velocity and position vectors:

$$\mathbf{v}(t) = \langle\, c_1,\ at+c_2\,\rangle \quad \text{and}$$

$$\mathbf{r}(t) = \langle\, c_1t+k_1,\ \tfrac{1}{2}at^2+c_2t+k_2\,\rangle$$

where $c_1,\ c_2,\ k_1,$ and k_2 are constants. Thus the position $(x(t),\ y(t))$ of the moving point is given by

$$x(t) = c_1 t + k_1 \quad \text{and} \quad y(t) = \tfrac{1}{2}at^2 + c_2 t + k_2.$$

If $c_1 = 0$ then the point moves in a straight line. Otherwise, we solve the first of these equations for t and substitute in the second:

$$t = \frac{x - k_1}{c_1};$$

$$\begin{aligned}
y &= \frac{1}{2}a\left(\frac{x-k_1}{c_1}\right)^2 + c_2 \cdot \frac{x-k_1}{c_1} + k_2 \\
&= \frac{a}{2c_1^2}(x^2 - 2k_1 x + k_1^2) + \frac{c_2}{c_1}(x - k_1) + k_2 \\
&= \frac{a}{2c_1^2}x^2 + \left(\frac{c^2}{c_1} - \frac{ak_1}{c_1^2}\right)x + \left(\frac{ak_1^2}{2c_1^2} - \frac{c_2 k_1}{c_1} + k_2\right) \\
&= Ax^2 + Bx + C
\end{aligned}$$

where $A,\ B,$ and C are constants. If $A \neq 0$ then the trajectory of the particle is a parabola. If $A = 0$ it is a straight line.

C12S05.060: If the acceleration of the particle is $\mathbf{a} = \langle\, 0,\ 0,\ 0\,\rangle$, then we first find its velocity and position vectors:

$$\begin{aligned}
&\mathbf{v}(t) = \langle\, c_1,\ c_2,\ c_3\,\rangle \quad \text{and} \\
&\mathbf{r}(t) = \langle\, c_1 t + k_1,\ c_2 t + k_2,\ c_3 t + k_3\,\rangle
\end{aligned}$$

where c_i and k_i are constants for $1 \leqq i \leqq 3$. Hence the position $(x,\ y,\ z)$ of the moving particle is given by

$$x = c_1 t + k_1, \quad y = c_2 t + k_2, \quad z = c_3 t + k_3.$$

If $c_1 = c_2 = c_3 = 0$ then the particle remains at a single point without motion. Otherwise these are Cartesian equations of a straight line, and that is the trajectory of the particle. Note also that its speed is given by

$$v(t) = |\mathbf{v}(t)| = \sqrt{c_1^2 + c_2^2 + c_3^2}\ ,$$

a constant.

C12S05.061: Given: $\mathbf{r}(t) = \langle\, r\cos\omega t,\ r\sin\omega t\,\rangle$. Part (a):

$$\begin{aligned}
&\mathbf{v}(t) = \langle\, -r\omega\sin\omega t,\ r\omega\cos\omega t\,\rangle = r\omega\langle\, -\sin\omega t,\ \cos\omega t\,\rangle. \quad \text{So} \\
&\mathbf{r}(t) \cdot \mathbf{v}(t) = r^2\omega(-\sin\omega t\cos\omega t + \sin\omega t\cos\omega t) = 0,
\end{aligned}$$

and therefore $\mathbf{r}$ and $\mathbf{v}$ are always perpendicular. Therefore $\mathbf{v}$ is always tangent to the circle. Moreover, the speed of motion is

$$v(t) = r\omega\sqrt{\sin^2\omega t + \cos^2\omega t} = r\omega.$$

Part (b): $\mathbf{a}(t) = r\omega^2\langle -\cos\omega t,\ -\sin\omega t\,\rangle = -\omega^2\mathbf{r}(t)$. Therefore $\mathbf{a}$ and $\mathbf{r}$ are always parallel and have opposite directions (because $-\omega^2 < 0$). Finally, the scalar acceleration is

$$a(t) = |\mathbf{a}(t)| = |-\omega^2| \cdot |\mathbf{r}(t)| = r\omega^2.$$

C12S05.062: Because $\mathbf{F} = k\mathbf{r}$, $\mathbf{r}$ and $\mathbf{a}$ are parallel. So

$$D_t(\mathbf{r} \times \mathbf{v}) = (\mathbf{r} \times \mathbf{a}) + (\mathbf{v} \times \mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Therefore $\mathbf{r} \times \mathbf{v} = \mathbf{C}$, a constant vector. Consequently the vector $\mathbf{r}$ is always perpendicular to the constant vector $\mathbf{C}$. This holds for every point on the trajectory of the particle, and thus every point on the trajectory lies in the plane through the origin with normal vector $\mathbf{C}$.

C12S05.063: With north the direction of the positive x-axis, west the direction of the positive y-axis, and upward the direction of the positive z-axis, the baseball has acceleration $\mathbf{a}(t) = \langle\, 0.1,\ 0,\ -32\,\rangle$, initial velocity $\mathbf{v}_0 = \langle\, 0,\ 0,\ 160\,\rangle$, and initial position $\mathbf{r}_0 = \langle\, 0,\ 0,\ 0\,\rangle$. It follows that its position vector is

$$\mathbf{r}(t) = \left\langle\ \frac{1}{20}t^2,\ 0,\ 160t - 16t^2\ \right\rangle.$$

The ball returns to the ground at that positive value of t for which the z-component of $\mathbf{r}$ is zero; that is, $t = 10$. At that time the x-component of $\mathbf{r}$ is 5, so the ball lands 5 feet north of the point from which it was thrown.

C12S05.064: We assume that the baseball is hit directly down the left-field foul line, that this line coincides with the positive y-axis, and that its direction is due north. We also assume that the acceleration due to spin is directed due east, in the direction of the positive x-axis. Then the acceleration vector of the baseball is $\mathbf{a}(t) = \langle\, 2,\ 0,\ -32\,\rangle$. Its initial velocity is $\mathbf{v}_0 = \langle\, 0,\ 96\cos 15^\circ,\ 96\sin 15^\circ\,\rangle$, so its velocity vector is

$$\mathbf{v}(t) = \left\langle 2t,\ 24\left(1 + \sqrt{3}\right)\sqrt{2}\,,\ 24\left(-1 + \sqrt{3}\right)\sqrt{2}\ - 32t\ \right\rangle.$$

The initial position of the baseball is $\mathbf{r}_0 = \langle\, 0,\ 0,\ 0\,\rangle$, so the baseball has position vector

$$\mathbf{r}(t) = \left\langle\ t^2,\ 24t\left(1 + \sqrt{3}\right)\sqrt{2}\,,\ 24t\left(-1 + \sqrt{3}\right)\sqrt{2}\ - 16t^2\ \right\rangle.$$

The ball strikes the ground when the z-component of $\mathbf{r}$ is zero; that is, when

$$t = \frac{3}{2}\left(\sqrt{6}\ - \sqrt{2}\right) \approx 1.552914.$$

At this time the x-component of $\mathbf{r}(t)$ is $18 - 9\sqrt{3}\ \approx 2.411543$, so the ball hits the ground just under 2 ft 5 in. from the foul line.

C12S05.065: In the "obvious" coordinate system, the acceleration of the projectile is $\mathbf{a}(t) = \langle\, 2,\ 0,\ -32\,\rangle$; its initial velocity is $\mathbf{v}_0 = \langle\, 0,\ 200,\ 160\,\rangle$ and its initial position is $\mathbf{r}_0 = \langle\, 0,\ 0,\ 384\,\rangle$. Hence its velocity and position vectors are

$$\mathbf{v}(t) = \langle\, 2t,\ 200,\ 160 - 32t\,\rangle \quad \text{and} \quad \mathbf{r}(t) = \langle\, t^2,\ 200t,\ 384 - 160t - 16t^2\,\rangle.$$

The projectile strikes the ground at that positive value of t from which the z-component of $\mathbf{r}$ is zero:

$$16t^2 - 160t - 384 = 0; \qquad t^2 - 10t - 24 = 0;$$

$$(t-12)(t+2) = 0; \qquad t = 12 \quad (\textit{not} \;\; t = -2).$$

When $t = 12$, the position of the projectile is $\mathbf{r}(12) = \langle\, 144,\ 2400,\ 0\,\rangle$, so it lands 2400 ft north and 144 ft east of the firing position. The projectile reaches its maximum altitude when the z-component of $\mathbf{v}(t)$ is zero; that is, when $t = 5$. Its position then is $\mathbf{r}(5) = \langle\, 25,\ 1000,\ 784\,\rangle$, so its maximum altitude is 784 ft.

C12S05.066: Situate the gun at the origin, north the positive y-direction, east the positive x-direction, upward the positive z-direction. Assume that the gun is fired at time $t = 0$ (seconds) with angle α of elevation and lateral deviation θ measured counterclockwise from the positive y-axis. Note that θ will be rather close to zero. If T is the time of impact of a shell at the point $(0, 5000, 0)$, it is then easy to derive the equations

$$\begin{aligned}
&T = 6000\cos\alpha\,\sin\theta,\\
&T\cos\alpha\,\cos\theta = 10, \quad \text{and}\\
&4T = 125\sin\alpha.
\end{aligned}$$

To obtain a first approximation to a solution, assume that $\theta = 0$. The previous equations then imply that

$$\sin 2\alpha = \frac{16}{25} \qquad \text{and} \qquad T = 10\sec\alpha.$$

The first of these equations has two first-quadrant solutions:

$$\alpha \approx 19.89590975^\circ \qquad \text{and} \qquad \alpha \approx 70.10409025^\circ.$$

The corresponding values of T are

$$T \approx 10.63476324 \qquad \text{and} \qquad T \approx 29.38476324 \quad \text{(seconds)}.$$

One may now continue in a very pragmatic way: Fire the gun due north with the smaller value of α. It's easy to show that the shell won't clear the hill. So the larger value of α must be used in any case. If the gun is fired due north with the larger value of α, the shell will strike the ground at

$$(x, y, z) \approx (71.95535922, 5000, 0)\,.$$

So swivel the gun counterclockwise through an angle of

$$\theta = \arctan\left(\frac{71.95535922}{5000}\right) \approx 0.8244907643^\circ$$

and *really* fire it this time. The results:

$$T = 29.38476324 \qquad \text{and point of impact} \qquad (x, y, z) \approx (0.0074499339,\ 4999.482323,\ 0.000001).$$

This is certainly close enough! You can also verify that the shell easily clears the hill unless the hill has an abnormal shape (the shell reaches a maximum altitude of more than 3450 ft).

A more sophisticated solution might proceed as follows. Obtain the approximate values of α and T using $\theta = 0$. Beginning with those values, iterate the following versions of the first equations (this is a method of repeated substitution):

$$T = \frac{125}{4}\sin\alpha,$$

$$\theta = \arcsin\left(\frac{T}{6000\cos\alpha}\right),$$

$$\alpha = \arccos\left(\frac{10}{T\cos\theta}\right).$$

A few iterations of these equations, in the order given, results in convergence to the values

$$T \approx 29.38430462,$$

$$\alpha \approx 70.10161954^\circ, \qquad \text{and}$$

$$\theta \approx 0.8244650319^\circ.$$

The point of impact is $(-0.0000000078,\ 5000,\ 0.0000011)$ if these values are used. The errors in x and z are undoubtedly roundoff errors.

Section 12.6

C12S06.001: We first compute

$$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{64 + 36\cos^2 2t + 36\sin^2 2t} = \sqrt{100} = 10.$$

Therefore the length of the graph is

$$s = \int_0^\pi 10\, dt = \Big[10t\Big]_0^\pi = 10\pi.$$

C12S06.002: Here we have

$$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{1 + 2t^2 + t^4} = t^2 + 1.$$

Hence the length of the graph is

$$s = \int_0^1 (t^2 + 1)\, dt = \left[\frac{t^3}{3} + t\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}.$$

C12S06.003: First,

$$\begin{aligned}
v(t) &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{(6e^t\cos t - 6e^t\sin t)^2 + (6e^t\cos t + 6e^t\sin t)^2 + (17e^t)^2} \\
&= \sqrt{289e^{2t} + 36e^{2t}\cos^2 t - 72e^{2t}\sin t\cos t + 36e^{2t}\sin t + 36e^{2t}\cos^2 t + 72e^{2t}\sin t\cos t + 36e^{2t}\sin^2 t} \\
&= \sqrt{289e^{2t} + 36e^{2t} + 36e^{2t}} = \sqrt{361e^{2t}} = 19e^t.
\end{aligned}$$

Therefore the arc length is

$$s = \int_0^1 19e^t\, dt = \Big[19e^t\Big]_0^1 = 19e - 19 = 19(e-1) \approx 32.647354740722.$$

C12S06.004: Here we compute

$$v(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{t^2 + t^{-2} + 2} = t + \frac{1}{t}.$$

Therefore the length of the arc is

$$s = \int_1^2 \left(t + \frac{1}{t}\right) dt = \left[\frac{t^2}{2} + \ln t\right]_1^2 = 2 + \ln 2 - \frac{1}{2} = \frac{3}{2} + \ln 2 \approx 2.193147180560.$$

C12S06.005: First,

$$\begin{aligned}
v(t) &= \sqrt{(3t\cos t + 3\sin t)^2 + (3\cos t - 3t\sin t)^2 + (4t)^2} \\
&= \sqrt{16t^2 + 9\cos^2 t + 9\sin^2 t + 9t^2\cos^2 t + 9t^2\sin^2 t} = \sqrt{9 + 25t^2}\,.
\end{aligned}$$

Then the substitutions $u = 5t,\ a = 3,$ and formula 44 of the endpapers of the text yields arc length

$$s = \int_0^{4/5} \sqrt{9 + 25t^2}\ dt = \left[\frac{t}{2}\sqrt{9+25t^2} + \frac{9}{10}\left(\ln 5t + \sqrt{9+25t^2}\right)\right]_0^{4/5}$$

$$= 2 + \frac{9}{10}\ln 9 - \frac{9}{10}\ln 3 = \frac{20 + 9\ln 3}{10} \approx 2.988751059801.$$

By comparison, *Mathematica* 3.0 yields

$$s = \left[\frac{t}{2}\sqrt{9+25t^2} + \frac{9}{10}\sinh^{-1}\left(\frac{5t}{3}\right)\right]_0^{4/5},$$

which can be simplified to the same answer using identities found in Section 7.6.

C12S06.006: First we compute

$$v(t) = \sqrt{(2e^t)^2 + (-e^{-t})^2 + 2^2} = \sqrt{4e^{2t} + 4 + e^{-2t}} = 2e^t + e^{-t}.$$

Then the arc length in question is

$$s = \int_0^1 \left(2e^t + e^{-t}\right)\ dt = \left[2e^t - e^{-t}\right]_0^1 = 2e - e^{-1} - 1 \approx 4.068684215747.$$

C12S06.007: By Eq. (13) of the text,

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{|6x|}{(1+9x^4)^{3/2}},$$

and therefore $\kappa(0) = 0$.

C12S06.008: By Eq. (13) of the text,

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{|6x|}{(1+9x^4)^{3/2}},$$

and therefore

$$\kappa(-1) = \frac{6}{10^{3/2}} = \frac{3\sqrt{10}}{50} \approx 0.1897366596.$$

C12S06.009: By Eq. (13) of the text,

$$\kappa(x) = \frac{|\cos x|}{(1+\sin^2 x)^{3/2}},$$

and thus $\kappa(0) = 1$.

C12S06.010: By Eq. (12) of the text,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{|1\cdot 2 - 0|}{[1^2 + (2t+3)^2]^{3/2}},$$

and consequently

$$\kappa(2) = \frac{2}{(1+49)^{3/2}} = \frac{2}{50^{3/2}} = \frac{2\sqrt{50}}{2500} = \frac{\sqrt{2}}{250} \approx 0.005656854248.$$

C12S06.011: By Eq. (12) of the text,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{|-20\sin^2 t - 20\cos^2 t|}{[25\sin^2 t + 16\cos^2 t]^{3/2}} = \frac{20}{(16 + 9\sin^2 t)^{3/2}},$$

so that

$$\kappa\left(\frac{\pi}{4}\right) = \frac{20}{\left(16 + \frac{9}{2}\right)^{3/2}} = \frac{40\sqrt{82}}{1681} \approx 0.2154761484.$$

C12S06.012: By Eq. (12) of the text,

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{|15\sinh^2 t - 15\cosh^2 t|}{(25\sinh^2 t + 9\cosh^2 t)^{3/2}} = \frac{15}{(25\sinh^2 t + 9\cosh^2 t)^{3/2}}.$$

Therefore

$$\kappa(0) = \frac{15}{9^{3/2}} = \frac{15}{27} = \frac{5}{9}.$$

C12S06.013: Given: $y = e^x$. By Eq. (13) of the text, the curvature at x is

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}.$$

Because $\kappa(x) > 0$ for all x, $\kappa(x) \to 0$ as $x \to \pm\infty$, and κ is continuous on the set of all real numbers, there is a maximum value. Next,

$$\kappa'(x) = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}; \quad \kappa'(x) = 0 \quad \text{when} \quad e^{2x} = \frac{1}{2}: \quad x = -\frac{1}{2}\ln 2.$$

Answer: The maximum curvature of the graph of $y = e^x$ occurs at the point

$$\left(-\frac{1}{2}\ln 2, \ \frac{1}{2}\sqrt{2}\right).$$

The curvature there is

$$\frac{\frac{1}{2}\sqrt{2}}{\left(1 + \frac{1}{2}\right)^{3/2}} = \frac{\sqrt{2}}{2} \cdot \frac{2^{3/2}}{3^{3/2}} = \frac{2\sqrt{3}}{9} \approx 0.3849001794597505.$$

C12S06.014: Given: $y = \ln x$. By the result in the solution of Problem 13, there is a unique point where the curvature of the graph is maximal and it is

$$\left(\frac{1}{2}\sqrt{2}, \ -\frac{1}{2}\ln 2\right).$$

Alternatively, by Eq. (12), the curvature at x is

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \frac{x^{-2}}{(1 + x^{-2})^{3/2}} = \frac{x^{-2} \cdot x^3}{(x^2 + 1)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}.$$

Because $\kappa(x) > 0$ for all $x > 0$, $\kappa(x) \to 0$ as $x \to +\infty$ and as $x \to 0^+$, and κ is continuous on $(0, +\infty)$, there is a maximum value. To find it,

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - x \cdot 2x \cdot \frac{3}{2}(x^2 + 1)^{1/2}}{(x^2 + 1)^3} = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}; \quad \kappa'(x) = 0 \quad \text{when} \quad x = \frac{1}{2}\sqrt{2}\,.$$

C12S06.015: Given: $x = 5\cos t$, $y = 3\sin t$. By Eq. (12) of the text, the curvature at $(x(t),\, y(t))$ is given by

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{|15\sin^2 t + 15\cos^2 t|}{(25\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{15}{(9 + 16\sin^2 t)^{3/2}}.$$

Nothing is lost by restriction of t to the interval $[0,\, 2\pi]$, and $\kappa(t)$ is continuous there, so κ has both a global maximum value and a global minimum value in that interval. To find them,

$$\kappa'(t) = \frac{-15 \cdot \frac{3}{2}(9 + 16\sin^2 t)^{1/2} \cdot 32\sin t\cos t}{(9 + 16\sin^2 t)^3} = -\frac{720\sin t\cos t}{(9 + 16\sin^2 t)^{5/2}}.$$

Because $\kappa'(t) = 0$ at every integral multiple of $\pi/2$, we check these critical points (and only these):

$$\kappa(0) = \frac{5}{9} = \kappa(\pi) = \kappa(2\pi);$$

$$\kappa(\pi/2) = \frac{3}{25} = \kappa(3\pi/2).$$

Therefore the maximum curvature of the graph of the given parametric equations is $\frac{5}{9}$, which occurs at $(5,\, 0)$ and at $(-5,\, 0)$ (corresponding to $t = 0$ and $t = \pi$); the minimum curvature is $\frac{3}{25}$ and occurs at $(0,\, 3)$ and at $(0,\, -3)$.

C12S06.016: Given: $xy = 1$. By Eq. (13) of the text, the curvature at x is given by

$$\kappa(x) = \frac{|2x^{-3}|}{(1 + x^{-4})^{3/2}} = \frac{2|x^3|}{(x^4 + 1)^{3/2}}.$$

Because $\kappa(x) \to 0$ as $x \to \pm\infty$ and as $x \to 0$ and κ is continuous on $(-\infty,\, 0)$ and on $(0,\, +\infty)$, κ has a global maximum on each of these two intervals. By symmetry it suffices to consider only the case in which $x > 0$. Then

$$\kappa'(x) = \frac{6x^2(x^4 + 1)^{3/2} - 2x^3 \cdot \frac{3}{2}(x^4 + 1)^{1/2} \cdot 4x^3}{(x^4 + 1)^3} = \frac{6x^2(x^4 + 1) - 12x^6}{(x^4 + 1)^{5/2}} = \frac{6x^2(1 - x^4)}{(x^4 + 1)^{5/2}};$$

$\kappa'(x) = 0$ when $x = 1$. By symmetry, in the case $x < 0$ we find that $\kappa'(x) = 0$ when $x = -1$. Answer: The maximum curvature occurs at the two points $(-1,\, -1)$ and $(1,\, 1)$ and the curvature there is $\frac{1}{2}\sqrt{2}\,$.

C12S06.017: Let $\mathbf{r}(t) = \langle\, t,\, t^3 \,\rangle$. For the purpose of determining the direction of $\mathbf{N}$, note that the graph of $y = x^3$ is concave downward at and near the given point $(-1,\, -1)$. Then

$$\mathbf{v}(t) = \langle\, 1,\, 3t^2 \,\rangle;$$

$$\mathbf{v}(-1) = \langle\, 1,\, 3\,\rangle;$$

$$\mathbf{T}(-1) = \left\langle\, \frac{\sqrt{10}}{10},\, \frac{3\sqrt{10}}{10} \,\right\rangle;$$

$$\mathbf{N}(-1) = \left\langle\, \frac{3\sqrt{10}}{10},\, -\frac{\sqrt{10}}{10} \,\right\rangle.$$

C12S06.018: Let $\mathbf{r}(t) = \langle\, t^3,\, t^2\,\rangle$. For the purpose of determining the direction of $\mathbf{N}$, note that the graph of the given parametric equations is concave downward at and near the given point $(-1,\, 1)$. Then

$$\mathbf{v}(t) = \langle\, 3t^2,\, 2t\,\rangle; \qquad \mathbf{v}(-1) = \langle\, 3,\, -2\,\rangle;$$

$$\mathbf{T}(-1) = \left\langle\, \frac{3\sqrt{13}}{13},\, -\frac{2\sqrt{13}}{13} \,\right\rangle; \qquad \mathbf{N}(-1) = \left\langle\, -\frac{2\sqrt{13}}{13},\, -\frac{3\sqrt{13}}{13} \,\right\rangle.$$

C12S06.019: Let $\mathbf{r}(t) = \langle\, 3\sin 2t,\, 4\cos 2t\,\rangle$. For the purpose of determining the direction of $\mathbf{N}$, note that the graph of the given parametric equations is concave downward at and near the given point for which $t = \pi/6$. Then

$$\mathbf{v}(t) = \langle\, 6\cos 2t,\, -8\sin 2t\,\rangle; \qquad \mathbf{v}(\pi/6) = \langle\, 3,\, -4\sqrt{3}\,\rangle;$$

$$\mathbf{T}(\pi/6) = \left\langle\, \frac{\sqrt{57}}{19},\, -\frac{4\sqrt{19}}{19} \,\right\rangle; \qquad \mathbf{N}(\pi/6) = \left\langle\, -\frac{4\sqrt{19}}{19},\, -\frac{\sqrt{57}}{19} \,\right\rangle.$$

C12S06.020: Let $\mathbf{r}(t) = \langle\, t - \sin t,\, 1 - \cos t\,\rangle$. For the purpose of determining the direction of $\mathbf{N}$, note that the graph of the given parametric equations is concave downward at and near the given point for which $t = \pi/2$. Then

$$\mathbf{v}(t) = \langle\, 1 - \cos t,\, \sin t\,\rangle; \qquad \mathbf{v}(\pi/2) = \langle\, 1,\, 1\,\rangle;$$

$$v(t) = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}; \qquad v(\pi/2) = \sqrt{2};$$

$$\mathbf{T}(\pi/2) = \left\langle\, \frac{\sqrt{2}}{2},\, \frac{\sqrt{2}}{2} \,\right\rangle; \qquad \mathbf{N}(\pi/2) = \left\langle\, \frac{\sqrt{2}}{2},\, -\frac{\sqrt{2}}{2} \,\right\rangle.$$

C12S06.021: Let $\mathbf{r}(t) = \langle\, \cos^3 t,\, \sin^3 t\,\rangle$. For the purpose of determining the direction of $\mathbf{N}$, note that the graph of the given parametric equations is concave upward at and near the given point for which $t = 3\pi/4$. Then

$$\mathbf{v}(t) = \langle\, -3\cos^2 t\sin t,\, 3\sin^2 t\cos t\,\rangle; \qquad \mathbf{v}(3\pi/4) = \left\langle\, -\frac{3\sqrt{2}}{4},\, -\frac{3\sqrt{2}}{4} \,\right\rangle; \qquad v(3\pi/4) = \frac{3}{2};$$

$$\mathbf{T}(3\pi/4) = \left\langle\, -\frac{\sqrt{2}}{2},\, -\frac{\sqrt{2}}{2} \,\right\rangle; \qquad \mathbf{N}(3\pi/4) = \left\langle\, -\frac{\sqrt{2}}{2},\, \frac{\sqrt{2}}{2} \,\right\rangle.$$

C12S06.022: We adjoin third component zero to form $\mathbf{r}(t) = \langle\, 3\sin \pi t,\, 3\cos \pi t,\, 0\,\rangle$. Then

$$\mathbf{v}(t) = \langle\, 3\pi\cos\pi t,\ -3\pi\sin\pi t,\ 0\,\rangle \qquad \mathbf{a}(t) = \langle\, -3\pi^2\sin\pi t,\ -3\pi^2\cos\pi t,\ 0\,\rangle,$$

and $v(t) \equiv 3\pi$. By Eq. (26) of the text,

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{0}{v} = 0,$$

and by Eq. (28),

$$a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{v} = \frac{1}{3\pi}\,|\langle\, 0,\ 0,\ -9\pi^3\,\rangle| = 3\pi^2.$$

C12S06.023: We adjoin third component zero to form $\mathbf{r}(t) = \langle\, 2t+1,\ 3t^2-1,\ 0\,\rangle$. Then $\mathbf{v}(t) = \langle\, 2,\ 6t,\ 0\,\rangle$, $\mathbf{a}(t) = \langle\, 0,\ 6,\ 0\,\rangle$, and $v(t) = (36t^2+4)^{1/2}$. By Eq. (26) of the text,

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{36t}{(36t^2+4)^{1/2}} = \frac{18t}{\sqrt{9t^2+1}},$$

and by Eq. (28),

$$a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{v} = \frac{1}{v}\cdot|\langle\, 0,\ 0,\ 12\,\rangle| = \frac{6}{\sqrt{9t^2+1}}.$$

C12S06.024: We adjoin third component zero to form $\mathbf{r}(t) = \langle\, \cosh 3t,\ \sinh 3t,\ 0\,\rangle$. Then

$$\mathbf{v}(t) = \langle\, 3\sinh 3t,\ 3\cosh 3t,\ 0\,\rangle \qquad \mathbf{a}(t) = \langle\, 9\cosh 3t,\ 9\sinh 3t,\ 0\,\rangle,$$

and $v(t) = \sqrt{9\sinh^2 3t + 9\cosh^2 3t} = 3\sqrt{\sinh^2 3t + \cosh^2 3t}$. Next,

$$\mathbf{v}\cdot\mathbf{a} = 27\sinh 3t\cosh 3t + 27\cosh 3t\sinh 3t = 54\sinh 3t\cosh 3t,$$

so by Eq. (26) of the text,

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v} = \frac{18\sinh 3t\cosh 3t}{\sqrt{\sinh^2 3t + \cosh^2 3t}} = \frac{9\sinh 6t}{\sqrt{\cosh 6t}}$$

(we used various hyperbolic identities from Section 6.9 to simplify some of these answers). Next,

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\sinh 3t & 3\cosh 3t & 0 \\ 9\cosh 3t & 9\sinh 3t & 0 \end{vmatrix} = \langle\, 27\sinh^2 3t - 27\cosh^2 3t,\ 0,\ 0\,\rangle = \langle\, -27,\ 0,\ 0\,\rangle.$$

Hence by Eq. (28),

$$a_N = \frac{27}{3\sqrt{\sinh^2 3t + \cosh^2 3t}} = \frac{9}{\sqrt{\cosh 6t}}.$$

C12S06.025: We adjoin third component zero to form $\mathbf{r}(t) = \langle\, t\cos t,\ t\sin t,\ 0\,\rangle$. Then

$$\mathbf{v}(t) = \langle\, \cos t - t\sin t,\ \sin t + t\cos t,\ 0\,\rangle \qquad \mathbf{a}(t) = \langle\, -2\sin t - t\cos t,\ 2\cos t - t\sin t,\ 0\,\rangle,$$

and $v(t) = \sqrt{\cos^2 t + t^2 \sin^2 t + \sin^2 t + t^2 \cos^2 t} = \sqrt{t^2 + 1}$. Next,

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = -2\sin t \cos t + 2t\sin^2 t - t\cos^2 t + t^2 \sin t \cos t + 2\sin t \cos t - t\sin^2 t + 2t\cos^2 t - t^2 \sin t \cos t$$
$$= 2t - t = t.$$

So by Eq. (26) of the text,

$$a_T = \frac{\mathbf{v(t) \cdot a(t)}}{v(t)} = \frac{t}{\sqrt{t^2+1}},$$

Next,

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}$$

$$= \langle\, 0,\ 0,\ 2\cos^2 t - t\sin t \cos t - 2t\sin t\cos t + t^2\sin^2 t + 2\sin^2 t + 2t\sin t \cos t + t\sin t\cos t + t^2\cos^2 t\,\rangle$$
$$= \langle\, 0,\ 0,\ t^2 + 2\,\rangle.$$

Therefore by Eq. (28),

$$a_N = \frac{|\,\mathbf{v}(t) \times \mathbf{a}(t)\,|}{v(t)} = \frac{t^2+2}{\sqrt{t^2+1}}$$

C12S06.026: We adjoin third component zero to form $\mathbf{r}(t) = \langle\, e^t \sin t,\ e^t \cos t,\ 0\,\rangle$. Then

$$\mathbf{v}(t) = \langle\, e^t(\cos t + \sin t),\ e^t(\cos t - \sin t),\ 0\,\rangle \qquad v(t) = e^t\sqrt{\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t} = e^t\sqrt{2},$$

and

$$\mathbf{a}(t) = \langle\, e^t(\sin t + \cos t + \cos t - \sin t),\ e^t(\cos t - \sin t - \sin t - \cos t),\ 0\,\rangle = e^t\langle\, -2\sin t,\ 2\cos t,\ 0\,\rangle.$$

Next,

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = e^{2t}(-2\sin t\cos t + 2\sin^2 t + 2\sin t \cos t + 2\cos^2 t) = 2e^{2t},$$

so by Eq. (26) of the text,

$$a_T = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{v(t)} = \frac{2e^{2t}}{e^t\sqrt{2}} = e^t\sqrt{2}.$$

Also

$$\mathbf{v}(t) \times \mathbf{a}(t) = e^{2t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t - \sin t & \sin t + \cos t & 0 \\ -2\sin t & 2\cos t & 0 \end{vmatrix}$$

$$= e^{2t}\langle\, 0,\ 0,\ 2\cos^2 t - 2\sin t\cos t + 2\sin^2 t + 2\sin t\cos t\,\rangle = e^{2t}\langle\, 0,\ 0,\ 2\,\rangle.$$

Thus by Eq. (28),

$$a_N = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)} = \frac{2e^{2t}}{e^t\sqrt{2}} = e^t\sqrt{2}\,.$$

C12S06.027: Given $x^2 + y^2 = a^2$, implicit differentiation yields

$$2x + 2y\frac{dy}{dx} = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

Differentiation of both sides of the last equation with respect to x then yields

$$\frac{d^2y}{dx^2} = \frac{x\dfrac{dy}{dx} - y}{y^2} = \frac{xy\dfrac{dy}{dx} - y^2}{y^3} = \frac{-x^2 - y^2}{y^3} = -\frac{a^2}{y^3}.$$

Next, Eq. (13) yields

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \frac{a^2}{|y|^3(1 + x^2y^{-2})^{3/2}}.$$

If $y \neq 0$, then

$$\kappa(x) = \frac{a^2}{(y^2 + x^2)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}.$$

If $y = 0$, then $x \neq 0$; interchange the roles of x and y and work with dx/dy to obtain the same result. In this case, instead of Eq. (13) use:

$$\kappa(y) = \frac{|x''(y)|}{\left[1 + (x'(y))^2\right]^{3/2}}.$$

C12S06.028: Let $\mathbf{r}(t) = \left\langle \frac{3}{2}t^2,\ \frac{4}{3}t^3 \right\rangle$. Then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, 3t,\ 4t^2 \,\rangle, \qquad \text{so that} \qquad \mathbf{v}(1) = \langle\, 3,\ 4 \,\rangle.$$

Hence the unit tangent vector corresponding to $t = 1$ is $\mathbf{T}(1) = \left\langle \frac{3}{5},\ \frac{4}{5} \right\rangle$ and the unit normal vector corresponding to $t = 1$ is $\mathbf{N}(1) = \left\langle -\frac{4}{5},\ \frac{3}{5} \right\rangle$. Finally,

$$\frac{41}{5}\mathbf{T}(1) + \frac{12}{5}\mathbf{N}(1) = \frac{41}{5}\left\langle \frac{3}{5},\ \frac{4}{5} \right\rangle + \frac{12}{5}\left\langle -\frac{4}{5},\ \frac{3}{5} \right\rangle = \langle\, 3,\ 8 \,\rangle.$$

C12S06.029: Let $x(t) = t$, $y(t) = 1 - t^2$, and $\mathbf{r}(t) = \langle\, x(t), y(t) \,\rangle$. Then $\mathbf{r}'(t) = \langle\, 1, -2t \,\rangle$, so that $\mathbf{r}'(0) = \langle\, 1, 0 \,\rangle = \mathbf{T}(1)$. The graph of the given equation is concave down everywhere, so that the unit normal vector corresponding to $t = 0$ is $\mathbf{N} = \langle\, 0, -1 \,\rangle$. Next, $\mathbf{a}(t) = \mathbf{r}''(t) = \langle\, 0, -2 \,\rangle$, and so $\mathbf{a}(0)$ is the same. By Eq. (12) the curvature is

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[v(t)]^3} = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Because $\kappa(0) = 2$, the radius of the osculating circle at $(0, 1)$ is $\frac{1}{2}$, and by Eq. (16) the position vector of the center of that circle is

$$\mathbf{r}(0) + \frac{1}{2}\mathbf{N}(0) = \left\langle 0,\ \frac{1}{2} \right\rangle.$$

Therefore an equation of the osculating circle is

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

C12S06.030: We let *Mathematica* 3.0 find the equation of the osculating circle at the point (0, 1) of the graph of $y = e^x$. First we let

```
x[t_] := t; y[t_] := Exp[t]; r[t_] := {x[t], y[t]}
```

Recall that vectors such as $\mathbf{r}(t)$ are enclosed in French braces. Then we used Eq. (12) to define the curvature function κ:

```
kappa[t_] := (Abs[x'[t]*y''[t] - x''[t]*y'[t]])/(((x'[t])∧2 + (y'[t])∧2)∧(3/2))
```

To find the unit tangent vector at (0, 1), we computed

```
r'[0]
```

and found that $\mathbf{v}(0) = \langle\, 1,\, 1\,\rangle$. Hence the unit tangent vector we need is

```
utan = r'[0]/Sqrt[r'[0].r'[0]]
```

—that is, $\mathbf{T}(1) = \left\langle\, \frac{1}{2}\sqrt{2}\,,\ \frac{1}{2}\sqrt{2}\,\right\rangle$. The graph of $y = e^x$ is concave upward everywhere, so the unit normal at (0, 1) is

```
unorm = { -1/Sqrt[2], 1/Sqrt[2] }
```

—thus $\mathbf{N}(0) = \left\langle\, -\frac{1}{2}\sqrt{2}\,,\ \frac{1}{2}\sqrt{2}\,\right\rangle$. The acceleration is

```
a = r''[0]
```

and thus $\mathbf{a}(0) = \langle\, 0,\, 1\,\rangle$. To find the curvature at (0, 1), we asked for `kappa[t]`, and found it to be

$$\kappa(t) = \frac{e^t}{(1 + e^{2t})^{3/2}}.$$

Therefore $\kappa(0) = \frac{1}{4}\sqrt{2}$, and so the radius of the osculating circle is $\rho = 2\sqrt{2}$. Its center has position vector

```
r[0] + (2*Sqrt[2])*unorm
```

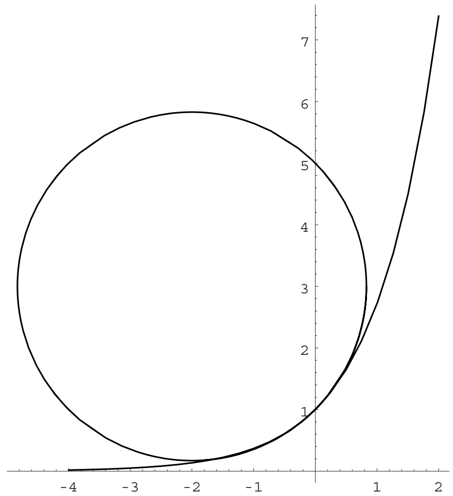
which turns out to be $\langle\, -2,\, 3\,\rangle$, and thus an equation of the osculating circle is $(x+2)^2 + (y-3)^2 = 8$. To check this answer, we wrote parametric equations of the osculating circle:

```
xx[t_] := - 2 + 2*Sqrt[2]*Cos[Pi*t]; yy[t_] := 3 + 2*Sqrt[2]*Sin[Pi*t]
```

and then plotted the graphs of $y = e^x$ and the osculating circle simultaneously by entering the command

```
ParametricPlot[ {{ x[t], y[t] }, { xx[t], yy[t] }}, { t, -4, 2 },
                  AspectRatio -> Automatic ];
```

the resulting graph is shown next.



C12S06.031: Let $x(t) = t$ and $y(t) = t^{-1}$; let $\mathbf{r}(t) = \langle x(t),\, y(t) \rangle$. Then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 1,\; -\frac{1}{t^2} \right\rangle \quad \text{and} \quad v(t) = |\mathbf{v}(t)| = \frac{\sqrt{t^4+1}}{t^2}.$$

Thus the unit tangent and unit normal vectors at (1, 1) are

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{v(1)} = \left\langle \frac{\sqrt{2}}{2},\; -\frac{\sqrt{2}}{2} \right\rangle \quad \text{and} \quad \mathbf{N}(1) = \left\langle \frac{\sqrt{2}}{2},\; \frac{\sqrt{2}}{2} \right\rangle,$$

respectively. By Eq. (12) of the text, the curvature at $(x(t),\, y(t))$ is

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{2t^3}{(t^4+1)^{3/2}},$$

so the curvature at (1, 1) is $\kappa(1) = \frac{1}{2}\sqrt{2}$. Therefore the osculating circle at (1, 1) has radius $\rho = \sqrt{2}$. By Eq. (16) of the text, the position vector of the center of that circle is

$$\mathbf{r}(1) + \left(\sqrt{2}\right) \cdot \mathbf{N}(1) = \langle 2,\, 2 \rangle.$$

Therefore an equation of the osculating circle is $(x-2)^2 + (y-2)^2 = 2$.

C12S06.032: Let $x(t) = t$, $y(t) = 2t - 1$, $z(t) = 3t + 5$, and $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t) \rangle$. Then

$$\mathbf{v}(t) = \langle 1,\, 2,\, 3 \rangle, \qquad \mathbf{a}(t) = \mathbf{0}, \qquad \text{and} \qquad \kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{[v(t)]^3} \equiv 0.$$

Well, of course: The curvature of a straight line should be zero at every point.

C12S06.033: Given $\mathbf{r}(t) = \langle t,\, \sin t,\, \cos t \rangle$, we first compute

$$\mathbf{v}(t) = \langle 1,\, \cos t,\, -\sin t \rangle, \quad v(t) = \sqrt{1 + \cos^2 t + \sin^2 t} \equiv \sqrt{2}, \quad \text{and} \quad \mathbf{a}(t) = \langle 0,\, -\sin t,\, -\cos t \rangle.$$

Then

$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = \langle -\cos^2 t - \sin^2 t,\ \cos t,\ -\sin t\,\rangle = \langle -1,\ \cos t\ \ -\sin t\,\rangle.$$

Therefore, by Eq. (27), the curvature is

$$\kappa(t) = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{[v(t)]^3} = \frac{\sqrt{2}}{2\sqrt{2}} \equiv \frac{1}{2}.$$

C12S06.034: Given $\mathbf{r}(t) = \langle\, t,\, t^2,\, t^3\,\rangle$, we first compute

$$\mathbf{v}(t) = \langle\, 1,\, 2t,\, 3t^2\,\rangle, \quad v(t) = \sqrt{1+4t^2+9t^4}, \quad \text{and} \quad \mathbf{a}(t) = \langle\, 0,\, 2,\, 6t\,\rangle.$$

Then

$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle\, 12t^2 - 6t^2,\ -6t,\ 2\,\rangle = \langle\, 6t^2,\ -6t,\ 2\,\rangle.$$

Therefore, by Eq. (27),

$$\kappa(t) = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{[v(t)]^{3/2}} = \frac{\sqrt{36t^4+36t^2+4}}{(9t^4+4t^2+1)^{1/2}} = \frac{2\sqrt{9t^4+9t^2+1}}{(9t^4+4t^2+1)^{3/2}}.$$

C12S06.035: Given: $\mathbf{r}(t) = \langle\, e^t\cos t,\ e^t\sin t,\ e^t\,\rangle$. Then

$$\begin{aligned}
&\mathbf{v}(t) = \langle\, e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ e^t\,\rangle,\\
&v(t) = e^t(\cos^2 t + \sin^2 t + \sin^2 t + \cos^2 t + 1)^{1/2} = e^t\sqrt{3}, \qquad \text{and}\\
&\mathbf{a}(t) = \langle e^t(\cos t - \sin t - \sin t - \cos t),\ e^t(\sin t + \cos t + \cos t - \sin t),\ e^t\,\rangle = e^t\langle\, -2\sin t,\ 2\cos t,\ 1\,\rangle.
\end{aligned}$$

Therefore

$$\mathbf{v}(t)\times\mathbf{a}(t) = e^{2t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t - \sin t & \sin t + \cos t & 1 \\ -2\sin t & 2\cos t & 0 \end{vmatrix} = e^{2t}\langle\, \sin t - \cos t,\ -\sin t - \cos t,\ 2\,\rangle.$$

Thus

$$|\mathbf{v}(t)\times\mathbf{a}(t)| = e^{2t}\sqrt{\sin^2 t + \cos^2 t + \sin^2 t + \cos^2 t + 4} = e^{2t}\sqrt{6},$$

and therefore

$$\kappa(t) = \frac{e^{2t}\sqrt{6}}{3e^{2t}\sqrt{3}} = \frac{\sqrt{2}}{3}\,e^{-t}.$$

C12S06.036: Given: $\mathbf{r}(t) = \langle\, t\sin t,\ t\cos t,\ t\,\rangle$. Then

$$\mathbf{v}(t) = \langle\, \sin t + t\cos t,\ \cos t - t\sin t,\ 1\,\rangle, \qquad v(t) = (\sin^2 t + t^2\cos^2 t + \cos^2 t + t^2 \sin^2 t + 1)^{1/2} = \sqrt{t^2+2}\,,$$

and $\mathbf{a}(t) = \langle\, 2\cos t - t\sin t,\ -2\sin t - t\cos t,\ 0\,\rangle$. Moreover,

$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t + t\cos t & \cos t - t\sin t & 1 \\ 2\cos t - t\sin t & -2\sin t - t\cos t & 0 \end{vmatrix}$$

$$= \langle\, 2\sin t + t\cos t,\ 2\cos t - t\sin t,$$

$$-2\sin^2 t - t\sin t\cos t - 2t\sin t\cos t - t^2\cos^2 t - 2\cos^2 t + t\sin t\cos t + 2t\sin t\cos t - t^2\sin^2 t\,\rangle$$

$$= \langle\, 2\sin t + t\cos t,\ 2\cos t - t\sin t,\ -2 - t^2\,\rangle.$$

Therefore

$$|\mathbf{v}(t)\times\mathbf{a}(t)| = (4\sin^2 t + 4t\sin t\cos t + t^2\cos^2 t + 4\cos^2 t - 4t\sin t\cos t + t^2\sin^2 t + t^4 + 4t^2 + 4)^{1/2}$$

$$= (t^4 + 4t^2 + 4 + 4 + t^2)^{1/2} = (t^4 + 5t^2 + 8)^{1/2}.$$

Thus by Eq. (27),

$$\kappa(t) = \frac{(t^4 + 5t^2 + 8)^{1/2}}{(t^2+2)^{3/2}}.$$

C12S06.037: Let $x(t) = t,\ y(t) = 2t - 1,\ z(t) = 3t + 5,$ and $\mathbf{r}(t) = \langle\, x(t),\ y(t),\ z(t)\,\rangle$. It follows that $\mathbf{v}(t) = \langle\, 1,\ 2,\ 3\,\rangle$ and $\mathbf{a}(t) = \mathbf{0}$. Hence $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$ and $\mathbf{v}(t)\times\mathbf{a}(t) = \mathbf{0}$, and therefore $a_T = 0 = a_N$.

C12S06.038: Given $\mathbf{r}(t) = \langle\, t,\ \sin t,\ \cos t\,\rangle$, we first compute

$$\mathbf{v}(t) = \langle\, 1,\ \cos t,\ -\sin t\,\rangle, \quad v(t) = \sqrt{1 + \cos^2 t + \sin^2 t} \equiv \sqrt{2}, \quad \text{and} \quad \mathbf{a}(t) = \langle\, 0,\ -\sin t,\ -\cos t\,\rangle.$$

Thus $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$, so that $a_T = 0$. Also

$$\mathbf{v}(t)\times\mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = \langle\, -1,\ \cos t,\ -\sin t\,\rangle,$$

so that $|\mathbf{v}(t)\times\mathbf{a}(t)| = \sqrt{2}$. Therefore

$$a_N = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{v(t)} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

C12S06.039: Using *Mathematica* 3.0, we let

```
r[t_] := { t, t∧2, t∧3 }; v[t_] := r′[t]
```

so that $\mathbf{v}(t) = \langle\, 1,\, 2t,\, 3t^2 \,\rangle$, and we let

```
speed[t_] := Sqrt[v[t].v[t]]
```

(using the fact that $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$), so that $v(t) = (1 + 4t^2 + 9t^4)^{1/2}$. We also defined

```
a[t_] := r''[t]
```

so that $\mathbf{a}(t) = \langle\, 0,\, 2,\, 6t \,\rangle$. Next we computed

```
v[t].a[t]
```

$4t + 18t^3$

and then, by Eq. (26),

```
asubT = v[t].a[t]/speed[t]
```

so that $a_T = \dfrac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$. Next,

```
p[t_] := Cross[v[t], a[t]]
```

yielded $\mathbf{v}(t) \times \mathbf{a}(t) = \langle\, 6t^2,\, -6t,\, 2 \,\rangle$. Then

```
magp[t_] := Sqrt[p[t].p[t]]
```

revealed that $|\mathbf{v}(t) \times \mathbf{a}(t)| = (4 + 36t^2 + 36t^4)^{1/2}$. Hence, by Eq. (28),

```
asubN = magp[t]/speed[t]
```

—that is, $a_N = \dfrac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{1 + 4t^2 + 9t^4}}$.

C12S06.040: Using *Mathematica* 3.0, we let

```
r[t_] := { Exp[t]*Cos[t], Exp[t]*Sin[t], Exp[t] }; v[t_] := r'[t]
```

so that $\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, e^t(\cos t - \sin t),\ e^t(\cos t + \sin t),\ e^t \,\rangle$. Then

```
speed[t_] := Sqrt[v[t].v[t]]
```

let us know that $v(t) = \sqrt{e^{2t} + (e^t \cos t - e^t \sin t)^2 + (e^t \cos t + e^t \sin t)^2}$. And then the obvious command `Simplify[speed[t]]` elicited the response

$\sqrt{3}\,\sqrt{e^{2t}}$

so we redefined

```
speed[t_] := Sqrt[3]*Exp[t]
```

Then we let

```
a[t_] := r''[t]
```

and thereby discovered that $\mathbf{a}(t) = \langle -2e^t \sin t,\ 2e^t \cos t,\ e^t \rangle$. Then we used Eq. (26) of the text to find a_T:

```
asubT = (v[t].a[t])/speed[t]
```

$$\frac{e^{-t}((e^{2t} - 2e^t \sin t)(e^t \cos t - e^t \sin t) + (2e^t \cos t)(e^t \cos t + e^t \sin t))}{\sqrt{3}}$$

```
Simplify[asubT]
```

$\sqrt{3}\, e^t$

—that is, $a_T = \sqrt{3}\, e^t$. Next we used Eq. (28) to find a_N:

```
p[t_] := Cross[ v[t], a[t] ]
```

revealed that

$$\mathbf{v}(t) \times \mathbf{a}(t) = \langle\, e^{2t}(\sin t - \cos t),\ -e^{2t}(\sin t + \cos t),\ 2e^{2t}(\cos^2 t + \sin^2 t)\,\rangle,$$

and then `Simplify[p[t]]` yielded

$$\mathbf{v}(t) \times \mathbf{a}(t) = \langle\, e^{2t}(\sin t - \cos t),\ -e^{2t}(\sin t + \cos t),\ 2e^{2t}\,\rangle$$

Next we entered

```
Simplify[ Sqrt[ p[t].p[t] ] ]
```

and defined the result to be `magp[t]`:

```
magp[t_] := Sqrt[6]*Exp[2*t]
```

Finally, Eq. (28) yielded the value of a_N:

```
asubN = magp[t]/speed[t]
```

gave the response $a_N = \sqrt{2}\, e^t$.

C12S06.041: Beginning with $\mathbf{r}(t) = \langle\, t \sin t,\ t \cos t,\ t\,\rangle$, we found:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, t \cos t + \sin t,\ \cos t - t \sin t,\ 1\,\rangle,$$

$$v(t) = |\mathbf{v}(t)| = \sqrt{1 + (t \cos t + \sin t)^2 + (\cos t - t \sin t)^2} = \sqrt{t^2 + 2}\,,$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle\, 2 \cos t - t \sin t,\ -t \cos t - 2 \sin t,\ 0\,\rangle,$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = (-t \cos t - 2 \sin t)(\cos t - t \sin t) + (t \cos t + \sin t)(2 \cos t - t \sin t) = t,$$

$$a_T = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{v(t)} = \frac{t}{\sqrt{t^2 + 2}},$$

$$\mathbf{v}(t) \times \mathbf{a}(t) = \langle\, t \cos t + 2 \sin t,\ 2 \cos t - t \sin t,\ -(t^2 + 2)\,\rangle,$$

$$|\mathbf{v}(t) \times \mathbf{a}(t)| = \sqrt{t^4 + 5t^2 + 8}, \quad \text{and}$$

$$a_N = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)} = \frac{\sqrt{t^4 + 5t^2 + 8}}{\sqrt{t^2 + 2}}.$$

C12S06.042: Given $\mathbf{r}(t) = \langle\, t,\ t^2,\ t^3 \,\rangle$, we will compute the unit tangent vector $\mathbf{T}(t)$ using its definition in Eq. (17) and the principal unit normal vector $\mathbf{N}(t)$ by means of Eq. (29). We find:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, 1,\ 2t,\ 3t^2 \,\rangle,$$

$$v(t) = |\mathbf{v}(t)| = \sqrt{9t^4 + 4t^2 + 1},$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle\, 0,\ 2,\ 6t \,\rangle,$$

$$a_T(t) = \frac{\mathbf{v}(t) \cdot \mathbf{a}(t)}{v(t)} = \frac{4t + 18t^3}{\sqrt{9t^4 + 4t^2 + 1}},$$

$$a_N(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)} = \frac{|\langle\, 6t^2,\ -6t,\ 2 \,\rangle|}{\sqrt{9t^4 + 4t^2 + 1}} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{9t^4 + 4t^2 + 1}},$$

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{[v(t)]^3} = \frac{|\langle\, 6t^2,\ -6t,\ 2 \,\rangle|}{(9t^4 + 4t^2 + 1)^{3/2}} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(9t^4 + 4t^2 + 1)^{3/2}},$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{9t^4 + 4t^2 + 1}} \langle\, 1,\ 2t,\ 3t^2 \,\rangle, \quad \text{and}$$

$$\mathbf{N}(t) = \frac{\mathbf{a} - a_T \mathbf{T}}{a_N} = \frac{1}{\sqrt{(9t^4 + 4t^2 + 1)(36t^4 + 36t^2 + 4)}} \langle\, -4t - 18t^3,\ 1 - 9t^4,\ 6t + 12t^3 \,\rangle.$$

Then substitution of $t = 1$ yields

$$\mathbf{T} = \left\langle \frac{\sqrt{14}}{14},\ \frac{2\sqrt{14}}{14},\ \frac{3\sqrt{14}}{14} \right\rangle \quad \text{and} \quad \mathbf{N} = \left\langle -\frac{11\sqrt{266}}{266},\ -\frac{8\sqrt{266}}{266},\ \frac{9\sqrt{266}}{266} \right\rangle.$$

C12S06.043: Given $\mathbf{r}(t) = \langle\, t,\ \sin t,\ \cos t \,\rangle$, we will compute the unit tangent vector $\mathbf{T}$ using its definition in Eq. (17), then the principal unit normal vector $\mathbf{N}$ by means of Eq. (29). We find:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, 1,\ \cos t,\ -\sin t \,\rangle,$$

$$\mathbf{v}(0) = \langle\, 1,\ 1,\ 0 \,\rangle,$$

$$v(0) = \sqrt{2},$$

$$\mathbf{T}(0) = \left\langle \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 0 \right\rangle,$$

$$\mathbf{a}(t) = \langle\, 0,\ -\sin t,\ -\cos t \,\rangle,$$

$$\mathbf{a}(0) = \langle\, 0,\ 0,\ -1 \,\rangle,$$

$$a_T(0) = \frac{\mathbf{v}(0) \cdot \mathbf{a}(0)}{v(0)} = 0,$$

$$
\begin{aligned}
a_N(0) &= \frac{|\,\mathbf{v}(0)\times\mathbf{a}(0)\,|}{v(t)} = 1,\\
\kappa(0) &= \frac{|\,\mathbf{v}(0)\times\mathbf{a}(0)\,|}{[v(t)]^3} = \frac{1}{2}, \qquad \text{and}\\
\mathbf{N}(0) &= \frac{\mathbf{a}-a_T\mathbf{T}}{a_N} = \langle\, 0,\ 0,\ -1\,\rangle
\end{aligned}
$$

C12S06.044: Given $\mathbf{r}(t) = \langle\, 6e^t\cos t,\ 6e^t\sin t,\ 17e^t\,\rangle$, we will compute the unit tangent vector $\mathbf{T}$ using its definition in Eq. (17), then the principal unit normal vector $\mathbf{N}$ by means of Eq. (29). We find:

$$
\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \langle\, 6e^t(\cos t-\sin t),\ 6e^t(\cos t+\sin t),\ 17e^t\,\rangle,\\
\mathbf{v}(0) &= \langle\, 6,\ 6,\ 17\,\rangle,\\
v(0) &= 19,\\
\mathbf{T}(0) &= \left\langle \frac{6}{19},\ \frac{6}{19},\ \frac{17}{19}\right\rangle,\\
\mathbf{a}(t) &= \langle\, -12e^t\sin t,\ 12e^t\cos t,\ 17e^t\,\rangle,\\
\mathbf{a}(0) &= \langle\, 0,\ 12,\ 17\,\rangle,\\
a_T(0) &= \frac{\mathbf{v}(0)\cdot\mathbf{a}(0)}{v(0)} = 19,\\
a_N(0) &= \frac{|\,\mathbf{v}(t)\times\mathbf{a}(t)\,|}{v(t)} = 6\sqrt{2},\\
\kappa(0) &= \frac{|\,\mathbf{v}(0)\times\mathbf{a}(0)\,|}{[v(0)]^3} = \frac{6\sqrt{2}}{361}, \qquad \text{and}\\
\mathbf{N}(0) &= \frac{\mathbf{a}-a_T\mathbf{T}}{a_N} = \left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 0\right\rangle.
\end{aligned}
$$

C12S06.045: The process of computing $\mathbf{T}$ and $\mathbf{N}$ can be carried out almost automatically in *Mathematica* 3.0. Given $\mathbf{r}(t) = \langle\, e^t\cos t,\ e^t\sin t,\ e^t\,\rangle$, we entered the following commands:

```
v0 = r′[0]
        {1, 1, 1}

a0 = r″[0]
        {0, 2, 1}

speed = Sqrt[v0.v0]
        √3

asubT = v0.a0/speed
        √3

vcrossa = Cross[ v0, a0 ]
        {-1, -1, 2}
```

```
asubN = (Sqrt[ vcrossa.vcrossa ])/speed
```

$\sqrt{2}$

```
kappa = asubN/(speed*speed)
```

$\dfrac{\sqrt{2}}{3}$

```
utan = v0/speed
```

$\{\ \dfrac{1}{\sqrt{3}},\ \dfrac{1}{\sqrt{3}},\ \dfrac{1}{\sqrt{3}}\ \}$

```
unorm = (a0 - asubT*utan)/asubN
```

$\{\ -\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}},\ 0\ \}$

Thus we see that

$$\mathbf{T}(0) = \left\langle \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3} \right\rangle \quad \text{and} \quad \mathbf{N}(0) = \left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 0 \right\rangle.$$

C12S06.046: Given $\mathbf{r}(t) = \langle\, a\cos\omega t,\ a\sin\omega t,\ bt\,\rangle$, we find—successively—that

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \langle\, -a\omega\sin\omega t,\ a\omega\cos\omega t,\ b\,\rangle, \\
v(t) &= |\,\mathbf{v}(t)\,| = \sqrt{a^2\omega^2\cos^2\omega t + a^2\omega^2\sin^2\omega t + b^2} = \sqrt{a^2\omega^2 + b^2}\,, \\
\mathbf{a}(t) &= \mathbf{v}'(t) = \langle -a\omega^2\cos\omega t,\ -a\omega^2\sin\omega t,\ 0\,\rangle, \\
a_T(t) &= \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)} = \frac{0}{v(t)} \equiv 0, \\
\mathbf{v}(t)\times\mathbf{a}(t) &= \langle\, ab\omega^2\sin\omega t,\ -ab\omega^2\cos\omega t,\ a^2\omega^3\,\rangle, \\
|\,\mathbf{v}(t)\times\mathbf{a}(t)\,| &= \sqrt{a^2\omega^4(a^2\omega^2+b^2)} = a\omega^2\sqrt{a^2\omega^2+b^2}\,, \\
a_N(t) &= \frac{|\,\mathbf{v}(t)\times\mathbf{a}(t)\,|}{v(t)} \equiv a\omega^2, \\
\kappa(t) &= \frac{|\,\mathbf{v}(t)\times\mathbf{a}(t)\,|}{[v(t)]^3} = \frac{a\omega^2}{a^2\omega^2+b^2}, \\
\mathbf{T}(t) &= \frac{1}{\sqrt{a^2\omega^2+b^2}}\,\langle -a\omega\sin\omega t,\ a\omega\cos\omega t,\ b\,\rangle, \\
\mathbf{N}(t) &= \frac{\mathbf{a}(t) - a_T(t)\mathbf{T}(t)}{a_N(t)} = \langle\, -\cos\omega t,\ -\sin\omega t,\ 0\,\rangle.
\end{aligned}$$

C12S06.047: Because

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{16+144+9} = 13,$$

we see that the arc length is given by $s = 13t$, and therefore the arc-length parametrization of the given curve is

$$x(s) = 2 + \frac{4s}{13}, \quad y(s) = 1 - \frac{12s}{13}, \quad z(s) = 3 + \frac{3s}{13}.$$

C12S06.048: Because

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{4\cos^2 t + 4\sin^2 t} = 2,$$

we see that the arc length is given by $s = 2t$, and therefore the arc-length parametrization of the given curve is

$$x(s) = 2\cos\frac{s}{2}, \quad y(s) = 2\sin\frac{s}{2}, \quad z(s) = 0.$$

C12S06.049: Because

$$\frac{ds}{dt} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{25} = 5,$$

we see that the arc length is given by $s = 5t$, and therefore the arc-length parametrization of the given curve is

$$x(s) = 3\cos\frac{s}{5}, \quad y(s) = 3\sin\frac{s}{5}, \quad z(s) = \frac{4s}{5}.$$

C12S06.050: We begin by letting $\mathbf{r}(t) = \langle\, t,\ f(t),\ 0\,\rangle$. Then $\mathbf{v}(t) = \langle\, 1,\ f'(t),\ 0\,\rangle$, and consequently $v(t) = \sqrt{1 + [f'(t)]^2}$. Next, $\mathbf{a}(t) = \langle\, 0,\ f''(t),\ 0\,\rangle$, and so

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = \langle\, 0,\ 0,\ f''(t)\,\rangle.$$

Therefore $|\mathbf{v}(t) \times \mathbf{a}(t)| = |f''(t)|$. So by Eq. (27),

$$\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{[v(t)]^3} = \frac{|f''(t)|}{(1 + [f'(t)]^2)^{3/2}}.$$

C12S06.051: By Newton's second law of motion, the acceleration of the particle is a scalar multiple of the force acting on the particle, so the force and acceleration vectors are parallel. Therefore the acceleration vector $\mathbf{a}$ is normal to the velocity vector $\mathbf{v}$. But then,

$$D_t(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{v} = 0 + 0 = 0,$$

and therefore $\mathbf{v} \cdot \mathbf{v} = K$, a constant. Hence the speed $v(t) = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is also constant.

C12S06.052: Beginning with Eq. (20), derive as in the text Eqs. (21)–(25). Then by Eq. (25),

$$a^2 = |\mathbf{a}|^2 = (a_N)^2 + (a_T)^2, \quad \text{so that} \quad a^2 - (a_T)^2 = (a_N)^2.$$

Then by Eq. (23),

$$\kappa = \frac{\kappa v^2}{v^2} = \frac{a_N}{v^2} = \frac{\sqrt{a^2 - (a_T)^2}}{v^2}.$$

Next, if we are working with vectors with two components and we let $\mathbf{r}(t) = \langle\, x(t),\, y(t)\,\rangle$, then $a_T = v'(t)$ by Eq. (23) and $\mathbf{a}(t) = \langle\, x''(t),\, y''(t)\,\rangle$. Hence

$$\frac{\sqrt{a^2 - (a_T)^2}}{v^2} = \frac{\sqrt{[x''(t)]^2 + [y''(t)]^2 - [v'(t)]^2}}{[x'(t)]^2 + [y'(t)]^2}.$$

C12S06.053: Given $x(t) = \cos t + t\sin t$ and $y(t) = \sin t - t\cos t$, we first compute

$$\begin{aligned}
[v(t)]^2 &= [x'(t)]^2 + [y'(t)]^2 \\
&= (t\cos t + \sin t - \sin t)^2 + (\cos t - \cos t + t\sin t)^2 = t^2\cos^2 t + t^2\sin^2 t = t^2,
\end{aligned}$$

$$v'(t) = 1,$$

$$[x''(t)]^2 = (\cos t - t\sin t)^2,$$

$$[y''(t)]^2 = (\sin t + t\cos t)^2, \quad \text{and}$$

$$[x''(t)]^2 + [y''(t)]^2 = \cos^2 t + \sin^2 t + t^2\sin^2 t + t^2\cos^2 t = t^2 + 1.$$

Then the formula in Problem 52 yields

$$\kappa(t) = \frac{\sqrt{[x''(t)]^2 + [y''(t)]^2 - [v'(t)]^2}}{[x'(t)]^2 + [y'(t)]^2} = \frac{\sqrt{t^2 + 1 - 1}}{t^2} = \frac{1}{|t|}.$$

C12S06.054: The *Mathematica* 3.0 command

```
Solve[ D[ x∧3 + (y[x])∧3 == 3∗x∗y[x], x ], y′[x] ]
```

asks *Mathematica* to differentiate the equation $x^3 + y^3 = 3xy$ implicitly with respect to x, then solve for $y'(x)$; its response (after slight simplifications) is

$$y'(x) = \frac{y(x) - x^2}{[y(x)]^2 - x} = \frac{y - x^2}{y^2 - x}, \tag{1}$$

exactly what we obtained by hand. To find the second derivative, use the command

```
Solve[ D[ x∧3 + (y[x])∧3 == 3∗x∗y[x], {x,2} ], y′′[x] ]
```

and the response—after slight simplifications—will be

$$y''(x) = \frac{2(x - y'(x) + y(x)[y'(x)]^2)}{x - [y(x)]^2}. \tag{2}$$

We entered these two commands, substituted the result in Eq. (1) for $y'(x)$ in Eq. (2), then used the `Simplify` command to find that

$$y''(x) = \frac{2xy(1 + x^3 - 3xy + y^3)}{(x - y^2)^3} = \frac{2xy}{(x - y^2)^3}.$$

It was then an easy matter to find that at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have

$$\frac{dy}{dx} = -1 \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{32}{3},$$

and it then follows from Eq. (13) of the text that the curvature at that point is $\kappa = \frac{8}{3}\sqrt{3}$. The unit normal there is clearly

$$\mathbf{N} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle,$$

and then Eq. (16) gives the center of the osculating circle to be $\left(\frac{21}{16}, \frac{21}{16}\right)$. The radius of the circle is $1/\kappa = \frac{3}{16}\sqrt{2}$, so its equation is

$$\left(x - \frac{21}{16}\right)^2 + \left(y - \frac{21}{16}\right)^2 = \left(\frac{3\sqrt{2}}{16}\right)^2;$$

we used the *Mathematica* commands `Expand` and `Simplify` to write this equation in the alternative form $8x^2 + 8y^2 - 21x - 21y + 27 = 0$.

C12S06.055: The six conditions listed in the statement of the problem imply, in order, that

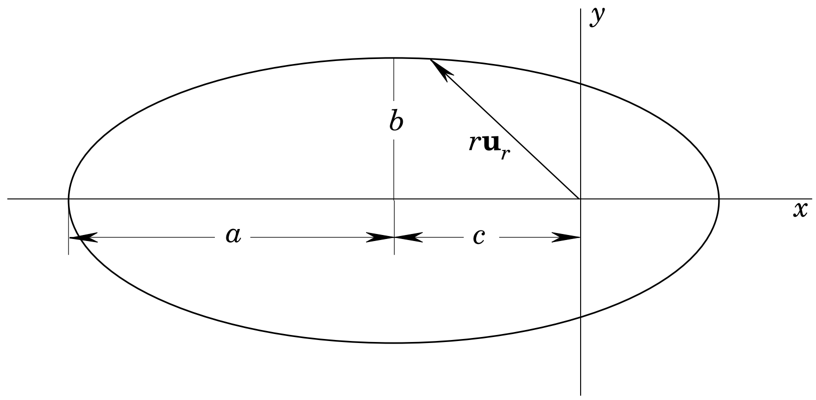
$$\begin{aligned}
0 &= F, \\
1 &= A + B + C + D + E + F, \\
0 &= E, \\
1 &= 5A + 4B + 3C + 2D + E, \\
0 &= 2D, \qquad \text{and} \\
0 &= 20A + 12B + 6C + 2D.
\end{aligned}$$

The last two equations were obtained by observing—via Eq. (13)—that curvature zero is equivalent to $y''(x) = 0$. Simultaneous solution of these equations yields $A = 3$, $B = -8$, $C = 6$, and $D = E = F = 0$. Answers: $y = 3x^5 - 8x^4 + 6x^3$; because $\kappa(x)$ is zero where the curved track meets the straight tracks, because their derivatives agree at the junctions, and because

$$\kappa(x) = \frac{|60x^3 - 96x^2 + 36x|}{[1 + (15x^4 - 32x^3 + 18x^2)^2]^{3/2}}$$

is continuous on $[0, 1]$, the normal forces on the train negotiating this transitional section of track will change continuously from zero to larger values and then back to zero. Thus there will be no abrupt change in the lateral forces on the train.

C12S06.056: See the following figure for the meanings of the symbols.



Part (a):

$$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \quad \text{and} \quad v = |\mathbf{v}| = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}.$$

At the nearest and farthest points of the orbit, $dr/dt = 0$. Hence

$$v = \left| r\frac{d\theta}{dt} \right| = r\frac{d\theta}{dt}.$$

Part (b): Note first that $\dfrac{1}{2}r^2\dfrac{d\theta}{dt}$ is constant. Moreover,

$$\int_0^T \frac{1}{2}r^2\frac{d\theta}{dt}\,dt = \pi ab.$$

Therefore

$$\frac{T}{2}r^2\frac{d\theta}{dt} = \pi ab, \qquad \text{and thus} \qquad r\frac{d\theta}{dt} = \frac{2\pi ab}{rT}.$$

Consequently, at the nearest and farthest points of the orbit, $v = \dfrac{2\pi ab}{rT}$ by part (a).

C12S06.057: Conversion into miles yields the semimajor axis of the orbit of Mercury to be $a = 35973972$. With eccentricity $e = 0.206$ and period $T = 87.97$ days, we use

$$b^2 = a^2(1 - e^2)$$

to find that $b \approx 35202402$. Then the formula $c^2 = a^2 - b^2$ yields $c \approx 7410638$. So at perihelion the speed of Mercury is

$$\frac{2\pi ab}{(a-c)T} \approx 3166628$$

miles per day. We divide by $24 \cdot 3600$ to convert this answer to 36.650789 miles per second. Replace $a - c$ with $a + c$ to find that its speed at aphelion is approximately 24.129956 miles per second.

C12S06.058: Conversion into miles yields the semimajor axis of the Earth's orbit to be $a = 92956000$. With eccentricity $e = 0.0167$ and period $T = 365.249$ days, we use

$$b^2 = a^2(1 - e^2)$$

to find that $b \approx 92943037$. Then the formula $c^2 = a^2 - b^2$ yields $c \approx 1552365$. So at perihelion the speed of Earth is

$$\frac{2\pi ab}{(a-c)T} \approx 1626004$$

miles per day. We divide by $24 \cdot 3600$ to convert this answer to 18.819493 miles per second. Replace $a - c$ with $a + c$ to find that the speed at aphelion is approximately 18.201246 miles per second.

C12S06.059: We are given that the semimajor axis of the Moon's orbit is $a = 238900$ (miles). With eccentricity $e = 0.055$ and period $T = 27.32$ days, we use

$$b^2 = a^2(1 - e^2)$$

to find that $b \approx 238538$. Then the formula $c^2 = a^2 - b^2$ yields $c \approx 13139$. So at perigee the speed of the Moon is

$$\frac{2\pi ab}{(a-c)T} \approx 58053$$

miles per day. We divide by $24 \cdot 3600$ to convert this answer to 0.671911 miles per second. Replace $a - c$ with $a + c$ to find that the speed at apogee is approximately 0.601854 miles per second.

C12S06.060: Let T_1 be the period of the Moon, a_1 the semimajor axis of its orbit, T the period of the artificial satellite, and a the semimajor axis of its orbit. From the data in Problem 59, we know that $T_1 = 27.32$ (days) and that $a_1 = 238900$ (mi). We are given $a = 10000$, and Kepler's third law of planetary motion (Eq. (44)) implies that

$$\frac{(T_1)^2}{(a_1)^3} = \frac{T^2}{a^3}; \qquad \text{that is,} \qquad \frac{(27.32)^2}{(238900)^3} = \frac{T^2}{(10000)^3},$$

which we solve for $T \approx 0.233968$ (days). Then the formula $b^2 = a^2(1 - e^2)$ with $e = 0.5$ yields $b = 8660.25$; next, the formula $c^2 = a^2 - b^2$ yields $c = 5000$. Hence at perigee the speed of the satellite is

$$v = \frac{2\pi ab}{(a-c)T} \approx 465140$$

miles per day; we divide by $24 \cdot 3600$ to convert this answer to approximately 5.383569 miles per second. Replace $a - c$ with $a + c$ to find that the speed of the satellite at apogee is approximately 1.794523 miles per second.

C12S06.061: Equation (44), applied to the Earth-Moon system with units of miles and days, yields $(27.32)^2 = \gamma \cdot (238900)^3$. For a satellite with period $T = \frac{1}{24}$ (of a day—one hour), it yields $T^2 = \gamma \cdot r^3$ where r is the radius of the orbit of the satellite. Divide the second of these equations by the first to eliminate γ:

$$\frac{T^2}{(27.32)^2} = \frac{r^3}{(238900)^3},$$

so that

$$r^3 = \frac{(238900)^3}{(24)^2 \cdot (27.32)^2}, \qquad \text{and thus} \qquad r \approx 3165.35$$

miles, about 795 miles below the surface of the Earth. So it can't be done.

C12S06.062: Equation (44), when applied to the Earth-Sun system with units of miles and years, yields $(1)^2 = \gamma \cdot (92956000)^3$. Applied to the Jupiter-Sun system it yields $(11.86)^2 = \gamma \cdot a^3$ where a is the semimajor axis of Jupiter's orbit. Division of the second of these equations by the first yields

$$a^3 = (92956000)^3 \cdot (11.86)^2,$$

and hence the semimajor axis of the Jovian orbit is $a \approx 483430322$ miles.

C12S06.063: With the usual meaning of the symbols, the data given in the problem tell us that $a+c = 4960$ and $a - c = 4060$, which we solve for $a = 4510$ (units are in days and miles). Let T be the period of the satellite in its orbit. Equation (44), applied to the Earth-Moon system, then to the Earth-satellite system, yields

$$(27.32)^2 = \gamma \cdot (238900)^3 \quad \text{and} \quad T^2 = \gamma \cdot (4510)^2,$$

which we solve for $T \approx 0.0708632854$. Multiply by 24 to convert this answer to approximately 1.7007188486 hours—about 1 h 42 min 2.588 s.

C12S06.064: Part (a): We begin with Eq. (40),

$$r = \frac{pe}{1 + e\cos\theta}$$

and differentiate both sides with respect to t, remembering that $\theta = \theta(t)$. Thus

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{h}{r^2} \cdot \frac{dr}{d\theta} = \frac{h}{r^2} \cdot \frac{pe^2 \sin\theta}{(1 + e\cos\theta)^2} = \frac{h\sin\theta}{p} \cdot \frac{p^2e^2}{r^2(1+\cos\theta)^2} = \frac{h\sin\theta}{p} \cdot \frac{r^2}{r^2} = \frac{h\sin\theta}{p}.$$

Part (b): Another differentiation with respect to t then yields

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \left[\frac{d}{d\theta}\left(\frac{dr}{dt}\right)\right] \cdot \frac{d\theta}{dt} = \frac{h}{r^2} \cdot \frac{h\cos\theta}{p} = \frac{h^2\cos\theta}{pr^2}.$$

Part (c): First we solve

$$r = \frac{pe}{1 + e\cos\theta} \quad \text{for} \quad \cos\theta = \frac{pe - r}{re}.$$

Then substitution in the result in part (b) yields

$$\frac{d^2r}{dt^2} = \frac{h^2\cos\theta}{pr^2} = \frac{h^2(pe - r)}{pr^3e} = \frac{h^2}{r^2}\left(\frac{pe - r}{pre}\right) = \frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{pe}\right).$$

C12S06.065: We will use Eqs. (37) and (41), which are—respectively—

$$r^2\frac{d\theta}{dt} = h \quad \text{(constant)} \quad \text{and} \quad \frac{d^2r}{dt^2} = \frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{pe}\right).$$

But we begin with Eq. (42),

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r.$$

Substitution of Eqs. (37) and (41) then yields

$$\begin{aligned}
\mathbf{a} &= \left[\frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{pe}\right) \;-\; \frac{1}{r^3}\left(r^2\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r \\
&= \left[\frac{h^2}{r^2}\left(\frac{1}{r} - \frac{1}{pe}\right) \;-\; \frac{1}{r^3}h^2\right]\mathbf{u}_r = \left[\frac{h^2}{r^3} - \frac{h^2}{per^2} - \frac{h^2}{r^3}\right]\mathbf{u}_r = -\frac{h^2}{per^2}\,\mathbf{u}_r.
\end{aligned}$$

C12S06.066: If $\theta = \theta(t)$, $\mathbf{u}_r = \langle \cos\theta, \sin\theta \rangle$, and $\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta \rangle$, then

$$\begin{aligned}
\frac{d\mathbf{u}_r}{dt} &= \langle -\sin\theta, \cos\theta \rangle \frac{d\theta}{dt} = \mathbf{u}_\theta \frac{d\theta}{dt} \quad \text{and} \\
\frac{d\mathbf{u}_\theta}{dt} &= \langle -\cos\theta, -\sin\theta \rangle \frac{d\theta}{dt} = -\mathbf{u}_r \frac{d\theta}{dt}.
\end{aligned}$$

C12S06.067: We begin with Eq. (33),

$$\mathbf{v} = \frac{dr}{dt}\,\mathbf{u}_r + r\frac{d\theta}{dt}\,\mathbf{u}_\theta,$$

and differentiate both sides with respect to t:

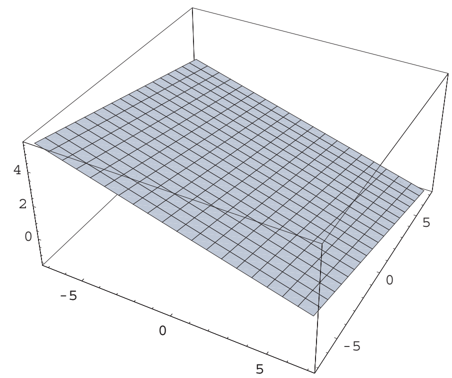
$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{v}}{dt} &= \left(\frac{d^2r}{dt^2}\,\mathbf{u}_r + \frac{dr}{dt}\cdot\frac{d\mathbf{u}_r}{dt}\right) + \left(\frac{dr}{dt}\cdot\frac{d\theta}{dt}\,\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\,\mathbf{u}_\theta + r\frac{d\theta}{dt}\cdot\frac{d\mathbf{u}_\theta}{dt}\right) \\
&= \frac{d^2r}{dt^2}\,\mathbf{u}_r + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\,\mathbf{u}_\theta + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\,\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\,\mathbf{u}_\theta - r\left(\frac{d\theta}{dt}\right)^2\mathbf{u}_r \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r \;+\; \left[\frac{1}{r}\cdot\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\mathbf{u}_\theta.
\end{aligned}$$

Section 12.7

C12S07.001: The graph of $3x + 2y + 10z = 20$ is a plane with intercepts $x = \frac{20}{3}$, $y = 10$, and $z = 2$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { x, y, (20 - 3*x - 2*y)/10 }, { x, -7, 7 }, { y, -7, 7 } ];
```

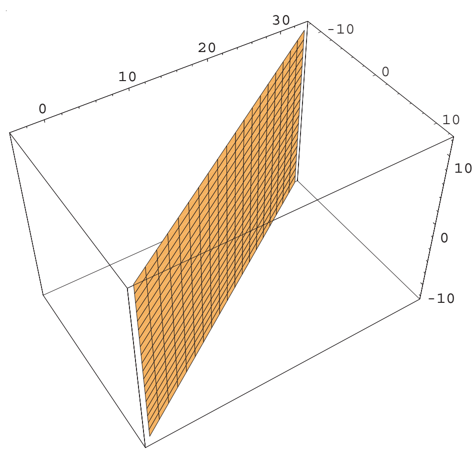
is shown next.



C12S07.002: The graph of $3x + 2y = 30$ is a plane perpendicular to the xy-plane; that is, parallel to the z-axis. Its intercepts are (10, 0, 0) and (0, 15, 0). The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { x, (30 - 3*x)/2, z }, { x, -12, 12 }, { z, -12, 12 },
                    ViewPoint → { 1.7, -1.1, 2 } ];
```

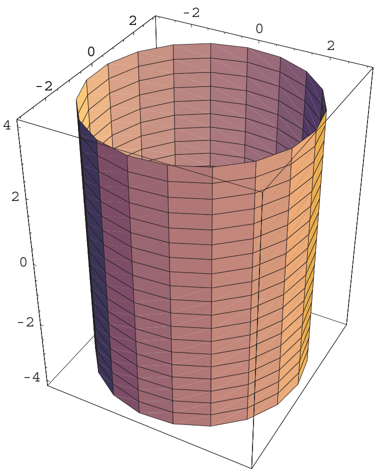
is shown next.



C12S07.003: The graph of $x^2 + y^2 = 9$ is a circular cylinder of radius 3 with axis the z-axis. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 3*Cos[t], 3*Sin[t], z }, { t, 0, 2*Pi }, { z, -4, 4 } ];
```

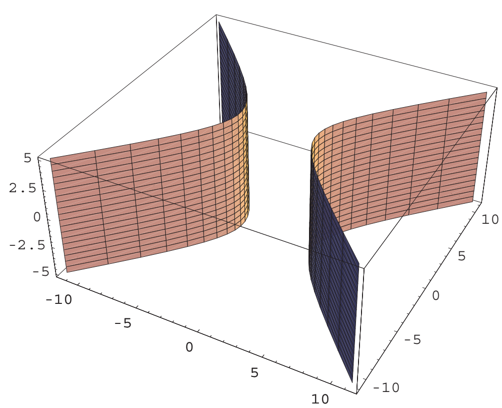
is shown next.



C12S07.004: The graph of $y^2 = x^2 - 9$ is a cylinder whose rulings are lines parallel to the z-axis. It meets the xy-plane in the hyperbola with equation $x^2 - y^2 = 9$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ 3*Cosh[t], 3*Sinh[t], z }, { -3*Cosh[t], 3*Sinh[t], z }},
                        { t, -2, 2 }, { z, -5, 5 } ];
```

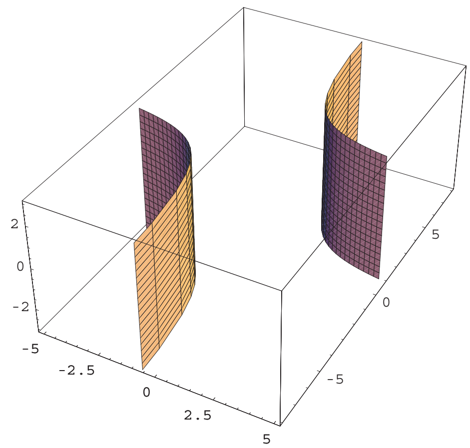
is shown next.



C12S07.005: The graph of $xy = 4$ is a cylinder whose rulings are parallel to the z-axis. It meets the xy-plane in the hyperbola with equation $xy = 4$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ x, 4/x, z }, { -x, -4/x, z }}, { x, 1/3, 5 }, { z, -3, 3 } ];
```

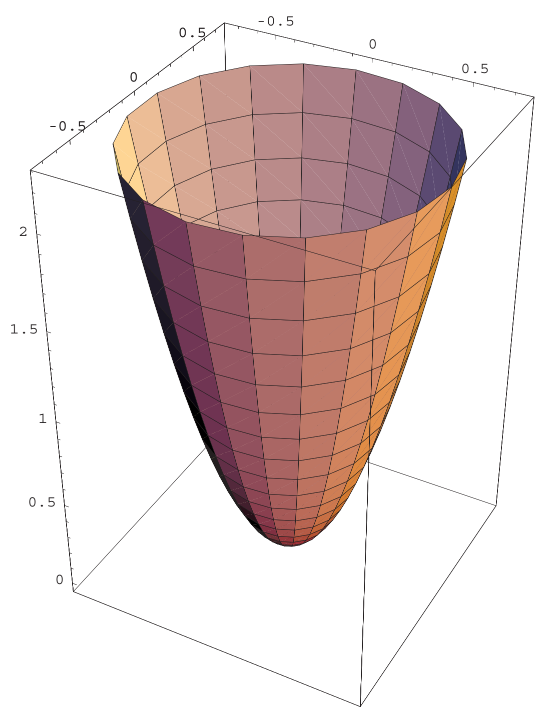
is shown next.



C12S07.006: The given equation $z = 4x^2 + 4y^2$ has the polar form $z = 4r^2$, which shows that its graph is a surface of revolution around the z-axis. The surface meets the xz-plane in the parabola $z = 4x^2$, so the surface is a circular paraboloid. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], 4*r*r }, { r, 0, 0.75 }, { t, 0, 2*Pi } ];
```

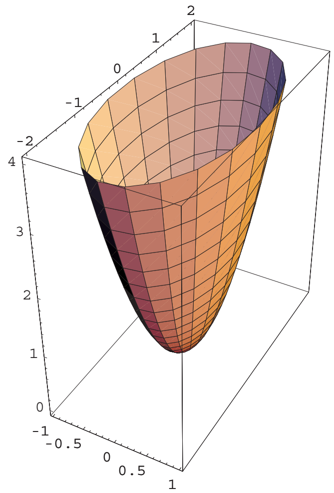
is shown next.



C12S07.007: The graph of the equation $z = 4x^2 + y^2$ is an elliptic paraboloid with axis the z-axis and vertex at the origin. It is elliptic because, if $z = a^2$, then the horizontal cross section there has equation $4x^2 + y^2 = a^2$. It is a paraboloid because it meets every vertical plane containing the z-axis in a parabola; for example, it meets the xz-plane in the parabola with equation $z = 4x^2$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], 2*r*Sin[t], 4*r*r }, { r, 0, 1 }, { t, 0, 2*Pi } ];
```

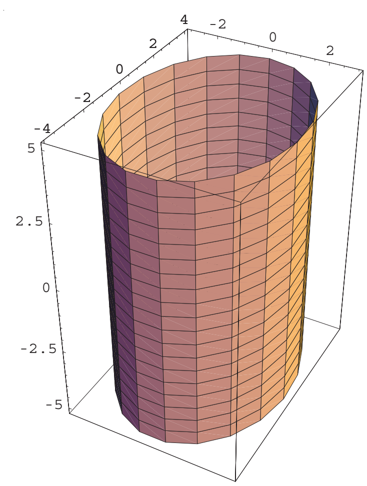
is shown next.



C12S07.008: The graph of the equation $4x^2 + 9y^2 = 36$ is an elliptical cylinder with axis the x-axis. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 3*Sin[t], 4*Cos[t], z }, { t, 0, 2*Pi }, { z, -5, 5 },
                    AspectRatio → Automatic ];
```

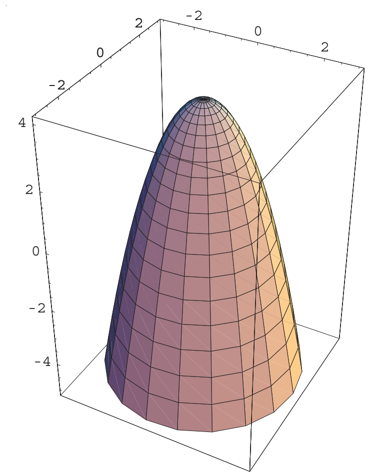
is shown next.



C12S07.009: The given equation $z = 4 - x^2 - y^2$ has the polar coordinates form $z = 4 - r^2$, so the graph is a surface of revolution around the z-axis. The graph meets the xz-plane in the parabola $z = 4 - x^2$, so the graph is a circular paraboloid, opening downward, with axis the z-axis, and vertex at (0, 0, 4). The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], 4 - r∧2 }, { r, 0, 3 }, { t, 0, 2*Pi } ];
```

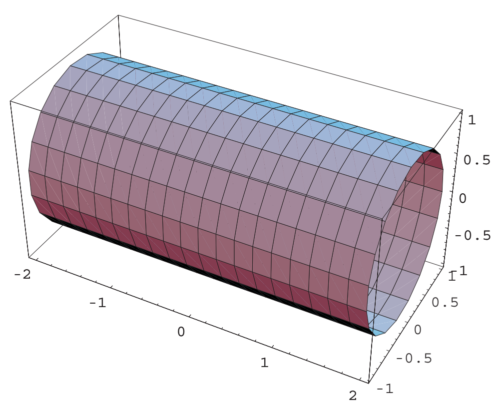
is shown next.



C12S07.010: The graph of the equation $y^2 + z^2 = 1$ is a circular cylinder of radius 1 with axis the x-axis. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { x, Cos[t], Sin[t] }, { t, 0, 2*Pi }, { x, -2, 2 } ];
```

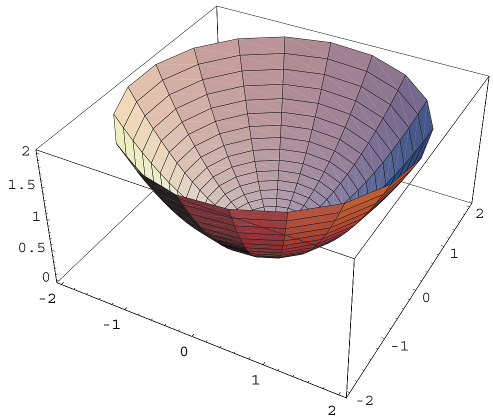
is shown next.



C12S07.011: The given equation $2z = x^2 + y^2$ has the polar form $z = \frac{1}{2}r^2$, so the graph is a surface of revolution around the z-axis. It meets the yz-plane in the parabola with equation $z = \frac{1}{2}x^2$, so the surface is a circular paraboloid; its axis is the z-axis and its vertex is at the origin. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], (r*r)/2 }, { t, 0, 2*Pi }, { r, 0, 2 } ];
```

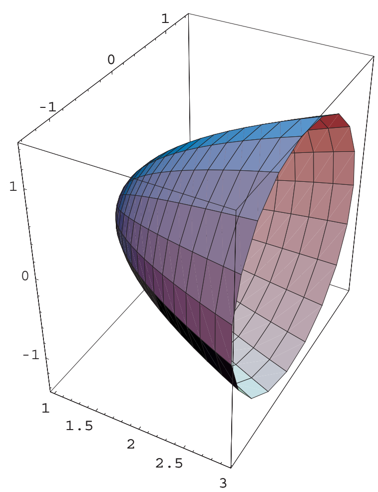
is shown next.



C12S07.012: The graph of the equation $x = 1+y^2+z^2$ is the graph of the equation $z = 1+x^2+y^2 = 1+r^2$ rotated 90°, so the surface is a circular paraboloid opening along the positive x-axis and with its vertex at (1, 0, 0). The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 1 + r*r, r*Cos[t], r*Sin[t] }, { t, 0, 2*Pi }, { r, 0, 1.4 } ];
```

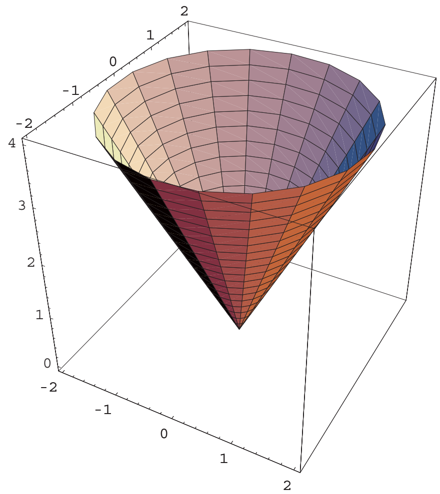
is shown next.



C12S07.013: The polar form of the given equation $z^2 = 4(x^2+y^2)$ is $z = \pm 2r$, so the graph is a surface of revolution around the z-axis. Because z is proportional to r, the graph consists of both nappes of a circular cone with axis the z-axis and vertex at the origin. To produce the graph of the upper nappe, we used the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], 2*r }, { t, 0, 2*Pi }, { r, 0, 2 } ];
```

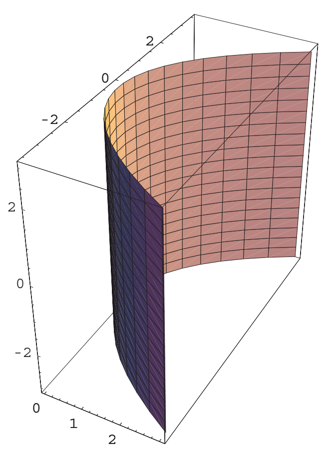
and the result is shown next.



C12S07.014: The graph of the equation $y^2 = 4x$ is a cylinder parallel to the z-axis. It is a parabolic cylinder because it intersects the xy-plane in the parabola with equation $y^2 = 4x$. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { y*y/4, y, z }, { y, -3.5, 3.5 }, { z, -3, 3 } ];
```

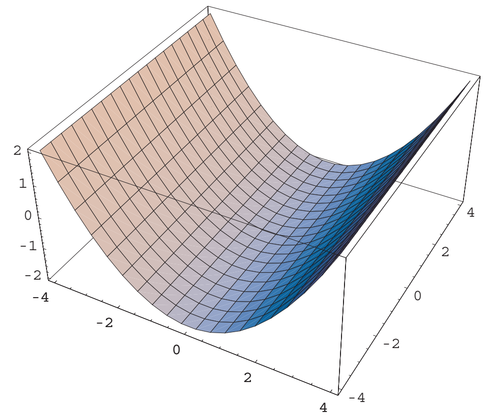
is shown next.



C12S07.015: The graph of the equation $x^2 = 4z + 8$ is a cylinder parallel to the y-axis. It is a parabolic cylinder because its trace in the xz-plane is the parabola with equation $z = (x^2 - 8)/4$. It opens upward and its lowest points consist of line $z = -2$, $x = 0$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { x, y, (x*x - 8)/4 }, { x, -4, 4 }, { y, -4, 4 } ];
```

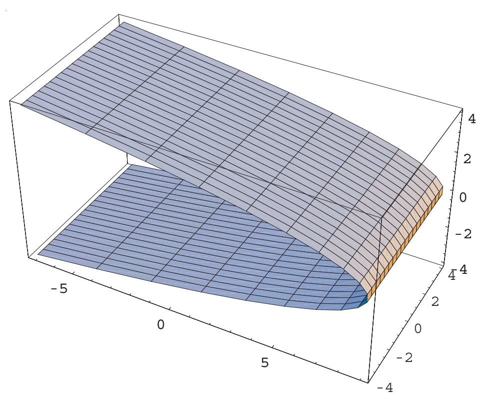
is shown next.



C12S07.016: The graph of the equation $x = 9 - z^2$ is a cylinder parallel to the y-axis. It is a parabolic cylinder because its trace in the xz-plane is the parabola with equation $x = 9 - z^2$. The graph produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 9 - z*z, y, z }, { y, -4, 4 }, { z, -4, 4 } ];
```

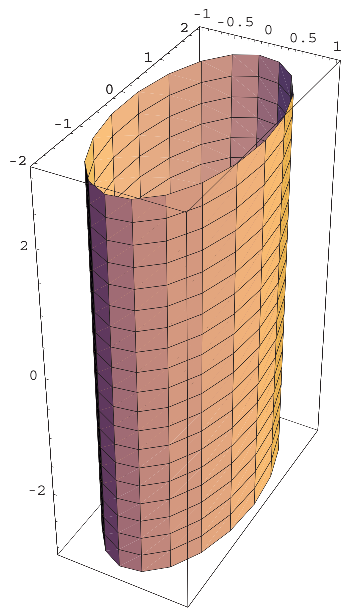
is shown next.



C12S07.017: The graph of the equation $4x^2 + y^2 = 4$ is a cylinder parallel to the z-axis. It is an elliptical cylinder because its trace in any horizontal plane is the ellipse with equation $4x^2 + y^2 = 4$. The graph of this surface produced by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { Cos[t], 2*Sin[t], z }, { t, 0, 2*Pi }, { z, -3, 3 } ];
```

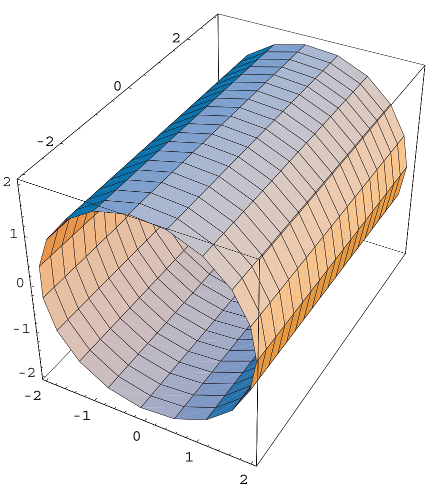
is shown next.



C12S07.018: The graph of the equation $x^2 + z^2 = 4$ is a cylinder parallel to the y-axis. It is a circular cylinder because its trace in the xz-plane is a circle with center at the origin and radius 2. The *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*Cos[t], y, 2*Sin[t] }, { t, 0, 2*Pi }, { y, -3, 3 } ];
```

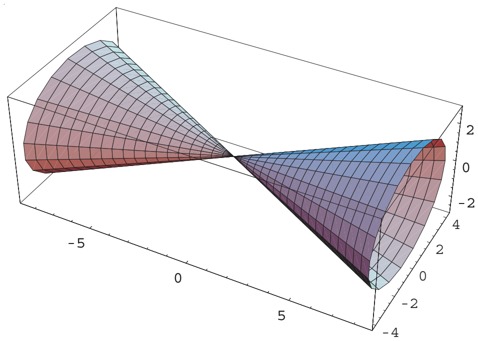
generates the graph of this surface, shown next.



C12S07.019: The graph of the equation $x^2 = 4y^2 + 9z^2$ is an elliptical cone with axis the x-axis. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ 6*r, 3*r*Cos[t], 2*r*Sin[t] }, { -6*r, 3*r*Cos[t], 2*r*Sin[t] }},
                    { r, 0, 1.4 }, { t, 0, 2*Pi } ];
```

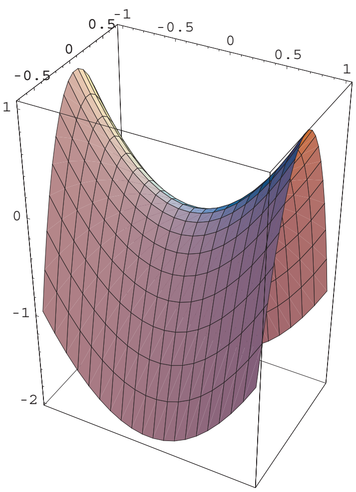
is shown next.



C12S07.020: The graph of the equation $z = x^2 - 4y^2$ is a hyperbolic paraboloid (see Example 13 of Section 12.7). The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { x, y, x*x - 4*y*y }, { x, -1, 1 }, { y, -0.7, 0.7 } ];
```

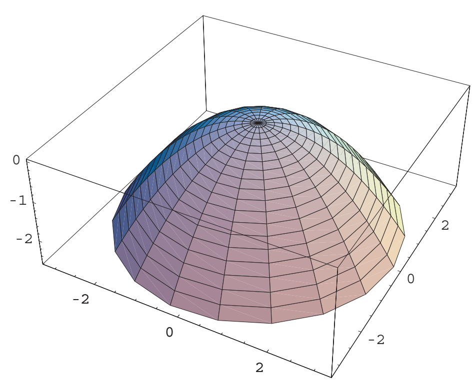
is shown next.



C12S07.021: The polar form of the given equation $x^2 + y^2 + 4z = 0$ is $z = -\frac{1}{4}r^2$, so its graph is a paraboloid opening downward, with axis the negative z-axis and vertex at the origin. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], -r*r/4 }, { t, 0, 2*Pi }, { r, 0, 3.2 } ];
```

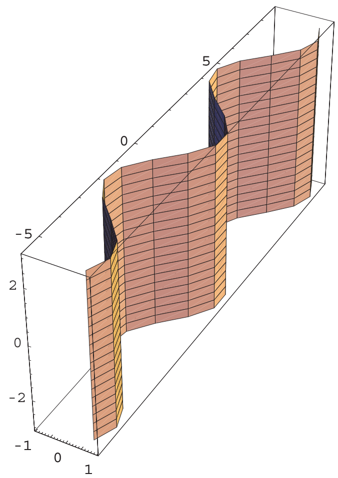
is shown next.



C12S07.022: The graph of the equation $x = \sin y$ is a cylinder parallel to the z-axis. It meets the xy-plane in the curve $x = \sin y$. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { Sin[y], y, z }, { y, -7*Pi/4, 11*Pi/4 }, { z, -3, 3 } ];
```

is shown next.

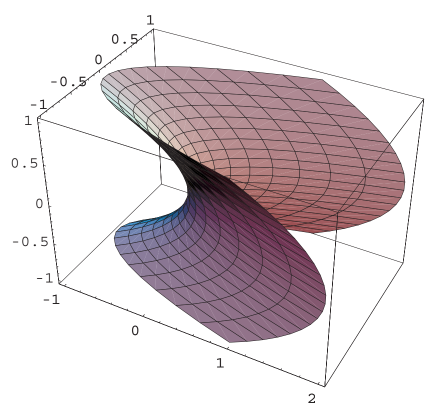


C12S07.023: There may be two versions of this problem, depending on which printing and which version (with or without matrices) of the textbook you are using. The intended version and its solution are given first.

The graph of the equation $x = 2y^2 - z^2$ is a hyperbolic paraboloid with saddle point at the origin. It meets the xz-plane in the parabola $x = -z^2$ with vertex at the origin and opening to the left; it meets the xy-plane in the parabola $x = 2y^2$ with vertex at the origin and opening to the right. The surface meets each plane parallel to the yz-plane in both branches of a hyperbola (except for the yz-plane itself, which it meets in a pair of straight lines that meet at the origin—a degenerate hyperbola). The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*y*y - z*z, y, z }, { y, -1, 1 }, { z, -1, 1 } ];
```

is shown next.



The other version of this problem is the result of a typograhical error discovered too late in the production process to be corrected in the first printing of the version of the textbook "with matrices." Its solution follows.

Given the equation $z = 2y^2 - z^2$, we complete the square in z to obtain

$$2y^2 - z^2 - z = 0;$$

$$2y^2 - z^2 - z - \frac{1}{4} = -\frac{1}{4};$$

$$2y^2 - \left(z + \frac{1}{2}\right)^2 = -\frac{1}{4}.$$

The graph of this equation is a cylinder with rulings parallel to the x-axis (because x is missing from the equation). It meets the yz-plane in the hyperbola with equation

$$\left(z + \frac{1}{2}\right)^2 - 2y^2 = \frac{1}{4}.$$

Thus the graph is a hyperbolic cylinder parallel to the x-axis.

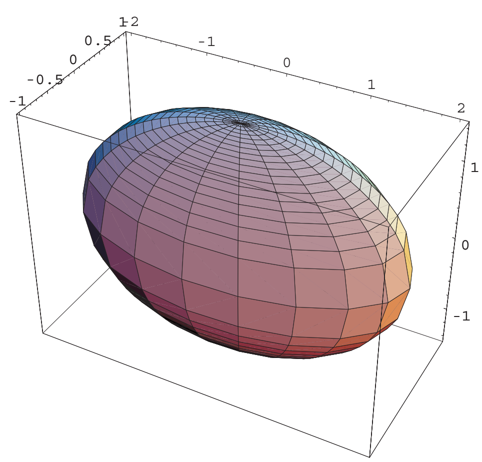
C12S07.024: The graph of the equation

$$x^2 + 4y^2 + 2z^2 = 4; \qquad \text{that is,} \qquad \frac{x^2}{4} + y^2 + \frac{z^2}{2} = 1$$

is an ellipsoid centered at the origin. Its x-intercepts are $(\pm 2, 0)$, its y-intercepts are $(0, \pm 1, 0)$, and its z-intercepts are $\left(0, 0, \pm\sqrt{2}\,\right)$. Any plane perpendicular to any coordinate axis and meeting that axis strictly between the intercepts meets the ellipsoid in a non-circular ellipse. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ 2*r*Cos[t], r*Sin[t], Sqrt[2 - 2*r*r] },
                   { 2*r*Cos[t], r*Sin[t], -Sqrt[2 - 2*r*r] }},
                   { t, 0, 2*Pi }, { r, 0, 1 } ];
```

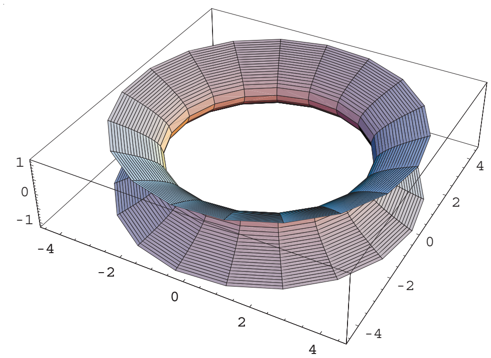
is shown next.



C12S07.025: The graph of the equation $x^2 + y^2 - 9z^2 = 9$ is a hyperboloid of one sheet with axis the z-axis. It meets the xy-plane in a circle of radius 3 and meets parallel planes in larger circles. It meets every plane containing the z-axis in both branches of a hyperbola. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r*Cos[t], r*Sin[t], Sqrt[(r*r - 9)/9] },
                   { r*Cos[t], r*Sin[t], -Sqrt[(r*r - 9)/9] }},
                   { r, 3, 4.5 }, { t, 0, 2*Pi } ];
```

is shown next.



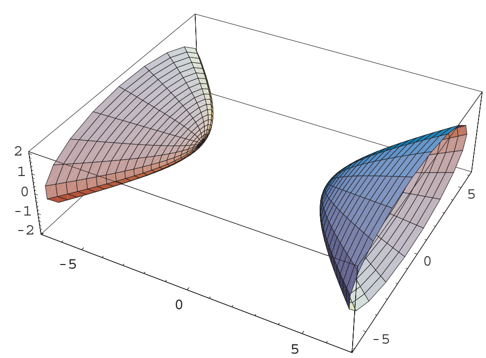
C12S07.026: The graph of the equation $x^2 - y^2 - 9z^2 = 9$ is a hyperboloid of two sheets centered at the origin and with its vertices on the x-axis at $(\pm 3,\ 0,\ 0)$. Each plane perpendicular to the x-axis and meeting it at a point x for which $|x| > 3$ intersects the hyperboloid in an ellipse centered on the x-axis. A plane perpendicular to either of the other coordinate axes meets it in both branches of a hyperbola. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ Sqrt[9*r*r + 9], 3*r*Cos[t], r*Sin[t] },

                   { -Sqrt[9*r*r + 9], 3*r*Cos[t], r*Sin[t] }},

                   { r, 0, 2 }, { t, 0, 2*Pi } ];
```

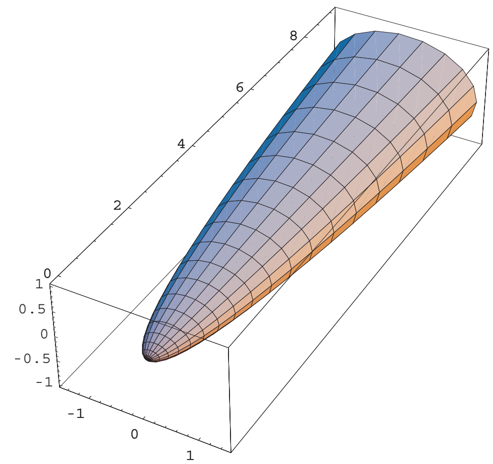
is shown next.



C12S07.027: The graph of the equation $y = 4x^2 + 9z^2$ is an elliptic paraboloid opening in the positive y-direction, with axis the nonnegative y-axis and vertex at the origin. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 3*r*Cos[t], 36*r*r, 2*r*Sin[t] }, { r, 0, 1/2 }, { t, 0, 2*Pi } ];
```

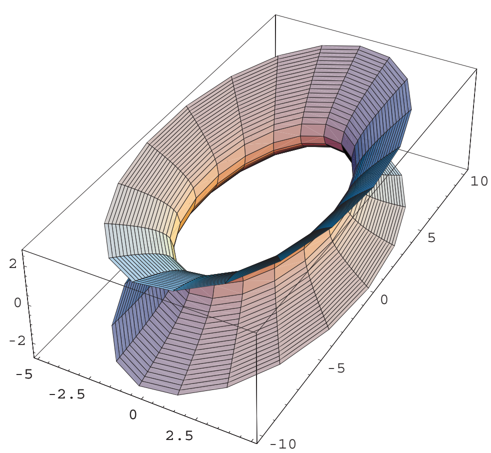
is shown next.



C12S07.028: The graph of the equation $y^2 + 4x^2 - 9z^2 = 36$ is a hyperboloid of one sheet with axis the z-axis. Planes normal to the z-axis meet the surface in ellipses. Planes containing the z-axis meet it in both branches of a hyperbola. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r*Cos[t], 2*r*Sin[t], Sqrt[(4*r*r - 36)/9] },
                   { r*Cos[t], 2*r*Sin[t], -Sqrt[(4*r*r - 36)/9] }},
                   { t, 0, 2*Pi }, { r, 3, 5 } ];
```

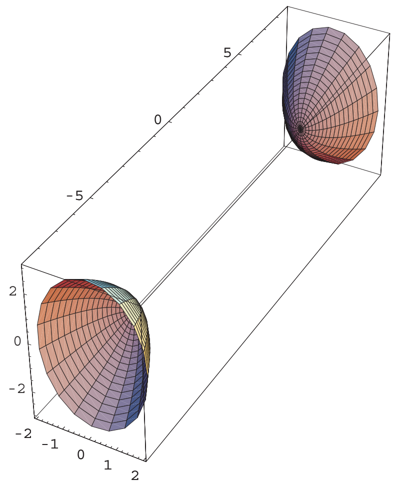
is shown next.



C12S07.029: The graph of the equation $y^2 - 9x^2 - 4z^2 = 36$ is a hyperboloid of two sheets with axis the y-axis, center the origin, and intercepts $(0, \pm 6, 0)$. Planes containing the y-axis meet it in both branches of a hyperbola. Planes normal to the y-axis and outside the intercepts meet it in ellipses. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ 2*r*Cos[t], 6*Sqrt[r*r + 1], 3*r*Sin[t] },
                   { 2*r*Cos[t], -6*Sqrt[r*r + 1], 3*r*Sin[t] }},
                   { t, 0, 2*Pi }, { r, 0, 1 } ];
```

is shown next.



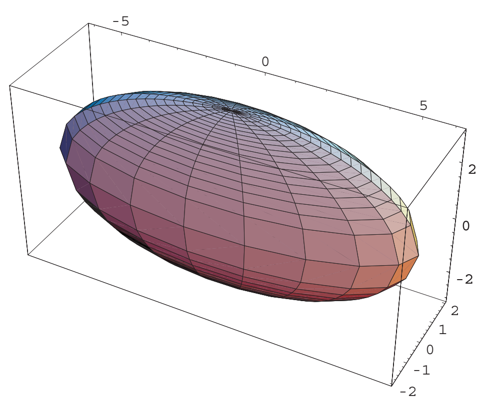
C12S07.030: The graph of the equation $x^2 + 9y^2 + 4z^2 = 36$ is an ellipsoid centered at the origin. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ 3*r*Cos[t], r*Sin[t], Sqrt[(36 - 9*r*r)/4] },

                  { 3*r*Cos[t], r*Sin[t], -Sqrt[(36 - 9*r*r)/4] }},

                  { r, 0, 2 }, { t, 0, 2*Pi } ];
```

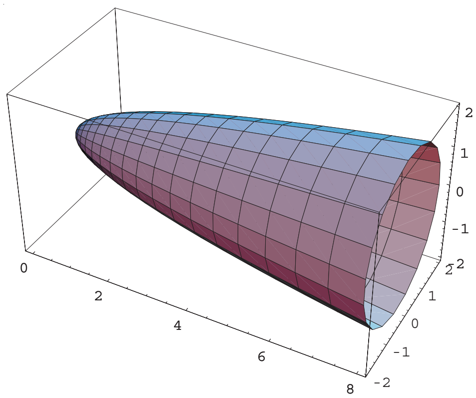
is shown next.



C12S07.031: The graph of the curve $x = 2z^2$ (in the xz-plane) is to be rotated around the x-axis. To obtain an equation of the resulting surface, replace z with $(y^2 + z^2)^{1/2}$ to obtain $x = 2(y^2 + z^2)$. The surface is a circular paraboloid opening along the positive x-axis. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*r*r, r*Cos[t], r*Sin[t] }, { t, 0, 2*Pi }, { r, 0, 2 } ];
```

is shown next.



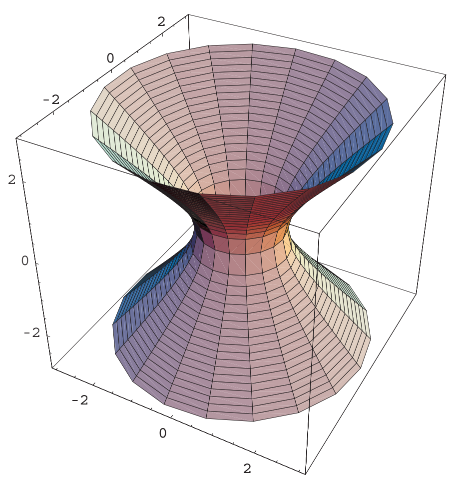
C12S07.032: The curve $4x^2 + 9y^2 = 36$ (in the xy-plane) is to be rotated around the y-axis. To obtain an equation of the resulting surface, replace x^2 with $x^2 + z^2$ to obtain $4x^2 + 4z^2 + 9y^2 = 36$. The surface is an ellipsoid (actually, an oblate spheroid) centered at the origin. To see its graph, enter the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r*Cos[t], Sqrt[(36 - 4*r*r)/9], r*Sin[t] },
                   { r*Cos[t], -Sqrt[(36 - 4*r*r)/9], r*Sin[t] }},
                   { t, 0, 2*Pi }, { r, 0, 3 } ];
```

C12S07.033: The curve $y^2 - z^2 = 1$ (in the yz-plane) is to be rotated around the z-axis. To obtain an equation of the resulting surface, replace y^2 with $x^2 + y^2$ to obtain $x^2 + y^2 - z^2 = 1$. The surface is a circular hyperboloid of one sheet with axis the z-axis. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r*Cos[t], r*Sin[t], Sqrt[r*r - 1] },
                   { r*Cos[t], r*Sin[t], -Sqrt[r*r - 1] }},
                   { r, 1, 3 }, { t, 0, 2*Pi } ];
```

is shown next.

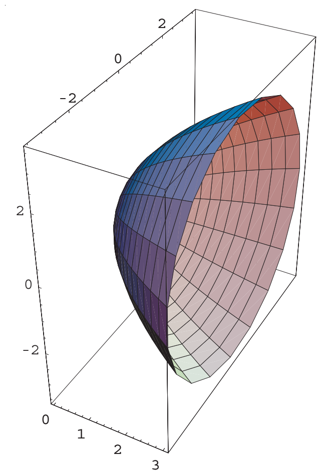


C12S07.034: The curve $z = 4 - x^2$ (in the xz-plane) is to be rotated around the z-axis. To obtain an equation of the resulting surface, replace x^2 with $x^2 + y^2$ to obtain $z = 4 - x^2 - y^2$. See the solution of Problem 9 of this section for further discussion of this surface.

C12S07.035: The curve $y^2 = 4x$ (in the xy-plane) is to be rotated around the x-axis. Replace y^2 with $y^2 + z^2$ to obtain an equation of the resulting surface: $y^2 + z^2 = 4x$. The surface is a circular paraboloid opening along the positive x-axis, with axis that axis, and with vertex at the origin. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*r/4, r*Cos[t], r*Sin[t] }, { t, 0, 2*Pi }, { r, 0, 3.5 } ];
```

is shown next.



C12S07.036: The curve with equation $yz = 1$ (in the yz-plane) is to be rotated around the z-axis. To obtain an equation of this surface, replace y with $\pm\sqrt{x^2 + y^2}$ to obtain $z^2(x^2 + y^2) = 1$. To see this surface, enter the *Mathematica* 3.0 command

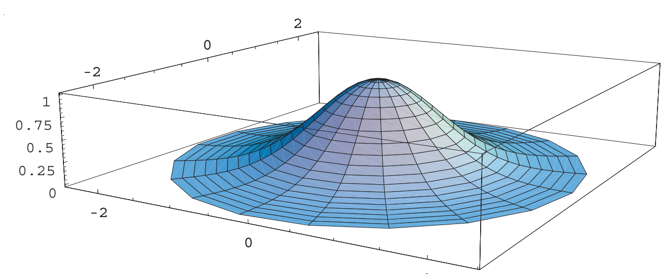
```
ParametricPlot3D[ {{ r*Cos[t], r*Sin[t], 1/r }, { r*Cos[t], r*Sin[t], -1/r }},
                    { t, 0, 2*Pi }, { r, 1/6, 4 }, ViewPoint → { 1.3, -2.2, 0.6 } ];
```

The change in `ViewPoint` is necessary; otherwise the top half of the figure completely conceals the bottom half.

C12S07.037: Given: the curve with equation $z = \exp(-x^2)$ in the xz-plane, to be rotated around the z-axis. To obtain an equation of the resulting surface, replace x^2 with $x^2 + y^2$ to obtain $z = \exp\left(-x^2 - y^2\right)$. The graph of this surface generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], Exp[-r*r] }, { t, 0, 2*Pi }, { r, 0, 2.4 },
                   AspectRatio → Automatic, ViewPoint → { 1.3, -2.2, 0.6 } ];
```

is shown next.

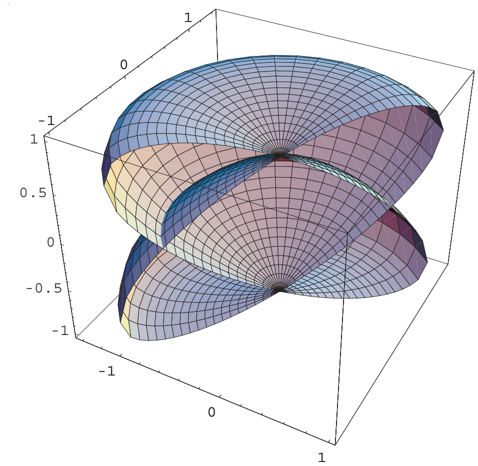


C12S07.038: Given: $y^2 - 2yz + 2z^2 = 1$. This is an ellipse in the yz-plane; in the appropriately rotated (through about 0.447 radians, about 25.6°) uv-coordinate system, its equation takes the form $(0.38)u^2 + (2.62)v^2 = 1$ (the coefficients are approximate). If this ellipse is rotated around the z-axis, the resulting

surface has equation obtained from that of the ellipse by replacement of y with $\pm\sqrt{x^2+y^2}$. The resulting equation is

$$x^2 + y^2 \pm 2z\sqrt{x^2+y^2} + 2z^2 = 1.$$

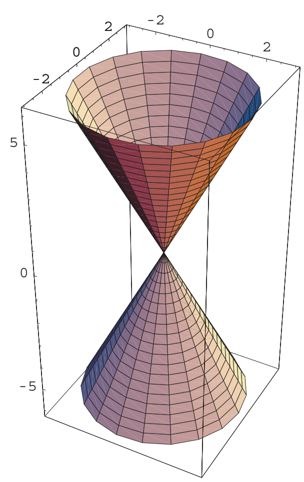
Half of the surface is shown next.



C12S07.039: The curve with equation $z = 2x$ (in the xz-plane) is to be rotated around the z-axis. To obtain an equation of the surface thereby generated, replace x with $\pm\sqrt{x^2+y^2}$. The resulting equation is $z^2 = 4(x^2+y^2)$ and its graph consists of both nappes of a right circular cone with vertices at the origin and axis the z-axis. The graph generated by the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r*Cos[t], r*Sin[t], 2*r }, { r*Cos[t], r*Sin[t], -2*r }},
                    { t, 0, 2*Pi }, { r, 0, 3 } ];
```

is shown next.



C12S07.040: The curve with equation $z = 2x$ (in the xz-plane) is to be rotated around the x-axis. To obtain an equation of the surface it generates, replace z with $\pm\sqrt{y^2+z^2}$. An equation of the surface is thus $4x^2 = y^2 + z^2$. The surface consists of both nappes of a cone with vertices at the origin and axis the x-axis. To generate its graph, execute the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{ r/2, r*Cos[t], r*Sin[t] }, { -r/2, r*Cos[t], r*Sin[t] }},
                    { t, 0, 2*Pi }, { r, 0, 3 } ];
```

C12S07.041: The graph of $x^2 + 4y^2 = 4$ is an elliptical cylinder with centerline the z-axis. Thus its traces in horizontal planes are ellipses with semiaxes 2 and 1.

C12S07.042: The graph of $x^2 + 4y^2 + 4z^2 = 4$ is an ellipsoid (actually, a prolate spheroid). Its traces in horizontal planes near the origin are ellipses centered on the z-axis.

C12S07.043: The graph of $x^2 + 4y^2 + 4z^2 = 4$ is an ellipsoid. A plane parallel to the yz-plane is perpendicular to the x-axis and thus has an equation of the form $x = a$. So the trace of the ellipsoid in such a plane has equations

$$x = a, \quad 4y^2 + 4z^2 = 4 - a^2.$$

Thus the trace is a circle if $|a| < 2$, a single point if $|a| = 2$, and the empty set if $|a| > 2$.

C12S07.044: The graph of $z = 4x^2 + 9y^2$ is an elliptical paraboloid opening upward with axis the z-axis and vertex at the origin. A horizontal plane has equation $z = a$ and thus the trace in such a plane has equations

$$z = a, \quad 4x^2 + 9y^2 = a.$$

Thus the trace is an ellipse if $a > 0$, a single point if $a = 0$, and the empty set if $a < 0$.

C12S07.045: The graph of $z = 4x^2 + 9y^2$ is an elliptical paraboloid opening upward with axis the z-axis and vertex at the origin. A plane parallel to the yz-plane has equation $x = a$ and thus the trace in such a plane has equations

$$x = a, \quad z = 4a^2 + 9y^2.$$

Thus the trace is a parabola opening upward with vertex at $(a, 0, 4a^2)$.

C12S07.046: The graph of $z = xy$ meets the horizontal plane $z = a$ in the curve with equations

$$z = a, \quad y = \frac{a}{x}.$$

Thus the trace is a hyperbola with asymptotes lying in the xz-plane and the yz-plane, except if $a = 0$ then the trace consists of the x- and y-axes.

C12S07.047: A plane containing the z-axis has an equation of the form $ax + by = 0$. So if $b \neq 0$ then the intersection of the hyperbolic paraboloid $z = xy$ with such a plane has equations

$$y = -\frac{a}{b}x, \quad z = -\frac{a}{b}x^2.$$

Hence the trace is a parabola opening downward if a and b have the same sign, opening upward if they have opposite sign, and is a horizontal line if $a = 0$ (or if $b = 0$). The surface itself resembles the one shown in Fig. 12.7.22 rotated 45° around the z-axis.

C12S07.048: Given: the surface with equation $x^2 - y^2 + z^2 = 1$, a hyperboloid of one sheet. Its traces in horizontal planes (where $z = a$, a constant) are hyperbolas with equations of the form

$$z = a, \quad x^2 - y^2 = 1 - a^2.$$

(If $a = \pm 1$ then the traces are pairs of intersecting straight lines.) Its traces in planes parallel to the yz-plane, where $x = a$, a constant, have equations of the form

$$x = a, \quad z^2 - y^2 = 1 - a^2,$$

so the traces are similar to those in the previous case. But its traces in planes parallel to the xz-plane, where $y = a$ (a constant), have equations of the form

$$y = a, \quad x^2 + z^2 = a^2 + 1,$$

and thus are circles centered on the y-axis, having minimum radius 1 when $a = 0$ but larger radius the larger $|a|$ becomes. The hyperboloid resembles the one shown in Fig. 12.7.17, except that its axis is the y-axis rather than the z-axis.

C12S07.049: The triangles OAC and OBC in Fig. 12.7.1 are congruent because the angles OCA and OCB are right angles, $|OA| = |OB|$ because they are both radii of the same sphere, and $|OC| = |OC|$. So the triangles have matching side-side-angle in the same order.

C12S07.050: The intersection I of the two surfaces $x = 1-y^2$ and $x = y^2+z^2$ satisfies both these equations simultaneously, and thus $1 - y^2 = y^2 + z^2$; that is, $2y^2 + z^2 = 1$. Hence I lies on the elliptical cylinder with equation $2y^2 + z^2 = 1$. Thus the projection of I into the yz-plane lies on the ellipse E with equations $x = 0$, $2y^2 + z^2 = 1$. Care is needed here. You can't simply set $x = 0$ and simplify the other two equations; you won't get all the points on the ellipse. To project the intersection of two surfaces into a plane, first find the equation of the cylinder perpendicular to the plane and containing the intersection. Moreover, at this point all we actually know is that the projection lies *on* the ellipse E. We need to show that the projection of I is *all* of E. Let $(0, y, z)$ be a point of E. Then $2y^2 + z^2 = 1$. Let $x = 1 - y^2$. Then

$$y^2 + z^2 = 2y^2 + z^2 + 1 - y^2 - 1 = 1 + 1 - y^2 - 1 = 1 - y^2 = x.$$

Therefore the point (x, y, z) lies on I. Hence every point of E is the image of a point of I under the vertical projection into the yz-plane. This proves that the projection of the intersection of the given surfaces into the yz-plane is an ellipse.

C12S07.051: The intersection I of the plane $z = y$ and the paraboloid $z = x^2+y^2$ satisfies both equations, and thus lies on the surface with equation

$$y = x^2 + y^2; \quad \text{that is,} \quad x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

This surface is a circular cylinder perpendicular to the xy-plane. Hence the projection of I into that plane lies on the circle C with equations

$$z = 0, \quad x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

It remains to show that all of C is obtained by projecting I into the xy-plane. Suppose that $(x, y, 0)$ lies on C. Then $y = x^2 + y^2$. Let $z = y$. Then

$$x^2 + y^2 = y = z,$$

and therefore (x, y, z) satisfies the equations of both the plane and the paraboloid. Therefore (x, y, z) lies on I. This proves that the projection of I into the xy-plane is *all* of C. Therefore the projection of I into the xy-plane is a circle.

C12S07.052: The intersection I of the paraboloids with equations $y = 2x^2 + 3z^2$ and $y = 5 - 3x^2 - 2z^2$ satisfies both equations, and therefore lies on the surface with equation

$$2x^2 + 3z^2 = 5 - 3x^2 - 2z^2; \quad \text{that is,} \quad x^2 + z^2 = 1.$$

This surface is a circular cylinder normal to the xz-plane. Therefore the projection of I into that plane lies on the circle C with equations

$$y = 0, \quad x^2 + z^2 = 1.$$

It remains to show that every point of C is thereby obtained. Suppose that $(x, 0, z)$ lies on C. Then $x^2 + z^2 = 1$. Let $y = 2x^2 + 3z^2$. Then

$$\begin{aligned} 5 - 3x^2 - 2z^2 &= 5 - 5x^2 - 5z^2 + 2x^2 + 3z^2 \\ &= 5 - 5(x^2 + z^2) + 2x^2 + 3z^2 = 5 - 5 + 2x^2 + 3z^2 = 2x^2 + 3z^2 = y. \end{aligned}$$

Therefore (x, y, z) lies on both paraboloids, and thus the point (x, y, z) on their intersection I projects onto the point $(x, 0, z)$ of the circle C. That is, *all* of C is obtained by projection of I into the xz-plane. This proves that the projection of I into that plane is indeed a circle.

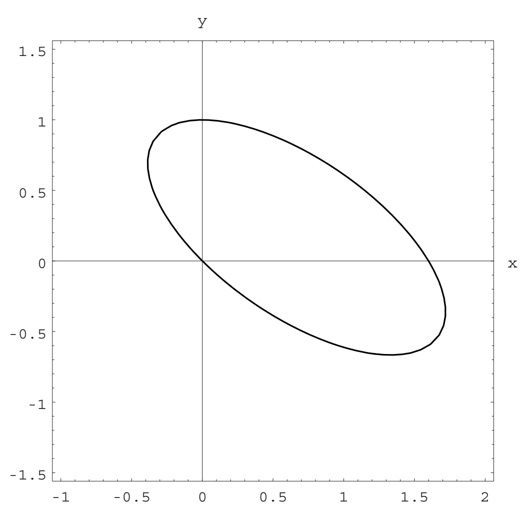
C12S07.053: The plane with equation $x + y + z = 1$ and the ellipsoid with equation $x^2 + 4y^2 + 4z^2 = 4$ have intersection that satisfies both equations, so it lies on the surface with equation

$$x^2 + 4y^2 + 4(1 - x - y)^2 = 4; \quad \text{that is,} \quad 5x^2 + 8xy + 8y^2 - 8x - 8y = 0.$$

This surface is a cylinder normal to the xy-plane, so its projection into the xy-plane—which is the projection of the intersection into the xy-plane—has the same equation. By earlier discussions of such equations, the projection must be a conic section. But it cannot be a hyperbola or a parabola because it is clearly a closed curve. (It is not any of the degenerates cases of a conic section because it contains the two points $(0, 0)$ and $(0, 1)$.) Therefore it is an ellipse. To be absolutely certain of this, the *Mathematica* 3.0 command

```
ContourPlot[ 5*x*x + 8*x*y + 8*y*y - 8*x - 8*y, { x, -1, 2 }, { y, -1.5, 1.5 },
   Axes → True, AxesOrigin → (0,0), AxesLabel → { x, y }, Contours → 3,
   ContourShading → False, PlotPoints → 47, PlotRange → { -0.01, 0.01 } ];
```

generates a plot of this curve, and it's shown next.



Section 12.8

C12S08.001: The formulas in (3) immediately yield

$$x = r\cos\theta = 1\cdot\cos\frac{\pi}{2} = 0, \qquad y = r\sin\theta = 1\cdot\sin\frac{\pi}{2} = 1, \qquad z = 2.$$

Answer: $(0,\ 1,\ 2)$.

C12S08.002: The formulas in (3) yield $(x,\ y,\ z) = (0,\ -3,\ -1)$.

C12S08.003: If $(r,\ \theta,\ z) = (2,\ 3\pi/4,\ 3)$, then $(x,\ y,\ z) = (r\cos\theta,\ r\sin\theta,\ z) = \big(-\sqrt{2},\ \sqrt{2},\ 3\big)$.

C12S08.004: If $(r,\ \theta,\ z) = (3,\ 7\pi/6,\ -1)$, then $(x,\ y,\ z) = (r\cos\theta,\ r\sin\theta,\ z) = \big(-\frac{3}{2}\sqrt{3},\ -\frac{3}{2},\ -1\big)$.

C12S08.005: If $(r,\ \theta,\ z) = (2,\ \pi/3,\ -5)$, then $(x,\ y,\ z) = (r\cos\theta,\ r\sin\theta,\ z) = \big(1,\ \sqrt{3},\ -5\big)$.

C12S08.006: If $(r,\ \theta,\ z) = (4,\ 5\pi/3,\ 6)$, then $(x,\ y,\ z) = (r\cos\theta,\ r\sin\theta,\ z) = \big(2,\ -2\sqrt{3},\ 6\big)$.

C12S08.007: Given $(\rho,\ \phi,\ \theta) = (2,\ 0,\ \pi)$, the equations in (6) immediately yield

$$x = \rho\sin\phi\,\cos\theta = 0, \qquad y = \rho\sin\phi\,\sin\theta = 0, \qquad z = \rho\cos\phi = 2.$$

Answer: $(0,\ 0,\ 2)$.

C12S08.008: If $(\rho,\ \phi,\ \theta) = (3,\ \pi,\ 0)$, then

$$(x,\ y,\ z) = (\rho\sin\phi\,\cos\theta,\ \rho\sin\phi\,\sin\theta,\ \rho\cos\phi) = (0,\ 0,\ -3).$$

C12S08.009: If $(\rho,\ \phi,\ \theta) = (3,\ \pi/2,\ \pi)$, then

$$(x,\ y,\ z) = (\rho\sin\phi\,\cos\theta,\ \rho\sin\phi\,\sin\theta,\ \rho\cos\phi) = (-3,\ 0,\ 0).$$

C12S08.010: If $(\rho,\ \phi,\ \theta) = (4,\ \pi/6,\ 2\pi/3)$, then

$$(x,\ y,\ z) = (\rho\sin\phi\,\cos\theta,\ \rho\sin\phi\,\sin\theta,\ \rho\cos\phi) = \left(-1,\ \sqrt{3},\ 2\sqrt{3}\right).$$

C12S08.011: If $(\rho,\ \phi,\ \theta) = (2,\ \pi/3,\ 3\pi/2)$, then

$$(x,\ y,\ z) = (\rho\sin\phi\,\cos\theta,\ \rho\sin\phi\,\sin\theta,\ \rho\cos\phi) = \left(0,\ -\sqrt{3},\ 1\right).$$

C12S08.012: If $(\rho,\ \phi,\ \theta) = (6,\ 3\pi/4,\ 4\pi/3)$, then

$$(x,\ y,\ z) = (\rho\sin\phi\,\cos\theta,\ \rho\sin\phi\,\sin\theta,\ \rho\cos\phi) = \left(-\tfrac{3}{2}\sqrt{2},\ -\tfrac{3}{2}\sqrt{6},\ -3\sqrt{2}\right).$$

C12S08.013: If $P(x,\ y,\ z) = P(0,\ 0,\ 5)$, then P has cylindrical coordinates

$$r = \sqrt{x^2+y^2} = 0, \qquad \theta \ \text{(therefore) arbitrary, and} \qquad z = 5.$$

So one correct answer is $(0,\ 0,\ 5)$. Also P has spherical coordinates

$$\rho = \sqrt{x^2 + y^2 + z^2} = \pm 5, \qquad \phi = 0 \ \ (\text{if } \rho > 0), \qquad \theta \ \ (\text{therefore}) \text{ arbitrary.}$$

Thus one correct answer is $(5,\ 0,\ 0)$. Another is $(-5,\ \pi,\ \pi/2)$.

C12S08.014: If $P(x,\ y,\ z) = P(0,\ 0,\ -3)$, then P has cylindrical coordinates

$$r = \sqrt{x^2 + y^2} = 0, \qquad \theta \ \ (\text{therefore}) \text{ arbitrary, and} \qquad z = -3.$$

So one correct answer is $(0,\ 0,\ -3)$. Also P has spherical coordinates

$$\rho = \sqrt{x^2 + y^2 + z^2} = \pm 3, \quad \phi = \pi \ \ (\text{if } \rho > 0), \qquad \theta \ \ (\text{therefore}) \text{ arbitrary.}$$

Thus one correct answer is $(3,\ \pi,\ 0)$. Another—perhaps more natural—is $(-3,\ 0,\ 0)$.

C12S08.015: Cylindrical $\big(\sqrt{2},\ \pi/4,\ 0\big)$, spherical $\big(\sqrt{2},\ \pi/2,\ \pi/4\big)$.

C12S08.016: Cylindrical $\big(2\sqrt{2},\ -\pi/4,\ 0\big)$, spherical $\big(2\sqrt{2},\ \pi/2,\ -\pi/4\big)$.

C12S08.017: Given: the point with Cartesian coordinates $P(1,\ 1,\ 1)$. Its cylindrical coordinates are $\big(\sqrt{2},\ \pi/4,\ 1\big)$. To find the spherical coordinates of P, we compute

$$\rho = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \qquad \text{and} \qquad \theta = \frac{\pi}{4}.$$

To find ϕ, Fig. 12.8.10 makes it clear that

$$\cos\phi = \frac{z}{\rho} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3},$$

and hence

$$\phi = \arccos\left(\frac{\sqrt{3}}{3}\right) \approx 0.9553166181245093;$$

that's approximately 54.7356103162°, about $54^\circ\ 44'\ 8.197''$. Thus the spherical coordinates of P are

$$\left(\sqrt{3},\ \cos^{-1}\frac{\sqrt{3}}{3},\ \frac{\pi}{4}\right).$$

Other ways to express ϕ are

$$\phi = \sin^{-1}\frac{\sqrt{6}}{3} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\sqrt{2}\right).$$

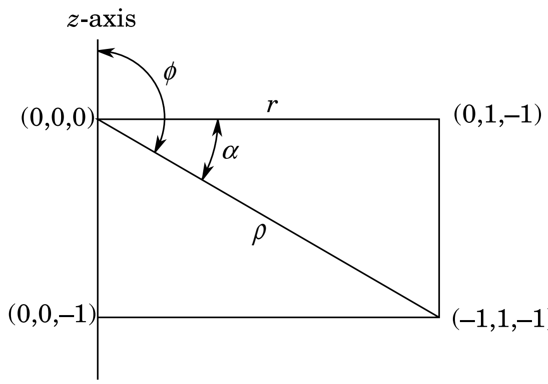
C12S08.018: Cylindrical $\big(\sqrt{2},\ 3\pi/4,\ -1\big)$, spherical $\big(\sqrt{3},\ \frac{1}{2}\pi + \arccos\big(\frac{1}{3}\sqrt{6}\,\big),\ 3\pi/4\big)$. A simpler way to express ϕ is

$$\phi = \frac{\pi}{2} + \arctan\left(\frac{\sqrt{2}}{2}\right),$$

which is approximately $125^\circ\ 15'\ 51.8''$. Thus the spherical coordinates of $P(-1,\ 1,\ -1)$ are approximately $(1.732051,\ 2.186276,\ 2.356194)$ (with the angles in the last answer measured in radians). To find ϕ, see the

following figure, showing the vertical plane containing the z-axis and the point $(-1,\ 1,\ -1)$. The figure shows that $r = \sqrt{2}$ and that $\rho = \sqrt{3}$. Also

$$\phi = \alpha + \frac{\pi}{2} \quad \text{where} \quad \tan\alpha = \frac{1}{r} = \frac{\sqrt{2}}{2}, \quad \sin\alpha = \frac{1}{\rho} = \frac{\sqrt{3}}{3}, \quad \text{and} \quad \cos\alpha = \frac{r}{\rho} = \frac{\sqrt{6}}{3}.$$



C12S08.019: The given point P having the Cartesian coordinates $(2,\ 1,\ -2)$ has cylindrical coordinates $\left(\sqrt{5},\ \tan^{-1}\left(\frac{1}{2}\right),\ -2\right)$. To find its spherical coordinates, it's clear that $\rho = 3$. Imagine the vertical plane that contains the z-axis and $P(2,\ 1,\ -2)$. If you mark the points $O(0,\ 0,\ 0)$ and $Q(2, 1, 0)$, draw the z-axis and the triangle OAP, then the angle ϕ is the obtuse angle reaching from the positive z-axis to the hypotenuse of triangle OAP. Let α be the acute angle of that triangle at O. Note that OA has length $r = \sqrt{5}$, that OP has length $\rho = 3$, and that AP has length $-z = 2$. Because

$$\phi = \alpha + \frac{\pi}{2}, \quad \sin\alpha = \frac{2}{3}, \quad \text{and} \quad \cos\alpha = \frac{\sqrt{5}}{3},$$

it now follows that

$$\cos\phi = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha = -\frac{2}{3}.$$

Therefore the spherical coordinates of P are

$$\left(3,\ \cos^{-1}\left(-\frac{2}{3}\right),\ \tan^{-1}\left(\frac{1}{2}\right)\right) \approx (3,\ 2.300523983822,\ 0.463647609001).$$

C12S08.020: The cylindrical coordinates are $\left(\sqrt{5},\ \pi + \tan^{-1}\left(\frac{1}{2}\right),\ -2\right)$. The spherical coordinates are

$$\left(3,\ \frac{\pi}{2} + \sin^{-1}\frac{2}{3},\ \pi + \tan^{-1}\frac{1}{2}\right).$$

The evaluation of ϕ is the most difficult; use a figure resembling that shown in the solution of Problem 18.

C12S08.021: The cylindrical coordinates of $P(3,\ 4,\ 12)$ are $\left(5,\ \arctan\left(\frac{4}{3}\right),\ 12\right)$ and its spherical coordinates are $\left(13,\ \arcsin\left(\frac{5}{13}\right),\ \arctan\left(\frac{4}{3}\right)\right)$.

C12S08.022: The cylindrical coordinates of $P(-2,\ 4,\ -12)$ are $\left(2\sqrt{5},\ \arccos\left(-\frac{1}{5}\sqrt{5}\right),\ -12\right)$; its spherical coordinates are $\left(2\sqrt{41},\ \frac{1}{2}\pi + \arccos\left(\frac{1}{41}\sqrt{205}\right),\ \arccos\left(-\frac{1}{5}\sqrt{5}\right)\right)$. Finding ϕ is the most difficult; draw a figure similar to the one used in the solution of Problem 18.

C12S08.023: The graph of the cylindrical equation $r = 5$ is the cylinder of radius 5 with axis the z-axis.

C12S08.024: The graph of the cylindrical *or* spherical equation $\theta = 3\pi/4$ is the vertical plane with Cartesian equation $x + y = 0$.

C12S08.025: The graph of the cylindrical *or* spherical equation $\theta = \pi/4$ is the vertical plane with Cartesian equation $y = x$.

C12S08.026: The graph of the spherical equation $\rho = 5$ is the spherical surface of radius 5 centered at the origin.

C12S08.027: The graph of the spherical equation $\phi = \pi/6$ consists of both nappes of the cone with Cartesian equation $z^2 = 3x^2 + 3y^2$ if ρ may be negative. If $\rho \geqq 0$, then the graph consists of the upper nappe alone.

C12S08.028: The graph of the spherical equation $\phi = 5\pi/6$ consists of both nappes of the cone with Cartesian equation $z^2 = 3x^2 + 3y^2$ if ρ may be negative. If $\rho \geqq 0$, then the graph consists of the lower nappe alone.

C12S08.029: The graph of the spherical equation $\phi = \pi/2$ is the xy-plane.

C12S08.030: If ρ is allowed to be negative, then the graph of the spherical equation $\phi = \pi$ is the z-axis. If $\rho \geqq 0$, then the graph is the nonpositive z-axis.

C12S08.031: The cylindrical equation $z^2 + 2r^2 = 4$ has the same graph as does the Cartesian equation $2x^2 + 2y^2 + z^2 = 4$. It is an ellipsoid (actually, a prolate spheroid) with intercepts $(\pm\sqrt{2}, 0, 0)$, $(0, \pm\sqrt{2}, 0)$, and $(0, 0, \pm 2)$.

C12S08.032: The cylindrical equation $z^2 - 2r^2 = 4$ has the same graph as does the Cartesian equation $z^2 - 2x^2 - 2y^2 = 4$. Hence its graph is a hyperboloid of two sheets with axis the z-axis and vertices $(0, 0, \pm 2)$. It resembles the one shown in Fig. 12.7.19.

C12S08.033: The graph of the cylindrical equation $r = 4\cos\theta$ is the same as the graph of the Cartesian equation

$$x^2 + y^2 = 4x; \qquad \text{that is,} \qquad (x-2)^2 + y^2 = 4.$$

Therefore its graph is a circular cylinder parallel to the z-axis, of radius 2, and centerline the line with Cartesian equations $x = 2$, $y = 0$.

C12S08.034: The spherical equation $\rho = 4\cos\phi$ takes the Cartesian form

$$x^2 + y^2 + z^2 = 4z; \qquad \text{that is,} \qquad x^2 + y^2 + (z-2)^2 = 4.$$

Thus the graph is a spherical surface with center $(0, 0, 2)$ and radius 2; thus the south pole of the sphere is located at the origin.

C12S08.035: The cylindrical equation $r^2 - 4r + 3 = 0$ can be written in the form

$$(r-1)(r-3) = 0,$$

so that $r = 1$ or $r = 3$. The graph consists of all points that satisfy either equation. Hence the graph consists of two concentric circular cylinders, each with axis the z-axis; their radii are 1 and 3.

C12S08.036: The graph of the spherical equation $\rho^2 - 4\rho + 3 = 0$ can be written in the form

$$(\rho - 1)(\rho - 3) = 0,$$

so that $\rho = 1$ or $\rho = 3$. The graph consists of all points that satisfy either of these equations. Hence the graph consists of two concentric spherical surfaces, both centered at the origin, and of radii 1 and 3, respectively.

C12S08.037: The cylindrical equation $z^2 = r^4$ can be written in Cartesian form as

$$z = \pm(x^2 + y^2), \qquad \text{so that} \qquad z = x^2 + y^2 \quad \text{or} \quad z = -(x^2 + y^2).$$

The graph consists of all points that satisfy either of the last two equations, hence it consists of two circular paraboloids, each with axis the z-axis, vertex at the origin; one opens upward and the other opens downward.

C12S08.038: The graph of the spherical equation $\rho^3 + 4\rho = 0$ can be written in the form $\rho(\rho^2 + 4) = 0$, and the only solution of this equation is $\rho = 0$. Hence the graph of the given equation consists of the single point (0, 0, 0).

C12S08.039: The Cartesian equation $x^2 + y^2 + z^2 = 25$ has cylindrical form $r^2 + z^2 = 25$ and spherical form $\rho^2 = 25$; that is, $\rho = \pm 5$ (but the graph of $\rho = -5$ coincides with the graph of $\rho = 5$, so $\rho = 5$ is also a correct answer).

C12S08.040: The Cartesian equation $x^2 + y^2 = 2x$ takes the cylindrical form $r^2 = 2r\cos\theta$, which has the same graph as the slightly simpler $r = 2\cos\theta$. Its spherical form is

$$\begin{aligned} \rho^2 \sin^2\phi\, \cos^2\theta + \rho^2 \sin^2\phi\, \sin^2\theta &= 2\rho \sin\phi\, \cos\theta; \\ \rho^2 \sin^2\phi &= 2\rho \sin\phi\, \cos\theta; \end{aligned}$$

the last equation holds when $\rho\sin\phi = 0$ and when $\rho\sin\phi = 2\cos\theta$. The former is the special case of the latter when $\theta = \pi/2$. Hence the spherical equation can be simplified to $\rho\sin\phi = 2\cos\theta$.

C12S08.041: The Cartesian equation $x + y + z = 1$ takes the cylindrical form

$$r\cos\theta + r\sin\theta + z = 1$$

and the spherical form

$$\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi = 1.$$

C12S08.042: The Cartesian equation $x + y = 4$ takes the cylindrical form $r\cos\theta + r\sin\theta = 4$ and the spherical form $\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta = 4$.

C12S08.043: The Cartesian equation $x^2 + y^2 + z^2 = x + y + z$ takes the cylindrical form

$$r^2 + z^2 = r\cos\theta + r\sin\theta + z$$

and the spherical form

$$\rho^2 = \rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta + \rho\cos\phi.$$

The common factor ρ may be cancelled from both sides of the last equation without loss of any points on the graph.

C12S08.044: The Cartesian equation $z = x^2 - y^2$ takes the cylindrical form

$$z = r^2 \cos^2 \theta - r^2 \sin^2 \theta = r^2 \cos 2\theta$$

and the spherical form

$$\rho \cos \phi = \rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta;$$

$$\rho \cos \phi = (\rho \sin \phi)^2 (\cos^2 \theta - \sin^2 \theta);$$

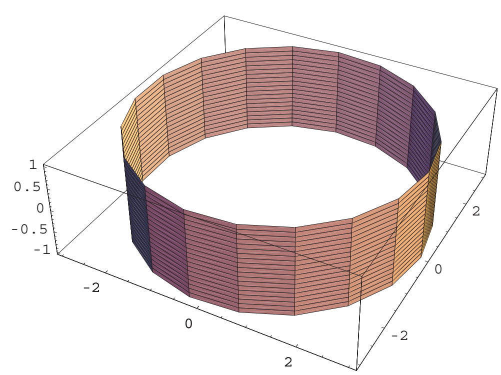
$$\rho \cos \phi = (\rho \sin \phi)^2 \cos 2\theta;$$

$$\cos \phi = \rho \sin^2 \phi \cos 2\theta.$$

C12S08.045: The surface is the part of the cylinder of radius 3 and centerline the z-axis that lies between the planes $z = -1$ and $z = 1$. The *Mathematica* 3.0 command

```
ParametricPlot3D[ { 3*Cos[t], 3*Sin[t], z }, { t, 0, 2*Pi }, { z, -1, 1 } ];
```

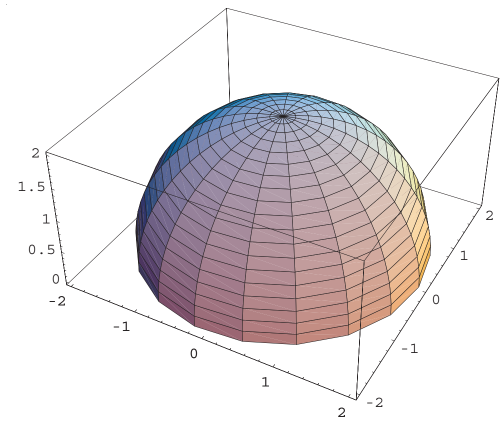
generates the graph, shown next.



C12S08.046: The surface is the hemisphere of radius 2 and center the origin that lies on and above the xy-plane. The *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*Sin[phi]*Cos[t], 2*Sin[phi]*Sin[t], 2*Cos[phi] },
                  { t, 0, 2*Pi }, { phi, 0, Pi/2 } ];
```

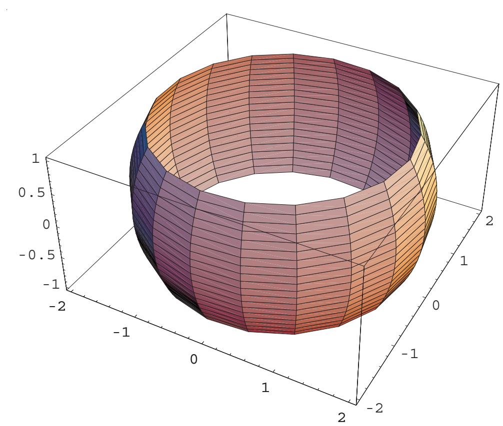
generates the graph, shown next.



C12S08.047: The surface is the part of the sphere of radius 2 and center the origin that lies between the two horizontal planes $z = -1$ and $z = 1$. The *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*Sin[phi]*Cos[t], 2*Sin[phi]*Sin[t], 2*Cos[phi] },
                    { t, 0, 2*Pi }, { phi, Pi/3, 2*Pi/3 } ];
```

generates the graph, shown next.



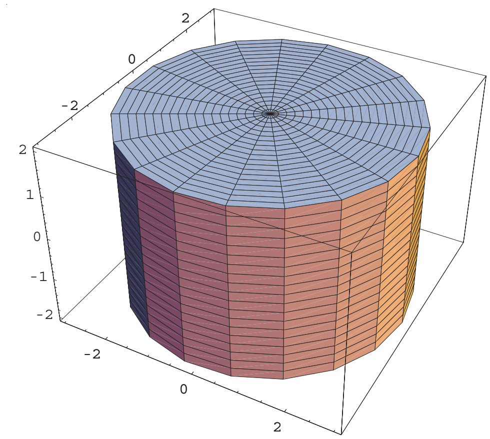
C12S08.048: The solid is the part of the solid cylinder with radius 3 and centerline the z-axis that lies between the horizontal planes $z = -2$ and $z = 2$. To generate an image, we will create its curved side and its top separately, then display them simultaneously. For the curved side, we use the *Mathematica* 3.0 command

```
f1 = ParametricPlot3D[ { 3*Cos[t], 3*Sin[t], z }, { t, 0, 2*Pi }, { z, -2, 2 } ];
```

For the top surface, we use

```
f2 = ParametricPlot3D[ { r*Cos[t], r*Sin[t], 2 }, { r, 0, 3 }, { t, 0, 2*Pi } ];
```

Finally, the command `Show[f1, f2]` produces what appears to be the solid cylinder, and the result is shown next.



C12S08.049: The solid lies between the horizontal planes $z = -2$ and $z = 2$, is bounded on the outside by the cylinder of radius 3 with centerline the z-axis, and is bounded on the inside by the cylinder of radius 1 with centerline the z-axis. To generate an image, we will create the inner and outer bounding surfaces, the top, and display them simultaneously. For the outer cylinder, we use the *Mathematica* 3.0 command

```
f1 = ParametricPlot3D[ { 3*Cos[t], 3*Sin[t], z }, { t, 0, 2*Pi }, { z, −2, 2 } ];
```

For the inner cylinder, we use

```
f2 = ParametricPlot3D[ { 1*Cos[t], 1*Sin[t], z }, { t, 0, 2*Pi }, { z, −2, 2 } ];
```

For the top, we use

```
f3 = ParametricPlot3D[ { r*Cos[t], r*Sin[t], 2 }, { t, 0, 2*Pi }, { r, 1, 3 } ];
```

Finally, the command `Show[ f1, f2, f3 ]` produces an image of the solid. The infamous "out of memory" message prohibits us from displaying it.

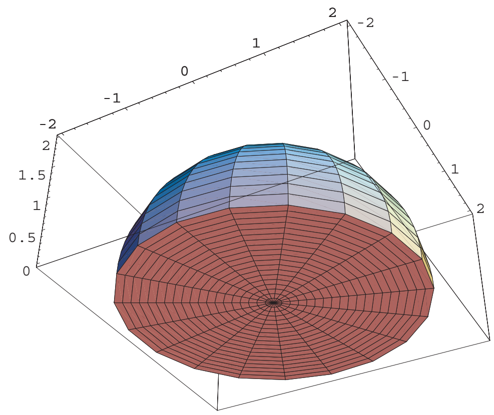
C12S08.050: The solid is bounded below by the xy-plane and above by the hemisphere of radius 2 centered at the origin. To generate an image, we will create the hemisphere and its base separately and display them simultaneously. We also want to view the resulting figure from below because, viewed from above, it would be indistinguishable from the hemispherical surface of Problem 46. We begin with the *Mathematica* 3.0 command

```
f1 = ParametricPlot3D[ { 2*Sin[phi]*Cos[t], 2*Sin[phi]*Sin[t], 2*Cos[phi] },
                       { phi, 0, Pi/2 }, { t, 0, 2*Pi },
                       ViewPoint -> { 1.3, −2.4, −2,0 } ];
```

followed by

```
f2 = ParametricPlot3D[ { rho*Cos[t], rho*Sin[t], 0 }, { rho, 0, 2 },
                       { t, 0, 2*Pi }, ViewPoint -> { 1.3, −2.4, −2.0 } ];
```

Then the command `Show[ f1, f2 ]` produces an image of the hemispherical solid, shown next.



C12S08.051: The solid described by the spherical inequality $3 \leqq \rho \leqq 5$ is the region between two concentric spheres centered at the origin, one of radius 3 and the other of radius 5. We will display the half of this solid for which $y \geqq 0$ (otherwise, you wouldn't be able to tell that the solid has a hollow core). We use *Mathematica* 3.0 to draw the outer hemisphere, then the inner hemisphere, then make it appear "solid" by covering the gap between the two with an annulus. Here are the commands:

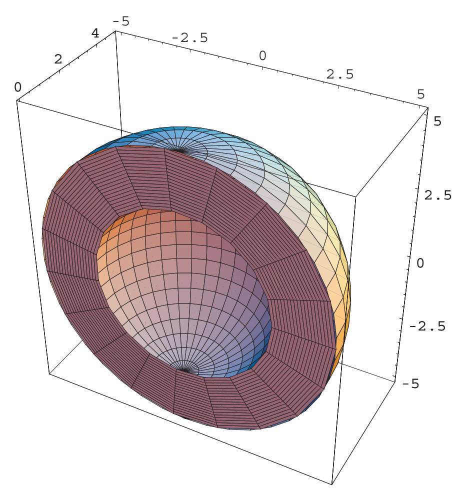
```
f1 = ParametricPlot3D[ { 5*Sin[phi]*Cos[t], 5*Sin[phi]*Sin[t], 5*Cos[phi] },
                       { phi, 0, Pi }, { t, 0, Pi } ];

f2 = ParametricPlot3D[ { 3*Sin[phi]*Cos[t], 3*Sin[phi]*Sin[t], 3*Cos[phi] },
                       { phi, 0, Pi }, { t, 0, Pi } ];

f3 = ParametricPlot3D[ { r*Cos[t], 0, r*Sin[t] }, { r, 3, 5 }, { t, 0, 2*Pi } ];

Show[ f1, f2, f3 ];
```

The figure is next.



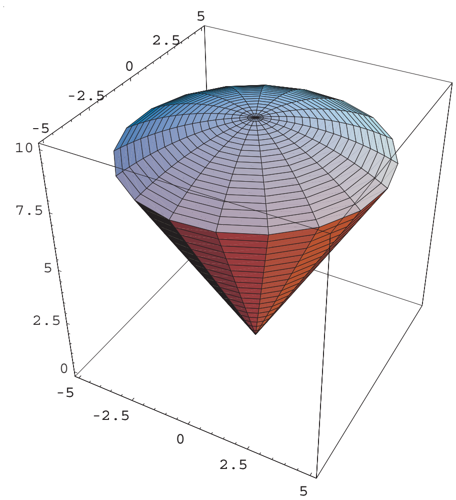
C12S08.052: The solid is bounded below by a right circular cone, vertex at the origin, axis the z-axis, opening upward; it is bounded above by part of the surface of a sphere of radius 10 centered at the origin. To see it, execute the *Mathematica* 3.0 commands

```
f1 = ParametricPlot3D[ { 10*Sin[phi]*Cos[t], 10*Sin[phi]*Sin[t], 10*Cos[phi] },
                       { t, 0, 2*Pi }, { phi, 0, Pi/6 } ];

f2 = ParametricPlot3D[ { rho*(1/2)*Cos[t], rho*(1/2)*Sin[t], rho*Cos[Pi/6] },
                       { t, 0, 2*Pi }, { rho, 0, 10 } ];

Show[ f1, f2 ];
```

The resulting image is shown next.



C12S08.053: Given $z = x^2$, a curve in the xz-plane to be rotated around the z-axis, replace x^2 with x^2+y^2 to obtain $z = x^2 + y^2$. In cylindrical coordinates, the equation is $z = r^2$.

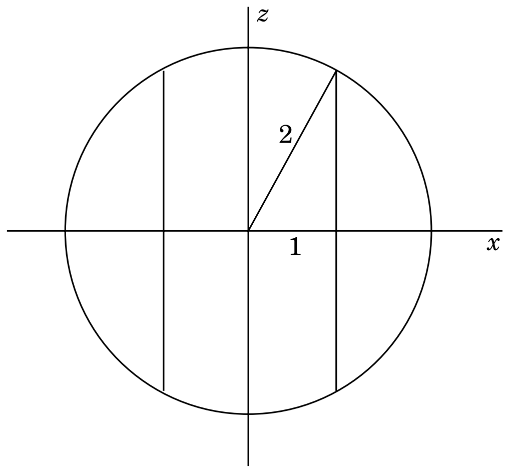
C12S08.054: Given $y^2 - z^2 = 1$, a curve in the yz-plane to be rotated around the z-axis, replace y^2 with $x^2 + y^2$ to obtain $x^2 + y^2 - z^2 = 1$. In cylindrical coordinates, the equation is $r^2 - z^2 = 1$.

C12S08.055: A central cross section of the sphere-with-hole, shown after this solution, makes it clear that the figure is described in cylindrical coordinates by $1 \leqq r \leqq 2$ and $0 \leqq \theta \leqq 2\pi$; the third cylindrical coordinate z is determined by the cylindrical equation of the sphere, $r^2 + z^2 = 4$, so that

$$-\sqrt{4 - r^2} \leqq z \leqq \sqrt{4 - r^2}\,.$$

The figure also makes it clear that the spherical coordinates of the figure satisfy the inequalities $0 \leqq \theta \leqq 2\pi$ and $\pi/3 \leqq \phi \leqq 2\pi/3$. The value of the third spherical coordinate ρ is $\sec\phi$ on the surface of the cylinder

and 2 on the surface of the sphere, and hence $\sec\phi \leqq \rho \leqq 2$.



C12S08.056: Take $\rho = 3960$ for the radius of the earth throughout this solution. The spherical coordinates of Atlanta are then $(\rho,\ \phi,\ \theta)$ where

$$\phi = \frac{5\pi}{16} \qquad \text{and} \qquad \theta = \frac{689\pi}{450}.$$

Then the formulas in (6) yield its Cartesian coordinates:

$$x_1 \approx 321.303374687189, \quad y_1 \approx -3276.905308046423, \quad z_1 \approx 2200.058122757625.$$

Similarly, the angular spherical coordinates of San Francisco are

$$\phi = \frac{2611\pi}{9000} \qquad \text{and} \qquad \theta = \frac{11879\pi}{9000},$$

and hence its Cartesian coordinates are

$$x_2 \approx -1677.985673523340, \quad y_2 \approx -2642.043377610385, \quad z_2 \approx 2426.019552739741.$$

Thus the straight-line distance between the two cities is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \approx 2109.802108457354.$$

Now we use the triangle with vertices at O (the center of the earth), A (Atlanta), and S (San Francisco) and the law of cosines to find the angle α between OA and OS:

$$d^2 = \rho^2 + \rho^2 - 2\rho^2 \cos\alpha;$$

$$\cos\alpha = \frac{2\rho^2 - d^2}{2\rho^2} \approx 0.858073636081;$$

$$\alpha \approx 0.539289738320.$$

Therefore the great circle distance from Atlanta to San Francisco is $\rho\alpha \approx 2135.587363747416$ miles, approximately 3436.894710322722 kilometers.

C12S08.057: Take $\rho = 3960$ for the radius of the earth throughout this solution. The spherical coordinates of Fairbanks are then $(\rho,\ \phi,\ \theta)$ where

$$\phi = \frac{7\pi}{50} \quad \text{and} \quad \theta = \frac{4243\pi}{3600}.$$

Then the formulas in (6) yield its Cartesian coordinates:

$$x_1 \approx -1427.537970311664, \quad y_1 \approx -897.229805845151, \quad z_1 \approx 3583.115127765437.$$

Similarly, the angular spherical coordinates of St. Petersburg are

$$\phi = \frac{1003\pi}{6000} \quad \text{and} \quad \theta = \frac{3043\pi}{18000},$$

and hence its Cartesian coordinates are

$$x_2 \approx 1711.895024130100, \quad y_2 \approx 1005.568095522826, \quad z_2 \approx 3426.346192611774.$$

Thus the straight-line distance between the two cities is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \approx 3674.405513694586.$$

Now we use the triangle with vertices at O (the center of the earth), F (Fairbanks), and S (St. Petersburg) and the law of cosines to find the angle α between OF and OS:

$$d^2 = \rho^2 + \rho^2 - 2\rho^2 \cos\alpha;$$

$$\cos\alpha = \frac{2\rho^2 - d^2}{2\rho^2} \approx 0.569519185572;$$

$$\alpha \approx 0.964875538222.$$

Therefore the great circle distance from Fairbanks to St. Petersburg is $\rho\alpha \approx 3820.907131357640$ miles, approximately 6149.153966407630 kilometers.

C12S08.058: Flying the 62nd parallel means that the spherical angle ϕ remains $7\pi/45$. With $\rho = 3960$ (miles) as the radius of the earth, the distance from this parallel to the xy-plane is then $z = \rho\cos\phi \approx 3496.472467721351$. So the radius of the circular arc flown by the airplane will be given by

$$r = \sqrt{\rho^2 - z^2} \approx 1859.107388632127.$$

The angular difference in longitude of the two cities is 178.28°, so the distance flown by the airplane will be

$$r \cdot (178.28) \cdot \frac{\pi}{180} \approx 5784.748336823967$$

miles.

C12S08.059: Take $r = 3960$ as the radius of the earth. Recall from the solution of Problem 57 that the Cartesian coordinates of Fairbanks are $F(x_1, y_1, z_1)$ where

$$x_1 \approx -1427.537970311664, \quad y_1 \approx -897.229805845151, \quad z_1 \approx 3583.115127765437$$

and that those of St. Petersburg are $S(x_2, y_2, z_2)$ where

$$x_2 \approx 1711.895024130100, \quad y_2 \approx 1005.568095522826, \quad z_2 \approx 3426.346192611774.$$

Thus a normal to the plane $\mathcal{P}$ containing the center of the earth O and the two points F and P is

$$\mathbf{n} = \langle\, n_1,\, n_2,\, n_3 \,\rangle = \overrightarrow{OF} \times \overrightarrow{OS} = \langle\, -6677286.184221,\ 11025156.247493,\ 100476.602035 \,\rangle.$$

The intersection of this plane with the surface of the earth is the great-circle route between F and S. The point on this route closest to the north pole will be the point where the plane $\mathcal{Q}$ containing the z-axis and normal to $\mathcal{P}$ intersects the route. (We are in effect constructing the geodesics from F and P to the north pole and bisecting the angle between them with $\mathcal{Q}$.) Observe that $\mathcal{Q}$ has an equation of the form $Ax + By = 0$, thus a normal of the form $\mathbf{m} = \langle\, A,\, B,\, 0 \,\rangle$ having the property that $\mathbf{m} \cdot \mathbf{n} = 0$. Indeed, we choose $B = 1$ and solve for A to find that $A \approx -6.056409573098$. Then we asked *Mathematica* to solve for the point $C(x,\, y,\, z)$ on the plane's great circle route closest to the north pole:

```
Solve[ { A*x + B*y == 0, n1*x + n2*y + n3*z == 0, x*x + y*y + z*z == r*r },
        { x, y, z } ]
```

The response was $x \approx -6.620560438727$, $y \approx -40.096825620356$, and $z \approx 3959.791460765913$ (and their negatives, which we rejected). Let $N(0,\, 0,\, r)$ denote the north pole. The straight-line distance d between C and N is given by

$$d = \sqrt{x^2 + y^2 + (z - r)^2} \approx 40.640260013511.$$

Then the law of cosines will tell us the angle α between OC and ON:

$$\begin{aligned} &d^2 = r^2 + r^2 - 2r^2 \cos\alpha; \\ &\cos\alpha = \frac{2r^2 - d^2}{2r^2} \approx 0.999947338577; \\ &\alpha \approx 0.010262736960. \end{aligned}$$

Thus the great-circle distance from C to N is $r\alpha \approx 40.640438363451$ (miles).

C12S08.060: Either:

$$0 \leqq x \leqq H, \qquad 0 \leqq r \leqq \frac{R}{H}z, \qquad 0 \leqq \theta \leqq 2\pi \qquad \text{or}$$

$$0 \leqq r \leqq R, \qquad \frac{H}{R}r \leqq z \leqq H, \qquad 0 \leqq \theta \leqq 2\pi.$$

C12S08.061: The cone has the following description in spherical coordinates:

$$0 \leqq \rho \leqq \sqrt{R^2 + H^2}\,, \qquad 0 \leqq \theta \leqq 2\pi, \qquad \phi = \arctan\left(\frac{R}{H}\right).$$

C12S08.062: Take $\rho = 3960$ as the radius of the earth in this solution. By Example 8, the angular spherical coordinates of New York City are then

$$\phi = \frac{197\pi}{720} \quad \text{and} \quad \theta = \frac{143\pi}{90},$$

so by the equations in (6) its Cartesian coordinates are $N(x_1,\, y_1,\, z_1)$ where

$$x_1 \approx 826,900308254456, \quad y_1 \approx -2883.744078623090, \quad z_1 \approx 2584.928619752381.$$

The angular spherical coordinates of London are

$$\phi = \frac{77\pi}{360} \quad \text{and} \quad \theta = 0.$$

Hence its Cartesian coordinates are $L(x_2, y_2, z_2)$ where

$$x_2 \approx 2465.157961084973, \quad y_2 = 0, \quad z_2 \approx 3099.128301135559.$$

Let O denote the origin, at the center of the earth. Then

$$\mathbf{n} = \langle\, n_1, n_2, n_3 \,\rangle = 10^{-6}\left(\overrightarrow{ON} \times \overrightarrow{OL}\right) \approx \langle\, -8,937092887293,\ 3.809587218899,\ 7.108884673149 \,\rangle$$

is normal to the plane containing O, N, and L. Thus an equation of that plane is

$$n_1x + n_2y + n_3z = 0, \quad \text{so that} \quad y = -\frac{n_1x + n_3z}{n_2}.$$

We substituted for y in the equation $x^2 + y^2 + z^2 = \rho^2$ of the earth's surface, then solved for $z = g(x)$; we found that

$$g(x) = \frac{-n_1n_3x + \sqrt{n_2^4\rho^2 + n_2^2n_3^2r^2 - n_1^2n_2^2x^2 - n_2^4x^2 - n_2^2n_3^2x^2}}{n_2^2 + n_3^2}.$$

At this point we asked *Mathematica* 3.0 to find $g'(x)$ and solve $g'(x) = 0$:

```
g'[x]
```

$$(0.015373)\left(63.5328 - \frac{(2103.23)x}{\sqrt{1.48043 \times 10^{10} - (2103.23)x^2}}\right)$$

```
Solve[ g'[x] == 0, x ]
```

$$\{\{\ x \to 2151.18\ \}\}$$

```
g[ 2151.18 ]
```

3195.81

So the maximum z-coordinate of the plane's route is considerably larger than that of London; the plane does indeed fly north of the latitude of London in the great circle route from New York to London.

C12S08.063: Begin with the equation of the given circle,

$$(y-a)^2 + z^2 = b^2 \qquad (x = 0),$$

and replace y with r to get the cylindrical-coordinates equation

$$(r-a)^2 + z^2 = b^2$$

of the torus. Then substitute $\sqrt{x^2+y^2}$ for r and rationalize to get the rectangular-coordinates equation

$$4a^2(x^2+y^2) = (x^2+y^2+z^2+a^2-b^2)^2$$

of the torus. We recognize the combination $x^2 + y^2 = r^2$ and $x^2 + y^2 + z^2 = \rho^2$ and substitute to obtain

$$4a^2r^2 = (\rho^2 + a^2 - b^2)^2.$$

Finally, we take positive square roots and substitute $\rho \sin \phi$ for r to obtain the spherical-coordinates equation

$$2a\rho \sin \phi = \rho^2 + a^2 - b^2$$

of the torus. (—C.H.E.)

C12S08.064: We use *Mathematica* 3.0 to solve this problem.

```
n = 6;   m = 2;   a = 5;   b = 1;

rho = a + b*Sin[ n*phi ]*Cos[ m*theta ];

r = rho*Sin[ phi ];

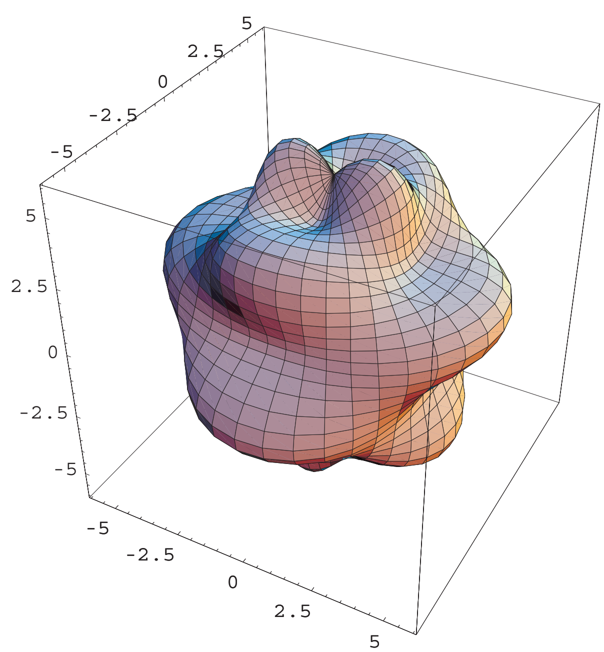
x = r*Cos[ theta ];

y = r*Sin[ theta ];

z = rho*Cos[ phi ];

ParametricPlot3D[ { x, y, z }, { theta, 0, 2*Pi }, { phi, 0, Pi },
                    PlotPoints → { 40, 40 } ];
```

The result is shown next. (—C.H.E.)



Chapter 12 Miscellaneous Problems

C12S0M.001: Note that $\overrightarrow{PM} = \overrightarrow{MQ}$. Hence

$$\frac{1}{2}\left(\overrightarrow{AP} + \overrightarrow{AQ}\right) = \frac{1}{2}\left(\overrightarrow{AM} - \overrightarrow{PM} + \overrightarrow{AM} + \overrightarrow{MQ}\right) = \overrightarrow{AM}.$$

C12S0M.002: Recall first that

$$\text{comp}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.$$

Therefore

$$\mathbf{a}_{\perp}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}_{\|}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{b} - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}\cdot\mathbf{b}}{|\mathbf{b}|} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} = 0.$$

Therefore $\mathbf{a}_{\perp}$ is perpendicular to $\mathbf{b}$.

C12S0M.003: Given the distinct points P and Q in space, let L be the straight line through both. Then a vector equation of L is $\mathbf{r}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}$. First suppose that R is a point on L. Then for some scalar t,

$$\overrightarrow{OR} = \overrightarrow{OP} + t\overrightarrow{PQ} = \overrightarrow{OP} + t\left(\overrightarrow{OQ} - \overrightarrow{OP}\right) = (1-t)\overrightarrow{OP} + t\overrightarrow{OQ} = a\overrightarrow{OP} + b\overrightarrow{OQ}$$

where $a = 1 - t$ and $b = t$; moreover, $a + b = 1$, as desired. Next suppose that there exist scalars a and b such that $a + b = 1$ and

$$\overrightarrow{OR} = a\overrightarrow{OP} + b\overrightarrow{OQ}.$$

Let $t = b$, so that $1 - t = a$. Then

$$\overrightarrow{OR} = (1-t)\overrightarrow{OP} + t\overrightarrow{OQ} = \overrightarrow{OP} + t\left(\overrightarrow{OQ} - \overrightarrow{OP}\right) = \overrightarrow{OP} + t\overrightarrow{PQ}.$$

Therefore R lies on the line L. Consequently an alternative form of the vector equation of L is

$$\mathbf{r}(t) = t\overrightarrow{OP} + (1-t)\overrightarrow{OQ}.$$

C12S0M.004: Suppose first that P, Q, and R are collinear. By the result in Problem 3, there exists scalars a and b such that $a + b = 1$ and $\overrightarrow{OR} = a\overrightarrow{OP} + b\overrightarrow{OQ}$. Let $c = -1$. Then

$$a\overrightarrow{OP} + b\overrightarrow{OQ} + c\overrightarrow{OR} = \mathbf{0} \qquad \text{and} \qquad a + b + c = 1 - 1 = 0.$$

Next suppose that there exists scalars a, b, and c not all zero such that $a + b + c = 0$ and

$$a\overrightarrow{OP} + b\overrightarrow{OQ} + c\overrightarrow{OR} = \mathbf{0}.$$

Without loss of generality suppose that $c \neq 0$. Then

$$\overrightarrow{OR} = -\frac{a}{c}\,\overrightarrow{OP} - \frac{b}{c}\,\overrightarrow{OQ}$$

and

$$-\frac{a}{c}-\frac{b}{c}=\frac{c}{c}=1.$$

Therefore, by the result in Problem 3, R lies on the straight line containing P and Q.

C12S0M.005: Given $P(x_0,\,y_0),\ Q(x_1,\,y_1),$ and $R(x_2,\,y_2),$ we think of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as vectors in space, so that

$$\overrightarrow{PQ}=\langle\,x_1-x_0,\ y_1-y_0,\ 0\,\rangle \quad \text{and} \quad \overrightarrow{PR}=\langle\,x_2-x_0,\ y_2-y_0,\ 0\,\rangle.$$

By Eq. (10) in Section 12.3, the area A of the triangle PQR is half the magnitude of $\overrightarrow{PQ}\times\overrightarrow{OR}$. But

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1-x_0 & y_1-y_0 & 0\\ x_2-x_0 & y_2-y_0 & 0\end{vmatrix}=\big[(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)\big]\mathbf{k}.$$

Therefore

$$A=\frac{1}{2}\,|(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)|.$$

C12S0M.006: A vector equation of the line is

$$\mathbf{r}(t)=\overrightarrow{OP_1}+t\mathbf{v}=\langle\,1,\,-1,\,0\,\rangle+t\langle\,2,\,-1,\,3\,\rangle.$$

Therefore the line has parametric equations

$$x=1+2t,\quad y=-1-t,\quad z=3t,\quad -\infty<t<+\infty$$

and symmetric equations

$$\frac{x-1}{2}=-y-1=\frac{z}{3}.$$

C12S0M.007: A vector equation of the line is

$$\mathbf{r}(t)=\overrightarrow{OP_1}+t\overrightarrow{P_1P_2}=\langle\,1,\,-1,\,2\,\rangle+t\langle\,2,\,3,\,-3\,\rangle.$$

Therefore the line has parametric equations

$$x=1+2t,\quad y=-1+3t,\quad z=2-3t,\quad -\infty<t<+\infty$$

and symmetric equations

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-2}{-3}.$$

C12S0M.008: Because the plane has normal vector $\mathbf{n}=\mathbf{i}+\mathbf{j}$, it has an equation of the form $x+y=D$ where D is a constant. Because the point $P(3,\,-5,\,1)$ lies on the plane, $D=3-5=-2$. Hence an equation of the plane is $x+y+2=0$.

C12S0M.009: The line L_1 with symmetric equations $x - 1 = 2(y + 1) = 3(z - 2)$ contains the points $P_1(1, -1, 2)$ and $Q_1(7, 2, 4)$, and thus L_1 is parallel to the vector $\mathbf{u}_1 = \langle 6, 3, 2 \rangle$. The line L_2 with symmetric equations $x - 3 = 2(y - 1) = 3(z + 1)$ contains the points $P_2(3, 1, -1)$ and $Q_2(9, 4, 1)$, and thus L_2 is parallel to the vector $\mathbf{u}_2 = \langle 6, 3, 2 \rangle = \mathbf{u}_1$. Therefore L_1 and L_2 are parallel. The vector $\mathbf{v} = \overrightarrow{P_1P_2} = \langle 2, 2, -3 \rangle$ lies in the plane $\mathcal{P}$ containing L_1 and L_2, as does $\mathbf{u}_1$, and therefore a normal to $\mathcal{P}$ is

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & 2 \\ 2 & 2 & -3 \end{vmatrix} = \langle -13, 22, 6 \rangle.$$

Therefore an equation of $\mathcal{P}$ has the form $13x - 22y - 6z = D$. The point $P_1(1, -1, 2)$ lies in the plane $\mathcal{P}$, and therefore $D = 13 \cdot 1 - 22 \cdot (-1) - 6 \cdot 2 = 23$. Therefore an equation of $\mathcal{P}$ is $13x - 22y - 6z = 23$.

C12S0M.010: For $i = 1$ and $i = 2$, the line L_i passes through the point $P_i(x_i, y_i, z_i)$ and is parallel to the vector $\mathbf{v}_i = \langle a_i, b_i, c_i \rangle$. Let L be the line through P_1 parallel to $\mathbf{v}_2$. Now, by definition, the lines L_1 and L_2 are skew lines if and only if they are not coplanar. But L_1 and L_2 are coplanar if and only if the plane determined by L_1 and L is the same as the plane determined by L_2 and L; that is, if and only if the plane determined by L_1 and L is the same as the plane that contains the segment P_1P_2 and the line L. Thus L_1 and L_2 are coplanar if and only if $\overrightarrow{P_1P_2}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ are coplanar, and this condition is equivalent (by Theorem 4 in Section 12.3) to the condition

$$\left| \overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \right| = 0.$$

That is, L_1 and L_2 are coplanar if and only if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

This establishes the conclusion in Problem 10.

C12S0M.011: Given the four points $A(2, 3, 2)$, $B(4, 1, 0)$, $C(-1, 2, 0)$, and $D(5, 4, -2)$, we are to find an equation of the plane containing A and B and parallel to $\overrightarrow{CD} = \langle 6, 2, -2 \rangle$. Because $\overrightarrow{AB} = \langle 2, -2, -2 \rangle$ is not parallel to $\overrightarrow{CD}$, a vertor normal to the plane is

$$\mathbf{m} = \overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -2 \\ 6 & 2 & -2 \end{vmatrix} = \langle 8, -8, 16 \rangle;$$

to simplify matters slightly we will use the parallel vector $\mathbf{n} = \langle 1, -1, 2 \rangle$. Therefore the plane has a Cartesian equation of the form $x - y + 2z = K$. Because $A(2, 3, 2)$ lies in the plane, $K = 2 - 3 + 2 \cdot 2 = 3$. Answer: $x - y + 2z = 3$.

C12S0M.012: Given the four points $A(2, 3, 2)$, $B(4, 1, 0)$, $C(-1, 2, 0)$, and $D(5, 4, -2)$, we are to find points P on the line L_1 through A and B and Q on the line L_2 through C and D such that the line L

containing P and Q is perpendicular to both L_1 and L_2. There are several ways to do this; here is one solution. The lines L_1 and L_2 have vector equations

$$L_1: \quad \mathbf{r}_1(s) = \overrightarrow{OA} + s\overrightarrow{AB} = \langle\, 2,\ 3,\ 2\rangle + s\langle 2,\ -2,\ -2\,\rangle \qquad \text{and}$$

$$L_2: \quad \mathbf{r}_2(t) = \overrightarrow{OC} + t\overrightarrow{CD} = \langle\, -1,\ 2,\ 0\,\rangle + t\langle\, 6,\ 2,\ -2\,\rangle.$$

Hence the points P and Q have coordinates

$$P(2+2s,\ 3-2s,\ 2-2s) \quad \text{and} \quad Q(-1+6t,\ 2+2t,\ -2t)$$

for some choice of the parameters s and t. Therefore the vector

$$\mathbf{u} = \overrightarrow{PQ} = \langle\, 6t-2s-3,\ 2t+2s-1,\ -2t+2s-2\,\rangle$$

for some choice of s and t. Now we impose the condition that $\mathbf{u}$ must be perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{CD}$ by requiring that $\mathbf{u}$ is parallel to

$$\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -2 \\ 6 & 2 & -2 \end{vmatrix} = \langle\, 8,\ -8,\ 16\,\rangle;$$

it will be simpler to impose the condition that $\mathbf{u}$ is parallel to $\mathbf{n} = \langle\, 1,\ -1,\ 2\,\rangle$. We do so by requiring that $\mathbf{u} \times \mathbf{n} = \mathbf{0}$:

$$\mathbf{u} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t-2s-3 & 2t+2s-1 & -2t+2s-2 \\ 1 & -1 & 2 \end{vmatrix} = \langle\, 2t+6s-4,\ -14t+6s+4,\ -8t+4\,\rangle = \mathbf{0}.$$

It follows that $s = t = \frac{1}{2}$, so that $P = (3,\ 2,\ 1)$ and $Q = (2,\ 3,\ -1)$. The perpendicular distance between L_1 and L_2 is therefore $d = |PQ| = \sqrt{6}$.

C12S0M.013: A vector normal to the plane $\mathcal{P}$ is $\mathbf{n} = \langle\, a,\ b,\ c\,\rangle$ and $\overrightarrow{OP_0} = \langle\, x_0,\ y_0,\ z_0\,\rangle$ connects the origin to $\mathcal{P}$. Hence the distance D from the origin to $\mathcal{P}$ is the length of the projection of $\overrightarrow{OP_0}$ in the direction of $\mathbf{n}$. But the projection of $\overrightarrow{OP_0}$ in the direction of $\mathbf{n}$ is

$$\frac{\overrightarrow{OP_0} \cdot \mathbf{n}}{|\mathbf{n}|} = \frac{ax_0 + by_0 + cz_0}{\sqrt{a^2+b^2+c^2}} = \frac{-d}{\sqrt{a^2+b^2+c^2}}.$$

Therefore $D = \dfrac{|d|}{\sqrt{a^2+b^2+c^2}}$.

C12S0M.014: Let Q_1 be the foot of the perpendicular from the point $P_1(x_1,\ y_1,\ z_1)$ to the plane $\mathcal{P}$ with equation $ax + by + cz + d = 0$; also let the point Q be determined by the vector equation

$$\overrightarrow{OQ} = \overrightarrow{OQ_1} - \overrightarrow{OP_1}.$$

Then Q is the foot of the perpendicular from the origin O to the plane with equation

$$a(x + x_1) + b(y + y_1) + c(z + z_1) + d = 0$$

(a translate of the origin plane by $-\overrightarrow{OP_1}$, using the translation principle)—that is,

$$ax + bx + cz + d_1 = 0 \quad \text{where} \quad d_1 = ax_1 + by_1 + cz_1 + d.$$

Hence, by the result in Problem 13,

$$D = \left|\overrightarrow{P_1Q_1}\right| = \left|\overrightarrow{OQ}\right| = \frac{|d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

C12S0M.015: The planes have the common normal $\mathbf{n} = \langle 2, -1, 2 \rangle$, a point on the first plane $\mathcal{P}_1$ is $P_1(2, 0, 0)$, and a point on the second plane $\mathcal{P}_2$ is $P_2(7, 1, 0)$. So a vector that connects the two planes is $\mathbf{c} = \overrightarrow{P_1P_2} = \langle 5, 1, 0 \rangle$. So the distance D between $\mathcal{P}_1$ and $\mathcal{P}_2$ is the absolute value of the projection of $\mathbf{c}$ in the direction of $\mathbf{n}$; that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{c}|}{|\mathbf{n}|} = \frac{9}{3} = 3.$$

C12S0M.016: Given $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, a vector tangent to its trajectory is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The plane at $(1, 1, 1)$ normal to the trajectory thus has normal vector $\mathbf{v}(1) = \langle 1, 2, 3 \rangle$, and hence an equation of this plane is $x + 2y + 3z = 6$.

C12S0M.017: Given the isosceles triangle with sides AB and AC of equal length and M the midpoint of the third side BC, let $\mathbf{u} = \overrightarrow{BM} = \overrightarrow{MC}$ and let $\mathbf{v} = \overrightarrow{AM}$. Then $\overrightarrow{AB} = \mathbf{v} - \mathbf{u}$ and $\overrightarrow{AC} = \mathbf{v} + \mathbf{u}$. Now

$$\begin{aligned}
|\mathbf{v} - \mathbf{u}| &= |\mathbf{v} + \mathbf{u}|, \quad \text{so} \\
(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) &= (\mathbf{v} + \mathbf{u}) \cdot (\mathbf{v} + \mathbf{u}); \\
\mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} &= \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}.
\end{aligned}$$

Therefore $\mathbf{v} \cdot 2\mathbf{u} = 0$, and therefore the segment AM is perpendicular to the segment BC.

C12S0M.018: Name the vertices of the rhombus A, B, C, and D in counterclockwise order. Then let $\mathbf{u} = \overrightarrow{AB} = \overrightarrow{DC}$ and $\mathbf{v} = \overrightarrow{BC} = \overrightarrow{AD}$. Thus the diagonal from A to C is represented by the vector $\mathbf{v} + \mathbf{u}$ and the diagonal from B to D is represented by the vector $\mathbf{v} - \mathbf{u}$. Moreover,

$$(\mathbf{v} + \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$$

because $|\mathbf{v}| = |\mathbf{u}|$. Therefore $\mathbf{v} + \mathbf{u}$ is perpendicular to $\mathbf{v} - \mathbf{u}$, and so the diagonals of a rhombus are mutually perpendicular.

C12S0M.019: Given $\mathbf{a}(t) = \langle \sin t, -\cos t \rangle$, the velocity vector is

$$\mathbf{v}(t) = \langle c_1 - \cos t, \ c_2 - \sin t \rangle.$$

But $\mathbf{v}(0) = \langle -1, 0 \rangle = \langle c_1 - 1, c_2 \rangle$, and it follows that $c_1 = c_2 = 0$. So $\mathbf{v}(t) = \langle -\cos t, -\sin t \rangle$. Hence the position vector of the particle is

$$\mathbf{r}(t) = \langle\, k_1 - \sin t,\ k_2 + \cos t\,\rangle.$$

But $\langle\, 0,\, 1\,\rangle = \mathbf{r}(0) = \langle\, k_1,\, k_2 + 1\,\rangle$, so that $k_1 = k_2 = 0$. Thus $\mathbf{r}(t) = \langle\, -\sin t,\, \cos t\,\rangle$. The parametric equations of the motion of the particle are $x(t) = -\sin t$, $y(t) = \cos t$. Because $x^2 + y^2 = 1$ for all t, the particle moves in a circle of radius 1 centered at the origin.

C12S0M.020: Given $\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$, suppose that $\mathbf{r}(t) = \langle\, x(t),\, y(t)\,\rangle$. Then

$$x''(t) = -\omega^2 x(t) \quad \text{and} \quad y''(t) = -\omega^2 y(t).$$

Following the *Suggestion* given in the statement of the problem,

$$x(t) = A_1\cos\omega t + B_1\sin\omega t \quad \text{and} \quad y(t) = A_2\cos\omega t + B_2\sin\omega t.$$

We are given $\mathbf{r}(0) = \langle\, p,\, 0\,\rangle$, and it follows that $A_1 = p$ and $A_2 = 0$. So

$$x(t) = p\cos\omega t + B_1\sin\omega t \quad \text{and} \quad y(t) = B_2\sin\omega t.$$

Next,

$$x'(t) = -p\omega\sin\omega t + B_1\omega\cos\omega t \quad \text{and} \quad y'(t) = B_2\omega\cos\omega t.$$

We are given $\mathbf{v}(0) = \langle\, 0,\, q\omega\,\rangle$, and it follows that $B_1 = 0$ and $B_2 = q$. Therefore

$$x(t) = p\cos\omega t \quad \text{and} \quad y(t) = q\sin\omega t.$$

Then, because

$$\left[\frac{x(t)}{p}\right]^2 + \left[\frac{y(t)}{q}\right]^2 = 1,$$

the trajectory of the particle is the ellipse centered at the origin and with semiaxes of lengths p and q on the coordinate axes.

C12S0M.021: The trajectory of the projectile fired by the gun is given by

$$x_1(t) = (320\cos\alpha)t, \quad y_1(t) = -16t^2 + (320\sin\alpha)t.$$

The trajectory of the moving target is given by

$$x_2(t) = 160 + 80t, \quad y_2(t) \equiv 0.$$

We require the angle of elevation α so that, at some time $T > 0$,

$$y_1(T) = 0 \quad \text{and} \quad x_1(T) = x_2(T); \quad \text{that is,}$$

$$-16T^2 + (320\sin\alpha)T = 0 \quad \text{and} \quad (320\cos\alpha)T = 160 + 80T.$$

It follows that $T = 20\sin\alpha$ and, consequently, that

$$6400\sin\alpha\,\cos\alpha = 160 + 1600\sin\alpha;$$

$$40\sin\alpha\,\cos\alpha = 1 + 10\sin\alpha.$$

Let $u = \sin\alpha$. Then the last equation yields

$$\begin{aligned}
40u\sqrt{1-u^2} &= 1 + 10u; \\
1600u^2(1-u^2) &= 1 + 20u + 100u^2; \\
1600u^4 - 1500u^2 + 20u + 1 &= 0.
\end{aligned}$$

The graph of the last equation shows solutions between 0 and $\pi/2$ near 0.03 and 0.96. Five iterations of Newton's method with these starting values yields the two solutions

$$u_1 \approx 0.0333580866275847 \quad \text{and} \quad u_2 \approx 0.9611546539773131.$$

These correspond to the two angles

$$\begin{aligned}
&\alpha_1 \approx 0.033364276320 \quad (\text{about } 1^\circ\ 54'\ 41.876'') \quad \text{and} \\
&\alpha_2 \approx 1.291155565633 \quad (\text{about } 73^\circ\ 58'\ 39.953'').
\end{aligned}$$

C12S0M.022: With the gun and the foot of the hill located at the origin and the hill in the first quadrant, the parametric equations of motion of the projectile fired by the gun (at time $t = 0$) are

$$x(t) = (v_0\cos\alpha)t, \qquad y(t) = -\frac{1}{2}gt^2 + (v_0\sin\alpha)t$$

where α is the angle of elevation of the gun from the horizontal. The range of the projectile is

$$R(\alpha) = \frac{2\sqrt{3}}{3}x(T)$$

where T is the time at which it strikes the hill; we also require $x(T) = y(T)\sqrt{3}$, which leads to

$$\begin{aligned}
(v_0\cos\alpha)T &= \left(-\frac{1}{2}gT^2 + Tv_0\sin\alpha\right)\sqrt{3}; \\
v_0\cos\alpha &= \left(-\frac{1}{2}gT + v_0\sin\alpha\right)\sqrt{3}; \\
T &= \frac{2v_0}{g}\left(\sin\alpha - \frac{\sqrt{3}}{3}\cos\alpha\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
R(\alpha) &= \frac{2\sqrt{3}}{3}x(T) = \frac{4(v_0)^2\sqrt{3}}{3g}\left(\sin\alpha\,\cos\alpha - \frac{\sqrt{3}}{3}\cos^2\alpha\right). \\
R'(\alpha) &= \frac{4(v_0)^2\sqrt{3}}{3g}\left(\cos 2\alpha + \frac{\sqrt{3}}{3}\sin 2\alpha\right).
\end{aligned}$$

Now $R'(\alpha) = 0$ when

$$\sin 2\alpha = -\sqrt{3}\,\cos 2\alpha;$$

$$\tan 2\alpha = -\sqrt{3} \qquad (0 \leqq 2\alpha \leqq \pi);$$

$$2\alpha = \frac{2\pi}{3}.$$

Therefore the angle α that maximizes the range of the projectile is $\pi/3$.

C12S0M.023: Let $\mathbf{r}(t) = \langle\, t,\ t^2,\ \frac{4}{3}t^{3/2}\,\rangle$. Then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, 1,\ 2t,\ 2t^{1/2}\,\rangle,$$

so that $\mathbf{v}(1) = \langle\, 1,\ 2,\ 2\,\rangle$. Also

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle\, 0,\ 2,\ t^{-1/2}\,\rangle,$$

and thus $\mathbf{a}(1) = \langle\, 0,\ 2,\ 1\,\rangle$. Moreover,

$$\mathbf{v}(1)\times\mathbf{a}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = \langle\, -2,\ -1,\ 2\,\rangle,$$

and thus $|\mathbf{v}(1)\times\mathbf{a}(1)| = 3$. Thus

$$a_T(1) = \frac{\mathbf{v}(1)\cdot\mathbf{a}(1)}{v(1)} = \frac{6}{3} = 2,$$

$$\kappa(1) = \frac{|\mathbf{v}(1)\times\mathbf{a}(1)|}{[v(1)]^3} = \frac{3}{27} = \frac{1}{9}, \qquad \text{and}$$

$$a_N(1) = \frac{|\mathbf{v}(1)\times\mathbf{a}(1)|}{v(1)} = \frac{3}{3} = 1.$$

C12S0M.024: We saw in the solution of Problem 23 that at the point $\left(1,\ 1,\ \frac{4}{3}\right)$ we have

$$\mathbf{v}(1) = \langle\, 1,\ 2,\ 2\,\rangle, \quad \mathbf{a}(1) = \langle\, 0,\ 2,\ 1\,\rangle, \quad v(1) = 3, \quad a_T(1) = 2, \quad \text{and} \quad a_N(1) = 1.$$

Hence

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{v(1)} = \left\langle \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3} \right\rangle$$

and therefore

$$\mathbf{N}(1) = \frac{\mathbf{a}(1) - a_T(1)\mathbf{T}(1)}{a_N(1)} = \langle\, 0,\ 2,\ 1\,\rangle - 2\left\langle \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3} \right\rangle = \left\langle -\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3} \right\rangle.$$

Thus a normal to the osculating plane at the point $\left(1,\ 1,\ \frac{4}{3}\right)$ is

$$\mathbf{n} = 3\mathbf{T}(1) \times 3\mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{vmatrix} = \langle -6, -3, 6 \rangle.$$

Therefore an equation of the osculating plane at $\left(1, 1, \frac{4}{3}\right)$ is $6x + 3y - 6z = 1$.

C12S0M.025: A vector normal to the plane is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \langle b_1c_2 - b_2c_1, \ a_2c_1 - a_1c_2, \ a_1b_2 - a_2b_1 \rangle.$$

Hence the plane has equation

$$(b_1c_2 - b_2c_1)(x - x_0) + (a_2c_1 - a_1c_2)(y - y_0) + (a_1b_2 - a_2b_1)(z - z_0) = 0.$$

But this is exactly the equation you obtain when the matrix in the equation

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

given in Problem 25 is expanded along its first row.

C12S0M.026: Because $\mathbf{r}'(t_0)$ and $\mathbf{r}''(t_0)$ are coplanar with $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$, we may use the former in place of the latter to construct—via their cross product—a vector normal to the osculating plane. The scalar triple product in the equation given in Problem 26 can be written as a three-by-three determinant, and when one does so, one obtains the equation in Problem 25.

C12S0M.027: Let $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. Then

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad \text{so that} \quad \mathbf{r}'(1) = \langle 1, 2, 3 \rangle.$$

Also

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \quad \text{and so} \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

Thus by the results in Problems 25 and 26, an equation of the osculating plane to the twisted cubic $\mathbf{r}(t)$ at the point $\mathbf{r}(1)$ is

$$\begin{vmatrix} x - 1 & y - 1 & z - 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 0.$$

Expansion of this determinant along its first row yields the equation of the osculating plane in more conventional form:

$$6(x-1) - 6(y-1) + 2(z-1) = 0; \qquad \text{that is,} \qquad 3x - 3y + z = 1.$$

C12S0M.028: From $x = r\cos\theta$ we find that

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - (r\sin\theta)\frac{d\theta}{dt};$$

from $y = r\sin\theta$ we derive

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + (r\cos\theta)\frac{d\theta}{dt}.$$

Then

$$\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \left(\frac{dr}{dt}\cos\theta - (r\sin\theta)\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\sin\theta + (r\cos\theta)\frac{d\theta}{dt}\right)^2 \\
&= \left(\frac{dr}{dt}\right)^2\cos^2\theta - 2(r\cos\theta\,\sin\theta)\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + (r^2\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2 \\
&\qquad + \left(\frac{dr}{dt}\right)^2\sin^2\theta + 2(r\sin\theta\,\cos\theta)\cdot\frac{dr}{dt}\cdot\frac{d\theta}{dt} + (r^2\cos^2\theta)\left(\frac{d\theta}{dt}\right)^2 \\
&= \left(\frac{dr}{dt}\right)^2 + \left(r\cdot\frac{d\theta}{dt}\right)^2.
\end{aligned}$$

Then the desired result follows immediately from Eq. (2) in Section 12.6.

C12S0M.029: Equation (6) in Section 12.8, with $\rho = 1$, yields

$$x = \sin\phi\,\cos\theta, \quad y = \sin\phi\,\sin\theta, \quad z = \cos\phi.$$

Mathematica 3.0 can solve this problem for us.

```
x[t_] := Sin[phi[t]]*Cos[theta[t]]
y[t_] := Sin[phi[t]]*Sin[theta[t]]
z[t_] := Cos[phi[t]]
term1 = (x'[t])∧2
  (Cos[phi[t]] Cos[theta[t]] phi'[t] - Sin[phi[t]] Sin[theta[t]] theta'[t])^2
term2 = (y'[t])∧2
  (Cos[phi[t]] Sin[theta[t]] phi'[t] + Cos[theta[t]] Sin[phi[t]] theta'[t])^2
term3 = (z'[t])∧2
  Sin[phi[t]]^2 phi't]^2
Simplify[ term1 + term2 + term3 ]
  phi'[t]^2 + Sin[phi[t]]^2 theta'[t]^2
```

That is,

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 = \left(\frac{d\phi}{dt}\right)^2+(\sin^2\phi)\left(\frac{d\theta}{dt}\right)^2.$$

Then the result in Problem 13 follows from Eq. (2) in Section 12.6.

C12S0M.030: Part (a): Beginning with $\mathbf{B}\cdot\mathbf{T}=0$, we differentiate both sides with respect to arc length s to find that

$$0=\frac{d\mathbf{B}}{ds}\cdot\mathbf{T}+\mathbf{B}\cdot\frac{d\mathbf{T}}{ds}=\frac{d\mathbf{B}}{ds}\cdot\mathbf{T}+\mathbf{B}\cdot\kappa\mathbf{N}=\frac{d\mathbf{B}}{ds}\cdot\mathbf{T}.$$

Therefore $\mathbf{T}$ and $\dfrac{d\mathbf{B}}{ds}$ are perpendicular.

Part (b): We begin with $\mathbf{B}\cdot\mathbf{B}=1$. Then differentiation of both sides with respect to s yields

$$0=\mathbf{B}\cdot\frac{d\mathbf{B}}{ds}+\frac{d\mathbf{B}}{ds}\cdot\mathbf{B}=2\mathbf{B}\cdot\frac{d\mathbf{B}}{ds}.$$

Therefore $\mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds}$ are perpendicular.

Part (c): Because both $\mathbf{N}$ and $d\mathbf{B}/ds$ are perpendicular to both $\mathbf{T}$ and $\mathbf{B}$ and because the latter two are not parallel, the former two *are* parallel. That is, there exists a number τ such that

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

C12S0M.031: The helix of Example 7 of Section 12.1 has position vector

$$\mathbf{r}(t)=\langle\, a\cos\omega t,\ a\sin\omega t,\ bt\,\rangle.$$

(Assume that a, b, and ω are all positive.) In that example we found that the velocity and acceleration vectors are

$$\mathbf{v}(t)=\langle\, -a\omega\sin\omega t,\ a\omega\cos\omega t,\ b\,\rangle \quad\text{and}\quad \mathbf{a}(t)=\langle\, -a\omega^2\cos\omega t,\ -a\omega^2\sin\omega t,\ 0\,\rangle.$$

We also found that the speed is given by $v(t)=\sqrt{a^2\omega^2+b^2}$. Note that $\mathbf{v}(t)\cdot\mathbf{a}(t)=0$ and that

$$\mathbf{v}(t)\times\mathbf{a}(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\omega\sin\omega t & a\omega\cos\omega t & b\\ -a\omega^2\cos\omega t & -a\omega^2\sin\omega t & 0\end{vmatrix}=\langle\, ab\omega^2\sin\omega t,\ -ab\omega^2\cos\omega t,\ a^2\omega^3\,\rangle.$$

By Eq. 26 of Section 12.6, the tangential component of acceleration is zero: $a_T=0$. Next,

$$|\,\mathbf{v}(t)\times\mathbf{a}(t)\,|=\sqrt{a^2b^2\omega^4+a^4\omega^6}=a\omega^2\sqrt{b^2+a^2\omega^2}\,.$$

So by Eq. (28) in Section 12.6, the normal component of acceleration is

$$a_N=\frac{|\,\mathbf{v}(t)\times\mathbf{a}(t)\,|}{v(t)}=a\omega^2.$$

Then the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \frac{1}{\sqrt{a^2\omega^2 + b^2}} \langle -a\omega \sin \omega t,\ a\omega \cos \omega t,\ b \rangle .$$

Thus by Eq. (29) of Section 12.6, the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{a} - a_T \mathbf{T}}{a_N} = \frac{1}{a\omega^2} \left(\langle -a\omega^2 \cos \omega t,\ -a\omega^2 \sin \omega t,\ 0 \rangle - 0 \cdot \mathbf{T} \right) = \langle -\cos \omega t,\ -\sin \omega t,\ 0 \rangle.$$

Therefore the unit binormal vector is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{a^2\omega^2 + b^2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\omega \sin \omega t & a\omega \cos \omega t & b \\ -\cos \omega t & -\sin \omega t & 0 \end{vmatrix} = \frac{1}{\sqrt{a^2\omega^2 + b^2}} \langle b \sin \omega t,\ -b \cos \omega t,\ a\omega \rangle.$$

Then

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt} \cdot \frac{dt}{ds} = \frac{1}{v(t)} \cdot \frac{d\mathbf{B}}{dt} = \frac{b\omega}{a^2\omega^2 + b^2} \langle \cos \omega t,\ \sin \omega t,\ 0 \rangle = -\frac{b\omega}{a^2\omega^2 + b^2} \mathbf{N}.$$

Therefore, by definition, the torsion is

$$\tau = \frac{b\omega}{a^2\omega^2 + b^2}.$$

C12S0M.032: If the terminal point of $\mathbf{r}(t)$ lies in a fixed plane with unit normal $\mathbf{n}$, then either $\mathbf{B} = \mathbf{n}$ or $\mathbf{B} = -\mathbf{n}$, so that $d\mathbf{B}/ds = \mathbf{0}$. Therefore τ is identically zero for such a curve because $|\mathbf{N}| = 1 \neq 0$.

C12S0M.033: The spherical surface with center (0, 0, 1) (in Cartesian coordinates) and radius 1 has Cartesian equation $x^2 + y^2 + (z-1)^2 = 1$; that is,

$$x^2 + y^2 + z^2 = 2z.$$

In spherical coordinates, this equation takes the form $\rho^2 = 2\rho \cos \phi$. No points on the surface are lost by cancellation of ρ from the last equation (take $\phi = \pi/2$), so a slightly simpler spherical equation is $\rho = 2 \cos \phi$.

C12S0M.034: Replace y with the cylindrical coordinate r to obtain $(r-1)^2 + z^2 = 1$. Simplify this to $r^2 + z^2 = 2r$, then convert to rectangular coordinates:

$$x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2}\ .$$

C12S0M.035: Replace y^2 with $x^2 + y^2$ to obtain a Cartesian equation of the surface of revolution:

$$(x^2 + y^2 + z^2)^2 = 2(z^2 - x^2 - y^2).$$

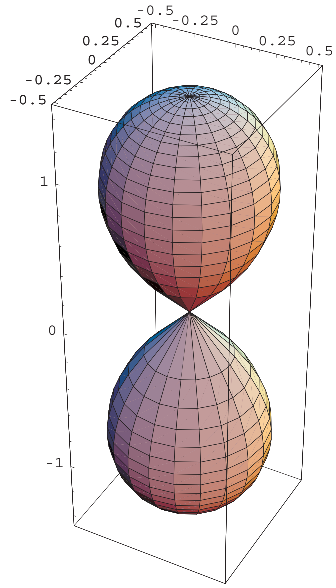
In cylindrical coordinates this equation takes the form

$$(r^2 + z^2)^2 = 2(z^2 - r^2),$$

and thus its spherical form is

$$\rho^4 = 2\rho^2(\cos^2\phi - \sin^2\phi); \qquad \text{that is,} \qquad \rho^2 = 2\cos 2\phi.$$

(No points on the graph are lost by cancellation of ρ^2: Take $\phi = \pi/2$.) The graph of this equation is next.



C12S0M.036: $A = |\mathbf{a} \times \mathbf{b}|$, so $\mathbf{a} \times \mathbf{b} = \mathbf{i}(A\cos\alpha) + \mathbf{j}(A\cos\beta) + \mathbf{k}(A\cos\gamma)$ where α, β, and γ are the direction cosines of $\mathbf{a} \times \mathbf{b}$. By assumption, the (signed) projections of A into the three coordinate planes are $A_x = A\cos\alpha$, $A_y = A\cos\beta$, and $A_z = A\cos\gamma$. Hence $\mathbf{a} \times \mathbf{b} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$, and therefore

$$A^2 = |\mathbf{a} \times \mathbf{b}|^2 = (A_x)^2 + (A_y)^2 + (A_z)^2.$$

C12S0M.037: If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then by Eq. (17) in Section 12.3,

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix};$$

similar results hold for $\mathbf{j} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{k} \cdot (\mathbf{a} \times \mathbf{b})$. Hence by the result in Problem 36,

$$A^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2.$$

C12S0M.038: It suffices to show that the intersection of the elliptical cylinder

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad (z \text{ arbitrary})$$

with the plane $Ax + By + Cz = 0$ (where $C \neq 0$) is an ellipse. This is done in the solution of Problem 39.

Rotation of Axes and Second-Degree Curves

In Section 10.6 we studied the second-degree equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

which contains no xy-term. We found that the graph is always a conic section, apart from exceptional cases of the following types:

$$\begin{aligned}
2x^2 + 3y^2 &= -1 && \text{(no locus)},\\
2x^2 + 3y^2 &= 0 && \text{(a single point)},\\
(2x-1)^2 &= 0 && \text{(a straight line)},\\
(2x-1)^2 &= 1 && \text{(two parallel lines)},\\
x^2 - y^2 &= 0 && \text{(two intersecting lines)}.
\end{aligned}$$

We may therefore say that the graph of Eq. (1) is a conic section, possibly **degenerate**. If either A or C is zero (but not both), then the graph is a parabola. It is an ellipse if $AC > 0$, a hyperbola if $AC < 0$ (by results in Section 10.6).

Let us assume that $AC \neq 0$. Then we can determine the particular conic section represented by Eq. (1) by completing squares; that is, we write Eq. (1) in the form

$$A(x-h)^2 + C(y-k)^2 = G. \tag{2}$$

This equation can be simplified further by a **translation of coordinates** to a new $x'y'$-coordinate system centered at the point (h, k) in the old xy-system. The relation between the old and new coordinates is

$$x' = x - h, \quad y' = y - k; \quad \text{that is,} \quad x = x' + h, y = y' + k. \tag{3}$$

In the new $x'y'$-coordinate system, Eq. (2) takes the simpler form

$$A(x')^2 + C(y')^2 = G, \tag{2$'$}$$

from which it is clear whether we have an ellipse, a hyperbola, or a degenerate case.

We now turn to the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{4}$$

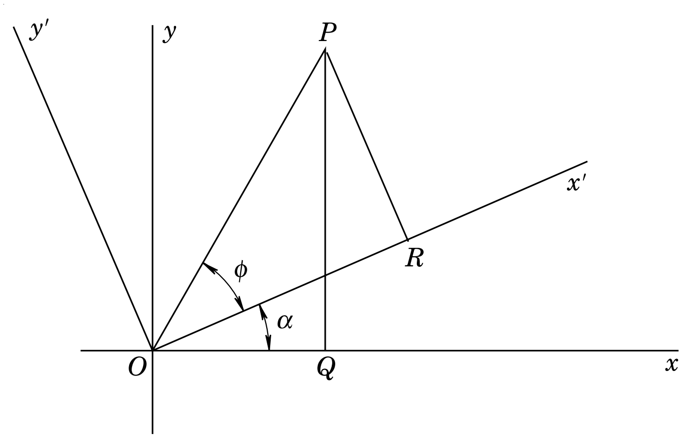
Note the presence of the "cross-product," or xy-, term. In order to recognize its graph, we need to change to a new $x'y'$-coordinate system obtained by a **rotation of axes**.

We obtain the $x'y'$-axes from the xy-axes by a rotation through an angle α in the counterclockwise direction. The next figure shows that

$$x = OQ = OP\cos(\phi + \alpha) \quad \text{and} \quad y = PQ = OP\sin(\phi + \alpha). \tag{5}$$

Similarly,

$$x' = OR = OP\cos\phi \quad \text{and} \quad y' = PR = OP\sin\phi. \tag{6}$$



Recall the addition formulas

$$\cos(\phi + \alpha) = \cos\phi\,\cos\alpha - \sin\phi\,\sin\alpha,$$

$$\sin(\phi + \alpha) = \sin\phi\,\cos\alpha + \cos\phi\,\sin\alpha.$$

With the aid of these identities and the substitution of the equations in (6) into those in (5), we obtain this result:

Equations for Rotation of Axes:

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha, \\ y &= x'\sin\alpha + y'\cos\alpha. \end{aligned} \tag{7}$$

These equations express the old xy-coordinates of the point P in terms of its new $x'y'$-coordinates and the rotation angle α.

Example 1: The xy-axes are rotated through an angle of $\alpha = 45°$. Find the equation of the curve $2xy = 1$ in the new coordinates x' and y'.

Solution: Because $\cos 45° = \sin 45° = \frac{1}{2}\sqrt{2}$, the equations in (7) yield

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

The original equation $2xy = 1$ then becomes

$$(x')^2 + (y')^2 = 1.$$

So, in the $x'y'$-coordinate system, we have a hyperbola with $a = b = 1$ (in the notation of Section 10.8), $c = \sqrt{2}$, and foci $\left(\pm\sqrt{2}, 0\right)$. In the original xy-coordinate system, its foci are $(1, 1)$ and $(-1, -1)$ and its asymptotes are the x- and y-axes. A hyperbola of this form, one which has equation $xy = k$, is called a **rectangular** hyperbola (because its asymptotes are perpendicular). ◀

Example 1 suggests that the cross-product term Bxy in Eq. (4) may disappear upon rotation of the coordinate axes. One can, indeed, always choose an appropriate angle α of rotation so that, in the new coordinate system, there is no xy-term.

To determine the appropriate rotation angle, we substitute the equations in (7) for x and y into the general second-degree equation in (4). We obtain the following new second-degree equation:

$$A'(x')^2 + B'x'y' + C'(y')^2 + D'x' + E'y' + F' = 0. \tag{8}$$

The new coefficients are given in terms of the old ones and the angle α by the following equations:

$$\begin{aligned}
A' &= A\cos^2\alpha + B\cos\alpha\,\sin\alpha + C\sin^2\alpha,\\
B' &= B(\cos^2\alpha - \sin^2\alpha) + 2(C-A)\sin\alpha\,\cos\alpha,\\
C' &= A\sin^2\alpha - B\sin\alpha\,\cos\alpha + C\cos^2\alpha,\\
D' &= D\cos\alpha + E\sin\alpha,\\
E' &= -D\sin\alpha + E\cos\alpha, \qquad \text{and}\\
F' &= F.
\end{aligned} \tag{9}$$

Now suppose that an equation of the form in (4) is given, with $B \neq 0$. We simply choose α so that $B' = 0$ in the list of new coefficients in (9). Then Eq. (8) will have no cross-product term, and we can identify and sketch the curve with little trouble in the $x'y'$-coordinate system. But is it really easy to choose such an angle α?

It is. Recall that

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \text{and} \qquad \sin 2\alpha = 2\sin\alpha\,\cos\alpha.$$

So the equation for B' in (9) may be written

$$B' = B\cos 2\alpha + (C-A)\sin 2\alpha.$$

Thus we can cause B' to be zero by choosing α to be that (unique) acute angle such that

$$\cot 2\alpha = \frac{A-C}{B}. \tag{10}$$

If we plan to use the equations in (9) to calculate the coefficients in the transformed Eq. (8), we shall need the values of $\sin\alpha$ and $\cos\alpha$ that follow from Eq. (10). It is sometimes convenient to calculate these values directly from $\cot 2\alpha$, as follows. Draw a right triangle containing an acute angle 2α with opposite side B and adjacent side $A-C$, so that Eq. (10) is satisfied. Then the numerical value of $\cos 2\alpha$ can be read directly from this triangle. Because the cosine and cotangent are both positive in the first quadrant and both negative in the second quadrant, we give $\cos 2\alpha$ the same sign as $\cot 2\alpha$. Then we use the half-angle formulas to find $\sin\alpha$ and $\cos\alpha$:

$$\sin\alpha = \left(\frac{1-\cos 2\alpha}{2}\right)^{1/2}, \qquad \cos\alpha = \left(\frac{1+\cos 2\alpha}{2}\right)^{1/2}. \tag{11}$$

Once we have the values of $\sin\alpha$ and $\cos\alpha$, we can compute the coefficients in the resulting Eq. (8) by means of the equations in (9). Alternatively, it's frequently simpler to get Eq. (8) directly by substitution of the equations in (9), with the numerical values of $\sin\alpha$ and $\cos\alpha$ obtained from Eq. (11), into Eq. (4).

Example 2: Determine the graph of the equation

$$73x^2 - 72xy + 52y^2 - 30x - 40y - 75 = 0.$$

Solution: We begin with Eq. (10) and find that $\cot 2\alpha = -\frac{7}{24}$, so that $\cos 2\alpha = -\frac{7}{25}$. Thus

$$\sin\alpha = \left(\frac{1-\left(-\frac{7}{25}\right)}{2}\right)^{1/2} = \frac{4}{5}, \qquad \cos\alpha = \left(\frac{1+\left(-\frac{7}{25}\right)}{2}\right)^{1/2} = \frac{3}{5}.$$

Then, with $A = 73$, $B = -72$, $C = 52$, $D = -30$, $E = -40$, and $F = -75$, the equations in (9) yield

$$\begin{aligned} A' &= 25, & D' &= -50, \\ B' &= 0 \quad \text{(this was the point)}, & E' &= 0, \\ C' &= 100, & F' &= -75. \end{aligned}$$

Consequently the equation in the new $x'y'$-coordinate system, obtained by rotation through an angle of $\alpha = \arcsin\left(\frac{4}{5}\right) \approx 53.13°$, is

$$25(x')^2 + 100(y')^2 - 50x' = 75.$$

If you prefer, you could obtain this equation by substitution of

$$x = \frac{3}{5}x' - \frac{4}{5}y', \quad y = \frac{4}{5}x' + \frac{3}{5}y'$$

in the original equation.

By completing the square in x' we finally obtain

$$25(x'-1)^2 + 100(y')^2 = 100,$$

which we put into the standard form

$$\frac{(x'-1)^2}{4} + \frac{(y')^2}{1} = 1.$$

Thus the original curve is an ellipse with major semiaxis 2, minor semiaxis 1, and center (1, 0) in the $x'y'$-coordinate system. ◀

Example 2 illustrates the general procedure for finding the graph of a second-degree equation. First, if there is a cross-product (xy) term, rotate axes to eliminate it. Then translate axes as necessary to reduce the equation to the standard form of a parabola, ellipse, or hyperbola (or one of the degenerate cases of a conic section).

There *is* a test by which the nature of the curve may be discovered without actually carrying out the transformations described here. This test derives from the fact that, whatever the angle α of rotation, the equations in (9) imply that

$$(B')^2 - 4A'C' = B^2 - 4AC \tag{12}$$

(it is easy to verify this for yourself). Thus the **discriminant** $B^2 - 4AC$ is an *invariant* under any rotation of axes. If α is so chosen that $B' = 0$, then the left-hand side of Eq. (12) is simply $-4A'C'$. Because A' and C' are the coefficients of the squared terms, our earlier discussion of Eq. (1) now applies. It follows that the graph will be:

- a *parabola* if $B^2 - 4AC = 0$,

- an *ellipse* if $B^2 - 4AC < 0$, and
- a *hyperbola* if $B^2 - 4AC > 0$.

Of course, degenerate cases may occur.

Here are some examples:

- $x^2 + 2xy + y^2 = 1$ is a (degenerate) parabola,
- $x^2 + xy + y^2 = 1$ is an ellipse, and
- $x^2 + 3xy + y^2 = 1$ is a hyperbola.

C12S0M.039: First Solution Without loss of generality, we may assume that the ellipse in the xy-plane is centered at the origin, and by rotation (if necessary) that the plane containing the curve K has equation $z = by + c$ (that is, parallel to the x-axis). Then the ellipse has an equation of the form

$$Ax^2 + Bxy + Cy^2 + D = 0$$

where $B^2 - 4AC < 0$ (by the preceding discussion of rotation of axes). Parametrize the plane with a uv-coordinate system that projects vertically onto the xy-coordinate system in the plane:

$$u = x, \qquad v = y\sqrt{1 + b^2}\ .$$

This parametrization preserves distance; that is,

$$\sqrt{x^2 + y^2 + (z - c)^2}$$

on the plane is

$$\sqrt{u^2 + v^2} = \sqrt{x^2 + (1 + b^2)y^2} = \sqrt{x^2 + b^2y^2 + y^2} = \sqrt{x^2 + (z - c)^2 + y^2}\ .$$

Now suppose that the point (x, y, z) is on the intersection of the plane and the cylinder. The equation

$$Ax^2 + Bxy + Cy^2 + D = 0$$

takes the form

$$Au^2 + Bu\frac{v}{\sqrt{1 + b^2}} + C\frac{v^2}{1 + b^2} + D = 0;$$

that is,

$$Au^2 + \frac{B}{\sqrt{1 + b^2}}uv + \frac{C}{1 + b^2}v^2 + D = 0.$$

Is this the equation of an ellipse? We check the discriminant:

$$\left(\frac{B}{\sqrt{1 + b^2}}\right)^2 - \frac{4AC}{1 + b^2} = \frac{B^2 - 4AC}{1 + b^2} < 0.$$

Because the descriminant is negative, the curve K is an ellipse.

Note: If you have access to the version of the sixth edition of the textbook *with early transcendentals functions and matrices*, you can use instead the following much simpler solution of this problem.

C12S0M.039: Second Solution Without loss of generality, we may suppose that the elliptical cylinder E with vertical sides has equation $z = ax^2 + by^2$ where a and b are both positive. We may also suppose that the nonvertical plane P has equation $z = px + qy + r$ where the coefficients have the property that the intersection C of P and E is a closed curve. Substitution of the equation of P for z in the equation of E shows that C lies in the vertical cylinder with equation

$$ax^2 - px + by^2 - qy = r, \tag{1}$$

and the equation of the vertical projection D of C into the xy-plane has the same equation (together with $z = 0$). The matrix associated with the form in Eq. (1) is

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

and has eigenvalues $\lambda_1 = a$ and $\lambda_2 = b$. Hence, in an appropriately rotated and translated uv-coordinate system, D has equation

$$au^2 + bv^2 = s.$$

Because D is a closed curve, $s > 0$; thus D is an ellipse. Therefore, by the result in Problem 38, C itself is also an ellipse.

C12S0M.040: The projection into the xy-plane of the intersection K has the equation $a^2x^2 - Ax + b^2y^2 - By = 0$. We may also assume that $a > 0$ and $b > 0$. It follows by completing squares that this projection is either empty, a single point, or an ellipse. In the latter case, it follows from Problem 38 that K is an ellipse.

C12S0M.041: The intersection of $z = Ax + By$ with the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

has the simultaneous equations

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \frac{A^2x^2 + 2ABxy + B^2y^2}{c^2} = 1, \quad z = Ax + By.$$

The first of these two equations is the equation of the projection of the intersection into the xy-plane; write it in the form

$$Px^2 + Qxy + Ry^2 = 1$$

and show that its discriminant is negative—this shows that the projection is an ellipse (see the discussion of rotation of axes immediately preceding the solution of Problem 39). It then follows from the result in Problem 38 that the intersection itself must be an ellipse. (Of course, if the plane is tangent to the ellipse or misses it altogether, then the intersection is not an ellipse—it is either empty or else consists of a single point.)

C12S0M.042: IF f'' is continuous and the graph of f has an inflection point at $(a, f(a))$, then $f''(a) = 0$. Hence by Eq. (13) in Section 12.6, the curvature of the graph of f is zero at that point.

C12S0M.043: By Eq. (13) in Section 12.6, the curvature of $y = \sin x$ at the point $(x, \sin x)$ is

$$\kappa(x) = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}.$$

The curvature is minimal when it is zero, and this occurs at every integral multiple of π. To maximize the curvature, note that the numbers that maximize the numerator in the curvature formula—the odd integral multiples of $\pi/2$—also minimize the denominator and thereby maximize the curvature itself.

C12S0M.044: By Eq. (12) in Section 12.6, the curvature of the hyperbola at $(x(t),\, y(t))$ is

$$\begin{aligned}\kappa(t) &= \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{|\sinh^2 t - \cosh^2 t|}{(\sinh^2 t + \cosh^2 t)^{3/2}}\\ &= \frac{1}{(\sinh^2 t + \cosh^2 t)^{3/2}} = \frac{1}{(\cosh 2t)^{3/2}}.\end{aligned}$$

For maximal curvature, minimize the denominator; when $\cosh 2t$ is minimal, $t = 0$ and $(x,\, y) = (1,\, 0)$. At no point is the curvature minimal because at no point is $\cosh 2t$ maximal.

C12S0M.045: With $\mathbf{r}(t) = \langle\, t\cos t,\ t\sin t\,\rangle$, we have

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= \langle\, \cos t - t\sin t,\ t\cos t + \sin t\,\rangle;\\ \mathbf{v}(\pi/2) &= \langle\, -\pi/2,\ 1\,\rangle;\\ v(\pi/2) &= \frac{1}{2}\sqrt{\pi^2 + 4}\,;\\ \mathbf{T}(\pi/2) &= \frac{1}{\sqrt{\pi^2+4}}\langle\, -\pi,\ 2\,\rangle.\end{aligned}$$

Because the curve turns left as t increases,

$$\mathbf{N}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{\pi^2+4}}\langle\, -2,\ -\pi\,\rangle.$$

C12S0M.046: By Eq. (12) in Section 12.6, the curvature is

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[(x'(t))^2 + (y'(t))^2]^{3/2}} = \frac{ab\sin^2 t + ab\cos^2 t}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}.$$

Because $a > b > 0$, the curvature will be maximal when the last denominator is minimal, which will occur when $\sin t = 0$ and $\cos t = \pm 1$. Thus it will be maximal when t is an integral multiple of π; the corresponding points on the ellipse are its vertices $(\pm a,\, 0)$. The curvature will be minimal when the denominator is maximal, and this occurs when $\sin t = \pm 1$ and $\cos t = 0$. So the curvature is minimal when t is an odd integral multiple of $\pi/2$, and the corresponding points on the ellipse are $(0,\, \pm b)$. If you prefer to use the derivative to maximize and minimize the curvature, you should find that

$$\kappa'(t) = \frac{3ab(a^2 - b^2)\sin t\, \cos t}{(a^2\sin^2 t + b^2\cos^2 t)^{5/2}},$$

that $\kappa'(t) = 0$ at every integral multiple of $\pi/2$, and that

$$\kappa(0) = \kappa(\pi) = \frac{a}{b^2} \quad \text{while} \quad \kappa(\pi/2) = \kappa(3\pi/2) = \frac{b}{a^2}.$$

C12S0M.047: We will ask *Mathematica* 3.0 to solve this problem (but we will include intermediate computations so you can check your work).

```
x[t_] := r[t]*Cos[t]

y[t_] := r[t]*Sin[t]

x′[t]

    -r[t] Sin[t] + Cos[t] r′[t]

y′[t]

    Cos[t] r[t] + Sin[t] r′[t]

x″[t]

    -Cos[t] r[t] - 2 Sin[t] r′[t] + Cos[t] r″[t]

y″[t]

    -r[t] Sin[t] + 2 Cos[t] r′[t] + Sin[t] r″[t]

num = Simplify[ x′[t]*y″[t] - x″[t]*y′[t] ]

    r[t]^2 + 2 r′[t]^2 - r[t] r″[t]

den1 = Expand[ (x′[t])^2 ]

    r[t]^2 r[t]^2 - 2 Cos[t] r[t] Sin[t] r′[t] + Cos[t]^2 r′[t]^2

den2 = Expand[ (y′[t])^2 ]

    Cos[t]^2 r[t]^2 + 2 Cos[t] r[t] Sin[t] r′[t] + Sin[t]^2 r′[t]^2

den = Simplify[ den1 + den2 ]

    r[t]^2 + r′[t]^2
```

Finally, when we asked for `Abs[num]/den∧(3/2)`, the response was

$$\frac{|\,(r(t))^2 + 2(r'(t))^2 - r(t)r''(t)\,|}{[(r(t))^2 + (r'(t))^2]^{3/2}}.$$

C12S0M.048: Substitution of $r(\theta) = \theta$ in the formula of Problem 47 yields

$$\kappa(\theta) = \frac{|\theta^2 + 2 - 0|}{(\theta^2 + 1)^{3/2}} = \frac{\theta^2 + 2}{(\theta^2 + 1)^{3/2}}.$$

Because $\kappa(\theta) \approx 1/\theta$ if θ is large positive or large negative, it is clear that $\kappa(\theta) \to 0$ as $\theta \to \pm\infty$.

C12S0M.049: The function f is said to be *odd* if $f(-x) = -f(x)$ for all x; f is said to be *even* if $f(-x) = f(x)$ for all x. Because $y(x) = Ax + Bx^3 + Cx^5$ is an odd function, the condition $y(1) = 1$ will imply that $y(-1) = -1$. Because $y'(x) = A + 3Bx^2 + 5Cx^4$ is an even function, the condition $y'(1) = 0$ will imply that $y'(-1) = 0$ as well. Because the graph of y is symmetric around the origin (every odd function has this property), the condition that the curvature is zero at (1, 1) will imply that it is also zero at $(-1, -1)$. By Eq. (13) of Section 12.6, the curvature at x is

$$\kappa(x) = \frac{|\,6Bx + 20Cx^3\,|}{\left[1 + (A + 3Bx^2 + 5Cx^4)^2\right]^{3/2}},$$

so the curvature at (1, 1) will be zero when $6B + 20C = 0$. Thus we obtain the simultaneous equations

$$
\begin{aligned}
A + \ \ B + \quad C \ &= 1,\\
A + 3B + \ 5C \ &= 0,\\
3B + 10C \ &= 0.
\end{aligned}
$$

These equations are easy to solve for $A = \frac{15}{8}$, $B = -\frac{5}{4}$, and $C = \frac{3}{8}$. Thus an equation of the connecting curve is

$$
y(x) = \frac{15}{8}x - \frac{5}{4}x^3 + \frac{3}{8}x^5.
$$

C12S0M.050: The given plane $\mathcal{P}$ through the origin has equation $Ax + By + Cz = 0$ where $ABC \neq 0$, and $\mathcal{P}$ intersects the sphere $x^2 + y^2 + z^2 = R^2$ in a great circle K. Take two points S and T on this great circle equally distant from, but close to, the "north pole" $N(0, 0, R)$ of the sphere. Draw the arc of the great circle connecting S with N and the arc of the great circle connecting T with N. The plane bisector of the angle between these arcs at N (the angle between their tangents at N) will meet the great circle K at its highest point. So all we need is to construct a plane $\mathcal{Q}$ containing the z-axis and normal to $\mathcal{P}$. Now $\mathcal{Q}$ has an equation of the form $Dx + Ey = 0$, so a normal to $\mathcal{Q}$ is $\mathbf{n} = \langle D, E, 0 \rangle$. The points $(0, 0, 0)$ and $(B, -A, 0)$ lie in $\mathcal{P}$, so the vector $\mathbf{u} = \langle B, -A, 0 \rangle$ is parallel to $\mathcal{P}$. So $\mathcal{Q}$ will be normal to $\mathcal{P}$ provided that $\mathbf{n}$ and $\mathbf{u}$ are parallel; that is, if there is a scalar λ such that

$$
\mathbf{n} = \lambda\mathbf{u}; \qquad \text{that is,} \qquad \langle D, E, 0 \rangle = \lambda\langle B, -A, 0 \rangle.
$$

It is simplest to choose $\lambda = 1$, so that $D = B$ and $E = -A$. This implies that the plane $\mathcal{Q}$ has equation $Bx - Ay = 0$.

To find the highest point (the point with greatest z-coordinate) on K, we now solve simultaneously the equations of the sphere, $\mathcal{P}$, and $\mathcal{Q}$. The *Mathematica* 3.0 command

```
Solve[ { x*x + y*y + z*z == r*r, a*x + b*y + c*z == 0, b*x == a*y }, { x, y, z } ]
```

produces the two solutions

$$
x_1 = \frac{ACR}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}}, \quad y_1 = \frac{BCR}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}},
$$
$$
z_1 = -\frac{(A^2 + B^2)R}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}}
$$

and

$$
x_2 = -\frac{ACR}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}}, \quad y_2 = -\frac{BCR}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}},
$$
$$
z_z = \frac{(A^2 + B^2)R}{\sqrt{(A^2 + B^2)(A^2 + B^2 + C^2)}}.
$$

Assuming that $R > 0$, the coordinates of the highest point on K are (x_2, y_2, z_2). (If you prefer, exactly the same answer can be obtained by using the method of Lagrange multipliers of Section 13.9.)

C12S0M.051: Without loss of generality we may suppose that the tetrahedron lies in the first octant with the solid right angle at the origin. Let the coordinates of its other three vertices be $(a, 0, 0)$, $(0, b, 0)$, and

$(0,\,0,\,c)$. Then the vectors $\mathbf{a} = \langle\, a,\,0,\,0\,\rangle$, $\mathbf{b} = \langle\, 0,\,b,\,0\,\rangle$, and $\mathbf{c} = \langle\, 0,\,0,\,c\,\rangle$ form three of the edges of the tetrahedron and the other three edges are $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{b}$, and $\mathbf{a}-\mathbf{c}$. The area A of the triangle with these three edges is half the magnitude of the cross product of $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$, and

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \langle\, bc,\; ac,\; ab\,\rangle.$$

Therefore

$$A^2 = \frac{1}{4}(a^2b^2 + a^2c^2 + b^2c^2).$$

Part (a): The triangular face of the tetrahedron that lies in the xz plane has area $B = \frac{1}{2}ac$, the triangle in the xy plane has area $C = \frac{1}{2}ab$, and the triangle in the yz-plane has area $D = \frac{1}{2}bc$. It follows immediately that

$$A^2 \;=\; B^2 + C^2 + D^2.$$

Part (b): This is a generalization of the Pythagorean theorem to three dimensions. How would you generalize it to higher dimensions?

Section 13.2

C13S02.001: Because $f(x, y) = 4 - 3x - 2y$ is defined for all x and y, the domain of f is the entire two-dimensional plane.

C13S02.002: Because $x^2 + 2y^2 \geqq 0$ for all x and y, the domain of $f(x, y) = \sqrt{x^2 + 2y^2}$ is the entire two-dimensional plane.

C13S02.003: If either x or y is nonzero, then $x^2 + y^2 > 0$, and so $f(x, y)$ is defined—but not if $x = y = 0$. Hence the domain of f consists of all points (x, y) in the plane other than the origin.

C13S02.004: If $x \neq y$ then the denominator in $f(x, y)$ is nonzero, and thus $f(x, y)$ is defined—but not if $x = y$. So the domain of f consists of all those points (x, y) in the plane for which $y \neq x$.

C13S02.005: The real number z has a unique cube root $z^{1/3}$ regardless of the value of z. Hence the domain of $f(x, y) = (y - x^2)^{1/3}$ consists of all points in the xy-plane.

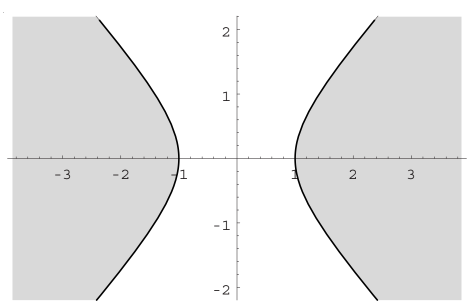
C13S02.006: The real number z has a unique cube root $z^{1/3}$ regardless of the value of z. But $\sqrt{2x}$ is real if and only if $x \geqq 0$. Therefore the domain of $f(x, y) = (2x)^{1/2} + (3y)^{1/3}$ consists of all those points (x, y) for which $x \geqq 0$.

C13S02.007: Because $\arcsin z$ is a real number if and only if $-1 \leqq z \leqq 1$, the domain of the given function $f(x, y) = \sin^{-1}(x^2 + y^2)$ consists of those points (x, y) in the xy-plane for which $x^2 + y^2 \leqq 1$; that is, the set of all points on and within the unit circle.

C13S02.008: Because $\arctan z$ is defined for every real number z, the only obstruction to the computation of $f(x) = \arctan(y/x)$ is the possibility that $x = 0$. This obstruction is insurmountable, and therefore the domain of f consists of all those points (x, y) in the xy-plane for which $x \neq 0$; that is, all points other than those on the y-axis.

C13S02.009: For every real number z, $\exp(z)$ is defined and unique. Therefore the domain of the given function $f(x, y) = \exp(-x^2 - y^2)$ consists of all points (x, y) in the entire xy-plane.

C13S02.010: Because $\ln z$ is a unique real number if and only if $z > 0$, the domain of $f(x, y) = \ln(x^2 - y^2 - 1)$ consists of those points (x, y) for which $x^2 - y^2 - 1 > 0$; that is, for which $y^2 < x^2 - 1$. This is the region bounded by the hyperbola with equation $x^2 - y^2 = 1$, shown shaded in the following figure; the bounding hyperbola itself is *not* part of the domain of f.



C13S02.0011: Because $\ln z$ is a unique real number if and only if $z > 0$, then domain of $f(x, y) = \ln(y - x)$ consists of those points (x, y) for which $y > x$. This is the region *above* the graph of the straight line with equation $y = x$ (the line itself is *not* part of the domain of f).

C13S02.012: Because $\sqrt{z}$ is a unique real number if and only if $z \geqq 0$, the domain of the given function $f(x, y) = \sqrt{4 - x^2 - y^2}$ consists of those points (x, y) for which $x^2 + y^2 \leqq 4$. That is, the domain consists of all those points (x, y) on and within the circle with center $(0, 0)$ and radius 2.

C13S02.013: If x and y are real numbers, then so are xy, $\sin xy$, and $1 + \sin xy$. Hence the only obstruction to computation of

$$f(x, y) = \frac{1 + \sin xy}{xy}$$

is the possibility of division by zero. So the domain of f consists of those points (x, y) for which $xy \neq 0$; that is, all points in the xy-plane other than those on the coordinate axes.

C13S02.014: If x and y are real numbers, then so are xy, $\sin xy$, $1+\sin xy$, and x^2+y^2. Hence the only obstruction to computation of

$$f(x, y) = \frac{1 + \sin xy}{x^2 + y^2}$$

is the possibility of division by zero. So the domain of f consists of those points (x, y) for which $x^2+y^2 \neq 0$; that is, all points in the xy-plane other than the origin.

C13S02.015: If x and y are real numbers, then so are xy and $x^2 - y^2$. So the only obstruction to the computation of

$$f(x, y) = \frac{xy}{x^2 - y^2}$$

is the possibility that $x^2 - y^2 = 0$. If so, then $f(x, y)$ is undefined, and therefore the domain of f consists of those points (x, y) for which $x^2 \neq y^2$. That is, the domain of f consists of those points in the xy-plane other than the two straight lines with equations $y = x$ and $y = -x$.

C13S02.016: The only obstruction to the computation of

$$f(x, y, z) = \frac{1}{\sqrt{z - x^2 - y^2}}$$

is the possibility that the radicand is negative or the denominator is zero. Thus the domain consists of those points (x, y, z) in space for which $x^2 + y^2 < z$. These are the points strictly above the circular paraboloid with equation $z = x^2 + y^2$ (such a paraboloid is shown in Fig. 12.3.15).

C13S02.017: If w is any real number, then $\exp(w)$ is a unique real number. So the only obstruction to the computation of

$$f(x, y, z) = \exp\left(\frac{1}{x^2 + y^2 + z^2}\right)$$

is the possibility that $x^2 + y^2 + z^2 = 0$. If so, then f is undefined, and therefore its domain consists of all those point (x, y, z) in space other than the origin $(0, 0, 0)$

C13S02.018: If w is a positive real number, then $\ln w$ is a unique real number, but not otherwise. So the domain of $f(x, y, z) = \ln xyz$ consists of those points (x, y, z) in space for which $xyz > 0$. These are the points for which either:

- $x > 0$, $y > 0$, and $z > 0$;

- $x > 0,\ y < 0,$ and $z < 0$;
- $x < 0,\ y > 0,$ and $z < 0$; or
- $x < 0,\ y < 0,$ and $z > 0$.

So the domain of f consists of those points $(x,\ y,\ z)$ strictly within the first octant and those strictly within three other octants.

C13S02.019: If w is a positive real number, then $\ln w$ is a unique real number, but not otherwise. So the domain of $f(x,\ y,\ z) = \ln(z - x^2 - y^2)$ consists of those points $(x,\ y,\ z)$ in space for which $x^2 + y^2 < z$. These are the points strictly above the circular paraboloid with equation $z = x^2 + y^2$; such a paraboloid is shown in Fig. 12.3.15.

C13S02.020: If $-1 \leqq w \leqq 1$, then $\sin^{-1} w$ is a unique real number, but not otherwise. Hence the domain of $f(x,\ y,\ z) = \arcsin(3 - x^2 - y^2 - z^2)$ consists of those points $(x,\ y,\ z)$ in space for which

$$-1 \leqq 3 - x^2 - y^2 - z^2 \leqq 1;$$

$$-1 \leqq x^2 + y^2 + z^2 - 3 \leqq 1;$$

$$2 \leqq x^2 + y^2 + z^2 \leqq 4.$$

Thus the domain of f consists of all points on or between the two spherical surfaces centered at $(0,\ 0,\ 0)$, the inner with radius $\sqrt{2}$, the outer with radius 2.

C13S02.021: The graph is the horizontal (parallel to the xy-plane) plane passing through the point $(0,\ 0,\ 10)$.

C13S02.022: The graph is the plane $z = x$ parallel to the y-axis; its trace in the xz-plane is the straight line with equations $z = x,\ y = 0$; the graph also contains the y-axis.

C13S02.023: The graph of $f(x,\ y) = x + y$ is the plane with equation $z = x + y$. The trace of this plane in the xy-plane is the straight line with equations $z = 0,\ y = -x$. Its trace in the xz-plane is the straight line with equations $y = 0,\ z = x$ and its trace in the yz-plane is the straight line with equations $x = 0,\ z = y$.

C13S02.024: If $z = \sqrt{x^2 + y^2}$, then $z^2 = x^2 + y^2$ (while $z \geqq 0$). Thus the graph of $f(x) = \sqrt{x^2 + y^2}$ is the upper nappe of the 45° circular cone with vertex at the origin and axis the nonnegative z-axis. See Example 7 of Section 12.3 and the top half of Fig. 12.3.11 there.

C13S02.025: The graph of $z = x^2 + y^2$ is a circular paraboloid with axis the nonnegative z-axis, opening upward, with its vertex at the origin. See Fig. 13.2.4.

C13S02.026: The graph of $z = 4 - x^2 - y^2$ is a circular paraboloid with axis the z-axis, opening downward, with its vertex at $(0,\ 0,\ 4)$. Its graph is shown in the solution of Problem 9 of Section 12.7 and a similar paraboloid appears in Fig. 13.2.7.

C13S02.027: If $z = \sqrt{4 - x^2 - y^2}$, then $z^2 = 4 - x^2 - y^2$ and $z \geqq 0$; that is, $x^2 + y^2 + z^2 = 4$ and $z \geqq 0$. Consequently the graph of $f(x,\ y) = \sqrt{4 - x^2 - y^2}$ is a hemisphere—the upper half of the sphere with radius 2 and center $(0,\ 0,\ 0)$.

C13S02.028: The graph of $z = 16 - y^2$ is a parabolic cylinder parallel to the x-axis; its trace in the yz-plane is the parabola with equations $z = 16 - y^2,\ x = 0$. It resembles the surface shown in Fig. 12.3.8.

C13S02.029: In cylindrical coordinates, the graph of $f(x, y) = 10 - \sqrt{x^2 + y^2}$ has the equation $z = 10 - r$ where $r \geqq 0$. Therefore the graph of f is the lower nappe of a circular cone with axis the z-axis and vertex at $(0, 0, 10)$.

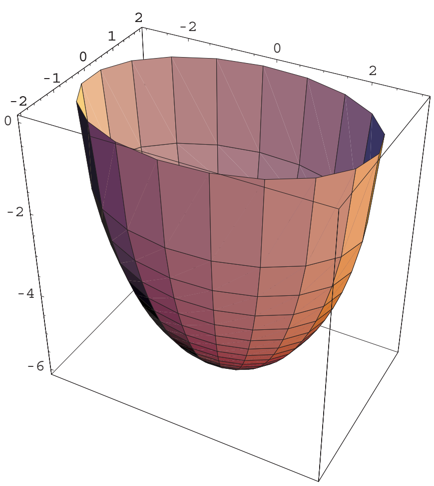
C13S02.030: Given: $f(x, y) = -\sqrt{36 - 4x^2 - 9y^2}$. The graph of f is, by definition, the graph of the equation $z = -\sqrt{36 - 4x^2 - 9y^2}$; that is,

$$4x^2 + 9y^2 + z^2 = 36, \quad z \geqq 0.$$

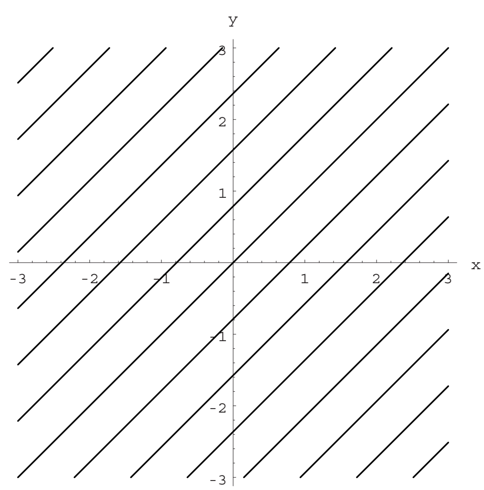
Thus the graph is the lower half of an ellipsoid with center at the origin and axes on the coordinate axes. Its intercepts are $(\pm 3, 0, 0)$, $(0, \pm 2, 0)$, and $(0, 0, -6)$. The *Mathematica* 3.0 command

```
ParametricPlot3D[ { 3*r*Cos[t], 2*r*Sin[t], -6*Sqrt[1 - r*r] },
                  { t, 0, 2*Pi }, { r, 0, 1 }, AspectRatio -> Automatic ];
```

generates its graph, shown next.



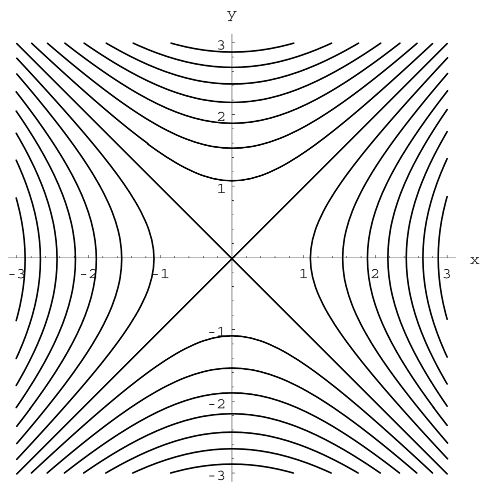
C13S02.031: The level curves of $f(x, y) = x - y$ are the straight lines $x - y = c$ where c is a constant. Some of these are shown next.



C13S02.032: The level curves of $f(x, y) = x^2 - y^2$ are rectangular hyperbolas (the level curve $x^2 - y^2 = 0$ is a degenerate hyperbola—two intersecting straight lines). The *Mathematica* 3.0 command

```
ContourPlot[ x*x - y*y, { x, -3, 3 }, { y, -3, 3 }, Axes -> True,
             AxesLabel -> { x, y }, AxesOrigin -> { 0, 0 }, Contours -> 15,
             ContourShading -> False, Frame -> False, PlotPoints -> 47 ];
```

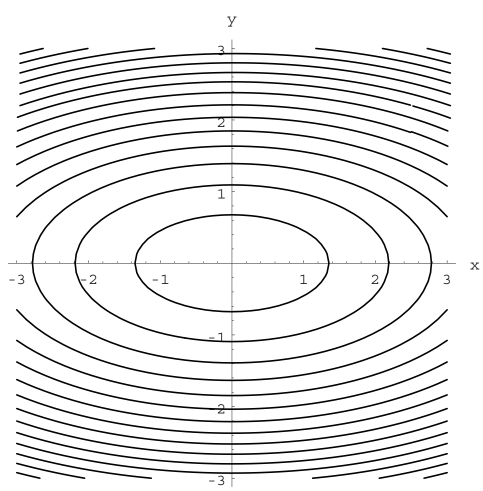
generates some of the level curves and the result is shown next.



C13S02.033: The level curves of $f(x, y) = 4x^2 + y^2$ are ellipses centered at the origin, with major axes on the x-axis and minor axes on the y-axis. The *Mathematica* 3.0 command

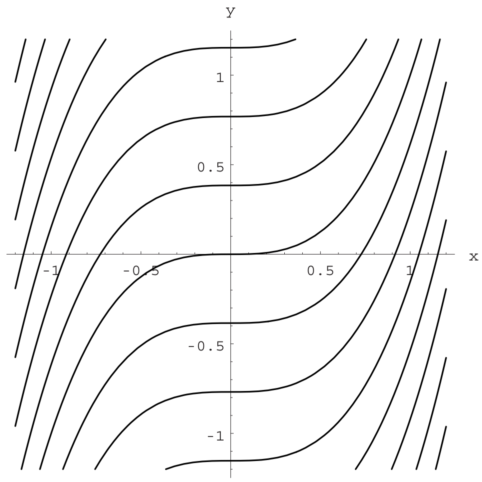
```
ContourPlot[ x*x + 4*y*y, { x, -3, 3 }, { y, -3, 3 }, Axes -> True,
             AxesLabel -> { x, y }, AxesOrigin -> { 0, 0 }, Contours -> 15,
             ContourShading -> False, Frame -> False, PlotPoints -> 47 ];
```

generates some of these level curves; the output resulting from the preceding command is shown next.



C13S02.034: The level curves of $f(x, y) = y - x^2$ are parabolas with equations of the form $y = x^2 + C$ where C is a constant. They open upward and their vertices lie on the y-axis.

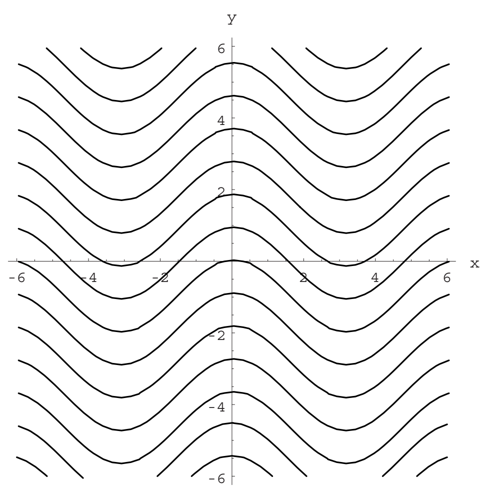
C13S02.035: The level curves of $f(x, y) = y - x^3$ are curves with the equation $y = x^3 + C$ for various values of C. Their inflection points all lie on the y-axis. A few of these level curves are shown next.



C13S02.036: The level curves of $f(x, y) = y - \cos x$ are curves with equations of the form $y = \cos x + C$ for various values of the constant C. The *Mathematica* 3.0 command

```
ContourPlot[ y - Cos[x], { x, -6, 6 }, { y, -6, 6 }, Axes -> True,
             AxesLabel -> { x, y }, AxesOrigin -> { 0, 0 }, Contours -> 15,
             ContourShading -> False, Frame -> False, PlotPoints -> 47 ];
```

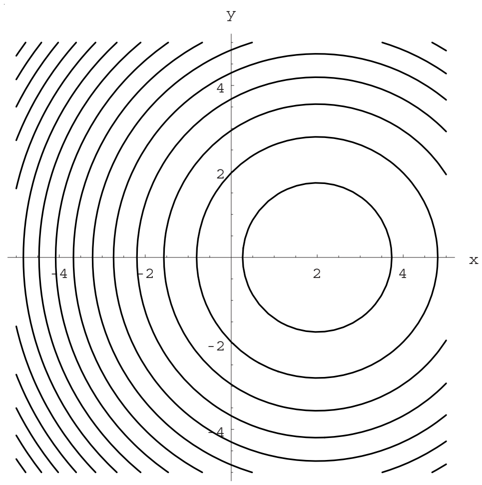
generates several of these level curves, as shown next.



C13S02.037: The level curves of $f(x, y) = x^2 + y^2 - 4x$ are circles centered at the point $(2, 0)$. The *Mathematica* 3.0 command

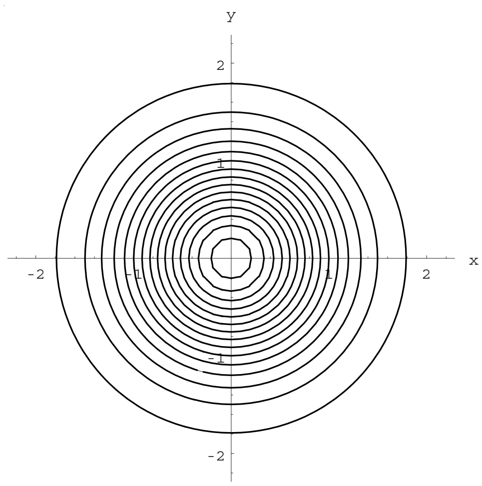
```
ContourPlot[ x*x + y*y - 4*x, { x, -5, 5 }, { y, -5, 5 }, Axes -> True,
             AxesLabel -> { x, y }, AxesOrigin -> { 0, 0 }, Contours -> 15,
             ContourShading -> False, Frame -> False, PlotPoints -> 47 ];
```

produces several of these level curves, shown next.



C13S02.038: The level curves of $f(x, y) = x^2 + y^2 - 6x + 4y + 7$ are circles centered at the point $(3, -2)$.

C13S02.039: Because $\exp(-x^2 - y^2)$ is constant exactly when $x^2 + y^2$ is constant, the level curves of $f(x, y) = \exp(-x^2 - y^2)$ are circles centered at the origin. When they are close the graph is steep; when they are far apart the slope of the graph is more moderate. Some level curves of f are shown next.



C13S02.040: Because

$$f(x, y) = \frac{1}{1 + x^2 + y^2}$$

is constant exactly when $x^2 + y^2$ is constant, the level curves of f are circles centered at the origin. Those of radii between 0.75 and 1.25 are quite close together; those with radii over 4 are quite far apart, as are those with radii less than 0.5.

C13S02.041: Because $f(x, y, z) = x^2 + y^2 - z$ is constant when $z = x^2 + y^2 + C$ (where C is a constant), the level surfaces of f are circular paraboloids with axis the z-axis, all opening upward and all having the same shape.

C13S02.042: Because $f(x, y, z) = z + \sqrt{x^2 + y^2}$ is constant when $z = C - (x^2 + y^2)^{1/2}$ (where C is a constant), the level surfaces of f are the lower nappes of circular cones with vertices on the z-axis.

C13S02.043: The function $f(x, y, z) = x^2 + y^2 + z^2 - 4x - 2y - 6z$ is constant when

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = C$$

for some nonnegative constant C. Therefore the level surfaces of f are spherical surfaces all centered at the point (2, 1, 3).

C13S02.044: Note first that $f(x, y, z) = z^2 - x^2 - y^2$ is constant exactly when

$$z^2 = x^2 + y^2 + C$$

for some constant C. If $C = 0$ then the level surface consists of both nappes of a circular cone with axis the z-axis and vertex at the origin. If $C < 0$ then the level surface is a circular hyperboloid of one sheet with axis the z-axis, similar to the one shown in Fig. 12.3.18. If $C > 0$ then it is a circular hyperboloid of two sheets, also with axis the z-axis, similar to the one shown in Fig. 12.3.20. If you made a movie showing the surface at time C as C varies from -5 to 5 it would be fascinating to watch. Such movies can be created with any of several computer algebra programs, including *Mathematica* and *Maple*. In *Maple V* (Release 5.1) the movie can be created using this command:

```
restart:with(plots):display(seq(implicitplot3d(z*z = x*x + y*y + t, x = -10..10,
y = -10..10, z = -5..5), t = -5..5), insequence = true);
```

If you have sufficient memory, you might consider changing the function to

```
z*z = x*x + y*y + t/2
```

and replacing `t = -5..5` with `t = -10..10` to increase the number of frames in the movie from 11 to 21. This should produce a "smoother" version of the movie.

C13S02.045: The function $f(x, y, z) = x^2 + 4y^2 - 4x - 8y + 17$ is constant exactly when $(x-2)^2 + 4(y-1)^2$ is a nonnegative constant. Hence the level surfaces of f are elliptical cylinders parallel to the z-axis and centered on the vertical line that meets the xy-plane at the point (2, 1, 0). The ellipse in which each such cylinder meets the xy-plane has major axis parallel to the x-axis, minor axis parallel to the y-axis, and the major axis is twice the length of the minor axis.

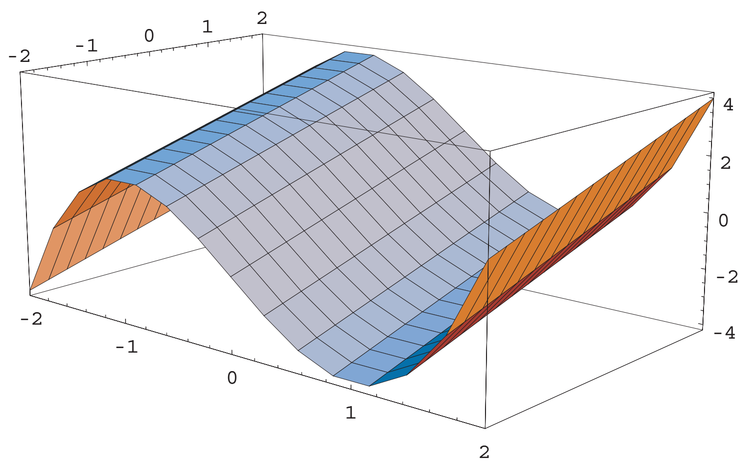
C13S02.046: The function $f(x, y, z) = x^2 + y^2 + 25$ is constant exactly when $x^2 + y^2$ is a nonnegative constant. Hence the level surfaces of f are circular cylinders concentric around the z-axis.

C13S02.047: The graph of $f(x, y) = x^3 + y^2$ should resemble vertical translates of $z = x^3$ in planes perpendicular to the y-axis and should resemble vertical translates of $z = y^2$ in planes perpendicular to the x-axis. Hence the graph must be the one shown in Fig. 13.2.32.

C13S02.048: The graph of $f(x, y) = 2x - y^2$ should resemble vertical translates of $z = 2x$ in planes perpendicular to the y-axis and resemble vertical translates of $z = -y^2$ in planes perpendicular to the x-axis. Hence the graph of f must be the one shown in Fig. 13.2.31.

C13S02.049: The graph of $f(x, y) = x^3 - 3x^2 + \frac{1}{2}y^2$ will resemble vertical translates of the cubic equation $z = x^3 - 3x^2 = x^2(x-3)$ in planes perpendicular to the y-axis and will resemble vertical translates of the parabola $z = \frac{1}{2}y^2$ in planes perpendicular to the x-axis. Therefore the graph of f does not appear among Figs. 13.3.27 through 13.3.32. It resembles slightly the graph in Fig.13.2.30, but that graph appears

to be linear in the y-direction, much as if it were the graph of $g(x, y) = \frac{1}{4}(5x^3 - 15x + 3y)$ instead. For comparison, the graph of $z = g(x, y)$ is shown next.



C13S02.050: The graph of $f(x, y) = x^2 - y^2$ should show us vertical translates of the parabola $z = x^2$ in planes perpendicular to the y-axis and vertical translates of the parabola $z = -y^2$ in planes perpendicular to the x-axis. Hence it must be the graph shown in Fig. 13.2.27.

C13S02.051: The graph of $f(x, y) = x^2 + y^4 - 4y^2$ will show vertical translates of the parabola $z = x^2$ in planes perpendicular to the y-axis and vertical translates of the quartic $z = y^4 - 4y^2$ in planes perpendicular to the x-axis. Thus this graph is the one shown in Fig. 13.2.28.

C13S02.052: The graph of $f(x, y) = 2y^3 - 3y^2 - 12y + x^2$ is shown in Fig. 13.2.29. The fact that the derivative of $g(y) = 2y^3 - 3y^2 - 12y$ is zero when $y = -1$ and when $y = 2$ accounts for the "waviness" of the figure in the y-direction.

C13S02.053: Figure 13.2.33 has the level curves shown in Fig. 13.2.41. Note how the level curves are close together where the graph is steep.

C13S02.054: Figure 13.2.34 has the level curves shown in Fig. 13.2.39. The latter clearly shows three peaks surrounded by level ground.

C13S02.055: Figure 13.2.35 has the level curves shown in Fig. 13.2.42. Moving outward from the center, Fig. 13.2.34 is alternately steep and almost flat; this is reflected in the behavior of the level curves, alternately close and far apart.

C13S02.056: Figure 13.2.36 has the level curves shown in Fig. 13.2.40. The latter indicates two peaks, or two deep holes, or one of each.

C13S02.057: Figure 13.2.57 has the level curves shown in Fig. 13.2.44. The latter indicates three peaks in a row, or two peaks with a deep hole between them, or some variation of this idea.

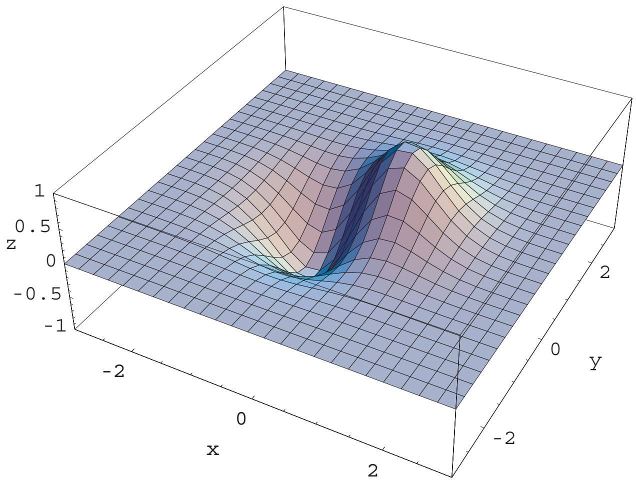
C13S02.058: Figure 13.2.38 has the level curves shown in Fig. 13.2.43. The latter indicates four peaks, or two peaks and two holes, or some variation of this theme.

C13S02.059: The *Mathematica* 3.0 commands

```
a = 2; b = 1;
```

```
Plot3D[ (a*x + b*y)*Exp[ −x∧2 − y∧2 ], { x, −3, 3 }, { y, −3, 3 },
    PlotRange → { −1, 1 }, PlotPoints → { 25, 25 },
    AxesLabel → { "x", "y", "z" } ];
```

produce the graph shown next.



Plots with various values of a and b (not both zero) indicate that the surface always has one pit and one peak, both lying on the same straight line in the xy-plane. The values of a and b determine the orientation of this line and the distances of the pit and the peak from the origin. —C.H.E.

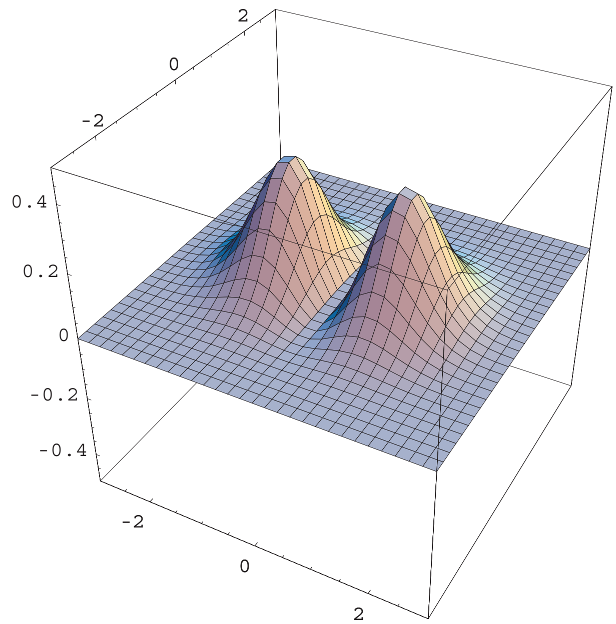
C13S02.060: The values of the parameters a, b, and c determine whether the graph is a rotated paraboloid that is elliptic opening upward, elliptic opening downward, or hyperbolic. Without loss of generality, we may assume that $b = 0$ so the paraboloid is not rotated. The *Mathematica* 3.0 commands

```
a = 1; b = 0; c = 0;

z = (a*x∧2 + b*x*y + c*y∧2)*Exp[ −x∧2 − y∧2 ];

Plot3D[ z, { x, −3, 3 }, { y, −3, 3 },
PlotPoints → { 30, 30 }, PlotRange → { −0.5, 0.5 },
BoxRatios → { 1, 1, 1 } ];
```

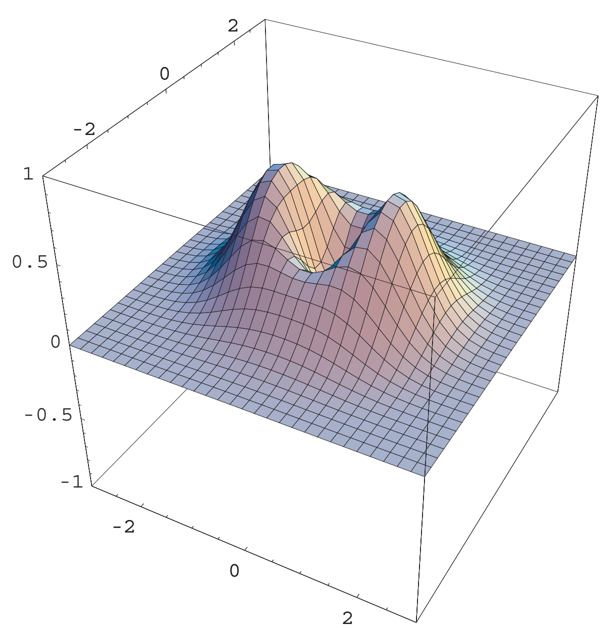
produce the following graph.



If the graph of the quadratic form is a parabolic paraboloid, we appear to see two peaks and no passes. Next, the *Mathematica* 3.0 commands `a = 2; b = 0; c = 1;`

```
z = (a∗x∧2 + b∗x∗y + c∗y∧2)∗Exp[ −x∧2 − y∧2 ];
Plot3D[ z, { x, −3, 3 }, { y, −3, 3 },
PlotPoints → { 30, 30 }, PlotRange → { −1, 1 },
BoxRatios → { 1, 1, 1 } ];
```

produce the following graph.



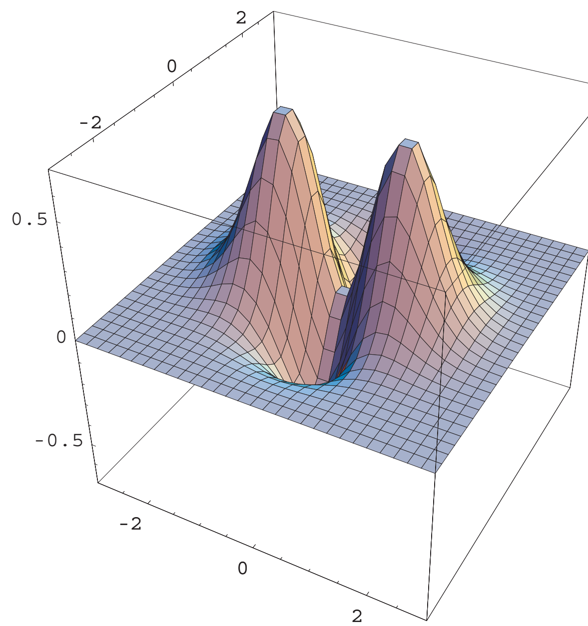
Hence if the graph of the quadratic form is an elliptic paraboloid, we appear to see two peaks separated by a pit and two passes. But if the paraboloid is circular, the pits and peaks coalesce, and we see a pit surrounded by a circular ridge. Finally, the *Mathematica* commands

```
a = 2; b = 0; c = −1;

z = a∗x∧2 + b∗x∗y + c∗y∧2)∗Exp[ −x∧2 − y∧2 ];

Plot3D[ z, { x, −3, 3 }, { y, −3, 3 },
PlotPoints → { 30, 30 }, PlotRange → { −0.7, 0.7 },
BoxRatios → { 1, 1, 1 } ];
```

produce the following graph.



So if the graph of the quadratic form is a hyperbolic paraboloid, we appears to see two peaks and two pits surrounding a pass. —C.H.E.

You should also try the case `a = 1; b = 0; c = 1` to see what happens.

C13S02.061: The *Mathematica* 3.0 commands

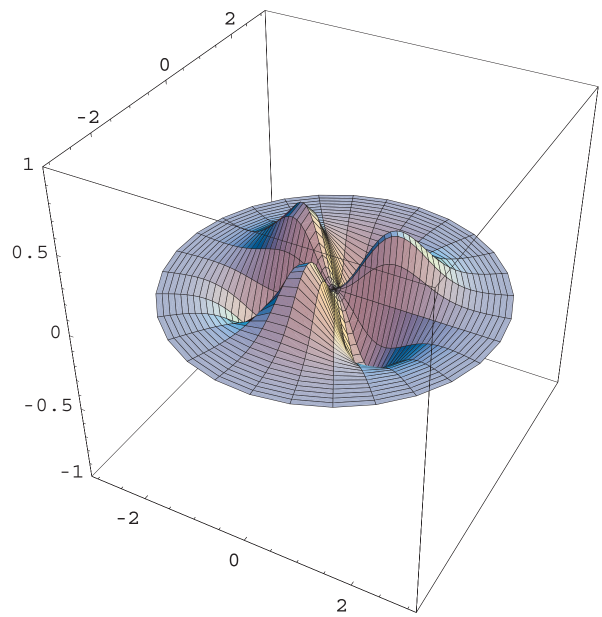
```
n = 3;

x = r∗Cos[t]; y = r∗Sin[t];

z = (r∧2)∗(Sin[ n∗t ])∗Exp[ −r∧2 ];

ParametricPlot3D[ { x, y, z }, { r, 0, 3 }, { t, 0, 2∗Pi },
  PlotPoints → { 30, 30 }, PlotRange → { −1, 1 },
  BoxRatios → { 1, 1, 1 } ];
```

produce the following graph.



Apparently n peaks and n pits alternately surround the origin. —C.H.E.

C13S02.062: The *Mathematica* 3.0 commands

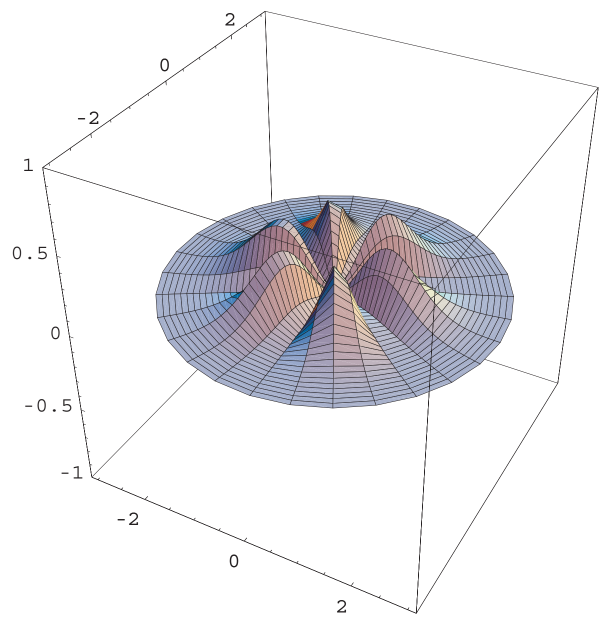
```
n = 3;

x = r*Cos[t]; y = r*Sin[t];

z = (r∧2)*((Cos[ n*t ])∧2)*Exp[ −r∧2 ];

ParametricPlot3D[ { x, y, z }, { r, 0, 3 }, { t, 0, 2*Pi },
  PlotPoints → { 30, 30 }, PlotRange → { −1, 1 },
  BoxRatios → { 1, 1, 1 } ];
```

produce the following graph.



Apparently $2n$ peaks (separated by passes) surround the origin. —C.H.E.

Section 13.3

C13S03.001: The limit laws in Eqs. (5), (6), and (7) and the result in Example 4 yield

$$\lim_{(x,y)\to(0,0)} \left(7 - x^2 + 5xy\right)$$
$$= \left(\lim_{(x,y)\to(0,0)} 7\right) - \left(\lim_{(x,y)\to(0,0)} x\right)^2 + 5\left(\lim_{(x,y)\to(0,0)} x\right)\cdot\left(\lim_{(x,y)\to(0,0)} y\right) = 7 - 0^2 + 5\cdot 0\cdot 0 = 7.$$

C13S03.002: Because $f(x,\, y) = 3x^2 - 4xy + 5y^2$ is a polynomial, it is continuous everywhere; then, by definition of continuity,

$$\lim_{(x,y)\to(1,-2)} f(x,\, y) = f(1, -2) = 3 + 8 + 20 = 31.$$

C13S03.003: The product and composition of continuous functions is continuous where defined, hence $f(x,\, y) = e^{-xy}$ is continuous everywhere. Then, by definition of continuity,

$$\lim_{(x,y)\to(1,-1)} f(x,\, y) = f(1, -1) = e^1 = e.$$

C13S03.004: The sum, product, and quotient of continuous functions is continuous where defined, so

$$f(x,\, y) = \frac{x+y}{1+xy}$$

is continuous where $xy \neq -1$. Therefore

$$\lim_{(x,y)\to(0,0)} f(x,\, y) = \frac{0+0}{1+0\cdot 0} = \frac{0}{1} = 0.$$

C13S03.005: The sum, product, and quotient of continuous functions is continuous where defined, so

$$f(x,\, y) = \frac{5-x^2}{3+x+y}$$

is continuous where $x + y \neq -3$. Therefore

$$\lim_{(x,y)\to(0,0)} f(x,\, y) = \frac{5-0^2}{3+0+0} = \frac{5}{3}.$$

C13S03.006: The sum, product, and quotient of continuous functions is continuous where defined, so

$$f(x,\, y) = \frac{9-x^2}{1+xy}$$

is continuous where $xy \neq -1$. Consequently

$$\lim_{(x,y)\to(2,3)} f(x,\, y) = \frac{9-2^2}{1+2\cdot 3} = \frac{5}{7}.$$

C13S03.007: The sum, product, and composition of continuous functions is continuous where defined, so

$$f(x,\, y) = \ln \sqrt{1 - x^2 - y^2}$$

is continuous if $x^2 + y^2 < 1$. Therefore

$$\lim_{(x,y)\to(0,0)} f(x,\, y) = f(0,\, 0) = \ln \sqrt{1 - 0^2 - 0^2} = \ln 1 = 0.$$

C13S03.008: The sum, product, and quotient of continuous functions is continuous where defined, so

$$f(x,\, y) = \ln \frac{1 + x + 2y}{3y^2 - x}$$

is continuous where

$$x \neq 3y^2 \qquad \text{and} \qquad \frac{1 + x + 2y}{3y^2 - x} > 0.$$

Hence

$$\lim_{(x,y)\to(2,-1)} f(x,\, y) = f(2, -1) = \ln \frac{1 + 2 - 2}{3 \cdot 1 - 2} = \ln 1 = 0.$$

C13S03.009: Let $z = x + 2y$ and $w = 3x + 4y$. Then $z \to 0$ and $w \to 0$ as $(x,\, y) \to (0,\, 0)$ by Example 4. Hence, by continuity of the natural exponential and cosine functions and the product law for limits,

$$\lim_{(x,y)\to(0,0)} e^{x+2y} \cos(3x + 4y) \;=\; e^0 \cos 0 \;=\; 1 \cdot 1 \;=\; 1.$$

C13S03.010: Convert to polar coordinates. Then by the limit laws of Chapter 2,

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x^2 + y^2)}{1 - x^2 - y^2} \;=\; \lim_{r\to 0} \frac{\cos r^2}{1 - r^2} \;=\; \frac{\cos 0}{1 - 0} \;=\; 1.$$

C13S03.011: Every polynomial is continuous everywhere, and hence every rational function (even of three variables) is continuous where its denominator is not zero. Therefore

$$f(x,\, y,\, z) = \frac{x^2 + y^2 + z^2}{1 - x - y - z}$$

is continuous where $x + y + z \neq 1$. Hence

$$\lim_{(x,y,z)\to(1,1,1)} f(x,\, y,\, z) = f(1, 1, 1) = \frac{1^2 + 1^2 + 1^2}{1 - 1 - 1 - 1} = -\frac{3}{2}.$$

C13S03.012: The sum, product, and composition of continuous functions is continuous where defined. Hence

$$f(x,\, y,\, z) = (x + y + z) \ln xyz$$

is continuous where $xyz > 0$. Hence

$$\lim_{(x,y,z)\to(1,1,1)} f(x,\, y,\, z) = f(1, 1, 1) = 3 \ln 1 = 3 \cdot 0 = 0.$$

C13S03.013: The sum, product, quotient, and composition of continuous functions is continuous where defined. Hence

$$f(x,\ y,\ z) = \frac{xy - z}{\cos xyz}$$

is continuous provided that xyz is not an odd integral multiple of $\pi/2$. Therefore

$$\lim_{(x,y,z)\to(1,1,0)} f(x,\ y,\ z) = f(1,1,0) = \frac{1\cdot 1 - 0}{\cos 0} = \frac{1}{1} = 1.$$

C13S03.014: The rational function

$$f(x,\ y,\ z) = \frac{x + y + z}{x^2 + y^2 + z^2}$$

is continuous wherever its denominator is not zero, so it is continuous at every point in space other than $(0,\ 0,\ 0)$. Thus

$$\lim_{(x,y,z)\to(2,-1,3)} f(x,\ y,\ z) = f(2,-1,3) = \frac{2 - 1 + 3}{2^2 + 1^2 + 3^2} = \frac{4}{14} = \frac{2}{7}.$$

C13S03.015: First,

$$f(z) = \tan\frac{3\pi z}{4}$$

is continuous provided that $3\pi z/4$ is not an odd integral multiple of $\pi/2$; that is, provided that z is not two-thirds of an odd integer. Hence $f(z)$ is continuous at $z = 1$. Next,

$$g(x,\ y) = \sqrt{xy}$$

is the composition of continuous functions, so $g(x,\ y)$ is continuous provided that $xy > 0$. So g is continuous at $(2,8)$. Finally, the product of continuous functions is continuous, so $h(x,\ y,\ z) = g(x,\ y)\cdot f(z)$ is continuous at $(2,8,1)$. Therefore

$$\lim_{(x,y,z)\to(2,8,1)} h(x,\ y,\ z) = h(2,8,1) = 16^{1/2}\cdot\tan\frac{3\pi}{4} = 4\cdot(-1) = -4.$$

C13S03.016: The sum, product, quotient, and composition of continuous functions is continuous where defined. Hence

$$f(x,\ y) = \arcsin\frac{xy}{\sqrt{x^2 + y^2}}$$

is continuous provided that $(x,\ y) \neq (0,0)$ and that

$$-1 < \frac{xy}{\sqrt{x^2 + y^2}} < 1.$$

Therefore

$$\lim_{(x,y)\to(1,-1)} f(x,\ y) = f(1,-1) = \arcsin\frac{1\cdot(-1)}{\sqrt{1+1}} = \arcsin\frac{-1}{\sqrt{2}} = -\frac{\pi}{4}.$$

C13S03.017: If $f(x, y) = xy$, then

$$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{xy + hy - xy}{h} = \lim_{h\to 0} \frac{hy}{h} = y \quad \text{and}$$

$$\lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} = \lim_{k\to 0} \frac{xy + kx - xy}{k} = \lim_{k\to 0} \frac{kx}{k} = x.$$

C13S03.018: If $f(x, y) = x^2 + y^2$, then

$$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + y^2 - x^2 - y^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = 2x \quad \text{and}$$

$$\lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} = \lim_{k\to 0} \frac{x^2 + (y+k)^2 - x^2 - y^2}{k} = \lim_{k\to 0} \frac{2ky + k^2}{k} = 2y.$$

C13S03.019: If $f(x, y) = xy^2 - 2$, then

$$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{(x+h)y^2 - 2 - xy^2 + 2}{h} = \lim_{h\to 0} \frac{hy^2}{h} = y^2 \quad \text{and}$$

$$\lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} = \lim_{k\to 0} \frac{x(y+k)^2 - 2 - xy^2 + 2}{k} = \lim_{k\to 0} \frac{2kxy + k^2x}{k} = 2xy.$$

C13S03.020: If $f(x, y) = x^2y^3 - 10$, then

$$f(x+h, y) - f(x, y) = (x+h)^2y^3 - 10 - x^2y^3 + 10 = 2hxy^3 + h^2y^3 \quad \text{and}$$

$$f(x, y+k) - f(x, y) = x^2(y+k)^3 - 10 - x^2y^3 + 10 = 3kx^2y^2 + 3k^2x^2y + k^3x^2.$$

Thus

$$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h\to 0} \frac{2hxy^3 + h^2y^3}{h} = 2xy^3 \quad \text{and}$$

$$\lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k} = \lim_{k\to 0} \frac{3kx^2y^2 + 3k^2x^2y + k^3x^2}{k} = 3x^2y^2.$$

C13S03.021: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{1 - xy}{1 + xy} = \frac{1 - 1\cdot 1}{1 + 1\cdot 1} = \frac{0}{2} = 0.$

C13S03.022: The limit

$$\lim_{(x,y)\to(2,-2)} \frac{4 - xy}{4 + xy}$$

does not exist because the numerator has limit 8 but the denominator has limit 0 as $(x, y) \to (2, -2)$.

C13S03.023: $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \frac{xyz}{yz + xz + xy} = \frac{1\cdot 1\cdot 1}{1\cdot 1 + 1\cdot 1 + 1\cdot 1} = \frac{1}{3}.$

C13S03.024: This limit does not exist. As $(x, y, z) \to (1, -1, 1)$, the numerator $yz + xz + xy$ has limit $-1 + 1 - 1 = -1 \neq 0$ but the denominator $1 + xyz$ has limit $1 - 1 = 0$.

C13S03.025: $\displaystyle\lim_{(x,y)\to(0,0)} \ln(1+x^2+y^2) = \ln(1+0+0) = \ln 1 = 0.$

C13S03.026: This limit does not exist. Let $z = 2 - x^2 - y^2$. Then $z \to 0$ through positive values as $(x, y) \to (0, 0)$, and hence

$$\lim_{(x,y)\to(1,1)} \ln(2 - x^2 - y^2) = \lim_{z\to 0^+} \ln z = -\infty.$$

C13S03.027: Convert to polar coordinates, then replace r^2 with z. Thus

$$\lim_{(x,y)\to(0,0)} \frac{\cot(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\cot r^2}{r^2} = \lim_{z\to 0^+} \frac{\cos z}{z \sin z},$$

which does not exist because $\cos z \to 1$ as $z \to 0^+$, but $z \sin z \to 0$. Because the last numerator is approaching $+\infty$ as $z \to 0^+$ but the last denominator is approaching zero through *positive* values, it is also correct to write

$$\lim_{(x,y)\to(0,0)} \frac{\cot(x^2+y^2)}{x^2+y^2} = +\infty.$$

C13S03.028: Because the sum and composition of continuous functions is continuous where defined,

$$\lim_{(x,y)\to(0,0)} \sin\big(\ln(1+x+y)\big) = \sin\big(\ln(1+0+0)\big) = \sin(\ln 1) = \sin 0 = 0.$$

C13S03.029: Convert to polar coordinates. Then

$$\lim_{(x,y)\to(0,0)} \exp\left(-\frac{1}{x^2+y^2}\right) = \lim_{r\to 0} \exp\left(-\frac{1}{r^2}\right) = \lim_{z\to\infty} e^{-z} = 0.$$

C13S03.030: Convert to polar coordinates. Then

$$\lim_{(x,y)\to(0,0)} \arctan\left(-\frac{1}{x^2+y^2}\right) = \lim_{r\to 0} \arctan\left(-\frac{1}{r^2}\right) = \lim_{z\to-\infty} \arctan z = -\frac{\pi}{2}.$$

C13S03.031: For continuity of $f(x, y) = \sqrt{x+y}$, we require $x + y > 0$. Thus f is continuous on the set of all points (x, y) that lie strictly above the graph of $y = -x$.

C13S03.032: For continuity of $f(x, y) = \arcsin(x^2 + y^2)$, we require

$$-1 < x^2 + y^2 < 1.$$

This condition will hold provided that $x^2 + y^2 < 1$, so f will be continuous on the points strictly within the unit circle centered at $(0, 0)$; that is, strictly within the circle with equation $x^2 + y^2 = 1$.

C13S03.033: For continuity of $f(x, y) = \ln(x^2 + y^2 - 1)$, we require that $x^2 + y^2 - 1 > 0$; that is, that $x^2 + y^2 > 1$. So f will be continuous on the points strictly outside the unit circle centered at $(0, 0)$; that is, strictly outside the circle with equation $x^2 + y^2 = 1$.

C13S03.034: For continuity of $f(x, y) = \ln(2x - y)$, we require that $2x - y > 0$ or, put another way, that $y < 2x$. Hence f will be continuous on the set of those points in the xy-plane that lie strictly below the straight line with equation $y = 2x$.

C13S03.035: Because the inverse tangent function is continuous on the set of all real numbers, the only discontinuity of

$$f(x,\,y) = \arctan\left(\frac{1}{x^2+y^2}\right)$$

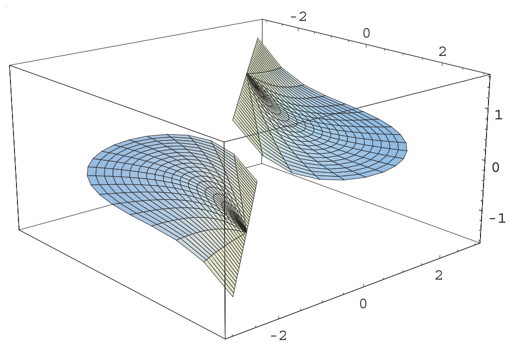
wlll occur when the denominator in the fraction is zero. Hence f is continuous on the set of all points in the xy-plane other than $(0,\,0)$. This discontinuity is removable because conversion to polar coordinates shows that

$$\lim_{(x,y)\to(0,0)} \arctan\left(\frac{1}{x^2+y^2}\right) = \lim_{r\to 0^+} \arctan\left(\frac{1}{r}\right) = \frac{\pi}{2}.$$

C13S03.036: Because the inverse tangent function is continuous on the set of all real numbers, the discontinuities of

$$f(x,\,y) = \arctan\left(\frac{1}{x+y}\right)$$

occur when (and only when) the denominator in the fraction is zero. Hence f is continuous on the set of all points in the xy-plane that do not lie on the line with equation $y = -x$. None of these discontinuities is removable. For example, as $(x,\,y) \to (0,\,0)$ through positive values along the line $y = x$, we find that $f(x,\,y) \to \pi/2$ because $1/(x+y) \to +\infty$. But as $(x,\,y) \to (0,\,0)$ through negative values along the line $y = x$, we find that $f(x,\,y) \to -\pi/2$ because $1/(x+y) \to -\infty$. The same argument holds for every other point on the line $y = -x$, so f has no limit at any of these points. Part of the graph of $z = f(x,\,y)$ is next.



C13S03.037: Using polar coordinates yields

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r} = \lim_{r\to 0} \frac{r^2\cos 2\theta}{r} = \lim_{r\to 0} r\cos 2\theta = 0.$$

C13S03.038: Use of polar coordinates yields

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0.$$

C13S03.039: Use of polar coordinates yields

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)^{3/2}} = \lim_{r\to 0} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^3} = \lim_{r\to 0} r(\cos^4\theta + \sin^4\theta) = 0.$$

C13S03.040: Using polar coordinates yields

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{\sin r}{r} = 1.$$

C13S03.041: Using spherical coordinates yields

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^3(\sin^2\phi\,\cos\phi\,\sin\theta\,\cos\theta)}{\rho^2} = \lim_{\rho\to 0} \rho(\sin^2\phi\,\cos\phi\,\sin\theta\,\cos\theta) = 0.$$

C13S03.042: Use of spherical coordinates yields

$$\lim_{(x,y,z)\to(0,0,0)} \arctan\frac{1}{x^2+y^2+z^2} = \lim_{\rho\to 0} \arctan\frac{1}{\rho^2} = \lim_{z\to\infty} \arctan z = \frac{\pi}{2}.$$

C13S03.043: The substitution $y = mx$ yields

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}.$$

This is the limit as $(x, y) \to (0, 0)$ along the straight line of slope m. Different values of m (such as 0 and 1) give different values for the limit (such as 1 and 0), and therefore the original limit does not exist (see the Remark following Example 9).

C13S03.044: The substitution $y = mx$ yields

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+x^2y^2+y^4} = \lim_{x\to 0} \frac{x^4(1-m^4)}{x^4(1+m^2+m^4)} = \lim_{x\to 0} \frac{1-m^4}{1+m^2+m^4} = \frac{1-m^4}{1+m^2+m^4}.$$

Hence if $(x, y) \to (0, 0)$ along the line $y = x$ (where $m = 0$) the limit is 1, whereas if $(x, y) \to (0, 0)$ along the line $y = x$ (where $m = 1$), the limit is zero. Therefore this limit does not exist.

C13S03.045: If $(x, y, z) \to (0, 0, 0)$ along the positive x-axis—where $y = z = 0$—we obtain

$$\lim_{x\to 0^+} \frac{x+y+z}{x^2+y^2+z^2} = \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Therefore the original limit does not exist.

C13S03.046: If $(x, y, z) \to (0, 0, 0)$ along the positive x-axis—where $y = z = 0$—we obtain

$$\lim_{x\to 0^+} \frac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim_{x\to 0^+} \frac{x^2}{x^2} = 1.$$

But if $(x, y, z) \to (0, 0, 0)$ along the positive z-axis—where $x = y = 0$—we obtain

$$\lim_{z\to 0^+} \frac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim_{z\to 0^+} \frac{-z^2}{z^2} = -1.$$

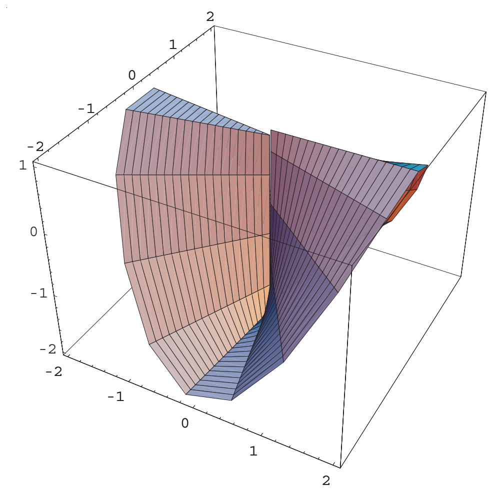
Therefore the original limit does not exist (see the Remark that follows Example 9).

C13S03.047: The graph that follows this solution suggests that as $(x, y) \to (0, 0)$ along straight lines of various slopes, the values of

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$$

range from -2 to 1, so that the limit in question does not exist. To be certain that it does not, let (x, y) approach $(0, 0)$ along the x-axis to get limit 1, then along the y-axis to get limit -2. The figure was generated by executing the *Mathematica* 3.0 command

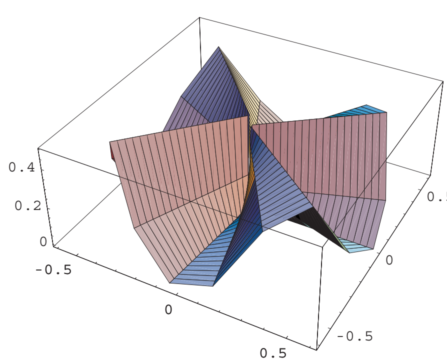
```
ParametricPlot3D[ { r*Cos[t], r*Sin[t], (Cos[t])∧2 - 2*(Sin[t])∧2 },
                  { t, 0, 2*Pi }, { r, 0.01, 2 } ];
```



C13S03.048: The graph that follows this solution suggests that as $(x, y) \to (0, 0)$ along straight lines of various slopes, the values of

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + y^4}$$

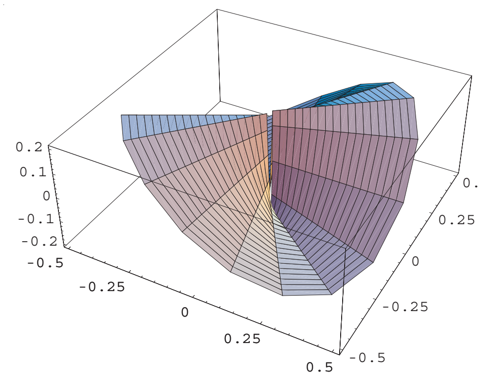
range from 0 to $\frac{1}{2}$, so that the limit in question does not exist. To be certain that it does not, let (x, y) approach $(0, 0)$ along the line $y = x$ to get limit $\frac{1}{2}$, then along the x-axis to get limit 0.



C13S03.049: The graph that follows this solution suggests that as $(x, y) \to (0, 0)$ along straight lines of various slopes, the values of

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2 + 3y^2}$$

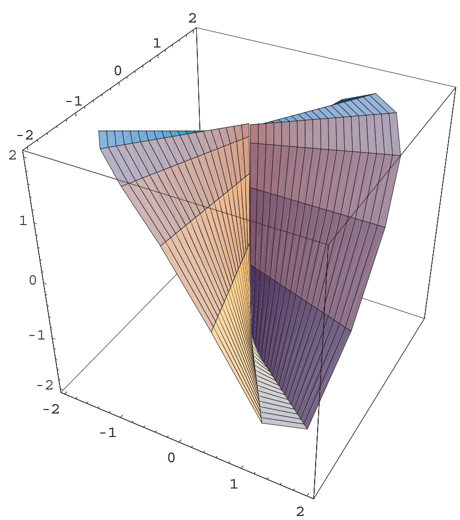
range from -0.2 to 0.2, so that the limit in question does not exist. To be sure that it does not, let (x, y) approach $(0, 0)$ along the line $y = x$ to get limit 0.2, then long the line $y = -x$ to get limit -0.2.



C13S03.050: The graph that follows this solution suggests that as $(x, y) \to (0, 0)$ along straight lines of various slopes, the values of

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 4xy + y^2}{x^2 + xy + y^2}$$

range from -2 to 2, so that the limit in question does not exist. To be certain of this, let (x, y) approach $(0, 0)$ along the line $y = x$ to get limit 2, then along the line $y = -x$ to get limit -2.



C13S03.051: Given:

$$f(x, y) = \frac{2x^2 y}{x^4 + y^2}.$$

Suppose that (x, y) approaches $(0, 0)$ along the nonvertical straight line with equation $y = mx$. If $m \neq 0$, then—on that line—

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} \frac{2mx^3}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{2mx}{x^2 + m^2} = \frac{0}{m^2} = 0.$$

Clearly if $m = 0$ the result is the same. And if (x, y) approaches $(0, 0)$ along the y-axis, then—on that line—

$$\lim_{(x,y)\to(0,0)} f(x,\,y) = \lim_{y\to 0} \frac{2\cdot 0\cdot y}{0+y^2} = 0.$$

Therefore as $(x,\,y)$ approaches $(0,\,0)$ along any straight line, the limit of $f(x,\,y)$ is zero. But on the curve $y = x^2$ we have

$$\lim_{(x,y)\to(0,0)} f(x,\,y) = \lim_{x\to 0} \frac{2x^4}{x^4+x^4} = 1,$$

and therefore the limit of $f(x,\,y)$ does not exist at $(0,\,0)$ (see the Remark following Example 9). For related paradoxical results involving functions of two variables, see Problem 60 of Section 13.5 and the miscellaneous problems of Chapter 13.

C13S03.052: Given:

$$f(x,\,y) = \begin{cases} \dfrac{x-y}{x^3-y} & \text{if} \quad y \neq x^3; \\[2ex] 1 & \text{if} \quad y = x^3. \end{cases}$$

Then as $(x,\,y)$ approaches $(1,\,1)$ along the horizontal line $y = 1$, we have

$$\lim_{(x,y)\to(1,1)} f(x,\,y) = \lim_{x\to 1} \frac{x-1}{x^3-1} = \lim_{x\to 1} \frac{1}{x^2+x+1} = \frac{1}{3}.$$

But as $(x,\,y)$ approaches $(1,\,1)$ along the vertical line $x = 1$, we have

$$\lim_{(x,y)\to(1,1)} f(x,\,y) = \lim_{y\to 1} \frac{1-y}{1-y} = 1.$$

Therefore $f(x,\,y)$ has no limit at $(x,\,y) = (1,\,1)$. Consequently f is not continuous there (and in fact we have shown that it has a nonremovable discontinuity there).

C13S03.053: Given:

$$f(x,\,y) = \frac{xy}{x^2+y^2}.$$

After we convert to polar coordinates, we have

$$f(r,\,\theta) = \frac{r^2\cos\theta\,\sin\theta}{r^2} = \cos\theta\,\sin\theta = \frac{1}{2}\sin 2\theta.$$

On the hyperbolic spiral $r\theta = 1$, we have $\theta \to +\infty$ as r approaches zero through positive values. Hence $f(r,\,\theta)$ takes on all values between $-\frac{1}{2}$ and $\frac{1}{2}$ infinitely often as $r \to 0^+$. Therefore, as we discovered in Example 9, $f(x,\,y)$ has no limit as $(x,\,y) \to (0,\,0)$.

C13S03.054: Let $z = xy$. By the discussions following Examples 4 and 5 and the "basic trigonometric limit" (Theorem 1 in Section 2.3),

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} = \lim_{z\to 0} \frac{\sin z}{z} = 1 = f(0,\,0).$$

Therefore f is continuous at $(0,\,0)$. By the discussion following Example 5, f is continuous at every other point of the xy-plane. Therefore f is continuous everywhere.

C13S03.055: Let $z = x^2 - y^2$. By the discussions following Examples 4 and 5 and the "basic trigonometric limit" (Theorem 1 in Section 2.3),

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 - y^2)}{x^2 - y^2} = \lim_{z\to 0} \frac{\sin z}{z} = 1 = f(0,\, 0).$$

Therefore f is continuous at $(0, 0)$. By the discussion following Example 5, f is continuous at every other point of the xy-plane. Therefore f is continuous everywhere.

C13S03.056: Let $w = xyz$. By the discussions following Examples 4 and 5 and the "basic trigonometric limit" (Theorem 1 in Section 2.3),

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sin xyz}{xyz} = \lim_{w\to 0} \frac{\sin w}{w} = 1 = f(0,\, 0,\, 0).$$

Therefore f is continuous at $(0, 0, 0)$. By the discussion following Example 5, f is continuous at every other point of xyz-space. Therefore f is continuous everywhere.

Section 13.4

C13S04.001: If $f(x, y) = x^4 - x^3y + x^2y^2 - xy^3 + y^4$, then

$$\frac{\partial f}{\partial x} = 4x^3 - 3x^2y + 2xy^2 - y^3 \quad \text{and}$$
$$\frac{\partial f}{\partial y} = -x^3 + 2x^2y - 3xy^2 + 4y^3.$$

C13S04.002: If $f(x, y) = x \sin y$, then

$$\frac{\partial f}{\partial x} = \sin y \quad \text{and} \quad \frac{\partial f}{\partial y} = x \cos y.$$

C13S04.003: If $f(x, y) = e^x(\cos y - \sin y)$, then

$$\frac{\partial f}{\partial x} = e^x(\cos y - \sin y) \quad \text{and} \quad \frac{\partial f}{\partial y} = -e^x(\cos y + \sin y).$$

C13S04.004: If $f(x, y) = x^2e^{xy}$, then

$$\frac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy} = xe^{xy}(xy + 2) \quad \text{and}$$
$$\frac{\partial f}{\partial y} = x^3e^{xy}.$$

C13S04.005: If $f(x, y) = \dfrac{x+y}{x-y}$, then

$$\frac{\partial f}{\partial x} = -\frac{2y}{(x-y)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{2x}{(x-y)^2}.$$

C13S04.006: If $f(x, y) = \dfrac{xy}{x^2+y^2}$, then

$$\frac{\partial f}{\partial x} = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

C13S04.007: If $f(x, y) = \ln(x^2 + y^2)$, then

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}.$$

C13S04.008: If $f(x, y) = (x - y)^{14}$, then

$$\frac{\partial f}{\partial x} = 14(x-y)^{13} \quad \text{and} \quad \frac{\partial f}{\partial y} = -14(x-y)^{13}.$$

C13S04.009: If $f(x, y) = x^y$, then

$$\frac{\partial f}{\partial x} = yx^{y-1} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^y \ln x.$$

C13S04.010: If $f(x, y) = \arctan xy$, then

$$\frac{\partial f}{\partial x} = \frac{y}{1 + x^2y^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x}{1 + x^2y^2}.$$

C13S04.011: If $f(x, y, z) = x^2y^3z^4$, then

$$\frac{\partial f}{\partial x} = 2xy^3z^4, \quad \frac{\partial f}{\partial y} = 3x^2y^2z^4, \quad \text{and} \quad \frac{\partial f}{\partial z} = 4x^2y^3z^3.$$

C13S04.012: If $f(x, y, z) = x^2 + y^3 + z^4$, then

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 3y^2, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = 4z^3.$$

C13S04.013: If $f(x, y, z) = e^{xyz}$, then

$$\frac{\partial f}{\partial x} = yze^{xyz}, \qquad \frac{\partial f}{\partial y} = xze^{xyz}, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = xye^{xyz}.$$

C13S04.014: If $f(x, y, z) = x^4 - 16yz$, then

$$\frac{\partial f}{\partial x} = 4x^3, \qquad \frac{\partial f}{\partial y} = -16z, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = -16y.$$

C13S04.015: If $f(x, y, z) = x^2e^y \ln z$, then

$$\frac{\partial f}{\partial x} = 2xe^y \ln z, \qquad \frac{\partial f}{\partial y} = x^2e^y \ln z, \qquad \text{and} \qquad \frac{\partial f}{\partial z} = \frac{x^2e^y}{z}.$$

C13S04.016: If $f(u, v) = (2u^2 + 3v^2)\exp(-u^2 - v^2)$, then

$$\frac{\partial f}{\partial u} = 4u\exp(-u^2 - v^2) - 2u(2u^2 + 3v^2)\exp(-u^2 - v^2) = -2u(2u^2 + 3v^2 - 2)\exp(-u^2 - v^2) \qquad \text{and}$$

$$\frac{\partial f}{\partial v} = 6v\exp(-u^2 - v^2) - 2v(2u^2 + 3v^2)\exp(-u^2 - v^2) = -2v(2u^2 + 3v^2 - 3)\exp(-u^2 - v^2).$$

C13S04.017: If $f(r, s) = \dfrac{r^2 - s^2}{r^2 + s^2}$. then

$$\frac{\partial f}{\partial r} = \frac{4rs^2}{(r^2 + s^2)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial s} = -\frac{4r^2s}{(r^2 + s^2)^2}.$$

C13S04.018: If $f(u, v) = e^{uv}(\cos uv + \sin uv)$, then

$$\frac{\partial f}{\partial u} = ve^{uv}(\cos uv + \sin uv) + ve^{uv}(\cos uv - \sin uv) = 2ve^{uv}\cos uv \qquad \text{and}$$

$$\frac{\partial f}{\partial v} = ue^{uv}(\cos uv + \sin uv) + ue^{uv}(\cos uv - \sin uv) = 2ue^{uv}\cos uv.$$

C13S04.019: If $f(u,\, v,\, w) = ue^v + ve^w + we^u$, then

$$\frac{\partial f}{\partial u} = we^u + e^v, \qquad \frac{\partial f}{\partial v} = ue^v + e^w, \qquad \text{and} \qquad \frac{\partial f}{\partial w} = e^u + ve^w.$$

C13S04.020: If $f(r,\, s,\, t) = (1 - r^2 - s^2 - t^2)\exp(-rst)$, then

$$\frac{\partial f}{\partial r} = -2re^{-rst} - st(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(r^2st + s^3t + st^3 - 2r - st),$$

$$\frac{\partial f}{\partial s} = -2se^{-rst} - rt(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(rs^2t + r^3t + rt^3 - 2s - rt), \qquad \text{and}$$

$$\frac{\partial f}{\partial t} = -2te^{-rst} - rs(1 - r^2 - s^2 - t^2)e^{-rst} = e^{-rst}(rst^2 + r^3s + rs^3 - 2t - rs).$$

C13S04.021: If $z(x,\, y) = x^2 - 4xy + 3y^2$, then

$$z_x(x,\, y) = 2x - 4y, \qquad z_y(x,\, y) = -4x + 6y,$$

$$z_{xy}(x,\, y) = -4, \qquad z_{yx}(x,\, y) = -4.$$

C13S04.022: If $z(x,\, y) = 2x^3 + 5x^2y - 6y^2 + xy^4$, then

$$z_x(x,\, y) = 6x^2 + 10xy + y^4, \qquad z_y(x,\, y) = 5x^2 - 12y + 4xy^3,$$

$$z_{xy}(x,\, y) = 10x + 4y^3, \qquad z_{yx}(x,\, y) = 10x + 4y^3.$$

C13S04.023: If $z(x,\, y) = x^2\exp(-y^2)$, then

$$z_x(x,\, y) = 2x\exp(-y^2), \qquad z_y(x,\, y) = -2x^2y\exp(-y^2),$$

$$z_{xy}(x,\, y) = -4xy\exp(-y^2), \qquad z_{yx}(x,\, y) = -4xy\exp(-y^2).$$

C13S04.024: If $z(x,\, y) = xy\exp(-xy)$, then

$$z_x(x,\, y) = y\exp(-xy) - xy^2\exp(-xy), \qquad z_y(x,\, y) = x\exp(-xy) - x^2y\exp(-xy),$$

$$\begin{aligned} z_{xy}(x,\, y) &= -xy\exp(-xy) - (xy-1)\exp(-xy) + xy(xy-1)\exp(-xy) \\ &= (x^2y^2 - 3xy + 1)\exp(-xy) = z_{yx}(x,\, y). \end{aligned}$$

C13S04.025: If $z(x,\, y) = \ln(x + y)$, then

$$z_x(x,\,y) = \frac{1}{x+y} = z_y(x,\,y) \quad \text{and} \quad z_{xy}(x,\,y) = -\frac{1}{(x+y)^2} = z_{yx}(x,\,y).$$

C13S04.026: If $z(x,\,y) = (x^3+y^3)^{10}$, then

$$z_x(x,\,y) = 30x^2(x^3+y^3)^9, \qquad z_y(x,\,y) = 30y^2(x^3+y^3)^9,$$
$$z_{xy}(x,\,y) = 810x^2y^2(x^3+y^3)^8, \qquad z_{yx}(x,\,y) = 810x^2y^2(x^3+y^3)^8,$$

C13S04.027: If $z(x,\,y) = e^{-3x}\cos y$, then

$$z_x(x,\,y) = -3e^{-3x}\cos y, \qquad z_y(x,\,y) = -e^{-3x}\sin y,$$
$$z_{xy}(x,\,y) = 3e^{-3x}\sin y, \qquad z_{yx}(x,\,y) = 3e^{-3x}\sin y.$$

C13S04.028: If $z(x,\,y) = (x+y)\sec xy$, then

$$\begin{aligned}
z_x(x,\,y) &= \sec xy + y(x+y)\sec xy\,\tan xy = (1 + xy\tan xy + y^2\tan xy)\sec xy,\\
z_y(x,\,y) &= \sec xy + x(x+y)\sec xy\,\tan xy = (1 + xy\tan xy + x^2\tan xy)\sec xy,\\
z_{xy}(x,\,y) &= (x^2y\sec^2 xy + xy^2\sec^2 xy + x\tan xy + 2y\tan xy)\sec xy\\
&\qquad + (1 + xy\tan xy + y^2\tan xy)x\sec xy\,\tan xy\\
&= (xy\sec^2 xy + xy\tan^2 xy + 2\tan xy)(x+y)\sec xy,\\
z_{yx}(x,\,y) &= (x^2y\sec^2 xy + xy^2\sec^2 xy + 2x\tan xy + y\tan xy)\sec xy\\
&\qquad + (1 + xy\tan xy + x^2\tan xy)y\sec xy\,\tan xy\\
&= (xy\sec^2 xy + xy\tan^2 xy + 2\tan xy)(x+y)\sec xy.
\end{aligned}$$

C13S04.029: If $z(x,\,y) = x^2\cosh\left(\dfrac{1}{y^2}\right)$, then

$$z_x(x,\,y) = 2x\cosh\left(\frac{1}{y^2}\right), \qquad z_y(x,\,y) = -\frac{2x^2}{y^3}\sinh\left(\frac{1}{y^2}\right),$$
$$z_{xy}(x,\,y) = -\frac{4x}{y^3}\sinh\left(\frac{1}{y^2}\right), \qquad z_{yx}(x,\,y) = -\frac{4x}{y^3}\sinh\left(\frac{1}{y^2}\right).$$

C13S04.030: If $z(x,\,y) = \sin xy + \arctan xy$, then

$$z_x(x,\,y) = \frac{y}{1+x^2y^2} + y\cos xy, \qquad z_y(x,\,y) = \frac{x}{1+x^2y^2} + x\cos xy,$$
$$z_{xy}(x,\,y) = \frac{1-x^2y^2}{(1+x^2y^2)^2} + \cos xy - xy\sin xy = z_{yx}(x,\,y).$$

C13S04.031: Given: $f(x, y) = x^2 + y^2$ and the point $P(3, 4, 25)$ on its graph. Then

$$f_x(x, y) = 2x; \qquad f_x(3, 4) = 6;$$

$$f_y(x, y) = 2y; \qquad f_y(3, 4) = 8.$$

Hence by Eq. (11) of Section 13.4, an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 25 = 6(x-3) + 8(y-4); \quad \text{that is,} \quad 6x + 8y - z = 25.$$

C13S04.032: Given: $f(x, y) = \sqrt{50 - x^2 - y^2}$ and the point $P(4, -3, 5)$ on its graph. Then

$$f_x(x, y) = -\frac{x}{\sqrt{50 - x^2 - y^2}}; \qquad f_y(x, y) = -\frac{y}{\sqrt{50 - x^2 - y^2}};$$

$$f_x(4, -3) = -\frac{4}{5}; \qquad f_y(4, -3) = \frac{3}{5}.$$

Hence by Eq. (11) of Section 13.4, an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 5 = -\frac{4}{5}(x-4) + \frac{3}{5}(y+3); \quad \text{that is,} \quad 4x - 3y + 5z = 50.$$

C13S04.033: Given: $f(x, y) = \sin \dfrac{\pi x y}{2}$ and the point $P(3, 5, -1)$ on its graph. Then

$$f_x(x, y) = \frac{\pi y}{2} \cos \frac{\pi x y}{2}; \qquad f_y(x, y) = \frac{\pi x}{2} \cos \frac{\pi x y}{2};$$

$$f_x(3, 5) = 0; \qquad f_y(3, 5) = 0.$$

The plane tangent to the graph of $z = f(x, y)$ at the point P is horizontal, so its has equation $z = -1$.

C13S04.034: Given: $f(x, y) = \dfrac{4}{\pi} \arctan xy$ and the point $P(1, 1, 1)$ on its graph. Then

$$f_x(x, y) = \frac{4y}{\pi(1 + x^2y^2)}; \qquad f_y(x, y) = \frac{4x}{\pi(1 + x^2y^2)};$$

$$f_x(1, 1) = \frac{2}{\pi}; \qquad f_y(1, 1) = \frac{2}{\pi}.$$

Then by Eq. (11), an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 1 = \frac{2}{\pi}(x-1) + \frac{2}{\pi}(y-1); \quad \text{that is,} \quad 2x + 2y - \pi z = 4 - \pi.$$

C13S04.035: Given: $f(x, y) = x^3 - y^3$ and the point $P(3, 2, 19)$ on its graph. Then $f_x(x, y) = 3x^2$ and $f_y(x\,y) = -3y^2$, so $f_x(3, 2) = 27$ and $f_y(3, 2) = -12$. So by Eq. (11) an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 19 = 27(x-3) - 12(y-2); \quad \text{that is,} \quad 27x - 12y - z = 38.$$

C13S04.036: The graph of the given equation $z = 3x + 4y$ is a plane, so it is its own tangent plane and the coordinates of the point of tangency don't matter. Answer: An equation of the plane tangent to the graph of $z = 3x + 4y$ at the point P is $z = 3x + 4y$.

C13S04.037: Given: $f(x, y) = xy$ and the point $P(1, -1, -1)$ on its graph. Then $f_x(x, y) = y$ and $f_y(x, y) = x$, so $f_x(1, -1) = -1$ and $f_y(1, -1) = 1$. By Eq. (11) an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is $z + 1 = -(x - 1) + (y - 1)$; that is, $x - y + z = 1$.

C13S04.038: Given: $f(x, y) = \exp(-x^2 - y^2)$ and the point $P(0, 0, 1)$ on its graph. Then

$$f_x(x, y) = -2x\exp(-x^2 - y^2); \qquad f_y(x, y) = -2y\exp(-x^2 - y^2);$$
$$f_x(0, 0) = 0; \qquad f_y(0, 0) = 0.$$

This plane is horizontal. Therefore its equation is $z = 1$.

C13S04.039: Given: $f(x, y) = x^2 - 4y^2$ and the point $P(5, 2, 9)$ on its graph. Then $f_x(x, y) = 2x$ and $f_y(x, y) = -8y$, so $f_x(5, 2) = 10$ and $f_y(5, 2) = -16$. By Eq. (11) an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 9 = 10(x - 5) - 16(y - 2); \quad \text{that is,} \quad 10x - 16y - z = 9.$$

C13S04.040: Given: $f(x, y) = (x^2 + y^2)^{1/2}$ and the point $P(3, -4, 5)$ on its graph. Then

$$f_x(x, y) = \frac{x}{(x^2 + y^2)^{1/2}}; \qquad f_y(x, y) = \frac{y}{(x^2 + y^2)^{1/2}};$$
$$f_x(3, -4) = \frac{3}{5}; \qquad f_y(3, -4) = -\frac{4}{5}.$$

By Eq. (11) an equation of the plane tangent to the graph of $z = f(x, y)$ at the point P is

$$z - 5 = \frac{3}{5}(x - 3) - \frac{4}{5}(y + 4); \quad \text{that is,} \quad 3x - 4y - 5z = 0.$$

C13S04.041: If $f_x(x, y) = 2xy^3$ and $f_y(x, y) = 3x^2y^2$, then

$$f_{xy}(x, y) = 6xy^2 = f_{yx}(x, y).$$

In Section 15.3 we will find that there must exist a function f having the given partial derivatives. Here we can find one by inspection; it is $f(x, y) = x^2y^3$.

C13S04.042: If $f_x(x, y) = 5xy + y^2$ and $f_y(x, y) = 3x^2 + 2xy$, then

$$f_{xy}(x, y) = 5x + 2y \neq 6x + 2y = f_{yx}(x, y).$$

Hence by the *Note* preceding and following Eq. (14), there can be no function $f(x, y)$ having the given first-order partial derivatives.

C13S04.043: If $f_x(x, y) = \cos^2 xy$ and $f_y(x, y) = \sin^2 xy$, then

$$f_{xy}(x, y) = -2x\sin xy\,\cos xy \neq -2y\sin xy\,\cos xy = f_{yx}(x, y).$$

By the *Note* preceding and following Eq. (14), there can be no function $f(x, y)$ having the given first-order partial derivatives.

C13S04.044: Given $f_x(x, y) = \cos x \sin y$ and $f_y(x, y) = \sin x \cos y$, we find that

$$f_{xy}(x, y) = \cos x \cos y = f_{yx}(x, y).$$

So it's not impossible that there exists a function $f(x, y)$ having the given first-order partial derivatives. Indeed, by inspection, one such function is $f(x, y) = \sin x \sin y$.

C13S04.045: The graph of $z = f(x, y)$ is shown in Fig. 13.4.14. The key to solving this group of six problems is first to locate f. Do this by sketching cross sections of each graph parallel to the xz- and yz-planes. This will make it easy to eliminate all but one candidate for the graph of f. Then the same sketches—the ones you created for the graph that turned out to be the graph of f—will quickly identify the other five graphs.

C13S04.046: The graph of $z = f_x(x, y)$ is shown in Fig. 13.4.17.

C13S04.047: The graph of $z = f_y(x, y)$ is shown in Fig. 13.4.13.

C13S04.048: The graph of $z = f_{xx}(x, y)$ is shown in Fig. 13.4.12.

C13S04.049: The graph of $z = f_{xy}(x, y)$ is shown in Fig. 13.4.15.

C13S04.050: The graph of $z = f_{yy}(x, y)$ is shown in Fig. 13.4.16.

C13S04.051: If m and n are positive integers and $f(x, y) = x^m y^n$, then $f_x(x, y) = mx^{m-1}y^n$ and $f_y(x, y) = nx^m y^{n-1}$. Hence $f_{xy}(x, y) = mnx^{m-1}y^{n-1} = f_{yx}(x, y)$.

C13S04.052: If $f(x, y) = e^{x+y}$, then $f_x(x, y) = e^{x+y}$ and $f_y(x, y) = e^{x+y}$. Therefore all higher-order partial derivatives must also be equal to e^{x+y}.

C13S04.053: If $f(x, y, z) = e^{xyz}$, then

$$f_x(x, y, z) = yze^{xyz}, \quad f_y(x, y, z) = xze^{xyz}, \quad \text{and} \quad f_z(x, y, z) = xye^{xyz}.$$

Therefore

$$\begin{aligned}
&f_{xx}(x, y, z) = y^2z^2e^{xyz}, \quad f_{xy}(x, y, z) = f_{yx}(x, y, z) = (xyz^2 + z)e^{xyz},\\
&f_{xz}(x, y, z) = f_{zx}(x, y, z) = (y + xy^2z)e^{xyz}, \quad f_{yz}(x, y, z) = f_{zy}(x, y, z) = (x + x^2yz)e^{xyz},\\
&f_{yy}(x, y, z) = x^2z^2e^{xyz}, \quad f_{zz}(x, y, z) = x^2y^2e^{xyz}, \quad \text{and}\\
&f_{xyz}(x, y, z) = (1 + 3xyz + x^2y^2z^2)e^{xyz}.
\end{aligned}$$

C13S04.054: If $g(x, y) = \sin xy$, then

$$\begin{aligned}
g_x(x, y) &= y\cos xy, & g_y(x, y) &= x\cos xy,\\
g_{xy}(x, y) &= \cos xy - xy\sin xy, & g_{yx}(x, y) &= \cos xy - xy\sin xy,
\end{aligned}$$

$$g_{xx}(x,\,y) = -y^2 \sin xy, \qquad g_{xxy}(x,\,y) = -xy^2 \cos xy - 2y \sin xy,$$

$$g_{xyx}(x,\;\; y) = -xy^2 \cos xy - 2y \sin xy, \qquad g_{yxx}(x,\,y) = -xy^2 \cos xy - 2y \sin xy.$$

C13S04.055: If $u(x,\,t) = \exp(-n^2 kt) \sin nx$ where n and k are constants, then

$$u_t(x,\,t) = -n^2 k \exp(-n^2 kt) \sin nx, \quad u_x(x,\,t) = n \exp(-n^2 kt) \cos nx,$$

$$\text{and} \quad u_{xx}(x,\,t) = -n^2 \exp(-n^2 kt) \sin nx.$$

Therefore $u_t = k u_{xx}$ for any choice of the constants k and n.

C13S04.056: If m and n are constants and

$$u(x,\,y,\,t) = \exp\big(-(m^2 + n^2)kt\big) \sin mx \,\cos ny,$$

then

$$u_t(x,\,y,\,t) = -k(m^2 + n^2) \exp\big(-(m^2 + n^2)kt\big) \sin mx \,\cos ny,$$

$$u_x(x,\,y,\,t) = m \exp\big(-(m^2 + n^2)kt\big) \cos mx \,\cos ny,$$

$$u_y(x,\,y,\,t) = -n \exp\big(-(m^2 + n^2)kt\big) \sin mx \,\sin ny,$$

$$u_{xx}(x,\,y,\,t) = -m^2 \exp\big(-(m^2 + n^2)kt\big) \sin mx \,\cos ny, \qquad \text{and}$$

$$u_{yy}(x,\,y,\,t) = -n^2 \exp\big(-(m^2 + n^2)kt\big) \sin mx \,\cos ny.$$

Therefore $u_t = k(u_{xx} + u_{yy})$ for any choice of the constants m and n.

C13S04.057: Part (a): If $y(x,\,t) = \sin(x + at)$ (where a is a constant), then

$$y_t(x,\,t) = a \cos(x + at), \qquad y_x(x,\,t) = \cos(x + at),$$

$$y_{tt}(x,\,t) = -a^2 \sin(x + at), \qquad y_{xx}(x,\,t) = -\sin(x + at).$$

Therefore $y_{tt} = a^2 y_{xx}$.

Part (b): If $y(x,\,t) = \cosh\big(3(x - at)\big)$, then

$$y_t(x,\,t) = -3a \sinh\big(3(x - at)\big), \qquad y_x(x,\,t) = 3 \sinh\big(3(x - at)\big),$$

$$y_{tt}(x,\,t) = 9a^2 \cosh\big(3(x - at)\big), \qquad y_{xx}(x,\,t) = 9 \cosh\big(3(x - at)\big).$$

Therefore $y_{tt} = a^2 y_{xx}$.

Part (c): If $y(x,\,t) = \sin kx \,\cos kat$ (where k is a constant), then

$$y_t(x,\,t) = -ka \sin kx \,\sin kat, \qquad y_x(x,\,t) = k \cos kx \,\cos kat,$$

$$y_{tt}(x,\,t) = -k^2 a^2 \sin kx \,\cos kat, \qquad y_{xx}(x,\,t) = -k^2 \sin kx \cos kat.$$

Therefore $y_{tt} = a^2 y_{xx}$.

C13S04.058: Part (a): If $u(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$, then

$$u_x(x, y) = \frac{x}{x^2 + y^2}, \qquad u_{xx}(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$u_y(x, y) = \frac{y}{x^2 + y^2}, \qquad u_{yy}(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Therefore $u_{xx} + u_{yy} = 0$.

Part (b): If $u(x, y) = \sqrt{x^2 + y^2}$, then

$$u_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \qquad u_{xx}(x, y) = \frac{y^2}{(x^2 + y^2)^{3/2}},$$

$$u_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}, \qquad u_{yy}(x, y) = \frac{x^2}{(x^2 + y^2)^{3/2}}.$$

Therefore $u_{xx} + u_{yy} \neq 0$, and thus u does not satisfy Laplace's equation.

Part (c): If $u(x, y) = \arctan(y/x)$, then

$$u_x(x, y) = -\frac{y}{x^2 + y^2}, \qquad u_{xx}(x, y) = \frac{2xy}{(x^2 + y^2)^2},$$

$$u_y(x, y) = \frac{x}{x^2 + y^2}, \qquad u_{yy}(x, y) = -\frac{2xy}{(x^2 + y^2)^2}.$$

Therefore $u_{xx} + u_{yy} = 0$, and thus u satisfies Laplace's equation.

Part (d): If $u(x, y) = e^{-x} \sin y$, then

$$u_x(x, y) = -e^{-x} \sin y, \qquad u_{xx}(x, y) = e^{-x} \sin y,$$

$$u_y(x, y) = e^{-x} \cos y, \qquad u_{yy}(x, y) = -e^{-x} \sin y.$$

Therefore $u_{xx} + u_{yy} = 0$, and thus u satisfies Laplace's equation.

C13S04.059: Given: f and g are twice-differentiable functions of a single variable, a is a constant, and $y(x, t) = f(x + at) + g(x - at)$. Then

$$y_t(x, t) = af'(x + at) - ag'(x - at), \qquad y_{tt}(x, t) = a^2 f''(x + at) + a^2 g''(x - at),$$

$$y_x(x, t) = f'(x + at) + g'(x - at), \qquad y_{xx}(x, t) = f''(x + at) + g''(x - at).$$

It's now clear that $y(x, t)$ satisfies the wave equation $y_{tt} = a^2 y_{xx}$.

C13S04.060: Given: The constant q and the function

$$\phi(x,\,y,\,z) = \frac{q}{\sqrt{x^2+y^2+z^2}}.$$

Then

$$\phi_x(x,\,y,\,z) = -\frac{qx}{(x^2+y^2+z^2)^{3/2}} \quad \text{and}$$

$$\phi_{xx}(x,\,y,\,z) = \frac{q(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^{5/2}}.$$

By the symmetries in ϕ among x, y, and z, it follows that

$$\phi_{yy}(x,\,y,\,z) = \frac{q(2y^2-x^2-z^2)}{(x^2+y^2+z^2)^{5/2}} \quad \text{and} \quad \phi_{zz}(x,\,y,\,z) = \frac{q(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^{5/2}}.$$

It is now clear that ϕ satisfies the three-dimensional Laplace equation $\phi_{xx}+\phi_{yy}+\phi_{zz}=0$.

C13S04.061: Given:

$$u(x,\,t) = T_0 + a_0 \exp\left(-x\sqrt{\omega/2k}\,\right)\cos\left(\omega t - x\sqrt{\omega/2k}\,\right)$$

where T_0, a_0, ω, and k are constants. First note that

$$u(0,\,t) = T_0 + a_0 e^0 \cos(\omega t - 0) = T_0 + a_0\cos\omega t.$$

Next,

$$u_t(x,\,t) = -a_0\omega\exp\left(-x\sqrt{\omega/2k}\,\right)\sin\left(\omega t - x\sqrt{\omega/2k}\,\right),$$

$$u_x(x,\,t) = -a_0\left(\sqrt{\omega/2k}\,\right)\exp\left(-x\sqrt{\omega/2k}\,\right)\left[\cos\left(\omega t - x\sqrt{\omega/2k}\,\right) - \sin\left(\omega t - x\sqrt{\omega/2k}\,\right)\right], \quad \text{and}$$

$$u_{xx}(x,\,t) = -\frac{a_0\omega}{k}\exp\left(-x\sqrt{\omega/2k}\,\right)\sin\left(\omega t - x\sqrt{\omega/2k}\,\right).$$

Therefore $y(x,\,t)$ satisfies the one-dimensional heat equation $u_t = k u_{xx}$.

C13S04.062: For simplicity replace R with u, R_1 with x, R_2 with y, and R_3 with z. Then

$$u = u(x,\,y,\,z) = (x^{-1}+y^{-1}+z^{-1})^{-1}.$$

Therefore

$$\frac{\partial u}{\partial x} = (-1)(x^{-1}+y^{-1}+z^{-1})^{-2}(-1)x^{-2} = x^{-2}(x^{-1}+y^{-1}+z^{-1})^{-2}.$$

Using the symmetry of the appearance of x, y, and z in the expression for u, we have

$$\frac{\partial u}{\partial y} = y^{-2}(x^{-1}+y^{-1}+z^{-1})^{-2} \quad \text{and}$$

$$\frac{\partial u}{\partial z} = z^{-2}(x^{-1}+y^{-1}+z^{-1})^{-2}.$$

Thus

$$u_x + u_y + u_z = (x^{-2} + y^{-2} + z^{-2})(x^{-1} + y^{-1} + z^{-1})^{-2},$$

and the solution is complete.

C13S04.063: Given $pV = nRT$ where n and R are constants, solve for p, V, and T in turn; then compute

$$\frac{\partial p}{\partial V} = -\frac{nRT}{V^2},$$
$$\frac{\partial V}{\partial T} = \frac{nR}{p}, \quad \text{and}$$
$$\frac{\partial T}{\partial p} = \frac{V}{nR}.$$

Then

$$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{nRT}{pV} = -1.$$

C13S04.064: Let $f(x, y) = \sqrt{x^2 + y^2}$. Then

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_x(a, b) = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and}$$
$$f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}, \quad f_y(a, b) = \frac{b}{\sqrt{a^2 + b^2}}.$$

By Eq. (11), an equation of the plane tangent to the cone $z = f(x, y)$ at the point (a, b) is

$$z - f(a, b) = \frac{a(x - a)}{\sqrt{a^2 + b^2}} + \frac{b(y - b)}{\sqrt{a^2 + b^2}},$$

which after simplifications becomes

$$z = \frac{ax + by}{\sqrt{a^2 + b^2}}.$$

Therefore this plane passes through the origin. Repeat the argument with $g(x, y) = -\sqrt{x^2 + y^2}$ for the case of the lower nappe of the cone.

C13S04.065: If $f(x, y) = x^2 + 2xy + 2y^2 - 6x + 8y$, then the equations

$$f_x(x, y) = 0, \quad f_y(x, y) = 0$$

are

$$2x + 2y - 6 = 0, \quad 2x + 4y = -8,$$

which have the unique solution $x = 10$, $y = -7$. So the surface that is the graph of $z = f(x, y)$ contains exactly one point at which the tangent plane is horizontal, and that point is $(10, -7, -58)$.

C13S04.066: Let $P(a,\, b,\, c)$ be a point on the paraboloid with equation $z = x^2 + y^2$. Note that $c = a^2 + b^2$, $z_x(a,\, b) = 2a$, and $z_y(a,\, b) = 2b$. Thus the plane tangent to the paraboloid at P has equation

$$2a(x - a) + 2b(y - b) = z - c.$$

Substitution of $a^2 + b^2$ for c then yields the equation

$$2ax + 2by = z + a^2 + b^2.$$

In the xy-plane we have $z = 0$, so the tangent plane meets the xy-plane in the line L with equation

$$2ax + 2by = a^2 + b^2.$$

Now $(x,\, y) = (a/2,\, b/2)$ satisfies this equation, so the point $Q(a/2,\, b/2)$ is on L. The slope of L is $-a/b$, so a normal to L has slope b/a. The segment OQ has this slope, so OQ is a perpendicular from the origin O to this line. Its length is $r = \frac{1}{2}\sqrt{a^2 + b^2}$, so L is tangent to the circle with center O and radius r. The equation of that circle is therefore $x^2 + y^2 = \frac{1}{4}(a^2 + b^2)$, and the solution is complete.

C13S04.067: Given: van der Waals' equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = (82.06)T$$

where p denotes pressure (in atm), V volume (in cm^3), and T temperature (in K). For CO_2, the empirical constants are $a = 3.59 \times 10^6$ and $b = 42.7$. We are also given $V = 25600$ when $p = 1$ and $T = 313$. Part (a): We differentiate with respect to p while holding T constant. We obtain

$$\begin{aligned}
&\left(1 - \frac{2a}{V^3}V_p\right)(V - b) + \left(p + \frac{a}{V^2}\right)\cdot V_p = 0;\\
&V - b - \frac{2a}{V^2}V_p + \frac{2ab}{V^3}V_p + pV_p + \frac{a}{V^2}V_p = 0;\\
&\left(\frac{a}{V^2} - \frac{2ab}{V^3} - p\right)V_p = V - b;\\
&\frac{aV - 2ab - pV^3}{V^3}V_p = V - b;\\
&V_p = \frac{V^3(V - b)}{aV - 2ab - pV^3}.
\end{aligned}$$

If p is changed from 1 to 1.1, then the resulting change in V will be predicted by

$$\Delta V \approx V_p\,\Delta p = \frac{1}{10}V_p,$$

and substitution of the data $V = 25600$, $p = 1$ yields $\Delta V \approx -2569.76$ (cm^3).

Part (b): We hold p constant and differentiate both sides of van der Waals' equation with respect to T. The result:

$$\begin{aligned}
&\left(p + \frac{a}{V^2}\right)V_T - \frac{2a}{V^3}(V - b)V_T = 82.06;\\
&\left(pV^3 + aV\right)V_T - 2a(V - b)V_T = (82.06)V^3;\\
&V_T = \frac{(82.06)V^3}{pV^3 - aV + 2ab}.
\end{aligned}$$

If T is changed from 313 to 314 while the pressure is maintained at the constant value $p = 1$, the resulting change in V is predicted by

$$\Delta V \approx V_T \,\Delta T = 1 \cdot V_T.$$

Substitution of the data $V = 25600$, $p = 1$, $T = 313$ yields $\Delta V \approx 82.5105$.

The exact values of the two answers are closer to -2347.60 and 69.9202. Part of the discrepancy is caused by the fact that $V = 25600$ is itself only an approximation; the true value is closer to 25669.920232. Reworking parts (a) and (b) with this value of V yields the different approximations -2557.17 and 82.5095.

C13S04.068: If $z(x, y) = \ln(\cos x) - \ln(\cos y)$, then

$$\begin{aligned} z_x(x, y) &= -\tan x, & z_y(x, y) &= \tan y, \\ z_{xx}(x, y) &= -\sec^2 x, & z_{xy}(x, y) &\equiv 0, \quad \text{and} \\ z_{yy}(x, y) &= \sec^2 y. \end{aligned}$$

Therefore

$$(1 + z_y^2)z_{xx} - zz_xz_yz_{xy} + (1 + z_x^2)z_{yy} = -\sec^2 x\, \sec^2 y - 0 + \sec^2 x\, \sec^2 y \equiv 0.$$

C13S04.069: Part (a): If $f(x, y) = \sin x\, \sinh(\pi - y)$, then

$$\begin{aligned} f_x(x, y) &= \cos x\, \sinh(\pi - y), & f_{xx}(x, y) &= -\sin x\, \sinh(\pi - y), \\ f_y(x, y) &= -\sin x\, \cosh(\pi - y), & f_{yy}(x, y) &= \sin x\, \sinh(\pi - y). \end{aligned}$$

Clearly f is harmonic.

Part (b): If $f(x, y) = \sinh 2x\, \sin 2y$, then

$$\begin{aligned} f_x(x, y) &= 2\cosh 2x\, \sin 2y, & f_{xx}(x, y) &= 4\sinh 2x\, \sin 2y, \\ f_y(x, y) &= 2\sinh 2x\, \cos 2y, & f_{yy}(x, y) &= -4\sinh 2x\, \sin 2y. \end{aligned}$$

Clearly f is harmonic.

Part (c): If $f(x, y) = \sin 3x\, \sinh 3y$, then

$$\begin{aligned} f_x(x, y) &= 3\cos 3x\, \sinh 3y, & f_{xx}(x, y) &= -9\sin 3x\, \sinh 3y, \\ f_y(x, y) &= 3\sin 3x\, \cosh 3y, & f_{yy}(x, y) &= 9\sin 3x\, \sinh 3y. \end{aligned}$$

Again it is clear that f is harmonic.

Part (d): If $f(x, y) = \sinh 4(\pi - x)\, \sin 4y$, then

$$\begin{aligned} f_x(x, y) &= -4\cosh 4(\pi - x)\, \sin 4y, & f_{xx}(x, y) &= 16\sinh 4(\pi - x)\, \sin 4y, \\ f_y(x, y) &= 4\sinh 4(\pi - x)\, \cos 4y, & f_{yy}(x, y) &= -16\sinh 4(\pi - y)\, \sin 4y. \end{aligned}$$

Therefore f is harmonic.

C13S04.070: The sum of two harmonic functions is harmonic. Here is a proof for the case of functions of two variables. Suppose that f and g are harmonic. Then

$$f_{xx} + f_{yy} = 0 \qquad \text{and} \qquad g_{xx} + g_{yy} = 0.$$

Let $h(x, y) = f(x, y) + g(x, y)$. Then

$$h_{xx} + h_{yy} = f_{xx} + g_{xx} + f_{yy} + g_{yy} = f_{xx} + f_{yy} + g_{xx} + g_{yy} = 0 + 0 = 0.$$

Therefore $f + g$ is harmonic. This concludes the proof. By induction, you may extend it to show that the sum of any finite number of harmonic functions is harmonic. This answers the question in Problem 70.

C13S04.071: If $f(x, y) = 100 + \dfrac{1}{100}(x^2 - 3xy + 2y^2)$, then

$$f_x(x, y) = \frac{1}{100}(2x - 3y), \qquad f_y(x, y) = \frac{1}{100}(4y - 3x),$$
$$f_x(100, 100) = -\frac{100}{100} = -1, \qquad f_y(100, 100) = \frac{100}{100} = 1.$$

Part (a): You will initially be descending at a 45° angle. Part (b): You will initially be ascending at a 45° angle.

C13S04.072: If $f(x, y) = 1000 + \dfrac{1}{1000}(3x^2 - 5xy + y^2)$, then

$$f_x(x, y) = \frac{1}{1000}(6x - 5y), \qquad f_y(x, y) = \frac{1}{1000}(2y - 5x),$$
$$f_x(150, 250) = -\frac{7}{20}, \qquad f_y(150, 250) = -\frac{1}{4}.$$

Part (a): You will initially be descending at an angle of $-\arctan\left(\frac{7}{20}\right)$ radians, about $-19°\ 17'\ 24.166''$. Part (b): You will initially be descending at an angle of $-\arctan\left(\frac{1}{4}\right)$ radians, approximately $-14°\ 2'\ 10.476''$.

C13S04.073: Given:

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0), \\[2ex] 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Part (a): To find the first-order partial derivatives of f at a point other than $(0, 0)$, we proceed normally:

$$f_x(x, y) = \frac{y^3 - x^2y}{(x^2 + y^2)^2} \qquad \text{and} \qquad f_y(x, y) = \frac{x^3 - xy^2}{(x^2 + y^2)^2}.$$

Clearly both are defined and continuous everywhere except possibly at the origin. Next,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

and

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

Therefore both f_x and f_y are defined everywhere.

Part (b): At points of the line $y = mx$ other than $(0, 0)$, we have

$$f_x(x, y) = f_x(x, mx) = \frac{m^3 - m}{(m^2 + 1)^2 x},$$

and hence (taking $m = 0$) we have $f_x(x, y) = f_x(x, 0) \equiv 0$, whereas (taking $m = 2$) we have

$$f_x(x, y) = f_x(x, 2x) = \frac{6}{25x},$$

so that f_x is not continuous at $(0, 0)$. Similarly, at points of the line $y = mx$ other than $(0, 0)$, we have

$$f_y(x, y) = f_y(x, mx) = \frac{(1 - m^2)}{(1 + m^2)^2 x},$$

and hence (taking $m = 1$) we have $f_y(x, y) = f_y(x, x) \equiv 0$, whereas (taking $m = 0$) we have

$$f_y(x, y) = f_y(x, 0) = \frac{1}{x}.$$

Hence f_y is also not continuous at $(0, 0)$.

Part (c): If $(x, y) \neq (0, 0)$, then differentiation of f_x with respect to x yields

$$f_{xx}(x, y) = \frac{2xy(x^2 - 3y^2)}{(x^2 + y^2)^3}.$$

Similarly,

$$f_{xy}(x, y) = f_{yx}(x, y) = \frac{6x^2y^2 - x^4 - y^4}{(x^2 + y^2)^3} \qquad \text{and} \qquad f_{yy}(x, y) = \frac{2xy(y^2 - 3x^2)}{(x^2 + y^2)^3}.$$

Therefore the second-order partial derivatives of f are all defined and continuous except possibly at the origin.

Part (d): Here we have

$$f_{xx}(0, 0) = \lim_{h \to 0} \frac{f_x(0+h, 0) - f_x(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

and, similarly, $f_{yy}(0, 0) = 0$. Hence both second-order partial derivatives f_{xx} and f_{yy} exist at the origin (but you can use polar coordinates to show that neither is continuous there). On the other hand,

$$f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, 0+k) - f_x(0, 0)}{k} = \lim_{k \to 0} \frac{1}{k^2},$$

which does not exist; $f_{yx}(0, 0)$ also does not exist by a similar computation.

C13S04.074: Given:

$$g(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0), \\[2ex] 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Part (a): At points (x, y) other than the origin, we have

$$g_x(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \quad \text{and} \quad g_y(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}.$$

Therefore both first-order partial derivatives of g are defined and continuous except possibly at $(0, 0)$. Also

$$g_x(0, 0) = \lim_{h \to 0} \frac{g(0+h, 0) - g(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0;$$

$g_y(0, 0) = 0$ by a similar computation. This establishes the result in Part (a).

Part (b): If $(x, y) \neq (0, 0)$, then the substitutions $x = r\cos\theta$, $y = r\sin\theta$ give us (with what we suppose is a clear but unconventional use of the notation)

$$g_x(r, \theta) = \frac{r^5\cos^4\theta\,\sin\theta + 4r^5\cos^2\theta\sin^3\theta - r^5\sin^5\theta}{r^4} = \frac{r}{4}(3\sin 3\theta - \sin 5\theta)$$

and

$$g_y(r, \theta) = \frac{r^5\cos^5\theta - 4r^5\cos^3\theta\,\sin^2\theta - r^5\sin^4\theta\,\cos\theta}{r^4} = \frac{r}{4}(3\cos 3\theta + \cos 5\theta)$$

(simplifications by *Mathematica* 3.0). It is now clear that both $g_x(x, y)$ and $g_y(x, y)$ approach zero as $(x, y) \to (0, 0)$, and therefore both g_x and g_y are continuous everywhere.

Part (c): If (x, y) is a point other than the origin, then

$$\begin{aligned}
g_{xx}(x, y) &= \frac{4xy^3(3y^2 - x^2)}{(x^2+y^2)^3},\\
g_{xy}(x, y) &= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = g_{yx}(x, y), \quad \text{and}\\
g_{yy}(x, y) &= \frac{4x^3y(y^2 - 3x^2)}{(x^2+y^2)^3}.
\end{aligned}$$

Therefore all four second-order partial derivatives of g are defined and continuous except possibly at the origin.

Part (d): Here we have

$$\begin{aligned}
g_{xx}(0, 0) &= \lim_{h \to 0} \frac{g_x(0+h, 0) - g_x(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0,\\
g_{yy}(0, 0) &= \lim_{k \to 0} \frac{g_y(0, 0+k) - g_y(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0,\\
g_{xy}(0, 0) &= \lim_{k \to 0} \frac{g_x(0, 0+k) - g_x(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1, \quad \text{and}\\
g_{yx}(0, 0) &= \lim_{h \to 0} \frac{g_y(0+h, 0) - g_y(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.
\end{aligned}$$

Therefore all four second-order partial derivatives of g exist at $(0, 0)$ but $g_{xy}(0, 0) \neq g_{yx}(0, 0)$.

Part (e): Let m be a constant and suppose that $(x, y) \neq (0, 0)$. Then

$$g_{xx}(x,\,y) \;=\; g_{xx}(x,\,mx) \;=\; \frac{4m^3(3m^2-1)}{(m^2+1)^3},$$

and thus (take $m\,=\,1$) $g_{xx}(x,\,x)\,\equiv\,1$. But (take $m\,=\,0$) $g_{xx}(x,\,0)\,\equiv\,0$. Hence g_{xx} is not continuous at $(0,\,0)$. Similarly,

$$g_{yy}(x,\,y) \;=\; g_{yy}(x,\,mx) \;=\; \frac{m(4m^2-3)}{(m^2+1)^3},$$

so (take $m\,=\,1$) $g_{yy}(x,\,x)\,\equiv\,\frac{1}{8}$. But (take $m\,=\,0$) $g_{yy}(x,\,0)\,\equiv\,0$. Therefore g_{yy} is not continuous at $(0,\,0)$. Finally,

$$g_{xy}(x,\,y) \;=\; g_{yx}(x,\,y) \;=\; g_{xy}(x,\,mx) \;=\; g_{yx}(x,\,mx) \;=\; \frac{1+9m^2-9m^4-m^6}{(m^2+1)^3},$$

and thus (take $m\,=\,1$) $g_{xy}(x,\,x)\,=\,g_{yx}(x,\,x)\,\equiv\,0$ but (take $m\,=\,0$) $g_{xy}(x,\,0)\,=\,g_{yx}(x,\,0)\,\equiv\,1$. Consequently neither g_{xy} nor g_{yx} is continuous at the origin. Observe how the result in Problem 74 illustrates the *Note* containing Eq. (16).

Section 13.5

C13S05.001: If $z = x - 3y + 5$, then $z_x \equiv 1$ and $z_y \equiv -3$. Hence there is no point on the graph at which the tangent plane is horizontal. Indeed, the graph of $z = x - 3y + 5$ is itself a plane with nonvertical normal vector $\langle 1, -3, -1 \rangle$, and that's another reason why no tangent plane is horizontal.

C13S05.002: If $z = 4 - x^2 - y^2$, then $z_x = -2x$ and $z_y = -2y$. Both vanish at $(0, 0)$, so there is exactly one point on the graph of $z = 4 - x^2 - y^2$ at which the tangent plane is horizontal; it is $(0, 0, 4)$.

C13S05.003: If $z = xy + 5$, then $z_x = y$ and $z_y = x$. Both vanish at $(0, 0)$, so there is exactly one point on the graph of $z = xy + 5$ at which the tangent plane is horizontal—$(0, 0, 5)$.

C13S05.004: If $z = x^2 + y^2 + 2x$, then $z_x = 2x + 2$ and $z_y = 2y$. Both vanish at $(-1, 0)$, so there is exactly one point on the graph at which the tangent plane is horizontal; namely, $(-1, 0, -1)$.

C13S05.005: If $z = f(x, y) = x^2 + y^2 - 6x + 2y + 5$, then $z_x = 2x - 6$ and $z_y = 2y + 2$. Both are zero only at the point $(3, -1)$, so the graph of $z = f(x, y)$ has a horizontal tangent plane at the point $(3, -1, -5)$.

C13S05.006: If $z = f(x, y) = 10 + 8x - 6y - x^2 - y^2$, then $z_x = -2x + 8$ and $z_y = -2y - 6$. Both vanish at $(4, -3)$, so the graph of $z = f(x, y)$ has a horizontal tangent plane at the point $(4, -3, 35)$.

C13S05.007: If $z = f(x, y) = x^2 + 4x + y^3$, then $z_x = 2x + 4$ and $z_y = 3y^2$. Both are zero at the point $(-2, 0)$, so the graph of $z = f(x, y)$ has exactly one horizontal tangent plane—the one that is tangent at the point $(-2, 0, -4)$.

C13S05.008: If $z = f(x, y) = x^4 + y^3 - 3y$, then $z_x = 4x^2$ and $z_y = 3y^2 - 3 = 3(y + 1)(y - 1)$. Thus there are two points where both partial derivatives are zero: $(0, -1)$ and $(0, 1)$. Therefore the graph of $z = f(x, y)$ has two horizontal tangent planes. One is at $(0, -1, 2)$, the other at $(0, 1, -2)$.

C13S05.009: If $z = f(x, y) = 3x^2 + 12x + 4y^3 - 6y^2 + 5$, then $z_x = 6x + 12$ and $z_y = 12y(y - 1)$. Both partial derivatives are zero at $(-2, 0)$ and $(-2, 1)$, so the graph of $z = f(x, y)$ has two horizontal tangent planes. One is tangent at the point $(-2, 0, -7)$ and the other is tangent at the point $(-2, 1, -9)$.

C13S05.010: If

$$z = f(x, y) = \frac{1}{1 - 2x + 2y + x^2 + y^2},$$

then

$$\frac{\partial f}{\partial x} = -\frac{2x - 2}{(1 - 2x + x^2 + 2y + y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{2y + 2}{(1 - 2x + x^2 + 2y + y^2)^2}.$$

Both partial derivatives are zero at $(1, -1)$, so the graph of $z = f(x, y)$ has one horizontal tangent plane, tangent to the graph at the point $(1, -1, -1)$.

C13S05.011: If $f(x, y) = (2x^2 + 3y^2)\exp(-x^2 - y^2)$, then

$$\frac{\partial f}{\partial x} = 4x\exp(-x^2 - y^2) - 2x(2x^2 + 3y^2)\exp(-x^2 - y^2) = -2x(2x^2 + 3y^2 - 2)\exp(-x^2 - y^2) \quad \text{and}$$

$$\frac{\partial f}{\partial y} = 6y\exp(-x^2 - y^2) - 2y(2x^2 + 3y^2)\exp(-x^2 - y^2) = -2y(2x^2 + 3y^2 - 3)\exp(-x^2 - y^2).$$

We note that $\exp(-x^2 - y^2)$ is never zero, so to find when both partial derivatives are zero, it is enough to solve simultaneously

$$x(2x^2 + 3y^2 - 2) = 0,$$

$$y(2x^2 + 3y^2 - 3) = 0.$$

One solution is obvious: $x = y = 0$. Next, if $x \neq 0$ but $y = 0$, then the first of these equations implies that $2x^2 = 2$, so we get the two critical points $(1, 0)$ and $(-1, 0)$. If $x = 0$ but $y \neq 0$, then the second of these equations implies that $3y^2 = 3$, so we obtain two more critical points, $(0, -1)$ and $(0, 1)$. There is no solution if both x and y are nonzero, for that would imply that $2x^2 + 3y^2 = 2$ *and* $2x^2 + 3y^2 = 3$. But are there five horizontal tangent planes? No, because two of them are tangent at two critical points. One plane is tangent to the graph of $z = f(x, y)$ at $(-1, 0, 2e^{-1})$ and $(1, 0, 2e^{-1})$, a second is tangent at $(0, -1, 3e^{-1})$ and $(0, 1, 3e^{-1})$, and the third is tangent at $(0, 0, 0)$.

C13S05.012: If $f(x, y) = 2xy\exp\left(-[4x^2 + y^2]/8\right)$, then

$$\begin{aligned} f_x(x, y) &= 2y\exp\left(-[4x^2 + y^2]/8\right) - 2x^2y\exp\left(-[4x^2 + y^2]/8\right) \\ &= -2y(x+1)(x-1)\exp\left(-[4x^2 + y^2]/8\right) \qquad \text{and} \\ f_y(x, y) &= 2x\exp\left(-[4x^2 + y^2]/8\right) - \frac{1}{2}xy^2\exp\left(-[4x^2 + y^2]/8\right) \\ &= -\frac{1}{2}x(y-2)(y+2)\exp\left(-[4x^2 + y^2]/8\right). \end{aligned}$$

Because $\exp\left(-[4x^2 + y^2]/8\right)$ is never zero, to find where both partials vanish it suffices to solve simultaneously the equations

$$\begin{aligned} (x+1)(x-1)y &= 0 \qquad \text{and} \\ x(y-2)(y+2) &= 0. \end{aligned}$$

A brief case argument reveals five critical points: $(-1, -2)$, $(-1, 2)$, $(0, 0)$, $(1, -2)$, and $(1, 2)$. But there are only three horizontal tangent planes because two of the planes are tangent to the graph of $z = f(x, y)$ at two different points. One is tangent at $(-1, -2, 4e^{-1})$ and $(1, 2, 4e^{-1})$, another is tangent at $(-1, 2, -4e^{-1})$ and $(1, -2, -4e^{-1})$, and the third is tangent to the graph at $(0, 0, 0)$.

C13S05.013: Given: $z = f(x, y) = x^2 - 2x + y^2 - 2y + 3$. Then

$$f_x(x, y) = 2x - 2 \qquad \text{and} \qquad f_y(x, y) = 2y - 2,$$

so both partials are zero at only one point: $(1, 1)$. So the graph of $z = f(x, y)$ has only one horizontal tangent plane; it is tangent at the point $(1, 1, 1)$. This is clearly the lowest point on the graph of f.

C13S05.014: If $z = f(x, y) = 6x - 8y - x^2 - y^2$, then

$$\frac{\partial f}{\partial x} = -2x + 6 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -2y - 8.$$

So both partials are zero at $(3, -4)$ and only there. Thus there is exactly one horizontal plane tangent to the graph of $z = f(x, y)$, and it is tangent to the graph at the point $(3, -4, 25)$. This is clearly the highest point on the graph of f.

C13S05.015: If $z = f(x, y) = 2x - x^2 + 2y^2 - y^4$, then

$$f_x(x, y) = -2x + 2 \quad \text{and} \quad f_y(x, y) = -4y^3 + 4y = -4y(y+1)(y-1).$$

Thus there are three critical points: $(1, -1)$, $(1, 0)$, and $(1, 1)$. But there are only two horizontal planes tangent to the graph of f because one is tangent at two points—namely, at $(1, -1, 2)$ and at $(1, 1, 2)$; the other horizontal tangent plane is tangent to the graph at $(1, 0, 1)$. The first two of these points are the equally high highest points on the graph of f.

C13S05.016: If $z = f(x, y) = 4xy - x^4 - y^4$, then

$$f_x(x, y) = 4(y - x^3) \quad \text{and} \quad f_y(x, y) = 4(x - y^3).$$

We find that both partial derivatives are zero at $(-1, -1)$, $(0, 0)$, and $(1, 1)$. But there are only two horizontal tangent planes, because one is tangent to the graph of f at the two points $(-1, -1, 2)$ and $(1, 1, 2)$; the other is tangent at the point $(0, 0, 0)$. The first two of these points are the equally high highest points on the graph of f.

C13S05.017: Given: $z = f(x, y) = 3x^4 - 4x^3 - 12x^2 + 2y^2 - 12y$, the following sequence of *Mathematica* 3.0 commands will find the points of tangency of all horizontal tangent planes. (Recall that % refers to the "last output.")

```
f[x_, y_ ] := 3*x^4 - 4*x^3 - 12*x^2 + 2*y^2 - 12*y
d1 = D[ f[x,y], x ]
```

$$-24x - 12x^2 + 12x^3$$

```
d2 = D[ f[x,y], y ]
```

$$-12 + 4y$$

```
Solve[ { d1 == 0, d2 == 0 }, { x, y } ];
```

$$\{\{x \to -1,\ y \to 3\},\ \{x \to 0,\ y \to 3\},\ \{x \to 2,\ y \to 3\}\}$$

```
f[x,y] /. %
```

$$\{-23,\ -18,\ -50\}$$

Thus there are three horizontal planes tangent to the graph of $z = f(x, y)$; the points of tangency are $(-1, 3, -23)$, $(0, 3, -18)$, and $(2, 3, -50)$. The last of these is the lowest point on the graph of f.

C13S05.018: If $z = f(x, y) = 3x^4 + 4x^3 + 6y^4 - 16y^3 + 12y^2$, then

$$f_x(x, y) = 12x^2 + 12x^3 = 12x^2(x+1) \quad \text{and} \quad f_y(x, y) = 24y - 48y^2 + 24y^3 = 24y(y-1)^2.$$

So the graph of $z = f(x, y)$ has four critical points: $(-1, 0)$, $(-1, 1)$, $(0, 0)$, and $(0, 1)$. And there are, indeed, four horizontal tangent planes; they are tangent at the four points $(-1, 0, -1)$, $(-1, 1, 1)$, $(0, 0, 0)$, and $(0, 1, 2)$. The first of these is the lowest point on the graph of f.

C13S05.019: If $f(x, y) = 2x^2 + 8xy + y^4$, then

$$f_x(x, y) = 4x + 8y = 4(x + 2y) \quad \text{and} \quad f_y(x, y) = 8x + 4y^3 = 4(2x + y^3).$$

Thus both partial derivatives are zero at the three points $(-4, 2)$, $(0, 0)$, and $(4, -2)$. But there are only two horizontal tangent planes because one is tangent to the graph of $z = f(x, y)$ at two points: $(-4, 2, -16)$ and $(4, -2, -16)$. The other plane is tangent to the graph at the origin. The two equally low lowest points on the graph of f are $(-4, 2, -16)$ and $(4, -2, -16)$.

Detail: Solving simultaneous nonlinear equations is an *ad hoc* procedure. One method that frequently works is to solve one equation for one of the variables, then substitute in the others. Here we begin with

$$4(x + 2y) = 0 \quad \text{and} \quad 4(2x + y^3) = 0.$$

We solve the first for $x = -2y$ and substitute in the second to obtain

$$-4y + y^3 = 0; \quad \text{that is,} \quad y(y + 2)(y - 2) = 0.$$

This yields the three solutions $y = -2$, $y = 0$, and $y = 2$, and the corresponding values of x are 4, 0, and -4.

C13S05.020: Given:

$$z = f(x, y) = \frac{1}{10 - 2x - 4y + x^2 + y^4}.$$

Then

$$f_x(x, y) = -\frac{2x - 2}{(10 - 2x + x^2 - 4y + y^4)^2} \quad \text{and} \quad f_y(x, y) = -\frac{4y^3 - 4}{(10 - 2x + x^2 - 4y + y^4)^2}.$$

Both partials vanish at $(1, 1)$, so there is exactly one horizontal plane tangent to the graph of $z = f(x, y)$. The point of tangency is $\left(1, 1, \frac{1}{6}\right)$. This is the highest point on the graph of f.

C13S05.021: If $z = f(x, y) = \exp(2x - 4y - x^2 - y^2)$, then

$$f_x(x, y) = (2 - 2x)\exp(2x - 4y - x^2 - y^2) \quad \text{and} \quad f_y(x, y) = -(2y + 4)\exp(2x - 4y - x^2 - y^2).$$

Hence both partial derivatives are zero when

$$2x - 2 = 0 \quad \text{and} \quad 2y + 4 = 0;$$

that is, at the single point $(1, -2)$. Hence there is exactly one horizontal plane tangent to the graph of $z = f(x, y)$; it is tangent at the point $(1, -2, e^5)$. This is the highest point on the graph of f.

C13S05.022: If $z = f(x, y) = (1 + x^2)\exp(-x^2 - y^2)$, then

$$f_x(x, y) = 2x\exp(-x^2 - y^2) - 2x(1 + x^2)\exp(-x^2 - y^2) = -2x^3\exp(-x^2 - y^2) \quad \text{and}$$
$$f_y(x, y) = -2y(1 + x^2)\exp(-x^2 - y^2).$$

Therefore both partials are zero when $x^3 = 0$ and $(1 + x^2)y = 0$; that is, only at $(0, 0)$. Thus there is exactly one horizontal plane tangent to the graph of $z = f(x, y)$; it is tangent at the point $(0, 0, 1)$. This is the highest point on the graph of f.

C13S05.023: The graph of $f(x, y) = x + 2y$ is a plane, so its maximum and minimum values on a polygonal region must occur at the vertices of the polygon. Here we have

$$f(-1,\,-1) = -3, \quad f(-1,\,1) = 1, \quad f(1,\,-1) = -1, \quad \text{and} \quad f(1,\,1) = 3.$$

At this point it is clear what are the maximum and minimum values of $f(x,\,y)$ on R.

C13S05.024: If $f(x,\,y) = x^2 + y^2 - x$, then

$$f_x(x,\,y) = 2x - 1 \qquad \text{and} \qquad f_y(x,\,y) = 2y,$$

so $\left(\frac{1}{2},\,0\right)$ is the only critical point of f. Because

$$f(x,\,y) = \left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{4},$$

the global minimum value of f on R is $f\left(\frac{1}{2},\,0\right) = -\frac{1}{4}$. The maximum value of f on R must occur on the boundary of R.

- On the lower edge of R, we have $f(x,\,-1) = x^2 - x + 1$, with minimum value $f\left(\frac{1}{2},\,-1\right) = \frac{3}{4}$ there and maximum value $f(-1,\,-1) = 3$.

- On the right-hand edge of R, we have $f(1,\,y) = y^2$, with minimum value $f(1,\,0) = 0$ and maximum value $f(1,\,-1) = f(1,\,1) = 1$.

- On the upper edge of R, we have $f(x,\,1) = x^2 - x + 1$, with minimum value $f\left(\frac{1}{2},\,1\right) = \frac{3}{4}$ there and maximum value $f(-1,\,1) = 3$.

- On the left-hand edge of R, we have $f(-1,\,y) = y^2$, with minimum value $f(-1,\,0) = 0$ and maximum value $f(-1,\,-1) = f(-1,\,1) = 3$.

Therefore the minimum value of f on R is $f\left(\frac{1}{2},\,0\right) = -\frac{1}{4}$; its maximum is $f(-1,\,-1) = f(-1,\,1) = 3$.

C13S05.025: Given: $f(x,\,y) = x^2 + y^2 - 2x$ on the triangular region R with vertices at $(0,\,0)$, $(2,\,0)$, and $(0,\,2)$. Then

$$f_x(x,\,y) = 2x - 2 \qquad \text{and} \qquad f_y(x,\,y) = 2y,$$

so the only critical point of f is $(1,\,0)$, which is a point of R. Because

$$f(x,\,y) = (x-1)^2 + y^2 - 1,$$

the global minimum value of f on R is $f(1,\,0) = -1$. The maximum value of f must occur on the boundary of R, which we explore next.

- On the lower edge of R, we have $f(x,\,0) = x^2 - 2x$, which must attain its maximum at one endpoint of that edge. Hence the maximum value of f there is $f(0,\,0) = f(2,\,0) = 0$.

- On the left-hand edge of R, we have $f(0,\,y) = y^2$, with maximum value $f(0,\,2) = 4$.

- On the diagonal edge of R, which has equation $y = 2 - x$, we have

$$f(x,\,2-x) = g(x) = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4,$$

and $g'(x) = 4x - 6$, so the minimum value of f there is $g\left(\frac{3}{2}\right) = -\frac{1}{2}$ and the maximum value there (because it must occur at an endpoint of the diagonal edge) is $f(0,\,2) = 4$.

In summary, the minimum value of f on R is $f(1,\,0) = -1$ and its maximum value is $f(0,\,2) = 4$.

C13S05.026: Given: $f(x,\,y) = x^2 + y^2 - x - y$ on the triangular region R of Problem 25. Then

$$f_x(x,\,y) = 2x - 1 \quad \text{and} \quad f_y(x,\,y) = 2y - 1,$$

so $\left(\frac{1}{2},\,\frac{1}{2}\right)$ is the only critical point of f. This point does lie within the region R, and

$$f(x,\,y) = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{2},$$

so the global minimum value of f on R is $f\left(\frac{1}{2},\,\frac{1}{2}\right) = -\frac{1}{2}$. The global maximum must occur on the boundary of R, which we study next.

- On the lower edge of R, we have $f(x,\,0) = x^2 - x$, which is minimized at the point $\left(\frac{1}{2},\,0\right)$, but the value of $f(x,\,y)$ there is $-\frac{1}{4}$, not the global minimum of f on R. The maximum value of f on this edge must occur at one of its endpoints, so the maximum of f on this edge is $f(2,\,0) = 2$.

- On the right-hand edge of R, we have $f(0,\,y) = y^2 - y$, and the analysis proceeds exactly as in the previous case; the maximum value of f there is $f(0,\,2) = 2$.

- On the diagonal edge of R, where $y = 2 - x$, we have

$$f(x,\,y) = f(x,\,2-x) = g(x) = x^2 + (2-x)^2 - x - 2 + x = 2x^2 - 4x + 2 = 2(x-1)^2,$$

so the minimum value of f on the diagonal is $g(1) = 0$ and its maximum value must occur at an endpoint of the diagonal. We have already examined the behavior of f at both those endpoints.

In summary, the global minimum value of f on R is $f\left(\frac{1}{2},\,\frac{1}{2}\right) = -\frac{1}{2}$ and the global maximum of f there is $f(2,\,0) = f(0,\,2) = 2$.

C13S05.027: Given: $f(x,\,y) = 2xy$ on the circular disk R described by the inequality $x^2 + y^2 \leqq 1$. Then $f_x(x,\,y) = 2y$ and $f_x(x,\,y) = 2x$, so the only critical point of f is $(0,\,0)$. On the line $y = x$ we have $f(x,\,x) = 2x^2$, but on the line $y = -x$ we have $f(x,\,-x) = -2x^2$. Therefore f does not have an extremum at $(0,\,0)$. (The graph of $z = 2xy$ is a hyperbolic paraboloid with a *saddle point* at $(0,\,0)$; to see its graph, rotate the graph shown in Fig. 13.10.1 45° around the z-axis.) Because f must have a global maximum and a global minimum on R, both must occur on its boundary.

- We describe the boundary $x^2 + y^2 = 1$ of R in polar coordinates: $r = 1,\ 0 \leqq \theta \leqq 2\pi$. Thus on the boundary, we have

$$f(x,\,y) = 2xy = g(\theta) = 2\sin\theta\,\cos\theta.$$

Then

$$g'(\theta) = 2\cos^2\theta - 2\sin^2\theta = 2\cos 2\theta,$$

and $g'(\theta) = 0$ when 2θ is an odd integral multiple of $\pi/2$; that is, when θ is an odd integral multiple of $\pi/4$. Now

$$g(\pi/4) = g(5\pi/4) = 1 \quad \text{and} \quad g(3\pi/4) = g(7\pi/4) = -1,$$

so we have discovered the global extrema of f on R.

Summary: The global maximum value of f is 1 and occurs at each of the two points $\left(\frac{1}{2}\sqrt{2},\,\frac{1}{2}\sqrt{2}\right)$ and $\left(-\frac{1}{2}\sqrt{2},\,-\frac{1}{2}\sqrt{2}\right)$. The global minimum value of f is -1 and occurs at each of the two points $\left(-\frac{1}{2}\sqrt{2},\,\frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{1}{2}\sqrt{2},\,-\frac{1}{2}\sqrt{2}\right)$.

C13S05.028: Given: $f(x, y) = xy^2$ on the circular disk R described by the inequality $x^2 + y^2 \leqq 3$. Then $f_x(x, y) = y^2$ and $f_y(x, y) = 2xy$, so both partial vanish when $y = 0$ (and x is arbitrary). Thus every point of the diameter of the disk R that lies on the x-axis is a critical point, and on this diameter the value of $f(x, y)$ is the constant 0. But $f(1, 1) > 0$ and $f(-1, 1) < 0$, so none of these critical points yields the maximum or the minimum value of f. They must therefore occur on the boundary of R.

- On the boundary of R, $y^2 = 3-x^2$, so $f(x, y)$ will have the same extrema as $g(x) = x(3-x^2) = 3x-x^3$. But $g'(x) = 3 - 3x^2$, so $g'(x) = 0$ when $x = \pm 1$. If so, then $y^2 = 2$, and thus we find four critical points, locations of possible extrema of f:

$$(x, y) = \left(1, \sqrt{2}\right), \text{ where } f(x, y) = 2; \qquad (x, y) = \left(1, -\sqrt{2}\right), \text{ where } f(x, y) = 2;$$

$$(x, y) = \left(-1, \sqrt{2}\right), \text{ where } f(x, y) = -2; \qquad (x, y) = \left(-1, -\sqrt{2}\right), \text{ where } f(x, y) = -2.$$

The domain of g is the closed interval $-\sqrt{3} \leqq x \leqq \sqrt{3}$, but at the endpoints of this interval $g(x) = 0$, so they do not yield extrema. Thus the global maximum of f is 2 and its global minimum is -2.

C13S05.029: The square of the distance between the plane and the origin is

$$f(x, y) = x^2 + y^2 + \left(\frac{169 - 12x - 4y}{3}\right)^2.$$

Setting both partials of $f(x, y)$ equal to zero yields the equations

$$2x - \frac{8}{3}(169 - 12x - 4y) = 0, \quad 2y - \frac{8}{9}(169 - 12x - 4y) = 0.$$

These equations are easiest to solve if you note first that they imply immediately that $6x = 18y$, because each is equal to $8(169 - 12x - 4y)$. It follows that their solution is $x = 12$, $y = 4$. The corresponding value of z on the given plane is 3, so the point of the plane closest to the origin is $(12, 4, 3)$. (The formula for f makes it plain that f has a global minimum and no maximum.) The distance between the plane and the origin is therefore $\sqrt{f(12, 4)} = 13$.

C13S05.030: The square of the distance between the plane and the point Q is

$$f(x, y) = (x - 9)^2 + (y - 9)^2 + (27 - 2x - 2y - 9)^2,$$

and setting both partials of f equal to zero yields the simultaneous equations

$$2(x - 9) - 4(18 - 2x - 2y) = 0, \quad 2(y - 9) - 4(18 - 2x - 2y) = 0.$$

These equations are easiest to solve if you first note that they immediately imply that $2(x - 9) = 2(y - 9)$, because each of these is equal to $4(18 - 2x - 2y)$. It follows that $x = y = 5$, and the corresponding z-coordinate on the plane is 7. So the point of the plane closest to Q is $(5, 5, 7)$. The distance between Q and the plane is 6.

C13S05.031: The square of the distance between the plane and the point Q is

$$f(x, y) = (x - 7)^2 + (y + 7)^2 + (49 - 2x - 3y)^2.$$

When we set both partials of f equal to zero, we get the simultaneous equations

$$2(x-7)-4(49-2x-3y)=0, \quad 2(y+7)-6(49-2x-3y)=0,$$

with solution $x = 15$, $y = 5$; the corresponding z-coordinate on the plane is 4. So the point on the plane closest to Q is $(15, 5, 4)$. The distance between the two is $4\sqrt{14}$.

C13S05.032: The square of the distance from the origin to the surface is

$$f(x, y) = x^2 + y^2 + \frac{64}{x^2y^2}.$$

When we equate both partial derivatives of f to zero, we obtain the equations

$$2x - \frac{128}{x^3y^2} = 0, \quad 2y - \frac{128}{x^2y^3} = 0.$$

It is easiest to solve these equations if you begin with the observation that they immediately imply that $2x^2 = 2y^2$. There are four solutions: $(-2, -2)$, $(-2, 2)$, $(2, -2)$, and $(2, 2)$. So the coordinates of the first-octant point P on the surface closest to the origin is $(2, 2, 2)$. The distance from P to the origin is $2\sqrt{3}$.

C13S05.033: The square of the distance from the origin to the point (x, y, z) of the surface is

$$f(x, y) = x^2 + y^2 + \frac{16}{x^4y^4}.$$

The equations $f_x(x, y) = 0 = f_y(x, y)$ are

$$2x - \frac{64}{x^5y^4} = 0, \quad 2y - \frac{64}{x^4y^5} = 0.$$

They are easiest to solve if you begin with the observation that they imply that $2x^2 = 2y^2$. There are four solutions—all possible combinations of $x = \pm\sqrt{2}$, $y = \pm\sqrt{2}$. It follows that the point on the surface in the first octant closest to the origin is $\left(\sqrt{2}, \sqrt{2}, 1\right)$; its distance from the origin is $\sqrt{5}$.

C13S05.034: The square of the distance between the point (x, y, z) of the surface and the origin is

$$f(x, y) = x^2 + y^2 + \frac{8}{x^4y^8},$$

and the equations $f_x(x, y) = 0 = f_y(x, y)$ take the form

$$2x - \frac{32}{x^5y^8} = 0, \qquad 2y - \frac{64}{x^4y^9} = 0; \qquad \text{that is,}$$

$$2x^6y^8 = 32, \qquad 2x^4y^{10} = 64.$$

It is easiest to solve these equations if you begin with their consequence $2x^6y^8 = x^4y^{10}$. The only positive solution is $x = 1$, $y = \sqrt{2}$. The corresponding value of z on the surface is $\frac{1}{2}\sqrt{2}$.

C13S05.035: We will find the maximum possible product of three *nonnegative* real numbers with sum 120—the reason in a moment. If x, y, and z are the three numbers, then we are to maximize xyz given $x + y + z = 120$. So we solve for z, substitute, and maximize

$$f(x, y) = xy(120 - x - y), \quad 0 \leqq x, \quad 0 \leqq y, \quad x + y \leqq 120.$$

Thus by allowing one or two of the numbers to be zero, the domain of f is now a closed and bounded subset of the plane—the triangle with two sides on the nonnegative coordinate axes and the third side part of the graph of $y = 120 - x$. Write $f(x, y) = 120xy - x^2y - xy^2$ and set both partial derivatives equal to zero to obtain

$$120y - 2xy - y^2 = 0, \quad 120x - 2xy - x^2 = 0.$$

Because neither x nor y is zero—that would minimize the product, not maximize it—we may cancel to obtain

$$120 - 2x - y = 0, \quad 120 - 2y - x = 0,$$

and it follows that $2x + y = x + 2y$, so that $y = x$, and then the equation $3x = 120$ yields $x = 40$ and $y = 40$. It follows that $z = 40$ as well. The maximum of $f(x, y)$ does not occur on the boundary of its domain, for $f(x, y) = 0$ there. Hence this lone interior critical point must yield the global maximum, which is $40 \cdot 40 \cdot 40 = 64000$.

C13S05.036: Let one set of four parallel edges of the box have length x each, another set length y each, and the third length z each. Then we are to maximize box volume $V = xyz$ given $4x + 4y + 4z = 6$; that is, $2x + 2y + 2z = 3$. Solve for z and substitute in V to obtain the function to be maximized:

$$V(x, y) = \frac{xy(3 - 2x - 2y)}{2}, \quad 0 \leqq x, \quad 0 \leqq y, \quad x + y \leqq \frac{3}{2}.$$

The domain of V is a closed and bounded subset of the plane and V is continuous there, so a global maximum exists. It does not occur on the boundary of the domain because $V(x, y) = 0$ there. So it must occur at an interior critical point. When both partials of V are set equal to zero, the resulting equations are

$$\frac{y}{2}(3 - 2x - 2y) - xy = 0, \quad \frac{x}{2}(3 - 2x - 2y) - xy = 0.$$

It follows that $y = x$ and then, from either of the preceding equations, that $y = x = \frac{1}{2}$. Next, $z = \frac{1}{2}$ as well, so—because this is the only interior critical point—the maximum volume of such a box is $V\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{8}$ (cubic meters).

C13S05.037: Let the dimensions of the box be x by y by z. We are to minimize total surface area $A = 2xy + 2xz + 2yz$ given $xyz = 1000$. Solve the latter equation for z and substitute in A to obtain the function to be minimized:

$$A(x, y) = \frac{2000}{x} + \frac{2000}{y} + 2xy, \quad 0 < x, \quad 0 < y.$$

Although A is continuous on its domain, the domain is neither closed nor bounded. But an argument similar to the one given in Example 7 makes it clear that A has a global minimum, so it must occur at a critical point of the domain. When we set both partial derivatives of A equal to zero, we obtain the equations

$$2y - \frac{2000}{x^2} = 0, \quad 2x - \frac{2000}{y^2} = 0.$$

These equations imply that $2x^2y = 2xy^2$, so that $y = x$. Then either of the two preceding equations implies that $x = 10 = y$. Finally, $xyz = 1000$ implies that $z = 10$ as well. This is the only critical point, so we have found the global minimum of A. To minimize the total surface area, make a cube of edge length 10.

C13S05.038: Let the bottom of the box have dimensions x by y, the front and back dimensions x by z, and the sides dimensions y by z. We are to minimize total surface area $A = xy + 2xz + 2yz$ given $xyz = 4000$. Solve the last equation for z and substitute in the area formula to obtain the function to be minimized:

$$A(x, y) = \frac{8000}{x} + \frac{8000}{y} + xy, \quad 0 < x, \quad 0 < y.$$

Although A is continuous on its domain, the domain is neither closed nor bounded. But an argument similar to the one given in Example 7 makes it clear that A has a global minimum, so it must occur at a critical point of the domain. When we set both partial derivatives of A equal to zero, we obtain the equations

$$y - \frac{8000}{x^2} = 0, \quad x - \frac{8000}{y^2} = 0,$$

and it follows immediately that $x^2y = xy^2$, so that $y = x$. Then either of the preceding equations yields $x = y = 20$. The corresponding value of z is 10. There is only one critical point, so the open-topped box of volume 4000 cm^3 and minimal surface area has base 20 cm by 20 cm and height 10 cm.

C13S05.039: Suppose that the dimensions of the base of the box are x by y, the front and back have dimensions x by z, and the sides have dimensions y by z. The cost of the base is then $6xy$ and the total cost of the other four sides is $2 \cdot 5xz + 2 \cdot 5yz$. So we are to minimize total cost $C = 6xy + 10xz + 10yz$ given $xyz = 600$. Solve the last equation for z and substitute in the cost expression to obtain the function to be minimized:

$$C(x, y) = \frac{6000}{x} + \frac{6000}{y} + 6xy, \quad 0 < x, \quad 0 < y.$$

By an argument similar to the one used in the solution of Example 7, $C(x, y)$ has a global minimum and it occurs at a critical point. When we set both partial derivatives of C equal to zero, we get the equations

$$6y - \frac{6000}{x^2} = 0, \quad 6x - \frac{6000}{y^2} = 0.$$

It follows immediately that $x^2y = xy^2$, so that $x = y$. Then either of the displayed equations yields $x = y = 10$. The corresponding value of z is 6, so the dimensions of the least expensive such box are these: base 10 inches by 10 inches, height 6 inches. It will cost \$18.00.

C13S05.040: Suppose that the base of the box has dimensions x by y and that the height of the box is z. Then the cost of the top and the bottom will be $3xy$ each and the total cost of the four sides will be $8xz + 8yz$. So we are to minimize total cost $C = 6xy + 8xz + 8yz$ given $xyz = 48$. Solve the last equation for z and substitute in the expression for the cost to obtain the function to be minimized:

$$C(x, y) = \frac{384}{x} + \frac{384}{y} + 6xy, \quad 0 < x, \quad 0 < y.$$

By an argument similar to the one used in Example 7, $C(x, y)$ has a global minimum and it occurs at a critical point. When we write $C_x(x, y) = 0$ and $C_y(x, y) = 0$, we obtain

$$6y - \frac{384}{x^2} = 0, \quad 6x - \frac{384}{y^2} = 0.$$

These equations imply that $x^2y = xy^2$, so that $y = x$. Then either displayed equation further implies that $x = y = 4$. The condition $xyz = 48$ then yields $z = 3$. We have found only one critical point, so we have minimized $C(x, y)$. The dimensions of the least expensive such box are these: base 4 ft by 4 ft, height 3 ft. This box will cost \$288.00.

C13S05.041: Let the base and top of the box have dimensions x by y, the front and back dimensions x by z, and the sides dimensions y by z. Then the top and base cost $3xy$ cents each, the front and back cost $6xz$ cents each, and the two sides cost $9yz$ cents each. Hence the total cost of the box will be $C = 6xy + 12xz + 18yz$. But $xyz = 750$, so that $z = 750/(xy)$. To substitute this into the formula for C and simplify, we use the *Mathematica* 3.0 command

```
6*x*y + 12*x*z + 18*y*z /. z -> 750/(x*y)
```

(Recall that `/.` translates roughly as "evaluate subject to.") The response is

$$\frac{13500}{x} + \frac{9000}{y} + 6xy.$$

Next we construct the total surface area function, the quantity to be minimized:

```
f[x_, y_] := 13500/x + 9000/y + 6*x*y
```

Then we compute both partial derivatives:

```
d1 = D[ f[x,y], x]
```

$$6y - \frac{13500}{x^2}$$

```
d2 = D[ f[x,y], y]
```

$$6x - \frac{9000}{y^2}$$

Then we set both partial derivatives equal to zero and solve simultaneously:

```
Solve[ { d1 == 0, d2 == 0 }, { x, y } ]
```

The response is

$$\{\{x \to 15,\ y \to 10\},\ \{x \to -15\,(-1)^{1/3},\ y \to -10\,(-1)^{1/3}\},\ \{x \to 15\,(-1)^{2/3},\ y \to 10\,(-1)^{2/3}\}\}$$

We ignore the two pairs of non-real roots and evaluate z:

```
750/(x*y) /. {x -> 15, y -> 10}
```

5

Finally, we evaluate f at (15, 10) to find the minimum cost:

```
f[15,10]
```

2700

The domain of f is not a closed and bounded set, but instead the interior of the entire first quadrant. Nevertheless, $f(x, y)$ has a global minimum at a critical point by an argument similar to the one used in Example 7. Because we have found only one critical point, we have found the global minimum as well. The box should have dimensions $x = 15$, $y = 10$, and $z = 5$ inches. It will cost \$27.00.

C13S05.042: Let the base and top of the box have dimensions x by y (units in meters, etc.), the front and back dimensions x by z, and the sides have dimensions y by z. Assume that the bottom costs 2 units per square meter and the other five sides cost 1 unit per square meter. Then the total cost of the box will be $C = 3xy + 2xz + 2yz$. Solve $xyz = 12$ for z and substitute in the expression for cost to obtain the quantity to be minimized:

$$C(x, y) = \frac{24}{x} + \frac{24}{y} + 3xy, \quad 0 < x, \quad 0 < y.$$

When both partial derivatives of C are set equal to zero, the result is the pair of simultaneous equations

$$3y - \frac{24}{x^2} = 0, \quad 3x - \frac{24}{y^2} = 0$$

with solution $x = 2$, $y = 2$. By an argument similar to the one in Example 7, we have found the location of the global minimum of $C(x, y)$. The corresponding value of z is 3. Hence the base of the box should measure 2 meters by 2 meters and its height should be 3 meters. Its total cost will be $C(2, 2) = 36$ (in whatever units cost per square meter is measured).

C13S05.043: Suppose that the base of the building measures x feet by y feet, that its front and back measure x by z, and that its two sides measure y by z. Then the total heating and cooling costs will be $C = 2xy + 4xz + 8yz$. Solve $xyz = 8000$ for z and substitute in the expression for cost to obtain the quantity to be minimized:

$$C(x, y) = \frac{64000}{x} + \frac{32000}{y} + 2xy, \quad 0 < x, \quad 0 < y.$$

Although the domain of C is not a closed and bounded subset of the plane, nevertheless $C(x, y)$ has a global minimum at a critical point by an argument similar to the one used in the solution of Example 7. When we set both partial derivatives of $C(x, y)$ equal to zero, we obtain

$$2y - \frac{64000}{x^2} = 0, \quad 2x - \frac{32000}{y^2} = 0,$$

having the only real solutions $x = 40$, $y = 20$. The corresponding value of z is 10, so the building should be 40 feet wide (in front), 20 feet deep, and 10 feet high. The annual heating and cooling costs will thereby have their minimum possible value, $C(40, 20) = 4800$ dollars per year.

C13S05.044: Suppose that the dimensions of the base and top of the aquarium are x by y (units are in inches, cents, etc.), that the front and back have dimensions x by z, and that the sides have dimensions y by z. Then the cost of the aquarium will be $C = 30xy + 10xz + 10yz$. Solve $xyz = 24000$ for z and substitute in the expression for the cost to get the quantity to be minimized:

$$C(x, y) = \frac{240000}{x} + \frac{240000}{y} + 30xy, \quad 0 < x, \quad 0 < y.$$

The domain of C is not a closed and bounded subset of the plane, but by an argument similar to the one used in Example 7, $C(x, y)$ has a global minimum value that occurs at a critical point in its domain. When we set both partials of C equal to zero, we obtain the simultaneous equations

$$30y - \frac{240000}{x^2} = 0, \quad 30x - \frac{240000}{y^2} = 0,$$

and the only real solution of these equations is $x = 20$, $y = 20$. Thus we have found the location of the global minimum value of $C(x, y)$. The corresponding value of z is 60, so the least expensive aquarium will

have base 20 inches by 20 inches and height 60 inches (an ideal shape for White Cloud Mountain tropical fish). Its total cost will be $C(20, 20)$ cents—a rather substantial \$360.00.

C13S05.045: Suppose that (x, y, z) is the vertex of the box that lies on the given plane with equation $x + 3y + 7z = 11$. We are to maximize the volume $V = xyz$ of the box. Solve the equation of the plane for z and subsitute to obtain

$$V(x, y) = \frac{xy\,(11 - x - 3y)}{7}, \quad 0 \leqq x, \quad 0 \leqq y, \quad x + 3y \leqq 11.$$

The domain of V is a closed and bounded subset of the xy-plane—it consists of the sides and interior of the triangle with vertices at $(0, 0)$, $(11, 0)$, and $\left(0, \frac{11}{3}\right)$. Therefore the continuous function (it's a polynomial) $V(x, y)$ has a global maximum on its domain. The maximum does not occur on the boundary because $V(x, y)$ is identically zero there. Hence the maximum occurs at an interior critical point. When we set the partial derivatives of V equal to zero, we get the simultaneous equations

$$\frac{(11 - x - 3y)y - xy}{7} = 0, \quad \frac{(11 - x - 3y)x - 3xy}{7} = 0.$$

To solve these equations, multiply through by 7 and factor to obtain

$$(11 - 2x - 3y)y = 0, \quad (11 - x - 6y)x = 0.$$

- If $x = 0$ and $y = 0$, we have one solution.
- If $x = 0$ and $y \neq 0$, then $y = \frac{11}{3}$.
- If $x \neq 0$ and $y = 0$, then $x = 11$.
- If $x \neq 0$ and $y \neq 0$, then $2x + 3y = 11$ and $x + 6y = 11$. It follows that $x = \frac{11}{3}$ and $y = \frac{11}{9}$.

Only the last of these solutions will produce a box of positive volume. The corresponding value of z is $\frac{11}{21}$. Thus we have found the maximizing values of x, y, and z. The maximum possible volume is

$$\frac{11}{3} \cdot \frac{11}{9} \cdot \frac{11}{21} = \frac{1331}{567} \approx 2.347442680776.$$

C13S05.046: Suppose that (x, y, z) is the vertex of the box that lies on the given plane with equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(a, b, and c are positive constants). We are to maximize the volume $V = xyz$ of the box. Solve the equation of the plane for z and substitute in the expression for V to obtain

$$V(x, y) = cxy\left(1 - \frac{x}{a} - \frac{y}{b}\right), \quad 0 \leqq x, \quad 0 \leqq y, \quad \frac{x}{a} + \frac{y}{b} \leqq 1.$$

Because the domain of V is a closed and bounded subset of the xy-plane and V is continuous there, it has a global maximum. Moreover, the maximum must occur at an interior critical point because $V(x, y)$ is identically zero on the boundary of its domain. We first find

$$V_x(x, y) = cy\left(1 - \frac{x}{a} - \frac{y}{b}\right) - \frac{cxy}{a} = \frac{cy}{ab}(ab - ay - 2bx) \quad \text{and}$$

$$V_y(x, y) = cx\left(1 - \frac{x}{a} - \frac{y}{b}\right) - \frac{cxy}{b} = \frac{cx}{ab}(ab - bx - 2ay).$$

The maximum does not occur when $x = 0$ or when $y = 0$, so we need only solve simultaneously

$$ab - ay - 2bx = 0 \quad \text{and} \quad ab - bx - 2ay = 0.$$

These equations imply that $ay + 2bx = bx + 2ay$, so that $bx = ay$. Then substitution of bx for ay in the equation $ab - bx - 2ay = 0$ yields

$$ab - bx - 2bx = 0; \quad \text{that is,} \quad x = \frac{a}{3}.$$

By symmetry, the corresponding values of y and z are

$$y = \frac{b}{3} \quad \text{and} \quad z = \frac{c}{3}.$$

Therefore the maximum value of the box in question is

$$\frac{a}{3} \cdot \frac{b}{3} \cdot \frac{c}{3} = \frac{abc}{27}.$$

C13S05.047: Suppose that the dimensions of the rectangular box are x by y by z (in inches). Without loss of generality we may suppose that $x \leqq y \leqq z$. Then the length of the box is z, so its girth is $2x + 2y$. We are to maximize box volume $V = xyz$ given the side condition $2x + 2y + z \leqq 108$; of course, the maximum occurs when $2x + 2y + z = 108$, so that is the side condition we use. Solve for z in the last equation and substitute in the expression for volume to obtain the function to be maximized:

$$V(x, y) = xy(108 - 2x - 2y), \quad 0 \leqq x, \quad 0 \leqq y, \quad x + y \leqq 54.$$

Now V is continuous ($V(x, y)$ is a polynomial) on its domain, a closed and bounded region in the xy-plane, so V has a global maximum there. The maximum does not occur on the boundary because $V(x, y)$ is identically zero on the boundary of its domain. So the global maximum occurs at an interior critical point where both partial derivatives are zero; that is,

$$(108 - 2x - 2y)y - 2xy = 0, \quad (108 - 2x - 2y)x - 2xy = 0.$$

We may cancel y from the first of these equations and x from the second because neither is zero at the maximum. Thus we are to solve

$$108 - 4x - 2y = 0, \quad 108 - 2x - 4y = 0.$$

It follows that $4x + 2y = 2x + 4y$, and thus that $x = y$. This implies in turn that $x = y = 18$ and $z = 36$. So the maximum volume of such a box is $18 \cdot 18 \cdot 36 = 11664$ cubic inches, exactly 6.25 cubic feet. If it were filled with osmium (the heaviest element known) it would weigh over 4338 kg, about 4.78 tons.

C13S05.048: Suppose that the cylindrical mailing tube has radius r and length h. Then its volume is $V = \pi r^2 h$.

- Case 1: The cylinder has a large height in comparison with its radius, so that its girth is clearly its circumference, $2\pi r$. This means that a measurement of the cylinder reveals that $h \geqq 2r$, so that the Post Office's definition of "length" is surely the height of the cylinder rather than one of its diameters. Then we are to maximize $V = \pi r^2 h$ given $2\pi rh + h = 108$. Then $h = 108 - 2\pi r$, and thus

$$V(r) = \pi r^2(108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3$$

with domain determined by the conditions that $r \geqq 0$ and $2r \leqq h$; the latter condition implies that

$$2r \leqq 108 - 2\pi r, \quad \text{so that} \quad 2r + 2\pi r \leqq 108: \quad r \leqq \frac{54}{\pi + 1} \approx 13.038462.$$

Next, $V'(r) = 216\pi r - 6\pi^2 r^2$; $V'(r) = 0$ when

$$r = \frac{36}{\pi} \approx 11.459156; \quad V\left(\frac{36}{\pi}\right) = \frac{46656}{\pi} \approx 14851.066$$

cubic inches is the maximum volume in this case. Note that this is somewhat greater than the maximum volume in Problem 47.

• Case 2: $2r \geqq h$. In this case the "girth" of the box is, strictly speaking, its greatest circumference in a plane perpendicular to its length. Its length is a diameter of the cylinder, so its girth is $4r + 2h$. Because its length is $2r$, we are to maximize box volume $V = \pi r^2 h$ subject to the side condition $6r + 2h = 108$. Because $h = 54 - 3r$, we maximize

$$V(r) = \pi r^2 (54 - 3r) = 54\pi r^2 - 3\pi r^3,$$

with domain determined by the conditions that $h \leqq 2r$ and $h \geqq 0$. The latter implies that $4r \leqq 108$, and thus the domain of V is now

$$\frac{54}{5} \leqq r \leqq \frac{108}{4}; \quad \text{that is,} \quad 10.8 \leqq r \leqq 25.2.$$

In this case $V'(r) = 0$ when $r = 12$; the corresponding value of h is 18 and the maximum volume of the box in this case is $V(12) = 2592\pi \approx 8143.008$ cubic inches.

Answer: Design the cylindrical box as indicated in Case 1 for maximum volume.

Note: See also Problems 25 and 26 in Section 3.6.

C13S05.049: Suppose that the upper corner of the box in the first octant meets the paraboloid at the point (x, y, z), so that $z = 1 - x^2 - y^2$. We are to maximize box volume $V = 2x \cdot 2y \cdot z$; that is,

$$V(x, y) = 4xy(1 - x^2 - y^2) = 4xy - 4x^3 y - 4xy^2, \quad 0 \leqq x, \quad 0 \leqq y, \quad x^2 + y^2 \leqq 1.$$

Because V is continuous ($V(x, y)$ is a polynomial) and its domain is a closed and bounded subset of the xy-plane, V has a global maximum—which does not occur on the boundary of its domain because $V(x, y)$ is identically zero there. Hence the maximum we seek occurs at an interior critical point. When we set the partial derivatives of V simultaneously equal to zero, we obtain the equations

$$4y(1 - x^2 - y^2) - 8x^2 y = 0, \quad 4x(1 - x^2 - y^2) - 8xy^2 = 0;$$

that is, because neither x nor y is zero at maximum box volume,

$$1 - 3x^2 - y^2 = 0 \quad \text{and } 1 - x^2 - 3y^2 = 0.$$

It follows in the usual way that $y = x$, and thus that $x = y = \frac{1}{2}$. The corresponding value of z is also $\frac{1}{2}$, so the dimensions of the box of maximum volume are 1 by 1 by $\frac{1}{2}$ and its volume is $\frac{1}{2}$.

C13S05.050: Place the box with its sides parallel to the coordinate planes and let (x, y, z) be the point where its upper corner in the first octant meets the hemisphere. Then $z = (R^2 - x^2 - y^2)^{1/2}$, so we are to maximize box volume

$$V(x, y) = 2x \cdot 2y \cdot (R^2 - x^2 - y^2)^{1/2}, \quad 0 \leqq x, \quad 0 \leqq y, \quad x^2 + y^2 \leqq R^2.$$

There is a global maximum because V is continuous on a closed and bounded subset of the xy-plane, and the maximum does not occur on the boundary because $V(x, y)$ is identically zero there. When we set both partials equal to zero, we obtain the equations

$$4y(R^2 - x^2 - y^2)^{1/2} = \frac{4x^2 y}{(R^2 - x^2 - y^2)^{1/2}}, \qquad 4x(R^2 - x^2 - y^2)^{1/2} = \frac{4xy^2}{(R^2 - x^2 - y^2)^{1/2}};$$

that is,

$$x^2 = R^2 - x^2 - y^2 \quad \text{and} \quad y^2 = R^2 - x^2 - y^2$$

because neither x nor y is zero at maximum box volume. It follows that $y = x$ and then that

$$x = y = z = \frac{R\sqrt{3}}{3},$$

and the maximum possible volume of the box is $\dfrac{4R^3\sqrt{3}}{9}$.

Plausibility check: The volume of the maximal box is approximately 37% of that of the hemisphere—a reasonable answer.

C13S05.051: Let r be the common radius of the two cones and the cylinder, h the height of the cylinder, and z the height of each cone. Note that the slant height of each cone is $(r^2 + z^2)^{1/2}$, so each has curved surface area

$$2\pi \cdot \frac{r}{2} \cdot (r^2 + z^2)^{1/2} = \pi r(r^2 + z^2)^{1/2}.$$

We are to minimize the total surface area

$$A = 2\pi r(r^2 + z^2)^{1/2} + 2\pi rh \tag{1}$$

of the buoy given fixed volume $V = \frac{2}{3}\pi r^2 z + \pi r^2 h$. We first solve this last equation for

$$h = \frac{3V - 2\pi r^2 z}{3\pi r^2},$$

then substitute in (1) to express A as a function of r and z:

$$A(r, z) = \frac{2V}{r} - \frac{4\pi rz}{3} + 2\pi r(r^2 + z^2)^{1/2}.$$

The domain of A is described by the inequalities

$$0 < r, \quad 0 \leqq z, \quad r^2 z \leqq \frac{3V}{2\pi},$$

and though it is neither closed nor bounded, it can be shown by an argument similar to the one in Example 7 that $A(r, z)$ has a global minimum that does not occur on the boundary of its domain (unless it occurs where $h = 0$ or where $z = 0$; we will attend to those possibilities later). Moreover, intuition and experience suggest that the minimal surface area will occur when the figure can be inscribed in a nearly spherical ellipsoid.

Next,

$$A_z(r,\, z) = \frac{2[3\pi rz - 2\pi r(r^2 + z^2)^{1/2}]}{3(r^2 + z^2)^{1/2}},$$

so $A_z(r,\, z) = 0$ when $3z = 2(r^2 + z^2)^{1/2}$. Therefore, when $A_z(r,\, z) = 0$, we have both $2r = z\sqrt{5}$ and $(r^2 + z^2)^{1/2} = \frac{3}{2}z$.

Also,

$$A_r(r,\, z) = \frac{2[6\pi r^4 + 3\pi r^2 z^2 - 3V(r^2 + z^2)^{1/2} - 2\pi r^2 z(r^2 + z^2)^{1/2}]}{3r^2(r^2 + z^2)^{1/2}},$$

so $A_r(r,\, z) = 0$ when

$$6\pi r^4 + 3\pi r^2 z^2 - 3V(r^2 + z^2)^{1/2} - 2\pi r^2 z(r^2 + z^2)^{1/2} = 0.$$

We substitute $\frac{3}{2}z$ for $(r^2 + z^2)^{1/2}$ in this last equation to find that when both partials vanish, also

$$\frac{9Vz}{2} = 6\pi r^4,$$

then we replace r with $\frac{1}{2}z\sqrt{5}$ to find that when both partials vanish,

$$25\pi z^3 = 12V.$$

Thus the minimum surface area seems to occur when

$$z = \left(\frac{12V}{25\pi}\right)^{1/3} \approx (0.534601847029)V^{1/3},$$

for which

$$r = \left(\frac{9V^2}{20\pi^2}\right)^{1/6} \approx (0.597703035427)V^{1/3}$$

and

$$h = \left(\frac{12V}{25\pi}\right)^{1/3} \approx (0.534601847029)V^{1/3}.$$

At these values of the variables, the surface area is

$$A = 5^{1/6}(18\pi V^2)^{1/3} \approx (5.019214931473)V^{2/3}.$$

The symmetry of the solution—that $h = z$—suggests that we have found the minimum, but we have yet to check the cases $h = 0$ and $z = 0$.

- If $z = 0$, then the buoy is a cylinder with radius r, height h, total surface area $A = 2\pi r^2 + 2\pi rh$, and fixed volume $V = \pi r^2 h$; we are to minimize its total surface area. We substitute

$$h = \frac{V}{\pi r^2}$$

in the surface area formula to obtain the function to be minimized:

$$A(r) = 2\pi r^2 + \frac{2V}{r}, \quad 0 < r < \infty.$$

Then

$$A'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2}.$$

Thus $A'(r) = 0$ when

$$r = r_0 = \left(\frac{V}{2\pi}\right)^{1/3} \approx (0.541926070139)V^{1/3}.$$

The corresponding value of $A(r)$ is a global minimum by the first derivative test, and it is

$$A(r_0) = (54\pi V^2)^{1/3} \approx (5.535810445932)V^{2/3}.$$

This is somewhat larger than the minimum we found in the case $z > 0$.

- If $h = 0$, then the buoy consists of two congruent right circular cones with their bases, circles of radii r, coinciding, and each cone of height z. The total volume of the two cones is

$$V = \frac{2}{3}\pi r^2 z,$$

which we solve for z and substitute in the surface area formula $A = 2\pi r(r^2 + z^2)^{1/2}$ to obtain the function to be minimized,

$$A(r) = 2\pi r\left(r^2 + \frac{9V^2}{4\pi^2 r^4}\right)^{1/2} = \frac{(4\pi^2 r^6 + 9V^2)^{1/2}}{r}, \quad 0 < r.$$

Now

$$A'(r) = \frac{8\pi^2 r^6 - 9V^2}{r^2(4\pi^2 r^6 + 9V^2)^{1/2}};$$

$A'(r) = 0$ when $8\pi^2 r^6 = 9V^2$, so that

$$r = r_0 = \frac{\sqrt{2}}{2}\left(\frac{3V}{\pi}\right)^{1/3} \approx (0.696319882685)V^{1/3}.$$

Then

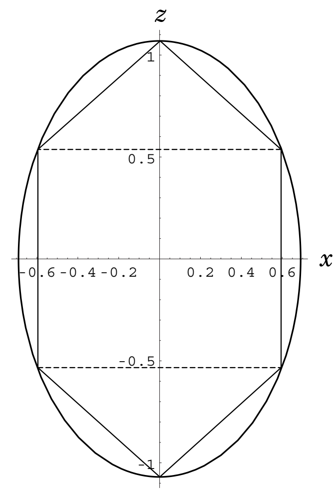
$$A(r_0) = 3^{7/6} \cdot \pi^{1/3} \cdot V^{2/3} \approx (5.276647566071)V^{2/3},$$

which is larger than the minimum found in the first part of this solution. Therefore the minimum possible surface area of the buoy is $5^{1/6}(18\pi V^2)^{1/3} \approx (5.019214931473)V^{2/3}$.

A final observation: The buoy of minimal possible surface area cannot be inscribed in a sphere. If $V = 1$ (say), then it can be inscribed in an ellipsoid (actually, a prolate spheroid) with approximate equation

$$\frac{x^2}{(0.69016802)^2} + \frac{y^2}{(0.69016802)^2} + \frac{z^2}{(1.06923700)^2} = 1$$

with the axis of symmetry of the buoy on the z-axis. A figure showing the elliptical cross-section of this ellipsoid in the xz-plane and the cross-section of the buoy in the xz plane (for the case $V = 1$) is next.



C13S05.052: Let $2x$ be the length of the base of the rectangle, y its height, and h the height of the isosceles triangle. We are to maximize the total area $A = 2xy + xh$ of the window given that it has perimeter 24; that is,

$$2x + 2y + 2\sqrt{x^2 + h^2} = 24, \qquad \text{so that} \qquad y = 12 - x - \sqrt{x^2 + h^2}\,. \tag{1}$$

Substitution for y in the area formula yields the quantity to be maximized:

$$A(x,\,h) = hx + 24x - 2x^2 - 2x\sqrt{x^2 + h^2}\,, \quad 0 \leqq x \leqq 6, \quad 0 \leqq h \leqq 12.$$

Because A is continuous on its domain and the latter is a closed and bounded subset of the xy-plane, $A(x,\,y)$ has a global maximum value. To find it, first write the equations $A_x(x,\,h) = 0 = A_h(x,\,h)$. These simplify to

$$\frac{(24 + h - 4x)\sqrt{x^2 + h^2} - 2h^2 - 4x^2}{\sqrt{x^2 + h^2}} = 0 \quad \text{and} \quad \frac{x\sqrt{x^2 + h^2} - 2hx}{\sqrt{x^2 + h^2}} = 0.$$

The second of these equations implies that $\sqrt{x^2 + h^2} = 2h$. It follows that $x^2 = 3h^2$, so that

$$h = \frac{x\sqrt{3}}{3}. \tag{2}$$

Moreover, substitution of $2h$ for $\sqrt{x^2 + h^2}$ in the first equation yields

$$\begin{aligned}
&2h^2 + 4x^2 = (24 + h - 4x)(2h) = 48h + 2h^2 - 8xh, \qquad \text{so that}\\
&x^2 = 12h - 2xh = h(12 - 2x).
\end{aligned}$$

Thus, by Eq. (2),

$$\begin{aligned}
&x^2 = \frac{x\sqrt{3}}{3}(12 - 2x) = 4x\sqrt{3} - \frac{2x^2\sqrt{3}}{3};\\
&3x^2 = 12x\sqrt{3} - 2x^2\sqrt{3};\\
&3x = 12\sqrt{3} - 2x\sqrt{3}
\end{aligned}$$

—the cancellation is possible because $x = 0$ does not maximize $A(x, h)$. Thus

$$\left(3 + 2\sqrt{3}\right)x = 12\sqrt{3};$$

$$x = \frac{12\sqrt{3}}{3 + 2\sqrt{3}} = 12\left(2 - \sqrt{3}\right)$$

after some routine arithmetic. Next,

$$h = \frac{x\sqrt{3}}{3} = \left(4\sqrt{3}\right) \cdot \left(2 - \sqrt{3}\right) = 4\left(2\sqrt{3} - 3\right).$$

Moreover,

$$y = 12 - x - \sqrt{x^2 + h^2} = 12 - x - 2h = 4\left(3 - \sqrt{3}\right)$$

after some more arithmetic. Finally, it appears that the maximum possible value of $A(x, h)$ is

$$A = 2xy + xh = 24\left(2 - \sqrt{3}\right) \cdot 4\left(3 - \sqrt{3}\right) + 12\left(2 - \sqrt{3}\right) \cdot 4\left(2\sqrt{3} - 3\right) = 144\left(2 - \sqrt{3}\right)$$

after yet another struggle with the arithmetic. There are two other possibilities for the global maximum to eliminate.

- If $h = 0$: Then we are to maximize the area of a rectangle of perimeter 24, and the largest such rectangle is a square of side 6 and area 36.

- If $y = 0$: We are to maximize the area of an isosceles triangle of perimeter 24. It is a routine single-variable calculus problem to show that the isosceles triangle of fixed perimeter and maximum area is equilateral; in this case, with each side of length 8 and total area $16\sqrt{3}$.

The maximum area in the first case is approximately 38.58468371. In the second case it is 36 and in the third case it is approximately 27.71281293. Hence the construction in the first case yields the window of largest area, with

$$x \approx 3.215390309173,$$

$$h \approx 1.856406460551,$$

$$y \approx 5.071796769724,$$

$$A \approx 38.584683710082.$$

C13S05.053: We want to minimize

$$f(x, y) = x^2 + (y-1)^2 + x^2 + y^2 + (x-2)^2 + y^2 = (x-2)^2 + 2x^2 + (y-1)^2 + 2y^2$$

with domain the entire xy-plane. An argument similar to the one used in Example 7 establishes that $f(x, y)$ has a global minimum value, which must be at a point where both partial derivatives are zero. These equations are $6x - 4 = 0$, $6y - 2 = 0$, with solution $x = \frac{2}{3}$, $y = \frac{1}{3}$. Because this is the only critical point, we have located the point that minimizes $f(x, y)$. It is $\left(\frac{2}{3}, \frac{1}{3}\right)$, and the value of $f(x, y)$ there is $\frac{10}{3}$.

It would seem somewhat more practical to find the point (x, y) such that the sum of the distances (not their squares) from (x, y) to the three points $(0, 1)$, $(0, 0)$, and $(2, 0)$ is a minimum. For example, where

should a power company be located to minimize the total length of its cables to industries located at the three points $(0, 1)$, $(0, 0)$, and $(2, 0)$? This is a much more difficult problem. The function to be minimized is

$$h(x, y) = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2},$$

and its partial derivatives are

$$f(x, y) = h_x(x, y) = \frac{x}{\sqrt{x^2 + (y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2 + y^2}} + \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and}$$

$$g(x, y) = h_y(x, y) = \frac{y-1}{\sqrt{x^2 + (y-1)^2}} + \frac{y}{\sqrt{(x-2)^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}.$$

It is doubtful that any present-day computer algebra system could solve the simultaneous equations

$$f(x, y) = 0, \qquad g(x, y) = 0 \tag{1}$$

exactly with a reasonable amount of memory and in a reasonable time. But recall Newton's method from the last section of Chapter 3. Techniques of this chapter can be used to extend Newton's method to two simultaneous equations in two unknowns, as in Eq. (1). Here is the extension.

The graph of $f(x, y) = 0$ is, generally, a curve in the xy-plane, as is the graph of $g(x, y) = 0$. The point where these curves meet is the simultaneous solution that we seek. If (x_0, y_0) is an initial "guess" for the simultaneous solution, then $(x_0, y_0, f(x_0, y_0))$ is a point on the surface $z = f(x, y)$ and $(x_0, y_0, g(x_0, y_0))$ is a point on the surface $z = g(x, y)$. The tangent planes to these respective surfaces at these respective points generally meet the xy-plane in a pair of lines, whose intersection should be close to the desired solution $(x_\star, y_\star)$ of the simultaneous equations in (1). Let (x_1, y_1) denote the intersection of these lines, and repeat the process with (x_0, y_0) replaced with (x_1, y_1). This leads to the pair of iterative formulas

$$x_{k+1} = x_k - \frac{f(x_k, y_k)g_y(x_k, y_k) - g(x_k, y_k)f_y(x_k, y_k)}{f_x(x_k, y_k)g_y(x_k, y_k) - g_x(x_k, y_k)f_y(x_k, y_k)},$$

$$y_{k+1} = y_k - \frac{g(x_k, y_k)f_x(x_k, y_k) - f(x_k, y_k)g_x(x_k, y_k)}{f_x(x_k, y_k)g_y(x_k, y_k) - g_x(x_k, y_k)f_y(x_k, y_k)}$$

for $k = 0, 1, 2, \ldots$.

We wrote a simple *Mathematica* 3.0 program to implement this procedure with the initial guess $(x_0, y_0) = (0.5, 0.5)$. Here are the results (rounded, of course):

$(x_1, y_1) = (0.201722209269, 0.369098300563)$, $\quad (x_2, y_2) = (0.250232659645, 0.283132873843)$,

$(x_3, y_3) = (0.254890752792, 0.304117462143)$, $\quad (x_4, y_4) = (0.254568996440, 0.304503089648)$,

$(x_5, y_5) = (0.254569313597, 0.304503701206)$, $\quad (x_6, y_6) = (0.254569313597, 0.304503701206)$,

and—to the number of digits shown—$(x_7, y_7) = (x_6, y_6)$. The sum of the distances from (x_6, y_6) to the three points $(0, 1)$, $(0, 0)$, and $(2, 0)$ is approximately 2.909312911180.

Warning: A much better "initial guess" is required than in the case of a function of a single variable. For example, with the initial guess $(x_0, y_0) = (2/3, 1/3)$ (the solution to the original version of Problem 53, which one would think would be a fairly good initial guess), $(x_6, y_6) \approx (1422.779, 732.866)$. Thus it is clear that Newton's method is not converging to the correct solution.

C13S05.054: We are to minimize

$$f(x,\,y) = (x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2 + (x-a_3)^2 + (y-b_3)^2$$

with domain the entire xy-plane. By an argument similar to that used in the solution of Example 7, there is a global minimum value and it occurs at a critical point at which $f_x(x,\,y) = 0$ and $f_y(x,\,y) = 0$. These equations are

$$2(x-a_1) + 2(x-a_2) + 2(x-a_3) = 0, \quad 2(y-b_1) + 2(y-b_2) + 2(y-b_3) = 0$$

with solution

$$x = \frac{a_1 + a_2 + a_3}{3}, \quad y = \frac{b_1 + b_2 + b_3}{3}.$$

Because the critical point is unique, we have found the point required in Problem 54. The value of $f(x,\,y)$ at this point is

$$\frac{2}{3}(a_1^2 - a_1a_2 + a_2^2 - a_1a_3 + a_3^2 - a_2a_3 + b_1^2 - b_1b_2 + b_2^2 - b_1b_3 + b_3^2 - b_2b_3).$$

In connection with Problem 53, consider the general problem of finding the point the sum of whose distances from three or more fixed points is minimal. This problem is extremely difficult (see the solution to Problem 53 for an approximation method when the numbers $\{a_i\}$ and $\{b_i\}$ are given). It is amusing that the exact solution can be accurately approximated in a few moments using an analog computer constructed from two thin sheets of clear rigid plastic (with coordinates inscribed on one), three small dowel rods each about 4 cm long, a little glue, and some soap-bubble solution. See the book *Plateau's Problem* by Frederick J. Almgren, Jr. (W. A. Benjamin, Inc., New York, 1966) for more information.

C13S05.055: Let $2x$ denote the length of the base of each isosceles triangle, z the height of each triangle, and y the distance between the triangles. We are to minimize the area $A = 2xz + 2y\sqrt{x^2+z^2}$ given the house has fixed volume $V = xyz$. Solve the latter equation for y and subsitute in the area formula to obtain the function

$$A(x,\,z) = 2xz + \frac{2V\sqrt{x^2+z^2}}{xz}, \quad 0 < x, \quad 0 < z.$$

An argument similar to the one used in the solution of Example 7 shows that A must have a global minimum value even though its domain is neither closed nor bounded. Therefore, if there is a unique critical point of A in its domain, that will be the location of its global minimum.

To find the critical point or points, we use *Mathematica* 3.0 and first define

```
a[x_, z_] := 2*x*z + (2*v*Sqrt[x*x + z*z])/(x*z)
```

Then we compute and simplify both partial derivatives.

```
d1 = D[ a[x,z], x ]
```

$$2z + \frac{2v}{z\sqrt{x^2+z^2}} - \frac{2v\sqrt{x^2+z^2}}{x^2z}$$

```
d1 = Together[d1]
```

$$\frac{2z\left(-v + x^2\sqrt{x^2+z^2}\,\right)}{x^2\sqrt{x^2+z^2}}$$

```
n1 = Numerator[ d1 ]
```

$$2z\left(-v + x^2\sqrt{x^2+z^2}\,\right)$$

```
d2 = D[ a[x,z], z ]
```

$$2x + \frac{2v}{x\sqrt{x^2+z^2}} - \frac{2v\sqrt{x^2+z^2}}{xz^2}$$

```
d2 = Together[ d2 ]
```

$$\frac{2x\left(-v + z^2\sqrt{x^2+z^2}\,\right)}{z^2\sqrt{x^2+z^2}}$$

```
n2 = Numerator[ d2 ]
```

$$2x\left(-v + z^2\sqrt{x^2+z^2}\,\right)$$

Because neither $x = 0$ nor $z = 0$ (because $V > 0$), we may cancel:

```
n1 = n1/(2*z) // Cancel
```

$$-v + x^2\sqrt{x^2+z^2}$$

```
n2 = n2/(2*x) // Cancel
```

$$-v + z^2\sqrt{x^2+z^2}$$

Both of the last expressions must be zero when both partial derivatives are set equal to zero, and it follows that $z = x$. We substitute this information into the numerator `n1` of $A_x(x,\,z)$ and set the result equal to zero.

```
n1 /.  z -> x
```

$$-v + \sqrt{2}\,(x^2)^{3/2}$$

```
Solve[ % == 0, x ]
```

And *Mathematica* returns six solutions, only two of which are real, and only one of the two is positive: We find that

$$x = z = \frac{V^{1/3}}{2^{1/6}}, \quad \text{and} \quad y = 2^{1/3} \cdot V^{1/3},$$

so that the minimum possible surface area is $3 \cdot 2^{2/3} \cdot V^{2/3}$.

C13S05.056: If the dimensions of the box are x by y by z, then $x^2 + y^2 + z^2 = L^2$. Hence we are to maximize box volume

$$V(x,\,y) = xy\sqrt{L^2 - x^2 - y^2}\,, \quad 0 \leqq x, \quad 0 \leqq y, \quad x^2 + y^2 \leqq L^2.$$

The domain D of V is the part of the disk with center $(0, 0)$ and radius L that lies in the first quadrant (including the boundary points of D). This is a closed and bounded subset of the xy-plane and V is continuous there, so $V(x, y)$ has a global maximum on D—and this maximum does not lie on the boundary of D because $V(x, y)$ is identically zero there. Thus we seek interior critical points of V. When we write the equations $V_x(x, y) = 0$ and $V_y(x, y) = 0$, we obtain

$$y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}} = 0,$$

$$x\sqrt{L^2 - x^2 - y^2} - \frac{x y^2}{\sqrt{L^2 - x^2 - y^2}} = 0.$$

Thus

$$y(L^2 - x^2 - y^2) - x^2 y = 0,$$

$$x(L^2 - x^2 - y^2) - x y^2 = 0.$$

The maximum does not occur when $x = 0$ nor when $y = 0$, so we may cancel without losing the desired solution:

$$(L^2 - x^2 - y^2) - x^2 = 0,$$

$$(L^2 - x^2 - y^2) - y^2 = 0.$$

It follows that $y^2 = x^2$, and hence (because both are positive) $y = x$. If we began anew and eliminated y rather than z, we would discover that $x = z$. Hence $x = y = z$, and so $3x^2 = L^2$; therefore

$$x = y = z = \frac{L\sqrt{3}}{3}$$

maximizes V and its maximum value is $\dfrac{\sqrt{3}}{9} \cdot L^3$.

C13S05.057: Let x, y, and z be the lengths of the edges of the three squares. We are to maximize and minimize their total area $A = x^2 + y^2 + z^2$ given the condition $4x + 4y + 4z = 120$; that is, $x + y + z = 30$. Using this side condition to eliminate z in the expression for A, we obtain the function

$$A(x, y) = x^2 + y^2 + (30 - x - y)^2, \quad 0 \leqq x, \quad 0 \leqq y, \quad x + y \leqq 30.$$

The domain D of A is the triangular region in the xy-plane with vertices at $(0, 0)$, $(30, 0)$, and $(0, 30)$, including the boundary segments. Hence D is a closed and bounded subset of the plane and A is continuous there (because $A(x, y)$, when expanded, is a polynomial). Therefore there is both a global maximum and a global minimum value of $A(x, y)$. We proceed in the usual way, first setting both partial derivatives equal to zero:

$$2x - 2(30 - x - y) = 0 \quad \text{and} \quad 2y - 2(30 - x - y) = 0.$$

It follows that $y = x$. By symmetry (rework the problem eliminating y rather than z) $z = x$, so that $x = y = z$. Therefore $x = y = z = 10$ may yield an extremum of A.

- On the boundary segment of D on which $y = 0$,

$$A(x, 0) = f(x) = x^2 + (30 - x)^2, \quad 0 \leqq x \leqq 30.$$

Methods of single-variable calculus yield the critical point $x = 15$; we have also the two endpoints of the domain of f to check.

- On the boundary segment of D on which $x = 0$,

$$A(0, y) = g(y) = y^2 + (30 - y)^2, \quad 0 \leqq y \leqq 30.$$

Methods of single-variable calculus yield the critical point $y = 15$; we have also the two endpoints of the domain of g to check.

- On the boundary segment of D with equation $x + y = 30$, we have $y = 30 - x$, so that

$$A(x, 30 - x) = h(x) = x^2 + (30 - x)^2, \quad 0 \leqq x \leqq 30.$$

Again, methods of single-variable calculus yield the critical point $x = 15$; the endpoints of this boundary segment will be checked in the other two cases.

Results:

$$\begin{aligned} A(10, 10) &= 300, & A(15, 0) &= 450, \\ A(0, 15) &= 450, & A(0, 0) &= 900, \\ A(30, 0) &= 900, & A(0, 30) &= 900, \quad \text{and} \\ A(15, 15) &= 450. \end{aligned}$$

Answer: For maximum total area, make only one square, measuring 30 cm on each side, with area 900 cm^2. For minimum total area, make three equal squares, each measuring 10 cm on each side, with total area 300 cm^2.

C13S05.058: Suppose that the edges of the cubes have lengths x, y, and z. We are to maximize and minimize total surface area $A = 6x^2 + 6y^2 + 6z^2$ given $x^3 + y^3 + z^3 = V$, a constant. Solve the last equation for

$$z = (V - x^3 - y^3)^{1/3}$$

and substitute in the area formula to obtain the function

$$A(x, y) = 6x^2 + 6y^2 + 6(V - x^3 - y^3)^{2/3}, \quad 0 \leqq x, \quad 0 \leqq y, \quad x^3 + y^3 \leqq V.$$

Then A is continuous on its domain, which is a closed and bounded subset of the xy-plane, and hence $A(x, y)$ has both a global maximum and a global minimum there. When we set both partials equal to zero, we get the equations

$$12x - \frac{12x^2}{(V - x^3 - y^3)^{1/3}} = 0 \quad \text{and} \quad 12y - \frac{12y^2}{(V - x^3 - y^3)^{1/3}} = 0. \tag{1}$$

There are several cases to consider.

- If $x \neq 0$, $y \neq 0$, and $x^3 + y^3 < V$ (the same as $z \neq 0$), then x and y can be cancelled from the equations in (1), and it follows immediately that $y = x$. Then the first of the equations in (1) implies that

$$(V - 2x^3)^{1/3} = x; \quad V - 2x^3 = x^3; \quad x = y = z = (V/3)^{1/3}$$

($z = x$ by symmetry). In this case the total surface area of the three cubes is $6 \cdot 3^{1/3} \cdot V^{2/3}$.

- If $x = 0$, $y \neq 0$, and $y^3 < V$ (the same as $z \neq 0$), then the second equation in (1) implies that

$$1 - \frac{y}{(V - y^3)^{1/3}} = 0; \quad V - y^3 = y^3; \quad y = (V/2)^{1/3}.$$

Rework the problem eliminating y instead of z to discover that $z = (V/2)^{1/3}$ as well, or simply substitute $x = 0$ and $y = (V/2)^{1/3}$ in $x^3 + y^3 + z^3 = V$ and solve for z. In this case the total surface area of the *two* cubes will be

$$6y^2 + 6z^2 = 12(V/2)^{2/3} = 6 \cdot 2^{1/3} \cdot V^{2/3},$$

less than in the first case.

- If $x \neq 0$, $y = 0$, and $x^2 < V$ (the same as $z \neq 0$), proceed exactly as in the previous case to obtain total surface area of the *two* cubes again $6 \cdot 2^{1/3} \cdot V^{2/3}$.

- If $x = 0$ and $y = 0$, then we get no information from the equations in (1). But in this case $z^3 = V$, so that $z = V^{1/3}$. There is only one cube, with total surface area $6V^{2/3}$, less than in any of the previous cases.

- If $z = 0$, then $x^3 + y^3 = V$ and the total surface area of the *two* (or fewer) cubes is $6x^2 + 6y^2$. Thus the maximum-minimum problem becomes a problem in single-variable calculus: Find the global maximum and minimum values of

$$f(x) = 6x^2 + 6(V - x^3)^{2/3}, \quad 0 \leqq x \leqq V^{1/3}.$$

Then

$$f'(x) = \frac{12x\left[(V - x^3)^{1/3} - x\right]}{(V - x^3)^{1/3}},$$

and $f'(x) = 0$ when $x = 0$ and when $x = (V/2)^{1/3}$. There is also the endpoint $x = V^{1/3}$ to check. If $x = 0$ then there is only one cube, of total surface area $6 \cdot V^{2/3}$. If $x = (V/2)^{1/3}$ then $y = x$; there are two equal cubes of total surface area $6 \cdot 2^{1/3} \cdot V^{2/3}$. If $x = V^{1/3}$ then there is only one cube, of total surface area $6 \cdot V^{2/3}$.

Answer: For maximum total surface area, make three equal cubes, each of edge length $(V/3)^{1/3}$ and total area $6 \cdot 3^{1/3} \cdot V^{2/3}$. For minimum total surface area, make only one cube, of edge length $V^{1/3}$ and surface area $6 \cdot V^{2/3}$.

Comment: This awkward method of solving Problem 58 is awkward precisely because it does not take advantage of the symmetries present in the original problem. For a more elegant (and much shorter) solution, use the method of Lagrange multipliers, which will be discussed in Section 13.9.

C13S05.059: Using the notation in Fig. 13.5.16, we have cross-sectional area

$$A(x,\,\theta) = (L - 2x)x\sin\theta + x^2\sin\theta\,\cos\theta, \quad 0 \leqq \theta \leqq \frac{\pi}{2}, \quad 0 \leqq x \leqq \frac{L}{2}.$$

to be maximized. When we set the partial derivatives of A equal to zero, we obtain the equations

$$(L - 2x)\sin\theta - 2x\sin\theta + 2x\sin\theta\,\cos\theta = 0, \quad (L - 2x)x\cos\theta + x^2\cos^2\theta - x^2\sin^2\theta = 0. \tag{1}$$

The first equation in (1) yields

$$(L - 2x) - 2x + 2x\cos\theta = 0 \qquad \text{or} \qquad \sin\theta = 0,$$

but the second of these can be rejected as $\theta = 0$ minimizes the cross-sectional area. The second equation in (1) yields

$$(L - 2x)\cos\theta + x\cos^2\theta - x\sin^2\theta = 0 \qquad \text{or} \qquad x = 0,$$

and the second of these can also be rejected because $x = 0$ minimizes the area. Thus we have the simultaneous equations

$$L - 2x - 2x + 2x\cos\theta = 0 \qquad \text{and} \qquad (L - 2x)\cos\theta + x\cos^2\theta - x\sin^2\theta = 0. \tag{2}$$

We solve the first of these for $L - 2x = 2x - 2x\cos\theta$ and substitute in the second to obtain

$$\begin{aligned}
&2x(1 - \cos\theta)\cos\theta + x\cos^2\theta - x\sin^2\theta = 0;\\
&2(1 - \cos\theta)\cos\theta + \cos^2\theta - \sin^2\theta = 0;\\
&2\cos\theta - 2\cos^2\theta + \cos^2\theta - 1 + \cos^2\theta = 0;\\
&\cos\theta = \frac{1}{2};\\
&\theta = \frac{\pi}{3}.
\end{aligned}$$

In the second step we used the fact that $x = 0$ minimizes A to cancel x with impunity; in the last step we used the domain of A to determine θ. In any case, the first equation in (2) lets us determine the maximizing value of x as well:

$$L - 4x + x = 0, \qquad \text{and thus} \qquad x = \frac{L}{3}.$$

We have found only one critical point, and the only endpoint that might produce a larger cross-sectional area occurs when $\theta = \pi/2$, in which case the first equation in (2) implies that

$$L - 4x = 0, \qquad \text{so that} \qquad x = \frac{L}{4}.$$

In the latter case the cross-sectional area of the gutter is $L^2/8$, but in the previous case, when $\theta = \pi/3$, we find that $A = \left(L^2\sqrt{3}\,\right)/12 \approx (0.1443)L^2$, so this is the maximum possible cross-sectional area of the gutter.

C13S05.060: Given: $f(x, y) = (y - x^2)(y - 3x^2)$. Part (a):

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{3h^4}{h} = \lim_{h\to 0} 3h^3 = 0$$

and

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{k^2}{k} = \lim_{k\to 0} k = 0.$$

Part (b): Suppose that m is a real number. Then

$$f(x,\, mx) = (mx - 3x^2)(mx - x^2) = 3x^4 - 4mx^3 + m^2x^2 = x^2(m - 3x)(m - x).$$

Therefore if $x \neq 0$ and

$$-\frac{m}{3} < x < \frac{m}{3}, \quad \text{then} \quad f(x,\, mx) > 0,$$

so—because $f(0,\, 0) = 0$—$f(x,\, y)$ has a local minimum at $(0,\, 0)$ on the line $y = mx$. On the vertical line through $(0,\, 0)$ (the y-axis), $f(x,\, y) > 0$ for all $(x,\, y)$ except that $f(0,\, 0) = 0$. Therefore $f(x,\, y)$ has a local minimum at $(0,\, 0)$ on every straight line that passes through $(0,\, 0)$.

Part (c): On the parabola $y = 2x^2$, we have

$$f(x,\, y) = f(x,\, 2x^2) = (2x^2 - x^2)(2x^2 - 3x^2) = -x^4,$$

and so $f(x,\, y)$ does not have a local minimum at $(0,\, 0)$ on the parabola $y = 2x^2$. (In fact, it has a global maximum at $(0,\, 0)$ on that curve.) Therefore $f(x,\, y)$ does not have a local minimum at $(0,\, 0)$ even though it has one on every straight line through $(0,\, 0)$.

C13S05.061: Given:

$$P(x) = -2x^2 + 12x + xy - y - 10,$$

$$Q(y) = -3y^2 + 18y + 2xy - 2x - 15.$$

Part (a): This is known as a *game of perfect information*—each player (manager) knows every strategy available to his opponent. Each manager computes

$$P'(x) = 12 - 4x + y, \quad Q'(y) = 18 + 2x - 6y,$$

sets both equal to zero (knowing that the other manager is doing the same), and solves for $x = 45/11$, $y = 48/11$. Thus each maximizes his profit knowing that the other manager is doing the same; indeed, if either player (manager) deviates from his *optimal strategy*, his profit will decrease and that of his opponent (the other manager) is likely to increase. With these values of x and y, the profits will be

$$P = \frac{2312}{121} \approx 19.107 \quad \text{and} \quad Q = \frac{4107}{121} \approx 33.942.$$

Part (b): After the merger and the agreement to maximize total profit, the new partners plan to maximize

$$R(x,\, y) = P(x) + Q(y) = -2x^2 - 3y^2 + 3xy + 10x + 17y - 25.$$

The junior partner computes both partial derivatives and sets both equal to zero:

$$10 - 4x + 3y = 0, \quad 17 + 3x - 6y = 0.$$

The simultaneous solution is

$$x = \frac{37}{5}, \quad y = \frac{98}{15}$$

for a combined profit of

$$\frac{1013}{15} \approx 67.533 > 53.050 \approx \frac{6419}{121} = \frac{2312}{121} + \frac{4107}{121}.$$

Thus the merger increases total profit.

C13S05.062: First we replace A with x, B with y, and C with z—purely for cosmetic reasons. The weekly profits may be written in the form

$$\begin{aligned}
\text{To AP:}&\quad p(x,\,y,\,z) = 1000x - x^2 - 2xy,\\
\text{To BQ:}&\quad q(x,\,y,\,z) = 2000y - 2y^2 - 4yz,\\
\text{To CR:}&\quad r(x,\,y,\,z) = 1500z - 3z^2 - 6xz.
\end{aligned}$$

Part (a): If the firms act independently, then p is really a function of x alone, q a function of y alone, and r a function of z alone. Again, in this three-person game of perfect information, each manager computes $p'(x)$, $q'(y)$, and $r'(z)$ and sets all three equal to zero:

$$1000 - 2x - 2y = 0, \qquad 2000 - 4y - 4z = 0, \qquad 1500 - 6x - 6z = 0.$$

The simultaneous solution is $x = 125,\ y = 375,$ and $z = 125$. The weekly profit of each company will be

$$\text{AP:}\quad \$15625, \qquad \text{BQ:}\quad \$281250, \qquad \text{CR:}\quad \$46875$$

for a total weekly profit of \$343750.

Part (b): Now firms AP and CR merge and act to maximize their total weekly profit; this fact is known to the management of BQ. The profit function for the new company APCR will be

$$s(x,\,y,\,z) = p(x,\,y,\,z) + q(x,\,y,\,z) = 1000x - x^2 - 2xy + 1500z - 6xz - 3z^2.$$

Note that, contrary to what the notation suggests, s is a function of x and z alone. The firms act as did the two in Problem 61: APCR computes $s_x,\ s_z,$ and $q'(y)$, sets all three equal to zero, and solves. BQ does the same. The results:

$$1000 - 2x - 2y - 6z = 0, \quad 1500 - 6x - 6z = 0, \quad 2000 - 4y - 4z = 0,$$

with solution $x = 500$, $y = 750$, $z = -250$. But this solution is not feasible; it contains a negative production number. The maximum for APCR must occur at a boundary point of its domain, which is $x \geqq 0$, $z \geqq 0$.

- If $z = 0$, then the profits are these:

$$\begin{aligned}
\text{To APCR:}&\quad s(x) = 1000x - x^2 - 2xy;\\
\text{To BQ:}&\quad q(y) = 2000y - 2y^2.
\end{aligned}$$

Now we solve $s'(x) = 0$ and $q'(y) = 0$ simultaneously:

$$1000 - 2x - 2y = 0, \quad 2000 - 4y = 0: \qquad y = 500, \quad x = 0.$$

The weekly profit of APCR will be \$0 and that of BQ will be \$500000. This is a loss for APCR of \$62500 per week and a weekly gain for BQ of \$218750. This is certainly not the maximum for APCR.

- If $x = 0$, then the profits are these:

$$\begin{aligned}
\text{To APCR:}&\quad s(z) = 1500z - 3z^2;\\
\text{To BQ:}&\quad q(y) = 2000y - 2y^2 - 4yz.
\end{aligned}$$

Now we solve $s'(z) = 0$ and $q'(y) = 0$ simultaneously:

$$1500 - 6z = 0, \quad 2000 - 4y - 4z = 0: \qquad y = 250, \quad z = 250.$$

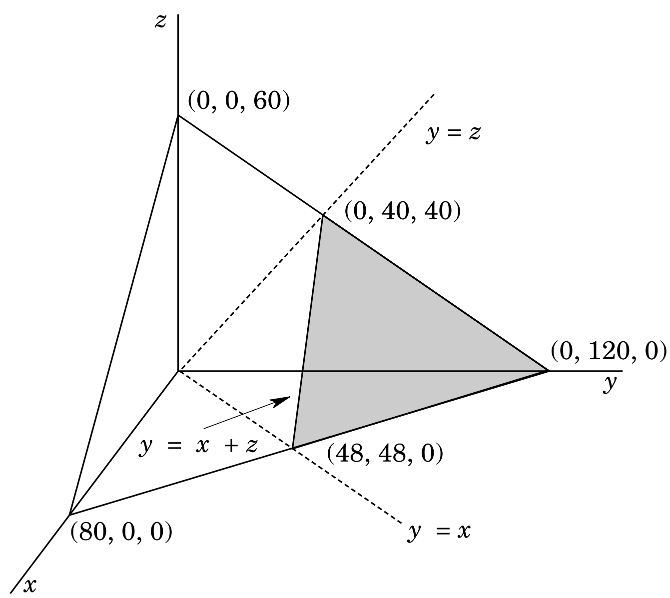
The weekly profit of APCR will be \$187500 and that of BQ will be \$125000. This is a weekly gain for APCR of \$125000 and a weekly loss for BQ of \$156250.

We need not consider the case $y = 0$ because BQ would never choose that alternative, inasmuch as BQ has a positive weekly profit in every case we have considered. *Answer:* After the merger, Ajax Products is in effect shut down and all products are produced by Conglomerate Resources. The newly merged firm will have a weekly profit of \$187500 and Behemoth Quicksilver will have a weekly profit of \$125000.

C13S05.063: Let x be the number of sheep, y the number of hogs, and z the number of head of cattle. Suppose that 60 cattle use 1 unit of land. Then each head of cattle uses $\frac{1}{60}$ units of land. By similar reasoning, each hog uses $\frac{1}{120}$ units of land and each sheep uses $\frac{1}{80}$ units of land. This leads to the side condition

$$\frac{x}{80} + \frac{y}{120} + \frac{z}{60} = 1$$

for each unit of land available. Let's write this in the simpler form $3x + 2y + 4z = 240$. An additional condition in the problem is that $y \geqq x + z$. We are now to maximize the profit P per unit of land, given by $P(x, y, z) = 10x + 8y + 20z$.



The domain of P is the triangular region shown shaded in the preceding figure. It is obtained as follows: Draw the part of the plane $3x+2y+4z = 240$ that lies in the first octant (because none of x, y, or z can be negative). The intersections of that plane with the positive coordinate planes are shown as solid lines. The intersection of the plane $y = x + z$ with the positive coordinate planes is shown as a pair of dashed lines. The condition $y \geqq x + z$ implies that only the part of the first plane to the *right* of the second may be used as the domain of P. Therefore we arrive at the shaded triangle as the domain of the profit function. Finally, because P is a linear function of x, y, and z, its maximum and minimum values occur at the vertices of the shaded triangle. Here, then, are the results:

Vertex:	(48, 48, 0);	Profit:	\$864	per unit of land.
Vertex:	(0, 120, 0);	Profit:	\$960	per unit of land.
Vertex:	(0, 40, 40);	Profit:	\$1120	per unit of land.

Were it not for the restriction of the state law mentioned in the problem, the farmer could maximize her profit per unit of land by raising only cattle: $P(0, 0, 60) = 1200$. *Answer:* Raise 40 hogs and 40 cattle per unit of land, but no sheep.

C13S05.064: Upon setting the partial derivatives of f equal to zero we get the equations

$$f_x(x, y) = ax + by = 0,$$

$$f_y(x, y) = bx + cy = 0,$$

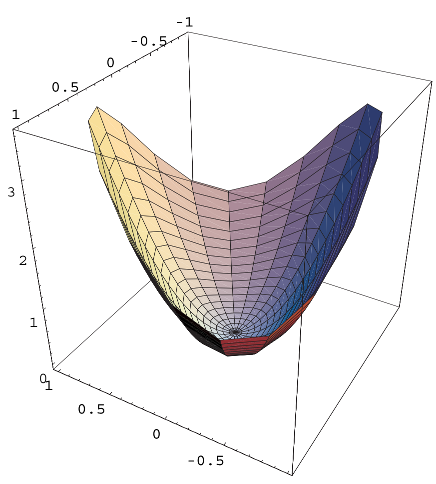
whose only solution is the point (0, 0), unless its coefficient determinant $ac-b^2$ vanishes. Computer graphing of various possibilities with $ac - b^2 = 0$ appears always to yield parabolic cylinders. Indeed, if $ac - b^2 = 0$ then a and c have the same sign and $b = \pm\sqrt{ac}$. Then we note that

$$f(x, y) = \begin{cases} \left(\sqrt{a}\,x \pm \sqrt{c}\,y\right)^2 & \text{if} \quad a, c > 0, \\ -\left(\sqrt{-a}\,x \pm \sqrt{-c}\,y\right)^2 & \text{if} \quad a, c < 0. \end{cases}$$

—C.H.E.

C13S05.065: Here is the case $a = 3,\ b = 1,\ c = 2$ (via *Mathematica* 3.0):

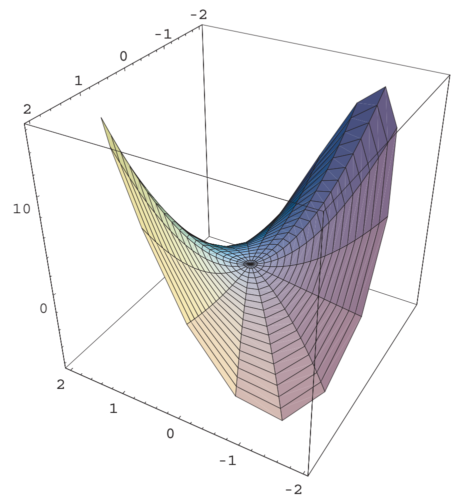
```
ParametricPlot3D[ { r∗Cos[t], r∗Sin[t], r*r*(a∗(Cos[t])∧2 + 2∗b∗Cos[t]∗Sin[t] +
      c∗(Sin[t])∧2) }, { r, 0, 1 }, { t, 0, 2∗Pi }, BosRatios → { 1, 1, 1 },
      ViewPoint → { −1.2, 2.1, 1.6 } ];
```



Here is the case $a = 1,\ b = 3,\ c = 1$:

```
ParametricPlot3D[ { r∗Cos[t], r∗Sin[t], r*r*(a∗(Cos[t])∧2 + 2∗b∗Cos[t]∗Sin[t] +
```

```
c*(Sin[t])∧2) }, { r, 0, 2 }, { t, 0, 2*Pi }, BoxRatios → { 1, 1, 1 },
ViewPoint → { −1.2, 2.1, 1.6 } ];
```



Space prohibits our further experimentation along these lines, but these two examples certainly support the conclusions given in the statement of Problem 65.

C13S05.066: Upon setting the partial derivatives of f equal to zero we get the equations

$$f_x(x, y) = 4x^3 + 4bxy^2 = 4x(x^2 + by^2) = 0,$$

$$f_y(x, y) = 4bx^2y + 4y^3 = 4y(bx^2 + y^2) = 0.$$

Obviously, the only solution is (0, 0) unless $b < 0$. In the latter case, if x and y are nonzero, then the other factors yield $by^2 = y^2/b$, so it follows that $b^2 = 1$ and hence that $b = -1$. Computer experimentation yields graphs that look like Fig. 13.5.7 if $b > -1$ (local minimum) and like Fig. 13.5.8 if $b < -1$ (saddle point). —C.H.E.

C13S05.067: Upon changing to polar coordinates we get $f(x, y) = x^4 + 2bx^2y^2 + y^4 = r^4 g(\theta)$ where

$$g(\theta) = \cos^4\theta + 2b\cos^2\theta\,\sin^2\theta + \sin^4\theta.$$

Upon differentiating and simplifying we find that

$$g'(\theta) = 4(b-1)(\cos^2\theta - \sin^2\theta)\cos\theta\,\sin\theta.$$

Hence the critical points of g are

- multiples of $\pi/2$, where $g(\theta) = 1$, and
- odd multiples of $\pi/4$, where $\cos^2\theta = \sin^2\theta = \frac{1}{2}$ so $g(\theta) = \frac{1}{4} + \frac{1}{2}b + \frac{1}{4} = \frac{1}{2}(b+1)$.

If $b > -1$ it follows that $g(\theta)$ is always positive, so $f(x, y) = r^4 g(\theta)$ is positive except at the origin. But if $b < -1$ it follows that $g(\theta)$ attains both positive and negative values, so $z = f(x, y) = r^4 g(\theta)$ exhibits a saddle point at the origin. —C.H.E.

C13S05.068: When we set the first-order partial derivatives of f equal to zero, we obtain the simultaneous equations

$$f_x(x, y, z) = 2x - 6y = 0,$$

$$f_y(x, y, z) = -6x + 2y + 2z = 0,$$

$$f_z(x, y, z) = 2y + 2z = 0.$$

The first and third equations yield $x = 3y$ and $z = -y$; when this information is substituted in the second equation, we find that $y = 0$. Hence the only critical point is at the origin, where we have $f(0, 0, 0) = 12$. Yet

$$f(1, 1, 0) = 8 < 12 \quad \text{and } f(1, -1, 0) = 20 > 12.$$

Therefore we cannot find the extrema of f merely by setting all partial derivatives equal to zero. If you examine the behavior of f on the two lines

$$y = x, \quad z = 0 \qquad \text{and} \qquad y = -x, \quad z = 0$$

through the origin, you will find that f has no extrema, global or local.

C13S05.069: When we set all first-order partial derivative of g equal to zero, we get the simultaneous equations

$$g_x(x, y, z) = 4x^3 - 16xy^2 = 0,$$
$$g_y(x, y, z) = 4y^3 - 16x^2y = 0,$$
$$g_z(x, y, z) = z = 0.$$

The only solution of these equations is $x = y = z = 0$, at which $g(0, 0, 0) = 12$. But

$$g(1, 1, 0) = 6 < 12 \quad \text{and} \quad g(0, 0, 2) = 28 > 12.$$

Therefore g has no global maximum or minimum at the origin. Examination of the behavior of $g(x, y, z)$ on the two lines

$$y = x, \quad z = 0 \qquad \text{and} \qquad x = 0, \quad y = 0$$

is enough to establish that g has no extrema, global or local. Thus one cannot conclude that g has an extremum at a point where all its partial derivatives are zero.

C13S05.070: Because $h_y(x, y) \equiv -2$, there is no point at which both partial derivatives of h are simultaneously zero. Examination of the behavior of h on the boundary of its domain

$$0 \leqq x, \quad 0 \leqq y, \quad x + y \leqq 1,$$

a triangle in the xy-plane, reveals that the global maximum value of h is 1 and the global minimum value of h is -1. Note that these extrema cannot discovered by setting the partial derivatives of h equal to zero and solving the resulting equations.

Section 13.6

C13S06.001: If $w = 3x^2 + 4xy - 2y^3$, then $dw = (6x + 4y)\,dx + (4x - 6y^2)\,dy$.

C13S06.002: If $w = \exp(-x^2 - y^2)$, then $dw = -2x\exp(-x^2 - y^2)\,dx - 2y\exp(-x^2 - y^2)\,dy$.

C13S06.003: If $w = \sqrt{1 + x^2 + y^2}$, then

$$dw = \frac{x}{\sqrt{1 + x^2 + y^2}}\,dx + \frac{y}{\sqrt{1 + x^2 + y^2}}\,dy = \frac{x\,dx + y\,dy}{\sqrt{1 + x^2 + y^2}}.$$

C13S06.004: If $w = xye^{x+y}$, then $dw = y(x+1)e^{x+y}\,dx + x(y+1)e^{x+y}\,dy$.

C13S06.005: If $w(x,\,y) = \arctan\left(\dfrac{x}{y}\right)$, then $dw = \dfrac{y\,dx - x\,dy}{x^2 + y^2}$.

C13S06.006: If $w = xz^2 - yx^2 + zy^2$, then $dw = (z^2 - 2xy)\,dx + (2yz - x^2)\,dy + (y^2 + 2xz)\,dz$.

C13S06.007: If $w = \ln(x^2 + y^2 + z^2)$, then

$$dw = \frac{2x\,dx}{x^2 + y^2 + z^2} + \frac{2y\,dy}{x^2 + y^2 + z^2} + \frac{2z\,dz}{x^2 + y^2 + z^2} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}.$$

C13S06.008: If $w = \sin xyz$, then $dw = yz\cos xyz\,dx + xz\cos xyz\,dy + xy\cos xyz\,dz$.

C13S06.009: If $w = x\tan yz$, then $dw = \tan yz\,dx + xz\sec^2 yz\,dy + xy\sec^2 yz\,dz$.

C13S06.010: If $w = xye^{uv}$, then $dw = ye^{uv}\,dx + xe^{uv}\,dy + xyve^{uv}\,du + xyue^{uv}\,dv$.

C13S06.011: If $w = e^{-xyz}$, then $dw = -yze^{-xyz}\,dx - xze^{-xyz}\,dy - xye^{-xyz}\,dz$.

C13S06.012: If $w = \ln(1 + rs)$, then

$$dw = \frac{s\,dr}{1 + rs} + \frac{r\,ds}{1 + rs} = \frac{s\,dr + r\,ds}{1 + rs}.$$

C13S06.013: If $w = u^2\exp(-v^2)$, then $dw = 2u\exp(-v^2)\,du - 2u^2v\exp(-v^2)\,dv$.

C13S06.014: If $w = \dfrac{s+t}{s-t}$, then

$$dw = -\frac{2t\,ds}{(s-t)^2} + \frac{2s\,dt}{(s-t)^2} = \frac{2s\,dt - 2t\,ds}{(s-t)^2}.$$

C13S06.015: If $w = \sqrt{x^2 + y^2 + z^2}$, then

$$dw = \frac{x\,dx}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\,dy}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\,dz}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}}.$$

C13S06.016: If $w = pqr\exp(-p^2 - q^2 - r^2)$, then

$$dw = (qr - 2p^2qr)\exp(-p^2 - q^2 - r^2)\ dp$$
$$+ (pr - 2pq^2r)\exp(-p^2 - q^2 - r^2)\ dq + (pq - 2pqr^2)\exp(-p^2 - q^2 - r^2)\ dr.$$

C13S06.017: If $w = f(x, y) = \sqrt{x^2 + y^2}$, then

$$dw = \frac{x\ dx\ + y\ dy}{\sqrt{x^2 + y^2}}.$$

Choose $x = 3$, $y = 4$, $dx = -0.03$, and $dy = 0.04$. Then

$$f(2.97, 4.04) \approx f(3, 4) + \frac{3 \cdot (-0.03) + 4 \cdot (0.04)}{\sqrt{3^2 + 4^2}} = \frac{2507}{500} = 5.014.$$

Compare with the true value of

$$f(2.97, 4.04) = \frac{\sqrt{10057}}{20} \approx 5.014229751417.$$

C13S06.018: If $w = f(x, y) = \sqrt{x^2 - y^2}$, then

$$dw = \frac{x\ dx}{\sqrt{x^2 - y^2}} - \frac{y\ dy}{\sqrt{x^2 - y^2}} = \frac{x\ dx\ - y\ dy}{\sqrt{x^2 - y^2}}.$$

Choose $x = 13$, $y = 5$, $dx = 0.2$, and $dy = -0.1$. Then

$$f(13.2, 4.9) \approx f(13, 5) + \frac{13 \cdot (0.2) + 5 \cdot (0.1)}{\sqrt{13^2 - 5^2}} = 12 + \frac{31}{120} = \frac{1471}{120} \approx 12.258333333333.$$

Compare with the true value of

$$f(13.2, 4.9) = \frac{\sqrt{15023}}{10} \approx 12.256834827964.$$

C13S06.019: If $w = f(x, y) = \dfrac{1}{1 + x + y}$, then

$$dw = -\frac{dx + dy}{(1 + x + y)^2}.$$

Choose $x = 3$, $y = 6$, $dx = 0.02$, and $dy = 0.05$. Then

$$f(3.02, 6.05) \approx f(3, 6) - \frac{0.02 + 0.05}{(1 + 3 + 6)^2} = \frac{1}{10} - \frac{7}{10000} = \frac{993}{10000} = 0.0993.$$

Compare with the true value of

$$f(3.02, 6.05) = \frac{100}{1007} \approx 0.0993048659384310.$$

C13S06.020: If $w = f(x, y, z) = \sqrt{xyz}$, then

$$dw = \frac{yz\ dx\ + xz\ dy\ + xy\ dz}{2\sqrt{xyz}}.$$

Choose $x = 1,\ y = 3,\ z = 3,\ dx = -0.1,\ dy = -0.1,$ and $dz = 0.1.$ Then

$$f(0.9,\ 2.9,\ 3.1) \approx f(1,\ 3,\ 3) + \frac{-9 \cdot (0.1) - 3 \cdot (0.1) + 3 \cdot (0.1)}{2\sqrt{9}} = 3 - \frac{3}{20} = \frac{57}{20} = 2.85.$$

Compare with the true value of

$$f(0.9,\ 2.9,\ 3.1) = \frac{3\sqrt{8990}}{100} \approx 2.8444683158720541.$$

C13S06.021: If $w = f(x,\ y,\ z) = \sqrt{x^2 + y^2 + z^2}$, then

$$dw = \frac{x\ dx\ + y\ dy\ + z\ dz}{\sqrt{x^2 + y^2 + z^2}}.$$

Choose $x = 3,\ y = 4,\ z = 12,\ dx = 0.03,\ dy = -0.04,$ and $dz = 0.05.$ Then

$$f(3.03,\ 3.96,\ 12.05) \approx f(3,\ 4,\ 12) + \frac{3 \cdot (0.03) - 4 \cdot (0.04) + 12 \cdot (0.05)}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$= 13 + \frac{53}{1300} = \frac{16953}{1300} \approx 13.040769230769.$$

Compare with the true value of

$$f(3.03,\ 3.96,\ 12.05) = \frac{\sqrt{68026}}{20} \approx 13.040897208398.$$

C13S06.022: If $w = f(x,\ y,\ z) = \dfrac{xyz}{x + y + z}$, then *Mathematica* 3.0 can find the linear approximation to $f(1.98,\ 3.03,\ 4.97)$ as follows:

```
f[x_, y_, z_] := x*y*z/(x + y + z)

D[f[x,y,z], x]*dx + D[f[x,y,z], y]*dy + D[f[x,y,z], z]*dz // Together
```

$$\frac{(yz^2 + y^2z)\ dx + (xz^2 + x^2z)\ dy + (xy^2 + x^2y)\ dz}{(x + y + z)^2}$$

```
% /.  { x → 2, y → 3, z → 5, dx → -2/100, dy → 3/100, dz → -3/100 }
```

$$-\frac{3}{250}$$

```
f[2, 3, 5]
```

$$3$$

Then we add the last two results:

```
% + %%
```

$$\frac{747}{250}$$

```
N[ %, 20 ]
```

2.9880000000000000000

Now we compare the approximation with the true value:

```
f[ 198/100, 303/100, 497/100 ]
```

$$\frac{14908509}{4990000}$$

```
N[ %, 20 ]
```

2.9876771543086172345

C13S06.023: If $w = f(x, y, z) = e^{-xyz}$, then

$$dw = -e^{-xyz}(yz\ dx + xz\ dy + xy\ dz).$$

Take $x = 1,\ y = 0,\ z = -2,\ dx = 0.02,\ dy = 0.03,$ and $dz = -0.02.$ Then

$$f(1.02,\ 0.03,\ -2.02) \approx f(1,\ 0,\ -2) - e^0(0 - 2 \cdot (0.03) + 0) = 1 + \frac{3}{50} = 1.06.$$

Compare with the exact value, which is

$$f(1.02,\ 0.03,\ -2.02) = \exp\left(\frac{15453}{250000}\right) \approx 1.0637623386083891.$$

C13S06.024: If $w = f(x, y) = (x - y)\cos 2\pi xy,$ then

$$dw = [\cos(2\pi xy) - 2\pi y(x - y)\sin(2\pi xy)]\ dx - [\cos(2\pi xy) + 2\pi x(x - y)\sin(2\pi xy)]\ dy.$$

Take $x = 1,\ y = 0.5,\ dx = 0.1,$ and $dy = -0.1.$ Then

$$f(1.1,\ 0.4) \approx f(1,\ 0.5) + (-1)\cdot(0.1) - (-1)\cdot(-0.1) = -\frac{1}{2} - \frac{1}{5} = -\frac{7}{10} = -0.7.$$

Compare with the exactly value, which is

$$f(1.1,\ 0.4) = \frac{7}{10}\cos\left(\frac{22\pi}{25}\right) \approx -0.6508435401217760.$$

C13S06.025: If $w = f(x, y) = \left(\sqrt{x} + \sqrt{y}\,\right)^2$, then

$$dw = \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}}\ dx + \frac{\sqrt{x} + \sqrt{y}}{\sqrt{y}}\ dy.$$

Take $x = 16,\ y = 100,\ dx = -1,$ and $dy = -1.$ Then

$$f(15,\ 99) \approx f(16,\ 100) + \frac{4 + 10}{4}\cdot(-1) + \frac{4 + 10}{10}\cdot(-1) = 196 - \frac{49}{10} = \frac{1911}{10} = 191.1.$$

By comparison, the exact value is

$$f(15,\ 99) = \left(\sqrt{15} + \sqrt{99}\right)^2 \approx 191.0713954719907741.$$

C13S06.026: If $w = f(x,\, y,\, z) = x^{1/2}y^{1/3}z^{1/4}$, then

$$dw = \frac{y^{1/3}z^{1/4}}{2x^{1/2}}\,dx + \frac{x^{1/2}z^{1/4}}{3y^{2/3}}\,dy + \frac{x^{1/2}y^{1/3}}{4z^{3/4}}\,dz.$$

Take $x = 25,\ y = 27,\ z = 16,\ dx = 1,\ dy = 1,$ and $dz = 1$. Then

$$f(26,\, 28,\, 17) \approx f(25,\, 27,\, 16) + \frac{3\cdot 4}{2\cdot 5} + \frac{5\cdot 4}{3\cdot 9} + \frac{5\cdot 3}{4\cdot 8} = 30 + \frac{6217}{4320} = \frac{135817}{4320} \approx 31.4391203703703704.$$

For comparison, the true value is

$$f(26,\, 28,\, 17) = 2^{7/6}\cdot 7^{1/3}\cdot 13^{1/2}\cdot 17^{1/4} \approx 31.4401721089687491.$$

C13S06.027: If $w = f(x,\, y) = \exp(x^2 - y^2)$, then

$$dw = 2x\exp(x^2 - y^2)\,dx - 2y\exp(x^2 - y^2)\,dy.$$

Take $x = 1,\ y = 1,\ dx = 0.1,$ and $dy = -0.1$. Then

$$f(1.1,\, 0.9) \approx f(1,\, 1) + 2\cdot(0.1) - 2\cdot(-0.1) = 1 + \frac{2}{5} = \frac{7}{5} = 1.4.$$

Compare with the true value, which is

$$f(1.1,\, 0.9) = e^{2/5} \approx 1.4918246976412703.$$

C13S06.028: If $w = f(x,\, y) = \dfrac{x^{1/3}}{y^{1/5}}$, then

$$dw = \frac{1}{3x^{2/3}y^{1/5}}\,dx - \frac{x^{1/3}}{5y^{6/5}}\,dy = \frac{5y\,dx - 3x\,dy}{15x^{2/3}y^{6/5}}.$$

Take $x = 27,\ y = 32,\ dx = -2,$ and $dy = -2$. Then

$$f(25,\, 30) \approx f(27,\, 32) + \frac{160\cdot(-2) - 81\cdot(-2)}{15\cdot 9\cdot 64} = \frac{3}{2} - \frac{79}{4320} = \frac{6401}{4320} \approx 1.4817129629629630.$$

For purposes of comparison, the true value is

$$f(25,\, 30) = \frac{5^{7/15}}{6^{1/5}} \approx 1.4810023646720941.$$

C13S06.029: If $w = f(x,\, y,\, z) = \sqrt{x^2 + y^2 + z^2}$, then

$$dw = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}}.$$

Take $x = 3,\ y = 4,\ z = 12,\ dx = 0.1,\ dy = 0.2,$ and $dz = -0.3$. Then

$$f(3.1,\, 4.2,\, 11.7) \approx f(3,\, 4,\, 12) + \frac{0.3 + 0.8 - 3.6}{13} = 13 - \frac{5}{26} = \frac{333}{26} \approx 12.8076923076923077.$$

The true value is

$$f(3.1,\,4.2,\,11.7) = \frac{\sqrt{16414}}{10} \approx 12.8117133904876283.$$

C13S06.030: If $w = f(x,\,y,\,z) = (x^2 + 2y^2 + 2z^2)^{1/3}$, then

$$dw = \frac{2x\;dx\, + 4y\;dy\, + 4z\;dz}{3(x^2 + 2y^2 + 2z^2)^{2/3}}.$$

Take $x = y = z = 5$, $dx = 0.1$, $dy = 0.2$, and $dz = 0.3$. Then

$$f(5.1,\,5.2,\,5.3) \approx f(5,\,5,\,5) + \frac{10\cdot(0.1) + 20\cdot(0.2) + 20\cdot(0.3)}{3(25+50+50)^{2/3}} = 5 + \frac{11}{75} = \frac{386}{75} \approx 5.1466666666666667.$$

For comparison purposes, the true value is

$$f(5.1,\,5.2,\,5.3) = \left(\frac{13627}{100}\right)^{1/3} \approx 5.1459640985125985.$$

C13S06.031: Given: The point $Q(1,\,2)$ on the curve $f(x,\,y) = 0$, where $f(x,\,y) = 2x^3 + 2y^3 - 9xy$. Then

$$df = (6x^2 - 9y)\;dx + (6y^2 - 9x)\;dy = 0.$$

Choose $x = 1$, $y = 2$, and $dx = 0.1$. Then

$$(6-18)\cdot\frac{1}{10} + (24-9)\;dy = 0;$$

$$dy = \frac{1}{15}\cdot\frac{12}{10} = \frac{12}{150} = \frac{2}{25} = 0.08.$$

So the point P on the curve $f(x,\,y) = 0$ near Q and with x-coordinate 1.1 has y-coordinate

$$y \approx 2 + \frac{2}{25} = \frac{52}{25} = 2.08.$$

The true value of the y-coordinate is approximately 2.0757642703016864.

C13S06.032: Given: The point $Q(2,\,4)$ on the curve $f(x,\,y) = 0$, where $f(x,\,y) = 4x^4 + 4y^4 - 17x^2y^2$. Then

$$df = (16x^3 - 34xy^2)\;dx + (16y^3 - 34x^2y)\;dy = 0.$$

Choose $x = 2$, $y = 4$, and $dy = -0.1$. Then

$$(128 - 1088)\;dx + (1024 - 544)(-0.1) = 0;$$

$$-\,960\;dx - 48 = 0;$$

$$dx = -\frac{1}{20} = -0.05.$$

So the point P on the curve $f(x,\,y) = 0$ near Q and having y-coordinate 3.9 therefore has x-coordinate $x \approx 2 - 0.05 = 1.95$. By some coincidence, the error in this approximation is zero: The point $P(3.9, 1.95)$ does lie on the curve $f(x,\,y) = 0$.

C13S06.033: Suppose that the base of the rectangle has length x and that its height is y. Then its area is $w = f(x, y) = xy$, and $dw = y\,dx + x\,dy$. Choose $x = 10$, $y = 15$, $dx = 0.1$, and $dy = 0.1$. Then $dw = 2.5$; this is the estimate of the maximum error in computing the area of the rectangle. The actual maximum error possible is $f(10.1, 15.1) - f(10, 15) = 2.51$.

C13S06.034: Part (a): The volume of the cylinder is $w = f(r, h) = \pi r^2 h$, so that

$$dw = 2\pi rh\,dr + \pi r^2\,dh.$$

Choose $r = 3$, $h = 9$, and $dr = dh = 0.1$. Then

$$dw = 54\pi \cdot (0.1) + 9\pi \cdot (0.1) = \frac{63\pi}{10} \approx 19.7920337176156974$$

is the estimate of the maximum error in computing the volume of the cylinder. The actual maximum error possible is

$$f(3.1, 9.1) - f(3, 9) = \frac{6451\pi}{1000} \approx 20.2664142083077562.$$

Part (b): The surface area of the cylinder is $w = f(r, h) = 2\pi rh + 2\pi r^2$, and thus

$$dw = (2\pi h + 4\pi r)\,dr + 2\pi r\,dh.$$

Choose $r = 3$, $h = 9$, and $dr = dh = 0.1$. Then

$$dw = (2\pi \cdot 9 + 4\pi \cdot 3) \cdot (0.1) + 2\pi \cdot 3 \cdot (0.1) = \frac{18\pi}{5} \approx 11.3097335529232557$$

is the estimate of the maximum error in computing the surface area of the cylinder. The actual maximum error possible is

$$f(3.1, 9.1) - f(3, 9) = \frac{91\pi}{25} = 11.4353972590668474.$$

C13S06.035: The volume of the cone is given by

$$w = f(r, h) = \frac{\pi}{3}r^2h, \qquad \text{so that} \qquad dw = \frac{2\pi}{3}rh\,dr + \frac{\pi}{3}r^2\,dh.$$

Choose $r = 5$, $h = 10$, and $dr = dh = 0.1$. Then

$$dw = \frac{2\pi}{3} \cdot 5 \cdot 10 \cdot (0.1) + \frac{\pi}{3} \cdot 5^2 \cdot (0.1) = \frac{25\pi}{6} \approx 13.0899693899574718$$

is an estimate of the maximum error in measuring the volume of the cylinder. The true value of the maximum error is

$$f(5.1, 10.1) - f(5, 10) = \frac{12701\pi}{3000} \approx 13.3004560977479880.$$

C13S06.036: If the dimensions of the box are x by y by z, then its total surface is $w = f(x, y, z) = 2xy + 2xz + 2yz$, and so

$$dw = 2(y + z)\,dx + 2(x + z)\,dy + 2(x + y)\,dz.$$

Choose $x = 10$, $y = 15$, $z = 20$, and $dx = dy = dz = 0.1$. Then

$$dw = (70 + 60 + 50) \cdot (0.1) = 18$$

is an estimate of the maximum error in measuring the total surface area of the box. The true value of the maximum error is

$$f(10.1,\ 15.1,\ 20.1) - f(10,\ 15,\ 20) = \frac{903}{50} = 18.06.$$

C13S06.037: If the sides of the field are x and y and the angle between them is θ, then the area of the field is given by

$$w = f(x,\ y,\ \theta) = \frac{1}{2}xy\sin\theta,$$

so that

$$dw = \frac{1}{2}y\sin\theta\ dx + \frac{1}{2}x\sin\theta\ dy + \frac{1}{2}xy\cos\theta\ d\theta.$$

If $x = 500,\ y = 700,\ \theta = \pi/6,\ dx = dy = 1,$ and $d\theta = \pi/720,$ then

$$dw = 350 \cdot \frac{1}{2} \cdot 1 + 250 \cdot \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 500 \cdot 700 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\pi}{720} = 300 + \frac{4375\pi\sqrt{3}}{36} \approx 961.2810182103919247$$

(in square feet) is an estimate of the maximum error in computing the area of the field. The true value of the maximum error is

$$f(501,\ 701,\ (\pi/6) + (\pi/720)) - f(500,\ 700,\ \pi/6) \approx 962.9622561829376760$$

(in square feet). The former amounts to approximately 0.0220679756246646 acres (there are 43560 square feet in one acre).

C13S06.038: We begin with the equation

$$V(T,\ p) = \frac{82.06T}{p}, \qquad \text{for which} \qquad dV = -\frac{4103(p\ dT\ -\ T\ dp)}{50p^2}.$$

When $p = 5,\ dp = -0.1,\ T = 300,$ and $dT = -20,$ we find that

$$dV = \frac{4103(5 \cdot (-20)\ -\ 300 \cdot (-0.1))}{50 \cdot 25} = -\frac{28721}{125} = -229.768.$$

The actual change in the volume is

$$V(280,\ 4.9) - V(300,\ 5) = -\frac{8206}{35} \approx -234.4571428571428571.$$

C13S06.039: The period T of a pendulum of length L is given (approximately) by

$$T = 2\pi\left(\frac{L}{g}\right)^{1/2}, \qquad \text{for which} \qquad dT = \left(\frac{g}{L}\right)^{1/2} \cdot \frac{\pi g\ dL - \pi L\ dg}{g^2}.$$

If $L = 2,\ dL = 1/12,\ g = 32,$ and $dg = 0.2,$ then

$$dT = \frac{17\pi}{1920} \approx 0.0278161849536596.$$

The true value of the increase in the period is

$$T(2 + 1/12,\ 32.2) - T(2,\ 32) \approx 0.0274043631738259.$$

C13S06.040: Given

$$T = 2\pi\left(\frac{L}{g}\right)^{1/2},$$

we compute

$$dT = 2\pi \cdot \frac{1}{2}\left(\frac{L}{g}\right)^{-1/2} \cdot \frac{g\,dL\ -\ L\,dg}{g^2} = \frac{\pi}{g^2}\left(\frac{g}{L}\right)^{1/2} \cdot (g\,dL\ -\ L\,dg).$$

Therefore

$$\begin{aligned}\frac{dT}{T} &= \frac{1}{2\pi}\left(\frac{g}{L}\right)^{1/2} \cdot \frac{\pi}{g^2}\left(\frac{g}{L}\right)^{1/2} \cdot (g\,dL\ -\ L\,dg)\\ &= \frac{1}{2} \cdot \frac{g}{L} \cdot \frac{1}{g^2} \cdot (g\,dL\ -\ L\,dg) = \frac{1}{2gL} \cdot (g\,dL\ -\ L\,dg) = \frac{1}{2} \cdot \left(\frac{dL}{L} - \frac{dg}{g}\right).\end{aligned}$$

C13S06.041: Given: $R(v_0,\ \alpha) = \dfrac{1}{32}(v_0)^2 \sin 2\alpha$, we first compute

$$dR = \frac{1}{16}\left(v_0\ \sin 2\alpha\ dv_0\ +\ (v_0)^2 \cos 2\alpha\ d\alpha\right).$$

Substitution of $v_0 = 400$, $dv_0 = 10$, $\alpha = \pi/6$, and $d\alpha = \pi/180$ yields

$$dR = 125\sqrt{3}\ + \frac{250\pi}{9} \approx 303.7728135458261405$$

as an estimate of the increase in the range. The true value of the increase is

$$R(410,\ (\pi/6) + (\pi/180)) - R(400,\ \pi/6) = -2500\sqrt{3}\ + \frac{42025}{8}\sin\left(\frac{31\pi}{90}\right) \approx 308.1070548148573585.$$

C13S06.042: Given

$$S = \frac{k}{wh^3},$$

we first compute

$$dS = -\frac{k}{w^2h^3}\,dw\ -\ \frac{3k}{wh^4}\,dh = -\frac{k}{wh^3}\left(\frac{1}{w}\,dw\ +\ \frac{3}{h}\,dh\right) = -S \cdot \left(\frac{1}{w}\,dw\ +\ \frac{3}{h}\,dh\right). \tag{1}$$

If $S = 1$ when $w = 2$ and $h = 4$, then

$$1 = \frac{k}{2 \cdot 64}, \quad \text{so that} \quad k = 128; \quad \text{thus} \quad S = \frac{128}{wh^3}.$$

To approximate the sag when $w = 2.1$ and $h = 4.1$, we take $w = 2,\ h = 4,$ and $dw = dh = 0.1$ in Eq. (1) to find that

$$dS = -1 \cdot \left(\frac{1}{2} \cdot (0.1) + \frac{3}{4} \cdot (0.1) \right) = -0.05 - 0.075 = -0.125,$$

and thus the sag will be approximately $1 - 0.125 = 0.875$ (inches). The true value is

$$S(2.1,\, 4.1) = \frac{128}{(2.1) \cdot (4.1)^3} \approx 0.88438039$$

inches.

C13S06.043: Part (a): If $(x,\, y) \to (0,\, 0)$ along the line $y = x$, then

$$\lim_{(x,y)\to(0,0)} f(x,\, y) = \lim_{x\to 0} f(x,\, x) = \lim_{x\to 0} 1 = 1.$$

But if $(x,\, y) \to (0,\, 0)$ along the line $y = 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,\, y) = \lim_{x\to 0} f(x,\, 0) = \lim_{x\to 0} 0 = 0.$$

Therefore f is not continuous at $(0,\, 0)$.

Part (b): We compute the partial derivatives of f at $(0,\, 0)$ by the definition:

$$f_x(0,\, 0) = \lim_{h\to 0} \frac{f(0+h,\, 0) - f(0,\, 0)}{h} = \lim_{h\to 0} \frac{f(h,\, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0;$$
$$f_y(0,\, 0) = \lim_{k\to 0} \frac{f(0,\, 0+k) - f(0,\, 0)}{k} = \lim_{k\to 0} \frac{f(0,\, k)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Therefore both f_x and f_y exist at $(0,\, 0)$ but f is not continuous at $(0,\, 0)$.

C13S06.044: The function $f(x,\, y) = \left(x^{1/3} + y^{1/3}\right)^3$ is continuous everywhere because it is the composition of the sum of continuous functions. At the origin we compute its partial derivative with respect to x as follows:

$$f_x(0,\, 0) = \lim_{h\to 0} \frac{f(0+h,\, 0) - f(0,\, 0)}{h} = \lim_{h\to 0} \frac{\left(h^{1/3}\right)^3}{h} = 1;$$

similarly, $f_y(0,\, 0) = 1$. So only the plane $z = x + y$ can approximate the graph of f at and near $(0,\, 0)$. But the line L_1 in the vertical plane $y = x$, through $(0,\, 0,\, 0)$, and tangent to the graph of f has slope

$$\lim_{x\to 0} \frac{f(x,\, x) - f(0,\, 0)}{x\sqrt{2}} = 4\sqrt{2}\,,$$

whereas the line L_2 in the vertical plane $y = x$, through $(0,\, 0,\, 0)$, and tangent to the graph of $z = x + y$ has slope $\sqrt{2}$. Because no plane through $(0,\, 0,\, 0)$ approximates the graph of f accurately near $(0,\, 0)$, the function f is not differentiable at $(0,\, 0)$.

C13S06.045: Given:

$$f(x,\, y) = y^2 + x^2 \sin\frac{1}{x}$$

if $x \neq 0$; $f(0, y) = y^2$. Then

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot h^2 \sin \frac{1}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

and

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{k^2}{k} = \lim_{k \to 0} k = 0.$$

Therefore the linear approximation to f at $(0, 0)$ can only be $z = 0$. Moreover,

$$0 \leqq f(x, y) \leqq x^2 + y^2 = g(x, y)$$

for all (x, y), and $z = g(x, y)$ has the tangent plane $z = 0$ at $(0, 0)$. Therefore $z = 0$ does approximate $f(x, y)$ accurately at and near $(0, 0)$. That is, f is differentiable at $(0, 0)$. But if $x \neq 0$, then

$$f_x(x, y) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

so $f_x(x, y)$ has no limit as $(x, y) \to (0, 0)$ along the x-axis. Therefore $f_x(x, y)$ is not continuous at $(0, 0)$, and thus f is not continuously differentiable at $(0, 0)$ even though it is differentiable there.

C13S06.046: Suppose that f is a function of a single variable. We are to show that $f'(a)$ exists if and only if f is differentiable at $x = a$ in the sense of Eq. (19), meaning that there exists a constant c such that

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - ch}{|h|} = 0 \tag{1}$$

and that, if Eq. (1) holds, then $c = f'(a)$. So let us suppose first that $f'(a)$ exists. Let $c = f'(a)$. Then

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

by definition, and

$$\lim_{h \to 0} \frac{ch}{h} = c = f'(a).$$

Consequently,

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - ch}{h} = \left(\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \right) - \left(\lim_{h \to 0} \frac{ch}{h} \right) = 0 - 0 = 0,$$

and therefore Eq. (1) holds as well.

Next suppose that

$$\lim_{h \to 0} \frac{f(a + h) - f(a) - ch}{|h|} = 0$$

for some constant c. Then

$$\lim_{h \to 0} \left| \frac{f(a + h) - f(a) - ch}{|h|} \right| = \left| \lim_{h \to 0} \frac{f(a + h) - f(a) - ch}{h} \right| = 0,$$

and thus

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - ch}{h} \;=\; 0.$$

It now follows that

$$\left(\lim_{h\to 0} \frac{f(a+h) - f(a) - ch}{h}\right) + \left(\lim_{h\to 0} \frac{ch}{h}\right) \;=\; 0 + c \;=\; c,$$

and thus

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \;=\; c.$$

That is, $f'(a)$ exists and $c \;=\; f'(a)$.

C13S06.047: Suppose that the function f of $n \geqq 2$ variables is differentiable at $\mathbf{a}$. Then there exists a constant vector $\mathbf{c} \;=\; \langle\, c_1,\, c_2,\, \dots\,,\, c_n \,\rangle$ such that

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h}}{|\mathbf{h}|} \;=\; 0.$$

Therefore

$$\left(\lim_{\mathbf{h}\to \mathbf{0}} |\mathbf{h}|\right) \cdot \left(\lim_{\mathbf{h}\to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h}}{|\mathbf{h}|}\right) \;=\; 0\cdot 0 \;=\; 0,$$

and therefore

$$\lim_{\mathbf{h}\to \mathbf{0}} \left[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h} \right] \;=\; 0.$$

But $\mathbf{c}\cdot\mathbf{h} \to 0$ as $\mathbf{h} \to 0$ because $\mathbf{c}$ is a constant vector. Therefore

$$\left(\lim_{\mathbf{h}\to \mathbf{0}} \left[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h} \right]\right) + \left(\lim_{\mathbf{h}\to \mathbf{0}} \left[\mathbf{c}\cdot\mathbf{h} + f(\mathbf{a})\right]\right) \;=\; 0 + 0 + f(\mathbf{a}) \;=\; f(\mathbf{a}),$$

and thus we see that

$$\lim_{\mathbf{h}\to \mathbf{0}} f(\mathbf{a}+\mathbf{h}) \;=\; f(\mathbf{a}).$$

Therefore f is continuous at $\mathbf{x} \;=\; \mathbf{a}$. That is, the function f is continuous wherever it is differentiable.

C13S06.048: Suppose that the function f of $n \geqq 2$ variables is differentiable at $\mathbf{a}$. Then there exists a constant vector $\mathbf{c} \;=\; \langle c_1,\, c_2,\, \dots\,,\, c_n \,\rangle$ such that

$$\lim_{\mathbf{h}\to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{c}\cdot\mathbf{h}}{|\mathbf{h}|} \;=\; 0. \tag{1}$$

Now Eq. (1) holds for *every* n-vector $\mathbf{h}$, and in particular, if i is a fixed integer between 1 and n, then Eq. (1) holds for the vector

$$\mathbf{h} \;=\; \langle\, 0,\, 0,\, \dots\,,\, 0,\, h,\, 0,\, \dots\,,\, 0 \,\rangle$$

having the nonzero scalar h as its ith entry and zeros for all other entries. Moreover, note that $\mathbf{h} \to \mathbf{0}$ is, in such a case, equivalent to $h \to 0$. Let $\mathbf{a} \;=\; \langle\, a_1,\, a_2,\, \dots\,,\, a_n \,\rangle$. Then Eq. (1) implies that

$$\lim_{h\to 0} \frac{f(a_1,\, a_2,\, \ldots\,,\, a_i+h,\, \ldots\,,\, a_n) - f(a_1,\, a_2,\, \ldots\,,\, a_i,\, \ldots\,,\, a_n) - \mathbf{c}\cdot\mathbf{h}}{h}$$
$$= \lim_{h\to 0} \frac{f(a_1,\, a_2,\, \ldots\,,\, a_i+h,\, \ldots\,,\, a_n) - f(a_1,\, a_2,\, \ldots\,,\, a_i,\, \ldots\,,\, a_n) - c_i h}{h} = 0.$$

It follows immediately that

$$\lim_{h\to 0} \frac{f(a_1,\, a_2,\, \ldots\,,\, a_i+h,\, \ldots\,,\, a_n) - f(a_1,\, a_2,\, \ldots\,,\, a_i,\, \ldots\,,\, a_n)}{h} = c_i.$$

Hence $D_i f(\mathbf{a})$ exists and is equal to c_i for $1 \leqq i \leqq n$. Moreover, this shows that the vector $\mathbf{c}$ in Eq. (19) of the text is unique.

Section 13.7

C13S07.001: If $w = \exp(-x^2 - y^2)$, $x = t$, and $y = t^{1/2}$, then

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\ &= -2x\exp(-x^2 - y^2) - yt^{-1/2}\exp(-x^2 - y^2) = -2t\exp(-t^2 - t) - \exp(-t^2 - t).\end{aligned}$$

Alternatively, $w = \exp(-t^2 - t)$, and hence

$$\frac{dw}{dt} = -(2t + 1)\exp(-t^2 - t).$$

C13S07.002: If $w = \dfrac{1}{u^2 + v^2}$, $u = \cos 2t$, and $v = \sin 2t$, then

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\ &= \frac{4u\sin 2t}{(u^2 + v^2)^2} - \frac{4v\cos 2t}{(u^2 + v^2)^2} = \frac{4\sin 2t\,\cos 2t - 4\sin 2t\,\cos 2t}{(\cos^2 2t + \sin^2 2t)^2} = 0.\end{aligned}$$

Alternatively, $w = \dfrac{1}{\cos^2 2t + \sin^2 2t} \equiv 1$, and hence $\dfrac{dw}{dt} \equiv 0$.

C13S07.003: If $w(x, y, z) = \sin xyz$, $x = t$, $y = t^2$, and $z = t^3$, then

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \\ &= yz\cos xyz + 2txz\cos xyz + 3t^2xy\cos xyz = t^5\cos t^6 + 2t^5\cos t^6 + 3t^5\cos t^6 = 6t^5\cos t^6.\end{aligned}$$

Alternatively, $w = \sin t^6$, and thus $\dfrac{dw}{dt} = 6t^5\cos t^6$.

C13S07.004: If $w(u, v, z) = \ln(u + v + z)$, $u = \cos^2 t$, $v = \sin^2 t$, and $z = t^2$, then

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} \\ &= -\frac{2\sin t\,\cos t}{u + v + z} + \frac{2\sin t\,\cos t}{u + v + z} + \frac{2t}{u + v + z} = \frac{2t}{u + v + z} = \frac{2t}{1 + t^2}.\end{aligned}$$

Alternatively, $w = \ln(t^2 + 1)$, and so $\dfrac{dw}{dt} = \dfrac{2t}{t^2 + 1}$.

C13S07.005: If $w(x, y, z) = \ln(x^2 + y^2 + z^2)$, $x = s - t$, $y = s + t$, and $z = 2(st)^{1/2}$, then

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} \\ &= \frac{2x}{x^2 + y^2 + z^2} + \frac{2y}{x^2 + y^2 + z^2} + \frac{2tz}{(st)^{1/2}(x^2 + y^2 + z^2)} = \frac{2(st)^{1/2}x + 2(st)^{1/2}y + 2tz}{(st)^{1/2}(x^2 + y^2 + z^2)} \\ &= \frac{2(s - t)(st)^{1/2} + 4t(st)^{1/2} + 2(s + t)(st)^{1/2}}{(st)^{1/2}[(s - t)^2 + 4st + (s + t)^2]} = \frac{2(2s + 2t)}{2s^2 + 4st + 2t^2} = \frac{2}{s + t}\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}\\
&= -\frac{2x}{x^2+y^2+z^2}+\frac{2y}{x^2+y^2+z^2}+\frac{2sz}{(st)^{1/2}(x^2+y^2+z^2)}=\frac{-2(st)^{1/2}x+2(st)^{1/2}y+2sz}{(st)^{1/2}(x^2+y^2+z^2)}\\
&= \frac{4s(st)^{1/2}-2(s-t)(st)^{1/2}+2(s+t)(st)^{1/2}}{(st)^{1/2}[(s-t)^2+4st+(s+t)^2]}=\frac{2(2s+2t)}{2s^2+4st+2t^2}=\frac{2}{s+t}.
\end{aligned}
$$

Alternatively,

$$
w(s,\,t)=\ln\big((s-t)^2+4st+(s+t)^2\big)=\ln(2s^2+4st+2t^2),
$$

and therefore

$$
\frac{\partial w}{\partial s}=\frac{2}{s+t}\qquad\text{and}\qquad\frac{\partial w}{\partial t}=\frac{2}{s+t}.
$$

C13S07.006: If $w(p,\,q,\,r)=pq\sin r,\;p=2s+t,\;q=s-t,\;$ and $r=st,\;$ then

$$
\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial p}\cdot\frac{\partial p}{\partial s}+\frac{\partial w}{\partial q}\cdot\frac{\partial q}{\partial s}+\frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial s}=2q\sin r+p\sin r+pqt\cos r\\
&= (s-t)t(2s+t)\cos st+2(s-t)\sin st+(2s+t)\sin st=(2s^2t-st^2-t^3)\cos st+(4s-t)\sin st
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial p}\cdot\frac{\partial p}{\partial t}+\frac{\partial w}{\partial q}\cdot\frac{\partial q}{\partial t}+\frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial t}=q\sin r-p\sin r+pqs\cos r\\
&= s(s-t)(2s+t)\cos st+(s-t)\sin st-(2s+t)\sin st=(2s^3-s^2t-st^2)\cos st-(s+2t)\sin st.
\end{aligned}
$$

Alternatively,

$$
w(s,\,t)=(s-t)(2s-t)\sin st,
$$

and thus

$$
\begin{aligned}
\frac{\partial w}{\partial s} &= (s-t)t(2s+t)\cos st+2(s-t)\sin st+(2s+t)\sin st\\
&= (2s^2t-st^2-t^3)\cos st+(4s-t)\sin st\qquad\text{and}\\
\frac{\partial w}{\partial t} &= s(s-t)(2s+t)\cos st+(s-t)\sin st-(2s+t)\sin st\\
&= (2s^3-s^2t-st^2)\cos st-(s+2t)\sin st.
\end{aligned}
$$

C13S07.007: If $w(u,\,v,\,z)=\sqrt{u^2+v^2+z^2}\,,\;u=3e^t\sin s,\;v=3e^t\cos s,\;$ and $z=4e^t,\;$ then

$$
\frac{\partial w}{\partial s}=\frac{3ue^t\cos s}{\sqrt{u^2+v^2+z^2}}-\frac{3ve^t\sin s}{\sqrt{u^2+v^2+z^2}}+0=\frac{3e^t(u\cos s-v\sin s)}{\sqrt{u^2+v^2+z^2}}=0
$$

because $u\cos s - v\sin s = 3e^t \sin s\cos s - 3e^t\cos s\sin s = 0$. But

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{3ue^t\sin s}{\sqrt{u^2+v^2+z^2}} + \frac{3ve^t\cos s}{\sqrt{u^2+v^2+z^2}} + \frac{4ze^t}{\sqrt{u^2+v^2+z^2}}\\ &= \frac{e^t(3u\sin s+3v\cos s+4z)}{\sqrt{u^2+v^2+z^2}} = \frac{e^t(16e^t+9e^t\cos^2 s+9e^t\sin^2 s)}{\sqrt{16e^{2t}+9e^{2t}\cos^2 s+9e^{2t}\sin^2 s}} = 5e^t.\end{aligned}$$

Alternatively,

$$w(s,\,t) = \sqrt{16e^{2t}+9e^{2t}\cos^2 s+9e^{2t}\sin^2 s} = 5e^t,$$

and therefore

$$\frac{\partial w}{\partial s} = 0 \quad \text{and} \quad \frac{\partial w}{\partial t} = 5e^t.$$

C13S07.008: If $w(x,\,y,\,z) = yz+zx+xy,\ x = s^2-t^2,\ y = s^2+t^2,$ and $z = s^2t^2,$ then

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s} = 2s(y+z)+2s(x+z)+2st^2(x+y)\\ &= 4s^3t^2+2s(s^2-t^2+s^2t^2)+2s(s^2+t^2+s^2t^2) = 4s^3(2t^2+1) \quad \text{and}\\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t} = -2t(y+z)+2t(x+z)+2s^2t(x+y)\\ &= 2t\big[(s^2+1)x+(s^2-1)y\big] = 2t\big[(s^2+1)(s^2-t^2)+(s^2-1)(s^2+t^2)\big] = 4t(s^4-t^2).\end{aligned}$$

Alternatively,

$$w(s,\,t) = s^2t^2(s^2-t^2)+s^2t^2(s^2+t^2)+(s^2-t^2)(s^2+t^2) = s^4+2s^4t^2-t^4,$$

and therefore

$$\frac{\partial w}{\partial s} = 4s^3+8s^3t^2 \quad \text{and} \quad \frac{\partial w}{\partial t} = 4s^4t-4t^3.$$

C13S07.009: Because $r(x,\,y,\,z) = \exp(yz+xz+xy),$ we find that

$$\frac{\partial r}{\partial x} = (y+z)\exp(yz+xz+xy), \qquad \frac{\partial r}{\partial y} = (x+z)\exp(yz+xz+xy),$$
$$\text{and} \qquad \frac{\partial r}{\partial z} = (x+y)\exp(yz+xz+xy).$$

C13S07.010: Because $r(x,\,y,\,z) = (x+y)(x+z)(y+z)-(x+y)^2-(x+z)^2-(y+z)^2,$ we find that

$$\begin{aligned}\frac{\partial r}{\partial x} &= 2xy+2xz+2yz-4x-2y-2z+y^2+z^2,\\ \frac{\partial r}{\partial y} &= 2xy+2xz+2yz-2x-4y-2z+x^2+z^2, \quad \text{and}\\ \frac{\partial r}{\partial z} &= 2xy+2xz+2yz-2x-2y-4z+x^2+y^2.\end{aligned}$$

C13S07.011: Because

$$r(x, y, z) = \sin\left(\frac{\sqrt{xy^2z^3}}{\sqrt{x+2y+3z}}\right),$$

we find that

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{(2y+3z)\sqrt{xy^2z^3}}{2x(x+2y+3z)^{3/2}}\cos\left(\frac{\sqrt{xy^2z^3}}{\sqrt{x+2y+3z}}\right),\\
\frac{\partial r}{\partial y} &= \frac{(x+y+3z)\sqrt{xy^2z^3}}{y(x+2y+3z)^{3/2}}\cos\left(\frac{\sqrt{xy^2z^3}}{\sqrt{x+2y+3z}}\right), \quad \text{and}\\
\frac{\partial r}{\partial z} &= \frac{3(x+2y+2z)\sqrt{xy^2z^3}}{2z(x+2y+3z)^{3/2}}\cos\left(\frac{\sqrt{xy^2z^3}}{\sqrt{x+2y+3z}}\right).
\end{aligned}$$

C13S07.012: Because $r(x, y, z) = \exp(xz - xy) + \exp(xy - yz) + \exp(yz - xz)$, we find that

$$\begin{aligned}
\frac{\partial r}{\partial x} &= y\exp(xy - yz) - z\exp(yz - xz) + (z - y)\exp(xz - xy),\\
\frac{\partial r}{\partial y} &= z\exp(yz - xz) - x\exp(xz - xy) + (x - z)\exp(xy - yz), \quad \text{and}\\
\frac{\partial r}{\partial z} &= x\exp(xz - xy) - y\exp(xy - yz) + (y - x)\exp(yz - xz).
\end{aligned}$$

C13S07.013: If $p = f(x, y)$, $x = x(u, v, w)$, and $y = y(u, v, w)$, then

$$\begin{aligned}
\frac{\partial p}{\partial u} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u},\\
\frac{\partial p}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}, \quad \text{and}\\
\frac{\partial p}{\partial w} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial w}.
\end{aligned}$$

C13S07.014: If $p = f(x, y, z)$, $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$, then

$$\begin{aligned}
\frac{\partial p}{\partial u} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u} \quad \text{and}\\
\frac{\partial p}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}.
\end{aligned}$$

C13S07.015: If $p = f(u, v, w)$, $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$, then

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x},\\
\frac{\partial p}{\partial y} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}, \quad \text{and}\\
\frac{\partial p}{\partial z} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}.
\end{aligned}$$

C13S07.016: If $p = f(v, w)$, $v = v(x, y, z, t)$, and $w = w(x, y, z, t)$, then

$$\frac{\partial p}{\partial x} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}, \qquad \frac{\partial p}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y},$$

$$\frac{\partial p}{\partial z} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}, \qquad \text{and} \qquad \frac{\partial p}{\partial t} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial t}.$$

C13S07.017: If $p = f(w)$ and $w = w(x, y, z, u, v)$, then

$$\frac{\partial p}{\partial x} = f'(w)\cdot\frac{\partial w}{\partial x}, \quad \frac{\partial p}{\partial y} = f'(w)\cdot\frac{\partial w}{\partial y}, \quad \frac{\partial p}{\partial z} = f'(w)\cdot\frac{\partial w}{\partial z}, \quad \frac{\partial p}{\partial u} = f'(w)\cdot\frac{\partial w}{\partial u}, \quad \text{and} \quad \frac{\partial p}{\partial v} = f'(w)\cdot\frac{\partial w}{\partial v}.$$

C13S07.018: If $p = f(x, y, u, v)$, $x = x(s, t)$, $y = y(s, t)$, $u = u(s, t)$, and $v = v(s, t)$, then

$$\frac{\partial p}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial s} \quad \text{and}$$

$$\frac{\partial p}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}.$$

C13S07.019: Let $F(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 1$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{2}{3}x^{-1/3}}{\frac{2}{3}z^{-1/3}} = -\frac{z^{1/3}}{x^{1/3}} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{2}{3}y^{-1/3}}{\frac{2}{3}z^{-1/3}} = -\frac{z^{1/3}}{y^{1/3}}.$$

C13S07.020: Let $F(x, y, z) = x^3 + y^3 + z^3 - xyz$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 - yz}{3z^2 - xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 - xz}{3z^2 - xy}.$$

C13S07.021: Let $F(x, y, z) = xe^{xy} + ye^{zx} + ze^{xy} - 3$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{xy} + xye^{xy} + yze^{zx} + yze^{xy}}{xye^{zx} + e^{xy}} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2e^{xy} + e^{zx} + xze^{xy}}{xye^{zx} + e^{xy}}.$$

C13S07.022: Let $F(x, y, z) = x^5 + xy^2 + yz - 5$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{5x^4 + y^2}{y} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + z}{y}.$$

C13S07.023: Let

$$F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{c^2 x}{a^2 z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{c^2 y}{b^2 z}.$$

C13S07.024: Let $F(x, y, z) = xyz - \sin(x + y + z)$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz - \cos(x+y+z)}{xy - \cos(x+y+z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz - \cos(x+y+z)}{xy - \cos(x+y+z)}.$$

C13S07.025: If $w = u^2 + v^2 + x^2 + y^2$, $u = x - y$, and $v = x + y$, then

$$\begin{aligned}
\frac{\partial w}{\partial x} &= 2u \cdot u_x + 2v \cdot v_x + 2x = 2(x-y) + 2(x+y) + 2x = 6x \quad \text{and} \\
\frac{\partial w}{\partial y} &= 2u \cdot u_y + 2v \cdot v_y + 2y = -2(x-y) + 2(x+y) + 2y = 6y.
\end{aligned}$$

C13S07.026: If $w = \sqrt{uvxy}$, $u = \sqrt{x-y}$, and $v = \sqrt{x+y}$, then

$$\begin{aligned}
w_x &= w_u u_x + w_v v_x + w_x \cdot 1 + w_y \cdot 0 \\
&= \frac{vxy}{4\sqrt{x-y}\,\sqrt{uvxy}} + \frac{uxy}{4\sqrt{x+y}\,\sqrt{uvxy}} + \frac{uvy}{2\sqrt{uvxy}} \\
&= \frac{uxy\sqrt{x-y} + vxy\sqrt{x+y} + 2uvy\sqrt{x^2-y^2}}{4\sqrt{x^2-y^2}\,\sqrt{uvxy}}.
\end{aligned}$$

Substitution of $\sqrt{x-y}$ for u and $\sqrt{x+y}$ for v finally yields

$$\frac{\partial w}{\partial x} = \frac{xy^2(2x^2 - y^2)}{2\left(xy\sqrt{x^2-y^2}\right)^{3/2}}.$$

Similarly,

$$\frac{\partial w}{\partial y} = \frac{x^2 y(x^2 - 2y^2)}{2\left(xy\sqrt{x^2-y^2}\right)^{3/2}}.$$

C13S07.027: If $w(u, v, x, y) = xy\ln(u+v)$, $u = (x^2+y^2)^{1/3}$, and $v = (x^3+y^3)^{1/2}$, then

$$\begin{aligned}
w_x &= w_u u_x + w_v v_x + w_x \cdot 1 + w_y \cdot 0 \\
&= \frac{2x^2 y}{3(u+v)(x^2+y^2)^{2/3}} + \frac{3x^3 y}{2(u+v)(x^3+y^3)^{1/2}} + y\ln(u+v).
\end{aligned}$$

Then substitution of the formulas for u and v yields

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{2x^2 y}{3(x^2+y^2)^{2/3}\left[\,(x^2+y^2)^{1/3} + (x^3+y^3)^{1/2}\,\right]} \\
&\quad + \frac{3x^3 y}{2(x^3+y^3)^{1/2}\left[\,(x^2+y^2)^{1/3} + (x^3+y^3)^{1/2}\,\right]} + y\,\ln\big((x^2+y^2)^{1/3} + (x^3+y^3)^{1/2}\big).
\end{aligned}$$

Similarly,

$$\frac{\partial w}{\partial y} = \frac{2xy^2}{3(x^2+y^2)^{2/3}\left[\,(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}\,\right]}$$
$$+\frac{3xy^3}{2(x^3+y^3)^{1/2}\left[\,(x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}\,\right]}+x\,\ln\big((\,x^2+y^2)^{1/3}+(x^3+y^3)^{1/2}\big).$$

C13S07.028: If $w(u,\,v,\,x,\,y)=uv-xy$,

$$u=\frac{x}{x^2+y^2},\qquad\text{and}\qquad v=\frac{y}{x^2+y^2}, \tag{1}$$

then

$$\begin{aligned} w_x &= w_u u_x + w_v v_x + w_x\cdot 1 + w_y\cdot 0\\ &= \frac{(y^2-x^2)v}{(x^2+y^2)^2}-\frac{2uxy}{(x^2+y^2)^2}-y. \end{aligned}$$

After substituting the formulas in (1) to eliminate u and v, the result is

$$\frac{\partial w}{\partial x}=\frac{y^3-3x^2y-x^6y-3x^4y^3-3x^2y^5-y^7}{(x^2+y^2)^3}.$$

Similarly,

$$\frac{\partial w}{\partial y}=\frac{x^3-3xy^2-xy^6-3x^3y^4-3x^5y^2-x^7}{(x^2+y^2)^3}.$$

C13S07.029: We differentiate the equation $x^2+y^2+z^2=9$ implicitly, first with respect to x, then with respect to y, and obtain

$$2x+2z\cdot z_x=0,\qquad 2y+2z\cdot z_y=0.$$

We substitute the coordinates of the point of tangency $P(1,\,2,\,2)$ and find that

$$2+4z_x=0\qquad\text{and}\qquad 4+4z_y=0,$$

and it follows that at P, $z_x=-\frac{1}{2}$ and $z_y=-1$. Hence an equation of the plane tangent to the given surface at the point P is

$$z-2=-\frac{1}{2}(x-1)-(y-2);$$

that is, $x+2y+2z=9$.

C13S07.030: Given: The surface with equation $x^2+2y^2+2z^2=14$ and the point $P(2,\,1,\,-2)$ on it. Implicit differentiation of the equation with respect to x and again with respect to y yields

$$2x+4z\cdot z_x=0,\qquad 4y+4z\cdot z_y=0.$$

Substitution of the coordinates of P then yields

$$4 - 8z_x = 0 \quad \text{and} \quad 4 - 8z_y = 0,$$

and thus $z_x = z_y = \frac{1}{2}$ at the point P. Hence an equation of the plane tangent to the surface at P is

$$z + 2 = \frac{1}{2}(x-2) + \frac{1}{2}(y-1);$$

that is, $x + y - 2z = 7$.

C13S07.031: Given: The surface with equation $x^3 + y^3 + z^3 = 5xyz$ and the point $P(2, 1, 1)$ on it. We differentiate the equation implicitly, first with respect to x, then with respect to y, and thereby obtain

$$3x^2 + 3z^2 \cdot z_x = 5yz + 5xy \cdot z_x, \qquad 3y^2 + 3z^2 \cdot z_y = 5xz + 5xy \cdot z_y.$$

Then we substitute the coordinates of P and find that

$$12 + 3z_x = 5 + 10z_x \quad \text{and} \quad 3 + 3z_y = 10 + 10z_y,$$

and it follows that $z_x = 1$ and $z_y = -1$ at the point P. Hence an equation of the plane tangent to the surface at P is

$$z - 1 = (x-2) - (y-1);$$

that is, $x - y - z = 0$.

C13S07.032: Given: The surface with equation $z^3 + (x+y)z^2 + x^2 + y^2 = 13$ and the point $P(2, 2, 1)$ on it. We first differentiate the equation with respect to x, then with respect to y, and find that

$$3z^2 \cdot z_x + 2(x+y)z \cdot z_x + z^2 + 2x = 0 \quad \text{and} \quad 3z^2 \cdot z_y + 2(x+y)z \cdot z_y + z^2 + 2y = 0.$$

Then substitution of the coordinates of P yields

$$3z_x + 8z_x + 1 + 4 = 0, \qquad 3z_y + 8z_y + 1 + 4 = 0.$$

It follows that $z_x = -\frac{5}{11} = z_y$ at the point P. Thus an equation of the plane tangent to the surface at P is

$$z - 1 = -\frac{5}{11}(x-2) - \frac{5}{11}(y-2);$$

that is, $5x + 5y + 11z = 31$.

C13S07.033: Suppose that the square base of the box measures x (inches) on each side and that its height is z. Suppose also that time t is measured in hours. Then the volume of the box is $V = x^2z$, and by the chain rule

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial z} \cdot \frac{dz}{dt} = 2xz \cdot (-3) + x^2 \cdot (-2).$$

Thus when $x = 24$ and $z = 12$, we have

$$\frac{dV}{dt} = 2 \cdot 24 \cdot 12 \cdot (-3) + 24^2 \cdot (-2) = -2880$$

cubic inches per hour; that is, $-\frac{5}{3}$ cubic feet per hour.

C13S07.034: Let x be the length of each edge of the square base of the box and let z be its height. Units: centimeters and minutes. We are given

$$\frac{dx}{dt} = 2 \quad \text{and} \quad \frac{dz}{dt} = -3,$$

and we are to find the rate of change $V'(t)$ of the volume of the box and the rate of change $A'(t)$ of its surface area when $x = z = 100$ (cm). Note that $V = x^2z$ and $A = 2x^2 + 4xz$. First,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial z} \cdot \frac{dz}{dt} = 2xz \cdot 2 + x^2 \cdot (-3),$$

and thus, when $x = z = 100$, we have

$$\frac{dV}{dt} = 40000 - 30000 = 10000$$

cubic centimeters per minute; that is, 0.01 cubic meters per minute. Next,

$$\frac{dA}{dt} = \frac{\partial A}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial A}{\partial z} \cdot \frac{dz}{dt} = (4x + 4z) \cdot 2 + 4x \cdot (-3),$$

so that, when $x = z = 100$,

$$\frac{dA}{dt} = 1600 - 1200 = 400$$

square centimeters per minute; that is, 0.04 square meters per minute.

C13S07.035: Let r denote the radius of the conical sandpile and h its height. Units: feet and minutes. We are given that, at the time t when when $h = 5$ and $r = 2$,

$$\frac{dh}{dt} = 0.4 \quad \text{and} \quad \frac{dr}{dt} = 0.7.$$

The volume of the sandpile is given by $V = \frac{1}{3}\pi r^2 h$, and thus

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} = \left(\frac{2}{3}\pi rh\right) \cdot \frac{7}{10} + \left(\frac{1}{3}\pi r^2\right) \cdot \frac{2}{5}.$$

Thus when $h = 5$ and $r = 2$,

$$\frac{dV}{dt} = \frac{20\pi}{3} \cdot \frac{7}{10} + \frac{4\pi}{3} \cdot \frac{2}{5} = \frac{26\pi}{5} \approx 16.336282$$

(cubic feet per minute).

C13S07.036: Units: centimeters and minutes. We are given

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \text{and} \quad \frac{dz}{dt} = -2.$$

The volume of the box is given by $V = xyz$ and its total surface area by $A = 2(xy + xz + yz)$. We are to find the rate of change of each at the time t at which $x = 300$, $y = 200$, and $z = 100$. First, the volume:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial V}{\partial z} \cdot \frac{dz}{dt} = yz + 2xz - 2xy.$$

Thus when $x = 300$, etc., we have

$$\frac{dV}{dt} = 200 \cdot 100 + 2 \cdot 300 \cdot 100 - 2 \cdot 300 \cdot 200 = -40000.$$

So the volume of the box is *decreasing* at the rate of 40000 cubic centimeters per minute; that is, at the rate of 0.04 cubic meters per minute. Next,

$$\frac{dA}{dt} = 2(y+z)\frac{dx}{dt} + 2(x+z)\frac{dy}{dt} + 2(x+y)\frac{dz}{dt}.$$

Then when $x = 300$, etc., we have

$$\frac{dA}{dt} = 2 \cdot 300 \cdot 1 + 2 \cdot 400 \cdot 2 - 2 \cdot 500 \cdot 2 = 200$$

square centimeters per minute. Thus the surface area of the box is *increasing* at the rate of 0.02 square meters per minute when $x = 300$.

C13S07.037: For this gas sample, we have $V = 10$ when $p = 2$ and $T = 300$. Substitution in the equation $pV = nRT$ yields $nR = \frac{1}{15}$. Moreover, with time t measured in minutes, we have

$$V = \frac{nRT}{p}, \qquad \text{so that} \qquad \frac{dV}{dt} = nR\left(\frac{1}{p} \cdot \frac{dT}{dt} - \frac{T}{p^2} \cdot \frac{dp}{dt}\right).$$

Finally, substitution of the data $nR = \frac{1}{15}$, $V = 10$, $p = 2$, $T = 300$, $dT/dt = 10$, and $dp/dt = 1$ yields the conclusion that the volume of the gas sample is decreasing at the rate of $\frac{13}{3}$ liters per minute at the time in question.

C13S07.038: From the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

and the given data $R_1 = R_2 = 100$, $R_3 = 200$, we find that $R = 40\ \Omega$. Also from this equation we derive (with time t in seconds)

$$\frac{1}{R^2} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt} + \frac{1}{R_3^2} \cdot \frac{dR_3}{dt}.$$

Then substitution of the previous data and the additional information that

$$\frac{dR_1}{dt} = \frac{dR_2}{dt} = 1 \quad \text{and} \quad \frac{dR_2}{dt} = -2$$

yields the result that R is increasing at the rate of $0.24\ \Omega$ per second at the time in question.

C13S07.039: Given: $x = h(y, z)$ satisfies the equation $F(x, y, z) = 0$. Thus $F(h(y, z), y, z) \equiv 0$, and so implicit differentiation with respect to y yields

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \cdot 0 = 0.$$

Thus if $F_x \neq 0$, we find that $\dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$.

C13S07.040: Suppose that $w = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$. Then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\,\cos\theta + \frac{\partial w}{\partial y}\,\sin\theta.$$

Thus

$$\left(\frac{\partial w}{\partial r}\right)^2 = (w_x)^2\cos^2\theta + 2w_x w_y \cos\theta\,\sin\theta + (w_y)^2\sin^2\theta.$$

Next,

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta).$$

Hence

$$\begin{aligned}\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 &= \frac{1}{r^2}(w_x)^2 r^2\sin^2\theta - \frac{2}{r^2}\,w_x w_y r^2\sin\theta\,\cos\theta + \frac{1}{r^2}(w_y)^2 r^2\cos^2\theta\\ &= (w_x)^2\sin^2\theta - 2w_x w_y\sin\theta\,\cos\theta + (w_y)^2\cos^2\theta.\end{aligned}$$

It follows immediately that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2.$$

C13S07.041: If $w = f(u)$ and $u = x + y$, then

$$\frac{\partial w}{\partial x} = f'(u)\cdot\frac{\partial u}{\partial x} = f'(u) = f'(u)\cdot\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y}.$$

C13S07.042: If $w = f(u)$ and $u = x - y$, then

$$\frac{\partial w}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u) \quad\text{ and }\quad \frac{\partial w}{\partial y} = f'(u)\frac{\partial u}{\partial y} = -f'(u),$$

which establishes the first result we were to prove. Moreover,

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x}\,f'(u) = f''(u)\frac{\partial u}{\partial x} = f''(u),\\ \frac{\partial^2 w}{\partial y^2} &= \frac{\partial}{\partial y}\big(-f'(u)\big) = -f''(u)\frac{\partial u}{\partial y} = f''(u), \quad\text{ and}\\ \frac{\partial^2 w}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\big(-f'(u)\big) = -f''(u)\frac{\partial u}{\partial x} = -f''(u).\end{aligned}$$

And this establishes the second equation.

C13S07.043: If $w = f(x,\,y)$, $x = u + v$, and $y = u - v$, then

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}.$$

Therefore

$$\begin{aligned}
\frac{\partial^2 w}{\partial u \partial v} &= \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right) \\
&= \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y \partial x}\cdot\frac{\partial y}{\partial u} - \frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial x}{\partial u} - \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial y}{\partial u} \\
&= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

C13S07.044: If $w = f(x, y)$, $x = 2u + v$, and $y = u - v$, then

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot 2 + \frac{\partial w}{\partial y}\cdot 1; \\
\frac{\partial^2 w}{\partial u^2} &= \frac{\partial}{\partial u}\left(2\,\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\right) \\
&= 2\left(\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y \partial x}\cdot\frac{\partial y}{\partial u}\right) + \frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial y}{\partial u} \\
&= 4\,\frac{\partial^2 w}{\partial x^2} + 2\,\frac{\partial^2 w}{\partial y \partial x} + 2\,\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} = 4\,\frac{\partial^2 w}{\partial x^2} + 4\,\frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}; \\
\frac{\partial^2 w}{\partial v^2} &= \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right) \\
&= \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 w}{\partial y \partial x}\cdot\frac{\partial y}{\partial v} - \frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial x}{\partial v} - \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial y}{\partial v} = \frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

Therefore

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 5\,\frac{\partial^2 w}{\partial x^2} + 2\,\frac{\partial^2 w}{\partial y \partial x} + 2\,\frac{\partial^2 w}{\partial y^2}.$$

C13S07.045: If $w = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$, then

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\,\cos\theta + \frac{\partial w}{\partial y}\,\sin\theta; \\
\frac{\partial^2 w}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\,\cos\theta + \frac{\partial w}{\partial y}\,\sin\theta\right) \\
&= \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial r}\,\cos\theta + \frac{\partial^2 w}{\partial y \partial x}\cdot\frac{\partial y}{\partial r}\,\cos\theta + \frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial x}{\partial r}\,\sin\theta + \frac{\partial^2 w}{\partial y^2}\cdot\frac{\partial y}{\partial r}\,\sin\theta \\
&= \frac{\partial^2 w}{\partial x^2}\,\cos^2\theta + \frac{\partial^2 w}{\partial y \partial x}\,\sin\theta\cos\theta + \frac{\partial^2 w}{\partial x \partial y}\,\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\,\sin^2\theta;
\end{aligned}$$

$$\frac{1}{r}\cdot\frac{\partial w}{\partial r} = \frac{1}{r}\cdot\frac{\partial w}{\partial x}\,\cos\theta + \frac{1}{r}\cdot\frac{\partial w}{\partial y}\,\sin\theta;$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta);$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left(-r \cdot \frac{\partial w}{\partial x}\,\sin\theta + r \cdot \frac{\partial w}{\partial y}\,\cos\theta\right) \\
&= -r\left(\frac{\partial w}{\partial x}\,\cos\theta + \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial x}{\partial \theta}\,\sin\theta + \frac{\partial^2 w}{\partial y\,\partial x} \cdot \frac{\partial y}{\partial \theta}\,\sin\theta\right) \\
&\qquad + r\left(-\frac{\partial w}{\partial y}\,\sin\theta + \frac{\partial^2 w}{\partial x\,\partial y} \cdot \frac{\partial x}{\partial \theta}\,\cos\theta + \frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial y}{\partial \theta}\,\cos\theta\right) \\
&= -r \cdot \frac{\partial w}{\partial x}\,\cos\theta + r^2 \cdot \frac{\partial^2 w}{\partial x^2}\,\sin^2\theta - r^2 \cdot \frac{\partial^2 w}{\partial y\,\partial x}\,\cos\theta\,\sin\theta \\
&\qquad - r \cdot \frac{\partial w}{\partial y}\,\sin\theta - r^2 \cdot \frac{\partial^2 w}{\partial x\,\partial y}\,\sin\theta\cos\theta + r^2 \cdot \frac{\partial^2 w}{\partial y^2}\,\cos^2\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 w}{\partial \theta^2} &= \frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial y\,\partial x}\sin\theta\cos\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta \\
&\quad + \frac{1}{r} \cdot \frac{\partial w}{\partial x}\cos\theta + \frac{1}{r} \cdot \frac{\partial w}{\partial y}\sin\theta - \frac{1}{r} \cdot \frac{\partial w}{\partial x}\cos\theta - \frac{1}{r} \cdot \frac{\partial w}{\partial y}\sin\theta \\
&\quad + \frac{\partial^2 w}{\partial x^2}\sin^2\theta + \frac{\partial^2 w}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 w}{\partial y\,\partial x}\sin\theta\cos\theta = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.
\end{aligned}$$

C13S07.046: Given

$$w = \frac{1}{r} f\Big(t - \frac{r}{a}\Big) \qquad \text{where} \qquad r = (x^2 + y^2 + z^2)^{1/2},$$

show that

$$w_{xx} + w_{yy} + w_{zz} = \frac{1}{a^2} w_{tt}.$$

This problem is best worked in spherical coordinates because of the spherical symmetry of the Laplacian, but it is stated in such a way to suggest that it should be worked in Cartesian coordinates. Thus we will follow that route, but we asked *Mathematica* 3.0 to help us with the complicated computations. First we expressed w in Cartesian form:

```
w[x_, y_, z_, t_] := (1/Sqrt[x*x + y*y + z*z])*f[t - (1/a)*Sqrt[x*x + y*y + z*z]]
```

Then we computed w_x:

```
D[w[x,y,z,t], x] // Together
```

$$-\frac{1}{a(x^2+y^2+z^2)^{3/2}}\left[-axf\bigg(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\bigg) - x(x^2+y^2+z^2)^{1/2} f'\bigg(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\bigg)\right]$$

Next we computed w_{xx}:

```
D[ w[x,y,z,t], {x,2} ]
```

$$\frac{3x^2}{(x^2+y^2+z^2)^{5/2}} f\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right) - \frac{1}{(x^2+y^2+z^2)^{3/2}} f\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right)$$
$$+ \frac{3x^2}{a(x^2+y^2+z^2)^2} f'\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right) - \frac{1}{a(x^2+y^2+z^2)} f'\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right)$$
$$+ \frac{x^2}{a^2(x^2+y^2+z^2)^{3/2}} f''\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right)$$

We then asked for $w_{xx} + w_{yy} + w_{zz}$ with the command

```
D[ w[x,y,z,t], {x,2} ] + D[ w[x,y,z,t], {y,2} ] + D[ w[x,y,z,t], {z,2} ];
```

but suppressed the output by ending the command with the semicolon. But when we asked *Mathematica* to `Simplify` the result, we obtained

$$\frac{1}{a^2(x^2+y^2+z^2)^{1/2}} f''\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right)$$

We compared this with $\dfrac{1}{a^2} w_{tt}$ by computing the latter:

```
(1/(a*a))*D[ w[x,y,z,t], {t,2} ]
```

$$\frac{1}{a^2(x^2+y^2+z^2)^{1/2}} f''\left(t - \frac{(x^2+y^2+z^2)^{1/2}}{a}\right)$$

and this is enough to show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{a^2} \cdot \frac{\partial^2 w}{\partial t^2}.$$

C13S07.047: Suppose that $w = f(r)$ where $r = (x^2+y^2+z^2)^{1/2}$. Then

$$\frac{\partial w}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} f'(r),$$

and thus

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}} f'(r)\right) \\
&= \frac{x}{(x^2+y^2+z^2)^{1/2}} \cdot \frac{\partial}{\partial x}\big(f'(r)\big) + f'(r) \cdot \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right) \\
&= -\frac{x^2 f'(r)}{(x^2+y^2+z^2)^{3/2}} + \frac{f'(r)}{(x^2+y^2+z^2)^{1/2}} + \frac{x^2 f''(r)}{x^2+y^2+z^2} \\
&= -\frac{x^2 f'(r)}{r^3} + \frac{f'(r)}{r} + \frac{x^2 f''(r)}{r^2}.
\end{aligned}$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = -\frac{y^2 f'(r)}{r^3} + \frac{f'(r)}{r} + \frac{y^2 f''(r)}{r^2} \quad \text{and} \quad \frac{\partial^2 w}{\partial z^2} = -\frac{z^2 f'(r)}{r^3} + \frac{f'(r)}{r} + \frac{z^2 f''(r)}{r^2}.$$

Hence

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{r^2 f'(r)}{r^3} + \frac{3f'(r)}{r} + \frac{r^2 f''(r)}{r^2}$$

$$= -\frac{f'(r)}{r} + \frac{3f'(r)}{r} + f''(r) = \frac{d^2 w}{dr^2} + \frac{2}{r} \cdot \frac{dw}{dr}.$$

C13S07.048: If $w = f(u) + g(v)$, $u = x - at$, and $v = x + at$, then

$$\frac{\partial w}{\partial t} = f'(u) \cdot \frac{\partial u}{\partial t} + g'(v) \cdot \frac{\partial v}{\partial t} = -af'(u) + ag'(v).$$

Hence

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\big(-af'(u) + ag'(v)\big) = a^2 f''(u) + a^2 g''(v) = a^2\big(f''(u) + g''(v)\big).$$

Also

$$\frac{\partial w}{\partial x} = f'(u) + g'(v)$$

because $u_x = v_x = 1$. For the same reason,

$$\frac{\partial^2 w}{\partial x^2} = f''(u) + g''(v).$$

Therefore

$$\frac{\partial^2 w}{\partial t^2} \ = \ a^2 \frac{\partial^2 w}{\partial x^2}.$$

C13S07.049: If $w = f(u, v)$, $u = x + y$, and $v = x - y$, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} \qquad \text{and}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}.$$

Therefore

$$\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u}\right)^2 - \left(\frac{\partial w}{\partial v}\right)^2.$$

C13S07.050: Suppose that $w = f(x, y)$, $x = e^u \cos v$, and $y = e^u \sin v$. Then

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial w}{\partial x}\, e^u \cos v + \frac{\partial w}{\partial y}\, e^u \sin v \qquad \text{and}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = -\frac{\partial w}{\partial x}\, e^u \sin v + \frac{\partial w}{\partial y}\, e^u \cos v.$$

Thus

$$\begin{aligned}(w_u)^2 + (w_v)^2 &= (w_x)^2 e^{2u} \cos^2 v + 2w_x w_y e^{2u} \sin v \, \cos v + (w_y)^2 e^{2u} \sin^2 v \\ &\qquad + (w_x)^2 e^{2u} \sin^2 v - 2w_x w_y e^{2u} \sin v \, \cos v + (w_y)^2 e^{2u} \cos^2 v \\ &= e^{2u}(w_x)^2 + e^{2u}(w_y)^2.\end{aligned}$$

Therefore

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2\right].$$

C13S07.051: We are given $w = f(x, y)$ and the existence of a constant α such that

$$x = u\cos\alpha - v\sin\alpha \quad \text{and} \quad y = u\sin\alpha + v\cos\alpha.$$

Then

$$\begin{aligned}w_u &= w_x x_u + w_y y_u = w_x \cos\alpha + w_y \sin\alpha; \\ w_v &= w_x x_v + w_y y_v = -w_x \sin\alpha + w_y \cos\alpha; \\ (w_u)^2 + (w_v)^2 &= (w_x)^2 \cos^2\alpha + 2w_x w_y \sin\alpha \, \cos\alpha + (w_y)^2 \sin^2\alpha \\ &\qquad + (w_x)^2 \sin^2\alpha - 2w_x w_y \sin\alpha \, \cos\alpha + (w_y)^2 \cos^2\alpha \\ &= (w_x)^2 + (w_y)^2.\end{aligned}$$

C13S07.052: If $w = f(u)$ where

$$u = \frac{x^2 - y^2}{x^2 + y^2},$$

then

$$xw_x + yw_y = xf'(u)u_x + yf'(u)u_y = xf'(u) \cdot \frac{4xy^2}{(x^2 + y^2)^2} - yf'(u) \cdot \frac{4x^2 y}{(x^2 + y^2)^2} = 0.$$

C13S07.053: Using the *Suggestion* and the notation in the equations in (17), we have

$$\left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{F_y}{F_x}\right) \cdot \left(-\frac{F_z}{F_y}\right) \cdot \left(-\frac{F_x}{F_z}\right) = -1.$$

C13S07.054: If $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ implicitly defines $z = f(x, y)$, $y = g(x, z)$, and $x = h(y, z)$, then:

$$\begin{aligned}2x\frac{\partial h}{\partial y} + 2y = 0, &\quad \text{so} \quad -\frac{y}{x} = \frac{\partial h}{\partial y} = \left(\frac{\partial x}{\partial y}\right)_z, \\ 2y\frac{\partial g}{\partial z} + 2z = 0, &\quad \text{so} \quad -\frac{z}{y} = \frac{\partial g}{\partial z} = \left(\frac{\partial y}{\partial z}\right)_x, \quad \text{and} \\ 2x + 2z\frac{\partial f}{\partial x} = 0, &\quad \text{so} \quad -\frac{x}{z} = \frac{\partial f}{\partial x} = \left(\frac{\partial z}{\partial x}\right)_y.\end{aligned}$$

Thus

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{y}{x}\right)\left(-\frac{z}{y}\right)\left(-\frac{x}{z}\right) = -1.$$

C13S07.055: If the equation $pV - nRT = 0$ implicitly defines the functions $T = f(p, V)$, $V = g(p, T)$, and $p = h(V, T)$, then

$$\begin{aligned} p + V\frac{\partial h}{\partial V} = 0, \quad &\text{so} \quad -\frac{p}{V} = \frac{\partial h}{\partial V} = \left(\frac{\partial p}{\partial V}\right)_T, \\ p\frac{\partial g}{\partial T} - nR = 0, \quad &\text{so} \quad \frac{nR}{p} = \frac{\partial g}{\partial T} = \left(\frac{\partial V}{\partial T}\right)_p, \quad \text{and} \\ V - nR\frac{\partial f}{\partial p} = 0, \quad &\text{so} \quad \frac{V}{nR} = \frac{\partial f}{\partial p} = \left(\frac{\partial T}{\partial p}\right)_V. \end{aligned}$$

Therefore

$$\left(\frac{\partial p}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial p}\right)_V = \left(-\frac{p}{V}\right)\left(\frac{nR}{p}\right)\left(\frac{V}{nR}\right) = -1.$$

C13S07.056: From the equation $F(p, V, T) = 0$ and results in Section 13.7, we find that

$$\frac{\partial V}{\partial p} = -\frac{F_p}{F_V}, \quad \frac{\partial V}{\partial T} = -\frac{F_T}{F_V}, \quad \frac{\partial p}{\partial V} = -\frac{F_V}{F_p}, \quad \text{and} \quad \frac{\partial p}{\partial T} = -\frac{F_T}{F_p}.$$

It now follows that

$$\frac{\alpha}{\beta} = -\frac{V_T}{V_p} = -\frac{F_T/F_V}{F_p/F_V} = -\frac{F_T}{F_p} = \frac{\partial p}{\partial T}.$$

C13S07.057: First note that

$$\frac{\partial p}{\partial T} = \frac{\alpha}{\beta} = \frac{1.8 \times 10^6}{3.9 \times 10^4} = \frac{600}{13}.$$

Hence an increase of 5° in the Celsius temperature multiplies the initial pressure (1 atm) by $\frac{3000}{13} \approx 230.77$, so the bulb will burst.

C13S07.058: The result

$$T'(u, v, w) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

follows immediately from the definition of derivative matrix.

C13S07.059: Here we have

$$T'(\rho,\,\phi,\,\theta) = \begin{bmatrix} \sin\phi\,\cos\theta & \rho\cos\phi\,\cos\theta & -\rho\sin\phi\,\sin\theta \\ \sin\phi\,\sin\theta & \rho\cos\phi\,\sin\theta & \rho\sin\phi\,\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} |T'(\rho,\,\phi,\,\theta)| &= \rho^2\sin^3\phi\,\sin^2\theta + \rho^2\sin\phi\,\cos^2\phi\,\cos^2\theta + \rho^2\cos^2\phi\,\sin\phi\,\sin^2\theta + \rho^2\sin^3\phi\,\cos^2\theta \\ &= \rho^2(\sin^3\phi + \sin\phi\,\cos^2\phi) = \rho^2\sin\phi. \end{aligned}$$

C13S07.060: According to the chain rule in Theorem 2, the partial derivatives of the composition

$$G(u,\,v,\,w) = F(T(u,\,v,\,w)) = F\big(x(u,\,v,\,w),\ y(u,\,v,\,w),\ z(u,\,v,\,w)\big)$$

are given in scalar notation by

$$\begin{aligned} G_u &= F_x x_u + F_y y_u + F_z z_u, \\ G_v &= F_x x_v + F_y y_v + F_z z_v, \\ G_w &= F_x x_w + F_y y_w + F_z z_w. \end{aligned}$$

In matrix notation, this means that the derivative matrix of G is given by

$$G' = \begin{bmatrix} G_u \\ G_v \\ G_w \end{bmatrix} = \begin{bmatrix} F_x x_u + F_y y_u + F_z z_u \\ F_x x_v + F_y y_v + F_z z_v \\ F_x x_w + F_y y_w + F_z z_w \end{bmatrix}.$$

Because the derivative matrices of F and T are given by

$$F' = [\,F_x \qquad F_y \qquad F_z\,] \qquad \text{and} \qquad T' = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix},$$

it follows upon calculating the matrix product that

$$F'T' = [\,F_x \qquad F_y \qquad F_z\,]\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} = \begin{bmatrix} F_x x_u + F_y y_u + F_z z_u \\ F_x x_v + F_y y_v + F_z z_v \\ F_x x_w + F_y y_w + F_z z_w \end{bmatrix} = G'.$$

C13S07.061: Here we compute

$$\begin{bmatrix} F_x & F_y & F_z \end{bmatrix} \begin{bmatrix} \sin\phi\,\cos\theta & \rho\cos\phi\,\cos\theta & -\rho\sin\phi\,\sin\theta \\ \sin\phi\,\sin\theta & \rho\cos\phi\,\sin\theta & \rho\sin\phi\,\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial \rho} & \dfrac{\partial w}{\partial \phi} & \dfrac{\partial w}{\partial \theta} \end{bmatrix},$$

which has first component

$$\frac{\partial w}{\partial \rho} = F_x \sin\phi\,\cos\theta + F_y \sin\phi\,\sin\theta + F_z \cos\phi,$$

second component

$$\frac{\partial w}{\partial \phi} = F_x\,\rho\cos\phi\,\cos\theta + F_y\,\rho\cos\phi\,\sin\theta - F_z\,\rho\sin\phi,$$

and third component

$$\frac{\partial w}{\partial \theta} = -F_x\,\rho\sin\phi\,\sin\theta + F_y\,\rho\sin\phi\,\cos\theta.$$

Section 13.8

C13S08.001: If $f(x, y) = 3x - 7y$ and $P(17, 39)$ are given, then

$$\nabla f(x, y) = \langle 3, -7 \rangle, \qquad \text{and thus} \qquad \nabla f(17, 39) = \langle 3, -7 \rangle.$$

C13S08.002: If $f(x, y) = 3x^2 - 5y^2$ and $P(2, -3)$ are given, then

$$\nabla f(x, y) = \langle 6x, -10y \rangle, \qquad \text{and thus} \qquad \nabla f(2, -3) = \langle 12, 30 \rangle.$$

C13S08.003: If $f(x, y) = \exp(-x^2 - y^2)$ and $P(0, 0)$ are given, then

$$\nabla f(x, y) = \langle -2x\exp(-x^2 - y^2), -2y\exp(-x^2 - y^2) \rangle, \qquad \text{and thus} \qquad \nabla f(0, 0) = \langle 0, 0 \rangle = \mathbf{0}.$$

C13S08.004: If $f(x, y) = \sin \frac{1}{4}\pi xy$ and $P(3, -1)$ are given, then

$$\nabla f(x, y) = \left\langle \tfrac{1}{4}\pi y \cos \tfrac{1}{4}\pi xy, \tfrac{1}{4}\pi x \cos \tfrac{1}{4}\pi xy \right\rangle, \qquad \text{and so} \qquad \nabla f(3, -1) = \left\langle \tfrac{1}{8}\pi\sqrt{2}, -\tfrac{3}{8}\pi\sqrt{2} \right\rangle.$$

C13S08.005: Given $f(x, y, z) = y^2 - z^2$ and $P(17, 3, 2)$, then

$$\nabla f(x, y, z) = \langle 0, 2y, -2z \rangle, \qquad \text{and therefore} \qquad \nabla f(17, 3, 2) = \langle 0, 6, -4 \rangle.$$

C13S08.006: Given $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ and $P(12, 3, 4)$, then

$$\nabla f(x, y, z) = \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle,$$

and therefore $\nabla f(12, 3, 4) = \langle \frac{12}{13}, \frac{3}{13}, \frac{4}{13} \rangle$.

C13S08.007: Given $f(x, y, z) = e^x \sin y + e^y \sin z + e^z \sin x$ and $P(0, 0, 0)$, then

$$\nabla f(x, y, z) = \langle e^z \cos x + e^x \sin y, e^x \cos y + e^y \sin z, e^y \cos z + e^z \sin x \rangle,$$

and therefore $\nabla f(0, 0, 0) = \langle 1, 1, 1 \rangle$.

C13S08.008: Given $f(x, y, z) = x^2 - 3yz + z^3$ and $P(2, 1, 0)$, then

$$\nabla f(x, y, z) = \langle 2x, -3z, 3z^2 - 3y \rangle, \qquad \text{and so} \qquad \nabla f(2, 1, 0) = \langle 4, 0, -3 \rangle.$$

C13S08.009: Given $f(x, y, z) = 2\sqrt{xyz}$ and $P(3, -4, -3)$, then

$$\nabla f(x, y, z) = \left\langle \frac{yz}{\sqrt{xyz}}, \frac{xz}{\sqrt{xyz}}, \frac{xy}{\sqrt{xyz}} \right\rangle, \qquad \text{and so} \qquad \nabla f(3, -4, -3) = \left\langle 2, -\frac{3}{2}, -2 \right\rangle.$$

C13S08.010: Given $f(x, y, z) = (2x - 3y + 5z)^5$ and $P(-5, 1, 3)$, then

$$\nabla f(x, y, z) = \langle 10(2x - 3y + 5z)^4, -15(2x - 3y + 5z)^4, 25(2x - 3y + 5z)^4 \rangle,$$

and therefore $\nabla f(-5, 1, 3) = \langle 160, -240, 400 \rangle$.

C13S08.011: Given $f(x, y) = x^2 + 2xy + 3y^2$, $P(2, 1)$, and $\mathbf{v} = \langle 1, 1 \rangle$, we first compute a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2} \right\rangle.$$

Also $\nabla f(x, y) = \langle 2x + 2y, 2x + 6y \rangle$, so $\nabla f(P) = \langle 6, 10 \rangle$. Therefore

$$D_{\mathbf{u}} f(P) = (\nabla f(P)) \cdot \mathbf{u} = \langle 6, 10 \rangle \cdot \left\langle \tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2} \right\rangle = 8\sqrt{2}.$$

C13S08.012: Given $f(x, y) = e^x \sin y$, $P\left(0, \tfrac{1}{4}\pi\right)$, and $\mathbf{v} = \langle 1, -1 \rangle$, we first compute a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\sqrt{2} \right\rangle.$$

Also $\nabla f(x, y) = \langle e^x \sin y, e^x \cos y \rangle$, so $\nabla f(P) = \langle \tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2} \rangle$. Therefore $D_{\mathbf{u}} f(P) = (\nabla f(P)) \cdot \mathbf{u} = 0$.

C13S08.013: Given $f(x, y) = x^3 - x^2 y + xy^2 + y^3$, $P(1, -1)$, and $\mathbf{v} = \langle 2, 3 \rangle$, we first compute a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \tfrac{2}{13}\sqrt{13}, \tfrac{3}{13}\sqrt{13} \right\rangle.$$

Also $\nabla f(x, y) = \langle 3x^2 - 2xy + y^2, 3y^2 + 2xy - x^2 \rangle$, so $\nabla f(P) = \langle 6, 0 \rangle$. Therefore

$$D_{\mathbf{u}} f(P) = (\nabla f(P)) \cdot \mathbf{u} = \langle 6, 0 \rangle \cdot \left\langle \tfrac{2}{13}\sqrt{13}, \tfrac{3}{13}\sqrt{13} \right\rangle = \tfrac{12}{13}\sqrt{13}.$$

C13S08.014: Given $f(x, y) = \arctan(y/x)$, $P(-3, 3)$, and $\mathbf{v} = \langle 3, 4 \rangle$, we can automate the computation of $D_{\mathbf{u}} f(P)$ using *Mathematica* 3.0 as follows. First we find the unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$ (remember that `v.v` is the way to compute $\mathbf{v} \cdot \mathbf{v}$):

```
v = {3, 4};

u = v/Sqrt[v.v]
```

$$\left\{ \frac{3}{5}, \frac{4}{5} \right\}$$

Then we define f and compute its gradient:

```
f[x_, y_] := ArcTan[y/x]

{D[f[x,y], x], D[f[x,y], y]} // Simplify
```

$$\left\{ -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\}$$

Then we evaluate $\nabla f(P)$ (recall that % refers to the "last output"):

```
% /.  {x → -3, y → 3}
```

$$\left\{-\frac{1}{6}, -\frac{1}{6}\right\}$$

Now we can compute $D_{\mathbf{u}}f(P) = (\boldsymbol{\nabla} f(P)) \cdot \mathbf{u}$:

```
%.u
```

$$-\frac{7}{30}$$

C13S08.015: Given: $f(x, y) = \sin x \cos y$, the point $P\left(\frac{1}{3}\pi, -\frac{2}{3}\pi\right)$, and the vector $\mathbf{v} = \langle 4, -3 \rangle$, we first construct a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle.$$

Next, $\boldsymbol{\nabla} f(x, y) = \langle \cos x \cos y, -\sin x \sin y \rangle$, and hence $\boldsymbol{\nabla} f(P) = \langle -\frac{1}{4}, \frac{3}{4} \rangle$. Therefore

$$D_{\mathbf{u}}f(P) = (\boldsymbol{\nabla} f(P)) \cdot \mathbf{u} = -\frac{13}{20}.$$

C13S08.016: Given $f(x, y, z) = xy + yz + zx$, the point $P(1, -1, 2)$, and the vector $\mathbf{v} = \langle 1, 1, 1 \rangle$, we first construct a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle.$$

Next, $\boldsymbol{\nabla} f(x, y, z) = \langle y + z, x + z, x + y \rangle$, and thus $\boldsymbol{\nabla} f(P) = \langle 1, 3, 0 \rangle$. Therefore

$$D_{\mathbf{u}}f(P) = \big(\boldsymbol{\nabla} f(P)\big) \cdot \mathbf{u} = \frac{4\sqrt{3}}{3}.$$

C13S08.017: Given $f(x, y, z) = \sqrt{xyz}$, the point $P(2, -1, -2)$, and the vector $\mathbf{v} = \langle 1, 2, -2 \rangle$, we first construct a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle.$$

Next,

$$\boldsymbol{\nabla} f(x, y, z) = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle,$$

and hence $D_{\mathbf{u}}f(P) = \big(\boldsymbol{\nabla} f(P)\big) \cdot \mathbf{u} = \left\langle \frac{1}{2}, -1, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle = -\frac{1}{6}.$

C13S08.018: We are given $f(x, y, z) = \ln(1 + x^2 + y^2 - z^2)$, the point $P(1, -1, 1)$, and the vector $\mathbf{v} = \langle 2, -2, -3 \rangle$. The first step is to construct a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}} \right\rangle.$$

Next,

$$\nabla f(x,\,y,\,z)=\left\langle\frac{2x}{1+x^2+y^2-z^2},\,\frac{2y}{1+x^2+y^2-z^2},\,-\frac{2z}{1+x^2+y^2-z^2}\right\rangle,$$

and thus $\nabla f(P)=\langle\,1,\,-1,\,-1\,\rangle$. Therefore

$$D_{\mathbf{u}}f(P)=\big(\nabla f(P)\big)\cdot\mathbf{u}=\frac{7}{\sqrt{17}}.$$

C13S08.019: Given $f(x,\,y,\,z)=\exp(xyz)$, the point $P(4,\,0,\,-3)$, and the vector $\mathbf{v}=\langle\,0,\,1,\,-1\,\rangle$, we first construct a unit vector with the same direction as $\mathbf{v}$:

$$\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left\langle 0,\,\frac{\sqrt{2}}{2},\,-\frac{\sqrt{2}}{2}\right\rangle.$$

Next, $\nabla f(x,\,y,\,z)=\exp(xyz)\langle\,yz,\,xz,\,xy\,\rangle$, and so $\nabla f(P)=\langle\,0,\,-12,\,0\,\rangle$. Therefore

$$D_{\mathbf{u}}f(P)=\big(\nabla f(P)\big)\cdot\mathbf{u}=-6\sqrt{2}\,.$$

C13S08.020: Given $f(x,\,y,\,z)=\sqrt{10-x^2-y^2-z^2}$, the point $P(1,\,1,\,-2)$, and the vector $\mathbf{v}=\langle\,3,\,4,\,-12\,\rangle$, we begin by constructing the unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$:

$$\mathbf{u}=\frac{\mathbf{v}}{|\mathbf{v}|}=\left\langle\frac{3}{13},\,\frac{4}{13},\,-\frac{12}{13}\right\rangle.$$

Then

$$\nabla f(x,\,y,\,z)=\left\langle-\frac{x}{\sqrt{10-x^2-y^2-z^2}},\,-\frac{y}{\sqrt{10-x^2-y^2-z^2}},\,-\frac{z}{\sqrt{10-x^2-y^2-z^2}}\right\rangle.$$

Therefore

$$D_{\mathbf{u}}f(P)=\big(\nabla f(P)\big)\cdot\mathbf{u}=\left\langle-\frac{1}{2},\,-\frac{1}{2},\,1\right\rangle\cdot\left\langle\frac{3}{13},\,\frac{4}{13},\,-\frac{12}{13}\right\rangle=-\frac{31}{26}.$$

C13S08.021: Given $f(x,\,y)=2x^2+3xy+4y^2$ and the point $P(1,\,1)$, we first compute

$$\nabla f(x,\,y)=\langle\,4x+3y,\,3x+8y\,\rangle.$$

So the direction in which f is increasing the most rapidly at P is $\nabla f(P)=\langle\,7,\,11\,\rangle$ and its rate of increase in that direction is $|\langle\,7,\,11\,\rangle|=\sqrt{170}\,$.

C13S08.022: Given $f(x,\,y)=\arctan\left(\dfrac{y}{x}\right)$ and the point $P(2,\,-3)$, we first compute

$$\nabla f(x,\,y)=\left\langle-\frac{y}{x^2+y^2},\,\frac{x}{x^2+y^2}\right\rangle.$$

So the direction in which f is increasing the most rapidly at P is

$$\mathbf{v}=\nabla f(P)=\left\langle\frac{3}{13},\,\frac{2}{13}\right\rangle,$$

and its rate of increase in that direction is $|\mathbf{v}| = \dfrac{\sqrt{13}}{13}$.

C13S08.023: Given $f(x, y) = \ln(x^2 + y^2)$ and the point $P(3, 4)$, we first compute

$$\nabla f(x, y) = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle.$$

Therefore the direction in which f is increasing the most rapidly at P is

$$\mathbf{v} = \nabla f(P) = \left\langle \frac{6}{25}, \frac{8}{25} \right\rangle,$$

and its rate of increase in that direction is $|\mathbf{v}| = \dfrac{2}{5}$.

C13S08.024: Given $f(x, y) = \sin(3x - 4y)$ and the point $P\left(\frac{1}{3}\pi, \frac{1}{4}\pi\right)$, we first compute

$$\nabla f(x, y) = \langle\, 3\cos(3x - 4y), -4\cos(3x - 4y)\,\rangle.$$

Therefore the direction in which f is increasing the most rapidly at P is $\mathbf{v} = \nabla f(P) = \langle\, 3, -4\,\rangle$ and its rate of increase in that direction is $|\mathbf{v}| = 5$.

C13S08.025: Given $f(x, y, z) = 3x^2 + y^2 + 4z^2$ and the point $P(1, 5, -2)$, we first compute

$$\nabla f(x, y, z) = \langle\, 6x, 2y, 8z\,\rangle.$$

Therefore the direction in which f is increasing the most rapidly at P is $\mathbf{v} = \nabla f(P) = \langle\, 6, 10, -16\,\rangle$ and its rate of increase in that direction is $|\mathbf{v}| = 14\sqrt{2}$.

C13S08.026: We are to find the direction in which $f(x, y, z) = \exp(x - y - z)$ is increasing the most rapidly at the point $P(5, 2, 3)$ and its rate of increase in that direction. Such computations can easily be carried out with computer algebra systems such as *Mathematica* 3.0. We first define f:

```
f[x_, y_, z_] := Exp[x - y - z]
```

Then we compute the gradient of f:

```
{D[f[x,y,z],x], D[f[x,y,z],y], D[f[x,y,z],z]}
```

$$\left\langle e^{x-y-z}, -e^{x-y-z}, -e^{x-y-z} \right\rangle$$

Then we evaluate the last output at P:

```
% /.  {x → 5, y → 2, z → 3}
```

$$\langle\, 1, -1, -1\,\rangle$$

The last output, $\nabla f(P)$, gives the direction in which f is increasing the most rapidly at P. Its magnitude is the rate of increase of f in that direction:

```
Sqrt[%.%]
```

$$\sqrt{3}$$

C13S08.027: We are given $f(x, y, z) = \sqrt{xy^2z^3}$ and the point $P(2, 2, 2)$. We first compute the gradient of f:

$$\nabla f(x, y, z) = \left\langle \frac{y^2z^3}{2\sqrt{xy^2z^3}}, \frac{xyz^3}{\sqrt{xy^2z^3}}, \frac{3xy^2z^2}{2\sqrt{xy^2z^3}} \right\rangle.$$

Thus the direction in which f is increasing the most rapidly at P is $\nabla f(P) = \langle 2, 4, 6 \rangle$ and its rate of increase in that direction is $|\nabla f(P)| = 2\sqrt{14}$.

C13S08.028: Given: $f(x, y, z) = \sqrt{2x + 4y + 6x}$ and the point $P(7, 5, 5)$. We first compute the gradient of f:

$$\nabla f(x, y, z) = \left\langle \frac{1}{\sqrt{2x + 4y + 6x}}, \frac{2}{\sqrt{2x + 4y + 6x}}, \frac{3}{\sqrt{2x + 4y + 6x}} \right\rangle.$$

Hence the direction in which f is increasing the most rapidly at P is

$$\mathbf{v} = \nabla f(P) = \left\langle \frac{1}{8}, \frac{1}{4}, \frac{3}{8} \right\rangle$$

and its rate of increase in that direction is $|\mathbf{v}| = \frac{1}{8}\sqrt{14}$.

C13S08.029: Let $f(x, y) = \exp(25 - x^2 - y^2) - 1$. Then

$$\nabla f(x, y) = \langle -2x\exp(25 - x^2 - y^2), -2y\exp(25 - x^2 - y^2) \rangle,$$

so at $P(3, 4)$ we have $\nabla f(P) = \langle -6, -8 \rangle$, a vector normal to the graph of $f(x, y) = 0$ at the point P. Hence, as in Example 7, an equation of the line tangent to the graph at P is $-6(x - 3) - 8(y - 4) = 0$; simplified, this is $3x + 4y = 25$.

C13S08.030: Let $f(x, y) = 2x^2 + 3y^2 - 35$. Then $\nabla f(x, y) = \langle 4x, 6y \rangle$. Thus a vector normal to the graph of $f(x, y) = 0$ at the point $P(2, 3)$ is $\nabla f(P) = \langle 8, 18 \rangle$. Hence an equation of the line tangent to the graph at P is $8(x - 2) + 18(y - 3) = 0$; that is, $4x + 9y = 35$.

C13S08.031: Let $f(x, y) = x^4 + xy + y^2 - 19$. Then $\nabla f(x, y) = \langle 4x^3 + y, x + 2y \rangle$, so a vector normal to the graph of $f(x, y) = 0$ at the point $P(2, -3)$ is $\nabla f(P) = \langle 29, -4 \rangle$. So the tangent line at P has equation $29(x - 2) - 4(y + 3) = 0$; that is, $29x - 4y = 70$.

C13S08.032: Let $f(x, y, z) = 3x^2 + 4y^2 + 5z^2 - 73$. Then $\nabla f(x, y, z) = \langle 6x, 8y, 10z \rangle$, so a vector normal to the graph of $f(x, y, z) = 0$ at the point $P(2, 2, 3)$ is $\nabla f(P) = \langle 12, 16, 30 \rangle$. Therefore the plane tangent to the graph at P has equation

$$12(x - 2) + 16(y - 2) + 30(z - 3) = 0; \qquad \text{that is,} \qquad 6x + 8y + 15z = 73.$$

C13S08.033: Let $f(x, y, z) = x^{1/3} + y^{1/3} + z^{1/3} - 1$. Then

$$\nabla f(x, y, z) = \left\langle \frac{1}{3x^{2/3}}, \frac{1}{3y^{2/3}}, \frac{1}{3z^{2/3}} \right\rangle,$$

and thus a vector normal to the surface $f(x, y, z) = 0$ at the point $P(1, -1, 1)$ is

$$\nabla f(P) = \left\langle \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3} \right\rangle.$$

Therefore an equation of the plane tangent to the surface at P is

$$\frac{1}{3}(x-1) + \frac{1}{3}(y+1) + \frac{1}{3}(z-1) = 0; \qquad \text{that is,} \qquad x + y + z = 1.$$

C13S08.034: Let $f(x, y, z) = xyz + x^2 - 2y^2 + z^3 - 14$. Then $\nabla f(x, y, z) = \langle\, yz + 2x,\ xz - 4y,\ xy + 3z^2 \,\rangle$, so a vector normal to the surface $f(x, y, z) = 0$ at the point $P(5, -2, 3)$ is $\nabla f(P) = \langle\, 4,\ 23,\ 17 \,\rangle$. Therefore the plane tangent to this surface at P has equation

$$4(x-5) + 23(y+2) + 17(z-3) = 0; \qquad \text{that is,} \qquad 4x + 23y + 17z = 25.$$

C13S08.035: If u and v are differentiable functions of x and y and a and b are constants, then

$$\begin{aligned}
\nabla\big(au(x, y) + bv(x, y)\big) &= \left\langle \frac{\partial}{\partial x}\big(au(x, y) + bv(x, y)\big),\ \frac{\partial}{\partial y}\big(au(x, y) + bv(x, y)\big) \right\rangle \\
&= \langle\, au_x + bv_x,\ au_y + bv_y \,\rangle = \langle\, au_x,\ au_y \,\rangle + \langle\, bv_x,\ bv_y \,\rangle \\
&= a\,\langle\, u_x,\ u_y \,\rangle + b\,\langle\, v_x,\ v_y \,\rangle = a\nabla u(x, y) + b\nabla v(x, y).
\end{aligned}$$

C13S08.036: If u and v are differentiable function of x and y, then

$$\begin{aligned}
\nabla\big(u(x, y)\cdot v(x, y)\big) &= \left\langle \frac{\partial}{\partial x}(uv),\ \frac{\partial}{\partial y}(uv) \right\rangle \\
&= \langle\, u_x v + uv_x,\ u_y v + uv_y \,\rangle = \langle\, u_x v,\ u_y v \,\rangle + \langle\, uv_x,\ uv_y \,\rangle \\
&= v\cdot\langle\, u_x,\ u_y \,\rangle + u\cdot\langle\, v_x,\ v_y \,\rangle = u(x, y)\cdot\nabla v(x, y) + v(x, y)\cdot\nabla u(x, y).
\end{aligned}$$

C13S08.037: If u and v are differentiable functions of x and y and $v(x, y) \neq 0$, then

$$\begin{aligned}
\nabla\left(\frac{u(x, y)}{v(x, y)}\right) &= \left\langle \frac{\partial}{\partial x}\left(\frac{u}{v}\right),\ \frac{\partial}{\partial y}\left(\frac{u}{v}\right) \right\rangle = \left\langle \frac{vu_x - uv_x}{v^2},\ \frac{vu_y - uv_y}{v^2} \right\rangle \\
&= \left\langle \frac{vu_x}{v^2},\ \frac{vu_y}{v^2} \right\rangle - \left\langle \frac{uv_x}{v^2},\ \frac{uv_y}{v^2} \right\rangle = \frac{v\nabla u}{v^2} - \frac{u\nabla v}{v^2} = \frac{v\nabla u - u\nabla v}{v^2}.
\end{aligned}$$

C13S08.038: Suppose that n is a positive integer and that u is a differentiable function of x and y. Then

$$\begin{aligned}
\nabla(u^n) &= \left\langle \frac{\partial}{\partial x}(u^n),\ \frac{\partial}{\partial y}(u^n) \right\rangle = \langle\, nu^{n-1}u_x,\ nu^{n-1}u_y \,\rangle \\
&= nu^{n-1}\cdot\langle\, u_x,\ u_y \,\rangle = nu^{n-1}\cdot\nabla u.
\end{aligned}$$

C13S08.039: We know that $\mathbf{v} = \nabla f(P)$ gives the direction in which f is increasing the most rapidly at P. Then $\mathbf{v}$ is the direction in which $-f$ is decreasing the most rapidly at P. But $\nabla\big(-f(P)\big) = -\nabla f(P) = -\mathbf{v}$, so that $-\mathbf{v}$ is the direction in which $-f$ is increasing the most rapidly at P and, therefore, is the direction in which f is decreasing the most rapidly at P.

C13S08.040: Suppose that f is a differentiable function of the three independent variables x, y, and z. Then

$$D_{\mathbf{i}}f(x, y, z) = \big(\nabla f(x, y, z)\big) \cdot \mathbf{i} = \langle f_x, f_y, f_z \rangle \cdot \langle 1, 0, 0 \rangle = f_x(x, y, z),$$
$$D_{\mathbf{j}}f(x, y, z) = \big(\nabla f(x, y, z)\big) \cdot \mathbf{j} = \langle f_x, f_y, f_z \rangle \cdot \langle 0, 1, 0 \rangle = f_y(x, y, z), \quad \text{and}$$
$$D_{\mathbf{k}}f(x, y, z) = \big(\nabla f(x, y, z)\big) \cdot \mathbf{k} = \langle f_x, f_y, f_z \rangle \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z).$$

C13S08.041: Let $f(x, y) = Ax^2 + Bxy + Cy^2 - D$. Then

$$\nabla f(x, y) = \langle 2Ax + By, 2Cy + Bx \rangle,$$

so a vector normal to the graph of $f(x, y) = 0$ at the point $P(x_0, y_0)$ is

$$\nabla f(P) = \langle 2Ax_0 + By_0, 2Cy_0 + Bx_0 \rangle.$$

Hence, as in Example 7, an equation of the line tangent to the graph at P is

$$(2Ax_0 + By_0)(x - x_0) + (2Cy_0 + Bx_0)(y - y_0) = 0;$$
$$2Ax_0x + By_0x - 2A(x_0)^2 - Bx_0y_0 + 2Cy_0y + Bx_0y - 2C(y_0)^2 - Bx_0y_0 = 0;$$
$$2Ax_0x + By_0x + Bx_0y + 2Cy_0y = 2A(x_0)^2 + 2Bx_0y_0 + 2C(y_0)^2;$$
$$2(Ax_0)x + B(y_0x + x_0y) + 2(Cy_0)y = 2D;$$
$$(Ax_0)x + \frac{1}{2}B(y_0x + x_0y) + (Cy_0)y = D.$$

C13S08.042: Let $f(x, y, z) = Ax^2 + By^2 + Cz^2 - D$. then

$$\nabla f(x, y, z) = \langle 2Ax, 2By, 2Cz \rangle,$$

so a vector normal to the graph of $f(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is

$$\nabla f(P) = \langle 2Ax_0, 2By_0, 2Cz_0 \rangle.$$

Therefore an equation of the plane tangent to the graph at P is

$$2Ax_0(x - x_0) + 2By_0(y - y_0) + 2Cz_0(z - z_0) = 0;$$
$$(Ax_0)x + (By_0)y + (Cz_0)z = A(x_0)^2 + B(y_0)^2 + C(z_0)^2 = D.$$

C13S08.043: The equation of the paraboloid can be written in the form

$$H(x, y, z) = Ax^2 + By^2 - z = 0, \quad \text{and} \quad \nabla H(x, y, z) = \langle 2Ax, 2By, -1 \rangle.$$

A vector normal to the paraboloid at the point $P(x_0, y_0, z_0)$ is $\mathbf{n} = \langle 2Ax_0, 2By_0, -1 \rangle$, and hence the plane tangent to the paraboloid at P has an equation of the form

$$2Ax_0x + 2By_0y - z = d.$$

But the point P also lies on the plane, and hence

$$d = 2A(x_0)^2 + 2B(y_0)^2 - z_0 = 2(Ax_0^2 + By_0^2) - z_0 = 2z_0 - z_0 = z_0.$$

Hence an equation of the tangent plane is $2Ax_0x + 2By_0y - z = z_0$, and the result in Problem 43 follows immediately.

C13S08.044: Because $\mathbf{v}$ is not a unit vector, we must replace it with a unit vector having the same direction before we can use the formulas of this section. So we take

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle.$$

The gradient vector of f is

$$\nabla f = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k},$$

so $\nabla f(1, 2, 3) = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$. Hence

$$D_{\mathbf{u}} f(P) = \langle 5, 4, 3 \rangle \cdot \langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3} \rangle = \tfrac{7}{3}$$

(degrees per kilometer) for the desired range of change of temperature with respect to distance.

C13S08.045: In the solution of Problem 44 we calculated $\nabla f(P) = \langle 5, 4, 3 \rangle$, and the unit vector in the direction from P to Q is

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle.$$

Then

$$D_{\mathbf{u}} f(P) = \nabla f(P) \cdot \mathbf{u} = \langle 5, 4, 3 \rangle \cdot \langle \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3} \rangle = 7$$

(degrees per kilometer). Hence

$$\frac{dw}{dt} = \frac{dw}{ds} \cdot \frac{ds}{dt} = \left(7 \, \frac{\text{deg}}{\text{km}}\right)\left(2 \, \frac{\text{km}}{\text{min}}\right) = 14 \, \frac{\text{deg}}{\text{min}}$$

as the hawk's rate of change of temperature at P.

C13S08.046: The gradient vector is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = (0.006)x\mathbf{i} - (0.008)y\mathbf{j},$$

so

$$\nabla f(40, 30) = (0.24)\mathbf{i} - (0.24)\mathbf{j} = \left(0.24\sqrt{2}\right)\mathbf{u}.$$

The unit vector

$$\mathbf{u} = \frac{\nabla f(40, 30)}{|\nabla f(40, 30)|} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$$

points southeast (into the fourth quadrant); this is the direction in which the bumblebee should initially fly. And, according to Section 13.8, the directional derivative of f in this optimal direction is

$$D_{\mathbf{u}} f(40, 30) = |\nabla f(40, 30)| = (0.24)\sqrt{2} \approx 0.34$$

degrees per unit of distance.

C13S08.047: Part (a): If $W(x, y, z) = 50 + xyz$, then $\nabla W = \langle yz, xz, xy \rangle$, so at the point $P(3, 4, 1)$ we have $\nabla W(P) = \langle 4, 3, 12 \rangle$. The unit vector with the same direction as $\mathbf{v} = \langle 1, 2, 2 \rangle$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle,$$

and so the rate of change of temperature at P in the direction of $\mathbf{v}$ is

$$\big(\nabla W(P)\big) \cdot \mathbf{u} = \langle 4, 3, 12 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle = \frac{34}{3}.$$

Because distance is measured in feet, the units for this rate of change are degrees Celsius per foot.

Part (b): The maximal directional derivative of W at P is $|\nabla W(P)| = |\langle 4, 3, 12 \rangle| = 13$ and the direction in which it occurs is $\nabla W(P) = \langle 4, 3, 12 \rangle$.

C13S08.048: Part (a): If $W(x, y, z) = 100 - x^2 - y^2 - z^2$, then $\nabla W = \langle -2x, -2y, -2z \rangle$. The unit vector with the same direction as $\mathbf{v} = \langle 3, -4, 12 \rangle$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \frac{3}{13}, -\frac{4}{13}, \frac{12}{13} \right\rangle,$$

so the rate of change of W in the direction of $\mathbf{v}$ at the point $P(3, -4, 5)$ is

$$\big(\nabla W(P)\big) \cdot \mathbf{u} = \langle -6, 8, -10 \rangle \cdot \left\langle \frac{3}{13}, -\frac{4}{13}, \frac{12}{13} \right\rangle = -\frac{170}{13}.$$

Because distance is measured in meters, the units for this rate of change are degrees Celsius per meter.

Part (b): The maximal directional derivative of W at P is $|\nabla W(P)| = 10\sqrt{2}$ and the direction in which it occurs is $\nabla W(P) = \langle -6, 8, -10 \rangle$.

C13S08.049: Part (a): Given $f(x, y) = \frac{1}{10}(x^2 - xy + 2y^2)$, let

$$g(x, y, z) = z - f(x, y); \qquad \text{then} \qquad \nabla g(x, y, z) = \frac{1}{10}\langle y - 2x, x - 4y, 10 \rangle.$$

Thus a normal to the surface $z = f(x, y)$ at the point $P\left(2, 1, \frac{2}{5}\right)$ is $\frac{1}{10}\langle -3, -2, 10 \rangle$. Hence an equation of the plane tangent to this surface at P is

$$3x + 2y - 10z = 4; \qquad \text{that is,} \qquad z = \frac{3}{10}x + \frac{1}{5}y - \frac{2}{5}.$$

Part (b): Let Q denote the point $(2, 1)$ and R the point $(2.2, 0.9)$. Then $\mathbf{v} = \overrightarrow{QR} = \langle 0.2, -0.1 \rangle$. Thus an approximation to $f(2.2, 0.9)$ is

$$f(2, 1) + \big(\nabla f(Q)\big) \cdot \mathbf{v} = 0.4 + \langle 0.3, 0.2 \rangle \cdot \langle 0.2, -0.1 \rangle = \frac{11}{25} = 0.44.$$

The true value is $f(2.2, 0.9) = \frac{56}{125} = 0.448$.

C13S08.050: Let $F(x, y, z) = 2x^2 + 3y^2 - z$. The equation $F(x, y, z) = 0$ has the paraboloid as its graph. The vector $\mathbf{n} = \langle 4, -3, -1 \rangle$ is normal to the plane. All we require is that $\nabla F = \langle 4x, 6y, -1 \rangle$ is parallel to $\mathbf{n}$. This leads to $x = 1$, $y = -\frac{1}{2}$, and (because $F(x, y, z) = 0$) $z = \frac{11}{4}$. Thus an equation of the required plane is

$$4(x-1) - 3\left(y + \frac{1}{2}\right) - \left(z - \frac{11}{4}\right) = 0; \qquad \text{that is,} \qquad 16x - 12y - 4z = 11.$$

C13S08.051: Let $F(x, y, z) = z^2 - x^2 - y^2$ and $G(x, y, z) = 2x + 3y + 4z + 2$. Then the cone is the graph of $F(x, y, z) = 0$ and the plane is the graph of $G(x, y, z) = 0$. At the given point $P(3, 4, -5)$ we have

$$\nabla F(3, 4, -5) = \langle -6, -8, -10 \rangle \qquad \text{and} \qquad \nabla G(3, 4, -5) = \langle 2, 3, 4 \rangle.$$

Let $\mathcal{P}$ denote the plane normal to the ellipse (the intersection of the cone and the first plane) at the point P. Then a normal to $\mathcal{P}$ is

$$\mathbf{n} = \langle -6, -8, -10 \rangle \times \langle 2, 3, 4 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & -8 & -10 \\ 2 & 3 & 4 \end{vmatrix} = \langle -2, 4, -2 \rangle.$$

We will use instead the parallel vector $\langle 1, -2, 1 \rangle$. In the usual way we find that $\mathcal{P}$ has Cartesian equation $x - 2y + z + 10 = 0$.

C13S08.052: Let $F(x, y, z) = z^2 - x^2 - y^2$ and $G(x, y, z) = 2x + 3y + 4z + 2$. Then the cone is the graph of $F(x, y, z) = 0$ and the plane is the graph of $G(x, y, z) = 0$. Then a plane $\mathcal{P}$ normal to the ellipse at the point $P(x, y, z)$ will itself have normal vector

$$\mathbf{n} = \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2x & -2y & 2z \\ 2 & 3 & 4 \end{vmatrix} = \langle -8y - 6z,\ 8x + 4z,\ 4y - 6x \rangle.$$

The condition that the line tangent to the ellipse be horizontal implies that the third component of $\mathbf{n}$ must be zero, so that $\mathcal{P}$ is a vertical plane. Thus we obtain the three simultaneous equations that the highest and lowest points of the ellipse must satisfy:

$$4y - 6x = 0,$$

$$z^2 = x^2 + y^2, \qquad \text{and}$$

$$2x + 3y + 4z + 2 = 0.$$

They have exactly two simultaneous solutions, and thus we discover the answers:

$$\text{Low point:}\qquad x=\frac{52+16\sqrt{13}}{39}\approx\ \ 2.8125338566006110,$$

$$y=\frac{26+8\sqrt{13}}{13}\approx\ \ 4.2188007849009165,$$

$$z=\frac{-8-2\sqrt{13}}{3}\approx -5.0703675169759929.$$

$$\text{High point:}\qquad x=\frac{52-16\sqrt{13}}{39}\approx -0.1458671899339443,$$

$$y=\frac{26-8\sqrt{13}}{13}\approx -0.2188007849009165,$$

$$z=\frac{-8+2\sqrt{13}}{3}\approx -0.2629658163573405.$$

C13S08.053: Let $F(x,\,y,\,z)=x^2+y^2+z^2-r^2$ and $G(x,\,y,\,z)=z^2-a^2x^2-b^2y^2$. Then the sphere is the graph of $F(x,\,y,\,z)=0$ and the cone is the graph of $G(x,\,y,\,z)=0$. At a point where the sphere and the cone meet, these vectors are the normals to their tangent planes. To show that the tangent planes are perpendicular, it is sufficient to show that their normals are perpendicular. But

$$(\nabla F)\cdot(\nabla G)=\langle\,2x,\ 2y,\ 2z\,\rangle\cdot\langle\,-2a^2x,\ -2b^2y,\ 2z\,\rangle=-4a^2x^2-4b^2y^2+4z^2=4(z^2-a^2x^2-b^2y^2)=0$$

because $(x,\,y,\,z)$ lies on the cone. Therefore the tangent planes are perpendicular at every point of the intersection of the sphere and the cone.

C13S08.054: The equation of the ellipsoid may be written in the form

$$F(x,\,y,\,z)\ =\ x^2+y^2+2z^2-2\ =\ 0,$$

and $\nabla F(x,\,y,\,z)\ =\ \langle 2x,\,2y,\,4z\rangle$ is normal to the ellipsoidal surface at the point $(x,\,y,\,z)$. A normal at $P(a,\,b,\,c)$ is therefore $\mathbf{n}\ =\ \langle a,\,b,\,2c\rangle$, and in general there are four points on the ellipsoid with z-coordinate c: They are $(\pm a,\,\pm b,\,c)$. Thus there are four normal vectors in question, $\langle \pm a,\,\pm b,\,2c\rangle$, and consequently four tangent planes, with equations $\pm ax\pm by+2cz\ =\ d$ for some constant d (the same d for all four planes). These planes meet the z-axis where $x\ =\ y\ =\ 0$, and thus all four have the same z-intercept $z\ =\ d/(2c)$. (If $c\ =\ 0$, then all four tangent planes are vertical—parallel to the z-axis—and none meets the z-axis. If $a=b=0$ then there is only one tangent plane meeting the ellipsoidal surface at a point with z-coordinate c. If exactly one of a and b is zero, then there are only two tangent planes of the sort specified in Problem 54, but the same argument shows that both meet the z-axis at the same point.)

C13S08.055: The surface is the graph of the equation $G(x,\,y,\,z)\ =\ 0$ where

$$G(x,\,y,\,z)\ =\ xyz-1,\qquad\text{so that}\qquad \nabla G(x,\,y,\,z)\ =\ \langle yz,\,xz,\,xy\rangle.$$

Suppose that $P(a,\,b,\,c)$ is a point strictly within the first octant (so that $a,\ b,$ and c are all positive). Note that $abc\ =\ 1$. A vector normal to the surface at P is $\mathbf{n}\ =\ \langle bc,\,ac,\,ab\rangle$, and hence the plane tangent to the surface at P has equation

$$bcx+acy+abz\ =\ d$$

for some constant d. Moreover, because P is a point of the surface,

$$bca + acb + abc = d; \qquad \text{that is,} \qquad d = 3abc.$$

Hence an equation of the tangent plane is $bcx + acy + abz = 3abc$. The intercepts of the pyramid therefore occur at $(3a,\, 0,\, 0)$, $(0,\, 3b,\, 0)$, and $(0,\, 0,\, 3c)$. Therefore, because of the right angle at the origin, the pyramid has volume

$$V = \frac{1}{6}(3a)(3b)(3c) = \frac{27}{6}abc = \frac{9}{2},$$

independent of the choice of P, as we were to show.

C13S08.056: Part (a): Given $z = f(x,\, y) = 500 - (0.003)x^2 - (0.004)y^2$, we begin by constructing a vector $\mathbf{v} = \langle -1,\, 1 \rangle$ that points northwest, then the unit vector with the same direction:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\langle -\frac{\sqrt{2}}{2},\, \frac{\sqrt{2}}{2} \right\rangle.$$

Next, $\boldsymbol{\nabla} f(x,\, y) = \langle -(0.006)x,\, -(0.008)y \rangle$, so the value of the gradient at your position on the hill is $\mathbf{v} = \boldsymbol{\nabla} f(-100,\, -100) = \langle 0.6,\, 0.8 \rangle$. So your initial rate of climb in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}}f(-100,\, -100) = \mathbf{v} \cdot \mathbf{u} = -\frac{3}{10}\sqrt{2} + \frac{4}{10}\sqrt{2} = \frac{1}{10}\sqrt{2} \approx 0.1414213562$$

in units of feet per foot; that is, you initially climb at the rate of about 0.1414 feet upward for every foot you travel horizontally. Your initial angle of climb is

$$\arctan\left(\frac{1}{10}\sqrt{2}\right)$$

radians, approximately $8^\circ\ 2'\ 58.081''$ (a gentle slope).

Part (b): If instead you head northeast, repeat the previous calculations with the new unit vector

$$\mathbf{u} = \left\langle \frac{\sqrt{2}}{2},\, \frac{\sqrt{2}}{2} \right\rangle.$$

Your initial rate of climb in the new direction of $\mathbf{u}$ will be

$$D_{\mathbf{u}}f(-100,\, -100) = \mathbf{v} \cdot \mathbf{u} = \frac{3}{10}\sqrt{2} + \frac{4}{10}\sqrt{2} = \frac{7}{10}\sqrt{2} \approx 0.989949493661$$

feet per foot. The initial angle of climb will be $\arctan\left(\frac{7}{10}\sqrt{2}\right)$ radians, approximately $44^\circ\ 42'\ 38.241''$, an extremly steep climb, comparable to the last 30 meters up the north face of Rabun Bald.

C13S08.057: The hill is steepest in the direction of $\boldsymbol{\nabla} z(-100,\, -100) = \langle 0.6,\, 0.8 \rangle$. The slope of the hill in that direction is $|\langle 0.6,\, 0.8 \rangle| = 1$, so that your initial angle of climb would be 45°. The compass heading in the direction you are climbing is

$$\frac{\pi}{2} - \arctan\frac{4}{3}$$

radians, approximately $36^\circ\ 52'\ 11.632''$.

C13S08.058: Part (a): First we compute the gradient of

$$z = \frac{1000}{1 + (0.00003)x^2 + (0.00007)y^2}:$$

$$\nabla z = -\frac{1}{[1 + (0.00003)x^2 + (0.00007)y^2]^2} \langle (0.06)x,\ (0.14)y \rangle.$$

The slope of this hill at the point (100, 100, 500) in the northwest direction is

$$[\nabla z(100,\ 100)] \cdot \frac{\sqrt{2}}{2} \langle -1,\ 1 \rangle = \left\langle -\frac{3}{2},\ -\frac{7}{2} \right\rangle \cdot \left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} \right\rangle = -\sqrt{2} \approx -1.4142135623731.$$

Your initial angle of *descent* is thus approximately 1.4142 feet per foot and the angle of descent is approximately $54^\circ\ 44'\ 8.2''$.

Part (b): The slope of this hill at the point (100, 100, 500) in the northeast direction is

$$[\nabla z(100,\ 100)] \cdot \frac{\sqrt{2}}{2} \langle 1,\ 1 \rangle = \left\langle -\frac{3}{2},\ -\frac{7}{2} \right\rangle \cdot \left\langle \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} \right\rangle = -\frac{5\sqrt{2}}{2} \approx -3.535533905933.$$

Your initial rate of *descent* is thus approximately 3.5355 feet per foot and the angle of descent is approximately $74^\circ\ 12'\ 13.6''$, a very steep descent.

C13S08.059: Given

$$z = f(x,\ y) = \frac{1000}{1 + (0.00003)x^2 + (0.00007)y^2},$$

we first compute

$$\nabla f(x,\ y) = \left\langle -\frac{600\,000\,000x}{(100000 + 3x^2 + 7y^2)^2},\ -\frac{1\,400\,000\,000y}{(100000 + 3x^2 + 7y^2)^2} \right\rangle.$$

Hence to climb the most steeply, the initial direction should be

$$\nabla f(100,\ 100) = \left\langle -\frac{3}{2},\ -\frac{7}{2} \right\rangle,$$

and the initial rate of ascent will be $|\nabla f(100,\ 100)| = \frac{2}{2}\sqrt{58} \approx 3.807886553$ feet per foot. The initial angle of climb will be $\arctan\left(\frac{1}{2}\sqrt{58}\right) \approx 1.313982409$ radians, approximately $75^\circ\ 17'\ 8.327''$. The compass heading is

$$270^\circ - \arctan\left(\frac{7}{3}\right)^\circ \approx 203^\circ\ 11'\ 54.926''.$$

C13S08.060: Given

$$z = f(x,\ y) = 100 \exp\left(-\frac{x^2 + 3y^2}{701}\right),$$

we first compute

$$\nabla f(x,\ y) = \left\langle -\frac{200}{701} x \exp\left(-\frac{x^2 + 3y^2}{701}\right),\ -\frac{600}{701} y \exp\left(-\frac{x^2 + 3y^2}{701}\right) \right\rangle.$$

Part (a):

$$\mathbf{v} = \boldsymbol{\nabla} f(30,\, 20) = \left\langle -\frac{6000}{701} \exp\left(-\frac{2100}{701}\right),\ -\frac{12000}{701} \exp\left(-\frac{2100}{701}\right) \right\rangle$$
$$\approx \langle\, -0.427965138743,\ -0.855930277485\, \rangle$$

gives the initial direction in which you should head to climb the most steeply, and if you do so, your rate of climb will initially be $|\mathbf{v}| \approx 0.956959142228$ feet per foot. That will be at an angle of approximately $43^\circ\ 44'\ 24.196''$ from the horizontal. The initial heading will be approximately

$$270^\circ - \arctan\left(\frac{0.855930277485}{0.427965138743}\right)^\circ \approx 206^\circ\ 33'\ 54.184''.$$

Part (b): If you initially head west, with direction vector $\mathbf{u} = \langle\, -1,\, 0\, \rangle$, then your initial rate of ascent will be

$$\mathbf{v} \cdot \mathbf{u} = \frac{6000}{701} \exp\left(-\frac{2100}{701}\right) \approx 0.427965138743$$

feet per foot, so you will initially climb at an angle of approximately $23^\circ\ 10'\ 9.252''$ from the horizontal.

C13S08.061: Let

$$f(x,\, y) = \frac{1}{1000}(3x^2 - 5xy + y^2).$$

Then

$$\boldsymbol{\nabla} f(x,\, y) = \frac{1}{1000}\,\langle\, 6x - 5y,\ 2y - 5x\, \rangle\,,$$

and therefore $\mathbf{v} = \boldsymbol{\nabla} f(100,\, 100) = \langle\, \frac{1}{10},\, -\frac{3}{10}\, \rangle$.

Part (a): A unit vector in the northeast direction is $\mathbf{u} = \frac{1}{2}\,\langle\, \sqrt{2},\ \sqrt{2}\, \rangle$, so the directional derivative of f at $(100,\, 100)$ in the northeast direction is

$$\mathbf{v} \cdot \mathbf{u} = \langle\, \tfrac{1}{10},\, -\tfrac{3}{10}\, \rangle \cdot \tfrac{1}{2} \left\langle\, \sqrt{2},\ \sqrt{2}\, \right\rangle = -\tfrac{1}{10}\sqrt{2}.$$

Hence you will initially be descending the hill, and at an angle of $\arctan\left(\frac{1}{10}\sqrt{2}\,\right)$ below the horizontal, approximately $8^\circ\ 2'\ 58.081''$.

Part (b): A unit vector in the direction 30° north of east is $\mathbf{u} = \frac{1}{2}\,\langle\, \sqrt{3},\, 1\, \rangle$, so the directional derivative of f at $(100,\, 100)$ in the direction of $\mathbf{u}$ is

$$\langle\, \tfrac{1}{10},\, -\tfrac{3}{10}\, \rangle \cdot \tfrac{1}{2} \left\langle\, \sqrt{3},\, 1\, \right\rangle = -\frac{3 - \sqrt{3}}{20} \approx -0.06339746.$$

Hence you will initially be descending the hill, and at an angle of approximately $3^\circ\ 37'\ 56.665''$.

C13S08.062: Given: The two surfaces $f(x,\, y,\, z) = 0$ and $g(x,\, y,\, z) = 0$ both pass through the point P at which both $\boldsymbol{\nabla} f(P)$ and $\boldsymbol{\nabla} g(P)$ exist. Part (a): Suppose that the two surfaces are mutually tangent at P. Then their tangent planes there coincide. So their normal vectors at P are parallel. Therefore

$$\boldsymbol{\nabla} f(P) \times \boldsymbol{\nabla} g(P) = \mathbf{0}.$$

To prove the converse, simply reverse the steps in this argument.

Part (b): Suppose that the two surfaces are orthogonal at P. Then their tangent planes at P are perpendicular. So their normal vectors at P are perpendicular. Therefore

$$\nabla f(P) \cdot \nabla g(P) = 0.$$

To prove the converse, simply reverse the steps in this argument. Also see the solution of Problem 53.

C13S08.063: Because $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ are not collinear, neither is zero and neither is a scalar multiple of the other. Hence, as vectors, they are linearly independent, and this implies that the simultaneous equations

$$\begin{aligned} af_x(P) + bf_y(P) &= D_{\mathbf{u}} f(P), \\ cf_x(P) + df_y(P) &= D_{\mathbf{v}} f(P) \end{aligned}$$

have a unique solution for the values of $f_x(P)$ and $f_y(P)$. Thus $\nabla f(P) = \langle f_x(P), f_y(P) \rangle$ is uniquely determined, and therefore so is the directional derivative

$$D_{\mathbf{w}} f(P) = \nabla f(P) \cdot \mathbf{w}$$

in the direction of the arbitrary unit vector $\mathbf{w}$. —C.H.E.

C13S08.064: Obviously f is continuous at the origin because $f(x, y) \to 0$ as $(x, y) \to (0, 0)$. Next,

$$D_{\langle a,b \rangle} f(0, 0) = \lim_{t \to 0} \frac{f(at, bt) - f(0, 0)}{t} = \lim_{t \to 0} \frac{\left(\sqrt[3]{at} + \sqrt[3]{bt}\right)^3}{t} = \left(\sqrt[3]{a} + \sqrt[3]{b}\right)^3 \tag{1}$$

for all a and b. Thus every directional derivative exists. For instance, with $a = 1$ and $b = 0$ we find that $f_x(0, 0) = 1$, and with $a = 0$ and $b = 1$ we find that $f_y(0, 0) = 1$. Therefore $\nabla f(0, 0) = \langle 1, 1 \rangle$.

But if f were differentiable at the origin, it would follow with $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ that

$$D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = 1 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} = \frac{7}{5}.$$

But the calculation in Eq. (1) shows that

$$D_{\mathbf{u}} f(0, 0) = \left(\sqrt[3]{\tfrac{3}{5}} + \sqrt[3]{\tfrac{4}{5}}\right)^3 \approx 5.561701 \neq \frac{7}{5}.$$

Therefore f is not differentiable at the origin. —C.H.E.

Section 13.9

C13S09.001: Given $f(x, y) = 2x + y$ and the constraint $g(x, y) = x^2 + y^2 - 1 = 0$, the equation $\nabla f = \lambda \nabla g$ yields

$$2 = 2\lambda x \quad \text{and} \quad 1 = 2\lambda y,$$

so that $\lambda \neq 0$. Hence

$$\frac{1}{\lambda} = x = 2y, \quad \text{and thus} \quad 4y^2 + y^2 = 1.$$

Thus we have two solutions:

$$(x, y) = \left(\frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right) \quad \text{and} \quad (x, y) = \left(-\frac{2}{5}\sqrt{5}, -\frac{1}{5}\sqrt{5}\right).$$

Clearly the first maximizes $f(x, y)$ and the second minimizes $f(x, y)$. So the global maximum value of $f(x, y)$ is $\sqrt{5}$ and its global minimum value is $-\sqrt{5}$.

Note: The function f is continuous on the circle $x^2 + y^2 = 1$ and a continuous function defined on a closed and bounded subset of euclidean space (such as that circle) must have both a global maximum value and a global minimum value. We will use this argument to identify extrema when possible and without stating the argument explicitly in various solutions in this section.

C13S09.002: From the vector equation $\langle 1, 1 \rangle = \lambda \langle 2x, 8y \rangle$ we see that $\lambda \neq 0$, and hence $x = 4y$. Then the constraint takes the form $20y^2 = 1$, and thus there are two critical points:

$$\begin{aligned} (x, y) &= \left(\frac{2}{5}\sqrt{5}, \frac{1}{10}\sqrt{5}\right): \qquad & f(x, y) &= \frac{1}{2}\sqrt{5} \quad \text{(global maximum)};\\ (x, y) &= \left(-\frac{2}{5}\sqrt{5}, -\frac{1}{10}\sqrt{5}\right): \qquad & f(x, y) &= -\frac{1}{2}\sqrt{5} \quad \text{(global minimum)}. \end{aligned}$$

We identify the critical points using the fact that $f(x, y)$ is continuous everywhere, including the set of points on the ellipse $x^2 + 4y^2 = 1$, and a continuous function defined on a closed and bounded subset of euclidean space (such as this ellipse) must have both a global maximum and a global minimum. We will use this argument without explicitly stating it in various other solutions in this section.

C13S09.003: The Lagrange multiplier equation $\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$ yields the scalar equations $x = \lambda x$ and $-y = \lambda y$. To solve them with a minimum number of cases, note that multiplication of the first by y and the second by x yields

$$xy = \lambda xy = -xy, \quad \text{so that} \quad xy = 0.$$

If $x = 0$ then $y = \pm 2$; if $y = 0$ then $x = \pm 2$. So there are four critical points:

$$\begin{aligned} f(2, 0) &= 4: \quad \text{global maximum}; \qquad & f(-2, 0) &= 4: \quad \text{global maximum};\\ f(0, 2) &= -4: \quad \text{global minimum}; \qquad & f(0, -2) &= -4: \quad \text{global minimum}. \end{aligned}$$

C13S09.004: The Lagrange multipler equation $\langle 2x, 2y \rangle = \lambda \langle 2, 3 \rangle$ leads to the scalar equations $2x = 2\lambda$ and $2y = 3\lambda$, and it follows that $3x = 2y$. Substitution in the constraint yields the only critical point,

$\left(\frac{12}{13},\ \frac{18}{13}\right)$, and the value of $f(x,\ y)$ there is $\frac{36}{13}$, which is the global minimum value of f; there is no maximum value because

$$\lim_{x\to\infty} f(x,\ 0) = +\infty.$$

C13S09.005: The Lagrange multiplier equation $\langle\, y,\ x\,\rangle = \lambda\langle\, 8x,\ 18y\,\rangle$ yields the scalar equations $y = 8\lambda x$ and $x = 18\lambda y$. Multiply the first by $9y$ and the second by $4x$ to obtain

$$9y^2 = 72\lambda xy = 4x^2, \qquad \text{so that} \qquad 3y = \pm 2x.$$

The constraint equation takes the form $18y^2 = 36$, so that $y^2 = 2$. Thus there are four critical points. But $f(x,\ y)$ must have both a global maximum and a global minimum value on the ellipse $4x^2 + 9y^2 = 36$, so we may identify all of the critical points:

$$f\left(\tfrac{3}{2}\sqrt{2},\ \sqrt{2}\,\right) = 3: \quad \text{global maximum;} \qquad f\left(-\tfrac{3}{2}\sqrt{2},\ \sqrt{2}\,\right) = -3: \quad \text{global minimum;}$$

$$f\left(\tfrac{3}{2}\sqrt{2},\ -\sqrt{2}\,\right) = -3: \quad \text{global minimum;} \qquad f\left(-\tfrac{3}{2}\sqrt{2},\ -\sqrt{2}\,\right) = 3: \quad \text{global maximum.}$$

C13S09.006: The Lagrange multiplier method yields the equations $4x = \lambda x$ and $9y = \lambda y$, so that

$$4xy = \lambda xy = 9xy, \qquad \text{and therefore} \qquad xy = 0.$$

There are four critical points; $f(x,\ y)$ attains its global maximum value 9 at $(0,\ \pm 1)$ and its global minimum value 4 at $(\pm 1,\ 0)$.

C13S09.007: It is clear that $f(x,\ y,\ z) = x^2 + y^2 + z^2$ can have no global maximum value on the plane $g(x,\ y,\ z) = 3x + 2y + z - 6 = 0$, and almost as clear that there is a unique global minimum. The Lagrange multiplier equation

$$\langle\, 2x,\ 2y,\ 2z\,\rangle = \lambda\langle\, 3,\ 2,\ 1\,\rangle$$

then yields $4x = 6y = 12z = 6\lambda$, so that $x = 3z$ and $y = 2z$. Then the constraint takes the form $9z + 4z + z = 6$, and therefore the global minimum value of f on the plane $g(x,\ y,\ z) = 0$ is $f\left(\frac{9}{7},\ \frac{6}{7},\ \frac{3}{7}\right) = \frac{18}{7}$. We have also discovered three distinct positive rational numbers whose sum is equal to the sum of their squares.

C13S09.008: It is clear that the continuous function $f(x,\ y,\ z) = 3x + 2y + z$ must have both a global maximum and a global minimum on the spherical surface with equation $g(x,\ y,\ z) = x^2 + y^2 + z^2 - 1 = 0$. The Lagrange multiplier equation $\langle\, 3,\ 2,\ 1\,\rangle = \lambda\langle\, 2x,\ 2y,\ 2z\,\rangle$ yields first the information that $\lambda \neq 0$, and therefore

$$\frac{3}{\lambda} = 2x, \quad \frac{2}{\lambda} = 2y, \quad \text{and} \quad \frac{1}{\lambda} = 2z.$$

Therefore

$$\frac{6}{\lambda} = 4x = 6y = 12z;$$

$$2x = 3y = 6z;$$

$$x = 3z \quad \text{and} \quad y = 2z;$$

$$9z^2 + 4z^2 + z^2 = 1;$$

$$z^2 = \frac{1}{14}.$$

Results:

$$\text{Global minimum:} \quad f\left(-\tfrac{3}{14}\sqrt{14},\ -\tfrac{1}{7}\sqrt{14},\ -\tfrac{1}{14}\sqrt{14}\right) = -\sqrt{14};$$

$$\text{Global maximum:} \quad f\left(\tfrac{3}{14}\sqrt{14},\ \tfrac{1}{7}\sqrt{14},\ \tfrac{1}{14}\sqrt{14}\right) = \sqrt{14}.$$

C13S09.009: The continuous function $f(x, y, z) = x + y + z$ must have both a global maximum value and a global minimum value on the ellipsoidal surface with equation $g(x, y, z) = x^2 + 4y^2 + 9z^2 - 36 = 0$. The Lagrange multiplier method yields $\langle 1, 1, 1 \rangle = \lambda \langle 2x, 8y, 18z \rangle$, and thus $\lambda \neq 0$. Therefore

$$\frac{1}{\lambda} = 2x = 8y = 18z, \qquad \text{so that} \qquad x = 4y = 9z.$$

Substitution of $x = 9z$ and $y = \frac{9}{4}z$ in the equation $g(x, y, z) = 0$ then yields $z = \pm\frac{4}{7}$. Results:

$$\text{Global maximum:} \quad f\left(\tfrac{36}{7}, \tfrac{9}{7}, \tfrac{4}{7}\right) = 7;$$

$$\text{Global minimum:} \quad f\left(-\tfrac{36}{7}, -\tfrac{9}{7}, -\tfrac{4}{7}\right) = -7.$$

C13S09.010: Clearly the continuous function $f(x, y, z) = xyz$ must have both a global maximum and a global minimum on the spherical surface with equation $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$, and because $f(x, y, z)$ takes on both positive and negatives values on the surface, we may ignore the possibility that any of x, y, or z is zero. The Lagrange multiplier method yields

$$\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle;$$

$$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z;$$

$$2\lambda xyz = y^2z^2 = x^2z^2 = x^2y^2.$$

Therefore $x^2 = y^2 = z^2$, and then the constraint yields $3x^2 = 1$. There are eight critical points—all combinations of $x = \pm\frac{1}{3}\sqrt{3}$, $y = \pm\frac{1}{3}\sqrt{3}$, and $z = \pm\frac{1}{3}\sqrt{3}$. At four of these f attains its global maximum value $\frac{1}{9}\sqrt{3}$ and at the other four f attains its global minimum value $-\frac{1}{9}\sqrt{3}$.

C13S09.011: Clearly $f(x, y, z) = xy + 2z$ is continuous on the closed and bounded spherical surface $g(x, y, z) = x^2 + y^2 + z^2 - 36 = 0$, and hence f has both a global maximum and a global minimum there. The Lagrange multiplier equation is

$$\langle y, x, 2 \rangle = \lambda \langle 2x, 2y, 2z \rangle, \quad \text{and hence} \quad y = 2\lambda x, \quad x = 2\lambda y, \quad \text{and} \quad 2 = 2\lambda z.$$

Now $\lambda z = 1$, so $\lambda \neq 0$. If neither x nor y is zero, then

$$\frac{1}{\lambda} = \frac{2x}{y} = \frac{2y}{x} = z.$$

So $y^2 = x^2$ in this case. If $y = x$, then $z = 2$, so $2x^2 + 4 = 36$ and we obtain two critical points and the values $f(4,\, 4,\, 2) = 20$ and $f(-4,\, -4,\, 2) = 20$. If $y = -x$, then $z = -2$, and we obtain two critical points and the values $f(4,\, -4,\, -2) = -20$ and $f(-4,\, 4,\, -2) = -20$. Finally, if either of x and y is zero, then so is the other; $z^2 = 36$, and we obtain two more critical points and the values $f(0,\, 0,\, 6) = 12$ and $f(0,\, 0,\, -6) = -12$. Hence the global maximum value of $f(x,\, y,\, z)$ is 20 and its global minimum value is -20.

C13S09.012: Given: $f(x,\, y,\, z) = x - y + z$ on the surface $g(x,\, y,\, z) = x^2 - 6xy + y^2 - z = 0$. The Lagrange multiplier method yields the vector equation

$$\langle\, 1,\, -1,\, 1\,\rangle = \lambda\langle\, 2x - 6y,\ 2y - 6x,\ -1\,\rangle,$$

and it follows immediately that $\lambda = -1$. Consequently $2x - 6y = -1$ and $2y - 6x = 1$. These equations have the solution $x = -\frac{1}{8}$, $y = \frac{1}{8}$, and it follows that $z = \frac{1}{8}$. At this point the value of $f(x,\, y,\, z)$ is $-\frac{1}{8}$.

Have we found an extremum? If so, it is not a global extremum, because as $x \to +\infty$ while $y = 0$, $f(x,\, y,\, z) \to +\infty$; as $x \to +\infty$ along the line $y = x$, $f(x,\, y,\, z) \to -\infty$.

Substitution of $z = x^2 - 6xy + y^2$ in $f(x,\, y,\, z) = x - y + z$ yields

$$h(x,\, y) = x - y + x^2 - 6xy + y^2,$$

which necessarily has a critical point at $\left(-\frac{1}{8},\, \frac{1}{8}\right)$. Let $x = u - \frac{1}{8}$ and $y = v + \frac{1}{8}$. Then (after some algebra)

$$h(u,\, v) = u^2 - 6uv + v^2 - \frac{1}{8}.$$

The critical point has been shifted to the origin $(0,\, 0)$. On the line $u = v$, we have $h(u,\, v) = -4u^2 - \frac{1}{8}$, so the origin is not a local minimum. On the line $u = -v$, we have $h(u,\, v) = 8u^2 - \frac{1}{8}$, so the origin is not a local maximum. Hence the origin is not an extremum. Therefore $f(x,\, y,\, z)$ subject to the constraint $g(x,\, y,\, z) = 0$ has no extrema.

C13S09.013: The continuous function $f(x,\, y,\, z) = x^2y^2z^2$ clearly has a global maximum value and a global minimum value on the ellipsoidal surface $g(x,\, y,\, z) = x^2 + 4y^2 + 9z^2 - 27 = 0$. The Lagrange multiplier method yields the equations

$$xy^2z^2 = \lambda x, \quad x^2yz^2 = 4\lambda y, \quad \text{and} \quad x^2y^2z = 9\lambda z.$$

If any one of x, y, or z is zero, then $f(x,\, y,\, z) = 0$, and this is clearly the global minimum value of $f(x,\, y,\, z)$. Otherwise, $\lambda \neq 0$, and hence

$$x^2y^2z^2 = \lambda x^2 = 4\lambda y^2 = 9\lambda z^2,$$

and thus $x^2 = 4y^2 = 9z^2 = 9$; that is, $x^2 = 9$, $y^2 = \frac{9}{4}$, and $z^2 = 1$. Thus the global maximum value of $f(x,\, y,\, z)$ occurs at eight different critical points, and that maximum value is $9 \cdot \frac{9}{4} \cdot 1 = \frac{81}{4}$.

C13S09.014: The continuous function $f(x,\, y,\, z) = x^2 + y^2 + z^2$ clearly has both a global maximum value and a global minimum value on the closed and bounded surface with equation $g(x,\, y,\, z) = x^4 + y^4 + z^4 - 3 = 0$. The Lagrange multiplier method yields the equations

$$x = 2\lambda x^3, \quad y = 2\lambda y^3, \quad \text{and} \quad z = 2\lambda z^3.$$

Note first that $\lambda \neq 0$.

Case 1: Two of x, y, and z are zero. Then the third is $\pm 3^{1/4}$ and the value of $f(x,\, y,\, z)$ is $\sqrt{3}$.

Case 2: Exactly one of x, y, and z is zero. Then the other two have equal squares, and thus each is $\pm\left(\frac{3}{2}\right)^{1/4}$. In this case the value of $f(x, y, z)$ is $\sqrt{6}$.

Case 3: None of x, y, and z is zero. Then they have equal squares, so each is equal to ± 1. In this case the value of $f(x, y, z)$ is 3.

Summary: The global minimum value of $f(x, y, z)$ is $\sqrt{3}$ and its global maximum value is 3.

C13S09.015: We are to find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + z - 1 = 0$ and $h(x, y, z) = x + 2y + 3z - 6 = 0$. The Lagrange multiplier equation is

$$\langle 2x, 2y, 2z \rangle = \lambda\langle 1, 1, 1 \rangle + \mu\langle 1, 2, 3 \rangle,$$

from which we obtain the scalar equations $\lambda = 2x - \mu = 2y - 2\mu = 2z - 3\mu$, so that

$$2x + \mu = 2y \quad \text{and} \quad 2y + \mu = 2z, \quad \text{and thus}$$

$$\mu = 2(y - x) = 2(z - y): \quad 2y = x + z.$$

Thus we are to solve the following three *linear* equations in three unknowns:

$$x + y + z = 1,$$

$$x - 2y + z = 0,$$

$$x + 2y + 3z = 6.$$

We find that $x = -\frac{5}{3}$, $y = \frac{1}{3}$, and $z = \frac{7}{3}$. A geometric interpretation of this problem is to find the point on the intersection of two planes—a line—closest to and farthest from the origin. Hence there is no maximum but surely a minimum, and we have found it: $f\left(-\frac{5}{3}, \frac{1}{3}, \frac{7}{3}\right) = \frac{25}{3}$.

C13S09.016: We are to find the extrema of $f(x, y, z) = z$ given the constraints $g(x, y, z) = x^2 + y^2 - 1 = 0$ and $h(x, y, z) = 2x + 2y + z - 5 = 0$. Geometrically, this is the problem of finding the highest (maximum z-coordinate) and lowest points on the ellipse formed by the intersection of a plane and a vertical cylinder. Hence there will be a unique global maximum and a unique global minimum unless the plane is horizontal or vertical—and it is not. The Lagrange multiplier equation is

$$\langle 0, 0, 1 \rangle = \lambda\langle 2x, 2y, 0 \rangle + \mu\langle 2, 2, 1 \rangle,$$

which leads to the scalar equations

$$\lambda x + \mu = 0, \quad \lambda y + \mu = 0, \quad \text{and} \quad \mu = 1.$$

Therefore $\lambda x = \lambda y = -1$ and, because $\lambda \neq 0$, we find that $y = x$. Thus we are to solve simultaneously the following three *nonlinear* equations in three unknowns:

$$y = x,$$

$$x^2 + y^2 = 1,$$

$$2x + 2y + z = 5.$$

The second equation becomes $2x^2 = 1$, so that $x = \pm\frac{1}{2}\sqrt{2}$ and $y = x$.

Case 1: If $x = \frac{1}{2}\sqrt{2}$, then $y = x$ and $z = 5 - 2\sqrt{2}$. This is the global minimum value of $f(x, y, z) = z$ subject to the two constraints.

Case 2: If $x = -\frac{1}{2}\sqrt{2}$, then $y = x$ and $z = 5 + 2\sqrt{2}$. This is the global maximum value of $f(x, y, z) = z$ subject to the two constraints.

See Problems 47 and 48 of Section 13.8 for an alternative method of solving such high point/low point problems.

C13S09.017: We are to find the extrema of the function $f(x, y, z) = z$ subject to the *two* constraints $g(x, y, z) = x + y + z - 1 = 0$ and $h(x, y, z) = x^2 + y^2 - 1 = 0$. Geometrically, this is the problem of finding the highest (maximum z-coordinate) and lowest points on the ellipse formed by the intersection of a plane and a vertical cylinder. Hence there will be a unique global maximum and a unique global minimum unless the plane is horizontal or vertical—which it is not. The Lagrange multiplier equation is

$$\langle 0, 0, 1 \rangle = \lambda\langle 1, 1, 1 \rangle + \mu\langle 2x, 2y, 0 \rangle.$$

We thus obtain the three (simultaneous) scalar equations

$$2\mu x + \lambda = 0, \quad 2\mu y + \lambda = 0, \quad \lambda = 1.$$

Thus $2\mu x = 2\mu y = -1$ and so, because $\mu \neq 0$, we find that $y = x$. Then the second constraint implies that $2x^2 = 1$, and because $y = x$ there are only two cases:

Case 1: $x = \frac{1}{2}\sqrt{2} = y$, $z = 1 - \sqrt{2}$. This is the lowest point on the ellipse.

Case 2: $x = -\frac{1}{2}\sqrt{2} = y$, $z = 1 + \sqrt{2}$. This is the highest point on the ellipse.

C13S09.018: We are to maximize and minimize the function $f(x, y, z) = x$ subject to the *two* constraints $g(x, y, z) = x + y + z - 12 = 0$ and $h(x, y, z) = 4y^2 + 9z^2 - 36 = 0$. Here's the geometry: The plane $g(x, y, z) = 0$ and the elliptical cylinder $h(x, y\ z) = 0$ (with axis the x-axis) meet in an ellipse, and we are to find the points on this ellipse farthest from and closest to the yz-plane. At the conclusion we will see that both extrema are positive, so the ellipse lies entirely on one side of the yz-plane; we will not encounter the anomalous situation of the closest point having x-coordinate zero.

The Lagrange multiplier equation in vector form is

$$\langle 1, 0, 0 \rangle = \lambda\langle 1, 1, 1 \rangle + \mu\langle 0, 8y, 18z \rangle,$$

and it leads to the scalar equations

$$\lambda = 1 \quad \text{and} \quad 8\mu y + \lambda = 0 = 18\mu z + \lambda,$$

and thus $8\mu y = 18\mu z = -1$. It follows that $\mu \neq 0$, and so $4y = 9z$. We substitute $y = \frac{9}{4}z$ in the second constraint and find that

$$\frac{81}{4}z^2 + 9z^2 = 36, \quad \text{so that} \quad z = \pm\frac{4}{13}\sqrt{13}.$$

Case 1: If $z = \frac{4}{13}\sqrt{13}$, then $y = \frac{9}{13}\sqrt{13}$ and $x = 12 - \sqrt{13}$. The latter is the global minimum value of $f(x, y, z) = x$.

Case 2: If $z = -\frac{4}{13}\sqrt{13}$, then $y = -\frac{9}{13}\sqrt{13}$ and $x = 12 + \sqrt{13}$. The latter is the global maximum value of $f(x, y, z) = x$.

C13S09.019: We should find only one possible extremum because there is no point on the line farthest from the origin. We minimize $f(x, y) = x^2 + y^2$ given the constraint $g(x, y) = 3x + 4y - 100 = 0$. The Lagrange multiplier method yields

$$\langle 2x, 2y \rangle = \lambda \langle 3, 4 \rangle, \quad \text{so that} \quad 2x = 3\lambda \quad \text{and} \quad 2y = 4\lambda.$$

Thus $8x = 12\lambda = 6y$, so that $y = \frac{4}{3}x$. Substitution in the constraint yields $x = 12$ and $y = 16$. Answer: The point on the line $g(x, y) = 0$ closest to the origin is $(12, 16)$.

C13S09.020: The units here are cents and inches. Suppose that the base of the box has dimensions x by y and its height is z. Then its cost (in cents) will be

$$C(x, y, z) = 7xy + 10xz + 10yz,$$

which is to be minimized subject to the constraint $g(x, y, z) = xyz - 700 = 0$. The Lagrange multiplier method yields the scalar equations

$$7y + 10z = \lambda yz, \quad 7x + 10z = \lambda xz, \quad \text{and} \quad 10x + 10y = \lambda xy.$$

Multiply each equation by the "missing" variable to find that

$$\lambda xyz = 7xy + 10xz = 7xy + 10yz = 10xz + 10yz.$$

But x, y, and z are positive, and thus $y = x$ and $10z = 7x$, so that $z = \frac{7}{10}x$. Substitution in the constraint yields

$$700 = xyz = \frac{7}{10}x^3, \quad \text{so that} \quad x^3 = 1000.$$

Therefore $x = 10$, $y = 10$, and $z = 7$. The box of minimum cost has base 10 inches by 10 inches, height 7 inches, and will cost \$21.00.

C13S09.021: Please refer to Problem 29 of Section 13.5. We are to minimize

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{given} \quad g(x, y, z) = 12x + 4y + 3z - 169 = 0.$$

Geometrically, we are to find the point (x, y, z) on the plane with equation $g(x, y, z) = 0$ closest to the origin, hence we can be sure that a unique solution exists. The Lagrange multiplier method yields the scalar equations

$$2x = 12\lambda, \quad 2y = 4\lambda, \quad \text{and} \quad 2z = 3\lambda,$$

so that $2x = 6y = 8z = 12\lambda$, and thus $y = \frac{1}{3}x$ and $z = \frac{1}{4}x$. Then the constraint yields

$$12x + \frac{4}{3}x + \frac{3}{4}x = 169;$$

$$144x + 16x + 9x = 169 \cdot 12;$$

$$x = 12, \quad y = 4, \quad z = 3.$$

Answer: The point on the plane $g(x, y, z) = 0$ closest to the origin is $(12, 4, 3)$.

C13S09.022: Please refer to Problem 30 of Section 13.5. We are to minimize

$$f(x,\ y,\ z) = (x-9)^2 + (y-9)^2 + (z-9)^2 \quad \text{given} \quad g(x,\ y\ z) = 2x + 2y + z - 27 = 0.$$

Geometrically, we are to find the point $(x,\ y,\ z)$ on the plane with equation $g(x,\ y\ z) = 0$ closest to the point $P(9,\ 9,\ 9)$. Thus we can be sure that a unique global minimum of $f(x,\ y,\ z)$ exists and that there can be no maximum, so we anticipate a single critical point. The Lagrange multiplier method yields the scalar equations

$$2(x-9) = 2\lambda, \qquad 2(y-9) = 2\lambda, \qquad 4(z-9) = 2\lambda,$$

and thus $x - 9 = y - 9 = 2z - 18$. Hence $y = x$ and $z = \frac{1}{2}(x+9)$. Substitution in the constraint yields

$$2x + 2x + \frac{x+9}{2} = 27; \quad \text{i.e.,} \quad 9x + 9 = 54.$$

Therefore $x = 5$, $y = 5$, and $z = 7$. So the point on the plane $g(x,\ y,\ z) = 0$ closest to P is $(5,\ 5,\ 7)$.

C13S09.023: Please refer to Problem 31 of Section 13.5. We are to minimize

$$f(x,\ y,\ z) = (x-7)^2 + (y+7)^2 + z^2 \quad \text{given} \quad g(x,\ y,\ z) = 2x + 3y + z - 49 = 0.$$

This is the geometric problem of finding the point $(x,\ y,\ z)$ on the plane $g(x,\ y,\ z) = 0$ closest to the point $Q(7,\ -7,\ 0)$. Hence we can be sure that a unique minimum exists and, because there can be no maximum of $f(x,\ y\ z)$, we anticipate a single critical point. The Lagrange multiplier method yields the scalar equations

$$2(x-7) = 2\lambda, \qquad 2(y+7) = 3\lambda, \qquad 2z = \lambda.$$

It follows that $6\lambda = 6x - 42 = 4y + 28 = 12z$, and thus

$$x = \frac{12z+42}{6} = 2z + 7 \quad \text{and}$$
$$y = \frac{12z-28}{4} = 3z - 7.$$

Substitution in the constraint gives the equation

$$4z + 14 + 9z - 21 + z = 49, \quad \text{so that} \quad 14z = 56.$$

Therefore $z = 4$, $y = 5$, and $x = 15$. The point on the plane $g(x,\ y,\ z) = 0$ closest to Q is thus $(15,\ 5,\ 4)$.

C13S09.024: Please refer to Problem 32 of Section 13.5. We are to minimize

$$f(x,\ y,\ z) = x^2 + y^2 + z^2 \quad \text{given} \quad g(x,\ y,\ z) = xyz - 8 = 0.$$

This is the geometric problem of finding the point (or points) on the surface $xyz = 8$ that are closest to the origin. This is an unbounded surface, so we expect to find no global maximum of $f(x,\ y,\ z)$ on it, but we expect to find a global minimum, perhaps at several different critical points, but we seek only the closest point that lies in the first octant. The Lagrange multipler method yields the scalar equations

$$2x = \lambda yz, \qquad 2y = \lambda xz, \qquad 2z = \lambda xy,$$

and thus $\lambda xyz = 2x^2 = 2y^2 = 2z^2$. Because we are restricted to the first octant, we find that $x = y = z = 2$, so the point in the first octant on the given surface closest to the origin is $(2,\ 2,\ 2)$. Its distance from the origin is $\sqrt{12} = 2\sqrt{3}$.

C13S09.025: Please refer to Problem 33 of Section 13.5. We are to find the point (x, y, z) in the first octant and on the surface $g(x, y, z) = x^2y^2z - 4 = 0$ closest to the origin, so we minimize

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{subject to the constraint} \quad g(x, y, z) = 0.$$

The surface is unbounded, so we anticipate no maximum and only one minimum in the first octant. The Lagrange multiplier method yields the scalar equations

$$2x = 2\lambda xy^2z, \qquad 2y = 2\lambda x^2yz, \qquad 2z = \lambda x^2y^2.$$

It follows that $2\lambda x^2y^2z = 2x^2 = 2y^2 = 4z^2$, and hence that $x^2 = y^2 = 2z^2$. The restriction to the first octant then implies that $x = y = z\sqrt{2}$, and then the constraint $x^2y^2z = 4$ implies that

$$(2z^2)(2z^2)(z) = 4; \qquad 4z^5 = 4; \quad z = 1.$$

So $x = y = \sqrt{2}$. The point on $g(x, y, z) = 0$ in the first octant closest to the origin is $\left(\sqrt{2}, \sqrt{2}, 1\right)$. Its distance from the origin is $\sqrt{5}$.

C13S09.026: Please see Problem 34 of Section 13.5. We are to find the point in the first octant on the surface $g(x, y, z) = x^4y^8z^2 - 8 = 0$ closest to the origin $Q(0, 0, 0)$, so we minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to this constraint. Note that the surface $g(x, y, z) = 0$ is an unbounded surface, so there is no maximum of f; we expect to find only one critical point in the first octant. The Lagrange multiplier equations are

$$2x = 4\lambda x^3y^8z^2, \quad 2y = 8\lambda x^4y^7z^2, \quad \text{and} \quad 2z = 2\lambda x^4y^8z.$$

Consequently, $8\lambda x^4y^8z^2 = 4x^2 = 2y^2 = 8z^2$, so that $2x^2 = y^2 = 4z^2$. Hence $y^2 = 4z^2$ and $x^2 = 2z^2$, and then the constraint equation yields

$$8 = x^4y^8z^2 = (4z^4)(256z^8)(z^2), \quad \text{so that} \quad 128z^{14} = 1.$$

Hence $z^2 = \frac{1}{2}$, and so $z = \frac{1}{2}\sqrt{2}$. Also $x^2 = 2z^2 = 1$, so $x = 1$; $y^2 = 4z^2 = 2$, so that $y = \sqrt{2}$. Therefore the point on the surface closest to the origin is $\left(1, \sqrt{2}, \frac{1}{2}\sqrt{2}\right)$. Its distance from the origin is $\frac{1}{2}\sqrt{14}$.

C13S09.027: Please see Problem 35 of Section 13.5. We are to maximize $f(x, y, z) = xyz$ given $g(x, y, z) = x + y + z - 120 = 0$ and the additional condition that x, y, and z are all positive. The Lagrange multiplier equations are

$$\lambda = yz = xz = xy, \quad \text{and hence} \quad x = y = z = 40.$$

Therefore the maximum value of f is $f(40, 40, 40) = 40^3 = 64000$.

To establish that $f(x, y, z)$ actually has a global maximum value on its domain (the set of all triples of positive real numbers (x, y, z) such that $x+y+z = 120$), extend the domain slightly to include all triples of *nonnegative* triples of real numbers (x, y, z) such that $x+y+z = 120$. Then f is continuous on this domain, which is a closed and bounded subset of three-dimensional space (it is the part of the plane $x + y + z = 120$ that lies in the first octant or on the adjacent coordinate planes). Hence f has a global maximum there, and the maximum does not occur on the boundary because $xyz = 0$ if any of x, y, or z is zero. Hence this maximum must occur at an interior critical point, and in the previous paragraph we found only one such critical point. This is enough to establish that 64000 is the correct answer.

C13S09.028: Please refer to Problem 36 of Section 13.5. Suppose that the dimensions of the box are x by y by z (units are in meters). We are to maximize box volume $V(x, y, z) = xyz$ given the constraint

$g(x, y, z) = 4x + 4y + 4z - 6 = 0$. Note also that x, y, and z are all positive. The Lagrange multiplier equations are

$$\lambda = yz = xz = xy, \qquad \text{so that} \qquad x = y = z = \frac{1}{2},$$

and therefore the maximum volume of such a box is $\frac{1}{8}$ (cubic meters).

C13S09.029: Please see Problem 37 of Section 13.5. Suppose that the box has dimensions x by y by z (units are in inches). Then we are to minimize total surface area $A(x, y, z) = 2xy + 2xz + 2yz$ given $V(x, y, z) = xyz - 1000 = 0$. The Lagrange multipliers equations are

$$2(y + z) = \lambda yz, \quad 2(x + z) = \lambda xz, \quad \text{and} \quad 2(x + y) = \lambda xy;$$

note also that in the solution, all three of x, y, and z must be positive. Hence we can eliminate λ from all three of the previous equations:

$$\begin{aligned}
&\frac{\lambda}{2} = \frac{y+z}{yz} = \frac{x+z}{xz} = \frac{x+y}{xy};\\
&\frac{1}{z} + \frac{1}{y} = \frac{1}{z} + \frac{1}{x} = \frac{1}{y} + \frac{1}{x};\\
&x = y = z \qquad \text{and} \qquad xyz = 1000;\\
&x = y = z = 10.
\end{aligned}$$

Answer: The minimum possible surface area occurs when the box is a cube measuring 10 in. along each edge. The minimum possible surface area is 600 in.^{2}

C13S09.030: Please see Problem 38 of Section 13.5. Dimensions in this problem will be centimeters, square centimeters, etc. Suppose that the base of the box has dimensions x by y and that its height is z. We are to minimize total surface area

$$A(x, y, z) = xy + 2xz + 2yz \qquad \text{given} \qquad V(x, y, z) = xyz - 4000 = 0.$$

The Lagrange multiplier equations are

$$y + 2z = \lambda yz, \quad x + 2z = \lambda xz, \quad \text{and} \quad 2x + 2y = \lambda xy;$$

note also that in the solution, all three of x, y, and z are positive. Multiply each of the previous equations by the "missing" variable to obtain

$$\lambda xyz = xy + 2xz = xy + 2yz = 2xz + 2yz.$$

Hence $2xz = 2yz$, so that $y = x$; thus $x^2 + 2xz = 2xz + 2xz$, so that $x = y = 2z$. Because $xyz = 4000$, it not follows that $4z^3 = 4000$, so that $z = 10$ and $x = y = 20$. Therefore the box of minimum surface area has base 20 by 20 cm and height 10 cm.

C13S09.031: Please see Problem 39 of Section 13.5. Units in this problem will be cents and inches. Suppose that the bottom of the box has dimensions x by y and that its height is z. We are to minimize total cost $C(x, y, z) = 6xy + 10xz + 10yz$ given the constraint $V(x, y, z) = xyz - 600 = 0$. The Lagrange multiplier equations are

$$6y + 10z = \lambda yz, \quad 6x + 10z = \lambda xz, \quad \text{and} \quad 10x + 10y = \lambda xy;$$

also note that $x,\ y,$ and y are all positive. Multiply each of the three previous equations by the "missing" variable to find that

$$\lambda xyz = 6xy + 10xz = 6xy + 10yz = 10xz + 10yz,$$

and thus $x = y$ and $6y = 10z;$ that is, $x = y = \frac{5}{3}z$. Substitution in the constraint equation yields

$$600 = xyz = \frac{25}{9}z^3: \quad z^3 = \frac{5400}{25} = 216.$$

Therefore $z = 6$ and $x = y = 10$. Answer: The dimensions that minimize the total cost are these: base 10 in. by 10 in., height 6 in.; its total cost will be \$18.00.

C13S09.032: Please see Problem 40 of Section 13.5. Units in this problem wil be dollars and feet. Suppose that the box has bottom (and top) dimensions x by y and that its height is z. We are to minimize the total cost $C(x,\ y,\ z) = 6xy + 8xz + 8yz$ given the constraint $V(x,\ y,\ z) = xyz - 48 = 0$. The Lagrange multiplier equations are

$$6y + 8z = \lambda yz, \quad 6x + 8z = \lambda xz, \quad \text{and} \quad 8x + 8y = \lambda xy.$$

Note that all three variables are positive. So multiply each of the preceding equations by the "missing" variable to obtain

$$\lambda xyz = 6xy + 8xz = 6xy + 8yz = 8xz + 8yz,$$

then cancel with impunity to find that

$$x = y \quad \text{and} \quad 3x = 4z, \qquad \text{so that} \quad x = y = \frac{4}{3}z.$$

Then the constraint yields

$$48 = xyz = \frac{16}{9}z^3, \qquad \text{and thus} \quad z = 3 \quad \text{and} \quad x = y = 4.$$

So the cheapest box has base 4 ft by 4 ft and height 3 ft. (It will cost \$288.00!)

C13S09.033: Please refer to Problem 41 of Section 13.5. Suppose that the front of the box has width x and height z (units are in inches and cents) and that the bottom of the box measures x by y. We are to minimize total cost $C(x,\ y,\ z) = 6xy + 12xz + 18yz$ given the constraint $V(x,\ y,\ z) = xyz - 750 = 0$. The Lagrange multiplier equations are

$$6y + 12z = \lambda yz, \quad 6x + 18z = \lambda xz, \quad \text{and} \quad 12x + 18y = \lambda xy.$$

Note that $x,\ y,$ and z are positive. Multiply each of the three preceding equations by the "missing" variable to obtain

$$\lambda xyz = 6xy + 12xz = 6xy + 18yz = 12xz + 18yz.$$

It follows that $12xz = 18yz,$ so that $2x = 3y;$ also, $6xy = 12xz,$ so that $y = 2z$. Thus $x = \frac{3}{2}y = 3z$. Then substitution in the constraint equation yields

$$750 = xyz = 6z^3: \quad z^3 = 125,$$

and so $z = 5$, $y = 10$, and $x = 15$. The front of the cheapest box should be 15 in. wide and 5 in. high; its depth (from front to back) should be 10 in. (This box will cost \$27.00.)

C13S09.034: Please refer to Problem 42 of Section 13.5. This box has both top and bottom, but the bottom costs twice as much per square meter as the other five sides. Assume the latter cost 1 unit per square meter, that the bottom has dimensions x by y, and that the height of the box is z. Then we are to minimize total cost $C(x, y, z) = 3xy + 2xz + 2yz$ given the constraint $V(x, y, z) = xyz - 12 = 0$. The Lagrange multiplier equations are

$$3y + 2z = \lambda yz, \quad 3x + 2z = \lambda xz, \quad \text{and} \quad 2x + 2y = \lambda xy.$$

Note that all three of x, y, and z are positive. Multiply the first of the last three equation by x, the second by y, and the third by z to obtain

$$\lambda xyz = 3xy + 2xz = 3xy + 2yz = 2xz + 2yz,$$

and it follows that $x = y$ and $3y = 2z$. Thus $z = \frac{3}{2}y$, and substitution in the constraint equation yields

$$12 = xyz = x \cdot x \cdot \frac{3}{2}x: \quad x^3 = 8,$$

and therefore $x = 2$, $y = 2$, and $z = 3$. Thus the cheapest box will have base measuring 2 m by 2 m and height 3 m.

C13S09.035: We are to minimize $f(x, y, z) = x^2 + y^2 + z^2$ given (x, y, z) lies on the surface with equation $g(x, y, z) = xy + 5 - z = 0$. The Lagrange multiplier equations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad \text{and} \quad 2z = -\lambda.$$

Therefore $\lambda = -2z$, and thus $2x = -2yz$ and $2y = -2xz$. So

$$2x^2 = 2y^2 = -2xyz: \quad x^2 = y^2 = -xyz.$$

Case 1: $y = x$. Then $x^2 = -x^2 z$, so $x = 0$ or $z = -1$. In the latter case, the constraint equation yields $x^2 = -6$, so that case is rejected. We obtain only the critical point $(0, 0, 5)$.

Case 2: $y = -x$. Then $x^2 = x^2 z$, so $x = 0$ or $z = 1$. If $x = 0$, then we obtain only the critical point of Case 1. If $z = 1$ then the constraint equation yields $x^2 = 4$, and we obtain two more critical points: $(2, -2, 1)$ and $(-2, 2, 1)$.

Now $f(0, 0, 5) = 25$, $f(2, -2, 1) = 9$, and $f(-2, 2, 1) = 9$. Hence there are two points of the surface $g(x, y, z) = 0$ closest to the origin. They are $(2, -2, 1)$ and $(-2, 2, 1)$; each is at distance $\sqrt{9} = 3$ from the origin.

C13S09.036: The semiperimeter $s = \frac{1}{2}(x + y + z)$ of the triangle with sides x, y, and z is fixed. We maximize the *square* of its area A,

$$A^2 = f(x, y, z) = s(s - x)(s - y)(s - z) \qquad (\textit{Heron's formula}),$$

subject to the constraint $x + y + z = 2s$; note also that x, y, and z are nonnegative. The Lagrange multipler equations are

$$-s(s-y)(s-z)=\lambda, \quad -s(s-x)(s-z)=\lambda, \quad \text{and} \quad -s(s-x)(s-y)=\lambda,$$

and therefore

$$(s-x)(s-y)=(s-x)(s-z)=(s-y)(s-z).$$

Clearly $x < s$, $y < s$, and $z < s$ if the area is to be maximal. Thus

$$s-x=s-y=s-z, \quad \text{and hence} \quad x=y=z=\frac{2}{3}s:$$

The triangle of maximal area is equilateral. It has area

$$A=\sqrt{s(s-x)(s-y)(s-z)}=\sqrt{s\cdot\frac{1}{3}s\cdot\frac{1}{3}s\cdot\frac{1}{3}s}=\frac{\sqrt{3}}{9}s^2=\frac{\sqrt{3}}{4}x^2.$$

C13S09.037: See Fig. 13.9.9 of the text. There we see three small isosceles triangles, each with two equal sides of length 1 meeting at the center of the circle. Their total area

$$A=\frac{1}{2}\sin\alpha+\frac{1}{2}\sin\beta+\frac{1}{2}\sin\gamma$$

is the area of the large triangle, the quantity to be maximized given the constraint

$$g(\alpha,\,\beta,\,\gamma)=\alpha+\beta+\gamma-2\pi=0.$$

It is clear that, at maximum area, the center of the circle is *within* the large triangle or, at worst, on its boundary. Thus we have the additional restrictions

$$0 \leqq \alpha \leqq \pi, \quad 0 \leqq \beta \leqq \pi, \quad \text{and} \quad 0 \leqq \gamma \leqq \pi. \tag{1}$$

The Lagrange multiplier equations are

$$\lambda=\frac{1}{2}\cos\alpha=\frac{1}{2}\cos\beta=\frac{1}{2}\cos\gamma,$$

and the restrictions in (1) imply that $\alpha=\beta=\gamma=\frac{2}{3}\pi$. Hence the triangle of maximal area is equilateral (because it is equiangular). If x denotes the length of each side, then by the law of cosines

$$x^2=1^2+1^2-2\cdot1\cdot1\cdot\cos\frac{2\pi}{3}=3,$$

and so each side of the maximal-area triangle has length $\sqrt{3}$. The area of the maximal-area triangle is

$$A=\frac{3}{2}\sin\frac{2\pi}{3}=\frac{3}{2}\cdot\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{4}\approx 1.299038106,$$

and the ratio of this area to that of the circumscribed circle is

$$\frac{A}{\pi}\approx 0.413496672.$$

So the inscribed triangle of maximal area occupies about 41% of the area of the circle; this result meets the test of plausibility.

C13S09.038: We maximize and minimize the *square* of the distance of a point (x, y) of the ellipse from the origin; thus we maximize and minimize

$$f(x, y) = x^2 + y^2$$

subject to the constraint $g(x, y) = x^2 + xy + y^2 - 3 = 0$. The Lagrange multiplier equations are

$$2x = \lambda(2x + y) \quad \text{and} \quad 2y = \lambda(2y + x). \tag{1}$$

Solution 1 (in which we ignore the *Suggestion*): Clearly $\lambda \neq 0$. So if neither x nor y is zero, we have

$$\frac{1}{\lambda} = \frac{2x + y}{2x} = \frac{2y + x}{2y};$$

$$1 + \frac{y}{2x} = 1 + \frac{x}{2y};$$

$$2y^2 = 2x^2;$$

$$y^2 = x^2.$$

Case 1: $y = x$. Then the constraint equation yields $3x^2 = 3$, so we obtain the two critical points $(1, 1)$, and $(-1, -1)$.

Case 2: $y = -x$. Then the constraint equation yields $x^2 = 3$, so we obtain the two critical points $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$.

Case 3: $x = 0$. Then the constraint equation yields $y = \pm\sqrt{3}$, so we obtain the two critical points $(0, \sqrt{3})$ and $(0, -\sqrt{3})$,

Case 4: $y = 0$. In a manner similar to that in Case 3, we obtain the two critical points $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$.

The values of $f(x, y)$ at these eight critical points are 2, 2, 6, 6, 3, 3, 3, and 3. Thus the two points on the ellipse closest to the origin are $(1, 1)$ and $(-1, -1)$, each at distance $\sqrt{2}$. The two farthest from the origin are $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$, each at distance $\sqrt{6}$.

Solution 2 (in which we follow the *Suggestion*): We write the equations in (1) in the form

$$\begin{aligned} (2 - 2\lambda)x - \lambda y &= 0, \\ \lambda x - (2 - 2\lambda)y &= 0. \end{aligned}$$

Their solution (x, y) cannot be $(0, 0)$, and there must be a nontrivial solution. Hence the determinant of the preceding system is zero:

$$\begin{aligned} -(2 - 2\lambda)^2 + \lambda^2 &= 0; \\ -4 + 8\lambda - 4\lambda^2 + \lambda^2 &= 0; \\ 3\lambda^2 - 8\lambda + 4 &= 0; \\ (\lambda - 2)(3\lambda - 2) &= 0. \end{aligned}$$

Case 1: $\lambda = \frac{2}{3}$. Then (1) yields

$$\frac{2}{3}x - \frac{2}{3}y = 0,$$

so that $y = x$. This brings us to Case 1 of the previous solution.

Case 2: $\lambda = 2$. Then (1) yields

$$-2x - 2y = 0,$$

so that $y = -x$. This brings us to Case 2 of the previous solution.

The conclusions in the second solution are exactly the same as in the first solution, but the annoying Cases 3 and 4 of the first solution are avoided.

C13S09.039: We are to use the *Suggestion* in Problem 38 to find the point or points on the hyperbola $g(x, y) = x^2 + 12xy + 6y^2 - 130 = 0$ that are closest to the origin. To do so we minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = 0$. The Lagrange multiplier equations are

$$2x = \lambda(2x + 12y) \qquad \text{and} \qquad 2y = \lambda(12y + 12x). \tag{1}$$

When put into the form suggested in Problem 38, they become

$$\begin{aligned}(1-\lambda)x - 6\lambda y &= 0,\\ 6\lambda x + (6\lambda - 1)y &= 0.\end{aligned}$$

Because $(x, y) = (0, 0)$ is not a solution of this system and because it must therefore have a nontrivial solution, the determinant of this system must be zero:

$$\begin{aligned}(1-\lambda)(6\lambda - 1) + 36\lambda^2 &= 0;\\ 30\lambda^2 + 7\lambda - 1 &= 0;\\ (3\lambda + 1)(10\lambda - 1) &= 0.\end{aligned}$$

Case 1: $\lambda = -\frac{1}{3}$. Then (1) implies that

$$\frac{4}{3}x + 2y = 0: \quad y = -\frac{2}{3}x.$$

Substitution of the last equation for y into the constraint equation $g(x, y) = 0$ yields $x^2 + 30 = 0$, so there is no solution in Case 1.

Case 2: $\lambda = \frac{1}{10}$. Then (1) implies that

$$\frac{9}{10}x - \frac{6}{10}y = 0: \quad y = \frac{3}{2}x.$$

Substitution of the last equation for y into the constraint equation yields $x^2 = 4$, so we obtain two solutions in this case: $(-2, -3)$ and $(2, 3)$. These two points of the hyperbola are its points closest to $(0, 0)$, each at distance $\sqrt{13}$.

C13S09.040: We are to find the points of the ellipse $g(x, y) = 4x^2 + 9y^2 - 36 = 0$ closest to, and farthest from, the point $(1, 1)$. So we maximize and minimize $f(x, y) = (x-1)^2 + (y-1)^2$ subject to the constraint $g(x, y) = 0$. The Lagrange multiplier equations are

$$2(x-1) = 8\lambda x, \qquad 2(y-1) = 18\lambda y. \tag{1}$$

Anticipating some tough computations ahead, we let *Mathematica* 3.0 complete the solution. First we eliminate λ from the equations in (1).

```
Eliminate[ {2*(x - 1) == 8*lambda*x, 2*(y - 1) == 18*lambda*y}, lambda ]
```

$$x(4+5y) = 9y$$

Then we solve this equation for x (remember that "%" refers to the "last output"):

```
Solve[ %, x ]
```

$$x = \frac{9y}{4+5y}$$

(Actually, *Mathematica* returns

$$\left\{\left\{x \to \frac{9y}{4+5y}\right\}\right\},$$

but we are rewriting its output for clarity.) We substitute this expression for x in the constraint equation:

```
4*x*x + 9*y*y - 36 /. %
```

$$-36 + 9y^2 + \frac{324y^2}{(4+5y)^2}$$

```
Together[%]
```

$$\frac{9(25y^4 + 40y^3 - 48y^2 - 160y - 64)}{(4+5y)^2}$$

```
Numerator[%]
```

$$9(25y^4 + 40y^3 - 48y^2 - 160y - 64)$$

```
Cancel[ %/9 ]
```

$$25y^4 + 40y^3 - 48y^2 - 160y - 64$$

```
Solve[ % == 0, y ];
```

The semicolon suppresses the very complicated three-quarter page exact solution of the quartic equation. Even the command `Simplify[%]`, though effective, did not produce an answer simple enough for reproduction here. So we turned to numerical techniques:

```
N[%,30]
```

Mathematica returned two real solutions and two non-real complex solutions. The two real solutions are

$$y_1 \approx -0.4940738409036014 \qquad \text{and} \qquad y_2 \approx 1.8181224938702933.$$

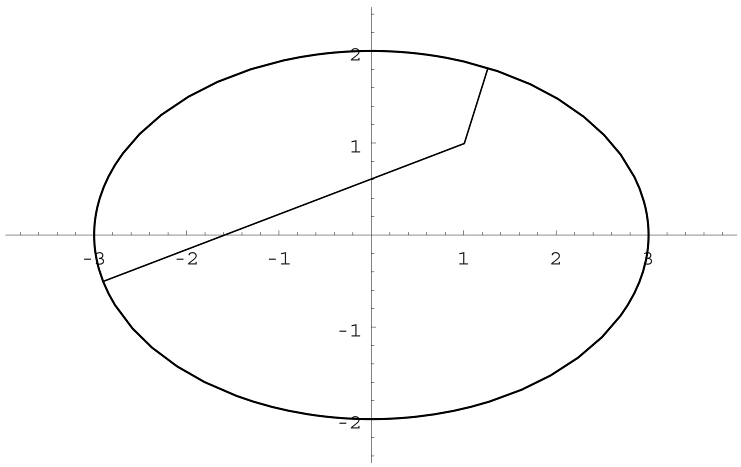
(Of course, the student can obtain these using Newton's method.) The corresponding values of x can be obtained from the previous equation

$$x = \frac{9y}{4+5y},$$

and we found that

$$x_1 \approx -2.9070182041747204 \qquad \text{and} \qquad x_2 \approx 1.2499875374924531.$$

The distance from $(x_1,\, y_1)$ to $(1,\, 1)$ is approximately 4.182947 and the distance from $(x_2,\, y_2)$ to $(1,\, 1)$ is approximately 0.855464, so the former is the point of the ellipse farthest from $(1,\, 1)$ and the latter is the point closest to $(1,\, 1)$. The ellipse and the line segments connecting $(1,\, 1)$ to each of $(x_1,\, y_1)$ and $(x_2,\, y_2)$ are shown next.



C13S09.041: The highest point has the largest z-coordinate; the lowest point, the smallest. Consequently we are to maximize and minimize

$$f(x,\, y\; z) = z \quad \text{given} \quad g(x,\, y,\, z) = x^2 + y^2 - 1 = 0 \quad \text{and} \quad h(x,\, y,\, z) = 2x + y - z - 4 = 0.$$

Because the ellipse is formed by the intersection of a plane with a vertical cylinder, there will be a unique highest point and a unique lowest point unless the plane is horizontal or vertical—and it is not; one of its normal vectors is $\mathbf{n} = \langle 2,\, 1,\, -1 \rangle$. So we expect to find exactly two critical points. The Lagrange multiplier equations are

$$2\lambda x + 2\mu = 0, \qquad 2\lambda y + \mu = 0, \qquad -\mu = 1,$$

and when the third is substituted into the first two, we find that

$$2\lambda x = -2 = 4\lambda y.$$

But then $\lambda \neq 0$, and therefore $x = 2y$. Substitution of this information into the first constraint yields $5y^2 = 1$ and thus leads to the following two cases.

Case 1: $y = \frac{1}{5}\sqrt{5}$. Then $x = \frac{2}{5}\sqrt{5}$, and the second constraint implies that $z = \sqrt{5} - 4$.

Case 2: $y = -\frac{1}{5}\sqrt{5}$. Then $x = -\frac{2}{5}\sqrt{5}$, and the second constraint implies that $z = -\sqrt{5} - 4$.

Therefore the lowest point on the ellipse is $\left(-\frac{2}{5}\sqrt{5},\, -\frac{1}{5}\sqrt{5},\, -\sqrt{5}\,-4\right)$ and the highest point on the ellipse is $\left(\frac{2}{5}\sqrt{5},\, \frac{1}{5}\sqrt{5},\, \sqrt{5}\,-4\right)$. See Problem 48 of Section 13.8 for an alternative method of solving such problems.

C13S09.042: We are to maximize and minimize $f(x, y, z) = z$ given the two constraint equations $g(x, y, z) = z^2 - x^2 - y^2 = 0$ and $h(x, y, z) = x + 2y + 3z - 3 = 0$. It is clear from the geometry of the problem that there are unique solutions, and we expect to find exactly two critical points. The Lagrange multiplier vector equation is

$$\langle\, 0,\, 0,\, 1\,\rangle = \lambda\langle\, -2x,\, -2y,\, 2z\,\rangle + \mu\langle\, 1,\, 2,\, 3\,\rangle,$$

which yields the scalar equations

$$-2\lambda x + \mu = 0, \qquad -2\lambda y + 2\mu = 0, \qquad \text{and} \qquad 2\lambda z + 3\mu = 1.$$

Note first that if $\lambda = 0$, then the first of these equations implies that $\mu = 0$, and then the third implies that $0 = 1$. Thus $\lambda \neq 0$. Then, because the first two equations imply that

$$4\lambda x = 2\mu = 2\lambda y, \qquad \text{it follows that} \qquad y = 2x.$$

Substitution of this information into the first constraint yields $z^2 = 5x^2$, and we obtain the following two cases.

Case 1: $z = x\sqrt{5}$. The second constraint implies that $5x + 3x\sqrt{5} = 3$, and thus that

$$x = \frac{3}{5 + 3\sqrt{5}} = \frac{9\sqrt{5}\,-15}{20}, \qquad y = \frac{9\sqrt{5}\,-15}{10}, \qquad \text{and} \qquad z = \frac{9 - 3\sqrt{5}}{4}.$$

Case 2: $z = -x\sqrt{5}$. The second constraint implies that $5x - 3x\sqrt{5} = 3$, and thus that

$$x = \frac{3}{5 - 3\sqrt{5}} = -\frac{9\sqrt{5}\,+15}{20}, \qquad y = -\frac{9\sqrt{5}\,+15}{10}, \qquad \text{and} \qquad z = \frac{9 + 3\sqrt{5}}{4}.$$

Case 1 gives the coordinates of the lowest point on the ellipse and Case 2 gives the coordinates of its highest point.

C13S09.043: We are to maximize and minimize $f(x, y, z) = x^2 + y^2 + z^2$ given the two constraints $g(x, y, z) = x^2 + y^2 - z^2 = 0$ and $h(x, y, z) = x + 2y + 3z - 3 = 0$. From the geometry of the problem we see that there will be a unique maximum distance and a unique minimum distance, but there may be more than two critical points. The Lagrange multiplier vector equation is

$$\langle\, 2x,\, 2y,\, 2z\,\rangle = \lambda\langle\, 2x,\, 2y,\, -2z\,\rangle + \mu\langle\, 1,\, 2,\, 3\,\rangle,$$

and the corresponding scalar equations are

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + 2\mu, \qquad \text{and} \qquad 2z = -2\lambda z + 3\mu.$$

It follows that

$$6\mu = 12x - 12\lambda x = 6y - 6\lambda y = 4z + 2\lambda z,$$

and hence $12x(1 - \lambda) = 6y(1 - \lambda)$.

Case 1: $\lambda = 1$. Then $6z = 0$, and so $z = 0$. Then the first constraint implies that $x = y = 0$, which contradicts the second constraint. This leaves only the second case.

Case 2: $\lambda \neq 1$. Then $y = 2x$. Then the first constraint takes the form $5x^2 = z^2$, and it is convenient to consider separately two subcases.

Case 2a: $z = x\sqrt{5}$. Then the second constraint yields

$$x = \frac{3}{20}\left(-5 + 3\sqrt{5}\right), \quad y = \frac{3}{10}\left(-5 + 3\sqrt{5}\right), \quad \text{and} \quad z = \frac{3}{4}\left(3 - \sqrt{5}\right).$$

Case 2b: $z = -x\sqrt{5}$. Then the second constraint yields

$$x = -\frac{3}{20}\left(5 + 3\sqrt{5}\right), \quad y = -\frac{3}{10}\left(5 + 3\sqrt{5}\right), \quad \text{and} \quad z = \frac{3}{4}\left(3 + \sqrt{5}\right).$$

The coordinates in Case 2a are those of the point closest to the origin; its distance from the origin is approximately 0.81027227. The coordinates of the point farthest from the origin are those given in Case 2b; the distance from this point to the origin is approximately 5.55368876.

C13S09.044: We are to minimize the function $f(x, y, z) = xy + 3xz + 7yz$ given the two constraint equations $g(x, y, z) = xyz - 12 = 0$ and $\frac{1}{6}x = \frac{1}{2}y$; we rewrite the latter in the form $h(x, y, z) = x - 3y = 0$. The Lagrange multiplier equations are

$$y + 3z = \lambda yz + \mu, \quad x + 7z = \lambda xz - 3\mu, \quad \text{and} \quad 3x + 7y = \lambda xy.$$

We also have the constraint equations

$$xyz = 12 \quad \text{and} \quad x = 3y.$$

Here we depart from the usual procedure of eliminating the multipliers because the last constraint equation is easy to use to eliminate x without complicating the other four equations. They become

$$y + 3z = \lambda yz + \mu, \qquad 3y + 7z = 3\lambda yz - 3\mu,$$
$$16y = 3\lambda y^2, \qquad 3y^2 z = 12.$$

Then, because $y \neq 0$, the equation $16y = 3\lambda y^2$ is equivalent to $3\lambda y = 16$, so that

$$\lambda = \frac{16}{3y}.$$

Then elimination of λ from the remaining three equations yields

$$y + 3z = \frac{16}{3}z + \mu, \qquad 3y + 7z = 16z - 3\mu, \qquad \text{and} \qquad y^2 z = 4.$$

Multiply the first of these by 3 and add the result to the second to eliminate μ:

$$6y + 16z = 32z, \quad \text{so that} \quad 3y = 8z; \quad \text{that is,} \quad z = \frac{3}{8}y.$$

Combine this with the equation $y^2 z = 4$ to find that

$$\frac{3}{8}y^3 = 4: \qquad y^3 = \frac{32}{3}, \quad \text{so that} \quad y = \frac{2}{3}\cdot(36)^{1/3} \approx 2.201284833.$$

It now follows that

$$z = \frac{3}{8}y = \frac{1}{4} \cdot (36)^{1/3} \approx 0.825481812 \qquad \text{and} \qquad x = 3y = 2 \cdot (36)^{1/3} \approx 6.603854498.$$

These are the dimensions that minimize the cost of the ice-cube tray shown in Fig. 13.9.10. Note that the ratio $x : y : z = 2 : \frac{2}{3} : \frac{1}{4}$ is in fact $24 : 8 : 3$, not quite the monolith's ratio of $9 : 4 : 1$. Well, that would have been too much to hope for.

C13S09.045: Suppose that $f(x,\, y,\, z)$ and $g(x,\, y,\, z)$ have continuous first-order partial derivatives. Suppose also that the maximum (or minimum) of $f(x,\, y,\, z)$ subject to the constraint

$$g(x,\, y,\, z) = 0$$

occurs at a point P at which $\boldsymbol{\nabla} g(P) \neq \mathbf{0}$. Prove that

$$\boldsymbol{\nabla} f(P) = \lambda \boldsymbol{\nabla} g(P)$$

for some number λ.

Proof: Parametrize the surface $g(x,\, y,\, z) = 0$ at and near the point P with a smooth function $\mathbf{r}(u,\, v)$ and in such a way that $\mathbf{r}_u(u,\, v)$ and $\mathbf{r}_v(u,\, v)$ are nonzero at and near P. Suppose that $\mathbf{r}(u_0, v_0) = \overrightarrow{OP}$. Then $f(\mathbf{r}(u,\, v))$ has a maximum (or minimum) at $(u_0,\, v_0)$. Hence

$$D_u(f(\mathbf{r}(u,\, v))) = 0 \quad \text{ and } \quad D_v(f(\mathbf{r}(u,\, v))) = 0.$$

Hence

$$\boldsymbol{\nabla} f(P) \cdot \mathbf{r}_u(u_0,\, v_0) = 0 \quad \text{ and } \quad \boldsymbol{\nabla} f(P) \cdot \mathbf{r}_v(u_0,\, v_0) = 0.$$

But $g(\mathbf{r}(u,\, v)) \equiv 0$, and hence

$$\boldsymbol{\nabla} g(P) \cdot \mathbf{r}_u(u_0,\, v_0) = 0 \quad \text{ and } \boldsymbol{\nabla} g(P) \cdot \mathbf{r}_v(u_0,\, v_0) = 0.$$

Therefore $\boldsymbol{\nabla} f(P)$ and $\boldsymbol{\nabla} g(P)$ are parallel to $\mathbf{r}_u(u_0,\, v_0) \times \mathbf{r}_v(u_0,\, v_0)$. Consequently, because $\boldsymbol{\nabla} g(P) \neq \mathbf{0}$, $\boldsymbol{\nabla} f(P) = \lambda \boldsymbol{\nabla} g(P)$ for some scalar λ. ◀

C13S09.046: If we project the ellipse into the xy-plane, we should obtain a circle or another ellipse. The center of the original ellipse will project onto the center of its image, and this will locate the center of the ellipse. Then we can find its minor and major semiaxes by minimizing and maximizing the distance of points of the ellipse from its center.

Every point in the intersection satisfies the equations $z = x^2 + y^2 = 12 - x - y$, and hence satisfies the equation $x^2 + x + y^2 + y = 12$; that is,

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 12 + \frac{1}{2} = \frac{25}{2}.$$

Hence the projection of the ellipse into the xy-plane is a circle with center $\left(-\frac{1}{2},\, -\frac{1}{2}\right)$. Because the center of the ellipse lies in the plane $x + y + z = 12$, its coordinates are $C\left(-\frac{1}{2},\, -\frac{1}{2},\, 13\right)$.

Next we maximize and minimize

$$f(x,\,y,\,z) = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + (z-13)^2$$

given the constraints $g(x,\,y,\,z) = x + y + z - 12 = 0$ and $h(x,\,y,\,z) = x^2 + y^2 - z = 0$. The Lagrange multiplier equations are

$$2x + 1 = \lambda + 2\mu x,$$

$$2y + 1 = \lambda + 2\mu y,$$

$$2z - 26 = \lambda - \mu.$$

We solve the third equation for $\lambda = \mu + 2z - 26$ and substitute for λ in the other two equations:

$$2x + 1 = \mu + 2z - 26 + 2\mu x,$$

$$2y + 1 = \mu + 2z - 26 + 2\mu y.$$

Subtract the second of these from the first to obtain $x - y = \mu(x - y)$. There are two cases to consider.

Case 1: $y = x$. Then the constraint equations yield

$$2x + z = 12, \qquad z = 2x^2;$$

$$2x^2 + 2x - 12 = 0;$$

$$x^2 + x - 6 = 0;$$

$$(x+3)(x-2) = 0.$$

Thus $x = -3$ or $x = 2$. Thus we obtain the two critical points $(-3,\,-3,\,18)$ and $(2,\,2,\,8)$.

Case 2: $y \neq x$. Then $\mu = 1$. The earlier equation $2x + 1 = \mu + 2z - 26 + 2\mu x$ now becomes

$$2x + 1 = 1 + 2z - 26 + 2x, \qquad \text{and thus} \qquad z = 13.$$

The constraint equations become $x + y = -1$ and $x^2 + y^2 = 13$. We solve the first of these for $y = -1 - x$ and substitute in the second to obtain $x^2 + x - 6 = 0$, exactly as in Case 1. But now we obtain two additional critical points: $(-3,\,2,\,13)$ and $(2,\,-3,\,13)$.

It turns out that the two critical points of Case 1 are at distance $\frac{5}{2}\sqrt{6} \approx 6.123724356958$ from C, whereas the two critical points of Case 2 are at distance $\frac{5}{2}\sqrt{2} \approx 3.535533905933$ from C. Therefore the major semiaxis of the ellipse has length $\frac{5}{2}\sqrt{6}$ and the minor semiaxis has length $\frac{5}{2}\sqrt{2}$.

C13S09.047: Given the constant P, we are to maximize

$$A(x,\,y,\,z) = \frac{1}{2}xy$$

given the constraints $g(x,\,y,\,z) = x + y + z - P = 0$ and $h(x,\,y,\,z) = x^2 + y^2 - z^2 = 0$. The Lagrange multiplier equations are

$$\frac{1}{2}y = \lambda + 2\mu x, \quad \frac{1}{2}x = \lambda + 2\mu y, \quad \text{and} \quad 0 = \lambda - 2\mu z.$$

We solve the last equation for $\lambda = 2\mu z$ and substitute in the other two to obtain

$$y = 4\mu z + 4\mu x,$$
$$x = 4\mu z + 4\mu y.$$

We subtract the second of these from the first to find that $y - x = -4\mu(y - x)$. There are two cases to consider.

Case 1: $y = x$. Then the constraint equations become $2x + z = P$ and $z^2 = 2x^2$. We solve the first for z and substitute in the second to obtain

$$\begin{aligned}
&(P - 2x)^2 = 2x^2;\\
&P - 2x = \pm x\sqrt{2};\\
&2x \pm x\sqrt{2} = P;\\
&x = \frac{P}{2 \pm \sqrt{2}}.
\end{aligned}$$

We must take the plus sign in the last denominator because $x \leqq P$. Thus we obtain our first critical point:

$$x = \frac{P}{2 + \sqrt{2}} = \frac{2 - \sqrt{2}}{2}P, \quad y = x, \quad z = P - 2x = \left(\sqrt{2} - 1\right)P.$$

In this case the area of the triangle is

$$A = \frac{1}{2}xy = \frac{\left(2 - \sqrt{2}\,\right)^2}{8}P^2 = \frac{3 - 2\sqrt{2}}{4}P^2 \approx (0.042893218813)P^2.$$

Case 2: $y \neq x$. Then $4\mu = -1$, so that $\mu = -\frac{1}{4}$. Our earlier equations

$$y = 4\mu(z + x) \quad \text{and} \quad x = 4\mu(z + y)$$

now become $y = -z - x$ and $x = -z - y$, each of which is impossible as, at maximum, x , y, and z are all positive.

Thus Case 1 is the only case that produces a critical point. We may conclude that $y = x$ and that the right triangle with fixed perimeter and maximum area is isosceles.

C13S09.048: We are to maximize

$$A(x,\, y,\, z,\, \alpha) = \frac{1}{2}xy\sin\alpha$$

given the constraints $x + y + z - P = 0$ and $x^2 + y^2 - 2xy\cos\alpha - z^2 = 0$. The Lagrange multiplier equations are

$$\begin{aligned}
\frac{1}{2}y\sin\alpha &= \lambda + 2\mu x - 2\mu y\cos\alpha,\\
\frac{1}{2}x\sin\alpha &= \lambda + 2\mu y - 2\mu x\cos\alpha,\\
0 &= \lambda - 2\mu z, \qquad \text{and}\\
\frac{1}{2}xy\cos\alpha &= 2\mu xy\sin\alpha.
\end{aligned}$$

The third of these equations implies that $\lambda = 2\mu z$. We substitute for λ in the other three equations and obtain

$$y \sin\alpha = 4\mu z + 4\mu x - 4\mu y \cos\alpha, \tag{1}$$

$$x \sin\alpha = 4\mu z + 4\mu y - 4\mu x \cos\alpha, \tag{2}$$

$$xy \cos\alpha = 4\mu xy \sin\alpha. \tag{3}$$

At maximum area, x and y are nonzero, so Eq. (3) implies that $\cos\alpha = 4\mu \sin\alpha$. Now subtract Eq. (2) from Eq. (1) to obtain

$$(y - x)\sin\alpha = -4\mu(y - x) - 4\mu(y - x)\cos\alpha.$$

There are two cases to consider.

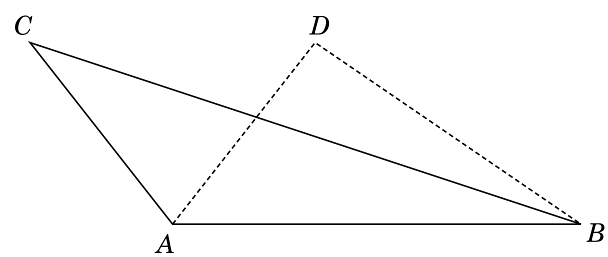
Case 1: $y \neq x$. Then

$$\begin{aligned} \sin\alpha &= -4\mu - 4\mu\cos\alpha = -4\mu(1 + \cos\alpha) \\ &= -4\mu(1 + 4\mu\sin\alpha) = -4\mu - 16\mu^2 \sin\alpha, \end{aligned}$$

and thus $(16\mu^2 + 1)\sin\alpha = -4\mu$. Thus

$$(16\mu^2 + 1)\sin^2\alpha = -4\mu\sin\alpha = -\cos\alpha,$$

and hence $\cos\alpha < 0$. But this implies that $\pi/2 < \alpha \leqq \pi$.



But α cannot be an obtuse angle at maximum area. See the preceding figure. If α is the angle CAB, replace α with $\pi - \alpha$ and ensure that AC and AD have the same length. Then triangle ABD has the same area as triangle ABC but smaller perimeter. By enlarging triangle ABD until it has perimeter P, you will obtain a triangle with perimeter P and area larger than that of triangle ABC. This is why, at maximum, α must be an acute angle. Thus we have shown that Case 1 is impossible. This leaves only

Case 2: $y = x$. Now repeat the earlier argument with y, z, and the angle β between them to show that $y = z$ as well.

Therefore the triangle with fixed perimeter P and maximum area is equilateral.

C13S09.049: The hexagon in Fig. 13.9.13 is the union of four congruent trapezoids, one in each quadrant. The area of the trapezoid in the first quadrant is $\frac{1}{2}(1 + y)x$, so the total area of the hexagon is

$$A(x,\, y) = 2x(1 + y),$$

which we are to maximize subject to the constraint $x^2+y^2-1=0$. Note also that $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$. The Lagrange multiplier equations are

$$2+2y=2\lambda x \qquad \text{and} \qquad 2x=2\lambda y,$$

which yield (multiply the first by y, the second by x)

$$2y+2y^2=2\lambda xy=2x^2, \quad \text{so that} \quad y^2+y=x^2.$$

Substitute for x^2 in the constraint equation to obtain $2y^2+y-1=0$, so that $(2y-1)(y+1)=0$. Thus $y=\frac{1}{2}$ because $y \neq -1$. So the only critical point is $\left(\frac{1}{2}\sqrt{3},\, \frac{1}{2}\right)$. To verify that the resulting hexagon is regular, it is sufficient (by the various symmetries in the figure) to verify that the distances from $\left(\frac{1}{2}\sqrt{3},\, \frac{1}{2}\right)$ to $(0,\, 1)$ and from $\left(\frac{1}{2}\sqrt{3},\, \frac{1}{2}\right)$ to $\left(\frac{1}{2}\sqrt{3},\, -\frac{1}{2}\right)$ are equal. They are; each distance is 1.

C13S09.050: We are to maximize the volume

$$V(x,\, y)=2\pi x^2y+2\cdot\frac{1}{3}\cdot\pi x^2(1-y);$$

to simplify the notation slightly, we maximize instead

$$f(x,\, y)=\frac{3}{2\pi}V(x,\, y)=3x^2y+x^2-x^2y=2x^2y+x^2$$

subject to the constraint $x^2+y^2-1=0$. Note also that $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$. The Lagrange multiplier equations are

$$4xy+2x=2\lambda x \qquad \text{and} \qquad 2x^2=2\lambda y,$$

and it follows that $4xy^2+2xy=2\lambda xy=2x^3$. But $x \neq 0$ at maximum volume, and hence $2y^2+y=x^2$. Substitution for x^2 in the constraint leads to

$$3y^2+y-1=0, \quad \text{so that} \quad y=\frac{-1\pm\sqrt{13}}{6},$$

and of course the plus sign is to be chosen. Then

$$x^2=1-y^2=\frac{11+\sqrt{13}}{18},$$

and hence, at maximum volume of the solid, the radius of the cylinder is

$$x=\frac{1}{3}\sqrt{\frac{11+\sqrt{13}}{2}}\approx 0.9007882744038995$$

and its height is

$$2y=\frac{-1+\sqrt{13}}{3}\approx 0.8685170918213398.$$

The maximum volume is approximately 3.175419716, which is just under 76% of the volume of the circumscribed sphere, so the answer is certainly plausible.

C13S09.051: We are to minimize $f(x,\, y)=x^2+y^2$ given the constraint $(x-1)^2-y=0$. The Lagrange multiplier equations are

$$2x = 2\lambda(x - 1) \qquad \text{and} \qquad 2y = -\lambda,$$

and elimination of λ leads to the equation $x = -2y(x-1)$. We combine this with the constraint equation and ask *Mathematica* 3.0 to solve the resulting two equations:

```
Solve[ { x == -2*y*(x - 1), y == (x - 1)∧2 }, { x, y } ]
```

The only real solution is

$$x = 1 + \frac{\left(-9+\sqrt{87}\,\right)^{1/3}}{6^{2/3}} - \frac{1}{\left[6\cdot\left(-9+\sqrt{87}\,\right)\right]^{1/3}} \approx 0.4102454876985416,$$

$$y = \frac{1}{36}\left\{-12 + \frac{6^{4/3}}{\left(-9+\sqrt{87}\,\right)^{2/3}} + \left[6\cdot\left(-9+\sqrt{87}\right)\right]^{2/3}\right\} \approx 0.3478103847799310.$$

C13S09.052: We are to maximize and minimize $f(x,\,y) = (x-3)^2 + (y-2)^2$ given the constraint $4x^2 + 9y^2 - 36 = 0$. The Lagrange multiplier equations are

$$2(x-3) = 8\lambda x \qquad \text{and} \qquad 2(y-2) = 18\lambda y.$$

We eliminate λ by multiplication of the first equation by $\frac{9}{2}y$ and the second by $2x$, which yields

$$9y(x-3) = 36\lambda xy = 4x(y-2);$$

thus we ask *Mathematica* 3.0 to solve simultaneously the equations

$$5xy + 8x = 27y \quad \text{and} \quad 4x^2 + 9y^2 = 36:$$

```
Solve[ { 5*x*y + 8*x == 27*y, 4*x*x + 9*y*y == 36 }, { x, y } ]
```

The computer algebra program returns two real solutions (and two complex non-real solutions). Their exact values are extremely long and complicated, so we provide only the numerical approximations to the two real solutions:

$$x_1 \approx 2.3558738622119243, \quad y_1 \approx 1.2382529530389304;$$

$$x_2 \approx -2.8814378699195849, \quad y_2 \approx -0.5567029137074361.$$

Hence the point of the ellipse closest to $(3,\,2)$ is $(x_1,\,y_1)$ and the point farthest from $(3,\,2)$ is $(x_2,\,y_2)$.

C13S09.053: We are to find the first-quadrant point of the hyperbola $xy = 24$ that is closest to the point $P(1,\,4)$, so we minimize $f(x,\,y) = (x-1)^2 + (y-4)^2$ subject to the constraint $xy = 24$ and the observation that $x > 0$ and $y > 0$. The Lagrange multiplier equations are

$$2(x-1) = \lambda y \quad \text{and} \quad 2(y-4) = \lambda x.$$

To eliminate λ, multiply the first equation by x and the second by y to obtain

$$2x(x-1) = \lambda xy = 2y(y-4).$$

We simplified this equation and asked *Mathematica* 3.0 to solve it simultaneously with the constraint equation:

```
Solve[ { x*x - x == y*y - y, x*y == 24 }, { x, y } ]
```

The resulting output is too long to reproduce here, but to evaluate the results numerically to 20 places we entered

```
N[ %, 20 ]
```

and obtained four pairs of solutions, only two of which were real:

$$x_1 = 4, \; y_1 = 6 \quad \text{and} \quad x_2 \approx -5.5338281384822297, \; y_2 \approx -4.3369615751352589.$$

Hence the point in the first quadrant on the hyperbola $xy = 24$ closest to P is $(4, 6)$. It seems very likely that (x_2, y_2) is the point on the hyperbola in the third quadrant closest to P.

C13S09.054: To find the point on the surface $xyz = 1$ closest to the point $P(1, 2, 3)$, we minimize $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ subject to the constraint $xyz = 1$. The Lagrange multiplier equations are

$$2(x-1) = \lambda yz, \quad 2(y-2) = \lambda xz, \quad \text{and} \quad 2(z-3) = \lambda xy.$$

To solve these using *Mathematica* 3.0, we entered the command

```
Solve[ { 2*(x - 1) == lambda*y*z, 2*(y - 2) == lambda*x*z,
          2*(z - 3) == lambda*x*y, x*y*z == 1 }.  { x, y, z, lambda } ]
```

The resulting output was far too long for inclusion here, but its numerical evaluation occupied only three-quarters of a page. There are twelve quadruples of solutions, only four of which are real. Omitting the values of λ, they are

$$\begin{array}{lll}
x_1 \approx 2.0677534914056975, & y_1 \approx -0.7910474616309738, & z_1 \approx -0.6113623587188331; \\
x_2 \approx -0.8475829693534350, & y_2 \approx 2.6018676160318058, & z_2 \approx -0.4534533163839419; \\
x_3 \approx 0.1760798739948590, & y_3 \approx 1.9246211375645630, & z_3 \approx 2.9508357067673743; \\
x_4 \approx -0.6647722146060670, & y_4 \approx -0.4514455938540451, & z_4 \approx 3.3321283557432110.
\end{array}$$

Next, evaluation of $f(x, y, z)$ at these points yielded

$$\begin{array}{ll}
f(x_1, y_1, z_1) \approx 21.9719815374570127, & f(x_2, y_2, z_2) \approx 15.7021472643159114, \\
f(x_3, y_3, z_3) \approx 0.6869434746674517, & f(x_4, y_4, z_4) \approx 8.8913612708384896.
\end{array}$$

Therefore (x_3, y_3, z_3) is the point on the surface $xyz = 1$ closest to the point P. Because the surface $xyz = 1$ contains points arbitrarily far from P, there can be no global maximum value of $f(x, y, z)$ there. To visualize the surface, note that its intersection with the horizontal plane $z = K$ consists of both branches of the hyperbola $xy = 1/K$ provided that $K \neq 0$. Thus the surface consists of four sheets, so the other three critical points we found are very likely local minima of $f(x, y, z)$ there.

C13S09.055: To find the point on the spherical surface with equation

$$(x-1)^2+(y-2)^2+(z-3)^2-36=0 \tag{1}$$

that are closest to and farthest from the origin, note that there do exist such points, and probably only one of each type, so we expect to find only two critical points. We maximize and minimize $f(x, y, z)=x^2+y^2+z^2$ subject to the constraint in Eq. (1). The Lagrange multiplier equations are

$$2x=2\lambda(x-1), \quad 2y=2\lambda(y-2), \quad \text{and} \quad 2z=2\lambda(z-3).$$

To solve these, we entered the *Mathematica* 3.0 command

```
Solve[ { x == lambda*(x - 1), y == lambda*(y - 2), z == lambda*(z - 3),
        (x - 1)∧2 + (y - 2)∧2 + (z - 3)∧2 == 36 }, { x, y, z, lambda } ]
```

As expected, there are two solutions. Omitting the values of λ, they are

$$x_1=\frac{7-3\sqrt{14}}{7}, \qquad y_1=\frac{14-6\sqrt{14}}{7}, \qquad z_1=\frac{21-9\sqrt{14}}{7} \quad \text{and}$$

$$x_2=\frac{7+3\sqrt{14}}{7}, \qquad y_2=\frac{14+6\sqrt{14}}{7}, \qquad z_2=\frac{21+9\sqrt{14}}{7}.$$

Their numerical values are

$$x_1 \approx -0.6035674514745463, \qquad y_1 \approx -1.2071349029490926, \qquad z_1 \approx -1.8107023544236389 \quad \text{and}$$

$$x_2 \approx 2.6035674514745463, \qquad y_2 \approx 5.2071349029490926, \qquad z_2 \approx 7.8107023544236389.$$

The first of these is obviously much closer to the origin than the second, so (x_1, y_1, z_1) is the point of the spherical surface closest to the origin and (x_2, y_2, z_2) is the point of the spherical surface farthest from the origin.

C13S09.056: We are given the ellipsoidal surface S with equation

$$4x^2+9y^2+z^2-36=0, \tag{1}$$

and we are to find the points of S closest to and farthest from the origin, so we maximize and minimize $f(x, y, z)=x^2+y^2+z^2$ subject to the constraint in Eq. (1). The Lagrange multiplier equations are

$$2x=8\lambda x, \qquad 2y=18\lambda y, \qquad 2z=2\lambda z;$$

that is,

$$x=4\lambda x, \qquad y=9\lambda y, \qquad z=\lambda z.$$

The instructions tell us to use a computer algebra system *as needed,* but none is needed here. Note first that $\lambda \neq 0$. If x and y are nonzero, then $4\lambda=1=9\lambda$, which is impossible. Similarly, x and z cannot both be nonzero, nor can both y and z. So at most one of x, y, and z is nonzero. Also, at *least* one of x, y, and z is nonzero. Hence there are only three cases.

Case 1: $x \neq 0,\ y = 0 = z$. Then $x = \pm 3$. The two critical points $(-3, 0, 0)$ and $(3, 0, 0)$ are at distance 3 from the origin.

Case 2: $y \neq 0,\ x = 0 = z$. Then $y = \pm 2$. The two critical points $(0, -2, 0)$ and $(0, 2, 0)$ are at distance 2 from the origin.

Case 3: $z \neq 0,\ x = 0 = y$. Then $z = \pm 6$. The two critical points $(0, 0, -6)$ and $(0, 0, 6)$ are at distance 6 from the origin.

Summary: The two points of Case 2 are closest to the origin; the two points of Case 3 are farthest from the origin. The two points of Case 1 are not even local extrema. You can verify the last assertion by examining the behavior of f near these points, first restricted to the xy-plane, then to the xz-plane.

C13S09.057: We are to find the points of the ellipse with equation $4x^2 + 9y^2 = 36$ closest to, and farthest from, the line with equation $x + y = 10$. What if the ellipse and the line intersect? If they do, then the equation $4x^2 + 9(10 - x)^2 = 36$ will have one or two real solutions. But this equation reduces to $13x^2 - 180x + 864 = 0$, which has discriminant

$$\Delta = 180^2 - 4 \cdot 13 \cdot 864 = -12528 < 0.$$

Because the quadratic has no real solutions, the ellipse and the line do not meet. To find the answers, we could assume that (x, y) is a point of the ellipse, that (u, v) is a point of the line, and maximize and minimize

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2$$

subject to the constraints $4x^2 + 9y^2 - 36 = 0$ and $u + v - 10 = 0$. The Lagrange multiplier equations are

$$\begin{aligned} 2(x - u) &= 8\lambda x, & 2(y - v) &= 18\lambda y, \\ -2(x - u) &= \mu, & -2(y - v) &= \mu. \end{aligned}$$

Mathematica 3.0 can solve the system of six simultaneous equations (the four Lagrange multiplier equations and the two constraint equations) exactly in a few tenths of a second, but this problem can be solved by hand. Observe that if the line is moved without rotation toward the ellipse, when it first touches the ellipse it will be touching the point of the ellipse closest to the original line and the moving line will be tangent to the ellipse at that point. As the line continues to move across the ellipse, it will last touch the ellipse at the point of the ellipse farthest from the original line and the moving line will be tangent to the ellipse at that point. Because the line has slope -1, all we need do is find the points of the ellipse where the tangent line has slope -1. Implicit differentiation of the equation of the ellipse yields

$$8x + 18y\frac{dy}{dx} = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = -\frac{8x}{18y} = -\frac{4x}{9y}.$$

The extrema occur when

$$\frac{dy}{dx} = -1; \qquad 4x = 9y; \qquad y = \frac{4}{9}x.$$

Substitution of the last equation for y in the equation of the ellipse yields the two solutions $x = \pm\frac{9}{13}\sqrt{13}$. So the points of the ellipse closest to, and farthest from, the line are (respectively)

$$\left(\frac{9}{13}\sqrt{13}, \frac{4}{13}\sqrt{13}\right) \quad \text{and} \quad \left(-\frac{9}{13}\sqrt{13}, -\frac{4}{13}\sqrt{13}\right).$$

(You can tell the maximum from the minimum by observing that the line is to the "northeast" of the ellipse.)

C13S09.058: Suppose that (x, y, z) is a point of the ellipsoid with equation $4x^2 + 9y^2 + z^2 = 36$ and that (u, v, w) is a point of the plane with equation $2x + 3y + z = 10$. Then we should maximimze and minimize

$$f(x, y, z, u, v, w) = (x - u)^2 + (y - v)^2 + (z - w)^2$$

subject to the constraints given by the equations of the ellipsoid and the line. The Lagrange multiplier equations and the constraint equations form a system of eight equations in eight unknowns, one of which is nonlinear; they are:

$$\begin{aligned} &2(x - u) = 8\lambda x, \qquad &&2(y - v) = 18\lambda y, \qquad &&2(z - w) = 2\lambda z, \\ &-2(u - x) = 2\mu, &&-2(y - v) = 3\mu, &&-2(z - 2) = \mu, \\ &4x^2 + 9y^2 + z^2 = 36, &&2u + 3v + w = 10. \end{aligned}$$

These equations are difficult to solve by hand, but a computer algebra program—even if unable to solve them exactly—could yield highly accurate approximations in seconds. A simpler alternative is to apply the condition that the ellipsoid's normal vector $\langle 8x, 18y, 2z \rangle$ must be a scalar multiple of the plane's normal vector $\langle 2, 3, 1 \rangle$. This gives the equations

$$8x = 2\lambda, \quad 18y = 3\lambda, \quad 2z = \lambda, \quad 4x^2 + 9y^2 + z^2 = 36,$$

whose two xyz-solutions are comparitively easy to find. Thus we find the two critical points

$$P_1\left(\sqrt{3}, \frac{2}{3}\sqrt{3}, 2\sqrt{3}\right) \quad \text{and} \quad P_2\left(-\sqrt{3}, -\frac{2}{3}\sqrt{3}, -2\sqrt{3}\right).$$

It may appear that these are the points of the ellipsoid nearest to and farthest from the plane.

But we must investigate the possibility that the plane and the ellipsoid intersect. Elimination of z from their equations gives the equation

$$4x^2 + 9y^2 + (10 - 2x - 3y)^2 = 36,$$

which can be simplified to

$$4x^2 + 9y^2 + 6xy - 20x - 30y + 32 = 0.$$

If we substitute $x = 2$ (for instance), we get the quadratic equation $9y^2 - 18y + 8 = 0$ with real solutions $y = \frac{2}{3}$ and $y = \frac{4}{3}$, which in turn give $z = 4$ and $z = 2$, respectively. Thus the ellipsoid and the plane meet; their intersection contains the points $\left(2, \frac{2}{3}, 4\right)$ and $\left(2, \frac{4}{3}, 2\right)$, among others. It follows (Why?) that the plane and ellipsoid intersect in an ellipse that evidently lies in the first octant. All the points on the ellipse are the points of ellipsoid closest to the plane; they are at distance zero from the plane. The point P_1 found previously is not the farthest point from the plane; it provides only a local maximum value for the distance function. The point P_2 is the point of the ellipsoid farthest from the plane.

C13S09.059: The sides of the box must be parallel to the coordinate planes. Let (x, y, z) be the upper vertex of the box that lies in the first octant, so that at maximum volume we have x, y, and z all positive. The box has width $2x$, depth $2y$, and height $z = 9 - x^2 - 2y^2$, so we are to maximize box volume $V(x, y, z) = 4xyz$ given the constraint $x^2 + 2y^2 + z - 9 = 0$. The Lagrange multiplier equations are

$$4yz = 2\lambda x, \quad 4xz = 4\lambda y, \quad \text{and} \quad 4xy = \lambda.$$

These equations (together with the constraint equation) are relatively easy to solve by hand. To solve them using *Mathematica* 3.0, we entered the command

```
Solve[ { 4*y*z == 2*lambda*x, 4*x*z == 4*lambda*y, 4*x*y == lambda,
          x*x + 2*y*y + z == 9 }, { x, y, z, lambda } ]
```

The computer returned nine solutions, but eight involve negative or zero values of x, y, or z; the only viable solution is

$$x = \frac{3}{2}, \quad y = \frac{3}{4}\sqrt{2}, \quad z = \frac{9}{2},$$

and therefore the box of maximum volume has volume $V = \dfrac{81}{4}\sqrt{2}$.

C13S09.060: We are given the plane $\mathcal{P}$ with equation $4x + 9y + z = 0$ and the elliptic paraboloid S with equation $2x^2 + 3y^2 - z = 0$. Because the paraboloid becomes arbitrarily steep as x and y take on very large (positive *or* negative) values, the intersection of $\mathcal{P}$ and S cannot be a parabola or a hyperbola; it must be an ellipse, so there is a unique highest point and a unique lowest point on the intersection. To find it, we maximize and minimize $f(x, y, z) = z$ subject to the constraints in the equations of $\mathcal{P}$ and S. The Lagrange multiplier equations are

$$0 = 4\lambda + 4\mu x, \quad 0 = 9\lambda + 6\mu y, \quad \text{and} \quad 1 = \lambda - \mu.$$

These equations are easy to solve by hand. First we show that $\mu \neq 0$. If $\mu = 0$ then the first of the previous equations implies that $\lambda = 0$, and this contradicts the third equation. Hence $\mu \neq 0$. Then the first two equations yield

$$-36\lambda = 36\mu x = 24\mu y, \quad \text{so that} \quad 3\mu x = 2\mu y.$$

Therefore $y = \frac{3}{2}x$. Substitution for y in the constraint equations yields

$$4x + \frac{27}{2}x + z = 0 \quad \text{and} \quad 2x^2 + \frac{27}{4}x^2 = z.$$

Elimination of z then yields

$$4x + \frac{27}{2}x + 2x^2 + \frac{27}{4}x^2 = 0;$$

$$16x + 54x + 8x^2 + 27x^2 = 0;$$

$$35x^2 + 70x = 0;$$

$$x^2 + 2x = 0.$$

If $x = 0$ then we obtain the critical point $(0, 0, 0)$, clearly the lowest point on the intersection. If $x = -2$ then we obtain the critical point $(-2, -3, 35)$, and this is the highest point on the ellipse.

C13S09.061: With $n = 3$ and $k = 2$ (for instance), the equations in (15) are

$$g_1(x_1, x_2, x_3, x_4) = 0,$$
$$g_2(x_1, x_2, x_3, x_4) = 0,$$
$$g_3(x_1, x_2, x_3, x_4) = 0$$

and the scalar component equations of the vector equation

$$\nabla f(x_1, x_2, x_3, x_4) = \lambda_1 \nabla g_1(x_1, x_2, x_3, x_4) + \lambda_2 \nabla g_2(x_1, x_2, x_3, x_4) + \lambda_3 \nabla g_3(x_1, x_2, x_3, x_4)$$

are

$$\begin{aligned}
D_1 f(x_1, x_2, x_3, x_4) &= \lambda_1 D_1 g_1(x_1, x_2, x_3, x_4) + \lambda_2 D_1 g_2(x_1, x_2, x_3, x_4) + \lambda_3 D_1 g_3(x_1, x_2, x_3, x_4),\\
D_2 f(x_1, x_2, x_3, x_4) &= \lambda_1 D_2 g_1(x_1, x_2, x_3, x_4) + \lambda_2 D_2 g_2(x_1, x_2, x_3, x_4) + \lambda_3 D_2 g_3(x_1, x_2, x_3, x_4),\\
D_3 f(x_1, x_2, x_3, x_4) &= \lambda_1 D_3 g_1(x_1, x_2, x_3, x_4) + \lambda_2 D_3 g_2(x_1, x_2, x_3, x_4) + \lambda_3 D_3 g_3(x_1, x_2, x_3, x_4),\\
D_4 f(x_1, x_2, x_3, x_4) &= \lambda_1 D_4 g_1(x_1, x_2, x_3, x_4) + \lambda_2 D_4 g_2(x_1, x_2, x_3, x_4) + \lambda_3 D_4 g_3(x_1, x_2, x_3, x_4).
\end{aligned}$$

Consequently we have altogether $3 + 4 = 7$ equations in the seven unknowns λ_1, λ_2, λ_3, x_1, x_2, x_3, and x_4. —C.H.E.

C13S09.062: The ith Lagrange multiplier equation is $1 = \lambda x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n$. Upon multiplying by x_i and using the constraint we get $x_i = \lambda$ for each i. Thus $x_1 = x_2 = \cdots = x_n$. Because each x_i is positive and their product is 1, it follows that the minimum value of f is attained with $x_i = 1$ for each i, $1 \leqq i \leqq n$. This shows that $x_1 + x_2 + \cdots + x_n \geqq n$. We get the arithmetic-geometric mean inequality almost immediately when we substitute

$$x_i = \frac{a_i}{\sqrt[n]{a_1 a_2 \cdots a_n}}.$$ —C.H.E.

C13S09.063: With

$$f(x, y) = x^2 + y^2 \qquad \text{and} \qquad g(x, y) = \frac{a}{x} + \frac{b}{y} - 1,$$

the Lagrange multiplier equations are

$$2x = -\frac{\lambda}{x^2} \quad \text{and} \quad 2y = -\frac{\lambda}{y^2},$$

and it follows that $2x^2 = -\lambda a$ and $2y^2 = -\lambda b$. Then division of the last equation by the one before it yields $y = xb^{1/3}a^{-1/3}$, and then substitution of this value in the constraint equation readily gives $x = a^{1/3}(a^{2/3} + b^{2/3})$. Thus $y = b^{1/3}(a^{2/3} + b^{2/3})$, and substitution of these values of x and y gives

$$L_{\min} = \sqrt{x^2 + y^2} = \left(a^{2/3} + b^{2/3}\right)^{3/2}.$$ —C.H.E.

C13S09.064: Part (a): Immediate. Part (b): With $a = b = c = 1$, we have $x = y = z$ by symmetry; hence the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

yields $x = y = z = 3$, and it follows that the minimum area is

$$A_{\min} = \sqrt{\frac{3^4 + 3^4 + 3^4}{4}} = \frac{9}{2}\sqrt{3}\,.$$

Part (c): We use *Mathematica.*

```
Clear[x, y, z, λ]
f = x∧2∗y∧2 + x∧2∗z∧2 + y∧2∗z∧2;
g = a/x + b/y + c/z − 1;
eq1 = g == 0;
eq2 = D[f, x] == λ∗D[g, x]
eq3 = D[f, y] == λ∗D[g, y]
eq4 = D[f, z] == λ∗D[g, z]
```

$$\begin{aligned} 2xy^2 + 2xz^2 &= -\frac{\lambda a}{x^2} \\ 2x^2y + 2yz^2 &= -\frac{\lambda b}{y^2} \\ 2x^2z + 2y^2z &= -\frac{\lambda c}{z^2} \end{aligned}$$

Now substitute *your* selected values of a, b, and c. To illustrate with $a = b = c = 1$, the equations we need to solve are these:

```
eqs = { eq1, eq2, eq3, eq4 } /. { a → 1, b → 1, c → 1 }
```

$$-1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0, \qquad 2xy^2 + 2xz^2 = -\frac{\lambda}{x^2}$$

$$2x^2y + 2yz^2 = -\frac{\lambda}{y^2}, \qquad 2x^2z + 2y^2z = -\frac{\lambda}{z^2}.$$

Because $x, y,\ z,\ a,\ b,$ and c are all positive, it's clear that λ is negative. So let's try the initial guesses $x = y = z = 2$ and $\lambda = -1$. (Generally, λ need not be estimated very accurately.)

```
soln = FindRoot[ eqs, { x, 2 }, { y, 2 }, { z, 2 }, { λ, −1 } ]
```

$$x = 3, \quad y = 3, \quad z = 3, \quad \lambda = -972.$$

```
{ Sqrt[ f/4 /. soln ], N[ (9/2)∗Sqrt[3] ] }
```

7.794228634051643, 7.794228634059946 —C.H.E.

C13S09.065: We write $P_1(x,\ y,\ z)$ and $P_2(u,\ v,\ w)$ because the two points are independent. *Mathematica* then yields the solution, as follows.

```
f = (x − u)∧2 + (y − v)∧2 + (z − w)∧2;
```

```
eq1 = 2*x + y + 2*z == 15;
eq2 = x + 2*y + 3*z == 30;
eq3 = u − v - 2 *w == 15;
eq4 = 3*u - 2*v - 3*w == 20;
eq5 = 2*(x −u) == 2*λ1 + λ2;
eq6 = 2*(y −v) == λ1 + 2*λ2;
eq7 = 2*(z −w) == 2*λ1 + 3*λ2;
eq8 = −2*(x − u) == λ3 + 3*λ4;
eq9 = −2*(y −v) == −λ3 − 2*λ4;
eq10 = −2*(z − w) == −2*λ3 − 3*λ4;
eqs = { eq1. eq2. eq3. eq4. eq5. eq6. eq7. eq8. eq9. eq10 };
vars = { x, y, z, u, v, w, λ1, λ2, λ3, λ4 };
soln = Solve[ eqs, vars ]
```

$$x = 7, \qquad y = 43, \qquad z = -21, \qquad u = 12, \qquad v = 41, \qquad w = -22,$$
$$\lambda_1 = -8, \qquad \lambda_2 = 6, \qquad \lambda_3 = -8, \qquad \lambda_4 = 6.$$

Thus the closest points are $P_1(7, 43, -21)$ on line L_1 and $P_2(12, 41, -22)$ on line L_2. —C.H.E.

Chapter 13 Miscellaneous Problems

C13S0M.001: Using polar coordinates, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^4 \sin^2\theta\, \cos^2\theta}{r^2} = \lim_{r\to 0} (r^2 \sin^2\theta\, \cos^2\theta) = 0$$

because $0 \leqq \sin^2\theta \leqq 1$ and $0 \leqq \cos^2\theta \leqq 1$ for all θ.

C13S0M.002: We convert to spherical coordinates:

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3-z^3}{x^2+y^2+z^2} &= \lim_{\rho\to 0} \frac{\rho^3 \sin^3\phi\, \cos^3\theta + \rho^3 \sin^3\phi\, \sin^3\theta - \rho^3 \cos^3\phi}{\rho^2} \\ &= \lim_{\rho\to 0} \rho(\sin^3\phi\, \cos^3\theta + \sin^3\phi\, \sin^2\theta - \cos^3\phi) = 0\end{aligned}$$

because $-1 \leqq \sin^3\zeta \leqq 1$ and $-1 \leqq \cos^3\zeta \leqq 1$ for all ζ.

C13S0M.003: First we note that $\displaystyle\lim_{(x,y)\to(0,0)} g(x,\,y) \neq 0$ because

$$\lim_{x\to 0} g(x,\,x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Therefore g is not continuous at $(0,\,0)$.

C13S0M.004: Using the very definition of partial derivative, we have

$$g_x(0,\,0) = \lim_{h\to 0} \frac{g(h,\,0)-g(0,\,0)}{h} = \lim_{h\to 0} \frac{h\cdot 0 - 0}{h(h^2+0^2)} = 0$$

and

$$g_y(0,\,0) = \lim_{k\to 0} \frac{g(0,\,k)-g(0,\,0)}{k} = \lim_{k\to 0} \frac{0\cdot k - 0}{k(0^2+k^2)} = 0.$$

Again we see that the existence of $g_x(a,\,b)$ and $g_y(a,\,b)$ is no guarantee of the continuity of g at $(a,\,b)$.

C13S0M.005: If $f_x(x,\,y) = 2xy^3 + e^x \sin y$, then

$$f(x,\,y) = x^2y^3 + e^x \sin y + g(y).$$

Hence

$$f_y(x,\,y) = 3x^2y^2 + e^x \cos y + g'(y) = 3x^2y^2 + e^x \cos y + 1.$$

Therefore $g'(y) = 1$, and thus $g(y) = y + C$. Thus every solution of this problem has the form

$$f(x,\,y) = x^2y^3 + e^x \sin y + y + C$$

where C is a constant.

C13S0M.006: If $f_x(x,\,y) = 6xy^2$, then $f(x,\,y) = 3x^2y^2 + g(y)$ where—as the notation indicates—$g(y)$ is a function of y alone. Therefore

$$f_y(x, y) = 6x^2y + g'(y) \neq 8x^2y$$

no matter what the choice of g. So there is no such function f having continuous second-order partial derivatives.

C13S0M.007: The paraboloid is a level surface of $f(x, y, z) = x^2 + y^2 - z$ with gradient $\langle 2a, 2b, -1 \rangle$ at the point $(a, b, a^2 + b^2)$. The normal line L through that point has vector equation

$$\langle x, y, z \rangle = \langle a, b, a^2 + b^2 \rangle + t\langle 2a, 2b, -1 \rangle$$

and thus parametric equations

$$x = 2at + a, \quad y = 2bt + b, \quad z = a^2 + b^2 - t.$$

Suppose that this line passes through the point $(0, 0, 1)$. Set $x = 0$, $y = 0$, and $z = 1$ in the parametric equations of L, solve the third equation for t, then substitute the result in the other two equations to find that

$$2a(a^2 + b^2) - a = 0 = 2b(a^2 + b^2) - b.$$

It now follows that $a^2 + b^2 = \frac{1}{2}$ or $a = 0 = b$. Therefore the points on the paraboloid where the normal vector points at $(0, 0, 1)$ are the origin $(0, 0, 0)$ and the points on the circle formed by the intersection of the paraboloid and the horizontal plane $z = \frac{1}{2}$.

C13S0M.008: Let $f(x, y, z) = \sin xy + \sin yz + \sin xz - 1$. Then the surface S in question is the level surface $f(x, y, z) = 0$ of f. Next we compute

$$\nabla f(x, y, z) = \langle y\cos xy + z\cos xz, \; x\cos xy + z\cos yz, \; y\cos yz + x\cos xz \rangle,$$

which is normal to S at the point (x, y, z). Hence a normal at $P\left(1, \frac{1}{2}\pi, 0\right)$ is $\langle 0, 0, 1 + \frac{1}{2}\pi \rangle$. Therefore an equation of the plane tangent to S at P is $z = 0$.

C13S0M.009: Let $f(x, y, z) = x^2 + y^2 - z^2$. Then $\nabla f(x, y, z) = \langle 2x, 2y, -2z \rangle$ is normal to the cone at (x, y, z). So a normal at $P(a, b, c)$ is $\mathbf{n} = \langle a, b, -c \rangle$. An equation of the line through P with direction $\mathbf{n}$ is

$$\langle x, y, z \rangle = \langle a, b, c \rangle + t\langle a, b, -c \rangle;$$

that is,

$$x = a + ta, \quad y = b + tb, \quad z = c - tc.$$

When $x = y = 0$, we have $t = -1$, and thus $z = 2c$. Hence $(0, 0, 2c)$ is the point where the line meets the z-axis. Thus the normal vector $\mathbf{n}$ (extended in length, if necessary) intersects the z-axis.

C13S0M.010: If

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right),$$

then

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{x^2 - 2kt}{8kt^2\sqrt{\pi kt}} \exp\left(-\frac{x^2}{4kt}\right), \\
\frac{\partial u}{\partial x} &= -\frac{x}{4kt\sqrt{\pi kt}} \exp\left(-\frac{x^2}{4kt}\right), \qquad \text{and} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{(x^2 - 2kt)}{8k^2t^2\sqrt{\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).
\end{aligned}$$

Therefore $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$.

C13S0M.011: If

$$u(x,\,y,\,t) = \frac{1}{4\pi kt} \exp\left(-\frac{x^2+y^2}{4kt}\right),$$

then

$$\begin{aligned}
u_t(x,\,y,\,t) &= \frac{x^2+y^2-4kt}{16k^2\pi t^3} \exp\left(-\frac{x^2+y^2}{4kt}\right), \\
u_x(x,\,y,\,t) &= -\frac{x}{8k^2\pi t^2} \exp\left(-\frac{x^2+y^2}{4kt}\right), \\
u_{xx}(x,\,y,\,t) &= \frac{x^2-2kt}{16k^3\pi t^3} \exp\left(-\frac{x^2+y^2}{4kt}\right), \\
u_y(x,\,y,\,t) &= -\frac{y}{8k^2\pi t^2} \exp\left(-\frac{x^2+y^2}{4kt}\right), \\
u_{yy}(x,\,y,\,t) &= \frac{y^2-2kt}{16k^3\pi t^3} \exp\left(-\frac{x^2+y^2}{4kt}\right), \qquad \text{and} \\
u_{xx}(x,\,y,\,t) + u_{yy}(x,\,y,\,t) &= \frac{x^2+y^2-4kt}{16k^3\pi t^3} \exp\left(-\frac{x^2+y^2}{4kt}\right).
\end{aligned}$$

Therefore $u_t = k(u_{xx} + u_{yy})$.

C13S0M.012: Given: $f(x,\,y,\,z) = (xyz)^{1/5}$. If $\mathbf{u} = \langle a,\,b,\,c\rangle$, then

$$D_{\mathbf{u}}f(0,\,0,\,0) = \lim_{t\to 0} \frac{(abct^3)^{1/5}}{t} = \lim_{t\to 0} \frac{(abc)^{1/5}}{t^{2/5}},$$

which exists if and only if $abc = 0$. —C.H.E.

C13S0M.013: If $\mathbf{r}(x,\,y) = \langle\, x,\,y,\,f(x,\,y)\,\rangle$, then

$$\mathbf{r}_x(x,\,y) = \langle\, 1,\,0,\,f_x(x,\,y)\,\rangle \quad \text{and} \quad \mathbf{r}_y(x,\,y) = \langle\, 0,\,1,\,f_y(x,\,y)\,\rangle,$$

and hence

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x,\,y) \\ 0 & 1 & f_y(x,\,y) \end{vmatrix} = \langle\, -f_x(x,\,y),\ -f_y(x,\,y),\ 1\,\rangle.$$

Let $g(x,\,y,\,z) = z - f(x,\,y)$. Then

$$\nabla g(x,\,y,\,z) = \langle\, -f_x(x,\,y),\ -f_y(x,\,y),\ 1\,\rangle = \mathbf{r}_x \times \mathbf{r}_y\,,$$

and therefore $\mathbf{r}_x \times \mathbf{r}_y$ is normal to the surface $z = f(x,\,y)$.

C13S0M.014: Suppose that the bottom of the box measures x by y (units are in centimeters) and that the box has height z. We are to maximize box volume $V(x,\,y,\,z) = xyz$ given the side condition $A(x,\,y,\,z) = xy + 2xz + 2yz - 300 = 0$. The Lagrange multiplier equations are

$$yz = \lambda(y + 2z), \quad xz = \lambda(x + 2z), \quad \text{and} \quad xy = \lambda(2x + 2y).$$

Because $\lambda \neq 0$ at maximum volume and x, y, and z are all positive, we may eliminate λ by writing

$$\frac{1}{\lambda} = \frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy},$$

and it follows (after obtaining the common denominator xyz) that

$$xy + 2xz = xy + 2yz = 2xz + 2yz.$$

Therefore at maximum, $x = y = 2z$. The condition $A(x,\,y,\,z) = 0$ then implies that $x = y = 10$ and $z = 5$. To maximize the volume of the box, its base should be a square 10 cm on a side and its height should be 5 cm. The maximum possible volume thereby obtained will be $V(10,\,10,\,5) = 500\ \text{cm}^3$.

C13S0M.015: Suppose that the bottom of the crate measures x by y (units are in feet and dollars) and that its height is z. We are to minimize its total cost $C(x,\,y,\,z) = 5xy + 2xz + 2yz$ given the constraint $V(x,\,y,\,z) = xyz - 60 = 0$. The Lagrange multiplier equations are

$$5y + 2z = \lambda yz, \quad 5x + 2z = \lambda xz, \quad \text{and} \quad 2x + 2y = \lambda xy.$$

Because x, y, and z are all positive (because of the constraint) and $\lambda \neq 0$, it follows that

$$\lambda xyz = 5xy + 2xz = 5xy + 2yz = 2xz + 2yz,$$

and thus $5x = 5y = 2z$, so that $x = y = \frac{2}{5}z$. Substitution in the constraint yields $z = 5 \cdot 3^{1/3}$, then $x = y = 2 \cdot 3^{1/3}$. The base of the shipping crate will be a square $2 \cdot 3^{1/3} \approx 2.884449914$ feet on each side and the height of the crate will be $5 \cdot 3^{1/3} \approx 7.21124785$ feet. (Its cost will be \$124.81!)

C13S0M.016: Let $(a,\,b,\,c)$ denote the point at which the plane is tangent to the surface $xyz = 1$; note that $abc = 1$. Let $f(x,\,y,\,z) = xyz - 1$; then $\nabla f(x,\,y,\,z) = \langle\, yz,\, xz,\, xy\,\rangle$, and hence a normal to the plane is

$$\nabla f(a,\,b,\,c) = \langle\, bc,\, ac,\, ab\,\rangle.$$

Therefore an equation of the plane is $bc(x - a) + ac(y - b) + ab(z - c) = 0$; that is, $bcx + acy + abz = 3$. Hence the intercepts of the plane are

$$x_0 = \frac{3}{bc}, \quad y_0 = \frac{3}{ac}, \quad \text{and} \quad z_0 = \frac{3}{ab}.$$

Because a pyramid is a special case of a cone, the volume of this pyramid is one-third the product of the area of its base and its height. Hence the volume of the pyramid is

$$V = \frac{1}{3} \cdot \frac{1}{2} x_0 y_0 z_0 = \frac{9}{2a^2b^2c^2} = \frac{9}{2}.$$

C13S0M.017: Given

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2},$$

we differentiate implicitly with respect to R_1 and thereby find that

$$-\frac{1}{R^2} \cdot \frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}, \quad \text{and thus} \quad \frac{\partial R}{\partial R_1} = \left(\frac{R}{R_1}\right)^2.$$

Similarly,

$$\frac{\partial R}{\partial R_2} = \left(\frac{R}{R_2}\right)^2.$$

Hence if the errors in measuring R_1 and R_2 are $\Delta R_1 = 3$ and $\Delta R_2 = 6$, we estimate the resulting error in computation of R as follows:

$$\begin{aligned} \Delta R &\approx \frac{\partial R}{\partial R_1} \cdot \Delta R_1 + \frac{\partial R}{\partial R_2} \cdot \Delta R_2 = \left(\frac{R}{R_1}\right)^2 \Delta R_1 + \left(\frac{R}{R_2}\right)^2 \Delta R_2 \\ &= \left(\frac{200}{300}\right)^2 \cdot 3 + \left(\frac{200}{600}\right)^2 \cdot 6 = \frac{4}{9} \cdot 3 + \frac{1}{9} \cdot 6 = 2. \end{aligned}$$

In fact, when $R_1 = 303$ and $R_2 = 606$, the value of R is exactly 202, so there is no error in this approximation.

C13S0M.018: Please refer to Problem 67 of Section 13.4, in which we are given van der Waals' equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = (82.06)T; \tag{1}$$

p, V, and T denote the pressure (in atm), volume (in cm^3), and temperature (in kelvins) of 1 mol of a gas. Let us first differentiate Eq. (1) implicitly with respect to p, then solve for V_p:

$$\begin{aligned} &\left(1 - \frac{2a}{V^3} \cdot V_p\right)(V - b) + \left(p + \frac{a}{V^2}\right) V_p = 0; \\ &V - \frac{2a}{V^2} V_p - b + \frac{2ab}{V^3} V_p + pV_p + \frac{a}{V^2} V_p = 0; \\ &\left(p + \frac{2ab}{V^3} - \frac{a}{V^2}\right) V_p = b - V; \\ &V_p = \frac{V^3(V - b)}{aV - 2ab - pV^3}. \end{aligned}$$

Similarly,

$$V_T = \frac{(82.06)V^3}{pV^3 - aV + 2ab}.$$

Now we use the linear approximation $\Delta V \approx dV = V_T\ \Delta T + V_p\ \Delta p$. We take $\Delta p = 0.1,\ p = 1,\ T = 313,$ $\Delta T = -10,\ a = 3.59 \times 10^6,\ b = 42.7,$ and $V = 25600$. We find that $\Delta V \approx -3394.86\ \text{cm}^3$. (The true value is approximately $-3098.264\ \text{cm}^3$.)

C13S0M.019: Suppose that $a,\ b,$ and c are positive. The ellipsoid with equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

and, therefore, semiaxes of lengths $a,\ b,$ and c, has volume $V = \frac{4}{3}\pi abc$. Let

$$V(x,\ y,\ z) = \frac{4}{3}\pi xyz.$$

Then

$$V_x(x,\ y,\ z) = \frac{4}{3}\pi yz, \quad V_y(x,\ y,\ z) = \frac{4}{3}\pi xz, \quad \text{and} \quad V_z(x,\ y,\ z) = \frac{4}{3}\pi xy.$$

Assume that errors of at most 1% are made in measuring $x,\ y,$ and z. Then the error in computing V will be at most

$$\Delta V \approx dV = V_x\ dx + V_y\ dy + V_z\ dz = \frac{4}{3}\pi yz \cdot \frac{x}{100} + \frac{4}{3}\pi xz \cdot \frac{y}{100} + \frac{4}{3}\pi xy \cdot \frac{z}{100} = \frac{1}{25}\pi xyz,$$

and hence the percentage error in computing V will be at most

$$100 \cdot \frac{\Delta V}{V} \approx 100 \cdot \frac{1}{25}\pi xyz \cdot \frac{3}{4\pi xyz} = 3;$$

that is, the maximum error is approximately 3%.

C13S0M.020: Assume that the first sphere S_1 is the level surface $f(x,\ y,\ z) = x^2 + y^2 + z^2 - a^2 = 0$ and that the second sphere is the level surface $g(x,\ y,\ z) = (x-c)^2 + y^2 + z^2 - b^2 = 0$. Set $z = 0$ to obtain the intersection of the two spheres with the xy-plane. The resulting two circles meet at two points, one of why (P, say) has coordinates

$$x = \frac{a^2 - b^2 + c^2}{2c}, \quad y = \sqrt{a^2 - x^2}, \quad z = 0.$$

Now $\nabla F = \langle\, 2x,\ 2y,\ 2z\,\rangle$ is normal to S_1 at P and $\nabla g = \langle\, 2(x-c),\ 2y,\ 2z\,\rangle$ is normal to S_2 at P. Because the angle θ between the two planes is the same as the angle between ∇f and ∇g, we find that

$$\cos\theta = \frac{(\nabla f)\cdot(\nabla g)}{|\,\nabla f\,|\cdot|\,\nabla g\,|} = \frac{a^2 - cx}{ab} = \frac{a^2 + b^2 - c^2}{2ab}.$$

C13S0M.021: The ellipsoidal surface S is the level surface

$$f(x,\ y,\ z) = x^2 + 4y^2 + 9z^2 - 16 = 0,$$

and hence a vector normal to S at the point (x, y, z) is $\mathbf{n} = \boldsymbol{\nabla} f = \langle 2x, 8y, 18z \rangle$. If $\mathbf{n}$ is parallel to the position vector $\langle x, y, z \rangle$, then $\mathbf{n}$ (extended if necessary) will pass through $(0, 0, 0)$. This leads to the equation

$$\langle 2x, 8y, 18z \rangle = \lambda \langle x, y, z \rangle;$$

that is,

$$2x = \lambda x, \quad 8y = \lambda y, \quad 18z = \lambda z. \tag{1}$$

Multiply both sides of the first equation by yz, the second by xz, and the third by xy to eliminate λ:

$$\lambda xyz = 2xyz = 8xyz = 18xyz.$$

We obtain a contradiction if $xyz \neq 0$, and thus at least one of x, y, and z is zero. If (say) $x \neq 0$, then multiply the second equation in (1) by z and the third by y to obtain $\lambda yz = 8yz = 18yz$. If $yz \neq 0$ we obtain a contradiction, so at least one of y and z is zero. Repeating with the other two cases, we conclude that at least two of x, y, and z are zero. Of course the third cannot be zero because of the condition $f(x, y, z) = 0$. Therefore there are six points on S at which the normal vector points toward the origin; they are $(\pm 4, 0, 0)$, $(0, \pm 2, 0)$, and $\left(0, 0, \pm\frac{4}{3}\right)$.

C13S0M.022: Write

$$w = w(u, v) = \int_u^v f(t)\, dt$$

where $u = g(x)$ and $v = h(x)$. Then $F(x) = w(u(x), v(x))$. Consequently

$$F'(x) = w_u u_x + w_v v_x = -f(u)g'(x) + f(v)h'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x).$$

C13S0M.023: Given: $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular unit vectors in space and f is a function of the three variables x, y, and z. Rename the unit vectors if necessary so that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ forms a right-handed triple. Then

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}, \quad \mathbf{b} \times \mathbf{c} = \mathbf{a}, \quad \text{and} \quad \mathbf{c} \times \mathbf{a} = \mathbf{b}. \tag{1}$$

Write $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Then

$$\begin{aligned}
&\mathbf{a} D_{\mathbf{a}} f + \mathbf{b} D_{\mathbf{b}} f + \mathbf{c} D_{\mathbf{c}} f = \mathbf{a}\big(\boldsymbol{\nabla} f\big) \cdot \mathbf{a} + \mathbf{b}\big(\boldsymbol{\nabla} f\big) \cdot \mathbf{b} + \mathbf{c}\big(\boldsymbol{\nabla} f\big) \cdot \mathbf{c} \\
&\quad = (a_1 f_x + a_2 f_y + a_3 f_z)\langle a_1, a_2, a_3 \rangle + (b_1 f_x + b_2 f_y + b_3 f_z)\langle b_1, b_2, b_3 \rangle + (c_1 f_x + c_2 f_y + c_3 f_z)\langle c_1, c_2, c_3 \rangle.
\end{aligned}$$

If the scalar multiplications and vector additions in the last line are carried out, the resulting vector will have first component

$$(a_1^2 + b_1^2 + c_1^2) f_x + (a_1 a_2 + b_1 b_2 + c_1 c_2) f_y + (a_1 a_3 + b_1 b_3 + c_1 c_3) f_z. \tag{2}$$

By Eq. (1),

$$c_3 = a_1 b_2 - a_2 b_1, \quad a_3 = b_1 c_2 - b_2 c_1, \quad \text{and} \quad b_3 = c_1 a_2 - c_2 a_1.$$

Hence substitution for a_3, b_3, and c_3 in Eq. (2) yields

$$(a_1^2+b_1^2+c_1^2)f_x+(a_1a_2+b_1b_2+c_1c_2)f_y+(a_1b_1c_2-a_1b_2c_1+b_1c_1a_2-b_1c_2a_1+c_1a_1b_2-c_1a_2b_1)f_z$$
$$=(a_1^2+b_1^2+c_1^2)f_x+(a_1a_2+b_1b_2+c_1c_2)f_y+0\cdot f_z.$$

Also by Eq. (1),

$$a_2=b_3c_1-b_1c_3,\quad b_2=c_3a_1-c_1a_3,\quad \text{and}\quad c_2=a_3b_1-a_1b_3.$$

Hence substitution for $a_2,\ b_2,$ and c_2 in the last expression yields

$$\begin{aligned}
&(a_1^2+b_1^2+c_1^2)f_x+(a_1a_2+b_1b_2+c_1c_2)f_y\\
&\qquad=(a_1^2+b_1^2+c_1^2)f_x+(a_1b_3c_1-a_1b_1c_3+b_1c_3a_1-b_1c_1a_3+c_1a_3b_1-c_1a_1b_3)f_y\\
&\qquad=(a_1^2+b_1^2+c_1^2)f_x+0\cdot f_y=(a_1^2+b_1^2+c_1^2)f_x.
\end{aligned}$$

Moreover, Eq. (1) implies that

$$a_1=b_2c_3-b_3c_2,\quad b_1=c_2a_3-c_3a_2,\quad \text{and}\quad c_1=a_2b_3-a_3b_2.$$

Hence substitution for *only one* of each of $a_1,\ b_1,$ and c_1 in the last expression yields

$$(a_1b_2c_3-a_1b_3c_2+b_1c_2a_3-b_1c_3a_2+c_1a_2b_3-c_1a_3b_2)f_x=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})f_x=f_x$$

by Eq. (17) of Section 11.3. Similarly, the second component of $\mathbf{a}D_{\mathbf{a}}f+\mathbf{b}D_{\mathbf{b}}f+\mathbf{c}D_{\mathbf{c}}f$ is f_y and its third component is f_z. Therefore $\nabla f=\mathbf{a}D_{\mathbf{a}}f+\mathbf{b}D_{\mathbf{b}}f+\mathbf{c}D_{\mathbf{c}}f$. ◀

C13S0M.024: Given: $\mathbf{R}=\langle\cos\theta,\,\sin\theta,\,0\rangle$ and $\mathbf{\Theta}=\langle-\sin\theta,\,\cos\theta,\,0\rangle$, if $f(x,\,y,\,z)=w(r,\,\theta,\,z)$ then

$$D_{\mathbf{R}}f=\big(\nabla f\big)\cdot\mathbf{R}=\langle f_x,\,f_y,\,f_z\rangle\cdot\langle\cos\theta,\,\sin\theta,\,0\rangle=f_x\cos\theta+f_y\sin\theta.$$

Then, using the facts that $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$w_r=f_xx_r+f_yy_r+f_zz_r=f_x\cos\theta+f_y\sin\theta.$$

Therefore, $D_{\mathbf{R}}f=w_r$. Next,

$$D_{\mathbf{\Theta}}f=\langle f_x,\,f_y,\,f_z\rangle\cdot\langle-\sin\theta,\,\cos\theta,\,0\rangle=-f_x\sin\theta+f_y\cos\theta$$

and

$$\frac{1}{r}w_\theta=\frac{1}{r}(f_xx_\theta+f_yy_\theta+f_zz_\theta)=\frac{1}{r}\left[f_x\cdot(-r\sin\theta)+f_y\cdot(r\cos\theta)\right]=-f_x\sin\theta+f_y\cos\theta.$$

Therefore $D_{\mathbf{\Theta}}f=\dfrac{1}{r}w_\theta$. Also,

$$D_{\mathbf{k}}f=w_zz_r+w_zz_\theta+w_zz_z=w_z.$$

Thus, by the result in Problem 23,

$$\nabla f=\mathbf{R}D_{\mathbf{R}}f+\mathbf{\Theta}D_{\mathbf{\Theta}}f+\mathbf{k}D_{\mathbf{k}}f=\frac{\partial w}{\partial r}\mathbf{R}+\frac{1}{r}\cdot\frac{\partial w}{\partial\theta}\mathbf{\Theta}+\frac{\partial w}{\partial z}\mathbf{k}.$$

C13S0M.025: If $f(x, y) = 500 - (0.003)x^2 - (0.004)y^2$, then

$$\nabla f(x, y) = \left\langle -\frac{3x}{500}, -\frac{4y}{500} \right\rangle, \quad \text{so} \quad \nabla f(-100, -100) = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

To maintain a constant altitude, you should move in a direction normal to the gradient vector; that is, you should initially move in the direction of either $\langle -4, 3 \rangle$ or $\langle 4, -3 \rangle$.

C13S0M.026: First, $\nabla f = -2k\langle x, 2y \rangle f(x, y)$, and the shark's path is described by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. The latter implies that the shark's direction is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$. That $\mathbf{v}$ is parallel to ∇f implies that $\langle x'(t), y'(t) \rangle$ is also parallel to $\langle x, 2y \rangle$. Therefore

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \lambda \langle x, 2y \rangle$$

for some scalar λ. Consequently

$$\lambda = \frac{1}{x} \cdot \frac{dx}{dt} = \frac{1}{2y} \cdot \frac{dy}{dt}.$$

Now forget λ and solve the differential equation:

$$C + \ln x = \frac{1}{2} \ln y;$$

$$\ln y = 2 \ln x + 2C;$$

$$y(x) = cx^2 : \qquad \text{a parabola.}$$

C13S0M.027: The given surface S is the level surface

$$f(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 1 = 0,$$

and

$$\nabla f(x, y, z) = \left\langle \frac{2}{3x^{1/3}}, \frac{2}{3y^{1/3}}, \frac{2}{3z^{1/3}} \right\rangle.$$

Hence the plane $\mathcal{P}$ tangent to S at the point $P(a, b, c)$ has normal vector

$$\mathbf{n} = \nabla f(a, b, c) = \left\langle \frac{2}{3a^{1/3}}, \frac{2}{3b^{1/3}}, \frac{2}{3c^{1/3}} \right\rangle,$$

and thus an equation of $\mathcal{P}$ is

$$\frac{x-a}{a^{1/3}} + \frac{y-b}{b^{1/3}} + \frac{z-c}{c^{1/3}} = 0.$$

Set x and y equal to zero to find the z-intercept of $\mathcal{P}$:

$$\frac{z-c}{c^{1/3}} = a^{2/3} + b^{2/3}, \quad \text{so that} \quad z = (a^{2/3} + b^{2/3} + c^{2/3})c^{1/3} = c^{1/3}.$$

Similarly, the x-intercept of $\mathcal{P}$ is $a^{1/3}$ and its y-intercept is $b^{1/3}$. So the sum of the squares of its intercepts is $a^{2/3} + b^{2/3} + c^{2/3} = 1$.

C13S0M.028: The given ellipse E with equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(a and b are positive and $a \neq b$) is the level curve

$$f(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 = 0,$$

and thus its normal vector at (x, y) is

$$\mathbf{n}(x, y) = \boldsymbol{\nabla} f(x, y) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2} \right\rangle.$$

This vector (extended, if necessary) will pass through the origin provided that it is parallel to the position vector $\mathbf{r}(x, y) = \langle x, y \rangle$; that is,

$$\left\langle \frac{2x}{a^2}, \frac{2y}{b^2} \right\rangle = \lambda \langle x, y \rangle$$

for some scalar λ. This leads to the simultaneous equations

$$\frac{2x}{a^2} = \lambda x \quad \text{and} \quad \frac{2y}{b^2} = \lambda y.$$

If $x \neq 0$ and $y \neq 0$, this leads to the contradiction that $a = b$. Hence one of x and y is zero (and they cannot both be zero). Hence there are four points on the ellipse E at which the normal vector, if extended, passes through the origin; they are the vertices $(\pm a, 0)$ and $(0, \pm b)$ of E.

C13S0M.029: Given:

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$; $f(0, 0) = 0$. Then

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0.$$

Also $f_y(0, 0) = 0$ by a very similar computation. If $(x, y) \neq (0, 0)$, then

$$f_x(x, y) = \frac{2xy^4}{(x^2 + y^2)^2} \quad \text{and} \quad f_y(x, y) = \frac{2x^4 y}{(x^2 + y^2)^2},$$

and (using polar coordinates)

$$\lim_{(x,y) \to (0,0)} f_x(x, y) = \lim_{r \to 0} \frac{2r^5 \cos\theta \sin^4\theta}{r^4} = 0,$$

so that f_x is continuous at $(0, 0)$ (as is f_y, by a similar argument). Hence both first-order partial derivatives are continuous everywhere, and therefore f is differentiable at the origin. Finally, $f(x, y) \geqq 0 = f(0, 0)$, so $0 = f(0, 0)$ is a local (indeed, the global) minimum value of $f(x, y)$. —C.H.E.

C13S0M.030: We minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = xy + 1 - z = 0$. The Lagrange multiplier equations are

$$2x = \lambda y, \quad 2y = \lambda x, \quad \text{and} \quad 2z = -\lambda.$$

The *Mathematica* 3.0 command

```
Eliminate[ { 2*x == lambda*y, 2*y == lambda*x, 2*z == -lambda }, { lambda } ]
```

yields the response (rewritten slightly, as usual)

$$x^2 = y^2 \quad \text{and} \quad yz = -x \quad \text{and} \quad xz = -y.$$

Then the command

```
Solve[ { x*x == y*y, y*z == -x, x*z == -y, g[x,y,z] == 0 }, {x, y, z} ]
```

produces three solutions, and the one involving only real numbers is $x = y = 0, \ z = 1$. Because there is no point on the surface $z = xy + 1$ farthest from the origin, we have found the unique closest point; it is $(0, 0, 1)$.

C13S0M.031: The absence of first-degree terms in its equation implies that the center of the given rotated ellipse $73x^2 + 72xy + 52y^2 = 100$ is the origin. Thus to find its semiaxes, we maximize and minimize $f(x, y) = x^2 + y^2$ subject to the constraint

$$g(x, y) = 73x^2 + 72xy + 52y^2 - 100 = 0.$$

The Lagrange multiplier equations are

$$2x = \lambda(146x + 72y) \quad \text{and} \quad 2y = \lambda(72x + 104y).$$

The methods of Problem 38 of Section 13.9 lead to the simultaneous equations

$$(1 - 73\lambda)x - 36\lambda y = 0,$$
$$36\lambda x + (52\lambda - 1)y = 0.$$

Because this system has a nontrivial solution, the determinant of its coefficient matrix must be zero:

$$\begin{vmatrix} 1 - 73\lambda & -36\lambda \\ 36\lambda & 52\lambda - 1 \end{vmatrix} = -2500\lambda^2 + 125\lambda - 1 = 0,$$

and hence $(100\lambda - 1)(25\lambda - 1) = 0$.

Case 1: $\lambda = \frac{1}{100}$. Then the simultaneous equations shown here lead to $3x = 4y$; then the constraint equation yields

$$(x_1, y_1) = \left(-\frac{4}{5}, -\frac{3}{5}\right) \quad \text{and} \quad (x_2, y_2) = \left(\frac{4}{5}, \frac{3}{5}\right).$$

Case 2: $\lambda = \frac{1}{25}$. Then the simultaneous equations shown here lead to $4x + 3y = 0$; then the constraint equation yields

$$(x_3, y_3) = \left(-\frac{6}{5}, \frac{8}{5}\right) \quad \text{and} \quad (x_4, y_4) = \left(\frac{6}{5}, -\frac{8}{5}\right).$$

Thus these four points are the endpoints of the semiaxes of the ellipse; the minor semiaxis (Case 1) has length 1 and the major semiaxis (Case 2) has length 2.

C13S0M.032: Assume (without loss of generality) that (1, 0, 0) is one endpoint of the longest chord of the sphere $x^2 + y^2 + z^2 = 1$. To find the other endpoint, we maximize

$$f(x,\, y,\, z) = (x-1)^2 + y^2 + z^2 \quad \text{given} \quad g(x,\, y,\, z) = x^2 + y^2 + z^2 - 1 = 0.$$

The Lagrange multiplier equations are

$$2(x-1) = 2\lambda x, \quad 2y = 2\lambda y, \quad 2z = 2\lambda z.$$

If either y or z is nonzero, this leads to $\lambda = 1$; then the first equation yields a contradiction. Hence $y = 0$, $z = 0$, and $x = \pm 1$. If $x = 1$ then we have found the shortest chord of the sphere; it has length zero. Hence the longest chord has endpoints (1, 0, 0) and $(-1, 0, 0)$ and its length is 2.

If you prefer not to assume that one endpoint of one of the longest chord is at (1, 0, 0), assume instead that it is at the point $(a,\, b,\, c)$ and let its other endpoint be $(x,\, y,\, z)$. Then maximize (and minimize)

$$f(x,\, y,\, z) = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad \text{given} \quad g(x,\, y,\, z) = x^2 + y^2 + z^2 - 1 = 0.$$

You can solve this problem quickly and easily with a computer algebra program. In *Mathematica* 3.0, after f and g are defined, proceed something like this.

```
delf = { D[f[x,y,z],x], D[f[x,y,z],y], D[f[x,y,z],z] }
```

$$\{\, 2(x-a),\ 2(y-b),\ 2(z-c) \,\}$$

```
delg = { D[g[x,y,z],x], D[g[x,y,z],y], D[g[x,y,z],z] }
```

$$\{\, 2x,\ 2y,\ 2z \,\}$$

```
Eliminate[ delf == lambda*delg, lambda }]
```

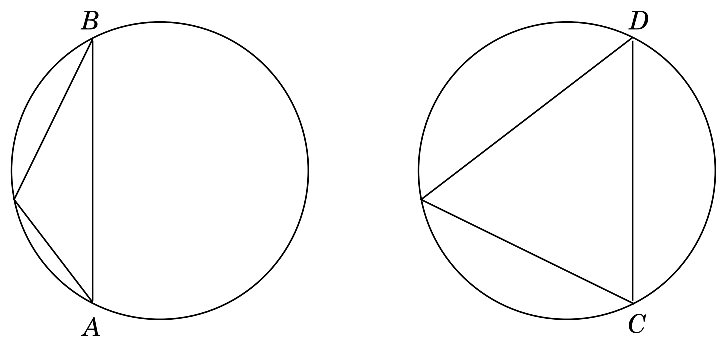
$$bz = cy \quad \text{and} \quad az = cx \quad \text{and} \quad ay = bx$$

```
Solve[ { b*z == c*y, a*z == c*x, a*y == b*x, g[a,b,c] == 0 }, { x, y, z } ]
```

$$\{\{\, x \to -a,\ y \to -b,\ z \to -c \,\},\ \{\, x \to a,\ y \to b,\ z \to c \,\}\}$$

C13S0M.033: Let us find both the maximum and the minimum perimeter of a triangle inscribed in the unit circle. The center of the circle will be within the triangle of maximum perimeter or on its boundary by the following argument. The following figure on the left shows a triangle with the center of the circle outside it. By moving the chord AB to the new position CD shown next on the right, you increase the perimeter of the circle and now the center of the circle is within the triangle. (Note: CD and AB are parallel and have the same length.) Similarly, the center of the circle is outside the triangle of minimum perimeter, for

if inside—as in the figure on the right—move the chord CD to position AB to obtain a triangle of smaller perimeter with the center of the circle now outside the triangle.



Thus to find the triangle of maximum perimeter, use the notation in Fig. 13.9.9 and maximize the perimeter

$$p(\alpha,\,\beta,\,\gamma) = \sqrt{2-2\cos\alpha}\ + \sqrt{2-2\cos\beta}\ + \sqrt{2-2\cos\gamma}\ = 2\sin\frac{\alpha}{2} + 2\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2}$$

given the constraint $g(\alpha,\,\beta,\,\gamma) = \alpha+\beta+\gamma-2\pi = 0$. The Lagrange multiplier equations are

$$\cos\frac{\alpha}{2} = \lambda, \qquad \cos\frac{\beta}{2} = \lambda, \qquad \cos\frac{\gamma}{2} = \lambda,$$

and therefore, because $f(x) = \cos\left(\frac{1}{2}x\right)$ is single-valued on the interval $0 \leqq x \leqq 2\pi$, we may conclude that $\alpha = \beta = \gamma$. Then the constraint equation implies that $\alpha = \beta = \gamma = \frac{2}{3}\pi$, and thus we find that the triangle of maximum perimeter inscribed in the unit circle is equilateral and its perimeter is

$$p\left(\frac{2}{3}\pi,\,\frac{2}{3}\pi,\,\frac{2}{3}\pi\right) = 3\sqrt{3}\ \approx 5.1961524227066319.$$

Next we seek the triangle of minimum perimeter. The figure resembles the one on the left in the preceding illustration. Let C denote the third vertex of the triangle. Let α be the central angle that subtends the arc BC, let β be the central angle that subtends the arc CA, and let γ be the central angle that subtends the *short* arc BA. Then we are to minimize the perimeter of the triangle, given by

$$p(\alpha,\,\beta,\,\gamma) = 2\sin\frac{\alpha}{2} + 2\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2}$$

(exactly as in the maximum-perimeter investigation); the constraint is now $\alpha+\beta-\gamma = 0$, and we note that $0 \leqq \gamma \leqq \pi$ as well. The Lagrange multiplier equations are

$$\cos\frac{\alpha}{2} = \lambda, \qquad \cos\frac{\beta}{2} = \lambda, \qquad \cos\frac{\gamma}{2} = -\lambda.$$

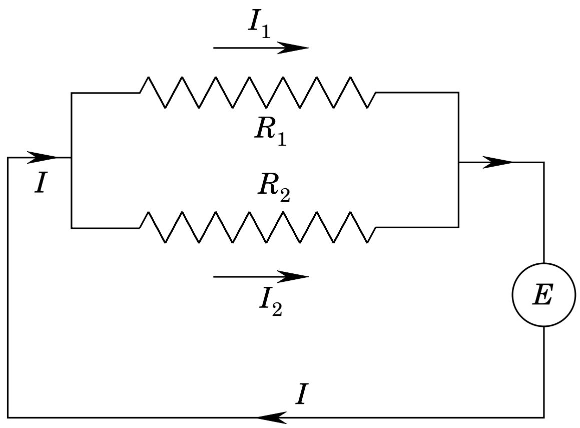
But all three of $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$ lie in the first quadrant, so these equations are impossible unless $\lambda = 0$, in which case $\alpha = \beta = \pi$ and $\gamma = 2\pi$ or, equivalently in this case, $\gamma = 0$. This implies that AC and BC are the same diameter of the circle and that AB is one endpoint of that diameter. Thus we obtain a degenerate triangle of perimeter $|AC| + |BC| = 4$. But this is not the minimum perimeter; the minimum occurs at a boundary point extremum missed by the Lagrange multiplier method.

To find the minimum, rewrite the perimeter function in the form

$$p(\alpha,\,\beta) = 2\sin\frac{\alpha}{2} + 2\sin\frac{\beta}{2} + 2\sin\frac{\alpha+\beta}{2}, \quad 0 \leqq \alpha, \;\; 0 \leqq \beta, \;\; \alpha+\beta \leqq \pi.$$

There is no point in setting both partial derivatives of p equal to zero; remember, we are seeking a boundary extremum. Hence we examine the boundary of the domain of $p(\alpha, \beta)$ to find the minimum. One possibility is that $\alpha = \beta = 0$. In this case $\gamma = 0$ as well. If so, then the triangle is totally degenerate; its three vertices all coincide at a single point on the circumference of the circle, its three sides all have length zero, and its perimeter is (finally, the global minimum) zero.

C13S0M.034: Please refer to the following circuit diagram.



Let $x = I_1$ and $y = I_2$. We minimize $f(x, y) = R_1x^2 + R_2y^2$ given the constraint $g(x, y) = x + y - I = 0$ (I, R_1, and R_2 are all constants). The Lagrange multiplier equations are

$$2R_1x = \lambda \qquad \text{and} \qquad 2R_2y = \lambda,$$

and thus

$$R_1x = R_2y = R_2(I - x);$$

$$R_1x + R_2x = R_2I;$$

$$I_1 = x = \frac{R_2I}{R_1 + R_2}; \qquad \text{similarly,}$$

$$I_2 = y = \frac{R_1I}{R_1 + R_2}.$$

The net resistance R of the two resistors in parallel satisfies the equation $E = IR$, but $E = R_1I_1 = R_2I_2$ (we are using Ohm's law and Kirchhoff's laws), and therefore

$$IR = \frac{R_1R_2I}{R_1 + R_2};$$

$$R = \frac{R_1R_2}{R_1 + R_2};$$

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1R_2} = \frac{1}{R_1} + \frac{1}{R_2}.$$

C13S0M.035: First we make sure that the line and the ellipse do not intersect. Substitution of $y = 2 - x$ in the equation of the ellipse yields

$$x^2 + 2(2-x)^2 = 1; \quad 3x^2 - 8x + 7 = 0.$$

The discriminiant of the quadratic is $\Delta = 64 - 84 < 0$, so the last equation has no real solutions. Therefore the ellipse and the line do not meet.

Next, following the *Suggestion,* we let

$$f(x, y, u, v) = (x-u)^2 + (y-v)^2$$

and maximize and minimize $f(x, y, u, v)$ subject to the two constraints

$$g(x, y, u, v) = x^2 + 2y^2 - 1 = 0 \quad \text{and} \quad h(x, y, u, v) = u + v - 2 = 0.$$

The Lagrange multiplier equations are

$$2(x-u) = 2\lambda x, \quad 2(y-v) = 4\lambda y, \quad 2(u-x) = \mu, \quad \text{and} \quad 2(v-y) = \mu.$$

Elimination of μ yields $u - x = v - y$. Then elimination of λ yields

$$2y(x-u) = x(y-v) \quad \text{and} \quad u + y = v + x.$$

Finally, solving the last two equations simultaneously with the two constraint equations yields the expected number of solutions—two. They are

$$x = -\frac{1}{3}\sqrt{6}, \quad y = -\frac{1}{6}\sqrt{6}, \quad u = \frac{12-\sqrt{6}}{6}, \quad v = \frac{12+\sqrt{6}}{6} \quad \text{and}$$
$$x = \frac{1}{3}\sqrt{6}, \quad y = \frac{1}{6}\sqrt{6}, \quad u = \frac{12+\sqrt{6}}{6}, \quad v = \frac{12-\sqrt{6}}{6}.$$

Because the ellipse is "southwest" of the line, the first solution listed here gives the point (x, y) of the ellipse farthest from the line and the second gives the point closest to the line.

C13S0M.036: Part (a): We maximize $f(x, y, z) = x + y + z$ subject to the constraint or side condition $g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$. The Lagrange multiplier equations are

$$1 = 2\lambda x, \quad 1 = 2\lambda y, \quad \text{and} \quad 1 = 2\lambda z.$$

Thus $\lambda \neq 0$, and therefore $x = y = z$. Substitution in the constraint yields two solutions:

$$x = y = z = -\frac{\sqrt{3}}{3}a \quad \text{and} \quad x = y = z = \frac{\sqrt{3}}{3}a.$$

Therefore the maximum value of $f(x, y, z)$ on the sphere is $a\sqrt{3}$.

Part (b): By Part (a), $|x + y + z| \leqq a\sqrt{3}$ for every point (x, y, z) on a sphere of radius a. Therefore

$$(x + y + z)^2 \leqq 3a^2 = 3(x^2 + y^2 + z^2)$$

for all points (x, y, z) on a sphere of radius a. Because a is an arbitrary nonnegative number and because every point (x, y, z) lies on some sphere, we may conclude that

$$(x + y + z)^2 \leqq 3(x^2 + y^2 + z^2)$$

for all real numbers $x,\ y,$ and $z.$

C13S0M.037: Let $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$ where n is a fixed positive integer. Suppose that a is a nonnegative real number. To maximize $f(x_1, x_2, \dots, x_n)$ subject to the constraint

$$g(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - a^2 = 0, \tag{1}$$

the Lagrange multiplier equations are

$$2\lambda x_i = 1, \quad 1 \leqq i \leqq n.$$

Hence $\lambda \neq 0,$ and thus $x_1 = x_2 = \cdots = x_n.$ Then the constraint equation implies that

$$nx_1^2 = a^2, \qquad \text{so that} \qquad x_1 = \pm\frac{a}{\sqrt{n}}.$$

Therefore the maximum value of f subject to the constraint in (1) is $nx_1 = a\sqrt{n}.$

It now follows that $|x_1+x_2+\ldots+x_n| \leqq a\sqrt{n}$ for every point $(x_1, x_2, \dots, x_n)$ satisfying Eq. (1). Therefore

$$(x_1 + x_2 + \cdots + x_n)^2 \leqq na^2 = n(x_1^2 + x_2^2 + \cdots + x_n^2)$$

for all numbers $x_1,\ x_2,\ \dots,\ x_n$ satisfying Eq. (1). But every such set of numbers satisfies Eq. (1) for some nonnegative value of $a,$ and a is arbitrary. Therefore

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leqq n\sum_{i=1}^{n} (x_i)^2 \tag{2}$$

for all n-tuples of real numbers $\{x_1,\ x_2,\ \dots,\ x_n\}.$ Note that equality holds *only* if $x_1 = x_2 = \cdots = x_n;$ this will be important in the solution of Problem 51. Finally, divide both sides in Eq. (2) by n^2 and take square roots to obtain the result

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leqq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

in Problem 37.

C13S0M.038: Given: $f(x, y) = xy - x - y$ on the plane triangle T with vertices at $(0, 0),\ (0, 1),$ and $(3, 0).$ When we set both partial derivatives equal to zero and solve, we obtain $(x, y) = (1, 1),$ which does not lie in $T.$ So the extrema—and there must be a global maximum and a global minimum—lie on the boundary of $T.$

• On the bottom of $T,$ we have $f(x, 0) = -x,$ so the only candidate for a maximum is $f(0, 0) = 0$ and the only candidate for a minimum is $f(3, 0) = -3.$

• On the left edge of $T,$ we have $f(0, y) = -y,$ so the only candidate for a maximum is $f(0, 0) = 0$ and the only candidate for a minimum is $f(0, 1) = -1.$

• On the inclined edge of $T,$ where $y = \frac{1}{3}(3 - x)$, we have

$$f\left(x,\ \frac{3-x}{3}\right) = \frac{x - x^2 - 3}{3},$$

a quadratic opening downward and with derivative $\frac{1}{3}(1 - 2x).$ The derivative is zero when $x = \frac{1}{2},$ so that $y = \frac{5}{6},$ and this point does lie on $T.$ So the only candidate for a maximum is $f\left(\frac{1}{2},\ \frac{5}{6}\right) = -\frac{11}{12};$ we have

already examined the behavior of f at the two endpoints of the inclined edge of T. Examination of the behavior of $f(x, y)$ on the vertical line $x = \frac{1}{2}$ establishes that there is no local extremum at $\left(\frac{1}{2}, \frac{5}{6}\right)$.

Therefore the maximum value of $f(x, y)$ on T is $f(0, 0) = 0$ and its minimum value is $f(3, 0) = -3$. At $(0, 1)$, $f(x, y)$ attains the local minimum value -1.

C13S0M.039: We are to find the extrema of $f(x, y, z) = x^2 - yz$ subject to the constraint or side condition $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. The Lagrange multiplier equations are

$$2x = 2\lambda x, \quad -z = 2\lambda y, \quad \text{and} \quad -y = 2\lambda z. \tag{1}$$

Thus

$$2xyz = 2\lambda xyz, \quad -xz^2 = 2\lambda xyz, \quad \text{and} \quad -xy^2 = 2\lambda xyz,$$

and hence $-xy^2 = -xz^2 = 2xyz$.

Case 1: $x = 0$. Multiply the second equation in (1) by z and the third by y to obtain $-z^2 = 2\lambda yz = -y^2$, so that $y^2 = z^2$. If $y = z$, then the constraint equation yields $y^2 = \frac{1}{2}$, and we obtain the two critical points

$$A\left(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right) \quad \text{and} \quad B\left(0, -\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right).$$

If $y = -z$, then the constraint equation yields $y^2 = \frac{1}{2}$, and we obtain two more critical points,

$$C\left(0, \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\right) \quad \text{and} \quad D\left(0, -\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right).$$

Case 2: $x \neq 0$. Then $y^2 = z^2 = -2yz$. If $z = y$ then $y^2 = -2y^2$, so that $y = z = 0$. If $z = -y$ then $y^2 = 2y^2$, and again $y = z = 0$. Then the constraint equation yields $x = \pm 1$, so in this case we obtain the two critical points

$$E(1, 0, 0) \quad \text{and} \quad F(-1, 0, 0).$$

The values of $f(x, y, z)$ at these six points are

$$f(A) = -\frac{1}{2} = f(B), \quad f(C) = \frac{1}{2} = f(D), \quad \text{and} \quad f(E) = f(F) = 1.$$

Therefore the maximum value of $f(x, y, z)$ is 1 and occurs at the two critical points E and F. Its minimum value is $-\frac{1}{2}$ and occurs at the two critical points A and B. Numerical experimentation with

$$h(y, z) = 1 - y^2 - z^2 + yz$$

for y and z near the critical points C and D shows that there is not even a local extremum at either of these points.

C13S0M.040: We seek the extrema of $f(x, y) = x^2y^2$ given the constraint $g(x, y) = x^2 + 4y^2 - 24 = 0$. The Lagrange multiplier equations are

$$2xy^2 = 2\lambda x \quad \text{and} \quad 2x^2y = 8\lambda y,$$

and elimination of λ yields the equation $x^3y = 4xy^3$.

Case 1: $x = 0$. Then the constraint equation yields the two critical points $A\left(0, \sqrt{6}\,\right)$ and $B\left(0, -\sqrt{6}\,\right)$.

Case 2: $y = 0$. Then the constraint equation yields the two critical points $C\left(2\sqrt{6}\,, 0\right)$ and $D\left(-2\sqrt{6}\,, 0\right)$.

Case 3: $x \neq 0$ and $y \neq 0$. Then the equation $x^3y = 4xy^3$ yields $x^2 = 4y^2$, so that $x = \pm 2y$. If $x = 2y$ then the constraint equation takes the form $y^2 = 3$, and we obtain the critical points $E\left(2\sqrt{3}\,, \sqrt{3}\,\right)$ and $F\left(-2\sqrt{3}\,, -\sqrt{3}\,\right)$. If $x = -2y$ then again $y^2 = 3$, and we obtain the critical points $G\left(2\sqrt{3}\,, -\sqrt{3}\,\right)$ and $H\left(-2\sqrt{3}\,, \sqrt{3}\,\right)$.

The corresponding values of $f(x, y)$ are zero at the critical points of Cases 1 and 2 and 36 at the critical points of Case 3. Hence the global maximum value of $f(x, y)$ is 36 and its global minimum value is zero.

C13S0M.041: If $f(x, y) = x^3y - 3xy + y^2$, then when we equate both partial derivatives to zero we obtain the equations

$$3y(x^2 - 1) = 0 \qquad \text{and} \qquad x^3 - 3x + 2y = 0.$$

The first equation holds when $x = \pm 1$ and when $y = 0$. Then the second equation yields the critical points $P(-1, -1)$, $Q(0, 0)$, $R\left(-\sqrt{3}\,, 0\right)$, $S\left(\sqrt{3}\,, 0\right)$, and $T(1, 1)$. In the notation of Theorem 2 of Section 13.10, we find that

$$A = 6xy, \quad B = 3x^2 - 3, \quad \text{and} \quad C = 2,$$

and application of Theorem 2 yields the following results:

At $P(-1, -1)$:	$A = 6$,	$B = 0$,	$C = 2$,	$\Delta = 12$,	$f(P) = -1$:	Local minimum;
At $Q(0, 0)$:	$A = 0$,	$B = -3$,	$C = 2$,	$\Delta = -9$,	$f(Q) = 0$:	Saddle point;
At $R\left(-\sqrt{3}\,, 0\right)$:	$A = 0$,	$B = 6$,	$C = 2$,	$\Delta = -36$,	$f(R) = 0$:	Saddle point;
At $S\left(\sqrt{3}\,, 0\right)$:	$A = 0$,	$B = 6$,	$C = 2$,	$\Delta = -36$,	$f(S) = 0$:	Saddle point;
At $T(1, 1)$:	$A = 6$,	$B = 0$,	$C = 2$,	$\Delta = 12$,	$f(T) = -1$:	Local minimum.

There are no global extrema; consider the behavior of $f(x, y)$ on the two lines $y = x$ and $y = -x$.

C13S0M.042: If $f(x, y) = x^2 + xy + y^2 - 6x + 2$, then when we equate both partial derivatives to zero we obtain the equations

$$2x + y - 6 = 0 \qquad \text{and} \qquad x + 2y = 0.$$

Thus the only critical point is $(4, -2)$. In the notation of Section 13.10, we have $A = 2$, $B = 1$, and $C = 2$, so that $\Delta = 3 > 0$. Hence by Theorem 2 of Section 13.10, $f(4, -2) = -10$ is a local minimum value of $f(x, y)$. In fact, this is the global minimum value of $f(x, y)$, demonstrated by the following computation:

$$\begin{aligned}
f(x, y) &= x^2 + xy + y^2 - 6x + 2 \\
&= \frac{1}{4}(4x^2 + 4xy + 4y^2 - 24x + 8) \\
&= \frac{1}{4}\big([x + 2y]^2 + 3x^2 - 24x + 8\big) \\
&= \frac{1}{4}\big([x + 2y]^2 + 3[x^2 - 8x + 16] - 40\big) \\
&= \frac{1}{4}\big([x + 2y]^2 + 3[x - 4]^2\big) - 10.
\end{aligned}$$

C13S0M.043: If $f(x, y) = x^3 - 6xy + y^3$, then the equations $f_x(x, y) = 0 = f_y(x, y)$ take the form

$$3x^2 - 6y = 0 \quad \text{and} \quad 3y^2 - 6x = 0.$$

Thus $2y = x^2$ and $y^2 = 2x$, so that $8x = x^4$. So the only two critical points are $(0, 0)$ and $(2, 2)$. In the notation of Section 13.10 we have $A = 6x$, $B = -6$, and $C = 6y$. Hence $\Delta = -36$ at $(0, 0)$; $A = 12$ and $\Delta = 108$ at $(2, 2)$. Therefore the graph of $z = f(x, y)$ has a saddle point at the origin and a local minimum at $(2, 2)$. There are no global extrema; examine the behavior of $f(x, 0)$.

C13S0M.044: If $f(x, y) = x^2y + xy^2 + x + y$, then the equations $f_x(x, y) = 0 = f_y(x, y)$ are

$$y^2 + 2xy + 1 = 0 \quad \text{and} \quad x^2 + 2xy + 1 = 0, \tag{1}$$

so that $y^2 = x^2$. If $y = x$ then neither of the equations in (1) has a real solution. If $y = -x$ then either of the equations in (1) yields $x^2 = 1$, and so there are two critical points: $(-1, 1)$ and $(1, -1)$. In the notation of Section 13.10 we have $A = 2y$, $B = 2x + 2y$, and $C = 2x$, and it follows from Theorem 2 of that section that $f(-1, 1) = 0$ and $f(1, -1) = 0$ are both saddle points. The graph of $z = f(x, y)$ has no extrema.

C13S0M.045: Given $f(x, y) = x^3y^2(1 - x - y)$, the equations $f_x(x, y) = 0 = f_y(x, y)$ become

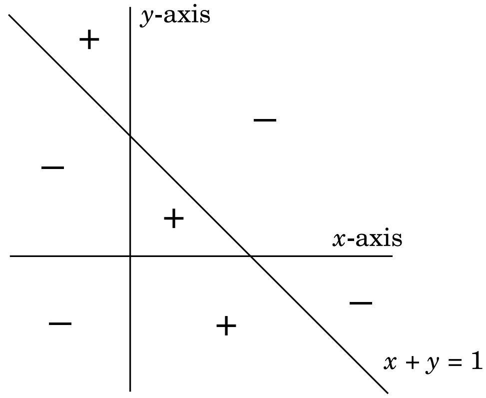
$$3x^2y^2(1 - x - y) - x^3y^2 = 0 \quad \text{and} \quad 2x^3y(1 - x - y) - x^3y^2 = 0. \tag{1}$$

Clearly both partial derivatives are zero at every point where $x = 0$ (the y-axis), at every point where $y = 0$ (the x-axis). Note that $f(x, y) = 0$ at *all* such points. Moreover, there is one additional critical point; if $x \neq 0$ and $y \neq 0$, then the equations in (1) may be simplified to

$$4x + 3y = 3 \quad \text{and} \quad 2x + 3y = 2,$$

with the unique solution $(x, y) = \left(\frac{1}{2}, \frac{1}{3}\right)$. At this critical point the value of $f(x, y)$ is $\frac{1}{432}$. Because we have infinitely many critical points, we will have to deal with them by *ad hoc* methods. The next diagram shows

the three lines where $f(x, y) = 0$. They divide the xy-plane into seven regions, and the sign of $f(x, y)$ is indicated on each.



It is clear from this diagram that no point on the line $x + y = 1$ is an extremum, but that there is a saddle point at $(0, 1)$. (Theorem 2 of Section 13.10 fails at this point because $\Delta = 0$ there.) It is also clear from the diagram that there is a local maximum at every point of the negative x-axis and at every point of the x-axis for which $x > 1$; there is a local minimum at every point of the x-axis for which $0 < x < 1$. Finally, no point of the y-axis is an extremum. The fact that $f\left(\frac{1}{2}, \frac{1}{3}\right)$ is positive shows that there is a local maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$, and this conclusion is supported by Theorem 2 of Section 13.10; at that point we have $A = -\frac{1}{9}, B = -\frac{1}{12}$, and $C = -\frac{1}{8}$. Finally, because $f(x, x) = x^5(1 - 2x)$, there are no global minima; because $f(x, -2x) = 4x^5(1 + x)$, there are no global maxima.

C13S0M.046: If $f(x, y) = x^4 - 2x^2 + y^2 + 4y + 3$, then when we equate both partial derivatives to zero we obtain the equations

$$4x(x^2 - 1) = 0 \quad \text{and} \quad 2y + 4 = 0.$$

The first equation holds when $x = 0$ and when $x = \pm 1$. Then the second equation yields the critical points $P(-1, -2)$, $Q(0, -2)$, and $R(1, -2)$. In the notation of Theorem 2 of Section 13.10, we find that

$$A = 12x^2 - 4, \quad B = 0, \quad \text{and} \quad C = 2,$$

and application of Theorem 2 yields the following results:

$$\begin{array}{llllll l}
\text{At } P(-1, -2): & A = 8, & B = 0, & C = 2, & \Delta = 16, & f(P) = -2: & \text{Local minimum;} \\
\text{At } Q(0, -2): & A = -4, & B = 0, & C = 2, & \Delta = -8, & f(Q) = -1: & \text{Saddle point;} \\
\text{At } R(1, -2): & A = 8, & B = 0, & C = 2, & \Delta = 16, & f(R) = -2: & \text{Local minimum.}
\end{array}$$

Because $f(x, y) = (x^2 - 1)^2 + (y + 2)^2 - 2$, the local minima are actually global minima. There are no maxima of any ilk.

C13S0M.047: If $f(x, y) = e^{xy} - 2xy$, then the equations $f_x(x, y) = 0 = f_y(x, y)$ take the form

$$(e^{xy} - 2)y = 0 \quad \text{and} \quad (e^{xy} - 2)x = 0.$$

Hence $(0, 0)$ is an isolated critical point and the points for which $e^{xy} = 2$ (the points on both branches of the hyperbola $xy = \ln 2$) are all critical points. Note that $f(0, 0) = 1$ and that $f(x, y) \equiv 2 - 2\ln 2 \approx 0.61370564$ for all points on the hyperbola $xy = \ln 2$. Next,

$$f_{xx}(x,\,y) = y^2e^{xy}, \quad f_{xy}(x,\,y) = (xy+1)e^{xy} - 2, \quad \text{and} \quad f_{yy}(x,\,y) = x^2e^{xy}.$$

Hence Theorem 2 of Section 13.10 yields $A = 0$, $B = -1$, and $C = 0$ at the critical point $(0, 0)$, and therefore it is a saddle point. It is easy to show that the global minimum value of $g(x) = e^x - 2x$ is $2 - 2\ln 2$, and therefore every point of the hyperbola $xy = \ln 2$ is a location of the global minimum value of $f(x, y)$. There are no other extrema.

C13S0M.048: If $f(x,\,y) = x^3 - y^3 + x^2 + y^2$, then the equations $f_x(x,\,y) = 0 = f_y(x,\,y)$ become

$$(3x+2)x = 0 \quad \text{and} \quad (2-3y)y = 0,$$

and thus there are four critical points. The results of an application of Theorem 2 of Section 13.10 to these points is next.

$$\begin{array}{llllll}
\text{At } \left(-\frac{2}{3},\,0\right): & A=-2, & B=0, & C=2, & f\left(-\frac{2}{3},\,0\right)=\dfrac{4}{27}: & \text{Saddle point;}\\
\text{At } \left(-\frac{2}{3},\,\frac{2}{3}\right): & A=-2, & B=0, & C=-2, & f\left(-\frac{2}{3},\,\frac{2}{3}\right)=\dfrac{8}{27}: & \text{Local maximum;}\\
\text{At } (0,\,0): & A=2, & B=0, & C=2, & f(0,\,0)=0: & \text{Local minimum;}\\
\text{At } \left(0,\,\frac{2}{3}\right): & A=2, & B=0, & C=-2, & f\left(0,\,\frac{2}{3}\right)=\dfrac{4}{27}: & \text{Saddle point.}
\end{array}$$

Because $f(x,\,-x) = 2x^3 + 2x^2$, there are no global extrema.

C13S0M.049: If $f(x,\,y) = (x-y)(xy-1)$, then when we equate both partial derivatives to zero we obtain

$$2xy - y^2 - 1 = 0 \quad \text{and} \quad 2xy - x^2 - 1 = 0. \tag{1}$$

Therefore $y^2 = x^2$. If $y = x$ then either equation in (1) yields $x^2 = 1$, and thus we obtain the two critical points $(-1,\,-1)$ and $(1,\,1)$. If $y = -x$ then each equation in (1) yields $3x^2 + 1 = 0$, and so there are no other critical points. When Theorem 2 of Section 13.10 is applied to the first critical point, we find that $A = -2$, $B = 0$, and $C = 2$; at the second critical point we find that $A = 2$, $B = 0$, and $C = -2$. Therefore both these points are saddle points and there are no extrema.

C13S0M.050: If $f(x,\,y) = (2x^2 + y^2)\exp(-x^2 - y^2)$, then the equations $f_x(x,\,y) = 0$ and $f_y(x,\,y) = 0$ become

$$(4x - 4x^3 - 2xy^2)\exp(-x^2-y^2) = 0 \quad \text{and} \quad (2y - 4x^2y - 2y^3)\exp(-x^2-y^2) = 0.$$

Cancel the factor $-2\exp(-x^2 - y^2)$ from both equations because it is never zero, then solve to find five critical points. Application of Theorem 2 of Section 13.10 then yields the following information.

$$\begin{array}{lllllll}
\text{At } P(-1,\,0): & A=-\dfrac{8}{e}, & B=0, & C=-\dfrac{2}{e}, & \Delta=\dfrac{16}{e^2}, & f(P)=\dfrac{2}{e}: & \text{Local maximum;}\\
\text{At } Q(0,\,0): & A=4, & B=0, & C=2, & \Delta=8, & f(Q)=0: & \text{Local minimum;}\\
\text{At } R(1,\,0): & A=-\dfrac{8}{e}, & B=0, & C=-\dfrac{2}{e}, & \Delta=\dfrac{16}{e^2}, & f(R)=\dfrac{2}{e}: & \text{Local maximum;}\\
\text{At } S(0,\,-1): & A=\dfrac{2}{e}, & B=0, & C=-\dfrac{4}{e}, & \Delta=-\dfrac{8}{e^2}, & f(S)=\dfrac{1}{e}: & \text{Saddle point;}\\
\text{At } T(0,\,1): & A=\dfrac{2}{e}, & B=0, & C=-\dfrac{4}{e}, & \Delta=-\dfrac{8}{e^2}, & f(T)=\dfrac{1}{e}: & \text{Saddle point.}
\end{array}$$

Because $f(x, y) > 0$ if $(x, y) \neq (0, 0)$ and because $f(x, y) \to 0$ as either $|x| \to 0$ or $|y| \to 0$ (or both), the three local extrema are actually global extrema. Please consider using a computer algebra system to plot level curves of f; for example, use the *Mathematica* 3.0 command

```
ContourPlot[ f[x,y], {x, -2, 2}, {y, -2, 2}, ContourShading → False,
             Contours → 19, PlotPoints → 101 ];
```

to see one of the most interesting level curve structures of this chapter.

C13S0M.051: Given the data $\{(x_i, y_i)\}$ for $i = 1, 2, 3, \ldots, n$ (where n is a positive integer), we are to minimize

$$f(m, b) = \sum_{i=1}^{n} \big[y_i - (mx_i + b)\big]^2.$$

When we equate both partial derivatives of f to zero, we obtain the equations

$$\sum_{i=1}^{n} \big(y_i - mx_i - b\big)x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \big(y_i - mx_i - b\big) = 0.$$

Rewrite the first of these in the form

$$b\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n}(x_i)^2 = \sum_{i=1}^{n} x_i y_i \tag{1}$$

and the second in the form

$$b\sum_{i=1}^{n} 1 + m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{2}$$

Let

$$P = \sum_{i=1}^{n} x_i, \quad Q = \sum_{i=1}^{n} y_i, \quad R = \sum_{i=1}^{n}(x_i)^2, \quad \text{and} \quad S = \sum_{i=1}^{n} x_i y_i.$$

Then Eqs. (1) and (2) take the form

$$Pb + Rm = S, \qquad nb + Pm = Q. \tag{3}$$

Because P, Q, R, and S are the results of experimental observations, it is highly unlikely that the determinant of coefficients in (3) is zero, and hence these equations will normally have a unique solution (although the experimenter should be wary if the determinant is near zero). Next,

$$f_{mm}(m, b) = 2\sum_{i=1}^{n}(x_i)^2 = 2R, \qquad f_{mb}(m, b) = 2\sum_{i=1}^{n} x_i = 2P, \qquad \text{and} \qquad f_{bb}(m, b) = 2n.$$

Application of Theorem 2 of Section 13.10 yields

$$A = 2R, \quad B = 2P, \quad C = 2n, \quad \text{and} \quad \Delta = 4Rn - 4P^2 = 4(Rn - P^2).$$

Hence our sole critical point will be a local minimum (and, by the geometry of the problem, a global minimum) provided that

$$P^2 < nR; \quad \text{that is,} \quad \left(\sum_{i=1}^{n} x_i\right)^2 < n\sum_{i=1}^{n}(x_i)^2.$$

The inequality in Miscellaneous Problem 37 is strict unless all the x_i are equal, and in an experimental problem in which the least squares method is used, they will not be equal. This establishes that the critical point is a global minimum.

C13S0M.052: Regard $\mathbf{y}$ as fixed and the constraint is

$$g(\mathbf{x}) = \left(\sum_{i=1}^{n} x_i^2\right) - 1 = 0.$$

Then the method of Lagrange multipliers gives $y_i = 2\lambda x_i$ for each i. Multiplication by x_i and subsequent summation yields $f(\mathbf{x}, \mathbf{y}) = 2\lambda$. Multiplication by y_i and subsequent summation gives $1 = 2\lambda f(\mathbf{x}, \mathbf{y})$. Therefore $\lambda = \frac{1}{2}$ and $f(\mathbf{x}, \mathbf{y}) = 1$ at a maximum. —C.H.E.

Section 14.1

C14S01.001: Part (a):

$$f(1,\,-2)\cdot 1 + f(2,\,-2)\cdot 1 + f(1,\,-1)\cdot 1 + f(2,\,-1)\cdot 1 + f(1,\,0)\cdot 1 + f(2,\,0)\cdot 1 = 198.$$

Part (b):

$$f(2,\,-1)\cdot 1 + f(3,\,-1)\cdot 1 + f(2,\,0)\cdot 1 + f(3,\,0)\cdot 1 + f(2,\,1)\cdot 1 + f(3,\,1)\cdot 1 = 480.$$

The average of the two answers is 339, fairly close to the exact value 312 of the integral.

C14S01.002: Part (a):

$$f(1,\,-1)\cdot 1 + f(2,\,-1)\cdot 1 + f(1,\,0)\cdot 1 + f(2,\,0)\cdot 1 + f(1,\,1)\cdot 1 + f(2,\,1)\cdot 1 = 144.$$

Part (b):

$$f(2,\,-2)\cdot 1 + f(3,\,-2)\cdot 1 + f(2,\,-1)\cdot 1 + f(3,\,-1)\cdot 1 + f(2,\,0)\cdot 1 + f(3,\,0)\cdot 1 = 570.$$

The average of the two answers is 357, fairly close to the exactly value 312 of the integral. The computations shown here can be automated in computer algebra systems. For example, in *Mathematica* 3.0, after defining $f(x,\,y) = 4x^3 + 6xy^2$, you could proceed as follows.

```
x[i_] := i + 1;  y[j_] := j - 2;  deltax = x[1] - x[0];  deltay = y[1] - y[0];

(* Part (a):  *)  xstar[i_] := x[i-1]; ystar[j_] := y[j]

Sum[ Sum[ f[xstar[i],ystar[j]]*deltax*deltay, {j, 1, 3}, {i, 1, 2} ]

   144

(* Part (b):  *)  xstar[i_] := x[i]; ystar[j_] := y[j-1]

Sum[ Sum[ f[xstar[i],ystar[j]]*deltax*deltay, {j, 1, 3}, {i, 1, 2} ]

   570
```

The idea is that to work another such problem, all you need to do is redefine f, `xstar` and `ystar`, and the limits on i and j.

C14S01.003: We omit Δx and Δy from the computation because each is equal to 1.

$$f\left(\tfrac{1}{2},\,\tfrac{1}{2}\right) + f\left(\tfrac{3}{2},\,\tfrac{1}{2}\right) + f\left(\tfrac{1}{2},\,\tfrac{3}{2}\right) + f\left(\tfrac{3}{2},\,\tfrac{3}{2}\right) = 8.$$

This is also the exact value of the iterated integral.

C14S01.004: We omit Δx and Δy from the computation because each is equal to 1.

$$f\left(\tfrac{1}{2},\,\tfrac{1}{2}\right) + f\left(\tfrac{3}{2},\,\tfrac{1}{2}\right) + f\left(\tfrac{1}{2},\,\tfrac{3}{2}\right) + f\left(\tfrac{3}{2},\,\tfrac{3}{2}\right) = 4.$$

This is also the exact value of the iterated integral. In a *Mathematica* 3.0 solution similar to the one in Problem 2, we would use

```
xstar[i_] := (x[i] + x[i-1])/2; ystar[j_] := (y[j] + y[j-1])/2
```

C14S01.005: The Riemann sum is

$$f(2,\,-1)\cdot 2+f(4,\,-1)\cdot 2+f(2,\,0)\cdot 2+f(4,\,0)\cdot 2=88.$$

The true value of the integral is $\dfrac{416}{3}\approx 138.666666666667.$

C14S01.006: We omit $\Delta x=1$ and $\Delta y=1$ from the computation.

$$f(1,\,1)+f(2,\,1)+f(1,\,2)+f(2,\,2)+f(1,\,3)+f(2,\,3)=43.$$

The true value of the integral is 26. The midpoint approximation gives the very close Riemann sum 25.

C14S01.007: We factor out of each term in the sum the product $\Delta x\cdot\Delta y=\frac{1}{4}\pi^2$. The Riemann sum then takes the form

$$\frac{1}{4}\pi^2\cdot\left[\,f\left(\tfrac{1}{4}\pi,\,\tfrac{1}{4}\pi\right)+f\left(\tfrac{3}{4}\pi,\,\tfrac{1}{4}\pi\right)+f\left(\tfrac{1}{4}\pi,\,\tfrac{3}{4}\pi\right)+f\left(\tfrac{3}{4}\pi,\,\tfrac{3}{4}\pi\right)\,\right]=\tfrac{1}{2}\pi^2\approx 4.935.$$

The true value of the integral is 4.

C14S01.008: We factor out of each term in the sum the product $\Delta x\cdot\Delta y=\frac{1}{6}\pi$. The Riemann sum then takes the form

$$\tfrac{1}{6}\pi\cdot\left[\,f\left(\tfrac{1}{4},\,\tfrac{1}{6}\pi\right)+f\left(\tfrac{3}{4},\,\tfrac{1}{6}\pi\right)+f\left(\tfrac{1}{4},\,\tfrac{1}{2}\pi\right)+f\left(\tfrac{3}{4},\,\tfrac{1}{2}\pi\right)+f\left(\tfrac{1}{4},\,\tfrac{5}{6}\pi\right)+f\left(\tfrac{3}{4},\,\tfrac{5}{6}\pi\right)\,\right]=\tfrac{1}{2}\pi\approx 1.571.$$

Mathematica 3.0 reports that the true value of the integral is

$$-\frac{1}{4}\,\texttt{CosIntegral[}4\pi\texttt{]}\ +\ \frac{1}{4}\,\texttt{(EulerGamma + Log[}4\pi\texttt{])}$$

and when we asked for a numerical value with the command `N[%]`, it returned the approximation

0.77858913775068568

C14S01.009: Because $f(x,\,y)=x^2y^2$ is increasing in both the positive x-direction and the positive y-direction on $[1,\,3]\times[2,\,5]$, $L\leqq M\leqq U$.

C14S01.010: Because $f(x,\,y)=\sqrt{100-x^2-y^2}$ is decreasing in both the positive x-direction and the positive y-direction on $[1,\,4]\times[2,\,5]$, $U\leqq M\leqq L$.

C14S01.011: We integrate first with respect to x, then with respect to y:

$$\int_0^2\int_0^4(3x+4y)\,dx\,dy=\int_0^2\left[\frac{3}{2}x^2+4xy\right]_0^4dy=\int_0^2(24+16y)\,dy=\Big[24y+8y^2\Big]_0^2=80.$$

C14S01.012: We integrate first with respect to x, then with respect to y:

$$\int_0^3\int_0^2x^2y\,dx\,dy=\int_0^3\left[\frac{1}{3}x^3y\right]_0^2dy=\int_0^3\frac{8}{3}y\,dy=\left[\frac{4}{3}y^2\right]_0^3=12.$$

C14S01.013: We integrate first with respect to y, then with respect to x:

$$\int_{-1}^{2}\int_{1}^{3}(2x-7y)\ dy\ dx = \int_{-1}^{2}\left[2xy-\frac{7}{2}y^2\right]_1^3 dx = \int_{-1}^{2}(4x-28)\ dx = \left[2x^2-28x\right]_{-1}^{2} = -48-30=-78.$$

C14S01.014: We integrate first with respect to y, then with respect to x:

$$\int_{-2}^{1}\int_{2}^{4} x^2y^3\ dy\ dx = \int_{-2}^{1}\left[\frac{1}{4}x^2y^4\right]_2^4 dx = \int_{-2}^{1} 60x^2\ dx = \left[20x^3\right]_{-2}^{1} = 180.$$

C14S01.015: We integrate first with respect to x, then with respect to y:

$$\int_0^3\int_0^3(xy+7x+y)\ dx\ dy = \int_0^3\left[xy+\frac{7}{2}x^2+\frac{1}{2}x^2y\right]_0^3 dy$$
$$= \int_0^3\left(\frac{3}{2}(5y+21)\right)\ dy = \left[\frac{1}{4}(15y^2+126y)\right]_0^3 = \frac{513}{4} = 128.25.$$

C14S01.016: We integrate first with respect to x, then with respect to y:

$$\int_0^2\int_2^4(x^2y^2-17)\ dx\ dy = \int_0^2\left[\frac{1}{3}x^3y^2-17x\right]_2^4 dy$$
$$= \int_0^2\frac{2}{3}(28y^2-51)\ dy = \left[\frac{2}{9}(28y^3-153y)\right]_0^2 = -\frac{164}{9} \approx -18.222222222222.$$

C14S01.017: We integrate first with respect to y, then with respect to x:

$$\int_{-1}^{2}\int_{-1}^{2}(2xy^2-3x^2y)\ dy\ dx = \int_{-1}^{2}\left[\frac{2}{3}xy^3-\frac{3}{2}x^2y^2\right]_{-1}^{2} dx$$
$$= \int_{-1}^{2}\frac{3}{2}(4x-3x^2)\ dx = \left[3x^2-\frac{3}{2}x^3\right]_{-1}^{2} = 0-\frac{9}{2} = -\frac{9}{2} = -4.5.$$

C14S01.018: We integrate first with respect to y, then with respect to x:

$$\int_1^3\int_{-3}^{-1}(x^3y-xy^3)\ dy\ dx = \int_1^3\left[\frac{1}{2}x^3y^2-\frac{1}{4}xy^4\right]_{-3}^{-1} dx$$
$$= \int_1^3(20x-4x^3)\ dx = \left[10x^2-x^4\right]_1^3 = 9-9=0.$$

C14S01.019: We integrate first with respect to x, then with respect to y:

$$\int_0^{\pi/2}\int_0^{\pi/2}\sin x\ \cos y\ dx\ dy = \int_0^{\pi/2}\left[-\cos x\ \cos y\right]_0^{\pi/2} dy$$
$$= \int_0^{\pi/2}\cos y\ dy = \left[\sin y\right]_0^{\pi/2} = 1-0=1.$$

C14S01.020: This is merely Problem 19 with x and y interchanged, so the answer should be the same.

$$\int_0^{\pi/2}\int_0^{\pi/2} \cos x\,\sin y\,dy\,dx = \int_0^{\pi/2}\Big[-\cos x\,\cos y\Big]_0^{\pi/2}\,dx$$
$$= \int_0^{\pi/2}\cos x\,dx = \Big[\,\sin x\,\Big]_0^{\pi/2} = 1 - 0 = 1.$$

C14S01.021: We integrate first with respect to y, then with respect to x:

$$\int_0^1\int_0^1 xe^y\,dy\,dx = \int_0^1\Big[xe^y\Big]_0^1\,dx$$
$$= \int_0^1 (ex - x)\,dx = \left[\frac{1}{2}(e-1)x^2\right]_0^1 = \frac{1}{2}(e-1) \approx 0.8591409142295226.$$

C14S01.022: We integrate first with respect to x, then with respect to y:

$$\int_0^1\int_{-2}^2 x^2e^y\,dx\,dy = \int_0^1\left[\frac{1}{3}x^3e^y\right]_{-2}^2\,dy$$
$$= \int_0^1 \frac{16}{3}e^y\,dy = \left[\frac{16}{3}e^y\right]_0^1 = \frac{16}{3}(e-1) \approx 9.1641697517815746.$$

C14S01.023: We integrate first with respect to y, then with respect to x:

$$\int_0^1\int_0^\pi e^x\sin y\,dy\,dx = \int_0^1\Big[-e^x\cos y\Big]_0^\pi\,dx$$
$$= \int_0^1 2e^x\,dx = \Big[\,2e^x\,\Big]_0^1 = 2e - 2 \approx 3.436563656918.$$

C14S01.024: We integrate first with respect to x, then with respect to y:

$$\int_0^1\int_0^1 e^{x+y}\,dx\,dy = \int_0^1\Big[e^{x+y}\Big]_0^1\,dy = \int_0^1\left(e^{y+1}-e^y\right)\,dy$$
$$= \Big[e^{y+1}-e^y\Big]_0^1 = (e^2-e)-(e-1) = (e-1)^2 \approx 2.9524924420125598.$$

C14S01.025: We integrate first with respect to x, then with respect to y:

$$\int_0^\pi\int_0^\pi (xy + \sin x)\,dx\,dy = \int_0^\pi\left[\frac{1}{2}x^2y - \cos x\right]_0^\pi\,dy = \int_0^\pi\left(2 + \frac{1}{2}\pi^2y\right)\,dy$$
$$= \left[2y + \frac{1}{4}\pi^2y^2\right]_0^\pi = \frac{1}{4}\left(\pi^4 + 8\pi\right) \approx 30.635458065680.$$

C14S01.026: We integrate first with respect to x, then with respect to y:

$$\int_0^{\pi/2}\int_0^{\pi/2}(y-1)\cos x\,dx\,dy = \int_0^{\pi/2}\Big[(y-1)\sin x\Big]_0^{\pi/2}\,dy = \int_0^{\pi/2}(y-1)\,dy$$

$$= \left[\frac{1}{2}y^2 - y\right]_0^{\pi/2} = \frac{1}{8}(\pi^2 - 4\pi) \approx -0.3370957766587268.$$

C14S01.027: We integrate first with respect to x, then with respect to y:

$$\int_0^{\pi/2}\int_1^e \frac{\sin y}{x}\,dx\,dy = \int_0^{\pi/2}\Big[(\ln x)\sin y\Big]_1^e\,dy$$

$$= \int_0^{\pi/2}\sin y\,dy = \Big[-\cos y\Big]_0^{\pi/2} = 0-(-1) = 1.$$

C14S01.028: We integrate first with respect to y, then with respect to x:

$$\int_1^e\int_1^e \frac{1}{xy}\,dy\,dx = \int_1^e\left[\frac{\ln y}{x}\right]_1^e\,dx = \int_1^e \frac{1}{x}\,dx = \Big[\ln x\Big]_1^e = 1-0 = 1.$$

C14S01.029: We integrate first with respect to x, then with respect to y:

$$\int_0^1\int_0^1\left(\frac{1}{x+1}+\frac{1}{y+1}\right)dx\,dy = \int_0^1\left[\frac{x}{y+1}+\ln(x+1)\right]_0^1 dy = \int_0^1\left(\frac{1}{y+1}+\ln 2\right)dy$$

$$= \Big[\ln(y+1) + y\ln 2\Big]_0^1 = 2\ln 2 - 0 = 2\ln 2 \approx 1.3862943611198906.$$

C14S01.030: We integrate first with respect to y, then with respect to x:

$$\int_1^2\int_1^3\left(\frac{x}{y}+\frac{y}{x}\right)dy\,dx = \int_1^2\left[\frac{y^2}{2x}+x\ln y\right]_1^3 dx = \int_1^2\left(\frac{4}{x}+x\ln 3\right)dx$$

$$= \left[\frac{1}{2}x^2\ln 3 + 4\ln x\right]_1^2 = 4\ln 2 + \frac{3}{2}\ln 3 \approx 4.4205071552419458.$$

C14S01.031: The first evaluation yields

$$\int_{-2}^2\int_{-1}^1(2xy-3y^2)\,dx\,dy = \int_{-2}^2\Big[x^2y - 3xy^2\Big]_{-1}^1\,dy$$

$$= \int_{-2}^2(-6y^2)\,dy = \Big[-2y^3\Big]_{-2}^2 = -16-16 = -32.$$

The second yields

$$\int_{-1}^1\int_{-2}^2(2xy-3y^2)\,dy\,dx = \int_{-1}^1\Big[xy^2 - y^3\Big]_{-2}^2\,dx$$

$$= \int_{-1}^1(-16)\,dx = \Big[-16x\Big]_{-1}^1 = -16-16 = -32.$$

C14S01.032: The first evaluation yields

$$\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \sin x\,\cos y\,dx\,dy = \int_{-\pi/2}^{\pi/2}\Big[-\cos x\,\cos y\Big]_0^{\pi}\,dy$$

$$= \int_{-\pi/2}^{\pi/2} 2\cos y\,dy = \Big[2\sin y\Big]_{-\pi/2}^{\pi/2} = 2-(-2) = 4.$$

The second yields

$$\int_0^{\pi}\int_{-\pi/2}^{\pi/2} \sin x\,\cos y\,dy\,dx = \int_0^{\pi}\Big[\sin x\,\sin y\Big]_{-\pi/2}^{\pi/2}\,dx$$

$$= \int_0^{\pi} 2\sin x\,dx = \Big[-2\cos x\Big]_0^{\pi} = 2-(-2) = 4.$$

C14S01.033: The first evaluation yields

$$\int_1^2\int_0^1 (x+y)^{1/2}\,dx\,dy = \int_1^2\left[\frac{2}{3}(x+y)^{3/2}\right]_0^1 dy = \int_1^2\left(\frac{2}{3}(y+1)^{3/2}-\frac{2}{3}y^{3/2}\right)dy$$

$$= \left[\frac{4}{15}(y+1)^{5/2}-\frac{4}{15}y^{5/2}\right]_1^2 = \frac{4}{15}\left(9\sqrt{3}\,-8\sqrt{2}\,+1\right) \approx 1.406599671769.$$

The second yields

$$\int_0^1\int_1^2 (x+y)^{1/2}\,dy\,dx = \int_0^1\left[\frac{2}{3}(x+y)^{3/2}\right]_1^2 dx = \int_0^1\left(\frac{2}{3}(x+2)^{3/2}-\frac{2}{3}(x+1)^{3/2}\right)dx$$

$$= \left[\frac{4}{15}(x+2)^{5/2}-\frac{4}{15}(x+1)^{5/2}\right]_0^1 = \frac{4}{15}\left(9\sqrt{3}\,-8\sqrt{2}\,+1\right).$$

C14S01.034: The first evaluation yields

$$\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dx\,dy = \int_0^{\ln 3}\Big[e^{x+y}\Big]_0^{\ln 2}\,dy = \int_0^{\ln 3}(2e^y-e^y)\,dy = \Big[e^y\Big]_0^{\ln 3} = 3-1=2.$$

The second yields

$$\int_0^{\ln 2}\int_0^{\ln 3} e^{x+y}\,dy\,dx = \int_0^{\ln 2}\Big[e^{x+y}\Big]_0^{\ln 3}\,dx = \int_0^{\ln 2}(3e^x-e^x)\,dx = \Big[2e^x\Big]_0^{\ln 2} = 4-2=2.$$

C14S01.035: We may assume that $n \geqq 1$ and, if you wish, even that n is a positive integer. Then

$$\int_0^1\int_0^1 x^ny^n\,dx\,dy = \int_0^1\left[\frac{x^{n+1}y^n}{n+1}\right]_0^1 dy = \int_0^1 \frac{y^n}{n+1}\,dy = \left[\frac{y^{n+1}}{(n+1)^2}\right]_0^1 = \frac{1}{(n+1)^2}.$$

Therefore

$$\lim_{n\to\infty}\int_0^1\int_0^1 x^ny^n\,dx\,dy = \lim_{n\to\infty}\frac{1}{(n+1)^2} = 0.$$

C14S01.036: Note that whatever the choice of $(x_i^\star,\, y_i^\star)$, $f(x_i^\star,\, y_i^\star) = k$, and hence $f(x_i^\star,\, y_i^\star)\,\Delta A_i$ is equal to the product of k and the area $a(R_i)$ of R_i for each i. Hence

$$\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star)\,\Delta A_i = \sum_{i=1}^{n} k \cdot a(R_i) = k\left(\sum_{i=1}^{n} a(R_i)\right) = k \cdot a(R) = k(b-a)(d-c).$$

C14S01.037: Let $a(R)$ denote the area of R. If $0 \leqq x \leqq \pi$ and $0 \leqq y \leqq \pi$, then $0 \leqq f(x,\, y) \leqq \sin \frac{1}{2}\pi = 1$. Hence every Riemann sum lies between $0 \cdot a(R)$ and $1 \cdot a(R)$. Therefore

$$0 \leqq \int_0^\pi \int_0^\pi \sin\sqrt{xy}\ dx\ dy \leqq a(R) = \pi^2 \approx 9.869604401.$$

The exact value of the integral is

$$\int_0^\pi \int_0^\pi \sin\sqrt{xy}\ dx\ dy = 4\int_0^\pi \frac{\sin t}{t}\ dt \approx 7.4077482079298646814442134806319654533832.$$

C14S01.038: The corresponding relation between Riemann sums is

$$\sum_{i=1}^{n} cf(x_i^\star,\, y_i^\star) \cdot \Delta A_i = c\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star) \cdot \Delta A_i.$$

C14S01.039: The corresponding relation among Riemann sums is

$$\sum_{i=1}^{n} \left[f(x_i^\star,\, y_i^\star) + g(x_i^\star,\, y_i^\star)\right] \cdot \Delta A_i = \left[\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star) \cdot \Delta A_i\right] + \left[\sum_{i=1}^{n} g(x_i^\star,\, y_i^\star) \cdot \Delta A_i\right].$$

C14S01.040: The corresponding relation between Riemann sums is this: If $f(x,\, y) \leqq g(x,\, y)$ at each point of R, then

$$\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star) \cdot \Delta A_i \ \leqq\ \sum_{i=1}^{n} g(x_i^\star,\, y_i^\star) \cdot \Delta A_i.$$

Section 14.2

C14S02.001: $\displaystyle\int_0^1\int_0^x (1+x)\,dy\,dx = \int_0^1 \Big[y+xy\Big]_{y=0}^x dx = \int_0^1 (x+x^2)\,dx = \left[\frac{1}{2}x^2+\frac{1}{3}x^3\right]_0^1 = \frac{5}{6}.$

C14S02.002: $\displaystyle\int_0^2\int_0^{2x} (1+y)\,dy\,dx = \int_0^2 \left[y+\frac{1}{2}y^2\right]_0^{2x} dx = \int_0^2 (2x+2x^2)\,dx = \left[x^2+\frac{2}{3}x^3\right]_0^2 = \frac{28}{3}.$

C14S02.003: $\displaystyle\int_0^1\int_y^1 (x+y)\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2+xy\right]_y^1 dy$

$$= \int_0^1 \left(\frac{1}{2}+y-\frac{3}{2}y^2\right) dy = \left[\frac{1}{2}(y+y^2-y^3)\right]_0^1 = \frac{1}{2}.$$

C14S02.004: $\displaystyle\int_0^2\int_{y/2}^1 (x+y)\,dx\,dy = \int_0^2 \left[\frac{1}{2}x^2+xy\right]_{y/2}^1 dy$

$$= \int_0^2 \left(\frac{1}{2}+y-\frac{5}{8}y^2\right) dy = \left[\frac{1}{2}y+\frac{1}{2}y^2-\frac{5}{24}y^3\right]_0^2 = \frac{4}{3}.$$

C14S02.005: $\displaystyle\int_0^1\int_0^{x^2} xy\,dy\,dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_0^{x^2} dx = \int_0^1 \frac{1}{2}x^5\,dx = \left[\frac{1}{12}x^6\right]_0^1 = \frac{1}{12}.$

C14S02.006: $\displaystyle\int_0^1\int_y^{\sqrt{y}} (x+y)\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2+xy\right]_y^{\sqrt{y}} dy$

$$= \int_0^1 \left(\frac{1}{2}y+y^{3/2}-\frac{3}{2}y^2\right) dy = \left[\frac{1}{4}y^2+\frac{2}{5}y^{5/2}-\frac{1}{2}y^3\right]_0^1 = \frac{3}{20}.$$

C14S02.007: $\displaystyle\int_0^1\int_x^{\sqrt{x}} (2x-y)\,dy\,dx = \int_0^1 \left[2xy-\frac{1}{2}y^2\right]_x^{\sqrt{x}} dx$

$$= \int_0^1 \left(-\frac{1}{2}x+2x^{3/2}-\frac{3}{2}x^2\right) dx = \left[-\frac{1}{4}x^2+\frac{4}{5}x^{5/2}-\frac{1}{2}x^3\right]_0^1 = \frac{1}{20}.$$

C14S02.008: $\displaystyle\int_0^2\int_{-\sqrt{2y}}^{\sqrt{2y}} (3x+2y)\,dx\,dy = \int_0^2 \left[\frac{3}{2}x^2+2xy\right]_{-\sqrt{2y}}^{\sqrt{2y}} dy$

$$= \int_0^2 \left(4\sqrt{2}\right) y^{3/2}\,dy = \left[\left(\frac{8}{5}\sqrt{2}\right) y^{5/2}\right]_0^2 = \frac{64}{5}.$$

C14S02.009: $\displaystyle\int_0^1\int_{x^4}^x (y-x)\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2-xy\right]_{x^4}^x dx$

$$= \int_0^1 \left(-\frac{1}{2}x^2+x^5-\frac{1}{2}x^8\right) dx = \left[-\frac{1}{6}x^3+\frac{1}{6}x^6-\frac{1}{18}x^9\right]_0^1 = -\frac{1}{18}.$$

C14S02.010: $\displaystyle\int_{-1}^2\int_{-y}^{y+2} (x+2y^2)\,dx\,dy = \int_{-1}^2 \left[\frac{1}{2}x^2+2xy^2\right]_{-y}^{y+2} dy$

$$= \int_{-1}^{2} (2 + 2y + 4y^2 + 4y^3)\,dy = \left[2y + y^2 + \frac{4}{3}y^3 + y^4\right]_{-1}^{2} = 36.$$

C14S02.011: $\displaystyle \int_0^1 \int_0^{x^3} e^{y/x}\,dy\,dx = \int_0^1 \left[xe^{y/x}\right]_0^{x^3} dx = \int_0^1 \left(x\exp(x^2) - x\right)\,dx$

$$= \left[\frac{1}{2}\left(\exp(x^2) - x^2\right)\right]_0^1 = \frac{e-2}{2} \approx 0.3591409142295226.$$

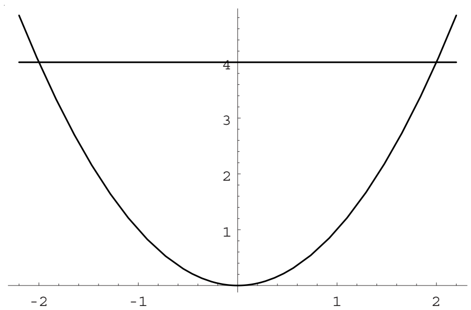
C14S02.012: $\displaystyle \int_0^{\pi} \int_0^{\sin x} y\,dy\,dx = \int_0^{\pi} \left[\frac{1}{2}y^2\right]_0^{\sin x} dx = \int_0^{\pi} \frac{1}{2}\sin^2 x\,dx = \left[\frac{1}{8}(2x - \sin 2x)\right]_0^{\pi} = \frac{\pi}{4}.$

C14S02.013: $\displaystyle \int_0^3 \int_0^y (y^2+16)^{1/2}\,dx\,dy = \int_0^3 \left[x(y^2+16)^{1/2}\right]_0^y dy$

$$= \int_0^3 y(y^2+16)^{1/2}\,dy = \left[\frac{1}{3}(y^2+16)^{3/2}\right]_0^3 = \frac{125}{3} - \frac{64}{3} = \frac{61}{3}.$$

C14S02.014: $\displaystyle \int_1^{e^2} \int_0^{1/y} e^{xy}\,dx\,dy = \int_1^{e^2} \left[\frac{1}{y}e^{xy}\right]_0^{1/y} dy = \int_1^{e^2} \frac{e-1}{y}\,dy = \Big[(e-1)\ln y\Big]_1^{e^2} = 2(e-1).$

C14S02.015: The following sketch of the graphs of $y = x^2$ and $y \equiv 4$ is extremely helpful in finding the limits of integration.



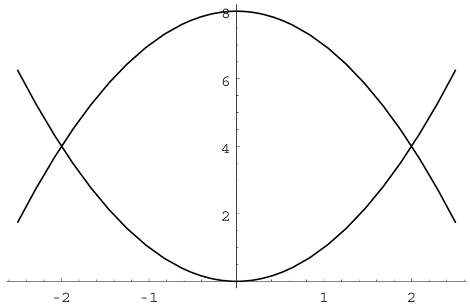
Answer:

$$\int_{-2}^{2} \int_{x^2}^{4} xy\,dy\,dx = \int_{-2}^{2} \left[\frac{1}{2}xy^2\right]_{x^2}^{4} dx = \int_{-2}^{2} \left(8x - \frac{1}{2}x^5\right)\,dx = \left[4x^2 - \frac{1}{12}x^6\right]_{-2}^{2} = \frac{32}{3} - \frac{32}{3} = 0.$$

C14S02.016: $\displaystyle \int_{-\sqrt{6}}^{\sqrt{6}} \int_{-4}^{2-x^2} x^2\,dy\,dx = \int_{-\sqrt{6}}^{\sqrt{6}} \left[x^2y\right]_{-4}^{2-x^2} dx = \int_{-\sqrt{6}}^{\sqrt{6}} (6x^2 - x^4)\,dx$

$$= \left[2x^3 - \frac{1}{5}x^5\right]_{-\sqrt{6}}^{\sqrt{6}} = \frac{48}{5}\sqrt{6} \approx 23.5151015307185097.$$

C14S02.017: The following diagram, showing the graphs of $y = x^2$ and $y = 8 - x^2$, is useful in finding the limits of integration. (Solve $y = x^2$ and $y = 8 - x^2$ simultaneously to find where the two curves cross.)

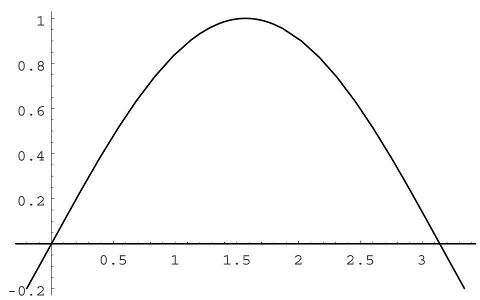


Answer:

$$\int_{-2}^{2}\int_{x^2}^{8-x^2} x\,dy\,dx = \int_{-2}^{2}\Big[xy\Big]_{x^2}^{8-x^2} dx = \int_{-2}^{2}(8x - 2x^3)\,dx = \left[4x^2 - \frac{1}{2}x^4\right]_{-2}^{2} = 8 - 8 = 0.$$

C14S02.018: $\displaystyle\int_{-1}^{1}\int_{y^2-1}^{1-y^2} y\,dx\,dy = \int_{-1}^{1}\Big[xy\Big]_{y^2-1}^{1-y^2} dy = \int_{-1}^{1}(2y - 2y^3)\,dy = \left[y^2 - \frac{1}{2}y^4\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$

C14S02.019: The following graph of $y = \sin x$ is helpful in determining the limits of integration.



Answer:

$$\int_0^{\pi}\int_0^{\sin x} x\,dy\,dx = \int_0^{\pi}\Big[xy\Big]_0^{\sin x} dx = \int_0^{\pi} x\sin x\,dx = \Big[\sin x - x\cos x\Big]_0^{\pi} = \pi.$$

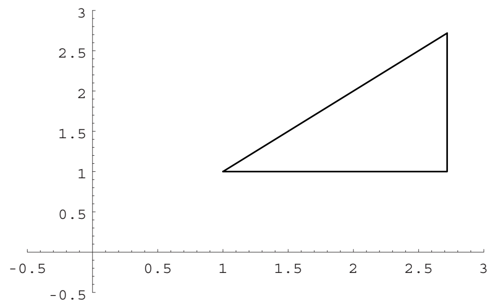
C14S02.020: To determine the order of integration and to determine the limits of integration, it is helpful to draw the domain of the double integral. The value of the integral is

$$\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} \sin x\,dy\,dx = \int_{-\pi/2}^{\pi/2}\Big[y\sin x\Big]_0^{\cos x} dx = \int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx = \left[\frac{1}{2}\sin^2 x\right]_{-\pi/2}^{\pi/2} = \frac{1}{2} - \frac{1}{2} = 0.$$

C14S02.021: A *Mathematica* 3.0 command to draw the domain of the double integral is

```
ParametricPlot[ {{t,1}, {E,t}, {t,t}}, {t,1,E}, AxesOrigin → {0,0},
                PlotRange → {{-0.5,3}, {-0.5,3}} ];
```

The graph produced by this command is shown next.



Answer:

$$\int_1^e \int_1^x \frac{1}{y}\,dy\,dx = \int_1^e \Big[\ln y\Big]_1^x\,dx = \int_1^e \ln x\,dx = \Big[-x + x\ln x\Big]_1^e = 0 - (-1) = 1.$$

C14S02.022: A *Mathematica* 3.0 command to draw the domain of the double integral is

```
ParametricPlot[ { Cos[t], Sin[t] }, { t, 0, Pi/2 }, AspectRatio → Automatic ];
```

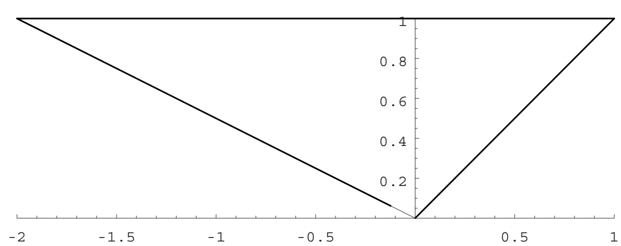
The value of the double integral is

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \int_0^1 \left[\frac{1}{2}xy^2\right]_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{1}{2}(x - x^3)\,dx = \left[\frac{1}{4}x^2 - \frac{1}{8}x^4\right]_0^1 = \frac{1}{8}.$$

C14S02.023: A *Mathematica* 3.0 command to draw the domain of the double integral is

```
Plot[ {x, -x/2, 1}, {x, -2, 1}, AspectRatio → Automatic,
        PlotRange → {{-2,1}, {0,1}} ];
```

the resulting figure is next.



The value of the double integral is

$$\int_0^1 \int_{-2y}^y (1-x)\,dx\,dy = \int_0^1 \left[x - \frac{1}{2}x^2\right]_{-2y}^y dy = \int_0^1 \left(3y + \frac{3}{2}y^2\right) dy = \left[\frac{1}{2}(3y^2 + y^3)\right]_0^1 = 2 - 0 = 2.$$

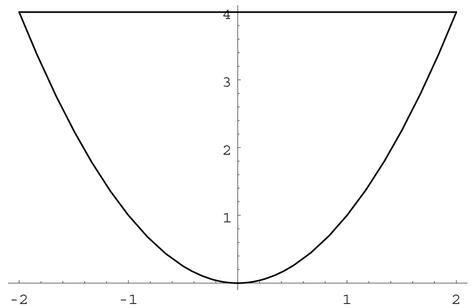
C14S02.024: The value of the double integral is

$$\int_0^3 \int_{2x}^{9-x} (9-y)\,dy\,dx = \int_0^3 \left[9y - \frac{1}{2}y^2\right]_{2x}^{9-x} dx = \int_0^3 \frac{3}{2}(x^2 - 12x + 27)\,dx = \left[\frac{1}{2}(x^3 - 18x^2 + 81x)\right]_0^3 = 54.$$

C14S02.025: The domain of the double integral can be plotted by executing the following *Mathematica* 3.0 command:

```
Plot[ {x*x, 4}, {x, -2, 2} ];
```

and the resulting figure is next.



When the order of integration is reversed, we obtain

$$\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} x^2 y \, dx \, dy = \int_0^4 \left[\frac{1}{3} x^3 y \right]_{-\sqrt{y}}^{\sqrt{y}} dy = \int_0^4 \frac{2}{3} y^{5/2} \, dy = \left[\frac{4}{21} y^{7/2} \right]_0^4 = \frac{512}{21} \approx 24.380952380952.$$

C14S02.026: To draw the domain of the double integral, execute the *Mathematica* 3.0 command

```
Plot[ {x, x∧4}, {x, 0, 1}, AspectRatio → Automatic ];
```

Reversal of the order of integration yields

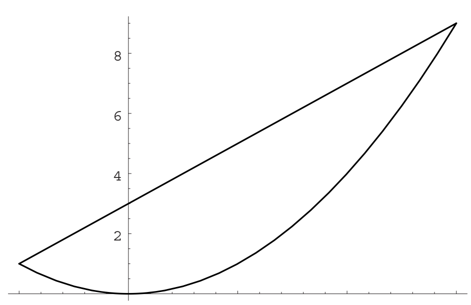
$$\int_0^1 \int_y^{y^{1/4}} (x-1) \, dx \, dy = \int_0^1 \left[\frac{1}{2} x^2 - x \right]_y^{y^{1/4}} dy$$

$$= \int_0^1 \frac{1}{2} (2y - y^2 + y^{1/2} - 2y^{1/4}) \, dy = \left[\frac{1}{2} y^2 - \frac{1}{6} y^3 + \frac{1}{3} y^{3/2} - \frac{4}{5} y^{5/4} \right]_0^1 = -\frac{2}{15}.$$

C14S02.027: The *Mathematica* 3.0 command

```
Plot[ { x*x, 2*x + 3 }, { x, -1, 3 } ];
```

produces a figure showing the domain of the double integral; it appears next.



When the order of integration is reversed, two integrals are required—one for the part of the region for which $0 \leqq y \leqq 1$, the other for the part when $1 \leqq y \leqq 9$. The reason is that the lower limit of integration for x changes where $y = 1$. The first integral is

$$I_1 = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} x\,dx\,dy = \int_0^1 \left[\frac{1}{2}x^2\right]_{-\sqrt{y}}^{\sqrt{y}} dy = \int_0^1 0\,dy = 0.$$

The second integral is

$$I_2 = \int_1^9 \int_{(y-3)/2}^{\sqrt{y}} x\,dx\,dy = \int_1^9 \left[\frac{1}{2}x^2\right]_{(y-3)/2}^{\sqrt{y}} dy$$

$$= \int_1^9 \frac{1}{8}(10y - 9 - y^2)\,dy = \left[\frac{1}{24}(15y^2 - 27y - y^3)\right]_1^9 = \frac{32}{3}.$$

C14S02.028: When the order of integration is reversed, two integrals are required. The one on the left half of the domain of the double integral is

$$I_1 = \int_{-4}^0 \int_{-\sqrt{4+x}}^{\sqrt{4+x}} y\,dy\,dx = \int_{-4}^0 \left[\frac{1}{2}y^2\right]_{-\sqrt{4+x}}^{\sqrt{4+x}} = \int_{-4}^0 0\,dx = 0$$

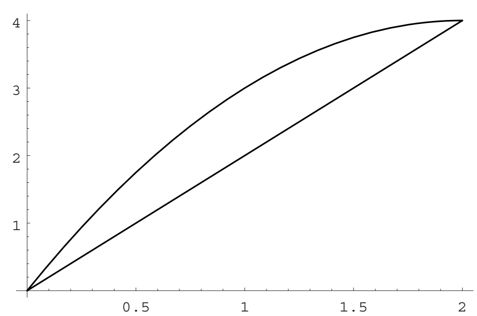
and the one of the right half of the domain is

$$I_2 = \int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} y\,dy\,dx = \int_0^4 \left[\frac{1}{2}y^2\right]_{-\sqrt{4-x}}^{\sqrt{4-x}} dx = \int_0^4 0\,dx = 0.$$

C14S02.029: To see the domain of the double integral, execute the *Mathematica* 3.0 command

```
Plot[ { 2*x, 4*x - x*x }, { x, 0, 2 } ];
```

the result is shown next.



When the order of integration is reversed, the given integral becomes

$$\int_0^4 \int_{2-\sqrt{4-y}}^{y/2} 1\,dx\,dy = \int_0^4 \Big[\,x\,\Big]_{2-\sqrt{4-y}}^{y/2} dy$$

$$= \int_0^4 \left(\frac{1}{2}y + (4-y)^{1/2} - 2\right) dy = \left[\frac{1}{4}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^4 = \frac{4}{3}.$$

C14S02.030: The domain of the given integral is bounded above by the line $y = x$, below by the x-axis, and on the right by the line $x = 1$. When the order of integration is reversed, we obtain

$$\int_0^1 \int_0^x \exp(-x^2)\,dy\,dx = \int_0^1 \Big[y\exp(-x^2) \Big]_0^x\,dx = \int_0^1 x\exp(-x^2)\,dx$$

$$= \left[-\frac{1}{2}\exp(-x^2) \right]_0^1 = \frac{e-1}{2e} \approx 0.3160602794142788.$$

Because the antiderivative

$$F(t) = \int_0^t \exp(-x^2)\,dx$$

is known to be nonelementary, the original integral cannot be evaluated by hand using the fundamental theorem of calculus.

C14S02.031: The domain of the given integral is bounded above by the line $y = \pi$, on the left by the y-axis, and on the right by the line $y = x$. When the order of integration is reversed, we obtain

$$\int_0^\pi \int_0^y \frac{\sin y}{y}\,dx\,dy = \int_0^\pi \left[\frac{x\sin y}{y} \right]_0^y dy = \int_0^\pi \sin y\,dy = \Big[-\cos y \Big]_0^\pi = 2.$$

If the improper integral is disturbing, merely define the integrand to have the value 1 at $y = 0$. Then it will be continuous and the integral will no longer be improper. Because the antiderivative

$$F(t) = \int_0^t \frac{\sin y}{y}\,dy$$

is known to be nonelementary, the only way to evaluate the given integral by hand is first to reverse the order of integration.

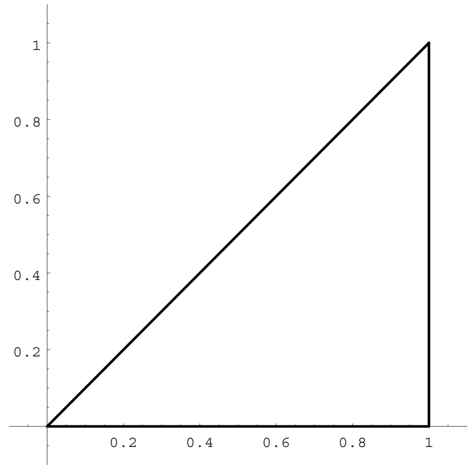
C14S02.032: The domain of the given integral is bounded above by the line $y = x$, on the right by the line $x = \sqrt{\pi}$, and below by the x-axis. When the order of integration is reversed, we obtain

$$\int_0^{\sqrt{\pi}} \int_0^x \sin x^2\,dy\,dx = \int_0^{\sqrt{\pi}} \Big[y\,\sin x^2 \Big]_0^x dx = \int_0^{\sqrt{\pi}} x\sin x^2\,dx = \left[-\frac{1}{2}\cos x^2 \right]_0^{\sqrt{\pi}} = \frac{1}{2} + \frac{1}{2} = 1.$$

C14S02.033: To generate a figure showing the domain of the given integral, use the *Mathematica* 3.0 command

```
ParametricPlot[ {{t,t}, {1,t}, {t,0}}, {t,0,1}, AspectRatio → Automatic,
          PlotRange → {{-0.1, 1.1}, {-0.1, 1.1}} ];
```

The result is shown next.



When the order of integration is reversed, we obtain

$$\int_0^1 \int_0^x \frac{1}{1+x^4}\,dy\,dx = \int_0^1 \left[\frac{y}{1+x^4}\right]_0^x dx = \int_0^1 \frac{x}{1+x^4}\,dx = \left[\frac{1}{2}\arctan x^2\right]_0^1 = \frac{\pi}{8}.$$

The integration can be carried out in the order given in the textbook, but finding the partial fraction decomposition of the integrand is long and complex if machine aid is not available.

C14S02.034: The domain of the given integral is the plane region bounded above by the graph of $y = \tan x$, below by the x-axis, and on the right by the line $x = \pi/4$. When the order of integration is reversed, the result is

$$\int_0^{\pi/4} \int_0^{\tan x} \sec x\,dy\,dx = \int_0^{\pi/4} \Big[\,y\,\sec x\,\Big]_0^{\tan x} dx = \int_0^{\pi/4} \sec x\,\tan x\,dx = \Big[\,\sec x\,\Big]_0^{\pi/4} = -1 + \sqrt{2}\,.$$

C14S02.035: We used *Mathematica* 3.0 in this problem. First we entered

```
Solve[ x∧3 + 1 == 3∗x∗x, x ]
```

and the computer returned the exact answers. Then we asked for the numerical values to 40 places, and we found that the curves intersect at the three points with approximate coordinates

$$(a,\,3a^2) \approx (-0.5320888862379561,\ 0.8493557485738457),$$

$$(b,\,3b^2) \approx (0.6527036446661393,\ 1.270661432813855), \quad \text{and}$$

$$(c,\,3c^2) \approx (2.8793852415718168,\ 24.8725781081447688).$$

We then entered the command

```
Plot[ {x∧3 + 1, 3∗x∗x}, {x, a, b}, PlotRange → {-1, 25} ];
```

and thereby discovered that the cubic graph is above the quadratic on $(a,\,b)$ but below it on $(b,\,c)$. Thus we needed to compute two integrals:

```
i1 = Integrate[ Integrate[ x, {y, 3*x*x, x^3 + 1} ], {x, a, b} ]
```

and

```
i2 = Integrate[ Integrate[ x, {y, x^3 + 1, 3*x*x} ], {x, b, c} ]
```

Results:

$$I_1 \approx 0.0276702414879754 \quad \text{and} \quad I_2 \approx 7.9240769481663325.$$

All the digits are correct or correctly rounded because we used at least 32 decimal digits in every computation. The answer, therefore, is

$$I_1 + I_2 \approx 7.9517471896543079.$$

C14S02.036: The *Mathematica* 3.0 command

```
Solve[ x^4 == x + 4, x ]
```

yielded the exact solution—two real, two complex non-real. We asked for the real solutions to 40 places and found that the two curves intersect in the two points

$$(a,\, a^4) \approx (-1.2837816658635382,\, 2.7162183341364618) \quad \text{and}$$

$$(b,\, b^4) \approx (1.5337511687552043,\, 5.5337511687552043).$$

Clearly the quartic lies under the linear graph on (a, b). Hence the only integral we need compute is

```
Integrate[ Integrate[ x, {y, x^4, x + 4} ], {x, a, b} ]
```

and the computer reported that its value is approximately 1.8930263071804474. All digits are correct or correctly rounded because we carried at least 32 decimal digits in every computation.

C14S02.037: We began with the *Mathematica* 3.0 command

```
Solve[ x*x - 1 == 1/(1 + x*x), x ]
```

and were rewarded with the exact solutions—two real, two complex non-real. The two real solutions are $a = -2^{1/4}$ and $b = 2^{1/4}$, so we used these exact values in the following computations. The double integral has the value

```
Integrate[ Integrate[ x, {y, x*x - 1, 1/(x*x + 1)}, {x, a, b} ]
```

$$\frac{1}{2}\left[1 - \sqrt{2} - \ln\left(1 + \sqrt{2}\,\right)\right] + \frac{1}{2}\left[-1 + \sqrt{2} + \ln\left(1 + \sqrt{2}\,\right)\right] = 0.$$

C14S02.038: We used *Mathematica* 3.0, beginning with the command

```
Solve[ x^4 - 16 == 2*x - x*x, x ]
```

and the computer returned the exact answer—two real solutions, two complex non-real solutions. We approximated the real solutions to 40 decimal digits; thus we found that the two curves cross at

$$(a,\, 2a - a^2) \approx (-1.7521717788841865,\, -6.5744495004865478) \quad \text{and}$$

$$(b,\, 2b - b^2) = (2,\, 0).$$

We carried at least 32 decimal digits in our computations, so the value of the integral given here is correct or correctly rounded:

```
Integrate[ Integrate[ x, {y, x∧4 - 16, 2*x - x*x}, {x, a, b} ] ]
```

8.8713485108009948311 61862.

C14S02.039: We used *Mathematica* 3.0 to automate Newton's method for solving the equation $x^2 = \cos x$:

```
f[x_] := x*x - Cos[x]
g[x_] := N[x - f[x]/f′[x], 60]
```

The function g carries out the iteration of Newton's method, carrying 60 digits in its computations. A graph indicated that the positive solution of $f(x) = 0$ is close to $4/5$, hence we entered the successive commands

```
g[4/5]
g[%]
```

(Recall that % refers to the "last output.")

```
g[%]
```

After six iterations the results agreed to over 50 decimal degits, and thus we find that the graphs cross at the two points

$$(b,\, b^2) \approx (0.8241323123025224,\, 0.6791940681811024) \quad \text{and } (a,\, a^2)$$

where $a = -b$. Hence the value of the double integral is

```
Integrate[ Integrate[ x, {y, x*x, Cos[x]}], {x, a, b} ]
```

0.0×10^{-58}

A moment's thought about Riemann sums reveals that the exact value of the integral is zero.

C14S02.040: We let $f(x) = x^2 - 2x - \sin x$. Clearly $f(0) = 0$. We used the function g of the solution of Problem 39 to implement Newton's method for finding the positive solution. Beginning with the initial approximation $x_0 = 2.2$, six iterations yielded 50-place accuracy; the graphs cross at $(0,\, 0)$ and at

$$(b,\, b^2 - 2b) \approx (2.3169342886237398,\, 0.7343159205529156).$$

The *Mathematica* 3.0 command

```
Integrate[ Integrate[ x, {y, x*x - 2*x, Sin[x]} ], {x, a, b} ]
```

3.394538444004254057156 3864737

yielded a good approximation to the value of the double integral. All digits shown are correct or correctly rounded because we carried at least 32 decimal digits in each computation.

C14S02.041: The integral is zero. Each term $f(x_i^\star,\, y_i^\star)\,\Delta A_i = x_i^\star\,\Delta A_i$ in every Riemann sum is cancelled by a similar term in which $x_i^\star$ has the opposite sign.

C14S02.042: By symmetry around both coordinate axes, the value of the integral is

$$4\int_0^1 \int_0^{1-x} x^2\,dy\,dx = 4\int_0^1 \Big[x^2 y\Big]_0^{1-x}\,dx = 4\int_0^1 (x^2 - x^3)\,dx = 4\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3}.$$

C14S02.043: Every term of the form $f(x_i^\star,\, y_i^\star)\,\Delta A_i$ in every Riemann sum is cancelled by a similar term in which $x_i^\star$ has the opposite sign (but $y_i^\star$ is the same). Therefore the value of the integral is zero.

C14S02.044: Clearly the double integral of y^2 over the square is the same as the double integral of x^2, so the answer is double the answer to Problem 42.

C14S02.045: Suppose that the rectangle R consists of those points $(x,\, y)$ for which both $a \leqq x \leqq b$ and $c \leqq y \leqq d$. Suppose that k is a positive constant and that f is a function continuous on R. Then

$$I = \iint_R f(x,\, y)\,dA$$

exists. Suppose that $\epsilon > 0$ is given. Then there exists a number $\delta_1 > 0$ such that, for every partition $\mathcal{P} = \{R_1,\, R_2,\, \dots\, R_n\}$ of R such that $|\mathcal{P}| < \delta_1$ and for every selection $(x_i^\star,\, y_i^\star)$ in R_i $(i = 1,\, 2,\, \dots,\, n)$,

$$\left|\sum_{i=1}^n f(x_i^\star,\, y_i^\star)\,\Delta A_i \;-\; I\right| < \frac{\epsilon}{2k}$$

(where ΔA_i is the area $a(R_i)$ of R_i). Consequently,

$$\left|k\sum_{i=1}^n f(x_i^\star,\, y_i^\star)\,\Delta A_i \;-\; kI\right| < \frac{\epsilon}{2}. \tag{1}$$

Moreover, kf is continuous on R, and hence

$$J = \iint_R kf(x,\, y)\,dA$$

exists. So there exists a number $\delta_2 > 0$ such that, for every partition $\mathcal{P} = \{R_1,\, R_2,\, \dots\, R_n\}$ of R such that $|\mathcal{P}| < \delta_2$ and every selection $(x_i^\star,\, y_i^\star)$ in R_i $(\,i = 1,\ 2,\ \dots\ n)$,

$$\left|\sum_{i=1}^n kf(x_i^\star,\, y_i^\star)\,\Delta A_i \;-\; J\right| < \frac{\epsilon}{2}; \qquad \text{that is,}$$

$$\left|k\sum_{i=1}^n f(x_i^\star,\, y_i^\star)\,\Delta A_i \;-\; J\right| < \frac{\epsilon}{2}.$$

Let δ be the minimum of δ_1 and δ_2. Then, for every partition $\mathcal{P}$ of R with $|\mathcal{P}| < \delta$ and every selection $(x_i^\star,\, y_i^\star)$ in R_i $(i = 1,\ 2,\ \dots\ n)$, we have both

$$\left| kI \;-\; k\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star)\,\Delta A_i \right| < \frac{\epsilon}{2} \qquad \text{(by (1)) and}$$

$$\left| k\sum_{i=1}^{n} f(x_i^\star,\, y_i^\star)\,\Delta A_i \;-\; J \right| < \frac{\epsilon}{2}.$$

Add the last two inequalities. Then, by the triangle inequality (Theorem 1 of Appendix A, page A-2),

$$|kI \;-\; J| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Because ϵ is an *arbitrary* positive number, this proves that $J = kI$; that is, we have shown that

$$\iint_R kf(x,\, y)\,dA \;=\; k\iint_R f(x,\, y)\,dA.$$

The proof is similar in the case $k < 0$, and if $k = 0$ there is nothing to prove. ◀

For a shorter proof, one that exploits both the continuity of f and the fact that R is a rectangle with sides parallel to the coordinate axes, choose a continuous function F such that $F_x = f$. Then choose a continuous function P such that $P_y = F$. Then

$$\begin{aligned}
\iint_R kf(x,\, y)\,dA &= \int_c^d \int_a^b kf(x,\, y)\,dx\,dy = \int_c^d \Big[kF(x,\, y) \Big]_a^b \,dy \\
&= \int_c^d [kF(b,\, y) - kF(a,\, y)]\,dy = \Big[kP(b,\, y) - kP(a,\, y) \Big]_c^d \\
&= kP(b,\, d) - kP(a,\, d) - kP(b,\, c) + kP(a,\, c) = k\,[P(b,\, d) - P(a,\, d) - P(b,\, c) + P(a,\, c)] \\
&= k\Big[P(b,\, y) - P(a,\, y) \Big]_c^d = k\int_c^d [F(b,\, y) - F(a,\, y)]\,dy \\
&= k\int_c^d \Big[F(x,\, y) \Big]_a^b \,dy = k\int_c^d \int_a^b f(x,\, y)\,dx\,dy = k\iint_R f(x,\, y)\,dA.
\end{aligned}$$

C14S02.046: Suppose that R is a plane rectangle with sides parallel to the coordinate axes, so that R consists of those points $(x,\, y)$ for which both $a \leqq x \leqq b$ and $c \leqq y \leqq d$ for some numbers a, b, c, and d. Suppose that f and g are functions both of which are continuous on R. Then there exist continuous functions F and G such that $F_x = f$ and $G_x = g$ on R. Moreover, there exist continuous functions P and Q such that $P_y = F$ and $Q_y = G$ on R. Then

$$\begin{aligned}
\iint_R f(x,\, y)\,dA &= \int_c^d \int_a^b f(x,\, y)\,dx\,dy = \int_c^d \Big[F(x,\, y) \Big]_a^b \,dy \\
&= \int_c^d [F(b,\, y) - F(a,\, y)]\,dy = \Big[P(b,\, y) - P(a,\, y) \Big]_c^d \\
&= P(b,\, d) - P(a,\, d) - P(b,\, c) + P(a,\, c).
\end{aligned}$$

Similarly,

$$\iint_R g(x,\,y)\,dA = Q(b,\,d) - Q(a,\,d) - Q(b,\,c) + Q(a,\,c).$$

Finally,

$$\begin{aligned}
\iint_R [f(x,\,y) + g(x,\,y)]\;dA &= \int_c^d \int_a^b [f(x,\,y) + g(x,\,y)]\;dx\;dy = \int_c^d \Big[F(x,\,y) + G(x,\,y)\Big]_a^b\;dy \\
&= \int_c^d [F(b,\,y) + G(b,\,y) - F(a,\,y) - G(a,\,y)]\;dy \\
&= \Big[P(b,\,y) + Q(b,\,y) - P(a,\,y) - Q(a,\,y)\Big]_c^d \\
&= P(b,\,d) + Q(b,\,d) - P(a,\,d) - Q(a,\,d) - P(b,\,c) - Q(b,\,c) + P(a,\,c) + Q(a,\,c).
\end{aligned}$$

We compare these three results; it follows immediately that

$$\iint_R [f(x,\,y) + g(x,\,y)]\;dA = \iint_R f(x,\,y)\,dA \;+\; \iint_R g(x,\,y)\,dA.$$

C14S02.047: Suppose that R is a plane rectangle with sides parallel to the coordinate axes, so that R consists of those points $(x,\,y)$ for which both $a \leqq x \leqq b$ and $c \leqq y \leqq d$ for some numbers $a,$ $b,$ $c,$ and $d.$ Suppose that f is continuous on R and that $m \leqq f(x,\,y) \leqq M$ for all $(x,\,y)$ in $R.$

Let $g(x,\,y) \equiv m$ and $h(x,\,y) \equiv M$ for $(x,\,y)$ in $R.$ Then $g(x,\,y) \leqq f(x,\,y) \leqq h(x,\,y)$ for all points $(x,\,y)$ in $R.$ Let $\mathcal{P} = \{R_1,\,R_2,\,\dots,\,R_n\}$ be a partition of R and let $(x_i^\star,\,y_i^\star)$ be a selected point in R_i for $1 \leqq i \leqq n.$ As usual, let $\Delta A_i = a(R_i)$ for $1 \leqq i \leqq n.$ Then for each integer $i,$ $1 \leqq i \leqq n,$ we have

$$g(x_i^\star,\,y_i^\star) \;\leqq\; f(x_i^\star,\,y_i^\star) \;\leqq\; h(x_i^\star,\,y_i^\star);$$

thus

$$g(x_i^\star,\,y_i^\star)\,\Delta A_i \;\leqq\; f(x_i^\star,\,y_i^\star)\,\Delta A_i \;\leqq\; h(x_i^\star,\,y_i^\star)\,\Delta A_i$$

for $1 \leqq i \leqq n.$ Add these inequalities to find that

$$\sum_{i=1}^n m \cdot \Delta A_i \;\leqq\; \sum_{i=1}^n f(x_i^\star,\,y_i^\star)\,\Delta A_i \;\leqq\; \sum_{i=1}^n M \cdot \Delta A_i.$$

Therefore

$$m \cdot \sum_{i=1}^n a(R_i) \;\leqq\; \sum_{i=1}^n f(x_i^\star,\,y_i^\star)\,\Delta A_i \;\leqq\; M \cdot \sum_{i=1}^n a(R_i);$$

that is,

$$m \cdot a(R) \;\leqq\; \sum_{i=1}^n f(x_i^\star,\,y_i^\star)\,\Delta A_i \;\leqq\; M \cdot a(R) \tag{1}$$

for every partition $\mathcal{P}$ of R and every selection $(x_i^\star,\,y_i^\star)$ for $\mathcal{P}.$ That is, the inequalities in (1) hold for every Riemann sum for f on $R.$ Because the double integral of f on R is the limit of such sums, we may conclude that

$$m \cdot a(R) \leqq \iint_R f(x, y)\, dA \leqq M \cdot a(R).$$

C14S02.048: Suppose that R_1 and R_2 are rectangles with sides parallel to the coordinate axes and that the right-hand edge of R_1 coincides with the left-hand edge of R_2. Suppose that R_1 consists of those points (x, y) in the plane for which $a \leqq x \leqq b$ and $r \leqq y \leqq s$ and that R_2 consists of those points (x, y) for which $b \leqq x \leqq c$ and $r \leqq y \leqq s$. Let R be the union of R_1 and R_2, so that R consists of those points (x, y) for which $a \leqq x \leqq c$ and $r \leqq y \leqq s$.

Suppose that f is continuous on R, and thus on R_1 and R_2. Choose a continuous function F such that $F_x = f$ on R and a function P such that $P_y = F$ on R. Note that these equations hold on R_1 and R_2 as well. Then

$$\iint_{R_1} f(x, y)\, dA = \int_r^s \int_a^b f(x, y)\, dx\, dy = \int_r^s \Big[F(x, y)\Big]_a^b\, dy = \int_r^s [F(b, y) - F(a, y)]\, dy$$
$$= \Big[P(b, y) - P(a, y)\Big]_r^s = P(b, s) - P(a, s) - P(b, r) + P(a, r).$$

Similarly,

$$\iint_{R_2} f(x, y)\, dA = P(c, s) - P(b, s) - P(c, r) + P(b, r)$$

and

$$\iint_R f(x, y)\, dA = P(c, s) - P(a, s) - P(c, r) + P(a, r).$$

Therefore

$$\iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$
$$= P(b, s) - P(a, s) - P(b, r) + P(a, r) + P(c, s) - P(b, s) - P(c, r) + P(b, r)$$
$$= P(c, s) - P(a, s) - P(c, r) + P(a, r) = \iint_R f(x, y)\, dA.$$

C14S02.049: Suppose that R is a rectangle with sides parallel to the coordinate axes, that $f(x, y) \leqq g(x, y)$ for all (x, y) in R, and that both

$$I = \iint_R f(x, y)\, dA \quad \text{and} \quad J = \iint_R g(x, y)\, dA$$

exist. Suppose by way of contradiction that $J < I$. Let $\epsilon = I - J$. Note that $\epsilon/3 > 0$. Choose $\delta > 0$ so small that if $\mathcal{P} = \{R_1, R_2, \dots, R_n\}$ is a partition of R with $|\mathcal{P}| < \delta$ and $(x_i^\star, y_i^\star)$ is a selection for $\mathcal{P}$ with $(x_i^\star, y_i^\star)$ in R_i for $1 \leqq i \leqq n$, then

$$\left| \sum_{i=1}^n f(x_i^\star, y_i^\star)\, \Delta A_i - I \right| < \frac{\epsilon}{3} \quad \text{and} \quad \left| \sum_{i=1}^n g(x_i^\star, y_i^\star)\, \Delta A_i - J \right| < \frac{\epsilon}{3}.$$

With such a partition and such a selection, note that

$$\sum_{i=1}^{n} g(x_i^\star,\, y_i^\star)\,\Delta A_i \;< J + \frac{\epsilon}{3} \;<\; I - \frac{\epsilon}{3} \;<\; \sum_{i=1}^{n} f(x_i^\star,\, y_i^\star)\,\Delta A_i,$$

and thus

$$\sum_{i=1}^{n} [\,f(x_i^\star,\, y_i^\star) \;-\; g(x_i^\star,\, y_i^\star)]\;\Delta A_i \;>\; 0.$$

Because $\Delta A_i > 0$ for $1 \leqq i \leqq n,$ it follows that

$$f(x_j^\star,\, y_j^\star) \;>\; g(x_j^\star,\, y_j^\star)$$

for some $j,\ 1 \leqq j \leqq n,$ contrary to hypothesis. Therefore $I \leqq J.$ ◀

C14S02.050: Let $m = f(x_0,\, y_0)$ and $M = f(x_1,\, y_1)$ where $(x_0,\, y_0)$ and $(x_1,\, y_1)$ are points of $R.$ Let $\mathbf{r}(t)$ be a continuous parametric curve lying entirely in R such that $\mathbf{r}(0) = (x_0,\, y_0)$ and $\mathbf{r}(1) = (x_1,\, y_1).$ By Eq. (8) of Section 14.2,

$$m \cdot a(R) \;\leqq\; \iint_R f(x,\, y)\; dA \;\leqq\; M \cdot a(R).$$

Note that $g(t) = f(\mathbf{r}(t)) \cdot a(R)$ is continuous on $[0,\, 1]$ and that $g(0) = m \cdot a(R)$ and $g(1) = M \cdot a(R).$ Therefore

$$g(\overline{t}) = \iint_R f(x,\, y)\; dA \tag{1}$$

for some $\overline{t}$ in $[0,\, 1].$ Let $(\overline{x},\, \overline{y}) = g(\overline{t}).$ Then $(\overline{x},\, \overline{y})$ is a point of $R,$ and by Eq. (1),

$$f(\overline{x},\, \overline{y}) \cdot a(R) \;=\; \iint_R f(x,\, y)\; dA.$$

C14S02.051: Recall that R is the region in the first quadrant bounded by the circle $x^2 + y^2 \,=\, 1$ and the coordinate axes. Hence

$$\iint_R (x + y)\; dA \;=\; \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} (x + y)\; dy\; dx$$

$$= \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_0^{\sqrt{1-x^2}} dx \;=\; \int_0^1 \left(x\sqrt{1 - x^2} \;+ \frac{1}{2}(1 - x^2) \right) dx$$

$$= \left[-\frac{1}{3}(1 - x^2)^{3/2} + \frac{1}{2}x - \frac{1}{6}x^3 \right]_0^1 \;=\; \frac{1}{2} - \frac{1}{6} + \frac{1}{3} \;=\; \frac{2}{3}.$$

C14S02.052: Here is one way to use *Mathematica* to solve this problem. First define $f(x,\, y) \,=\, xy$ and set $n \,=\, 5.$ Then the Riemann sum for the induced partition using the midpoint of each small rectangle (actually, a square) is

```
( 1/n∧2)*Sum[ f[ (i − 1/2)/n, (y − 1/2)/n ], { j, 1, n − 1 },
              { i, 1, IntegerPart[ Sqrt[ n∧2 − j∧2 ] ] } ]
```

$$\frac{207}{2500}$$

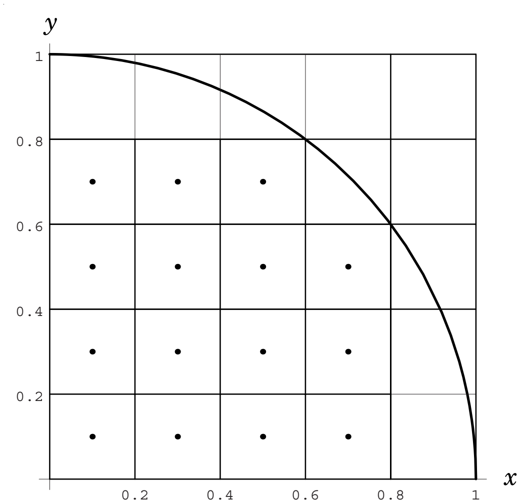
(You may need to substitute `Floor` for `IntegerPart` in the *Mathematica* command shown here.) So the midpoint approximation to the integral is 0.0828. Its actual value is

```
Integrate[ f[ x, y ], { y, 0, 1 }, { x, 0, Sqrt[ 1 − y∧2 ] } ]
```

$$\frac{1}{8}$$

Therefore the exact value of the integral is 0.125. You can use ideas illustrated by the limits of summation in the first command to verify the entries in the second column of the table in Fig. 14.2.3. —C.H.E.

C14S02.053: The domain of the integral and the partition using $n = 5$ subintervals of equal length in each direction is shown next.



The midpoints of the subrectangles of the inner partition are indicated with "bullets" in the figure. Let $f(x, y) = xy \exp\left(y^2\right)$. Then the corresponding midpoint sum for the given integral is

$$\begin{aligned} S &= \frac{1}{n^2}\Big[f(0.1,\,0.1) + f(0.3,\,0.1) + f(0.5, 0.1) + f(0.7, 0.1) + f(0.3,\,0.1) + f(0.3,\,0.3) \\ &\qquad + f(0.3,\,0.5) + f(0.3,\,0.7) + f(0.5,\,0.1) + f(0.5,\,0.3) + f(0.5, 0.5) + f(0.5,\,0.7) \\ &\qquad + f(0.7,\,0.1) + f(0.7,\,0.3) + f(0.7,\,0.5)\Big] \\ &= \frac{1}{25}\left[\frac{4}{25}e^{0.01} + \frac{12}{25}e^{0.09} + \frac{4}{5}e^{0.25} + \frac{63}{100}e^{0.49}\right] \approx 0.109696. \end{aligned}$$

The exact value of the integral is

$$\iint_R f(x,\,y)\,dA = \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} xy \exp\left(y^2\right)\,dx\,dy$$

$$= \int_0^1 \left[\frac{1}{2}x^2 y \exp\left(y^2\right)\right]_0^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{2}(y - y^3) \exp\left(y^2\right) dy$$

$$= \left[\frac{2-y^2}{4} \exp\left(y^2\right)\right]_0^1 = \frac{e-2}{4} \approx 0.17957045711476130884007186 78.$$

If you prefer the other order of integration—which avoids the integration by parts—it is

$$\iint_R f(x,\,y)\,dA = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} xy \exp\left(y^2\right) dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x \exp\left(y^2\right)\right]_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{1}{2}\left[x \exp\left(1-x^2\right) - x\right] dx$$

$$= \left[-\frac{1}{4}x^2 - \frac{1}{4}\exp\left(1-x^2\right)\right]_0^1 = \frac{e-2}{4}.$$

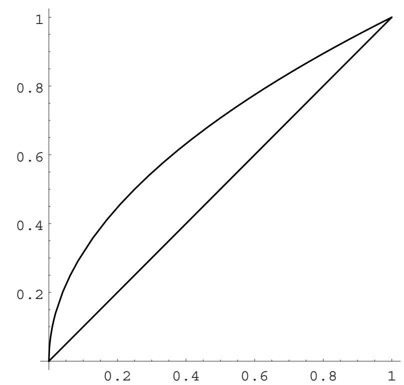
Section 14.3

C14S03.001: The area is

$$A = \int_{y=0}^{1} \int_{x=y^2}^{y} 1 \, dx \, dy = \int_{y=0}^{1} \left[x \right]_{x=y^2}^{y} dy$$
$$= \int_{y=0}^{1} (y - y^2) \, dy = \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_{y=0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

To find the limits of integration, it is very helpful to sketch the domain of the double integral. The figure is next; it was produced by *Mathematica* 3.0 via the command

```
Plot[ {x, Sqrt[x]}, {x, 0, 1}, AspectRatio→ Automatic ];
```

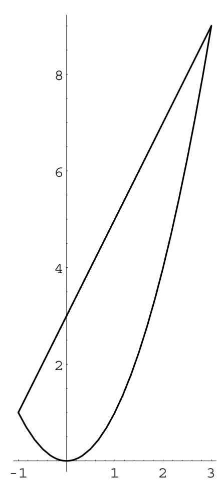


C14S03.002: The area is

$$A = \int_{0}^{1} \int_{y=x^4}^{y=x} 1 \, dy \, dx = \int_{0}^{1} (x - x^4) \, dx = \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_{0}^{1} = \frac{3}{10}.$$

C14S03.003: The graphs cross where $x^2 = 2x + 3$; that is, where $x = -1$ and where $x = 3$. A sketch of the domain of the integral is next; it was produced by *Mathematica* 3.0 via the command

```
Plot[ {x∗x, 2∗x + 3}, {x, -1, 3}, AspectRatio → Automatic ];
```



The area of the region is

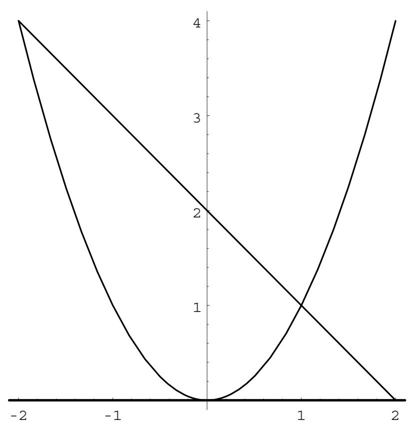
$$A = \int_{-1}^{3} \int_{y=x^2}^{2x+3} 1 \, dy \, dx = \int_{-1}^{3} (3 + 2x - x^2) \, dx = \left[3x + x^2 - \frac{1}{3}x^3 \right]_{-1}^{3} = \frac{32}{3}.$$

C14S03.004: The graphs cross where $2x + 3 = 6x - x^2$; that is, where $x = 1$ and where $x = 3$. The area they enclose is

$$A = \int_{1}^{3} \int_{y=2x+3}^{6x-x^2} 1 \, dy \, dx = \int_{1}^{3} (4x - 3 - x^2) \, dx = \left[2x^2 - 3x - \frac{1}{3}x^3 \right]_{1}^{3} = \frac{4}{3}.$$

C14S03.005: The graphs cross where $x^2 = 2 - x$; that is, where $x = -2$ and where $x = 1$. But the x-axis is also part of the boundary of the region in question, and hence the following figure is important to find not only the correct limits of integration, but indeed the very *region* whose area is sought. It was produced by the *Mathematica* 3.0 command

```
Plot[ {x*x, 2 - x}, {x, -2, 2}, AspectRatio → Automatic ];
```



(We enhanced the result using Adobe Illustrator.) The area of the region bounded by all three graphs is

$$A = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} 1 \, dx \, dy = \int_{0}^{1} (2 - \sqrt{y} - y) \, dy = \left[2y - \frac{2}{3}y^{3/2} - \frac{1}{2}y^2 \right]_{0}^{1} = \frac{5}{6}.$$

C14S03.006: The region is bounded on the northwest by the graph of $y = (x + 1)^2$, on the northeast by the graph of $y = (x - 1)^2$, and below by the x-axis. To avoid radicals we will integrate first with respect to y, then with respect to x, even though this entails computing two integrals. The area of the part of the region to the right of the y-axis is

$$A_1 = \int_{x=0}^{1} \int_{y=0}^{(x-1)^2} 1 \, dy \, dx = \int_{0}^{1} (x - 1)^2 \, dx = \left[x - x^2 + \frac{1}{3}x^3 \right]_{0}^{1} = \frac{1}{3}.$$

The area of the part of the region to the left of the y-axis is

$$A_2 = \int_{x=-1}^{0} \int_{y=0}^{(x+1)^2} 1 \, dy \, dx = \int_{-1}^{0} (x + 1)^2 \, dx = \left[x + x^2 + \frac{1}{3}x^3 \right]_{-1}^{0} = \frac{1}{3}.$$

Therefore the total area bounded by all three of the given curves is $A_1 + A_2 = \dfrac{2}{3}$.

C14S03.007: The graphs cross where $x^2 + 1 = 2x^2 - 3$; that is, where $x = -2$ and where $x = 2$. The area between them is

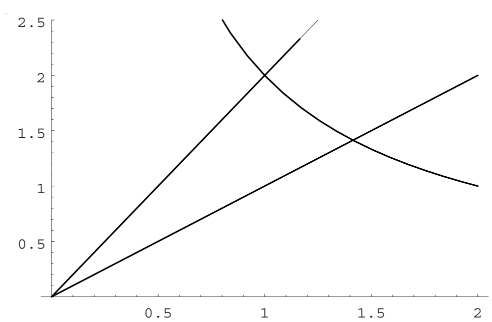
$$A = \int_{x=-2}^{2} \int_{y=2x^2-3}^{x^2+1} 1\,dy\,dx = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{32}{3}.$$

C14S03.008: The graphs cross where $x^2 + 1 = 9 - x^2$; that is, where $x = -2$ and where $x = 2$. The area between them is

$$A = \int_{x=-2}^{2} \int_{y=x^2+1}^{9-x^2} 1\,dy\,dx = \int_{-2}^{2} (8 - 2x^2)\,dx = \left[8x - \frac{2}{3}x^3\right]_{-2}^{2} = \frac{64}{3}.$$

C14S03.009: The part of the region that lies in the first quadrant is shown next; the figure was generated using the *Mathematica* 3.0 command

```
Plot[ {x, 2*x, 2/x}, {x, 0, 2}, PlotRange → {0, 2.5} ];
```



The bounding curves cross at the origin and at the points (1, 2), and $\left(\sqrt{2},\, \sqrt{2}\,\right)$. The area they bound (in the first quadrant) is

$$A = \int_{x=0}^{1} \int_{y=x}^{2x} 1\,dy\,dx + \int_{x=1}^{\sqrt{2}} \int_{y=x}^{2/x} 1\,dy\,dx = \int_0^1 x\,dx + \int_1^{\sqrt{2}} \left(\frac{2}{x} - x\right)\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1 + \left[2\ln x - \frac{1}{2}x^2\right]_1^{\sqrt{2}} = \frac{1}{2} + \ln 2 - \frac{1}{2} = \ln 2 \approx 0.6931471805599453094172\,32.$$

The region is symmetric around the origin, so the total area is $2\ln 2$.

C14S03.010: The curves cross where

$$x^2 = \frac{2}{1+x^2};$$

$$x^4 + x^2 - 2 = 0;$$

$$(x^2+2)(x^2-1) = 0;$$

thus where $x = -1$ and where $x = 1$. The area of the region they bound is

$$\int_{x=-1}^{1} \int_{y=x^2}^{2/(1+x^2)} 1\, dy\, dx = \int_{-1}^{1} \left(\frac{2}{1+x^2} - x^2 \right) dx = \left[2\arctan x - \frac{1}{3}x^3 \right]_{-1}^{1} = \pi - \frac{2}{3} \approx 2.474925986923.$$

C14S03.011: The volume is

$$V = \int_{x=0}^{1} \int_{y=0}^{1} (1+x+y)\, dy\, dx = \int_{x=0}^{1} \left[y + xy + \frac{1}{2}y^2 \right]_{y=0}^{1} dx = \int_{0}^{1} \left(x + \frac{3}{2} \right) dx = \left[\frac{3}{2}x + \frac{1}{2}x^2 \right]_{0}^{1} = 2.$$

C14S03.012: The volume is

$$V = \int_{y=0}^{2} \int_{x=0}^{3} (2x+3y)\, dx\, dy = \int_{y=0}^{2} \left[x^2 + 3xy \right]_{x=0}^{3} dy = \int_{0}^{2} (9+9y)\, dy = \left[9y + \frac{9}{2}y^2 \right]_{0}^{2} = 36.$$

C14S03.013: The volume is

$$V = \int_{y=0}^{2} \int_{x=0}^{1} (y+e^x)\, dx\, dy = \int_{y=0}^{2} \left[xy + e^x \right]_{x=0}^{1} dy = \int_{0}^{2} (e+y-1)\, dy = \left[ey + \frac{1}{2}y^2 - y \right]_{0}^{2} = 2e.$$

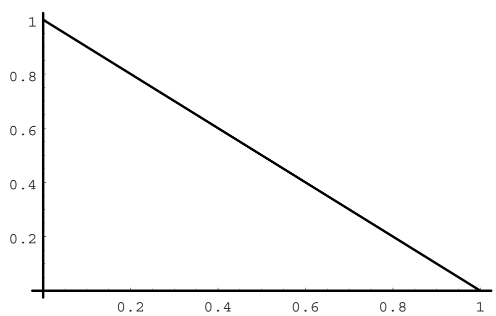
C14S03.014: The volume is

$$\begin{aligned} V &= \int_{x=0}^{\pi} \int_{y=0}^{\pi} (3+\cos x+\cos y)\, dy\, dx = \int_{x=0}^{\pi} \left[3y + y\cos x + \sin y \right]_{y=0}^{\pi} dx \\ &= \int_{0}^{\pi} (3\pi + \pi\cos x)\, dx = \left[3\pi x + \pi \sin x \right]_{0}^{\pi} = 3\pi^2 \approx 29.608813203268075856503473. \end{aligned}$$

C14S03.015: The domain of the integral can be drawn by using the *Mathematica* 3.0 command

```
Plot[ 1 - x, {x, 0, 1} ];
```

and the result is shown next.



The volume is

$$V = \int_{x=0}^{1} \int_{y=0}^{1-x} (x+y)\, dy\, dx = \int_{x=0}^{1} \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{1-x} dx = \int_{0}^{1} \frac{1}{2}(1-x^2)\, dx = \left[\frac{1}{2}x - \frac{1}{6}x^3 \right]_{0}^{1} = \frac{1}{3}.$$

C14S03.016: The volume is

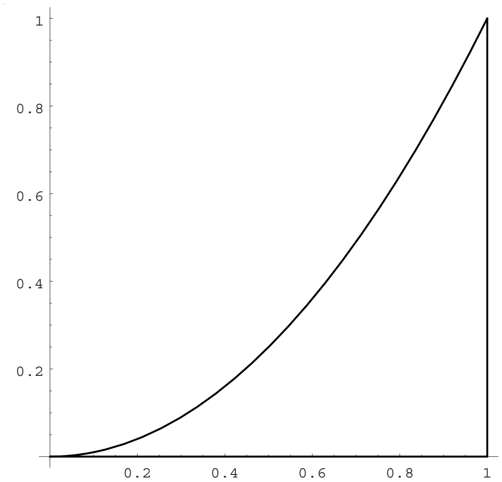
$$V = \int_{x=0}^{4} \int_{y=0}^{(4-x)/2} (3x + 2y)\ dy\ dx = \int_{x=0}^{4} \left[3xy + y^2 \right]_{y=0}^{(4-x)/2} dx$$

$$= \int_0^4 \left(4 + 4x - \frac{5}{4}x^2 \right) dx = \left[4x + 2x^2 - \frac{5}{12}x^3 \right]_0^4 = \frac{64}{3}.$$

C14S03.017: The domain of the integral can be drawn using the *Mathematica* 3.0 command

```
ParametricPlot[ {{1,t}, {t,0}, {t,t*t}}, {t,0,1}, AspectRatio → Automatic ];
```

and the result is shown next.



The volume is

$$V = \int_{x=0}^{1} \int_{y=0}^{x^2} (1 + x + y)\ dy\ dx = \int_{x=0}^{1} \left[y + xy + \frac{1}{2}y^2 \right]_{y=0}^{x^2} dx = \int_0^1 \left(x^2 + x^3 + \frac{1}{2}x^4 \right) dx$$

$$= \left[\frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{10}x^5 \right]_0^1 = \frac{41}{60} \approx 0.683333333333.$$

C14S03.018: The domain of the integral can be drawn using the *Mathematica* 3.0 command

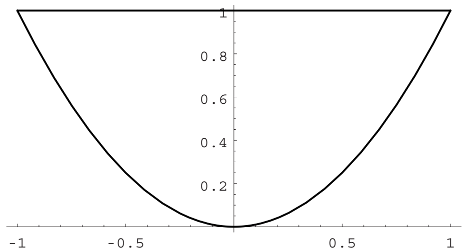
```
ParametricPlot[ {{0,t}, {t,1}, {Sqrt[t],t}}, {t,0,1}, AspectRatio → Automatic ];
```

and the volume of the solid is

$$V = \int_{x=0}^{1} \int_{y=x^2}^{1} (2x + y)\ dy\ dx = \int_{x=0}^{1} \left[2xy + \frac{1}{2}y^2 \right]_{y=x^2}^{1} dx = \int_0^1 \left(\frac{1}{2} + 2x - 2x^3 - \frac{1}{2}x^4 \right) dx$$

$$= \left[\frac{1}{2}x + x^2 - \frac{1}{2}x^4 - \frac{1}{10}x^5 \right]_0^1 = \frac{9}{10}.$$

C14S03.019: The domain of the integral is shown next.



The volume of the solid is

$$V = \int_{x=-1}^{1} \int_{y=x^2}^{1} x^2 \, dy \, dx = \int_{x=-1}^{1} \left[x^2 y \right]_{y=x^2}^{1} dx = \int_{-1}^{1} (x^2 - x^4) \, dx = \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^{1} = \frac{4}{15}.$$

C14S03.020: The domain of the integral can be drawn using the *Mathematica* 3.0 command

```
ParametricPlot[ {{t*t,t}, {4,t}}, {t, -2, 2} ];
```

and the volume of the solid is

$$V = \int_{y=-2}^{2} \int_{x=y^2}^{4} y^2 \, dx \, dy = \int_{y=-2}^{2} \left[xy^2 \right]_{x=y^2}^{4} dy = \int_{-2}^{2} (4y^2 - y^4) \, dy$$

$$= \left[\frac{4}{3}y^3 - \frac{1}{5}y^5 \right]_{-2}^{2} = \frac{128}{15} \approx 8.533333333333.$$

C14S03.021: The volume of the solid is

$$V = \int_{y=0}^{2} \int_{x=0}^{1} (x^2 + y^2) \, dx \, dy = \int_{y=0}^{2} \left[\frac{1}{3}x^3 + xy^2 \right]_{x=0}^{1} dy = \int_{0}^{2} \left(\frac{1}{3} + y^2 \right) dy = \left[\frac{1}{3}(y + y^3) \right]_{0}^{2} = \frac{10}{3}.$$

C14S03.022: The volume is

$$V = \int_{x=-2}^{1} \int_{y=x}^{2-x^2} (1 + x^2 + y^2) \, dy \, dx = \int_{x=-2}^{1} \left[y + x^2 y + \frac{1}{3}y^3 \right]_{y=x}^{2-x^2} dx$$

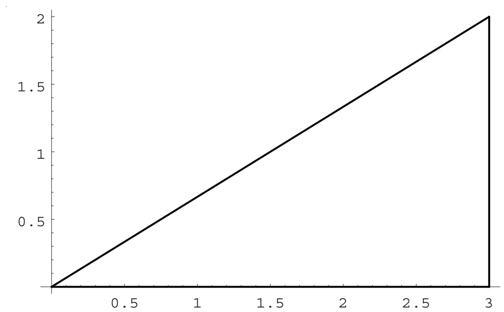
$$= \int_{-2}^{1} \frac{1}{3}(14 - 3x - 9x^2 - 4x^3 + 3x^4 - x^6) \, dx = \left[\frac{14}{3}x - \frac{1}{2}x^2 - x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 - \frac{1}{21}x^7 \right]_{-2}^{1}$$

$$= \frac{837}{70} \approx 11.9571428571428571.$$

C14S03.023: The domain of the integral can be drawn by executing the *Mathematica* 3.0 command

```
ParametricPlot[ {{3, 2*t/3}, {t, 0}, {t, 2*t/3}}, {t, 0, 3} ];
```

and the result is shown next.



The volume of the solid is

$$V = \int_{x=0}^{3} \int_{y=0}^{2x/3} (9 - x - y)\, dy\, dx = \int_{x=0}^{3} \left[9y - xy - \frac{1}{2}y^2 \right]_{y=0}^{2x/3} dx$$
$$= \int_0^3 \left(6x - \frac{8}{9}x^2 \right) dx = \left[3x^2 - \frac{8}{27}x^3 \right]_0^3 = 19.$$

C14S03.024: The volume of the solid is

$$V = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (10 + y - x^2)\, dy\, dx = \int_{x=0}^{1} \left[10y - x^2 y + \frac{1}{2}y^2 \right]_{y=x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 \left(10x^{1/2} + \frac{1}{2}x - 10x^2 - x^{5/2} + \frac{1}{2}x^4 \right) dx = \left[\frac{20}{3}x^{3/2} + \frac{1}{4}x^2 - \frac{10}{3}x^3 - \frac{2}{7}x^{7/2} + \frac{1}{10}x^5 \right]_0^1$$
$$= \frac{1427}{420} \approx 3.3976190476190476.$$

C14S03.025: The volume is

$$V = \int_{x=0}^{1} \int_{y=0}^{2-2x} (4x^2 + y^2)\, dy\, dx = \int_{x=0}^{1} \left[4x^2 y + \frac{1}{3}y^3 \right]_{y=0}^{2-2x} dx$$
$$= \int_0^1 \frac{8}{3}(1 - 3x + 6x^2 - 4x^3)\, dx = \left[\frac{8}{3}x - 4x^2 + \frac{16}{3}x^3 - \frac{8}{3}x^4 \right]_0^1 = \frac{4}{3}.$$

C14S03.026: The volume is

$$V = \int_{x=0}^{1} \int_{y=x^3}^{x^2} (2x + 3y)\, dy\, dx = \int_{x=0}^{1} \left[2xy + \frac{3}{2}y^2 \right]_{y=x^3}^{x^2} dx = \int_0^1 \left(2x^3 - \frac{1}{2}x^4 - \frac{3}{2}x^6 \right) dx$$
$$= \left[\frac{1}{2}x^4 - \frac{1}{10}x^5 - \frac{3}{14}x^7 \right]_0^1 = \frac{13}{70} \approx 0.1857142857142857.$$

C14S03.027: The volume is

$$V = \int_{x=0}^{2} \int_{y=0}^{(6-3x)/2} (6 - 3x - 2y)\ dy\ dx = \int_{x=0}^{2} \Big[6y - 3xy - y^2 \Big]_{y=0}^{(6-3x)/2} dx$$

$$= \int_0^2 \left(9 - 9x + \frac{9}{4}x^2 \right) dx = \left[9x - \frac{9}{2}x^2 + \frac{3}{4}x^3 \right]_0^2 = 6.$$

C14S03.028: The volume is

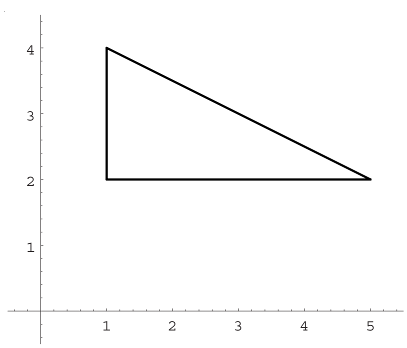
$$V = \int_{y=0}^{2} \int_{x=y/2}^{(4-y)/2} (8 - 4x - 2y)\ dx\ dy = \int_{y=0}^{2} \Big[8x - 2x^2 - 2xy \Big]_{x=y/2}^{(4-y)/2} dy$$

$$= \int_0^2 (2y^2 - 8y + 8)\ dy = \left[\frac{2}{3}y^3 - 4y^2 + 8y \right]_0^2 = \frac{16}{3}.$$

C14S03.029: The triangular domain of the integral can be drawn by executing the *Mathematica* 3.0 command

```
ParametricPlot[ {{1, 2 + 2*t}, {1 + 4*t, 2}, {1 + 4*t, 4 - 2*t}},
                {t, 0, 1}, PlotRange → {{-0.5, 5.5}, {-0.5, 4.5}},
                AspectRatio → Automatic, AxesOrigin → {0, 0} ];
```

and the result is shown next.



The volume of the solid is

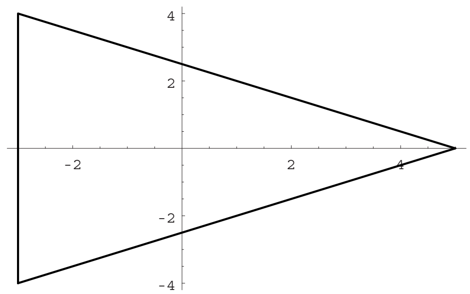
$$V = \int_{x=1}^{5} \int_{y=2}^{(9-x)/2} xy\ dy\ dx = \int_{x=1}^{5} \left[\frac{1}{2}xy^2 \right]_{y=2}^{(9-x)/2} dx$$

$$= \int_1^5 \left(\frac{65}{8}x - \frac{9}{4}x^2 + \frac{1}{8}x^3 \right) dx = \left[\frac{65}{16}x^2 - \frac{3}{4}x^3 + \frac{1}{32}x^4 \right]_1^5 = 24.$$

C14S03.030: To generate the triangular base of the solid, execute the *Mathematica* 3.0 command

```
ParametricPlot[ {{-3, -4 + 8*t}, {-3 + 8*t, 4 - 4*t}, {-3 + 8*t, -4 + 4*t}},
                {t, 0, 1} ];
```

and the result is shown next.



The top of the triangle has equation $y = \frac{1}{2}(5 - x)$ and the bottom has equation $y = \frac{1}{2}(x - 5)$. Hence the volume of the solid is

$$V = \int_{x=-3}^{5} \int_{y=(x-5)/2}^{(5-x)/2} (25 - x^2 - y^2)\, dy\, dx = \int_{x=-3}^{5} \left[25y - x^2 y - \frac{1}{3}y^3 \right]_{y=(x-5)/2}^{(5-x)/2} dx$$

$$= \int_{-3}^{5} \left(\frac{1375}{12} - \frac{75}{4}x - \frac{25}{4}x^2 + \frac{13}{12}x^3 \right) dx = \left[\frac{1375}{12}x - \frac{75}{8}x^2 - \frac{25}{12}x^3 + \frac{13}{48}x^4 \right]_{-3}^{5} = \frac{1792}{3}.$$

C14S03.031: The volume is

$$V = \int_{y=-1}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x+1)\, dx\, dy.$$

This integral can be evaluated exactly with a single command in *Mathematica* 3.0, but we will evaluate it one step at a time. As usual, the *Mathematica* output is rewritten slightly for more clarity.

```
Integrate[ x + 1, x ]
```

$$x + \frac{1}{2}x^2$$

```
(% /.  x → Sqrt[1 - y*y]) - (% /.  x → -Sqrt[1 - y*y])
```

$$2\sqrt{1 - y^2} + \frac{1}{2}(1 - y^2) + \frac{1}{2}(y^2 - 1)$$

```
Simplify[ % ]
```

$$2\sqrt{1 - y^2}$$

```
Integrate[ %, y ]
```

$$y\sqrt{1 - y^2} + \arcsin y$$

```
(% /.  y → 1) - (% /.  y → -1)
```

$$\pi$$

```
N[ %, 60 ]
```

3.14159265358979323846264338327950288419716939937510582097494

C14S03.032: The volume is

$$V = \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2)\ dy\ dx.$$

We used *Mathematica* 3.0 much as in the solution of Problem 31 to obtain the numerical value of V:

$$\begin{aligned} V &= \int_{x=-3}^{3} \left[9y - x^2y - \frac{1}{3}y^3 \right]_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx = \int_{-3}^{3} \frac{4}{9}(9-x^2)^{3/2}\ dx \\ &= \left[\frac{4}{3}(9-x^2)^{1/2} \left(\frac{45}{8}x - \frac{1}{4}x^3 \right) + \frac{81}{2} \arcsin\left(\frac{x}{3} \right) \right]_{-3}^{3} = \frac{81}{2}\pi \approx 127.2345024703866262. \end{aligned}$$

C14S03.033: The volume is

$$V = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{4 - x^2 - y^2}\ dy\ dx.$$

We used *Derive* 2.56 to evaluate this integral in a step-by-step fashion much as in the solution of Problem 31. Results:

$$\begin{aligned} V &= \int_{x=-1}^{1} \left[(4 - x^2) \arctan \frac{y}{\sqrt{4 - x^2 - y^2}} + y\sqrt{4 - x^2 - y^2}\ \right]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} \left[2(4 - x^2) \arctan \frac{\sqrt{3}\ \sqrt{1-x^2}}{3} + 2\sqrt{3}\ \sqrt{1-x^2}\ \right] dx \\ &= \left[\frac{2}{3}x(12 - x^2) \arctan \frac{\sqrt{3}\ \sqrt{1-x^2}}{3} + \frac{16}{3} \arctan \frac{\sqrt{3}\ (2x+1)}{3\sqrt{1-x^2}} \right. \\ &\qquad \left. + \frac{16}{3} \arctan \frac{\sqrt{3}\ (2x-1)}{3\sqrt{1-x^2}} - 4\sqrt{3}\ \arcsin x + \frac{2\sqrt{3}}{3}x\sqrt{1-x^2} \right]_{-1}^{1} \\ &= \frac{\pi}{3}\left(32 - 12\sqrt{3}\ \right) \approx 11.7447292674805137. \end{aligned}$$

In Section 14.4 we will find that this integral is quite easy to evaluate if we first convert to polar coordinates. If so, the integral takes the form

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2r\sqrt{4 - r^2}\ dr\ d\theta = \int_{\theta=0}^{2\pi} \left[-\frac{2}{3}(4 - r^2)^{3/2} \right]_{r=0}^{1} d\theta = \int_{0}^{2\pi} \left(\frac{16}{3} - 2\sqrt{3}\ \right) d\theta = 2\pi \left(\frac{16}{3} - 2\sqrt{3}\ \right).$$

C14S03.034: The volume is

$$V = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\sqrt{2 - x^2 - y^2}\ - (x^2 + y^2) \right] dy\ dx$$

We used *Derive* 2.56 to evaluate V, one step at a time much as in the solution of Problem 31. Results:

$$V = \int_{x=-1}^{1} \left[\frac{2-x^2}{2} \arctan\left(\frac{y}{\sqrt{2-x^2-y^2}} \right) + \frac{y\sqrt{2-x^2-y^2}}{2} - x^2y - \frac{1}{3}y^3 \right]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} \left[(2-x^2)\arctan\sqrt{1-x^2} + \frac{1-4x^2}{3}\sqrt{1-x^2} \right] dx$$

$$= \left[\frac{x(6-x^2)}{3} \arctan\sqrt{1-x^2} + \frac{7}{6}\arcsin x + \frac{2\sqrt{2}}{3}\arctan\left(\frac{x\sqrt{2}+1}{\sqrt{1-x^2}} \right) \right.$$

$$\left. + \frac{2\sqrt{2}}{3}\arctan\left(\frac{x\sqrt{2}-1}{\sqrt{1-x^2}} \right) + \frac{x}{3}(1-x^2)^{3/2} - \frac{x}{6}\sqrt{1-x^2} \right]_{-1}^{1}$$

$$= \frac{\pi}{6}\left(8\sqrt{2} - 7 \right) \approx 2.2586524883563962.$$

The volume would be much easier to evaluate using polar coordinates (as in Section 14.4, coming up next). We would thereby obtain

$$V = 2\pi \int_0^1 (r\sqrt{2-r^2} - r^3)\, dr = 2\pi \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4 \right]_0^1 = \frac{\pi}{6}\left(8\sqrt{2} - 7 \right).$$

C14S03.035: Given: the plane with Cartesian equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

cutting off a tetrahedron in the first octant. We set $z = 0$ and solve for

$$y = \frac{b}{a}(a-x);$$

the triangular region in the first quadrant bounded by this line and the coordinate axes is the domain of the volume integral. Hence the volume of the tetrahedron is

$$V = \int_{x=0}^{a} \int_{y=0}^{b(a-x)/a} c\left(1 - \frac{x}{a} - \frac{y}{b} \right) dy\, dx = \int_{x=0}^{a} \left[\frac{2abcy - 2bcxy - acy^2}{2ab} \right]_{y=0}^{b(a-x)/a} dx$$

$$= \int_0^a \frac{bc(a-x)^2}{2a^2}\, dx = \left[\frac{3a^2bcx - 3abcx^2 + bcx^3}{6a^2} \right]_0^a = \frac{abc}{6}.$$

C14S03.036: The volume is

$$V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x+h)\, dy\, dx = \int_{x=0}^{a} \Big[xy + hy \Big]_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx = \int_{-a}^{a} 2(x+h)(a^2-x^2)^{1/2}\, dx$$

$$= 2h\int_{-a}^{a} (a^2-x^2)^{1/2}\, dx + \int_{-a}^{a} 2x(a^2-x^2)^{1/2}\, dx = h \cdot \pi a^2 + \left[-\frac{2}{3}(a^2-x^2)^{3/2} \right]_{-a}^{a} = \pi a^2 h.$$

We evaluated the first integral in the second line by observing that it is the area of a semicircle of radius a.

C14S03.037: The volume is

$$V = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-y^2}\ dx\ dy = \int_{y=0}^{1} \left[x\sqrt{1-y^2}\ \right]_{x=0}^{\sqrt{1-y^2}} dy = \int_0^1 (1-y^2)\ dy = \left[y - \frac{1}{3}y^3 \right]_0^1 = \frac{2}{3}.$$

As the text indicates, the other order of integration provides more difficulties. You obtain

$$\begin{aligned}
V &= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-y^2}\ dy\ dx = \int_{x=0}^{1} \left[\frac{1}{2}y\sqrt{1-y^2}\ + \frac{1}{2}\arcsin y \right]_{y=0}^{\sqrt{1-x^2}} dx \\
&= \int_0^1 \left[\frac{1}{2}x\sqrt{1-x^2}\ + \frac{1}{2}\arcsin\left(\sqrt{1-x^2}\ \right) \right] dx \\
&= \left[\left(\frac{1}{6}x^2 - \frac{2}{3} \right)\sqrt{1-x^2}\ + \frac{1}{2}x\arcsin\left(\sqrt{1-x^2}\ \right) \right]_0^1 = \frac{2}{3}.
\end{aligned}$$

C14S03.038: The volume is

$$\begin{aligned}
V &= \int_{x=0}^{\pi} \int_{y=-\sin x}^{\sin x} 2\sin x\ dy\ dx = \int_{x=0}^{\pi} \Big[2y\ \sin x \Big]_{y=-\sin x}^{\sin x} dx \\
&= \int_0^{\pi} 4\sin^2 x\ dx = \Big[2x - \sin 2x \Big]_0^{\pi} = 2\pi \approx 6.2831853071795865.
\end{aligned}$$

C14S03.039: We integrate to find the volume of an eighth of the sphere, then multiply by 8. Thus the volume of a sphere of radius a is

$$V = 8\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\ dy\ dx.$$

Let $y = (a^2-x^2)^{1/2}\sin\theta$. Then $dy = (a^2-x^2)^{1/2}\cos\theta\ d\theta$. This substitution yields

$$\begin{aligned}
V &= 8\int_{x=0}^{a} \int_{\theta=0}^{\pi/2} \left[(a^2-x^2) - (a^2-x^2)\sin^2\theta \right]^{1/2} (a^2-x^2)^{1/2}\cos\theta\ d\theta\ dx \\
&= 8\int_{x=0}^{a} \int_{\theta=0}^{\pi/2} (a^2-x^2)\cos^2\theta\ d\theta\ dx \\
&= 8\int_{x=0}^{a/2} \int_{\theta=0}^{\pi/2} (a^2-x^2)\cdot\frac{1+\cos 2\theta}{2}\ d\theta\ dx = 8\int_{x=0}^{a} (a^2-x^2)\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_0^{\pi/2} dx \\
&= 8\int_0^a \frac{\pi}{4}(a^2-x^2)\ dx = 2\pi\left[a^2x - \frac{1}{3}x^3 \right]_0^a = 2\pi\cdot\frac{2}{3}a^3 = \frac{4}{3}\pi a^3.
\end{aligned}$$

C14S03.040: Given: the ellipsoid with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{1}$$

we assume that a, b, and c are all positive. Set $z = 0$ in Eq. (1) to find that the ellipsoid intersects the xy-plane in the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1; \qquad \text{that is,} \qquad y = \frac{b}{a}(a^2 - x^2)^{1/2}$$

(we take the positive root because we plan to integrate over the quarter of the ellipse that lies in the first quadrant). Finally, we solve Eq. (1) for

$$z = \frac{c}{ab}(a^2b^2 - b^2x^2 - a^2y^2)^{1/2}.$$

We integrate to find the volume of the eighth of the ellipsoid that lies in the first octant, then multiply by 8. Hence the volume of the ellipsoid is

$$V = 8\int_{x=0}^{a}\int_{y=0}^{(b/a)(a^2-x^2)^{1/2}} \frac{c}{ab}(a^2b^2 - b^2x^2 - a^2y^2)^{1/2}\;dy\;dx.$$

Let

$$y = \frac{b}{a}(a^2 - x^2)^{1/2}\sin\theta; \qquad \text{then} \qquad dy = \frac{b}{a}(a^2 - x^2)^{1/2}\cos\theta\;d\theta.$$

This substitution yields

$$\begin{aligned}
V &= 8\int_{x=0}^{a}\int_{\theta=0}^{\pi/2} \frac{c}{ab}\left[(a^2b^2 - b^2x^2)(1 - \sin^2\theta)\right]^{1/2} \cdot \frac{b}{a}(a^2 - x^2)^{1/2}\cos\theta\;d\theta \\
&= \frac{8bc}{a^2}\int_{x=0}^{a}\int_{\theta=0}^{\pi/2} (a^2 - x^2)\cos^2\theta\;d\theta\;dx = \frac{8bc}{a^2}\int_{x=0}^{a}\int_{\theta=0}^{\pi/2} (a^2 - x^2)\cdot\frac{1 + \cos 2\theta}{2}\;d\theta\;dx \\
&= \frac{8bc}{a^2}\int_{x=0}^{a} (a^2 - x^2)\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\theta=0}^{\pi/2} dx = \frac{8bc}{a^2}\int_{0}^{a} \frac{\pi}{4}(a^2 - x^2)\;dx \\
&= \frac{2\pi bc}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{2\pi bc}{a^2}\cdot\frac{2}{3}a^3 = \frac{4}{3}\pi abc.
\end{aligned}$$

C14S03.041: We integrate over the quarter-circle of radius 5 and center (0, 0) in the first quadrant, then multiply by 4. Hence the volume is

$$\begin{aligned}
V &= 4\int_{x=0}^{5}\int_{y=0}^{\sqrt{25-x^2}} (25 - x^2 - y^2)\;dy\;dx = 4\int_{x=0}^{5}\left[25y - x^2y - \frac{1}{3}y^3\right]_{y=0}^{\sqrt{25-x^2}} dx \\
&= \int_0^5 \frac{8}{3}(25 - x^2)^{3/2}\;dx = \left[\frac{8}{3}\sqrt{25 - x^2}\left(\frac{125}{8}x - \frac{1}{4}x^3\right) + 625\arcsin\left(\frac{x}{5}\right)\right]_0^5 \\
&= \frac{625}{2}\pi \approx 981.7477042468103870195760 57.
\end{aligned}$$

The techniques of Section 14.4 will transform this problem into one that is remarkably simple.

C14S03.042: When we solve the equations of the paraboloids simultaneously, we find that $x^2 + y^2 = 4$. Thus the intersection of the paraboloids is a curve that lies in this cylinder. So an appropriate domain for a double integral will be the circular disk of radius 2 centered at the origin. Therefore the volume of the intersection of the two paraboloids is

$$V=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12-3x^2-3y^2)\,dy\,dx=\int_{x=-2}^{2}\left[12y-3x^2y-y^3\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$
$$=\int_{-2}^{2}4(4-x^2)^{3/2}\,dx=\left[(10x-x^3)\sqrt{4-x^2}\,+24\arcsin\left(\frac{x}{2}\right)\right]_{-2}^{2}=24\pi\approx 75.398223686155.$$

C14S03.043: Suppose that the cylinder is the one with equation $x^2+z^2=R^2$ and that the square hole is centered on the z-axis and its sides are parallel to the coordinate planes. Thus the hole meets the xy-plane in the square with vertices at $\left(\pm\frac{1}{2}R,\ \pm\frac{1}{2}R\right)$. We will integrate over the quarter of that square that lies in the first quadrant, then multiply by 4. Hence the volume of material removed by the drill is

$$V=4\int_{x=0}^{R/2}\int_{y=0}^{R/2}2\sqrt{R^2-x^2}\,dy\,dx=4\int_{x=0}^{R/2}\left[2y\sqrt{R^2-x^2}\right]_{y=0}^{R/2}dx$$
$$=4\int_{0}^{R/2}R(R^2-x^2)^{1/2}\,dx=4\left[\frac{1}{2}Rx(R^2-x^2)^{1/2}-\frac{1}{2}R^3\arctan\left(\frac{x}{(R^2-x^2)^{1/2}}\right)\right]_{0}^{R/2}$$
$$=\left[\frac{1}{2}\sqrt{3}\,+2\arctan\left(\frac{1}{3}\sqrt{3}\right)\right]\cdot R^3=\frac{3\sqrt{3}\,+2\pi}{6}\cdot R^3\approx(1.913222954981)R^3.$$

C14S03.044: When the equations of the elliptical paraboloid and the parabolic cylinder are solved simultaneously, one consequence is that $x^2+4y^2=4$. Hence this elliptical cylinder contains the curve of intersection of the two surfaces, and the ellipse $x^2+4y^2=4$ in the xy-plane is an appropriate domain for a double integral. Hence the volume bounded by the two surfaces is

$$V=\int_{x=-2}^{2}\int_{y=-(1/2)(4-x^2)^{1/2}}^{(1/2)(4-x^2)^{1/2}}(4-x^2-4y^2)\,dy\,dx=\int_{x=-2}^{2}\left[4y-x^2y-\frac{4}{3}y^3\right]_{y=-(1/2)(4-x^2)^{1/2}}^{(1/2)(4-x^2)^{1/2}}dx$$
$$=\int_{-2}^{2}\frac{2}{3}(4-x^2)^{3/2}\,dx=\left[\frac{1}{6}(4-x^2)^{1/2}(10x-x^3)+4\arcsin\left(\frac{x}{2}\right)\right]_{-2}^{2}=4\pi\approx 12.566370614359.$$

C14S03.045: The region bounded by the parabolas $y=x^2$ and $y=8-x^2$ in the xy-plane is a suitable domain for a double integral that gives the volume of the solid. Hence the volume of the solid is

$$V=\int_{x=-2}^{2}\int_{y=x^2}^{8-x^2}(2x^2-x^2)\,dy\,dx=\int_{x=-2}^{2}\left[x^2y\right]_{y=x^2}^{8-x^2}dx=\int_{-2}^{2}(8x^2-2x^4)\,dx$$
$$=\left[\frac{8}{3}x^3-\frac{2}{5}x^5\right]_{-2}^{2}=\frac{256}{15}\approx 17.066666666667.$$

C14S03.046: We used *Mathematica* 3.0 in the usual way; the volume of the solid is

$$V=\int_{x=-\pi/2}^{\pi/2}\int_{y=-\cos x}^{\cos x}(4-x^2-y^2)\,dy\,dx=\int_{x=-\pi/2}^{\pi/2}\left[4y-x^2y-\frac{1}{3}y^3\right]_{y=-\cos x}^{\cos x}dx$$
$$=\int_{-\pi/2}^{\pi/2}\left((8-2x^2)\cos x-\frac{2}{3}\cos^3 x\right)dx=\frac{1}{18}\Big[207\sin x-72x\cos x-36x^2\sin x-\sin 3x\Big]_{-\pi/2}^{\pi/2}$$
$$=\frac{208-9\pi^2}{9}\approx 13.2415067100217525.$$

C14S03.047: We used *Mathematica* 3.0 in the usual way; the volume of the solid is

$$V = \int_{x=-\pi/2}^{\pi/2} \int_{y=-\cos x}^{\cos x} \cos y \, dy \, dx = \int_{x=-\pi/2}^{\pi/2} \left[\sin y \right]_{y=-\cos x}^{\cos x} dx$$

$$= \int_{-\pi/2}^{\pi/2} 2\sin(\cos x) \, dx \approx 3.57297496390010467337.$$

Mathematica reports that the exact value of the integral is

$$\texttt{4*HypergeometricPFQ}\left[\{1\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, -\frac{1}{4}\right].$$

C14S03.048: A *Mathematica* solution:

```
I1 = Integrate[ Sin[x]*Cos[y], { y, 0, Cos[x] } ]
```

$(\sin x)\sin(\cos x)$

```
V = 4*Integrate[ I1, { x, 0, Pi/2 } ]
```

$4[1 - \cos(1)]$

```
N[V]
```

1.83879 —C.H.E.

C14S03.049: A *Mathematica* solution:

```
eq1 = z == 2*x + 3;
eq2 = z == x∧2 + y∧2;
Eliminate[ { eq1, eq2 }, z ]
```

$y^2 = -x^2 + 2x + 3$

This is the circle $(x-1)^2 + y^2 = 4$ with center $(1, 0)$ and radius 2. Therefore the volume of the solid is

```
Integrate[ 3 + 2*x − x∧2 - y∧2, { x, −1, 3 },
                 { y, −Sqrt[ 3 + 2*x − x{∧2 ], Sqrt[ 3 + 2*x − x∧2 ] } ]
```

8π —C.H.E.

C14S03.050: A *Mathematica* solution:

```
eq1 == z == 4*x + 4*y;
eq2 = z == x∧2 + y∧2 − 1;
Eliminate[ { eq1, eq2 }, z ]
```

$-y^2 + 4y + 1 == x^2 - 4x$

This is the circle $(x-2)^2 + (y-2)^2 = 9$ with center $(2, 2)$ and radius 3. Hence the volume of the solid bounded by the two surfaces is

```
Integrate[ 1 + 4*x + 4*x - x∧2 - y∧2, { x, −1, 5 },
             { y, 2 − Sqrt[ 9 − (x − 2)∧2 ], 2 + Sqrt[ 9 − (x - 2)∧2 ] } ]
```

$$\frac{81\pi}{2}$$

—C.H.E.

The answer is correct in spite of the typographical error in the last *Mathematica* command, which should be

```
Integrate[ 1 + 4*x + 4*y - x∧2 - y∧2, { x, −1, 5 },
             { y, 2 − Sqrt[ 9 − (x − 2)∧2 ], 2 + Sqrt[ 9 − (x - 2)∧2 ] } ]
```

How do you explain that?

C14S03.051: A *Mathematica* solution:

```
eq1 = z == −16*x − 18*y;
eq2 = z == 11 − 4*x∧2 − 9*y∧2;
Eliminate[ { eq1, eq2 }, z ]
```

$$-9y^2 + 18y + 11 = 4x^2 - 16x$$

This is the ellipse $4(x-2)^2 + 9(y-1)^2 = 36$ with center $(2, 1)$ and semiaxes $a = 3$ and $b = 2$. Hence the volume of the solid is

```
Integrate[ 11 − 4*x∧2 − 9*y∧2 + 16*x + 18*y, { x, −1, 5 },
     { y, 1 − 1/3*Sqrt[ 36 − 4*(x − 2)∧2 ], 1 + 1/3*Sqrt[ 36 − 4*(x − 2)∧2 ] } ]
```

$$108\pi$$

—C.H.E.

C14S03.052: A *Mathematica* solution:

```
I1 = Simplify[ 8*Integrate[ Sqrt[ 4 − x∧2 − y∧2 ], y ] ]
```

$$4y\sqrt{4-x^2-y^2} - 4(x^2-4)\tan^{-1}\left(\frac{y}{\sqrt{4-x^2-y^2}}\right)$$

```
(I1 /. y → 1) − (I1 /. y → 0)
```

$$4\sqrt{3-x^2} - 4(x^2-4)\tan^{-1}\left(\frac{1}{\sqrt{3-x^2}}\right)$$

(Assuming that no one really wants to see the antiderivative of the preceding expression, let's jump immediately to the double integral that gives the volume of the hole.)

```
V = 8*Integrate[ Integrate[
                Sqrt[ 4 − x∧2 - y∧2 ], { y, 0, 1 } ], { x, 0, 1 } ]
```

$$\frac{4}{3}\left[2\sqrt{2} + 11\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 19\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - 8\tan^{-1}\left(\frac{5}{\sqrt{2}}\right)\right]$$

```
N[V]
```

14.5755

```
100*%/(4*Pi*8/3)*percent
```

43.4954 percent —C.H.E.

C14S03.053: First let's put the center of the sphere at the point $(-2, 0, 0)$. A *Mathematica* solution:

```
V = FullSimplify[ 2*Integrate[ Integrate[
            Sqrt[ 16 - (x + 2)^2 - y^2 ], { y, -1, 1 } ], { x, -1, 1 } ] ]
```

$$\frac{2}{3}\Big[6\sqrt{6} - 2\sqrt{14} + 29\cot^{-1}\left(\sqrt{6}\right) + 41\cot^{-1}\left(\sqrt{14}\right) - 47\csc^{-1}\left(\sqrt{15}\right) + 47\sin^{-1}\left(\sqrt{3/5}\right) + 20\tan^{-1}\left(\sqrt{3/2}\right) - 108\tan^{-1}\left(9\sqrt{3/2}\right) - 20\tan^{-1}\left(11/\sqrt{14}\right) + 108\tan^{-1}\left(19/\sqrt{14}\right)\Big]$$

```
N[V]
```

26.7782

```
100*%/(4*Pi*64/3)*percent
```

9.98878 percent —C.H.E.

Section 14.4

C14S04.001: The circle with center (0, 0) and radius $a > 0$ has polar description $r = a,\ 0 \leqq \theta \leqq 2\pi$. Therefore its area is

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{1}{2} a^2\ d\theta = \left[\frac{1}{2} a^2 \theta \right]_0^{2\pi} = \pi a^2.$$

C14S04.002: The circle with polar equation $r = 3 \sin \theta$ has area

$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{3\sin\theta} r\ dr\ d\theta = \int_{\theta=0}^{\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{3\sin\theta} d\theta = \int_0^{\pi} \frac{9}{2} \sin^2\theta\ d\theta$$

$$= \int_0^{\pi} \frac{9}{4}(1 - \cos 2\theta)\ d\theta = \frac{9}{8}\Big[2\theta - \sin 2\theta \Big]_0^{\pi} = \frac{9}{4}\pi.$$

C14S04.003: The area bounded by the cardioid with polar description $r = 1 + \cos\theta,\ 0 \leqq \theta \leqq 2\pi$, is

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{1+\cos\theta} d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \cos\theta + \frac{1}{2}\cos^2\theta \right) d\theta$$

$$= \frac{1}{8}\Big[6\theta + 8\sin\theta + \sin 2\theta \Big]_0^{2\pi} = \frac{3}{2}\pi.$$

C14S04.004: The area bounded by one loop of the four-leaved rose with polar equation $r = 2\cos 2\theta$ is

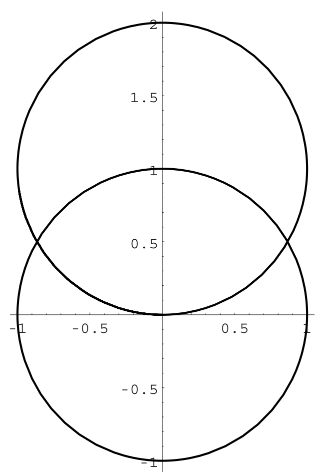
$$A = \int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{2\cos 2\theta} r\ dr\ d\theta = \int_{\theta=-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2 \right]_{r=0}^{2\cos 2\theta} d\theta = \int_{-\pi/4}^{\pi/4} 2\cos^2 2\theta\ d\theta$$

$$= \frac{1}{4}\Big[4\theta + \sin 4\theta \Big]_{-\pi/4}^{\pi/4} = \frac{1}{2}\pi \approx 1.5707963267948966.$$

C14S04.005: To see the two circles, execute the *Mathematica* 3.0 command

```
ParametricPlot[ {{Cos[t], Sin[t]}, {2*Sin[t]*Cos[t], 2*Sin[t]*Sin[t]}},
                {t, 0, 2*Pi}, AspectRatio → Automatic ];
```

the result is shown next.



To find where the two circles meet, solve their equations simultaneously:

$$2\sin\theta = 1; \qquad \sin\theta = \frac{1}{2}; \qquad \theta = \frac{1}{6}\pi, \quad \theta = \frac{5}{6}\pi.$$

To find the area between them, two integrals are required. Each is doubled because we are actually finding the area of the right half of the intersection of the circles.

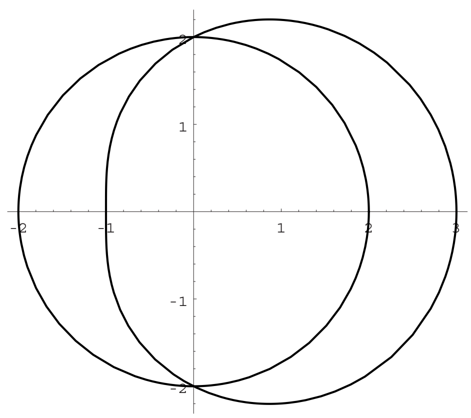
$$A_1 = 2\int_{\theta=0}^{\pi/6}\int_{r=0}^{2\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/6} 2\sin^2\theta\,d\theta = \Big[2\theta - \sin 2\theta\Big]_0^{\pi/6} = \frac{2\pi - 3\sqrt{3}}{6};$$

$$A_2 = 2\int_{\theta=\pi/6}^{\pi/2}\int_{r=0}^{1} r\,dr\,d\theta = \int_{\theta=\pi/6}^{\pi/2} 1\,d\theta = \frac{\pi}{3}.$$

Therefore the total area enclosed by both circles is

$$A = A_1 + A_2 = \frac{4\pi - 3\sqrt{3}}{6} \approx 1.2283696986087568.$$

C14S04.006: We are to find the area within the limaçon $r = 2 + \cos\theta$ and outside the circle $r = 2$. Their graphs are shown next.



To find where they intersect, solve their equations simultaneously for $\theta = \pm\frac{1}{2}\pi$. The area we seek can be found with a single integral:

$$A = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=2}^{2+\cos\theta} r\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2}\left[\frac{1}{2}r^2\right]_{r=2}^{2+\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{1}{2}(2+\cos\theta)^2 - 2\right) d\theta$$

$$= \frac{1}{8}\Big[2\theta + 16\sin\theta + \sin 2\theta\Big]_{-\pi/2}^{\pi/2} = \frac{\pi + 16}{4} \approx 4.7853981633974483.$$

C14S04.007: To find where the limaçon $r = 1 - 2\sin\theta$ passes through the origin, solve $1 - 2\sin\theta = 0$ for $\theta = \frac{1}{6}\pi$, $\theta = \frac{5}{6}\pi$. The small loop of the limaçon is generated by the values of θ between these two angles, and is thus given by

$$A = \int_{\theta=\pi/6}^{5\pi/6}\int_{r=0}^{1-2\sin\theta} r\,dr\,d\theta = \int_{\theta=\pi/6}^{5\pi/6}\left[\frac{1}{2}r^2\right]_{r=0}^{1-2\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}\left[\frac{1}{2}(1-2\sin\theta)^2\right] d\theta$$

$$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{5\pi/6} = \frac{2\pi - 3\sqrt{3}}{2} \approx 0.5435164422364773.$$

C14S04.008: In polar coordinates the integral becomes

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r^3 \, dr \, d\theta = 2\pi \left[\frac{1}{4} r^4 \right]_0^3 = \frac{81}{2}\pi \approx 127.2345024703866262.$$

Because the inner integral does not involve θ, it is constant with respect to θ. Therefore to integrate it with respect to θ over the interval $0 \leqq \theta \leqq 2\pi$, simply multiply the inner integral by 2π.

C14S04.009: In polar coordinates the integral takes the form

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^2 \, dr \, d\theta = 2\pi \left[\frac{1}{3} r^3 \right]_0^2 = \frac{16}{3}\pi \approx 16.7551608191455639.$$

Because the inner integral does not involve θ (either in the integrand or in the limits of integration), it is constant with respect to θ. Therefore to integrate it with respect to θ over the interval $0 \leqq \theta \leqq 2\pi$, simply multiply the inner integral by 2π. We will use this time-saving technique frequently and without further comment.

C14S04.010: Note that the domain of the integral is generated as θ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. In polar coordinates the integral is

$$\begin{aligned} I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} r^3 \, dr \, d\theta &= \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{4} r^4 \right]_{r=0}^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta \, d\theta \\ &= \frac{1}{8} \Big[12\theta + 8\sin 2\theta + \sin 4\theta \Big]_{-\pi/2}^{\pi/2} = \frac{3}{2}\pi \approx 4.71238898038468985769. \end{aligned}$$

If you prefer, the last integral in the first line of the display may be evaluated using Formula (113) of the endpapers:

$$\int_{-\pi/2}^{\pi/2} 4\cos^4\theta \, d\theta = 8 \int_0^{\pi/2} \cos^4\theta \, d\theta = 8 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = \frac{3}{2}\pi.$$

C14S04.011: Note that the domain of the integral is generated as θ varies from 0 to π. In polar coordinates the integral takes the form

$$\begin{aligned} I &= \int_{\theta=0}^{\pi} \int_{r=0}^{\sin\theta} (10 + 2r\cos\theta + 3r\sin\theta) \cdot r \, dr \, d\theta = \int_{\theta=0}^{\pi} \left[5r^2 + \frac{1}{3} r^3 (2\cos\theta + 3\sin\theta) \right]_{r=0}^{\sin\theta} d\theta \\ &= \int_0^{\pi} \left(5\sin^2\theta + \frac{2}{3}\sin^3\theta\cos\theta + \sin^4\theta \right) d\theta \\ &= \frac{1}{96} \Big[276\theta - 8\cos 2\theta + 2\cos 4\theta - 144\sin 2\theta + 3\sin 4\theta \Big]_0^{\pi} = \frac{23}{8}\pi \approx 9.03207887907065556058. \end{aligned}$$

The reduction formulas in Problems 53 and 54 of Section 7.3, and the formulas of Problem 58 there and Formula (113) of the endpapers, may be used if you prefer to avoid trigonometric identities.

C14S04.012: The polar form of the integral is

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (a^2 - r^2) \cdot r \, dr \, d\theta = 2\pi \left[\frac{1}{4}(2a^2 r^2 - r^4) \right]_0^a = \frac{\pi a^4}{2}.$$

C14S04.013: In polar form the given integral becomes

$$J = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{r}{1+r^2}\,dr\,d\theta = \frac{\pi}{2} \cdot \left[\frac{1}{2}\ln(1+r^2) \right]_0^1 = \frac{1}{4}\pi \ln 2 \approx 0.5443965225759005.$$

To evaluate the integral in Cartesian coordinates, you would first need to evaluate

$$\int_{x=0}^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\,dx = \frac{1}{\sqrt{1+y^2}} \arctan\left(\frac{\sqrt{1-y^2}}{\sqrt{1+y^2}} \right). \tag{1}$$

Then you would need to antidifferentiate the last expression in (1). Neither *Mathematica* 3.0 nor *Derive* 2.56 could do so.

C14S04.014: Conversion of the integral to polar coordinates yields

$$K = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{r}{\sqrt{4-r^2}}\,dr\,d\theta = \frac{\pi}{2} \cdot \left[-\sqrt{4-r^2} \right]_0^1 = \frac{\pi}{2} \cdot \left(2 - \sqrt{3} \right) \approx 0.4208936072384665.$$

It is possible to evaluate this integral in Cartesian coordinates. You should obtain

$$\begin{aligned}
K &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{4-x^2-y^2}}\,dy\,dx = \int_0^1 \left[\arctan\left(\frac{y}{\sqrt{4-x^2-y^2}} \right) \right]_0^{\sqrt{1-x^2}} dx \\
&= \int_0^1 \arctan\left(\frac{\sqrt{1-x^2}}{\sqrt{3}} \right) dx \\
&= \left[x \arctan\left(\frac{\sqrt{1-x^2}}{\sqrt{3}} \right) - \sqrt{3}\,\arcsin x - \arctan\left(\frac{1-2x}{\sqrt{3}\,\sqrt{1-x^2}} \right) + \arctan\left(\frac{1+2x}{\sqrt{3}\,\sqrt{1-x^2}} \right) \right]_0^1 \\
&= \frac{\pi}{2}\left(2 - \sqrt{3} \right).
\end{aligned}$$

We have concealed one complication: The evaluation in the next-to-last line requires l'Hôpital's rule.

C14S04.015: In polar coordinates the integral becomes

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} r^4\,dr\,d\theta = \frac{\pi}{2} \cdot \left[\frac{1}{5}r^5 \right]_0^2 = \frac{\pi}{2} \cdot \frac{32}{5} = \frac{16}{5}\pi \approx 10.0530964914873384.$$

This integral can be evaluated in Cartesian coordinates. You should obtain

$$\begin{aligned}
I &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)^{3/2}\,dy\,dx \\
&= \int_0^2 \left[\left(\frac{5}{8}x^2 y + \frac{1}{4}y^3 \right) \sqrt{x^2+y^2} + \frac{3}{8}x^4 \ln\left(y + \sqrt{x^2+y^2} \right) \right]_0^{\sqrt{4-x^2}} dx \\
&= \int_0^2 \left[\frac{5}{4}x^2\sqrt{4-x^2} + \frac{1}{2}(4-x^2)^{3/2} - \frac{3}{4}x^4 \ln x + \frac{3}{8}x^4 \ln\left(2 + \sqrt{4-x^2} \right) \right] dx \\
&= \left[\left(\frac{2}{5}x + \frac{3}{20}x^3 \right) \sqrt{4-x^2} + \frac{32}{5}\arcsin\left(\frac{x}{2} \right) - \frac{3}{20}x^5 \ln x + \frac{3}{40}x^5 \ln\left(2 + \sqrt{4-x^2} \right) \right]_0^2 = \frac{16}{5}\pi.
\end{aligned}$$

The final equality requires l'Hôpital's rule.

C14S04.016: The polar form of the integral is

$$J = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\csc\theta} r^3 \cos^2\theta \, dr \, d\theta = \int_{\theta=\pi/4}^{\pi/2} \left[\frac{1}{4} r^4 \cos^2\theta \right]_{r=0}^{\csc\theta} d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{4} \cot^2\theta \csc^2\theta \, d\theta$$

$$= \left[-\frac{1}{12} \cot^3\theta \right]_{\pi/4}^{\pi/2} = \frac{1}{12} \approx 0.0833333333333333.$$

This integral is easier to evaluate in Cartesian coordinates.

C14S04.017: In polar coordinates the integral becomes

$$K = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r \sin r^2 \, dr \, d\theta = \frac{\pi}{2} \cdot \left[-\frac{1}{2} \cos r^2 \right]_0^1 d\theta = \frac{\pi}{4}(1 - \cos 1) \approx 0.3610457246892050.$$

Exact evaluation of the given integral in Cartesian coordinates may be impossible. *Mathematica* 3.0 reports that

$$\int \sin(x^2 + y^2) \, dx = \sqrt{\frac{\pi}{2}} \left[(\cos y^2) \cdot \text{FresnelS}\left(x\sqrt{\frac{2}{\pi}} \right) + (\sin y^2) \cdot \text{FresnelC}\left(x\sqrt{\frac{2}{\pi}} \right) \right]$$

where

$$\text{FresnelS}(x) = \int_0^x \sin\left(\frac{\pi t^2}{2} \right) dt \quad \text{and} \quad \text{FresnelC}(x) = \int_0^x \cos\left(\frac{\pi t^2}{2} \right) dt.$$

C14S04.018: The polar form of the given integral is

$$I = \int_{\theta=0}^{\pi/4} \int_{r=\sec\theta}^{2\cos\theta} 1 \, dr \, d\theta = \int_{\theta=0}^{\pi/4} \left[r \right]_{r=\sec\theta}^{2\cos\theta} d\theta = \int_0^{\pi/4} (2\cos\theta - \sec\theta) \, d\theta$$

$$= \left[2\sin\theta - \ln(\sec\theta + \tan\theta) \right]_0^{\pi/4} = \sqrt{2} - \ln\left(1 + \sqrt{2}\right) \approx 0.5328399753535520.$$

This integral can also be evaluated in Cartesian coordinates. You should obtain

$$I = \int_1^2 \left[\ln\left(y + \sqrt{x^2 + y^2} \right) \right]_0^{\sqrt{2x - x^2}} dx = \int_1^2 \left[-\ln x + \ln\left(\sqrt{2x} + \sqrt{2x - x^2} \right) \right] dx$$

$$= \left[-\frac{\sqrt{2}\sqrt{2x - x^2}}{\sqrt{x}} - x\ln x + x\ln\left(\sqrt{2x} + \sqrt{2x - x^2} \right) \right]_1^2 = \sqrt{2} - \ln\left(1 + \sqrt{2}\right).$$

C14S04.019: The volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (2 + r\cos\theta + r\sin\theta) \cdot r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \left[r^2 + \frac{1}{3} r^3 (\cos\theta + \sin\theta) \right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \left(1 + \frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta \right) d\theta = \left[\theta - \frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta \right]_0^{2\pi} = 2\pi.$$

C14S04.020: The volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (2 + r\cos\theta)\cdot r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left[r^2 + \frac{1}{3}r^3\cos\theta \right]_{r=0}^{2} d\theta = \int_0^{2\pi} \left(4 + \frac{8}{3}\cos\theta \right) d\theta$$

$$= \left[4\theta + \frac{8}{3}\sin\theta \right]_0^{2\pi} = 8\pi \approx 25.1327412287182349.$$

C14S04.021: The volume is

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} (3 + r\cos\theta + r\sin\theta)\cdot r\,dr\,d\theta = \int_{\theta=0}^{\pi} \left[\frac{3}{2}r^2 + \frac{1}{3}r^3(\cos\theta + \sin\theta) \right]_{r=0}^{2\sin\theta} d\theta$$

$$= \int_0^{\pi} \left(6\sin^2\theta + \frac{8}{3}\sin^3\theta\cos\theta + \frac{8}{3}\sin^4\theta \right) d\theta = \frac{1}{12}\left[48\theta - 4\cos 2\theta + \cos 4\theta - 26\sin 2\theta + \sin 4\theta \right]_0^{\pi}$$

$$= \frac{1}{4} + \frac{48\pi - 3}{12} = 4\pi \approx 12.56637061435917295385.$$

C14S04.022: The volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} (1 + r\cos\theta)\cdot r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{2}r^2 + \frac{1}{3}r^3\cos\theta \right]_{r=0}^{1+\cos\theta} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}(1+\cos\theta)^2 + \frac{1}{3}(1+\cos\theta)^3\cos\theta \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\left(4 + 10\cos\theta + 8\cos^2\theta + 2\cos^3\theta + \cos 2\theta + 2\cos\theta\cos 2\theta + \cos^2\theta\cos 2\theta \right) d\theta$$

$$= \frac{1}{96}\left[132\theta + 200\sin\theta + 44\sin 2\theta + 8\sin 3\theta + \sin 4\theta \right]_0^{2\pi} = \frac{11}{4}\pi \approx 8.6393797973719314.$$

C14S04.023: We will find the volume of the sphere of radius a centered at the origin:

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 2r\sqrt{a^2 - r^2}\,dr\,d\theta = 2\pi \cdot \left[-\frac{2}{3}(a^2 - r^2)^{3/2} \right]_{r=0}^{a} = 2\pi \cdot \frac{2}{3}a^3 = \frac{4}{3}\pi a^3.$$

C14S04.024: When we solve the equations of the paraboloids simultaneously, we find that $x^2 + y^2 = 4$. Hence the curve in which the surfaces intersect lies on the cylinder $x^2 + y^2 = 4$. Thus the disk in the xy-plane bounded by the circle $x^2 + y^2 = 4$ is appropriate for the domain of the volume integral. So the volume of the solid bounded by the paraboloids is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (12 - 3r^2)\cdot r\,dr\,d\theta = 2\pi \cdot \left[6r^2 - \frac{3}{4}r^4 \right]_{r=0}^{2} = 24\pi \approx 75.3982236861550377.$$

C14S04.025: The volume of the solid is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (h + r\cos\theta) \cdot r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{2} r^2 h + \frac{1}{3} r^2 \cos\theta \right]_{r=0}^{a} d\theta = \int_0^{2\pi} \left(\frac{1}{2} a^2 h + \frac{1}{3} a^3 \cos\theta \right) d\theta$$

$$= \frac{1}{6} \left[3a^2 h\theta + 2a^3 \sin\theta \right]_0^{2\pi} = \pi a^2 h.$$

C14S04.026: The wedge is bounded above by the plane $z = x = r\cos\theta$, below by the plane $z = 0$, and on the side by the cylinder $r = 2$ for $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$. Hence the volume of the wedge is

$$V = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2} r^2 \cos\theta \, dr \, d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{3} r^3 \cos\theta \right]_{r=0}^{2} d\theta = \int_{-\pi/2}^{\pi/2} \frac{8}{3} \cos\theta \, d\theta$$

$$= \left[\frac{8}{3} \sin\theta \right]_{-\pi/2}^{\pi/2} = \frac{16}{3} \approx 5.3333333333333333.$$

C14S04.027: When we solve the equations of the paraboloids simultaneously, we find that one consequence is that $x^2 + y^2 = 1$. Thus the curve in which the paraboloids meet lies on that cylinder, and hence the circular disk $x^2 + y^2 \leqq 1$ in the xy-plane is a suitable domain for the double integral that yields the volume between the paraboloids. That volume is therefore

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4 - 4r^2) \cdot r \, dr \, d\theta = 2\pi \cdot \left[2r^2 - r^4 \right]_{r=0}^{1} d\theta = 2\pi \approx 6.2831853071795865.$$

C14S04.028: When we solve the equations of the paraboloids simultaneously, we find that one consequence is that $x^2 + y^2 = 1$. Thus the curve in which the paraboloids meet lies on that cylinder, and hence the circular disk $x^2 + y^2 \leqq 1$ in the xy-plane is a suitable domain for the double integral that yields the volume between the paraboloids. That volume is therefore

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 - r^2) \cdot r \, dr \, d\theta = 2\pi \cdot \left[\frac{1}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{1} d\theta = \frac{1}{2}\pi \approx 1.5707963267948966.$$

C14S04.029: When the equations of the sphere and the cone are solved simultaneously, one consequence is that $x^2 + y^2 = \frac{1}{2}a^2$. Therefore the circle in which the sphere and cone meet lies on the cylinder with that equation. Thus a suitable domain for the double integral that gives the volume in question is the circle in the xy-plane with equation $x^2 + y^2 = \frac{1}{2}a^2$. Hence the volume of the "ice-cream cone" is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{a/\sqrt{2}} \left(\sqrt{a^2 - r^2} - r \right) \cdot r \, dr \, d\theta = 2\pi \cdot \left[-\frac{1}{3} \left((a^2 - r^2)^{3/2} + r^3 \right) \right]_{r=0}^{a/\sqrt{2}}$$

$$= 2\pi \cdot \frac{1}{12} \left(4 - 2\sqrt{2} \right) a^3 = \frac{1}{3}\pi \left(2 - \sqrt{2} \right) a^3 \approx (0.6134341230070734)a^3.$$

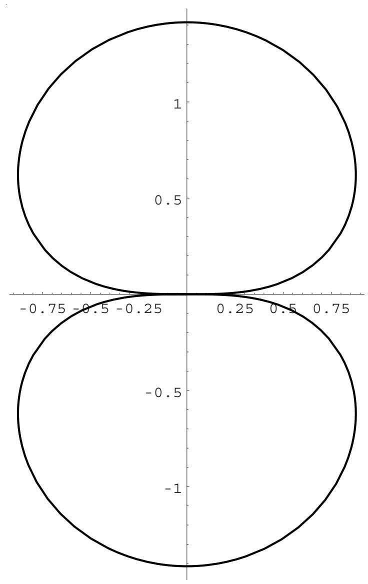
C14S04.030: The volume is

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{2a\sin\theta} r^3 \, dr \, d\theta = \int_{\theta=0}^{\pi} \left[\frac{1}{4} r^4 \right]_{r=0}^{2a\sin\theta} d\theta = \int_0^{\pi} 4a^4 \sin^4\theta \, d\theta$$

$$= \frac{1}{8} \left[a^4 (12\theta - 8\sin 2\theta + \sin 4\theta) \right]_0^{\pi} = \frac{3}{2}\pi a^4 \approx (4.7123889803846899)a^4.$$

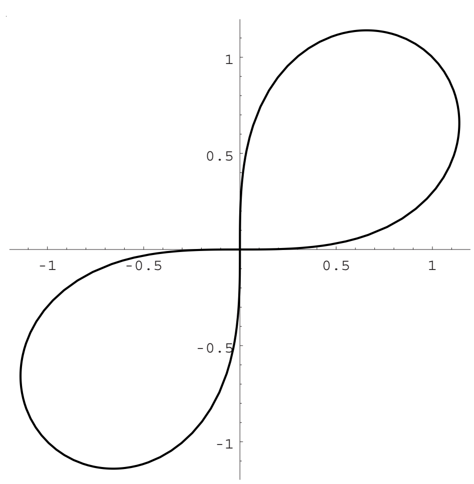
C14S04.031: The curve $r^2 = 2\sin\theta$ is not a lemniscate. The lemniscate was discovered in 1694 by Jacques Bernoulli (1654–1705) and has an equation of the form $r^2 = a^2\cos(2\theta - \omega)$ or of the form $r^2 = a^2\sin(2\theta - \omega)$ where ω and a are constants. The effect of ω is to rotate the graph through the angle ω and the effect of a is to magnify the graph. The curve given in Problem 31 has the shape shown in Figure 14.04.031A, shown next, and was generated by the *Mathematica* 3.0 command

```
ParametricPlot[ {{(Sqrt[2*Sin[t]])*Cos[t], (Sqrt[2*Sin[t]])*Sin[t]},
                {-(Sqrt[2*Sin[t]])*Cos[t], -(Sqrt[2*Sin[t]])*Sin[t]}},
                {t, 0, Pi}, PlotPoints → 43, AspectRatio → Automatic ];
```



It seems likely that there is a typographical error in this problem and the equation given for the lemniscate should be $r^2 = 2\sin 2\theta$. This curve has the shape shown in Figure 14.04.031B, shown next, and was generated by the *Mathematica* 3.0 command

```
ParametricPlot[ {{(Sqrt[2*Sin[2*t]])*Cos[t], (Sqrt[2*Sin[2*t]])*Sin[t]},
                {-(Sqrt[2*Sin[2*t]])*Cos[t], -(Sqrt[2*Sin[2*t]])*Sin[t]}},
                {t, 0, Pi/2}, PlotPoints → 43, AspectRatio → Automatic ];
```



We will solve the problem both ways. First, with the curve $r^2 = 2\sin\theta,$ we find the volume to be

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{\sqrt{2\sin\theta}} r^3\ dr\ d\theta = \int_{\theta=0}^{\pi} \left[\frac{1}{4}r^4\right]_{r=0}^{\sqrt{2\sin\theta}} d\theta = \int_0^{\pi} \sin^2\theta\ d\theta$$

$$= \frac{1}{4}\left[2\theta - \sin 2\theta\right]_0^{\pi} = \frac{1}{2}\pi \approx 1.5707963267948966.$$

Second, using the lemniscate with polar equation $r^2 = 2\sin 2\theta,$ we find the volume to be

$$V = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\sqrt{2\sin 2\theta}} r^3\ dr\ d\theta = \int_{\theta=0}^{\pi/2} \left[\frac{1}{4}r^4\right]_{r=0}^{\sqrt{2\sin 2\theta}} d\theta = \int_0^{\pi/2} \sin^2 2\theta\ d\theta$$

$$= \frac{1}{8}\left[4\theta - \sin 4\theta\right]_0^{\pi/2} = \frac{1}{4}\pi \approx 0.7853981633974483.$$

C14S04.032: The volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} 2r\sqrt{18 - 2r^2}\ dr\ d\theta = 2\pi \cdot \left[-\frac{1}{3}(18 - 2r^2)^{3/2}\right]_0^2$$

$$= \frac{4}{3}\pi\left(-5\sqrt{10} + 27\sqrt{2}\right) \approx 93.7131973350605099635.$$

The volume of a cylinder of the same radius as the given cylinder but with height $6\sqrt{2}$ (the major axis of the ellipsoid) is $24\pi\sqrt{2} \approx 106.6292$ and the volume of the ellipsoid itself is $36\pi\sqrt{2} \approx 159.9438$, so the answer we obtained is certainly plausible.

C14S04.033: Part(a): A cross section of Fig. 14.4.23 in the xz-plane reveals a right triangle with legs b and $a - h$ and hypotenuse a, and it follows immediately that $b^2 = 2ah - h^2$. Part (b): The volume of the spherical segment is

$$V = \int_0^{2\pi} \int_0^b r\left[\sqrt{a^2 - r^2} - (a - h)\right]\ dr\ d\theta = 2\pi\left[-\frac{1}{3}(a^2 - r^2)^{3/2} - \frac{1}{2}ar^2 + \frac{1}{2}hr^2\right]_0^b$$

$$= \frac{\pi}{3}\left[3r^2h - 3r^2a - 2(a^2 - r^2)^{3/2}\right]_0^b = \frac{\pi}{3}\left[3b^2h - 3b^2a - 2(a^2 - b^2)^{3/2} + 2a^3\right]$$

$$= \frac{\pi}{3}\left[3b^2h - 3b^2a - 2(a^2 - 2ah + h^2)^{3/2} + 2a^3\right] = \frac{\pi}{3}\left[3b^2h - 3b^2a - 2(a - h)^3 + 2a^3\right]$$

$$= \frac{\pi}{3}(3b^2h - 3b^2a + 6a^2h - 6ah^2 + 2h^3) = \frac{\pi}{3}\left[3b^2h - 3a(2ah - h^2) + 6a^2h - 6ah^2 + 2h^3\right]$$

$$= \frac{\pi}{3}(3b^2h - 6a^2h + 3ah^2 + 6a^2h - 6ah^2 + 2h^3) = \frac{\pi}{3}(3b^2h - 3ah^2 + 2h^3) = \frac{\pi}{3}h(3b^2 - 3ah + 2h^2).$$

Recall that $2ah = b^2 + h^2,$ so we may substitute $\frac{3}{2}b^2 + \frac{3}{2}h^2$ for $3ah$ in the last expression. Therefore the volume of the spherical cap is

$$V = \frac{1}{3}\pi h\left(3b^2 - \frac{3}{2}b^2 - \frac{3}{2}h^2 + 2h^2\right) = \frac{1}{6}\pi h(6b^2 - 3b^2 - 3h^2 + 4h^2) = \frac{1}{6}\pi h(3b^2 + h^2).$$

C14S04.034: We first convert the given integral to polar form and integrate over the quarter circle with polar description $0 \leqq \theta \leqq \frac{1}{2}\pi,\ 0 \leqq r \leqq a$. Then we let $a \to +\infty$. Thus we obtain

$$I_a = \int_0^{\pi/2} \int_0^a \frac{r}{(1+r^2)^2}\,dr\,d\theta = \frac{\pi}{2} \cdot \left[-\frac{1}{2(1+r^2)}\right]_0^a = \frac{\pi}{4}\left(1 - \frac{1}{1+a^2}\right).$$

Then

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \lim_{a\to\infty} I_a = \frac{\pi}{4}.$$

C14S04.035: Using the *Suggestion*, we have

$$2\pi(b-x)\,dA = 2\pi(b - r\cos\theta)r\,dr\,d\theta,$$

and hence the volume of the torus is

$$V = \int_0^{2\pi} \int_0^a 2\pi r(b - r\cos\theta)\,dr\,d\theta = \int_0^{2\pi} \left[b\pi r^2 - \frac{2}{3}\pi r^3\cos\theta\right]_0^a d\theta$$

$$= \left[\frac{1}{3}(3\pi a^2 b\theta - 2\pi a^3 \sin\theta)\right]_0^{2\pi} = 2\pi^2 a^2 b.$$

Read the *First Theorem of Pappus* in Section 14.5 and apply it to this circular disk of radius a rotated around a circle of radius b to obtain the same answer in a tenth of the time.

C14S04.036: The plane and the paraboloid meet in the circle $x^2+y^2=9,\ z=-3$, so the circular disk $x^2+y^2 \leqq 9,\ z=0$ is a suitable domain for the volume integral. The volume is therefore

$$V = \int_0^{2\pi} \int_0^3 (18-2r^2)\cdot r\,dr\,d\theta = 2\pi \cdot \left[9r^2 - \frac{1}{2}r^4\right]_0^3 = 81\pi \approx 254.4690049407732523.$$

C14S04.037: The plane and the paraboloid meet in a curve that lies on the cylinder $x^2+y^2=4$, so the circular disk $x^2+y^2 \leqq 4$ in the xy-plane is a suitable domain for the volume integral. The volume of the solid is thus

$$V = \int_0^{2\pi} \int_0^2 (4-r^2)\cdot r\,dr\,d\theta = 2\pi \cdot \left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 8\pi \approx 25.1327412287183459.$$

C14S04.038: The circular disk $x^2+y^2 \leqq 4$ in the xy-plane is a suitable domain for the volume integral. The volume of the solid is therefore

$$V = \int_0^{2\pi} \int_0^2 (3 + r\cos\theta + r\sin\theta)\cdot r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{3}{2}r^2 + \frac{1}{3}r^3(\cos\theta+\sin\theta)\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(6 + \frac{8}{3}[\cos\theta+\sin\theta]\right) d\theta = \left[6\theta - \frac{8}{3}\cos\theta + \frac{8}{3}\sin\theta\right]_0^{2\pi} = 12\pi \approx 37.6991118430775189.$$

C14S04.039: When we solve the equations of the paraboloids simultaneously, we find that $x^2+y^2=4$. Hence the curve in which the surfaces intersect lies on the cylinder $x^2+y^2=4$. Thus the disk in the xy-plane

bounded by the circle $x^2 + y^2 = 4$ is appropriate for the domain of the volume integral. So the volume of the solid bounded by the paraboloids is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (12 - 3r^2) \cdot r \, dr \, d\theta = 2\pi \cdot \left[6r^2 - \frac{3}{4} r^4 \right]_{r=0}^{2} = 24\pi \approx 75.3982236861550377.$$

C14S04.040: When we solve the equations of the paraboloid and the ellipsoid simultaneously, we find that their curve of intersection lies on the cylinder $x^2 + y^2 = 4$. Hence the circular disk $x^2 + y^2 \leqq 4$ in the xy-plane is a suitable domain for the volume integral. The volume is

$$V = \int_0^{2\pi} \int_0^2 \left(\sqrt{80 - 4r^2} - 2r^2 \right) \cdot r \, dr \, d\theta = 2\pi \cdot \left[-\frac{1}{2} r^4 - \frac{2}{3} (20 - r^2)^{3/2} \right]_0^2$$
$$= 2\pi \left(\frac{80}{3}\sqrt{5} - \frac{152}{3} \right) = \frac{16}{3}\pi \left(10\sqrt{5} - 19 \right) \approx 56.3087300917396928.$$

C14S04.041: A *Mathematica* solution:

```
V = Simplify[ 2*Integrate[ Integrate[
            r*Sqrt[ b∧2 - r∧2 ], { r, 0, a } ], { θ, 0, 2*Pi } ] ]
```

$$\frac{3}{2}\left(b^3 - (b^2 - a^2)^{3/2}\right)\pi$$

```
V1 = V /. { a → 2, b → 4 }
```

$$\frac{4}{3}\left(64 - 24\sqrt{3}\right)\pi$$

Percent of material removed:

```
N[ 100*V1/(4*Pi*64/3) ]
```

35.0481 —C.H.E.

C14S04.042: A *Maple* solution:

```
V := 2*int(int(r*sqrt(12-4*r*cos(t)-r∧2), r=0..1), t=0..2*Pi):
V := evalf(V);
```

$$V := 21.22150986$$

MATLAB gives

```
f = inline('r.*sqrt(12-4*r.*cos(t)-r.∧2','r','t');
V = 2*dblquad(f,0,1,0,2*pi)
```

$$V = 21.2215$$

The percent of material removed:

```
N[100*V/(4*Pi*64/3)]
```

7.91603 —C.H.E.

C14S04.043: A *Mathematica* solution for the case of a square hole:

```
n = 4;
V = 4*n*Integrate[ Integrate[ r*Sqrt[ 4 − r∧2 ],
            { r, 0, Cos[ Pi/n ]*Sec[ θ ] } ], { θ, 0, Pi/n } ];
```

(We suppress the output; it's not attractive.)

```
V = Chop[ N[V] ]
```

7.6562

Percent:

```
100*V/(4*Pi*8/3)
```

22.8473

Now for the pentagonal, hexagonal, heptagonal, and 17-sided holes. First, the pentagonal hole:

```
n = 5;
V = 4*n*Integrate[ Integrate[ r*Sqrt[ 4 − r∧2 ],
            { r, 0, Cos[ Pi/n ]*Sec[ θ ] } ], { θ, 0, Pi/n } ];
V = Chop[ N[V] ]
```

9.03688

Percent:

```
100*V/(4*Pi*8/3)
```

26.9675

Next, the hexagonal hole:

```
n = 6;
V = 4*n*Integrate[ Integrate[ r*Sqrt[ 4 − r∧2 ],
            { r, 0, Cos[ Pi/n ]*Sec[ θ ] } ], { θ, 0, Pi/n } ];
V = Chop[ N[V] ]
```

9.83041

Percent:

```
100*V/(4*Pi*8/3)
```

29.3355

For the heptagonal and 17-sided holes, we use *Maple* V Release 5.

```
n:=7;
V:=4*n*int(int(r*sqrt(4-r^2), r=0..cos(Pi/n)*sec(t)), t=0..Pi/n);
V:=evalf(V);
```

$V := 10.32346688$

Percent:

```
evalf(100*V/(4*Pi*8/3);
```

30.80682719

```
n:=17;
V:=4*n*int(int(r*sqrt(4-r^2), r=0..cos(Pi/n)*sec(t)), t=0..Pi/n);
V:=evalf(V);
```

$V := 11.49809060$

Percent:

```
evalf(100*V/(4*Pi*8/3);
```

34.31208666

—C.H.E.

Section 14.5

Note: To integrate positive integral powers of sines and cosines (and their products), there are effective techniques not illustrated in the text. They use the following identities, which are consequences of the Euler-DeMoivre formula

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

---

1. $\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$.

2. $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$.

3. $\sin^4\theta = \dfrac{1}{8}(3-4\cos 2\theta + \cos 4\theta)$.

4. $\sin^5\theta = \dfrac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$.

5. $\sin^6\theta = \dfrac{1}{32}(10-15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$.

6. $\sin^7\theta = \dfrac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$.

7. $\sin^8\theta = \dfrac{1}{128}(35-56\cos 2\theta + 28\cos 4\theta - 8\cos 6\theta + \cos 8\theta)$.

8. $\sin^9\theta = \dfrac{1}{256}(126\sin\theta - 84\sin 3\theta + 36\sin 5\theta - 9\sin 7\theta + \sin 9\theta)$.

9. $\sin^{10}\theta = \dfrac{1}{512}(126-210\cos 2\theta + 120\cos 4\theta - 45\cos 6\theta + 10\cos 8\theta - \cos 10\theta)$.

10. $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$.

11. $\cos^3\theta = \dfrac{1}{4}(3\cos\theta + \cos 3\theta)$.

12. $\cos^4\theta = \dfrac{1}{8}(3+4\cos 2\theta + \cos 4\theta)$.

13. $\cos^5\theta = \dfrac{1}{16}(10\cos\theta + 5\cos 3\theta + \cos 5\theta)$.

14. $\cos^6\theta = \dfrac{1}{32}(10+15\cos 2\theta + 6\cos 4\theta + \cos 6\theta)$.

15. $\cos^7\theta = \dfrac{1}{64}(35\cos\theta + 21\cos 3\theta + 7\cos 5\theta + \cos 7\theta)$.

16. $\cos^8\theta = \dfrac{1}{128}(35+56\cos 2\theta + 28\cos 4\theta + 8\cos 6\theta + \cos 8\theta)$.

17. $\cos^9\theta = \dfrac{1}{256}(126\cos\theta + 84\cos 3\theta + 36\cos 5\theta + 9\cos 7\theta + \cos 9\theta)$.

18. $\cos^{10}\theta = \dfrac{1}{512}(126 + 210\cos 2\theta + 120\cos 4\theta + 45\cos 6\theta + 10\cos 8\theta + \cos 10\theta)$.

19. $\sin^2\theta\cos^2\theta = \dfrac{1}{8}(1 - \cos 4\theta)$.

20. $\sin^3\theta\cos^2\theta = \dfrac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)$.

21. $\sin^4\theta\cos^2\theta = \dfrac{1}{32}(2 - \cos 2\theta - 2\cos 4\theta + \cos 6\theta)$.

22. $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(5\sin\theta + \sin 3\theta - 3\sin 5\theta + \sin 7\theta)$.

23. $\sin^6\theta\cos^2\theta = \dfrac{1}{128}(5 - 4\cos 2\theta - 4\cos 4\theta + 4\cos 6\theta - \cos 8\theta)$.

24. $\sin^2\theta\cos^3\theta = \dfrac{1}{16}(2\cos\theta - \cos 3\theta - \cos 5\theta)$.

25. $\sin^2\theta\cos^4\theta = \dfrac{1}{32}(2 + \cos 2\theta - 2\cos 4\theta - \cos 6\theta)$.

26. $\sin^2\theta\cos^5\theta = \dfrac{1}{64}(5\cos\theta - \cos 3\theta - 3\cos 5\theta - \cos 7\theta)$.

27. $\sin^2\theta\cos^6\theta = \dfrac{1}{128}(5 + 4\cos 2\theta - 4\cos 4\theta - 4\cos 6\theta - \cos 8\theta)$.

28. $\sin^3\theta\cos^3\theta = \dfrac{1}{32}(3\sin 2\theta - \sin 6\theta)$.

29. $\sin^4\theta\cos^3\theta = \dfrac{1}{64}(3\cos\theta - 3\cos 3\theta - \cos 5\theta + \cos 7\theta)$.

30. $\sin^5\theta\cos^3\theta = \dfrac{1}{128}(6\sin 2\theta - 2\sin 4\theta - 2\sin 6\theta + \sin 8\theta)$.

40. $\sin^6\theta\cos^3\theta = \dfrac{1}{256}(6\cos\theta - 8\cos 3\theta + 3\cos 7\theta - \cos 9\theta)$.

41. $\sin^4\theta\cos^4\theta = \dfrac{1}{128}(3 - 4\cos 4\theta + \cos 8\theta)$.

42. $\sin^5\theta\cos^4\theta = \dfrac{1}{256}(6\sin\theta + 4\sin 3\theta - 4\sin 5\theta - \sin 7\theta + \sin 9\theta)$.

43. $\sin^6\theta\cos^4\theta = \dfrac{1}{512}(6 - 2\cos 2\theta - 8\cos 4\theta + 3\cos 6\theta + 2\cos 8\theta - \cos 10\theta)$.

44. $\sin^5\theta\cos^5\theta = \dfrac{1}{512}(10\sin 2\theta - 5\sin 6\theta + \sin 10\theta)$.

45. $\sin^6\theta\cos^5\theta = \dfrac{1}{1024}(10\cos\theta - 10\cos 3\theta - 5\cos 5\theta + 5\cos 7\theta + \cos 9\theta - \cos 11\theta)$.

46. $\sin^6\theta\cos^6\theta = \dfrac{1}{2048}(10 - 15\cos 4\theta + 6\cos 8\theta - \cos 12\theta)$.

C14S05.001: By the symmetry principle, the centroid is at $(\overline{x}, \overline{y}) = (2, 3)$.

C14S05.002: By the symmetry principle, the centroid is at $(\overline{x}, \overline{y}) = (2, 3)$.

C14S05.003: By the symmetry principle, the centroid is at $(\overline{x}, \overline{y}) = (1, 1)$.

C14S05.004: The mass and moments are

$$m = \int_0^3 \int_0^{3-x} 1\,dy\,dx = \int_0^3 (3-x)\,dx = \left[3x - \frac{1}{2}x^2\right]_0^3 = \frac{9}{2},$$

$$M_y = \int_0^3 \int_0^{3-x} x\,dy\,dx = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = \frac{9}{2}, \quad \text{and}$$

$$M_x = \int_0^3 \int_0^{3-x} y\,dy\,dx = \int_0^3 \frac{1}{2}(3-x)^2\,dx = \left[\frac{1}{6}(x-3)^3\right]_0^3 = \frac{9}{2}.$$

Therefore the centroid is located at $(\overline{x}, \overline{y}) = (1, 1)$.

C14S05.005: The mass and moments are

$$m = \int_0^4 \int_0^{(4-x)/2} 1\,dy\,dx = \int_0^4 \frac{4-x}{2}\,dx = \left[2x - \frac{1}{4}x^2\right]_0^4 = 4,$$

$$M_y = \int_0^4 \int_0^{(4-x)/2} x\,dy\,dx = \int_0^4 \left(2x - \frac{1}{2}x^2\right)dx = \left[x^2 - \frac{1}{6}x^3\right]_0^4 = \frac{16}{3}, \quad \text{and}$$

$$M_x = \int_0^4 \int_0^{(4-x)/2} y\,dy\,dx = \int_0^4 \frac{1}{8}(4-x)^2\,dx = \left[\frac{1}{24}(x-4)^3\right]_0^4 = \frac{8}{3}.$$

Therefore the centroid is located at $(\overline{x}, \overline{y}) = \left(\frac{4}{3}, \frac{2}{3}\right)$.

C14S05.006: Here we have

$$m = \int_0^1 \int_y^{2-y} 1\,dx\,dy = \int_0^1 (2-2y)\,dy = \left[2y - y^2\right]_0^1 = 1 \quad \text{and}$$

$$M_x = \int_0^1 \int_y^{2-y} y\,dx\,dy = \int_0^1 (2y - 2y^2)\,dx = \left[y^2 - \frac{2}{3}y^3\right]_0^1 = \frac{1}{3}.$$

By symmetry, $\overline{x} = 1$. Therefore the centroid is located at $\left(1, \frac{1}{3}\right)$.

C14S05.007: The mass and moments are

$$m = \int_0^2 \int_0^{x^2} 1\,dy\,dx = \int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3},$$

$$M_y = \int_0^2 \int_0^{x^2} x\,dy\,dx = \int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4, \quad \text{and}$$

$$M_x = \int_0^2 \int_0^{x^2} y\,dy\,dx = \int_0^2 \frac{1}{2}x^4\,dx = \left[\frac{1}{10}x^5\right]_0^2 = \frac{16}{5}.$$

Therefore the centroid is $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{6}{5}\right)$.

C14S05.008: By symmetry, $M_y = 0$; next,

$$m = \int_{-3}^3 \int_{x^2}^9 1\,dy\,dx = \int_{-3}^3 (9 - x^2)\,dx = \left[9x - \frac{1}{3}x^3\right]_{-3}^3 = 36 \quad \text{and}$$

$$M_x = \int_{-3}^3 \int_{x^2}^9 y\,dy\,dx = \int_{-3}^3 \frac{1}{2}(81 - x^4)\,dx = \left[\frac{81}{2}x - \frac{1}{10}x^5\right]_{-3}^3 = \frac{972}{5}.$$

Therefore the centroid is located at $\left(0, \frac{27}{5}\right)$.

C14S05.009: By symmetry, $M_y = 0$. Next,

$$m = \int_{-2}^2 \int_{x^2-4}^0 1\,dy\,dx = \int_{-2}^2 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \frac{32}{3} \quad \text{and}$$

$$M_x = \int_{-2}^2 \int_{x^2-4}^0 y\,dy\,dx = \int_{-2}^2 -\frac{1}{2}(4 - x^2)^2\,dx = \left[\frac{4}{3}x^3 - 8x - \frac{1}{10}x^5\right]_{-2}^2 = -\frac{256}{15}.$$

Therefore the centroid is at the point $\left(0, -\frac{8}{5}\right)$.

C14S05.010: By symmetry, $M_y = 0$. Next,

$$m = \int_{-2}^2 \int_0^{x^2+1} 1\,dy\,dx = \int_{-2}^2 (x^2 + 1)\,dx = \left[\frac{1}{3}x^3 + x\right]_{-2}^2 = \frac{28}{3} \quad \text{and}$$

$$M_x = \int_{-2}^2 \int_0^{x^2+1} y\,dy\,dx = \int_{-2}^2 \frac{1}{2}(x^2 + 1)^2\,dx = \left[\frac{1}{10}x^5 + \frac{1}{3}x^3 + \frac{1}{2}x\right]_{-2}^2 = \frac{206}{15}.$$

Hence the centroid is located at the point $\left(0, \frac{103}{70}\right) \approx (0, 1.4714285714285714)$.

C14S05.011: The mass and moments are

$$m = \int_0^1 \int_0^{1-x} xy\,dy\,dx = \int_0^1 \frac{1}{2}(x - 2x^2 + x^3)\,dx = \frac{1}{24}\left[6x^2 - 8x^3 + 3x^4\right]_0^1 = \frac{1}{24},$$

$$M_y = \int_0^1 \int_0^{1-x} x^2y\,dy\,dx = \int_0^1 \left[\frac{1}{2}x^2y^2\right]_0^{1-x} dx = \int_0^1 \frac{1}{2}(x^2 - 2x^3 + x^4)\,dx$$

$$= \frac{1}{60}\left[10x^3 - 15x^4 + 6x^5\right]_0^1 = \frac{1}{60},$$

$$M_x = \int_0^1 \int_0^{1-x} xy^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}xy^3\right]_0^{1-x} dx = \int_0^1 \frac{1}{3}(x - 3x^2 + 3x^3 - x^4)\,dx$$

$$= \frac{1}{60}\left[10x^2 - 20x^3 + 15x^4 - 4x^5\right]_0^1 = \frac{1}{60}.$$

Therefore the centroid is located at the point $\left(\frac{2}{5}, \frac{2}{5}\right)$.

C14S05.012: Using *Mathematica* 3.0, the computations to find the mass m and the moments M_y and M_x of the lamina can be partially automated in such a way to reduce typing and make clear the individual steps in the solution, as follows. First we compute the appropriate antiderivatives with respect to y:

```
{Integrate[ x^2, y ], Integrate[ x^3, y ], Integrate[ x^2*y, y ]}
```

$$\left\{x^2y, \quad x^3y, \quad \frac{1}{2}x^2y^2\right\}$$

Then we evaluate these antiderivatives at $y = 1 - x$ and at $y = 0$:

```
(% /.  y → 1 - x) - (% /.  y → 0)
```

$$\left\{(1-x)x^2, \quad (1-x)x^3, \quad \frac{1}{2}(1-x)^2x^2\right\}$$

Now we integrate with respect to x:

```
Integrate[ %, x ]
```

$$\left\{\frac{1}{3}x^3 - \frac{1}{4}x^4, \quad \frac{1}{4}x^4 - \frac{1}{5}x^5, \quad \frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5\right\}$$

Then we evaluate these antiderivatives at $x = 0$ and at $x = 1$. This will yield, respectively, the values of m, M_y, and M_x.

```
(% /.  x → 1) - (% /.  x → 0)
```

$$\left\{\frac{1}{12}, \quad \frac{1}{20}, \quad \frac{1}{60}\right\}$$

The coordinates of the centroid are then

```
{ %[[2]]/%[[1]], %[[3]]/%[[1]] }
```

$$\left\{\frac{3}{5},\ \frac{1}{5}\right\}$$

Answer: Mass $m = \dfrac{1}{12}$, centroid $(\overline{x}, \overline{y}) = \left(\dfrac{3}{5}, \dfrac{1}{5}\right)$.

C14S05.013: The mass and moments of the lamina are

$$m = \int_{-2}^{2}\int_{0}^{4-x^2} y\,dy\,dx = \int_{-2}^{2} \frac{1}{2}(4-x^2)^2\,dx = \frac{1}{30}\Big[240x - 40x^3 + 3x^5\Big]_{-2}^{2} = \frac{256}{15},$$

$$M_y = \int_{-2}^{2}\int_{0}^{4-x^2} xy\,dy\,dx = \int_{-2}^{2} \frac{1}{2}x(4-x^2)^2\,dx = \frac{1}{12}\Big[48x^2 - 12x^4 + x^6\Big]_{-2}^{2} = 0, \quad \text{and}$$

$$M_x = \int_{-2}^{2}\int_{0}^{4-x^2} y^2\,dy\,dx = \int_{-2}^{2} \frac{1}{3}(4-x^2)^3\,dx = \left[\frac{64}{3}x - \frac{16}{3}x^3 + \frac{4}{5}x^5 - \frac{1}{21}x^7\right]_{-2}^{2} = \frac{4096}{105}.$$

Therefore the centroid of the lamina is $(\overline{x}, \overline{y}) = \left(0, \frac{16}{7}\right)$.

C14S05.014: The mass and moments of the lamina are

$$m = \int_{-3}^{3}\int_{0}^{9-y^2} x^2\,dx\,dy = \int_{-3}^{3} \frac{1}{3}(9-y^2)^3\,dy = \left[243y - 27y^3 + \frac{9}{5}y^5 - \frac{1}{21}y^7\right]_{-3}^{3} = \frac{23328}{35},$$

$$M_y = \int_{-3}^{3}\int_{0}^{9-y^2} x^3\,dx\,dy = \int_{-3}^{3} \frac{1}{4}(9-y^2)^4\,dy = \left[\frac{6561}{4}y - 243y^3 + \frac{243}{10}y^5 - \frac{9}{7}y^7 + \frac{1}{36}y^9\right]_{-3}^{3} = \frac{139968}{35},$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-y^2} x^2y\,dx\,dy = \int_{-3}^{3} \frac{1}{3}y(9-y^2)^3\,dy = \left[\frac{243}{2}y^2 - \frac{81}{4}y^4 + \frac{3}{2}y^6 - \frac{1}{24}y^8\right]_{-3}^{3} = 0.$$

Therefore the centroid of the lamina is $(\overline{x}, \overline{y}) = (6, 0)$.

C14S05.015: The mass and moments of the lamina are

$$m = \int_{0}^{1}\int_{x^2}^{\sqrt{x}} xy\,dy\,dx = \int_{0}^{1} \frac{1}{2}(x^2 - x^5)\,dx = \frac{1}{12}\Big[2x^3 - x^6\Big]_{0}^{1} = \frac{1}{12},$$

$$M_y = \int_{0}^{1}\int_{x^2}^{\sqrt{x}} x^2y\,dy\,dx = \int_{0}^{1} \frac{1}{2}(x^3 - x^6)\,dx = \frac{1}{56}\Big[7x^4 - 4x^7\Big]_{0}^{1} = \frac{3}{56},$$

$$M_x = \int_{0}^{1}\int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx = \int_{0}^{1} \frac{1}{3}(x^{5/2} - x^7)\,dx = \frac{1}{168}\Big[16x^{7/2} - 7x^8\Big]_{0}^{1} = \frac{3}{56}.$$

Hence the centroid of the lamina is at $(\overline{x}, \overline{y}) = \left(\frac{9}{14}, \frac{9}{14}\right)$.

C14S05.016: The mass and moments of the lamina are

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy\,dx = \int_0^1 \left(\frac{1}{3}x^{3/2} + x^{5/2} - x^4 - \frac{1}{3}x^6\right) dx$$

$$= \left[\frac{2}{15}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{1}{5}x^5 - \frac{1}{21}x^7\right]_0^1 = \frac{6}{35},$$

$$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x(x^2 + y^2)\,dy\,dx = \int_0^1 \left(\frac{1}{3}x^{5/2} + x^{7/2} - x^5 - \frac{1}{3}x^7\right) dx$$

$$= \left[\frac{2}{21}x^{7/2} + \frac{2}{9}x^{9/2} - \frac{1}{6}x^6 - \frac{1}{24}x^8\right]_0^1 = \frac{55}{504},$$

$$M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)y\,dy\,dx = \int_0^1 \left(\frac{1}{4}x^2 + \frac{1}{2}x^3 - \frac{1}{2}x^6 - \frac{1}{4}x^8\right) dx$$

$$= \left[\frac{1}{12}x^3 + \frac{1}{8}x^4 - \frac{1}{14}x^7 - \frac{1}{36}x^9\right]_0^1 = \frac{55}{504}.$$

Therefore the centroid of the lamina is at $(\overline{x}, \overline{y}) = \left(\frac{275}{432}, \frac{275}{432}\right)$.

C14S05.017: The mass and moments of the lamina are

$$m = \int_{-1}^1 \int_{x^2}^{2-x^2} y\,dy\,dx = \int_{-1}^1 (2 - 2x^2)\,dx = \left[2x - \frac{2}{3}x^3\right]_{-1}^1 = \frac{8}{3},$$

$$M_y = \int_{-1}^1 \int_{x^2}^{2-x^2} xy\,dy\,dx = \int_{-1}^1 (2x - 2x^3)\,dx = \left[x^2 - \frac{1}{2}x^4\right]_{-1}^1 = 0,$$

$$M_x = \int_{-1}^1 \int_{x^2}^{2-x^2} y^2\,dy\,dx = \int_{-1}^1 \left[\frac{1}{3}y^3\right]_{x^2}^{2-x^2} dx = \int_{-1}^1 \left(\frac{1}{3}(2 - x^2)^3 - \frac{1}{3}x^6\right) dx$$

$$= \left[\frac{8}{3}x - \frac{4}{3}x^3 + \frac{2}{5}x^5 - \frac{2}{21}x^7\right]_{-1}^1 = \frac{344}{105}.$$

Hence the centroid of the lamina is at $(\overline{x}, \overline{y}) = \left(0, \frac{43}{35}\right)$.

C14S05.018: The mass and moments of the lamina are

$$m = \int_1^e \int_0^{\ln x} 1\,dy\,dx = \int_1^e \ln x\,dx = \Big[-x + x\ln x\Big]_1^e = 1,$$

$$M_y = \int_1^e \int_0^{\ln x} x\,dy\,dx = \int_1^e x\ln x\,dx = \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^e = \frac{e^2+1}{4},$$

$$M_x = \int_1^e \int_0^{\ln x} y\,dy\,dx = \int_1^e \frac{1}{2}(\ln x)^2\,dx = \left[x - x\ln x + \frac{1}{2}x(\ln x)^2\right]_1^e = \frac{e-2}{2}.$$

(The reduction formula in Problem 51 of Section 8.3 (Section 7.3 of the "early transcendentals version") is helpful in evaluating the last integral.) The centroid of the lamina is located at the point

$$(\overline{x}, \overline{y}) = \left(\frac{e^2+1}{4}, \frac{e-2}{2}\right) \approx (2.0972640247326626, 0.3591409142295226).$$

C14S05.019: The mass and moments of the lamina are

$$m = \int_0^\pi \int_0^{\sin x} 1\ dy\ dx = \int_0^\pi \sin x\ dx = \Big[-\cos x\Big]_0^\pi = 2,$$

$$M_y = \int_0^\pi \int_0^{\sin x} x\ dy\ dx = \int_0^\pi x\sin x\ dx = \Big[\sin x - x\cos x\Big]_0^\pi = \pi,$$

$$M_x = \int_0^\pi \int_0^{\sin x} y\ dy\ dx = \int_0^\pi \frac{1}{2}\sin^2 x\ dx = \left[\frac{1}{8}(2x - \sin 2x)\right]_0^\pi = \frac{1}{4}\pi.$$

Therefore the centroid of the lamina is located at

$$(\overline{x},\ \overline{y}) = \left(\frac{\pi}{2},\ \frac{\pi}{8}\right) \approx (1.5707963267948966,\ 0.3926990816987242).$$

C14S05.020: The mass of the lamina is

$$m = \int_{x=-1}^1 \int_{y=0}^{\exp(-x^2)} |xy|\ dy\ dx = -\int_{x=-1}^0 \int_{y=0}^{\exp(-x^2)} xy\ dy\ dx + \int_0^1 \int_{y=0}^{\exp(-x^2)} xy\ dy\ dx$$

$$= 2\int_0^1 \int_{y=0}^{\exp(-x^2)} xy\ dy\ dx = 2\int_0^1 \frac{1}{2}x\exp(-2x^2)\ dx = \left[-\frac{1}{4}\exp(-2x^2)\right]_0^1 = \frac{e^2-1}{4e^2}.$$

By symmetry, $M_y = 0$. Finally,

$$M_x = \int_{x=-1}^1 \int_{y=0}^{\exp(-x^2)} |x|\cdot y^2\ dy\ dx = \int_{x=-1}^1 \frac{1}{3}|x|\cdot\exp(-3x^2)\ dx$$

$$= \frac{2}{3}\int_0^1 x\exp(-3x^2)\ dx = \frac{2}{3}\left[-\frac{1}{6}\exp(-3x^2)\right]_0^1 = \frac{1}{9}\left(1 - \frac{1}{e^3}\right) = \frac{e^3-1}{9e^3}.$$

Therefore the centroid of the lamina is located at the point

$$(\overline{x},\ \overline{y}) = \left(0,\ \frac{4(e^2+e+1)}{9e(e+1)}\right) \approx (0,\ 0.4884168976896321).$$

C14S05.021: The mass and moments of the lamina are

$$m = \int_0^a \int_0^a (x+y)\ dy\ dx = \int_0^a \left(\frac{1}{2}a^2 + ax\right)\ dx = \left[\frac{1}{2}(a^2x + ax^2)\right]_0^a = a^3,$$

$$M_y = \int_0^a \int_0^a x(x+y)\ dy\ dx = \int_0^a \left(\frac{1}{2}a^2x + ax^2\right)\ dx = \left[\frac{1}{4}a^2x^2 + \frac{1}{3}ax^3\right]_0^a = \frac{7}{12}a^4, \quad \text{and}$$

$$M_x = \int_0^a \int_0^a (x+y)y\ dy\ dx = \int_0^a \left(\frac{1}{3}a^3 + \frac{1}{2}a^2x\right)\ dx = \left[\frac{1}{3}a^3x + \frac{1}{4}a^2x^2\right]_0^a = \frac{7}{12}a^4.$$

Therefore its centroid is located at the point $(\overline{x},\ \overline{y}) = \left(\frac{7}{12}a,\ \frac{7}{12}a\right)$.

C14S05.022: The mass and moments of the lamina are

$$m = \int_0^a \int_0^{a-x} (x^2 + y^2)\,dy\,dx = \int_0^a \left(\frac{1}{3}a^3 - a^2x + 2ax^2 - \frac{4}{3}x^3\right)\,dx$$

$$= \left[\frac{1}{3}a^3x - \frac{1}{2}a^2x^2 + \frac{2}{3}ax^3 - \frac{1}{3}x^4\right]_0^a = \frac{1}{6}a^4,$$

$$M_y = \int_0^a \int_0^{a-x} x(x^2 + y^2)\,dy\,dx = \int_0^a \left(\frac{1}{3}a^3x - a^2x^2 + 2ax^3 - \frac{4}{3}x^4\right)\,dx$$

$$= \left[\frac{1}{6}a^3x^2 - \frac{1}{3}a^2x^3 + \frac{1}{2}ax^4 - \frac{4}{15}x^5\right]_0^a = \frac{1}{15}a^5, \quad \text{and}$$

$$M_x = \int_0^a \int_0^{a-x} (x^2 + y^2)y\,dy\,dx = \int_0^a \left(\frac{1}{4}a^4 - a^3x + 2a^2x^2 - 2ax^3 + \frac{3}{4}x^4\right)\,dx$$

$$= \left[\frac{1}{4}a^4x - \frac{1}{2}a^3x^2 + \frac{2}{3}a^2x^3 - \frac{1}{2}ax^4 + \frac{3}{20}x^5\right]_0^a = \frac{1}{15}a^5.$$

Therefore the centroid of the lamina is located at $(\overline{x}, \overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

C14S05.023: The mass and moments of the lamina are

$$m = \int_{-2}^{2} \int_{x^2}^{4} y\,dy\,dx = \int_{-2}^{2} \left(8 - \frac{1}{2}x^4\right)\,dx = \left[8x - \frac{1}{10}x^5\right]_{-2}^{2} = \frac{128}{5},$$

$$M_y = \int_{-2}^{2} \int_{x^2}^{4} xy\,dy\,dx = \int_{-2}^{2} \left(8x - \frac{1}{2}x^5\right)\,dx = \left[4x^2 - \frac{1}{12}x^6\right]_{-2}^{2} = 0, \quad \text{and}$$

$$M_x = \int_{-2}^{2} \int_{x^2}^{4} y^2\,dy\,dx = \int_{-2}^{2} \left(\frac{64}{3} - \frac{1}{3}x^6\right)\,dx = \left[\frac{64}{3}x - \frac{1}{21}x^7\right]_{-2}^{2} = \frac{512}{7}.$$

Thus the centroid of the lamina is located at the point $(\overline{x}, \overline{y}) = \left(0, \frac{20}{7}\right)$.

C14S05.024: The curves cross where $x^2 = 2x + 3$; that is, at $x = -1$ and at $x = 3$. Hence the mass and moments of the lamina are

$$m = \int_{-1}^{3} \int_{x^2}^{2x+3} x^2\,dy\,dx = \int_{-1}^{3} (3x^2 + 2x^3 - x^4)\,dx = \left[x^3 + \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_{-1}^{3} = \frac{96}{5},$$

$$M_y = \int_{-1}^{3} \int_{x^2}^{2x+3} x^3\,dy\,dx = \int_{-1}^{3} (3x^3 + 2x^4 - x^5)\,dx = \left[\frac{3}{4}x^4 + \frac{2}{5}x^5 - \frac{1}{6}x^6\right]_{-1}^{3} = \frac{544}{15}, \quad \text{and}$$

$$M_x = \int_{-1}^{3} \int_{x^2}^{2x+3} x^2y\,dy\,dx = \int_{-1}^{3} \left(\frac{9}{2}x^2 + 6x^3 + 2x^4 - \frac{1}{2}x^6\right)\,dx$$

$$= \left[\frac{3}{2}x^3 + \frac{3}{2}x^4 + \frac{2}{5}x^5 - \frac{1}{14}x^7\right]_{-1}^{3} = \frac{3616}{35}.$$

Therefore the centroid is at

$$(\overline{x}, \overline{y}) = \left(\frac{17}{9}, \frac{113}{21}\right) \approx (1.8888888888888889, 5.3809523809523810).$$

C14S05.025: The mass and moments are

$$m = \int_0^\pi \int_0^{\sin x} x\ dy\ dx = \int_0^\pi x \sin x\ dx = \left[\sin x - x \cos x \right]_0^\pi = \pi,$$

$$M_y = \int_0^\pi \int_0^{\sin x} x^2\ dy\ dx = \int_0^\pi x^2 \sin x\ dx = \left[2\cos x - x^2 \cos x + 2x \sin x \right]_0^\pi = \pi^2 - 4, \quad \text{and}$$

$$M_x = \int_0^\pi \int_0^{\sin x} xy\ dy\ dx = \int_0^\pi \frac{1}{2} x \sin^2 x\ dx = \left[\frac{1}{16} (2x^2 - \cos 2x - 2x \sin 2x) \right]_0^\pi = \frac{1}{8}\pi^2.$$

Consequently the centroid of the lamina is at

$$(\overline{x}, \overline{y}) = \left(\frac{\pi^2 - 4}{\pi}, \frac{\pi}{8} \right) \approx (1.8683531088546306, 0.3926990816987242).$$

C14S05.026: By symmetry, the moment M_y is zero, and thus $\overline{x} = 0$. The mass and other moment are

$$m = \int_{\theta=0}^\pi \int_{r=0}^a r^2 \sin\theta\ dr\ d\theta = \int_0^\pi \frac{1}{3} a^3 \sin\theta\ d\theta = \left[-\frac{1}{3} a^3 \cos\theta \right]_0^\pi = \frac{2}{3} a^3 \quad \text{and}$$

$$M_x = \int_{\theta=0}^\pi \int_{r=0}^a r^3 \sin^2\theta\ dr\ d\theta = \int_0^\pi \frac{1}{4} a^2 \cdot \frac{1 - \cos 2\theta}{2}\ d\theta = \frac{1}{8} a^4 \left[\theta - \sin\theta \cos\theta \right]_0^\pi = \frac{1}{8}\pi a^4.$$

Therefore the y-coordinate of the centroid is $\overline{y} = \dfrac{3}{16}\pi a$.

C14S05.027: The mass and moments are

$$m = \int_{\theta=0}^\pi \int_{r=0}^a r^2\ dr\ d\theta = \pi \cdot \left[\frac{1}{3} r^3 \right]_0^a = \frac{1}{3}\pi a^3,$$

$$M_y = \int_{\theta=0}^\pi \int_{r=0}^a r^3 \cos\theta\ dr\ d\theta = \int_0^\pi \frac{1}{4} a^4 \cos\theta\ d\theta = \left[\frac{1}{4} a^4 \sin\theta \right]_0^\pi = 0, \quad \text{and}$$

$$M_x = \int_{\theta=0}^\pi \int_{r=0}^a r^3 \sin\theta\ dr\ d\theta = \int_0^\pi \frac{1}{4} a^4 \sin\theta\ d\theta = \left[-\frac{1}{4} a^4 \cos\theta \right]_0^\pi = \frac{1}{2} a^4.$$

Therefore the centroid of the lamina is located at the point

$$(\overline{x}, \overline{y}) = \left(0, \frac{3a}{2\pi} \right) \approx \big(0, (0.47746483) \cdot a \big).$$

C14S05.028: The mass and moments are

$$m = \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} r^2 \, dr \, d\theta = \int_0^{2\pi} \frac{1}{3}(1+\cos\theta)^3 \, d\theta = \int_0^{2\pi} \left(\frac{1}{3} + \cos\theta + \cos^2\theta + \frac{1}{3}\cos^3\theta \right) d\theta$$

$$= \frac{1}{36} \Big[30\theta + 45\sin\theta + 9\sin 2\theta + \sin 3\theta \Big]_0^{2\pi} = \frac{5}{3}\pi,$$

$$M_y = \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} r^3 \cos\theta \, dr \, d\theta = \int_0^{2\pi} \frac{1}{4}(\cos\theta)(1+\cos\theta)^4 \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\cos\theta + \cos^2\theta + \frac{3}{2}\cos^3\theta + \cos^4\theta + \frac{1}{4}\cos^5\theta \right) d\theta$$

$$= \frac{1}{960} \Big[840\theta + 1470\sin\theta + 480\sin 2\theta + 145\sin 3\theta + 30\sin 4\theta + 3\sin 5\theta \Big]_0^{2\pi} = \frac{7}{4}\pi,$$

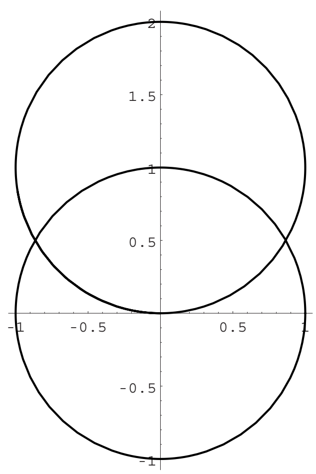
$$M_x = \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} r^3 \sin\theta \, dr \, d\theta = \int_0^{2\theta} \frac{1}{4}(\sin\theta)(1+\cos\theta)^4 \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\sin\theta + \sin\theta\cos\theta + \frac{3}{2}\sin\theta\cos^2\theta + \sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos^4\theta \right) d\theta$$

$$= -\frac{1}{320} \Big[210\cos\theta + 120\cos 2\theta + 45\cos 3\theta + 10\cos 4\theta + \cos 5\theta \Big]_0^{2\pi} = 0.$$

Therefore the centroid of the lamina is at $(\overline{x}, \overline{y}) = \left(\frac{21}{20}, 0\right)$.

C14S05.029: The following figure, generated by *Mathematica* 3.0, shows the two circles.



They cross where $2\sin\theta = 1$; that is, where $\theta = \frac{1}{6}\pi$ and where $\theta = \frac{5}{6}\pi$. The lamina in question is the region outside the lower circle and inside the upper circle. Its mass and moments are

$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^2 \sin\theta \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \left(\frac{8}{3}\sin^4\theta - \frac{1}{3}\sin\theta \right) d\theta$$

$$= \frac{1}{12}\Big[12\theta + 4\cos\theta - 8\sin 2\theta + \sin 4\theta \Big]_{\pi/6}^{5\pi/6} = \frac{8\pi + 3\sqrt{3}}{12},$$

$$M_y = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^3 \sin\theta\cos\theta \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \left(4\sin^5\theta\cos\theta - \frac{1}{4}\sin\theta\cos\theta \right) d\theta$$

$$= \frac{1}{48}\Big[6\cos 4\theta - \cos 6\theta - 12\cos 2\theta \Big]_{\pi/6}^{5\pi/6} = 0,$$

$$M_x = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^3 \sin^2\theta \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \left(4\sin^6\theta - \frac{1}{4}\sin^2\theta \right) d\theta$$

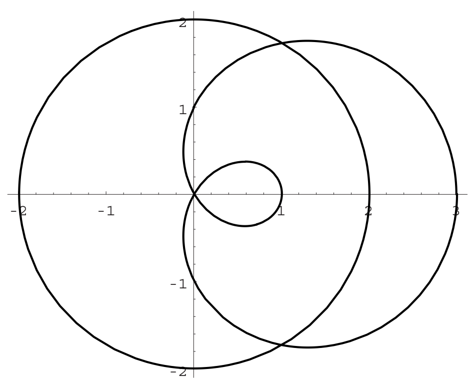
$$= \frac{1}{48}\Big[54\theta - 42\sin 2\theta + 9\sin 4\theta - \sin 6\theta \Big]_{\pi/6}^{5\pi/6} = \frac{12\pi + 11\sqrt{3}}{16}.$$

Thus its centroid is at the point

$$(\overline{x}, \overline{y}) = \left(0, \frac{36\pi + 33\sqrt{3}}{32\pi + 12\sqrt{3}} \right) \approx (0, 1.4034060567438982)$$

and its mass is approximately 2.5274078042854148.

C14S05.030: The limaçon and the circle are shown next, in a figure generated by *Mathematica* 3.0. We are to find the mass and centroid of the region within the limaçon and outside the circle given its density at (r, θ) is r.



The curves intersect where $1 + 2\cos\theta = 2$; that is, where $\theta = \pm\frac{1}{3}\pi$. The mass and moments of the lamina are

$$m = \int_{-\pi/3}^{\pi/3} \int_2^{1+2\cos\theta} r^2 \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{3}(1 + 2\cos\theta)^3 - \frac{8}{3} \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left(2\cos\theta + 4\cos^2\theta + \frac{8}{3}\cos^3\theta - \frac{7}{3} \right) d\theta = \frac{1}{9} \Big[36\sin\theta + 9\sin 2\theta + 2\sin 3\theta - 3\theta \Big]_{-\pi/3}^{\pi/3}$$

$$= \frac{45\sqrt{3} - 2\pi}{9} \approx 7.9621223370466546368 7,$$

$$M_y = \int_{-\pi/3}^{\pi/3} \int_2^{1+2\cos\theta} r^3 \cos\theta \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{4}(\cos\theta)(1 + 2\cos\theta)^4 - 4\cos\theta \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left(2\cos^2\theta + 6\cos^3\theta + 8\cos^4\theta + 4\cos^5\theta - \frac{15}{4}\cos\theta \right) d\theta$$

$$= \frac{1}{60} \Big[240\theta + 195\sin\theta + 150\sin 2\theta + 55\sin 3\theta + 15\sin 4\theta + 3\sin 5\theta \Big]_{-\pi/3}^{\pi/3} = \frac{160\pi + 327\sqrt{3}}{60},$$

$$M_x = \int_{-\pi/3}^{\pi/3} \int_2^{1+2\cos\theta} r^3 \sin\theta \, dr \, d\theta = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{4}(\sin\theta)(1 + 2\cos\theta)^4 - 4\sin\theta \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left(2\sin\theta\cos\theta + 6\sin\theta\cos^2\theta + 8\sin\theta\cos^3\theta + 4\sin\theta\cos^4\theta - \frac{15}{4}\sin\theta \right) d\theta$$

$$= \frac{1}{20} \Big[35\cos\theta - 30\cos 2\theta - 15\cos 3\theta - 5\cos 4\theta - \cos 5\theta \Big]_{-\pi/3}^{\pi/3} = 0.$$

Therefore the centroid of the lamina is at the point

$$(\overline{x}, \overline{y}) = \left(\frac{981\sqrt{3} + 480\pi}{900\sqrt{3} - 40\pi}, 0 \right) \approx (2.2377522671212835, 0).$$

C14S05.031: The polar moment of inertia of the lamina is

$$I_0 = \int_0^{2\pi} \int_0^a r^{n+3} \, dr \, d\theta = 2\pi \cdot \left[\frac{r^{n+4}}{n+4} \right]_0^a = \frac{2\pi a^{n+4}}{n+4}.$$

C14S05.032: The polar moment of inertia of the lamina is

$$I_0 = \int_0^{\pi} \int_0^a r^4 \sin\theta \, dr \, d\theta = \int_0^{\pi} \frac{1}{5} a^5 \sin\theta \, d\theta = \left[-\frac{1}{5} a^5 \cos\theta \right]_0^{\pi} = \frac{2}{5} a^5.$$

C14S05.033: The polar moment of inertia of the lamina is

$$I_0 = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} kr^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} 4k\cos^4\theta \, d\theta = \frac{1}{8} k \Big[12\theta + 8\sin 2\theta + \sin 4\theta \Big]_{-\pi/2}^{\pi/2} = \frac{3}{2}\pi k.$$

C14S05.034: See the figure that accompanies the solution of Problem 29. The polar moment of inertia of the lamina is

$$I_0 = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r^4 \sin\theta \, dr \, d\theta = \int_{\pi/6}^{5\pi/6} \frac{1}{5}(32\sin^6\theta - \sin\theta)\, d\theta$$

$$= \frac{1}{30}\Big[60\theta + 6\cos\theta - 45\sin 2\theta + 9\sin 4\theta - \sin 6\theta\Big]_{\pi/6}^{5\pi/6} = \frac{4\pi + 3\sqrt{3}}{3} \approx 5.9208410123552683.$$

C14S05.035: The polar moment of inertia of the lamina is

$$I_0 = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r^5 \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{6}\cos^3 2\theta \, d\theta = \frac{1}{144}\Big[9\sin 2\theta + \sin 6\theta\Big]_{-\pi/4}^{\pi/4} = \frac{1}{9}.$$

C14S05.036: In Problem 21 we found that the mass of the lamina is $m = a^3$. Next,

$$I_y = \int_0^a \int_0^a x^2(x+y)\, dy \, dx = \int_0^a \left(\frac{1}{2}a^2x^2 + ax^3\right) dx = \left[\frac{1}{6}a^2x^3 + \frac{1}{4}ax^4\right]_0^a = \frac{5}{12}a^5,$$

and $I_x = I_y$ by symmetry. Therefore $\hat{x} = \hat{y} = \frac{1}{6}a\sqrt{15}$.

C14S05.037: In Problem 23 we found that the mass of the lamina is $m = \frac{128}{5}$. Next,

$$I_y = \int_{-2}^{2} \int_{x^2}^{4} x^2 y \, dy \, dx = \int_{-2}^{2} \left(8x^2 - \frac{1}{2}x^6\right) dx = \left[\frac{8}{3}x^3 - \frac{1}{14}x^7\right]_{-2}^{2} = \frac{512}{21} \quad \text{and}$$

$$I_x = \int_{-2}^{2} \int_{x^2}^{4} y^3 \, dy \, dx = \int_{-2}^{2} \left(64 - \frac{1}{4}x^8\right) dx = \left[64x - \frac{1}{36}x^9\right]_{-2}^{2} = \frac{2048}{9}.$$

Therefore $\hat{x} = \frac{2}{21}\sqrt{105}$ and $\hat{y} = \frac{4}{3}\sqrt{5}$.

C14S05.038: In Problem 24 we found that the mass of the lamina is $m = \frac{96}{5}$. Next,

$$I_y = \int_{-1}^{3} \int_{x^2}^{2x+3} x^4 \, dy \, dx = \int_{-1}^{3} (3x^4 + 2x^5 - x^6)\, dx = \left[\frac{3}{5}x^5 + \frac{1}{3}x^6 - \frac{1}{7}x^7\right]_{-1}^{3} = \frac{8032}{105} \quad \text{and}$$

$$I_x = \int_{-1}^{3} \int_{x^2}^{2x+3} x^2y^2 \, dy \, dx = \int_{-1}^{3} \left(9x^2 + 18x^3 + 12x^4 + \frac{8}{3}x^5 - \frac{1}{3}x^8\right) dx$$

$$= \left[3x^3 + \frac{9}{2}x^4 + \frac{12}{5}x^5 + \frac{4}{9}x^6 - \frac{1}{27}x^9\right]_{-1}^{3} = \frac{84256}{135}.$$

Therefore

$$\hat{x} = \frac{\sqrt{1757}}{21} \approx 1.9960278014413988 \quad \text{and} \quad \hat{y} = \frac{\sqrt{2633}}{9} \approx 5.7014184936299995.$$

C14S05.039: In the solution of Problem 27 we found that the mass of the lamina is $m = \frac{1}{3}\pi a^3$. Next,

$$I_y = \int_0^{\pi} \int_0^a r^4 \cos^2 t \, dr \, d\theta = \int_0^{\pi} \frac{1}{5}a^5 \cos^2\theta \, d\theta = \left[\frac{1}{20}a^5(2\theta + \sin 2\theta)\right]_0^{\pi} = \frac{1}{10}\pi a^5 \quad \text{and}$$

$$I_x = \int_0^{\pi} \int_0^a r^4 \sin^2\theta \, dr \, d\theta = \int_0^{\pi} \frac{1}{5}a^5 \sin^2\theta d\theta = \left[\frac{1}{20}a^5(2\theta - \sin 2\theta)\right]_0^{\pi} = \frac{1}{10}\pi a^5.$$

Therefore $\hat{x} = \hat{y} = \frac{1}{10}a\sqrt{30}$.

C14S05.040: In the solution of Problem 33 we found that the mass of the lamina is $m = \pi k$. Next,

$$I_y = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} kr^3 \cos^2\theta \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} 4k\cos^6\theta \, d\theta$$

$$= \frac{1}{48}k\left[60\theta + 45\sin 2\theta + 9\sin 4\theta + \sin 6\theta\right]_{-\pi/2}^{\pi/2} = \frac{5}{4}\pi k;$$

$$I_x = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} kr^3 \sin^2\theta \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} 4k\cos^4\theta \sin^2\theta \, d\theta$$

$$= \frac{1}{48}k\left[12\theta + 3\sin 2\theta - 3\sin 4\theta - \sin 6\theta\right]_{-\pi/2}^{\pi/2} = \frac{1}{4}\pi k.$$

Therefore $\hat{x} = \frac{1}{2}\sqrt{5}$ and $\hat{y} = \frac{1}{2}$.

C14S05.041: The quarter of the circular disk $x^2 + y^2 \leqq a^2$ that lies in the first quadrant has the polar-coordinates description $0 \leqq r \leqq a$, $0 \leqq \theta \leqq \frac{1}{2}\pi$. If we assume that it has uniform density $\delta = 1$, then its mass is $m = \frac{1}{4}\pi a^2$. Next,

$$M_y = \int_0^{\pi/2} \int_0^a r^2 \cos\theta \, dr \, d\theta = \int_0^{\pi/2} \frac{1}{3}a^3 \cos\theta \, d\theta = \left[\frac{1}{3}a^2 \sin\theta\right]_0^{\pi/2} = \frac{1}{3}a^3,$$

and $M_x = M_y$ by symmetry. Therefore the centroid is located at the point

$$(\overline{x}, \overline{y}) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right) \approx \big([0.4244131815783876]a, [0.4244131815783876]a\big).$$

C14S05.042: Given: The quarter of the circular disk $x^2 + y^2 \leqq a^2$ that lies in the first quadrant, with centroid $(\overline{x}, \overline{y})$. Note that $\overline{x} = \overline{y}$ by symmetry. By the first theorem of Pappus,

$$2\pi\overline{y} \cdot \frac{1}{4}\pi a^2 = \frac{2}{3}\pi a^3,$$

which we solve for $\overline{y} = \dfrac{4a}{3\pi}$.

C14S05.043: The arc $x^2 + y^2 = r^2$ can be parametrized in this way:

$$x(t) = r\cos\theta, \quad y(t) = r\sin\theta, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

We may also assume that it has unit density. The arc length element is

$$ds = \sqrt{[x'(\theta)]^2 + [y'(\theta)]^2}\ d\theta = r\, d\theta,$$

and hence its mass is

$$m = \int_0^{\pi/2} r\, d\theta = \frac{1}{2}\pi r.$$

Its moment around the y-axis is

$$M_y = \int_0^{\pi/2} r^2 \cos\theta \, d\theta = \Big[r^2 \sin\theta \Big]_0^{\pi/2} = r^2.$$

Therefore, because $\overline{y} = \overline{x}$ by symmetry,

$$(\overline{x}, \overline{y}) = \left(\frac{2r}{\pi}, \frac{2r}{\pi} \right).$$

C14S05.044: By the second theorem of Pappus,

$$2\pi\overline{y} \cdot \frac{\pi r}{2} = 2\pi r^2,$$

and hence $\overline{x} = \overline{y} = \dfrac{2r}{\pi}$.

C14S05.045: We may assume that the triangle has unit density. The hypotenuse of this right triangle has equation

$$y = f(x) = h\left(1 - \frac{x}{r}\right),$$

and hence the mass and moments of the triangle are

$$\begin{aligned}
m &= \int_0^r \int_0^{f(x)} 1 \, dy \, dx = \int_0^r h\left(1 - \frac{x}{r}\right) dx = \left[hx - \frac{h}{2r}x^2 \right]_0^r = \frac{1}{2}hr,\\
M_y &= \int_0^r \int_0^{f(x)} x \, dy \, dx = \int_0^r hx\left(1 - \frac{x}{r}\right) dx = \left[\frac{h}{2}x^2 - \frac{h}{3r}x^3 \right]_0^r = \frac{1}{6}hr^2, \qquad \text{and}\\
M_x &= \int_0^r \int_0^{f(x)} y \, dy \, dx = \int_0^r \left(\frac{1}{2}h^2 - \frac{h^2}{r}x + \frac{h^2}{2r^2}x^2 \right) dx\\
&= \left[\frac{h^2}{2}x - \frac{h^2}{2r}x^2 + \frac{h^2}{6r^2}x^3 \right]_0^r = \frac{1}{6}h^2 r.
\end{aligned}$$

Therefore the centroid of the triangle is

$$C = (\overline{x}, \overline{y}) = \left(\frac{1}{3}r, \frac{1}{3}h \right).$$

The midpoint of the hypotenuse is the point $M\left(\frac{1}{2}r, \frac{1}{2}h\right)$ and the line from the origin to M has equation $y = hx/r$, so it is clear that C lies on this line and is two-thirds of the way from the origin to M.

C14S05.046: Here we simply observe that $2\pi\dfrac{r}{3} \cdot \dfrac{1}{2}rh = \dfrac{1}{3}\pi r^2 h$.

C14S05.047: Here we simply observe that $2\pi\dfrac{r}{2} \cdot L = \pi r L$.

C14S05.048: First we compute the mass and moments of the rectangular part of the lamina (we assume that it has unit density):

$$m = \int_0^{r_2} \int_0^h 1 \, dy \, dx = \int_0^{r_2} h \, dx = hr_2,$$

$$M_y = \int_0^{r_2} \int_0^h x \, dy \, dx = \int_0^{r_2} hx \, dx = \frac{1}{2} hr_2^2, \quad \text{and}$$

$$M_x = \int_0^{r_2} \int_0^h y \, dy \, dx = \int_0^{r_2} \frac{1}{2} h^2 \, dx = \frac{1}{2} h^2 r_2.$$

Next we compute the mass and moments of the triangular part of the lamina. First note that the equation of the diagonal side of the lamina is

$$y = \frac{h}{r_1 - r_2}(r_1 - x) = f(x).$$

Thus

$$m = \int_{r_2}^{r_1} \int_0^{f(x)} 1 \, dy \, dx = \int_{r_2}^{r_1} \frac{h(r_1 - x)}{r_1 - r_2} \, dx = \left[\frac{2hr_1 x - hx^2}{2(r_1 - r_2)} \right]_{r_2}^{r_1} = \frac{1}{2} h(r_1 - r_2);$$

$$M_y = \int_{r_2}^{r_1} \int_0^{f(x)} x \, dy \, dx = \int_{r_2}^{r_1} \frac{hr_1 x - hx^2}{r_1 - r_2} \, dx = \left[\frac{3hr_1 x^2 - 2hx^3}{6(r_1 - r_2)} \right]_{r_2}^{r_1} = \frac{1}{6} h(r_1 - r_2)(r_1 + 2r_2);$$

$$M_x = \int_{r_2}^{r_1} \int_0^{f(x)} y \, dy \, dx = \int_{r_2}^{r_1} \frac{h^2(r_1 - x)^2}{2(r_1 - r_2)^2} \, dx = \left[\frac{3h^2(r_1)^2 x - 3h^2 r_1 x^2 + h^2 x^3}{6(r_1 - r_2)^2} \right]_{r_2}^{r_1} = \frac{1}{6} h^2(r_1 - r_2).$$

Now moments, like masses, are additive. Thus the mass and moments of the entire lamina can be obtained by addition; they are

$$m = \frac{1}{2} h(r_1 + r_2), \quad M_y = \frac{1}{6} h(r_1^2 + r_1 r_2 + r_2^2), \quad \text{and} \quad M_x = \frac{1}{6} h^2(r_1 + 2r_2).$$

Therefore the centroid of the lamina is at

$$(\overline{x}, \, \overline{y}) \; = \; \left(\frac{r_1^2 + r_1 r_2 + r_2^2}{3(r_1 + r_2)}, \; \frac{h(r_1 + 2r_2)}{3(r_1 + r_2)} \right).$$

Finally, using the first theorem of Pappus, the volume generated by rotating the trapezoid around the y-axis—the volume of the conical frustum—is

$$V = 2\pi \overline{x} \cdot \frac{1}{2} h(r_1 + r_2) = \frac{1}{3} \pi h(r_1^2 + r_1 r_2 + r_2^2).$$

C14S05.049: The diagonal side of the trapezoid has centroid at its midpoint, so that $\overline{x} = \frac{1}{2}(r_1 + r_2)$. By the second theorem of Pappus, the curved surface area of the conical frustum—which is generated by the diagonal side—is

$$A = 2\pi \overline{x} \cdot L = \pi L(r_1 + r_2).$$

C14S05.050: The vertical line segment L in the xy-plane connecting the two points $(r, \, 0)$ and $(r, \, h)$ generates the curved side of the cylinder of radius r and height h when L is rotated around the y-axis. The

centroid of L has x-coordinate $\overline{x} = r$, so by the second theorem of Pappus the curved surface area of the cylinder is

$$A = 2\pi\overline{x} \cdot h = 2\pi rh.$$

This result also follows from the result in Problem 49 by letting $r_1 = r_2 = r$, in which case $L = h$, so that $A = \pi(r_1 + r_2)L = 2\pi rh$.

C14S05.051: First we compute the mass and moments of the rectangle (we assume that it has unit density):

$$m = \int_{-a}^{a} \int_{0}^{b} 1\ dy\ dx = \int_{-a}^{a} b\ dx = 2ab;$$

$$M_y = \int_{-a}^{a} \int_{0}^{b} x\ dy\ dx = \int_{-a}^{a} bx\ dx = \left[\frac{1}{2}bx^2\right]_{-a}^{a} = 0;$$

$$M_x = \int_{-a}^{a} \int_{0}^{b} y\ dy\ dx = \int_{-a}^{a} \frac{1}{2}b^2\ dx = ab^2.$$

Next we compute the mass and moments of the semicircle. Note that its curved boundary has equation $y = b + \sqrt{a^2 - x^2} = f(x)$. Because we have assumed unit density, its mass is $m = \frac{1}{2}\pi a^2$. By symmetry, $M_y = 0$. And

$$\begin{aligned} M_x &= \int_{-a}^{a} \int_{b}^{f(x)} y\ dy\ dx = \int_{-a}^{a} \left[\frac{1}{2}\left(b + \sqrt{a^2 - x^2}\right)^2 - \frac{1}{2}b^2\right] dx \\ &= \int_{-a}^{a} \left[b(a^2 - x^2)^{1/2} + \frac{1}{2}(a^2 - x^2)\right] dx = b\int_{-a}^{a} (a^2 - x^2)^{1/2}\ dx + \left[\tfrac{1}{2}a^2x - \frac{1}{6}x^3\right]_{-a}^{a} \\ &= b \cdot \frac{\pi a^2}{2} + a^3 - \frac{1}{3}a^3 = \frac{3\pi a^2 b + 4a^3}{6}. \end{aligned}$$

Moments, like masses, are additive. Thus we find the mass and moments for the entire lamina by adding those of the rectangle and semicircle:

$$m = \frac{4ab + \pi a^2}{2}, \quad M_y = 0, \quad \text{and} \quad M_x = \frac{6ab^2 + 3\pi a^2 b + 4a^3}{6}.$$

Therefore $\overline{x} = 0$ and

$$\overline{y} = \frac{6ab^2 + 3\pi a^2 b + 4a^3}{6} \cdot \frac{2}{4ab + \pi a^2} = \frac{6b^2 + 3\pi ab + 4a^2}{12b + 3\pi a}.$$

Finally, we apply the first theorem of Pappus to find the volume of the solid generated by rotation of the lamina around the x-axis:

$$V = 2\pi\overline{y} \cdot \frac{4ab + \pi a^2}{2} = \pi \cdot \frac{6b^2 + 3\pi ab + 4a^2}{12b + 3\pi a} \cdot (4ab + \pi a^2) = \frac{\pi a}{3} \cdot (6b^2 + 3\pi ab + 4a^2).$$

To check the answer, note that it has the correct domensions of volume and that in the extreme case $b = 0$, it becomes $\frac{4}{3}\pi a^3$, the correct formula for the volume of a sphere of radius a.

C14S05.052: The area of the region is

$$A = \int_0^h \int_0^{\sqrt{2py}} 1\ dx\ dy = \int_0^h \sqrt{2py}\ dy = \left[\frac{2}{3} y^{3/2} \sqrt{2p} \right]_0^h = \frac{2}{3} h^{3/2} \sqrt{2p} = \frac{2}{3} rh$$

(the last equality follows from the substitution of $\dfrac{r^2}{2h}$ for p). The moments are

$$M_y = \int_0^h \int_0^{\sqrt{2py}} x\ dx\ dy = \int_0^h py\ dy = \frac{h^2 p}{2} = \frac{r^2 h}{4} \qquad \text{and}$$

$$M_x = \int_0^h \int_0^{\sqrt{2py}} y\ dx\ dy = \int_0^h y^{3/2} \sqrt{2p}\ dy = \frac{2}{5} h^{5/2} \sqrt{2p} = \frac{2}{5} rh^2.$$

Therefore the centroid of the region is

$$(\overline{x},\ \overline{y}) = \left(\frac{3}{8} r,\ \frac{3}{5} h \right).$$

By the first theorem of Pappus, the volume of the paraboloid generated by rotating this region around the y-axis is therefore

$$V = 2\pi \overline{x} \cdot A = 2\pi \cdot \frac{3}{8} r \cdot \frac{2}{3} rh = \frac{1}{2} \pi r^2 h.$$

C14S05.053: The plate is a rectangle of area ab, area density δ, and mass m, and hence $m = ab\delta$. The moment of inertia with respect to the y-axis is

$$I_y = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \delta x^2\ dy\ dx = \int_{-a/2}^{a/2} \delta b x^2\ dx = \left[\frac{1}{3} \delta b x^3 \right]_{-a/2}^{a/2} = \frac{1}{12} \delta a^3 b = \frac{1}{12} ma^2.$$

By symmetry or by a very similar computation $I_x = \frac{1}{12} mb^2$. So, by the comment following Eq. (6) of the text,

$$I_0 = I_x + I_y = \frac{1}{12} m(a^2 + b^2).$$

C14S05.054: Note that the moments M_x and M_y are both zero because $(\overline{x},\ \overline{y}) = (0,\ 0)$. Therefore the moment of inertia of the region R with respect to the line perpendicular to the xy-plane at the point $(x_0,\ y_0)$ is

$$\begin{aligned} I &= \iint_R \left[(x - x_0)^2 + (y - y_0)^2 \right] \cdot \delta\ dA \\ &= \iint_R (x^2 + y^2) \cdot \delta\ dA - 2 \iint_R (x_0 x + y_0 y) \cdot \delta\ dA + \iint_R (x_0^2 + y_0^2) \cdot \delta\ dA \\ &= I_0 - 2x_0 \delta \iint_R x\ dA - 2y_0 \delta \iint_R y\ dA + (x_0^2 + y_0^2) \iint_R \delta\ dA \\ &= I_0 - 2x_0 \delta M_y - 2y_0 \delta M_x + (x_0^2 + y_0^2) m = I_0 + m(x_0^2 + y_0^2). \end{aligned}$$

C14S05.055: Suppose that the plane lamina L is the union of the two nonoverlapping laminae R and S. Let I_L, I_R, and I_S denote the polar moments of inertia of L, R, and S, respectively. Let $\delta(x, y)$ denote the density of L at the point (x, y). Then

$$I_L = \iint_L (x^2 + y^2)\delta(x, y)\, dA = \iint_R (x^2 + y^2)\delta(x, y)\, dA + \iint_S (x^2 + y^2)\delta(x, y)\, dA = I_R + I_S.$$

Next, let I_1 denote the polar moment of inertia of the lower rectangle in Fig. 14.5.25 and let I_2 denote the polar moment of inertia of the upper rectangle. Then

$$I_1 = \int_{-1}^{1}\int_0^3 (x^2 + y^2)\cdot k\, dy\, dx = k\int_{-1}^{1}(3x^2 + 9)\, dx = k\Big[x^3 + 9x\Big]_{-1}^{1} = 20k;$$

$$I_2 = \int_{-4}^{4}\int_3^4 (x^2 + y^2)\cdot k\, dy\, dx = k\int_{-4}^{4}\left[x^2 y + \frac{1}{3}y^3\right]_3^4 dx = k\int_{-4}^{4}\left(x^2 + \frac{37}{3}\right) dx$$

$$= k\left[\frac{1}{3}x^3 + \frac{37}{3}x\right]_{-4}^{4} = k\cdot\left(\frac{128}{3} + \frac{296}{3}\right) = \frac{424}{3}k.$$

Thus, by the first result in this solution, the polar moment of inertia of the `T`-shaped lamina is

$$I_0 = I_1 + I_2 = \frac{484}{3}k.$$

C14S05.056: Let k denote the [constant] density of the racquet. The area of the racquet is

$$A = \int_{\theta=-\pi/4}^{\pi/4} \frac{1}{2}r^2\, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos 2\theta\, d\theta = \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2},$$

and hence its mass is $m = \frac{1}{2}k$. Next we compute the moment of inertia I of the racquet with respect to the line $x = -1$:

$$I = 2\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} kr(1 + r\cos\theta)^2\, dr\, d\theta = k\int_0^{\pi/4}\left[r^2 + \frac{4}{3}r^3\cos\theta + \frac{1}{2}r^4\cos^2\theta\right]_0^{\sqrt{\cos 2\theta}} d\theta = J_1 + J_2 + J_3$$

where

$$J_1 = k\int_0^{\pi/4}\cos 2\theta\, d\theta = k\cdot\left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}k,$$

$$J_2 = k\int_0^{\pi/4}\frac{1}{2}\cos^2\theta\cos^2 2\theta\, d\theta = \frac{k}{96}\Big[12\theta + 9\sin 2\theta + 3\sin 4\theta + \sin 6\theta\Big]_0^{\pi/4} = \frac{k}{96}(3\pi + 8),$$

and

$$J_3 = \frac{4k}{3}\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\, d\theta.$$

To evaluate J_3 using *Mathematica* 3.0, the command

```
Integrate[ (4*k/3)*(Cos[2*t])∧(3/2)*Cos[t], t]
```

produces the antiderivative

$$\frac{4k}{3} \cdot \left[\frac{3 \arcsin(\sqrt{2}\ \sin t)}{8\sqrt{2}} + \sqrt{\cos 2t}\ \left(\frac{1}{4} \sin t + \frac{1}{8} \sin 3t \right) \right.$$

Then we evaluate the antiderivative at the limits of integration:

```
(% /.  t → Pi/4) - (% /.  t → 0)
```

$$\frac{k\pi}{4\sqrt{2}}$$

To evaluate J_3 by hand, note that

$$J_3 = \frac{4k}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \ d\theta.$$

The substitution $u = \arcsin\left(\sqrt{2}\ \sin\theta\right)$ then yields

$$\sin u = \sqrt{2}\ \sin\theta; \quad \sin^2 u = 2\sin^2\theta; \quad 1 - 2\sin^2\theta = 1 - \sin^2 u = \cos^2 u.$$

Moreover, $\cos u \ du = \sqrt{2}\ \cos\theta \ d\theta,$ and therefore

$$\cos\theta \ d\theta = \frac{1}{\sqrt{2}}\ \cos u \ du.$$

These substitutions yield

$$\begin{aligned} J_3 &= \frac{4k}{3} \int_{u=0}^{\pi/2} (\cos^3 u) \cdot \frac{\cos u}{\sqrt{2}}\ du = \frac{2k\sqrt{2}}{3} \int_0^{\pi/2} \left(\frac{1 + \cos 2u}{2} \right)^2 du \\ &= \frac{2k\sqrt{2}}{3} \cdot \frac{1}{4} \int_0^{\pi/2} \left(1 + 2\cos 2u + \frac{1 + \cos 4u}{2} \right) du \\ &= \frac{k\sqrt{2}}{6} \cdot \left[\frac{3}{2} u + \sin 2u + \frac{1}{8} \sin 4u \right]_0^{\pi/2} = \frac{k\sqrt{2}}{6} \cdot \frac{3}{2} \cdot \frac{\pi}{2} = \frac{\pi k\sqrt{2}}{8}. \end{aligned}$$

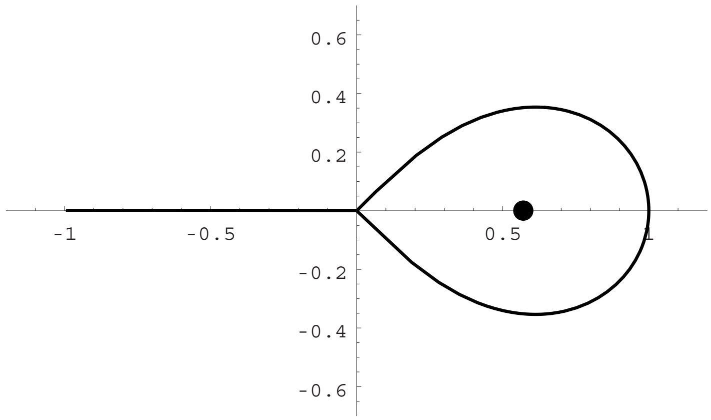
Therefore

$$I = J_1 + J_2 + J_3 = \frac{k}{96} \left(56 + 3\pi + 12\pi\sqrt{2}\ \right),$$

and hence the radius of gyration of the racquet with respect to the line $x = -1$ is

$$\hat{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{56 + 3\pi + 12\pi\sqrt{2}}{48}} \approx 1.5728117948615531.$$

The coordinates of the "sweet spot" are thus approximately (0.573, 0); this point is marked with a bullet in the following sketch of the racquet.



C14S05.057: The mass and moments are

$$m = \int_0^{\pi} \int_0^{2\sin\theta} r^2 \sin\theta \, dr \, d\theta = \int_0^{\pi} \frac{8}{3} \sin^4\theta \, d\theta = \frac{1}{12} \Big[12\theta - 8\sin 2\theta + \sin 4\theta \Big]_0^{\pi} = \pi;$$

$$M_y = \int_0^{\pi} \int_0^{2\sin\theta} r^3 \sin\theta \cos\theta \, dr \, d\theta = \int_0^{\pi} 4\sin^5\theta \cos\theta \, d\theta = \left[\frac{2}{3} \sin^6\theta \right]_0^{\pi} = 0;$$

$$M_x = \int_0^{\pi} \int_0^{2\sin\theta} r^3 \sin^2\theta \, dr \, d\theta = \int_0^{\pi} 4\sin^6\theta \, d\theta = \frac{1}{48} \Big[60\theta - 45\sin 2\theta + 9\sin 4\theta - \sin 6\theta \Big]_0^{\pi} = \frac{5}{4}\pi.$$

Therefore the centroid is at $(\overline{x}, \overline{y}) = \left(0, \frac{5}{4}\right)$.

C14S05.058: The mass and moments are

$$m = \int_0^{\pi} \int_0^{2\sin\theta} r^3 \sin\theta \, dr \, d\theta = \int_0^{\pi} 4\sin^5\theta \, d\theta = \frac{1}{60} \Big[-150\cos\theta + 25\cos 3\theta - 3\cos 5\theta \Big]_0^{\pi} = \frac{64}{15};$$

$$M_y = \int_0^{\pi} \int_0^{2\sin\theta} r^4 \sin\theta \cos\theta \, dr \, d\theta = \int_0^{\pi} \frac{32}{5} \sin^6\theta \cos\theta \, d\theta == \left[\frac{32}{35} \sin^7\theta \right]_0^{\pi} = 0;$$

$$M_x = \int_0^{\pi} \int_0^{2\sin\theta} r^4 \sin^2\theta \, dr \, d\theta = \int_0^{\pi} \frac{32}{5} \sin^7\theta \, d\theta$$

$$= \frac{1}{350} \Big[-1225\cos\theta + 245\cos 3\theta - 49\cos 5\theta + 5\cos 7\theta \Big]_0^{\pi} = \frac{1024}{175}.$$

Therefore the centroid is at the point $(\overline{x}, \overline{y}) = \left(0, \frac{48}{35}\right)$.

C14S05.059: The mass and moments are

$$m = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \cos\theta \; dr \; d\theta = \int_0^{\pi/2} \frac{8}{3} \cos^4\theta \; d\theta = \frac{1}{12}\Big[12\theta + 8\sin 2\theta + \sin 4\theta\Big]_0^{\pi/2} = \frac{1}{2}\pi;$$

$$M_y = \int_0^{\pi/2} \int_0^{2\cos\theta} r^3 \cos^2\theta \; dr \; d\theta = \int_0^{\pi/2} 4\cos^6\theta \; d\theta = \frac{1}{48}\Big[60\theta + 45\sin 2\theta + 9\sin 4\theta + \sin 6\theta\Big]_0^{\pi/2} = \frac{5}{8}\pi;$$

$$M_x = \int_0^{\pi/2} \int_0^{2\cos\theta} r^3 \sin\theta\cos\theta \; dr \; d\theta = \int_0^{\pi/2} 4\sin\theta\cos^5\theta \; d\theta = \Big[-\frac{2}{3}\cos^6\theta\Big]_0^{\pi/2} = \frac{2}{3}.$$

Therefore the centroid is located at $(\overline{x}, \overline{y}) = \left(\frac{5}{4}, \frac{4}{3\pi}\right)$.

C14S05.060: The mass and moments are

$$m = \int_0^{\pi/2} \int_0^{2\cos\theta} r^5 \cos^2\theta \sin^2\theta \; dr \; d\theta = \int_0^{\pi/2} \frac{32}{3} \cos^8\theta \sin^2\theta \; d\theta$$

$$= \frac{1}{2880}\Big[840\theta + 420\sin 2\theta - 120\sin 4\theta - 130\sin 6\theta - 45\sin 8\theta - 6\sin 10\theta\Big]_0^{\pi/2} = \frac{7}{48}\pi;$$

$$M_y = \int_0^{\pi/2} \int_0^{2\cos\theta} r^6 \cos^3\theta \sin^2\theta \; dr \; d\theta = \int_0^{\pi/2} \frac{128}{7} \cos^{10}\theta \sin^2\theta \; d\theta$$

$$= \frac{1}{6720}\Big[2520\theta + 1440\sin 2\theta - 225\sin 4\theta - 400\sin 6\theta - 195\sin 8\theta - 48\sin 10\theta - 5\sin 12\theta\Big]_0^{\pi/2} = \frac{3}{16}\pi;$$

$$M_x = \int_0^{\pi/2} \int_0^{2\cos\theta} r^6 \cos^2\theta \sin^3\theta \; dr \; d\theta = \int_0^{\pi/2} \frac{128}{7} \cos^9\theta \sin^3\theta \; d\theta$$

$$= \frac{1}{6720}\Big[-1080\cos 2\theta - 405\cos 4\theta + 20\cos 6\theta + 90\cos 8\theta + 36\cos 10\theta + 5\cos 12\theta\Big]_0^{\pi/2} = \frac{32}{105}.$$

Hence the centroid is located at the point $(\overline{x}, \overline{y}) = \left(\frac{9}{7}, \frac{512}{245\pi}\right) \approx (1.2857142857,\ 0.6652027009)$.

Section 14.6

Note: For some triple integrals, there are six possible orders of integration of the corresponding iterated integrals; for many triple integrals, there are two. To save space, in this section we do not show all possibilities, but only the one that seems most natural.

C14S06.001: The value of the triple integral is

$$I = \int_{z=0}^{1} \int_{y=0}^{3} \int_{x=0}^{2} (x + y + z)\ dx\ dy\ dz = \int_{z=0}^{1} \int_{y=0}^{3} \left[\frac{1}{2}x^2 + xy + xz \right]_{x=0}^{2} dy\ dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{3} (2 + 2y + 2z)\ dy\ dz = \int_{z=0}^{1} \left[2y + y^2 + 2yz \right]_{y=0}^{3} dz = \int_{0}^{1} (15 + 6z)\ dz = \left[15z + 3z^2 \right]_{0}^{1} = 18.$$

C14S06.002: The value of the triple integral is

$$J = \int_{z=0}^{\pi} \int_{y=0}^{\pi} \int_{x=0}^{\pi} xy \sin z\ dx\ dy\ dz = \int_{0}^{\pi} \int_{0}^{\pi} \frac{1}{2}\pi^2 y \sin z\ dy\ dz = \int_{0}^{\pi} \frac{1}{4}\pi^4 \sin z\ dz$$

$$= \left[-\frac{1}{4}\pi^4 \cos z \right]_{0}^{\pi} = \frac{1}{2}\pi^4 \approx 48.7045455170012186.$$

C14S06.003: The value of the triple integral is

$$K = \int_{z=-2}^{6} \int_{y=0}^{2} \int_{x=-1}^{3} xyz\ dx\ dy\ dz = \int_{z=-2}^{6} \int_{y=0}^{2} \left[\frac{1}{2}x^2 yz \right]_{x=-1}^{3} dy\ dz = \int_{z=-2}^{6} \int_{y=0}^{2} 4yz\ dy\ dz$$

$$= \int_{z=-2}^{6} \left[2y^2 z \right]_{y=0}^{2} dz = \int_{z=-2}^{6} 8z\ dz = \left[4z^2 \right]_{z=-2}^{6} = 128.$$

C14S06.004: The value of the triple integral is

$$I = \int_{z=-2}^{6} \int_{y=0}^{2} \int_{x=-1}^{3} (x + y + z)\ dx\ dy\ dz = \int_{z=-2}^{6} \int_{y=0}^{2} \left[\frac{1}{2}x^2 + xy + xz \right]_{x=-1}^{3} dy\ dz$$

$$= \int_{z=-2}^{6} \int_{y=0}^{2} (4 + 4y + 4z)\ dy\ dz = \int_{z=-2}^{6} \left[2y^2 + 4y + 4yz \right]_{y=0}^{2} dz = \int_{z=-2}^{6} (16 + 8z)\ dz$$

$$= \left[16z + 4z^2 \right]_{z=-2}^{6} = 256.$$

C14S06.005: The value of the triple integral is

$$J = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^2\ dz\ dy\ dx = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[x^2 z \right]_{z=0}^{1-x-y} dy\ dx = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2 - x^3 - x^2 y)\ dy\ dx$$

$$= \int_{x=0}^{1} \left[x^2 y - x^3 y - \frac{1}{2}x^2 y^2 \right]_{y=0}^{1-x} dx = \int_{0}^{1} \left(\frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4 \right) dx = \left[\frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5 \right]_{0}^{1} = \frac{1}{60}.$$

C14S06.006: The value of the triple integral is

$$K = \int_0^3 \int_0^{(6-2x)/3} \int_0^{6-2x-3y} (2x+3y)\ dz\ dy\ dx = \int_0^3 \int_0^{(6-2x)/3} \Big[2xz + 3yz\Big]_0^{6-2x-3y}\ dy\ dx$$

$$= \int_0^3 \int_0^{(6-2x)/3} (12x - 4x^2 + 18y - 12xy - 9y^2)\ dy\ dx$$

$$= \int_0^3 \Big[12xy - 4x^2y + 9y^2 - 6xy^2 - 3y^3\Big]_0^{(6-2x)/3}\ dx = \int_0^3 \left(12 - 4x^2 + \frac{8}{9}x^3\right)\ dx$$

$$= \left[12x - \frac{4}{3}x^3 + \frac{2}{9}x^4\right]_0^3 = 18.$$

C14S06.007: The value of the triple integral is

$$I = \int_{x=-1}^0 \int_{y=0}^2 \int_{z=0}^{1-x^2} xyz\ dz\ dy\ dx = \int_{x=-1}^0 \int_{y=0}^2 \left[\frac{1}{2}xyz^2\right]_{z=0}^{1-x^2}\ dy\ dx$$

$$= \int_{x=-1}^0 \int_{y=0}^2 \left(\frac{1}{2}xy - x^3y + \frac{1}{2}x^5y\right)\ dy\ dx = \int_{x=-1}^0 \left[\frac{1}{4}xy^2 - \frac{1}{2}x^3y^2 + \frac{1}{4}x^5y^2\right]_{y=0}^2\ dx$$

$$= \int_{x=-1}^0 (x - 2x^3 + x^5)\ dx = \frac{1}{6}\Big[3x^2 - 3x^4 + x^6\Big]_{x=-1}^0 = -\frac{1}{6}.$$

C14S06.008: The value of the triple integral is

$$J = \int_{x=-1}^1 \int_{y=-2}^2 \int_{z=0}^{4-y^2} (2y+z)\ dz\ dy\ dx = \int_{-1}^1 \int_{-2}^2 \left[2yz + \frac{1}{2}z^2\right]_0^{4-y^2}\ dy\ dx$$

$$= \int_{-1}^1 \int_{-2}^2 \left(8 + 8y - 4y^2 - 2y^3 + \frac{1}{2}y^4\right)\ dy\ dx = \int_{-1}^1 \left[8y + 4y^2 - \frac{4}{3}y^3 - \frac{1}{2}y^4 + \frac{1}{10}y^5\right]_{-2}^2\ dx$$

$$= \int_{-1}^1 \frac{256}{15}\ dx = \frac{512}{15} \approx 34.1333333333333333.$$

C14S06.009: The value of this triple integral is

$$K = \int_{-1}^1 \int_0^3 \int_{x^2}^{2-x^2} (x+y)\ dz\ dy\ dx = \int_{-1}^1 \int_0^3 \Big[xy + xz\Big]_{x^2}^{2-x^2}\ dy\ dx$$

$$= \int_{-1}^1 \int_0^3 (2x - 2x^3 + 2y - 2x^2y)\ dy\ dx = \int_{-1}^1 \Big[2xy - 2x^3y + y^2 - x^2y^2\Big]_0^3\ dx$$

$$= \int_{-1}^1 (9 + 6x - 9x^2 - 6x^3)\ dx = \left[9x + 3x^2 - 3x^3 - \frac{3}{2}x^4\right]_{-1}^1 = 12.$$

C14S06.010: The value of this triple integral is

$$I = \int_{-1}^{1} \int_{-2}^{2} \int_{y^2}^{8-y^2} z\,dz\,dy\,dx = \int_{-1}^{1} \int_{-2}^{2} \left[\frac{1}{2}z^2 \right]_{y^2}^{8-y^2} dy\,dx = \int_{-1}^{1} \int_{-2}^{2} (32 - 8y^2)\,dy\,dx$$

$$= \int_{-1}^{2} \left[32y - \frac{8}{3}y^3 \right]_{-2}^{2} dx = \int_{-1}^{1} \frac{256}{3}\,dx = \frac{512}{3} \approx 170.6666666666666667.$$

C14S06.011: The solid resembles the tetrahedron shown in Fig. 14.6.4 of the text. We don't have enough memory to transfer a *Mathematica*-generated graphic from Adobe Illustrator to this document, but you can see it if you execute the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{x, 0, 6 - 2*x - 3*y}, {x, y, 6 - 2*x - 3*y}},
   {x, 0, 3}, {y, 0, 2}, PlotRange → {0, 3}, {0, 2}, {0, 6}},
   AspectRatio → 1.0, ViewPoint → {3.5, -1.6, 3.0},
   LightSources → {{{1., 0., 1.}, RGBColor[1., 0., 0.]},
   {{1., 1., 1.}, RGBColor[0., 1., 0.]}, {{0., 1., 1.}, RGBColor[1., 0., 1.]}} ];
```

If you change the `Viewpoint` parameters, you will see that in an effort to conserve memory, we plotted only the diagonal face of the tetrahedron and the face that lies in the xz-plane. The volume of this tetrahedron is given by

$$V = \int_{x=0}^{3} \int_{y=0}^{(6-2x)/3} \int_{z=0}^{6-2x-3y} 1\,dz\,dy\,dx = \int_{x=0}^{3} \int_{y=0}^{(6-2x)/3} (6 - 2x - 3y)\,dy\,dx$$

$$= \int_{x=0}^{3} \left[6y - 2xy - \frac{3}{2}y^2 \right]_{y=0}^{(6-2x)/3} dx = \int_{x=0}^{3} \left(6 - 4x + \frac{2}{3}x^2 \right) dx = \left[6x - 2x^2 + \frac{2}{9}x^3 \right]_0^3 = 6.$$

C14S06.012: The volume is

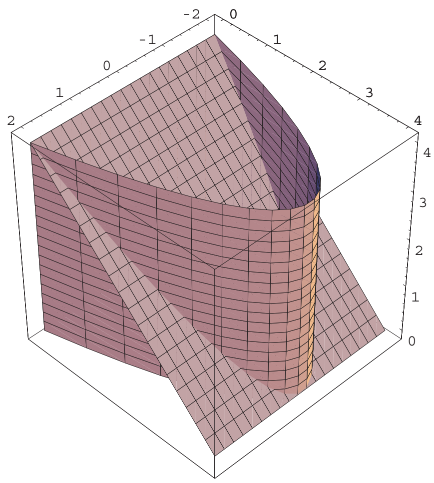
$$V = \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{y} 1\,dz\,dy\,dx = \int_{-2}^{2} \int_{x^2}^{4} y\,dy\,dx = \int_{-2}^{2} \left[\frac{1}{2}y^2 \right]_{x^2}^{4} dx$$

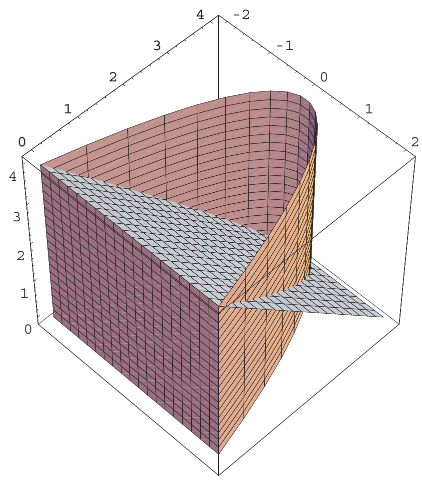
$$= \int_{-2}^{2} \left(8 - \frac{1}{2}x^4 \right) dx = \left[8x - \frac{1}{10}x^5 \right]_{-2}^{2} = \frac{128}{5} = 25.6.$$

C14S06.013: The view of the figure next, on the left, was generated with the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{u, 0, v}, {u, 4 - u*u, v}, {u, v, 4 - v}},
   {u, -2, 2}, {v, 0, 4}, AspectRatio → Automatic,
```

```
PlotRange → {{-2.2, 2.2}, {-0.2, 4.2}, { -0.2, 4.2}}, ViewPoint → {4, 4, 5} ];
```





Change the `ViewPoint` command parameters to `{4, -4, 7}` to see the figure from another angle (this view is shown above, on the right). The volume of the solid is

$$V = \int_{x=-2}^{2} \int_{y=0}^{4-x^2} \int_{z=0}^{4-y} 1\,dz\,dy\,dx = \int_{x=-2}^{2} \int_{y=0}^{4-x^2} (4-y)\,dy\,dx = \int_{x=-2}^{2} \left[4y - y^2\right]_{y=0}^{4-x^2} dx$$

$$= \int_{-2}^{2} \left(8 - \frac{1}{2}x^4\right) dx = \left[8x - \frac{1}{10}x^5\right]_{-2}^{2} = \frac{128}{5} = 25.6.$$

C14S06.014: The volume is

$$V = \int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} 1\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{1}{3}y^3\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{3}(1 - 3x + 6x^2 - 4x^3)\,dx = \frac{1}{6}\left[2x - 3x^2 + 4x^3 - 2x^4\right]_0^1 = \frac{1}{6}.$$

C14S06.015: It's not easy to get a clear three-dimensional plot of this solid. We got fair results with the *Mathematica* 3.0 command

```
ParametricPlot3D[ {{u, u*u, v}, {u*u, u, v}, {u*Sqrt[10], v, 10 - u*u - v*v}},
   {u, 0, 1}, {v, 0, 10}, AspectRatio → 0.8,
   PlotRange → {{0, Sqrt[10]}, {0, Sqrt[10]}, {0, 10}}, ViewPoint → {6, -4, 12} ];
```

If you have the time and computer memory, experiment with changing the `AspectRatio` and `ViewPoint` parameters until you get a clear view of the solid. Its volume is

$$V = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{10-x^2-y^2} 1\ dz\ dy\ dx = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (10 - x^2 - y^2)\ dy\ dx$$

$$= \int_{x=0}^{1} \left[\frac{1}{3}(30y - 3x^2y - y^3) \right]_{y=x^2}^{y=\sqrt{x}} dx = \int_{0}^{1} \left(10x^{1/2} - \frac{1}{3}x^{3/2} - 10x^2 - x^{5/2} + x^4 + \frac{1}{3}x^6 \right) dx$$

$$= \left[\frac{20}{3}x^{3/2} - \frac{2}{15}x^{5/2} - \frac{10}{3}x^3 - \frac{2}{7}x^{7/2} + \frac{1}{5}x^5 + \frac{1}{21}x^7 \right]_0^1 = \frac{332}{105} \approx 3.1619047619047619.$$

C14S06.016: The volume is

$$V = \int_{-3}^{-1} \int_{-2}^{2} \int_{z^2}^{8-z^2} 1\ dx\ dz\ dy = \int_{-3}^{-1} \int_{-2}^{2} (8 - 2z^2)\ dz\ dy = \int_{-3}^{-1} \left[8z - \frac{2}{3}z^3 \right]_{-2}^{2} dy$$

$$= \int_{-3}^{-1} \frac{64}{3}\ dy = \frac{128}{3} \approx 42.6666666666666667.$$

C14S06.017: The volume is

$$V = \int_{x=-2}^{2} \int_{z=x^2}^{4} \int_{y=0}^{4-z} 1\ dy\ dz\ dx = \int_{x=-2}^{2} \int_{z=x^2}^{4} (4 - z)\ dz\ dx = \int_{x=-2}^{2} \left[4z - \frac{1}{2}z^2 \right]_{z=x^2}^{4} dx$$

$$= \int_{x=-2}^{2} \left(8 - 4x^2 + \frac{1}{2}x^4 \right) dx = \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5 \right]_{-2}^{2} = \frac{256}{15} \approx 17.0666666666666667.$$

C14S06.018: The volume is

$$V = \int_{y=-1}^{1} \int_{z=y^2-1}^{1-y^2} \int_{x=0}^{1-z} 1\ dx\ dz\ dy = \int_{y=-1}^{1} \int_{z=y^2-1}^{1-y^2} (1 - z)\ dz\ dy = \int_{y=-1}^{1} \left[z - \frac{1}{2}z^2 \right]_{y^2-1}^{1-y^2} dy$$

$$= \int_{y=-1}^{1} (2 - 2y^2)\ dy = \left[2y - \frac{2}{3}y^3 \right]_{-1}^{1} = \frac{8}{3} \approx 2.6666666666666667.$$

C14S06.019: The volume is

$$V = \int_{y=0}^{1} \int_{z=y^2}^{\sqrt{y}} \int_{x=0}^{2-y-z} 1\ dx\ dz\ dy = \int_{y=0}^{1} \int_{z=y^2}^{\sqrt{y}} (2 - y - z)\ dz\ dy = \int_{y=0}^{1} \left[2z - yz - \frac{1}{2}z^2 \right]_{z=y^2}^{\sqrt{y}} dy$$

$$= \int_{y=0}^{1} \frac{1}{2}(4y^{1/2} - y - 2y^{3/2} - 4y^2 + 2y^3 + y^4)\ dy$$

$$= \frac{1}{60} \left[80y^{3/2} - 15y^2 - 24y^{5/2} - 40y^3 + 15y^4 + 6y^5 \right]_0^1 = \frac{11}{30} \approx 0.3666666666666667.$$

C14S06.020: The volume is

$$V = \int_0^2 \int_0^{2-x} \int_0^{4-x^2-z^2} 1\ dy\ dz\ dx = \int_0^2 \int_0^{2-x} (4 - x^2 - z^2)\ dz\ dx = \int_0^2 \left[4z - x^2 z - \frac{1}{3}z^3\right]_0^{2-x} dx$$

$$= \int_0^2 \frac{1}{3}(16 - 12x^2 + 4x^3)\ dx = \frac{1}{3}\left[16x - 4x^3 + x^4\right]_0^2 = \frac{16}{3} \approx 5.3333333333333333.$$

C14S06.021: Because the solid has constant density $\delta = 1$, its mass is

$$m = \int_{x=-2}^2 \int_{y=x^2}^4 \int_{z=0}^y 1\ dz\ dy\ dx = \int_{x=-2}^2 \int_{y=x^2}^4 y\ dy\ dx = \int_{x=-2}^2 \left[\frac{1}{2}y^2\right]_{y=x^2}^4 dx$$

$$= \int_{x=-2}^2 \left(8 - \frac{1}{2}x^4\right) dx = \left[8x - \frac{1}{10}x^5\right]_{-2}^2 = \frac{128}{5}.$$

Its moments are

$$M_{yz} = \int_{x=-2}^2 \int_{y=x^2}^4 \int_{z=0}^y x\ dz\ dy\ dx = \int_{x=-2}^2 \int_{y=x^2}^4 xy\ dy\ dx = \int_{x=-2}^2 \left[\frac{1}{2}xy^2\right]_{y=x^2}^4 dx$$

$$= \int_{x=-2}^2 \left(8x - \frac{1}{2}x^5\right) dx = \left[4x^2 - \frac{1}{12}x^6\right]_{-2}^2 = 0,$$

$$M_{xz} = \int_{x=-2}^2 \int_{y=x^2}^4 \int_{z=0}^y y\ dz\ dy\ dz = \int_{x=-2}^2 \int_{y=x^2}^4 \frac{1}{2}y^2\ dy\ dx = \int_{x=-2}^2 \left[\frac{1}{3}y^3\right]_{y=x^2}^4 dx$$

$$= \int_{x=-2}^2 \left(\frac{64}{3} - \frac{1}{3}x^6\right) dx = \left[\frac{64}{3}x - \frac{1}{21}x^7\right]_{-2}^2 = \frac{512}{7},$$

$$M_{xy} = \int_{x=-2}^2 \int_{y=x^2}^4 \int_{z=0}^y z\ dz\ dy\ dx = \int_{x=-2}^2 \int_{y=x^2}^4 \left[\frac{1}{2}z^2\right]_{z=0}^y dy\ dx$$

$$= \frac{1}{2}\int_{x=-2}^2 \int_{y=x^2}^4 y^2\ dy\ dx = \frac{1}{2}\int_{x=-2}^2 \left[\frac{1}{3}y^3\right]_{y=x^2}^4 dx = \int_{x=-2}^2 \left(\frac{32}{3} - \frac{1}{6}x^6\right) dx$$

$$= \left[\frac{32}{3}x - \frac{1}{42}x^7\right]_{-2}^2 = \frac{256}{7}.$$

Therefore the centroid of the solid is located at the point $(\overline{x}, \overline{y}, \overline{z}) = \left(0, \frac{20}{7}, \frac{10}{7}\right)$.

C14S06.022: Because the hemispherical solid has unit density $\delta = 1$, its mass is $m = \frac{2}{3}\pi R^3$. By symmetry, $\overline{x} = \overline{y} = 0$. It remains only to compute the moment M_{xy}. Let D denote the circular disk $x^2 + y^2 \leqq R^2$, $z = 0$. Then

$$M_{xy} = \iint_D \left(\int_{z=0}^{\sqrt{R^2-x^2-y^2}} z\ dz\right) dA = \iint_D \left[\frac{1}{2}z^2\right]_{z=0}^{\sqrt{R^2-x^2-y^2}} dA = \iint_D \frac{1}{2}(R^2 - x^2 - y^2)\ dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^R \frac{1}{2}(R^2 - r^2) \cdot r\ dr\ d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{4}R^2 r^2 - \frac{1}{8}r^4\right]_{r=0}^R d\theta = 2\pi \cdot \frac{1}{8}R^4 = \frac{1}{4}\pi R^4.$$

Therefore the centroid is located at the point $(\overline{x},\,\overline{y},\,\overline{z}) = \left(0,\ 0,\ \dfrac{3}{8}R\right).$

C14S06.023: Because the solid has unit density $\delta = 1$, its mass and moments are

$$m = \int_{x=-2}^{2}\int_{z=x^2}^{4}\int_{y=0}^{4-z} 1\ dy\ dz\ dx = \int_{x=-2}^{2}\int_{z=x^2}^{4}(4-z)\ dz\ dx = \int_{x=-2}^{2}\left[4z - \frac{1}{2}z^2\right]_{z=x^2}^{4}\ dx$$

$$= \int_{-2}^{2}\left(8 - 4x^2 + \frac{1}{2}x^4\right)\ dx = \left[8x - \frac{4}{3}x^3 + \frac{1}{10}x^5\right]_{-2}^{2} = \frac{256}{15};$$

$$M_{yz} = \int_{x=-2}^{2}\int_{z=x^2}^{4}\int_{y=0}^{4-z} x\ dy\ dz\ dx = \int_{x=-2}^{2}\int_{z=x^2}^{4}(4x - xz)\ dz\ dx = \int_{x=-2}^{2}\left[4xz - \frac{1}{2}xz^2\right]_{z=x^2}^{4}\ dx$$

$$= \int_{-2}^{2}\left(8x - 4x^3 + \frac{1}{2}x^5\right)\ dx = \left[4x^2 - x^4 + \frac{1}{12}x^6\right]_{=2}^{2} = 0;$$

$$M_{xz} = \int_{x=-2}^{2}\int_{z=x^2}^{4}\int_{y=0}^{4-z} y\ dy\ dz\ dx = \int_{x=-2}^{2}\int_{z=x^2}^{4}\frac{1}{2}(4-z)^2\ dz\ dx = \int_{x=-2}^{2}\left[8z - 2z^2 + \frac{1}{6}z^4\right]_{z=x^2}^{4}\ dx$$

$$= \int_{-2}^{2}\left(\frac{32}{3} - 8x^2 + 2x^4 - \frac{1}{6}x^6\right)\ dx = \left[\frac{32}{3}x - \frac{8}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{42}x^7\right]_{-2}^{2} = \frac{2048}{105};$$

$$M_{xy} = \int_{x=-2}^{2}\int_{z=x^2}^{4}\int_{y=0}^{4-z} z\ dy\ dz\ dx = \int_{x=-2}^{2}\int_{z=x^2}^{4}(4z - z^2)\ dz\ dx = \int_{x=-2}^{2}\left[2z^2 - \frac{1}{3}z^3\right]_{z=x^2}^{4}\ dx$$

$$= \int_{-2}^{2}\left(\frac{32}{3} - 2x^4 + \frac{1}{3}x^6\right)\ dx = \left[\frac{32}{3}x - \frac{2}{5}x^5 + \frac{1}{21}x^7\right]_{-2}^{2} = \frac{1024}{35}.$$

Therefore the centroid of the solid is located at the point $(\overline{x},\,\overline{y},\,\overline{z}) = \left(0,\ \dfrac{8}{7},\ \dfrac{12}{7}\right).$

C14S06.024: Because the solid has unit density $\delta = 1$, its mass is

$$m = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x^2} 1\ dz\ dy\ dx = \int_{-1}^{1}\int_{-1}^{1}(1-x^2)\ dy\ dx = \int_{-1}^{1}(2-2x^2)\ dx = \left[2x - \frac{2}{3}x^3\right]_{-1}^{1} = \frac{8}{3}.$$

By symmetry $\overline{x} = \overline{y} = 0$, and

$$M_{xy} = \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x^2} z\ dz\ dy\ dx = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{2}(1-x^2)^2\ dy\ dx = \int_{-1}^{1}(1-x^2)^2\ dx$$

$$= \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16}{15}.$$

Therefore the centroid of the solid is located at the point $(\overline{x},\,\overline{y},\,\overline{z}) = \left(0,\ 0,\ \dfrac{2}{5}\right).$

C14S06.025: Because the solid has unit density $\delta = 1$, its mass and moments are given by

$$m = \int_{x=-\pi/2}^{\pi/2} \int_{z=0}^{\cos x} \int_{y=0}^{1-z} 1\ dy\ dz\ dx = \int_{x=-\pi/2}^{\pi/2} \left[z - \frac{1}{2}z^2 \right]_{z=0}^{\cos x} dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\cos x - \frac{1}{2}\cos^2 x \right) dx = \int_0^{\pi/2} (2\cos x - \cos^2 x)\ dx = \frac{1}{4}\Big[-2x + 8\sin x - \sin 2x \Big]_0^{\pi/2} = \frac{8-\pi}{4};$$

$M_{yz} = 0 \quad$ (by symmetry);

$$M_{xz} = \int_{x=-\pi/2}^{\pi/2} \int_{z=0}^{\cos x} \int_{y=0}^{1-z} y\ dy\ dz\ dx = \int_{x=-\pi/2}^{\pi/2} \int_{z=0}^{\cos x} \frac{1}{2}(1-z)^2\ dz\ dx$$

$$= \int_{x=-\pi/2}^{\pi/2} \left[\frac{1}{2}z - \frac{1}{2}z^2 + \frac{1}{6}z^3 \right]_{z=0}^{\cos x} dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\cos x - \frac{1}{2}\cos^2 x + \frac{1}{6}\cos^3 x \right) dx$$

$$= \frac{1}{72}\Big[-18x + 45\sin x - 9\sin 2x + \sin 3x \Big]_{-\pi/2}^{\pi/2} = \frac{44-9\pi}{36};$$

$$M_{xy} = \int_{x=-\pi/2}^{\pi/2} \int_{z=0}^{\cos x} \int_{y=0}^{1-z} z\ dy\ dz\ dx = \int_{x=-\pi/2}^{\pi/2} \int_{z=0}^{\cos x} (z - z^2)\ dz\ dx = \int_{x=-\pi/2}^{\pi/2} \left[\frac{1}{2}z^2 - \frac{1}{3}z^3 \right]_{z=0}^{\cos x} dx$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\cos^2 x - \frac{1}{3}\cos^3 x \right) dx = \frac{1}{72}\Big[18x - 18\sin x + 9\sin 2x - 2\sin 3x \Big]_{-\pi/2}^{\pi/2} = \frac{9\pi - 16}{36}.$$

Therefore the centroid of the solid is located at the point

$$(\overline{x}, \overline{y}, \overline{z}) = \left(0, \frac{44-9\pi}{72-9\pi}, \frac{9\pi-16}{72-9\pi} \right) \approx (0, 0.359643831963, 0.280712336074).$$

C14S06.026: (See Problem 12.) The moment of inertia of the solid (with density $\delta = 1$) with respect to the z-axis is

$$I_z = \int_{-2}^{2} \int_{x^2}^{4} \int_0^y (x^2 + y^2)\ dz\ dy\ dx = \int_{-2}^{2} \int_{x^2}^{4} (x^2 y + y^3)\ dy\ dx = \int_{-2}^{2} \left[\frac{1}{2}x^2 y^2 + \frac{1}{4}y^4 \right]_{x^2}^{4} dx$$

$$= \int_{-2}^{2} \left(648x^2 - \frac{1}{2}x^6 - \frac{1}{4}x^8 \right) dx = \left[64x + \frac{8}{3}x^3 - \frac{1}{14}x^7 - \frac{1}{36}x^9 \right]_{-2}^{2} = \frac{15872}{63} \approx 251.936507936508.$$

C14S06.027: (See Problem 24.) The moment of inertia of the solid (with density $\delta = 1$) with respect to the y-axis is

$$I_y = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=0}^{1-x^2} (x^2 + z^2)\ dz\ dy\ dx = \int_{x=-1}^{1} \int_{y=-1}^{1} \left[x^2 z + \frac{1}{3}z^3 \right]_{z=0}^{1-x^2} dy\ dx$$

$$= \int_{x=-1}^{1} \int_{-1}^{1} \frac{1}{3}(1 - x^6)\ dy\ dx = \int_{-1}^{1} \frac{2}{3}(1-x^6)\ dx = \left[\frac{2}{3}x - \frac{2}{21}x^7 \right]_{-1}^{1} = \frac{8}{7}.$$

C14S06.028: Given: The solid cylinder $x^2 + y^2 \leqq R^2,\ 0 \leqq z \leqq H;$ we assume constant density δ rather than constant density 1. Let D denote the base of the cylinder—the circular disk $x^2 + y^2 \leqq R^2,\ z = 0.$ The moment of inertia of the cylinder with respect to the z-axis is

$$I_z = \iint_D \left(\int_{z=0}^{H} \delta(x^2+y^2)\,dz \right) dA = \delta H \iint_D (x^2+y^2)\,dA = \delta H \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r^3\,dr\,d\theta$$

$$= 2\pi\delta H \left[\frac{1}{4} r^4 \right]_0^R = \frac{1}{2}\pi\delta R^4 H.$$

Because the mass of the cylinder is $M = \delta\pi R^2 H$, the answer can also be expressed in the form

$$I_z = \frac{1}{2} M R^2 = M \cdot \left(\frac{R}{\sqrt{2}} \right)^2,$$

demonstrating that the answer has the correct dimensions—the product of mass and square of distance. For a physical interpretation of the last equation, the cylinder behaves for purposes of angular acceleration around the z-axis as if all its mass were concentrated at distance $\frac{1}{2}R\sqrt{2}$ from the z-axis.

C14S06.029: With unit density $\delta = 1$, the moment of inertia of the given tetrahedron with respect to the z-axis is

$$I_z = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} (x^2+y^2)\,dx\,dy\,dz = \int_{z=0}^{1} \int_{y=0}^{1-z} \left[\frac{1}{3}x^3 + xy^2 \right]_{x=0}^{1-y-z} dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1-z} \left(\frac{1}{3} - y + 2y^2 - \frac{4}{3}y^3 - z + 2yz - 2y^2 z + z^2 - yz^2 - \frac{1}{3}z^3 \right) dy\,dz$$

$$= \int_{z=0}^{1} \left[-\frac{1}{3}y^4 + \frac{2}{3}y^3(1-z) - \frac{1}{2}y^2(1-z)^2 + \frac{1}{3}y(1-z)^3 \right]_{y=0}^{1-z} dz$$

$$= \int_0^1 \frac{1}{6}(1 - 4z + 6z^2 - 4z^3 + z^4)\,dz = \frac{1}{30}\Big[5z - 10z^2 + 10z^3 - 5z^4 + z^5 \Big]_0^1 = \frac{1}{30}.$$

C14S06.030: Problem 54 of Section 14.5 provides an alternative method of solving this problem, one that is frequently simpler. But we will solve it by direct methods. The moment of inertia of the given solid cube of unit density with respect to the z-axis is

$$I_z = \int_{z=-1/2}^{1/2} \int_{y=3}^{4} \int_{x=-1/2}^{1/2} (x^2+y^2)\,dx\,dy\,dz = \int_{z=-1/2}^{1/2} \int_{y=3}^{4} \left[\frac{1}{3}x^3 + xy^2 \right]_{-1/2}^{1/2} dy\,dz$$

$$= \int_{z=-1/2}^{1/2} \int_{y=3}^{4} \left(\frac{1}{12} + y^2 \right) dy\,dz = \int_{-1/2}^{1/2} \left[\frac{1}{12}y + \frac{1}{3}y^3 \right]_3^4 dz = \int_{-1/2}^{1/2} \frac{149}{12}\,dz = \frac{149}{12} \approx 12.41666667.$$

C14S06.031: It should be clear that $\overline{x} = \overline{y} = 0$ by symmetry, but there is a slight suggestion in the wording of the problem that we should *prove* this more rigorously. Hence we will compute the mass and all three moments. Assuming unit density $\delta = 1$, we have

$$m = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{z=x^2+y^2}^{h} 1\,dz\,dy\,dx = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (h - x^2 - y^2)\,dy\,dx$$

$$= \int_{x=-\sqrt{h}}^{\sqrt{h}} \left[hy - x^2 y - \frac{1}{3}y^3 \right]_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} dx = \int_{x=-\sqrt{h}}^{\sqrt{h}} \frac{4}{3}(h-x^2)^{3/2}\,dx.$$

The substitution $x = \sqrt{h}\,\sin\theta,\ dx = \sqrt{h}\,\cos\theta\,d\theta$ then yields

$$m = \int_{\theta=-\pi/2}^{\pi/2} \frac{4}{3}h^2\cos^4\theta\,d\theta = \frac{1}{24}h^2\Big[12\theta + 8\sin 2\theta + \sin 4\theta\Big]_{-\pi/2}^{\pi/2} = \frac{1}{2}\pi h^2.$$

Next,

$$\begin{aligned}
M_{yz} &= \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{z=x^2+y^2}^{h} x\,dz\,dy\,dx = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (hx - x^3 - xy^2)\,dy\,dx \\
&= \int_{x=-\sqrt{h}}^{\sqrt{h}} \left[hxy - x^3y - \frac{1}{3}xy^3\right]_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} dx = \int_{-\sqrt{h}}^{\sqrt{h}} \frac{4}{3}x(h-x^2)^{3/2}\,dx \\
&= \left[-\frac{4}{15}(h-x^2)^{5/2}\right]_{-\sqrt{h}}^{\sqrt{h}} = 0;
\end{aligned}$$

$$\begin{aligned}
M_{xz} &= \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{z=x^2+y^2}^{h} y\,dz\,dy\,dx = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (hy - x^2y - y^3)\,dy\,dx \\
&= \int_{x=-\sqrt{h}}^{\sqrt{h}} \left[\frac{1}{2}hy^2 - \frac{1}{2}x^2y^2 - \frac{1}{4}y^4\right]_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} dx = \int_{-\sqrt{h}}^{\sqrt{h}} 0\,dx = 0;
\end{aligned}$$

$$M_{xy} = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \int_{z=x^2+y^2}^{h} z\,dz\,dy\,dx = \int_{x=-\sqrt{h}}^{\sqrt{h}} \int_{y=-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \left(\frac{1}{2}h^2 - \frac{1}{2}(x^2+y^2)^2\right) dy\,dx.$$

Now rewrite the last integral in polar coordinates. Then

$$M_{xy} = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{h}} \frac{1}{2}(h^2 - r^4)\cdot r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{4}h^2r^2 - \frac{1}{12}r^6\right]_{r=0}^{\sqrt{h}} d\theta = 2\pi\cdot\frac{1}{6}h^3 = \frac{1}{3}\pi h^3.$$

Thus the centroid of the parabolic segment is located at the point

$$(\overline{x},\,\overline{y},\,\overline{z}) \;=\; \left(0,\,0,\,\frac{2}{3}h\right),$$

and therefore the centroid is on the axis of symmetry of the segment, two-thirds of the way from the vertex $(0,\,0,\,0)$ to the base.

C14S06.032: We assume that the cone has unit density $\delta = 1$, that its vertex is at the origin, that its axis of symmetry lies on the nonnegative z-axis, and that it has radius R and height H. Then the equation of its base is $z = H$ and the equation of its curved side is

$$z = \frac{H}{R}\sqrt{x^2+y^2}\,.$$

Let D denote the circular disk $x^2 + y^2 \leqq R^2$ in the xy-plane. Then the mass of the cone is

$$m = \iint_D \left(\int_{z=(H/R)\sqrt{x^2+y^2}}^{H} 1\,dz\right) dA = \iint_D \left(H - \frac{H}{R}\sqrt{x^2+y^2}\right) dA.$$

Rewrite the double integral as an iterated integral in polar coordinates. Thus

$$m = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \left(H - \frac{H}{R} r \right) \cdot r \; dr \; d\theta = 2\pi \left[\frac{H}{2} r^2 - \frac{H}{3R} r^3 \right]_0^R = \frac{1}{3} \pi R^2 H.$$

Next, we could argue that the centroid lies on the axis of the cone by symmetry, but there is an implication in the statement of the problem that this should be proved rigorously. Hence we compute all three moments. First,

$$M_{yz} = \iint_D \left(\int_{z=(H/R)\sqrt{x^2+y^2}}^{H} x \; dz \right) dA = \iint_D \left(Hx - \frac{H}{R} x \sqrt{x^2+y^2} \right) dA.$$

Rewrite the double integral as an iterated integral in polar coordinates (don't forget to replace x with $r \cos \theta$). Thus

$$M_{yz} = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \left(Hr \cos \theta - \frac{H}{R} r^2 \cos \theta \right) \cdot r \; dr \; d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{3} Hr^3 \cos \theta - \frac{H}{4R} r^4 \cos \theta \right]_{r=0}^{R} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} R^3 H \cos \theta \; d\theta = \frac{1}{12} \left[R^3 H \sin \theta \right]_0^{2\pi} = 0.$$

Also,

$$M_{xz} = \iint_D \left(\int_{z=(H/R)\sqrt{x^2+y^2}}^{H} y \; dz \right) dA = \iint_D \left(Hy - \frac{H}{R} y \sqrt{x^2+y^2} \right) dA.$$

As before, rewrite the double integral as an iterated integral in polar coordinates (and don't forget the substitution of $r \sin \theta$ for y). So

$$M_{xz} = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \left(Hr \sin \theta - \frac{H}{R} r^2 \sin \theta \right) r \; d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{3} Hr^3 \sin \theta - \frac{H}{4R} r^4 \sin \theta \right]_{r=0}^{R} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} R^3 H \sin \theta \; d\theta = \left[-\frac{1}{12} R^3 H \cos \theta \right]_0^{2\pi} = 0.$$

Finally,

$$M_{xy} = \iint_D \left(\int_{z=(H/R)\sqrt{x^2+y^2}}^{H} z \; dz \right) dA = \iint_D \left(\frac{1}{2} H^2 - \frac{H^2}{2R^2} (x^2 + y^2) \right) dA.$$

As before, rewrite the double integral in polar form. Thus

$$M_{xy} = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \left(\frac{1}{2} H^2 - \frac{H^2}{2R^2} r^2 \right) \cdot r \; dr \; d\theta = 2\pi \cdot \left[\frac{H^2}{4} r^2 - \frac{H^2}{8R^2} r^4 \right]_0^R = \frac{1}{4} \pi R^2 H^2.$$

Therefore the centroid of the uniform solid right circular cone is located at the point

$$(\overline{x}, \, \overline{y}, \, \overline{z}) \;\; = \;\; \left(0, \, 0, \, \frac{3}{4} H \right),$$

on the axis of the cone and three-fourths of the way from the vertex to the base.

C14S06.033: Place the cube in the first octant with three of its faces in the coordinate planes, one vertex at (0, 0, 0), and the opposite vertex at $(a,\ a,\ a)$. With density $\delta = 1$, its moment of inertia with respect to the z-axis is

$$I_z = \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2+y^2)\ dx\ dy\ dz = \int_0^a\int_0^a\left[\frac{1}{3}x^3+xy^2\right]_0^a dy\ dz = \int_0^a\int_0^a\left(\frac{1}{3}a^3+ay^2\right) dy\ dz$$

$$= \int_0^a\left[\frac{1}{3}a^3y+\frac{1}{3}ay^3\right]_0^a dz = \int_0^a \frac{2}{3}a^4\ dz = a\cdot\frac{2}{3}a^4 = \frac{2}{3}a^5.$$

Because the mass of the cube is $m = a^3$, we see that $I_z = \frac{2}{3}ma^2$, which is dimensionally correct.

C14S06.034: The density at $P(x,\ y,\ z)$ is $\delta(x,\ y,\ z) = k(x^2+y^2+z^2)$ where k is a positive constant. It will not change the answer to assume that $k = 1$. Then the mass and moments of the cube are

$$m = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\ dz\ dy\ dx = \int_0^a\int_0^a\left(\frac{1}{3}a^3+ax^2+ay^2\right) dy\ dx$$

$$= \int_0^a \frac{1}{3}\left[a^3y+3ax^2y+ay^3\right]_0^a dx = \frac{1}{3}\int_0^a (2a^4+3a^2x^2)\ dx = \frac{1}{3}\left[2a^4x+a^2x^3\right]_0^a = a^5;$$

$$M_{yz} = \int_0^a\int_0^a\int_0^a (x^3+xy^2+xz^2)\ dz\ dy\ dx = \int_0^a\int_0^a\left(\frac{1}{3}a^3x+ax^3+axy^2\right) dy\ dx$$

$$= \int_0^a \frac{1}{3}\left[a^3xy+3ax^3y+axy^3\right]_0^a dx = \frac{1}{3}\int_0^a (2a^4x+3a^2x^3)\ dx = \frac{1}{3}\left[a^4x^2+\frac{3}{4}a^2x^4\right]_0^a = \frac{7}{12}a^6;$$

$$M_{xz} = M_{xy} = \frac{7}{12}a^6 \quad \text{by symmetry.}$$

Therefore the centroid of this cube is located at the point $(\overline{x},\ \overline{y},\ \overline{z}) = \left(\frac{7}{12}a,\ \frac{7}{12}a,\ \frac{7}{12}a\right).$

C14S06.035: With density $\delta(x,\ y,\ z) = k(x^2+y^2+z^2)$ at the point $(x,\ y,\ z)$ (k is a positive constant), the moment of inertia of the cube of Problem 34 with respect to the z-axis is

$$I_z = \int_0^a\int_0^a\int_0^a k(x^2+y^2+z^2)(x^2+y^2)\ dz\ dy\ dx$$

$$= k\int_0^a\int_0^a\left[x^4z+2x^2y^2z+y^4z+\frac{1}{3}x^2z^3+\frac{1}{3}y^2z^3\right]_0^a dy\ dx$$

$$= k\int_0^a\int_0^a\left(\frac{1}{3}a^3x^2+ax^4+\frac{1}{3}a^3y^2+2ax^2y^2+ay^4\right) dy\ dx$$

$$= k\int_0^a\left[\frac{1}{3}a^3x^2y+ax^4y+\frac{1}{9}a^3y^3+\frac{2}{3}ax^2y^3+\frac{1}{5}ay^5\right]_0^a dx$$

$$= k\int_0^a\left(\frac{14}{45}a^6+a^4x^2+a^2x^4\right) dx = k\left[\frac{14}{45}a^6x+\frac{1}{3}a^4x^3+\frac{1}{5}a^2x^5\right]_0^a = \frac{38}{45}ka^7.$$

C14S06.036: The density of the cube at the point $(x,\ y,\ z)$ is $\delta(x,\ y,\ z) = kz$ where k is a positive constant. Its mass and moments are

$$m = \int_0^1 \int_0^1 \int_0^1 kz\, dz\, dy\, dx = \int_0^1 \int_0^1 \frac{1}{2}k\, dy\, dx = 1 \cdot 1 \cdot \frac{1}{2}k = \frac{1}{2}k;$$

$$M_{yz} = \int_0^1 \int_0^1 \int_0^1 kxz\, dz\, dx\, dy = \int_0^1 \int_0^1 \frac{1}{2}kx\, dx\, dy = \int_0^1 \frac{1}{4}k\, dy = 1 \cdot \frac{1}{4}k = \frac{1}{4}k;$$

$$M_{xz} = \int_0^1 \int_0^1 \int_0^1 kyz\, dz\, dy\, dx = \int_0^1 \int_0^1 \frac{1}{2}ky\, dy\, dx = \int_0^1 \frac{1}{4}k\, dx = 1 \cdot \frac{1}{4}k = \frac{1}{4}k;$$

$$M_{xy} = \int_0^1 \int_0^1 \int_0^1 kz^2\, dz\, dy\, dx = \int_0^1 \int_0^1 \frac{1}{3}k\, dy\, dx = 1 \cdot 1 \cdot \frac{1}{3}k = \frac{1}{3}k.$$

Therefore its centroid is located at the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right)$.

C14S06.037: The moment of inertia of the cube of Problem 36 with respect to the z-axis is

$$I_z = \int_0^1 \int_0^1 \int_0^1 k(x^2 + y^2)z\, dz\, dy\, dx = \int_0^1 \int_0^1 \frac{1}{2}k(x^2 + y^2)\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2}kx^2y + \frac{1}{6}ky^3\right]_0^1 dx = \int_0^1 \left(\frac{1}{2}kx^2 + \frac{1}{6}k\right) dx = k \cdot \left[\frac{1}{6}x^3 + \frac{1}{6}x\right]_0^1 = \frac{1}{3}k.$$

C14S06.038: Assume that the sphere is centered at the origin, so that it consists of the points (x, y, z) for which $x^2 + y^2 + z^2 \leqq a^2$. Let D denote the circular disk $x^2 + y^2 \leqq a^2$ in the xy-plane. With constant density δ, the moment of inertia of the sphere with respect to the z-axis is

$$I_z = \iint_D \left(\int_{z=-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \delta(x^2 + y^2)\, dz\right) dA = \iint_D 2\delta(x^2 + y^2)\sqrt{a^2 - x^2 - y^2}\, dA.$$

Now rewrite the last double integral as an iterated integral in polar coordinates. The result:

$$I_z = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 2\delta r^3 (a^2 - r^2)^{1/2}\, dr\, d\theta = \int_{\theta=0}^{2\pi} \left[\delta \cdot \frac{6r^4 - 2a^2r^2 - 4a^4}{15} \cdot (a^2 - r^2)^{1/2}\right]_{r=0}^{a} d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{4}{15}\delta a^5\, d\theta = 2\pi \cdot \frac{4}{15}\delta a^5 = \frac{8}{15}\pi\delta a^5.$$

Because the mass of the sphere is $m = \frac{4}{3}\pi\delta a^3$, we see that $I_z = \frac{2}{5}ma^2$, which is dimensionally correct and of plausible magnitude.

C14S06.039: With constant density $\delta = 1$, the mass and moments are

$$m = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \int_{x=0}^{\sqrt{1-z^2}} 1\, dx\, dy\, dz = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \sqrt{1 - z^2}\, dy\, dz$$

$$= \int_{z=0}^{1} (1 - z^2)\, dz = \left[z - \frac{1}{3}z^3\right]_0^1 = \frac{2}{3};$$

$$M_{yz} = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \int_{x=0}^{\sqrt{1-z^2}} x \, dx \, dy \, dz = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \frac{1}{2}(1-z^2) \, dy \, dz$$

$$= \int_{z=0}^{1} \frac{1}{2}(1-z^2)^{3/2} \, dz = \left[\frac{5z-2z^3}{16} \cdot (1-z^2)^{1/2} + \frac{3}{16} \arcsin z \right]_0^1 = \frac{3}{32}\pi;$$

$$M_{xz} = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \int_{x=0}^{\sqrt{1-z^2}} y \, dx \, dy \, dz = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} y(1-z^2)^{1/2} \, dy \, dz$$

$$= \int_{z=0}^{1} \frac{1}{2}(1-z^2)^{3/2} \, dz = \left[\frac{5z-2z^3}{16} \cdot (1-z^2)^{1/2} + \frac{3}{16} \arcsin z \right]_0^1 = \frac{3}{32}\pi;$$

$$M_{xy} = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \int_{x=0}^{\sqrt{1-z^2}} z \, dx \, dy \, dz = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} z(1-z^2)^{1/2} \, dy \, dz$$

$$= \int_{z=0}^{1} (z-z^3) \, dz = \left[\frac{1}{2}z^2 - \frac{1}{4}z^4 \right]_0^1 = \frac{1}{4}.$$

Therefore the centroid of the solid is located at the point $\left(\frac{9}{64}\pi, \ \frac{9}{64}\pi, \ \frac{3}{8} \right)$.

C14S06.040: Assuming constant density $\delta = 1$, the moment of inertia of the solid of Problem 39 with respect to the z-axis is

$$I_z = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \int_{x=0}^{\sqrt{1-z^2}} (x^2+y^2) \, dx \, dy \, dz = \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \left[\frac{1}{3}x^3 + xy^2 \right]_{x=0}^{\sqrt{1-z^2}} dy \, dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{\sqrt{1-z^2}} \left(\frac{1}{3}(1-z^2)^{3/2} + y^2(1-z^2)^{1/2} \right) dy \, dz$$

$$= \int_{z=0}^{1} \left[\frac{1}{3}y(1-z^2)^{3/2} + \frac{1}{3}y^3(1-z^2)^{1/2} \right]_{y=0}^{\sqrt{1-z^2}} dz = \int_{z=0}^{1} \left[\frac{1}{3}y(1+y^2-z^2)(1-z^2)^{1/2} \right]_{y=0}^{\sqrt{1-z^2}} dz$$

$$= \int_0^1 \frac{2}{3}(1-z^2)^2 \, dz = \frac{2}{45} \left[15z - 10z^3 + 3z^5 \right]_0^1 = \frac{16}{45}.$$

C14S06.041: The given solid projects onto the circular disk D with radius 2 and center $(0, 0)$ in the xy-plane. Hence the volume of the solid is

$$V = \iint_D \left(\int_{z=2x^2+y^2}^{12-x^2-2y^2} 1 \, dz \right) dA = \iint_D (12 - 3x^2 - 3y^2) \, dA.$$

Rewrite the last double integral as an iterated integral in polar coordinates. Thus

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (12-3r^2) \cdot r \, dr \, d\theta = 2\pi \cdot \left[6r^2 - \frac{3}{4}r^4 \right]_0^2 = 24\pi \approx 75.3982236861550377.$$

C14S06.042: Note first that the two surfaces intersect in a curve that projects vertically onto the ellipse

$$\left(\frac{x-1}{2}\right)^2 + z^2 \;=\; 1$$

in the xz-plane. Hence the volume of the solid is

$$V = \int_{z=-1}^{1} \int_{x=1-2\sqrt{1-z^2}}^{1+2\sqrt{1-z^2}} \int_{y=x^2+4z^2}^{2x+3} 1\;dy\;dx\;dz = \int_{z=-1}^{1} \int_{x=1-2\sqrt{1-z^2}}^{1+2\sqrt{-z^2}} (3+2x-x^2-4z^2)\;dx\;dz$$

$$= \int_{z=-1}^{1} \left[3x + x^2 - \frac{1}{3}x^3 - 4xz^2\right]_{x=1-2\sqrt{1-z^2}}^{1+2\sqrt{1-z^2}} dz$$

$$= \int_{-1}^{1} \left[3\left(1+2\sqrt{1-z^2}\,\right) - 4z^2\left(1+2\sqrt{1-z^2}\,\right) + \left(1+2\sqrt{1-z^2}\,\right)^2 - \frac{1}{3}\left(1+2\sqrt{1-z^2}\,\right)^3\right.$$

$$\left. - 3\left(1-2\sqrt{1-z^2}\,\right) + 4z^2\left(1-2\sqrt{1-z^2}\,\right) - \left(1-2\sqrt{1-z^2}\,\right)^2 + \frac{1}{3}\left(1-2\sqrt{1-z^2}\,\right)^3\right] dz$$

$$= \int_{-1}^{1} \frac{32}{3}(1-z^2)^{3/2}\;dz = \left[\frac{4}{3}(5z-2z^3)\sqrt{1-z^2}\; + 4\arcsin z\right]_{-1}^{1} = 4\pi \approx 12.566370614359.$$

C14S06.043: Following the *Suggestion*, the volume is

$$V = \int_{z=0}^{1} \int_{y=-z/2}^{z/2} \int_{x=-\sqrt{z^2-4y^2}}^{\sqrt{z^2-4y^2}} 1\;dx\;dy\;dz = \int_{z=0}^{1} \int_{y=-z/2}^{z/2} 2\sqrt{z^2-4y^2}\;\;dy\;dz.$$

Let $y = \frac{1}{2}z\sin u,\;\; dy = \frac{1}{2}z\cos u\;du$. This substitution yields

$$V = \int_{z=0}^{1} \int_{u=-\pi/2}^{\pi/2} z^2\cos^2 u\;du\;dz = \int_{z=0}^{1} \left[\frac{1}{4}z^2(2u+\sin 2u)\right]_{u=-\pi/2}^{\pi/2} dz$$

$$= \int_{0}^{1} \frac{1}{2}\pi z^2\;dz = \left[\frac{1}{6}\pi z^3\right]_0^1 = \frac{1}{6}\pi \approx 0.5235987755982989.$$

Methods of single-variable calculus also succeed here. A horizontal cross section of the solid at $z = h$ $(0 < h \leqq 1)$ is an ellipse with equation $x^2 + 4y^2 = h^2$. This ellipse has major semiaxis of length h and minor semiaxis of length $\frac{1}{2}h$. Therefore its area is $\frac{1}{2}\pi h^2$. So, by the method of parallel cross sections (see Eq. (3) of Section 6.2), the volume of the solid is

$$\int_0^1 \frac{1}{2}\pi h^2\;dh = \left[\frac{1}{6}\pi h^3\right]_0^1 = \frac{1}{6}\pi.$$

C14S06.044: First interchange the roles of z and x. Thus we are to find the volume of the region bounded by the paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $z = 2 - y^2$. The two surfaces meet in a curve that projects vertically onto the circle $x^2 + y^2 = 1,\;\; z = 0$. Let D be the disk bounded by that circle. Then the volume of the solid bounded by the two surfaces is

$$V = \iint_D \left(\int_{z=2x^2+y^2}^{2-y^2} 1\;dz\right) dA = \iint_D (2-2x^2-2y^2)\;dA.$$

Rewrite the last double integral as an iterated integral in polar coordinates. Thus

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(2-2r^2)\cdot r\,dr\,d\theta = 2\pi\left[r^2 - \frac{1}{2}r^4\right]_0^1 = 2\pi\cdot\frac{1}{2} = \pi \approx 3.14159265358979323846.$$

C14S06.045: If the pyramid (tetrahedron) of Example 2 has density z at the point $(x,\,y,\,z)$, then its mass and moments are

$$m = \int_{x=0}^{2}\int_{y=0}^{(6-3x)/2}\int_{z=0}^{6-3x-2y} z\,dz\,dy\,dx = \int_0^2\int_0^{(6-3x)/2}\frac{1}{2}(6-3x-2y)^2\,dy\,dx$$

$$= \int_0^2\left[18y - 18xy + \frac{9}{2}x^2y - 6y^2 + 3xy^2 + \frac{2}{3}y^3\right]_0^{(6-3x)/2} dx$$

$$= \int_0^2\left[9(6-3x) - \frac{3}{2}(6-3x)^2 + \frac{1}{12}(6-3x)^3 - 9x(6-3x) + \frac{3}{4}x(6-3x)^2 + \frac{9}{4}x^2(6-3x)\right] dx$$

$$= \int_0^2\left(18 - 27x + \frac{27}{2}x^2 - \frac{9}{4}x^3\right) dx = \left[18x - \frac{27}{2}x^2 + \frac{9}{2}x^3 - \frac{9}{16}x^4\right]_0^2 = 9;$$

$$M_{yz} = \int_{x=0}^{2}\int_{y=0}^{(6-3x)/2}\int_{z=0}^{6-3x-2y} xz\,dz\,dy\,dx = \int_0^2\int_0^{(6-3x)/2}\frac{1}{2}x(6-3x-2y)^2\,dy\,dx$$

$$= \int_0^2\int_0^{(6-3x)/2}\left(18x - 18x^2 + \frac{9}{2}x^3 - 12xy + 6x^2y + 2xy^2\right) dy\,dx$$

$$= \int_0^2\left[\frac{9}{2}xy(x-2)^2 + 3xy^2(x-2) + \frac{2}{3}xy^3\right]_0^{(6-3x)/2} dx$$

$$= \int_0^2\left[9x(6-2x) - \frac{3}{2}x(6-3x)^2 + \frac{1}{12}x(6-3x)^3 - 9x^2(6-3x) + \frac{3}{4}x^2(6-3x)^2 + \frac{9}{4}x^3(6-3x)\right] dx$$

$$= \int_0^2\left(18x - 27x^2 + \frac{27}{2}x^3 - \frac{9}{4}x^4\right) dx = \left[9x^2 - 9x^3 + \frac{27}{8}x^4 - \frac{9}{20}x^5\right]_0^2 = \frac{18}{5};$$

$$M_{xz} = \int_{x=0}^{2}\int_{y=0}^{(6-3x)/2}\int_{z=0}^{6-3x-2y} yz\,dz\,dy\,dx = \int_0^2\int_0^{(6-3x)/2}\frac{1}{2}y(6-3x-2y)^2\,dy\,dx$$

$$= \int_0^2\int_0^{(6-3x)/2}\left(18y - 18xy + \frac{9}{2}x^2y - 12y^2 + 6xy^2 + 2y^3\right) dy\,dx$$

$$= \int_0^2\left[9y^2 - 9xy^2 + \frac{9}{4}x^2y^2 - 4y^3 + 2xy^3 + \frac{1}{2}y^4\right]_0^{(6-3x)/2} dx$$

$$= \int_0^2\left[\frac{9}{4}(6-3x)^2 - \frac{1}{2}(6-3x)^3 + \frac{1}{32}(6-3x)^4 - \frac{9}{4}x(6-3x)^2 + \frac{1}{4}x(6-3x)^3 + \frac{9}{16}x^2(6-3x)^2\right] dx$$

$$= \int_0^2\left(\frac{27}{2} - 27x + \frac{81}{4}x^2 - \frac{27}{4}x^3 + \frac{27}{32}x^4\right) dx = \left[\frac{27}{2}x - \frac{27}{2}x^2 + \frac{27}{4}x^3 - \frac{27}{16}x^4 + \frac{27}{160}x^5\right]_0^2 = \frac{27}{5};$$

$$M_{xy} = \int_{x=0}^{2}\int_{y=0}^{(6-3x)/2}\int_{z=0}^{6-3x-2y} z^2\,dz\,dy\,dx = \int_0^2\int_0^{(6-3x)/2}\frac{1}{3}(6-3x-2y)^3\,dy\,dx$$

$$= \int_0^2 \int_0^{(6-3x)/2} \left(72 - 108x + 54x^2 - 9x^3 - 72y + 72xy - 18x^2y + 24y^2 - 12xy^2 - \frac{8}{3}y^3\right) dy\ dx$$

$$= \int_0^2 \left[72y - 108xy + 54x^2y - 9x^3y - 36y^2 + 36xy^2 - 9x^2y^2 + 8y^3 - 4xy^3 - \frac{2}{3}y^4\right]_{y=0}^{(6-3x)/2} dx$$

$$= \int_0^2 \left(54 - 108x + 81x^2 - 27x^3 + \frac{27}{8}x^4\right) dx = \left[54x - 54x^2 + 27x^3 - \frac{27}{4}x^4 + \frac{27}{40}x^5\right]_0^2 = \frac{108}{5}.$$

Therefore the centroid of the pyramid is located at the point $\left(\frac{2}{5}, \frac{3}{5}, \frac{12}{5}\right)$.

C14S06.046: The mass and moments are

$$m = 2\int_{-1}^2 \int_{y^2}^{y+2} \int_0^{\sqrt{z-y^2}} 1\ dx\ dz\ dy = \int_{-1}^2 \int_{y^2}^{y+2} 2\sqrt{z-y^2}\ dz\ dy = \int_{-1}^2 \left[\frac{4}{3}(z-y^2)^{3/2}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^2 \frac{4}{3}(2+y-y^2)^{3/2}\ dy$$

$$= \left[\frac{1}{48}(-43 + 78y + 24y^2 - 16y^3)(2+y-y^2)^{1/2} + \frac{81}{32}\arcsin\left(\frac{2y-1}{3}\right)\right]_{-1}^2 = \frac{81}{32}\pi;$$

$$M_{yz} = 2\int_{-1}^2 \int_{y^2}^{y+2} \int_0^{\sqrt{z-y^2}} x\ dx\ dz\ dy = 2\int_{-1}^2 \int_{y^2}^{y+2} \frac{1}{2}(z-y^2)\ dz\ dy = 2\int_{-1}^2 \left[\frac{1}{4}z^2 - \frac{1}{2}y^2z\right]_{y^2}^{y+2} dy$$

$$= 2int_{-1}^2 \left(1 + y - \frac{3}{4}y^2 - \frac{1}{2}y^3 + \frac{1}{4}y^4\right) dy = 2\left[y + \frac{1}{2}y^2 - \frac{1}{4}y^3 - \frac{1}{8}y^4 + \frac{1}{20}y^5\right]_{-1}^2 = \frac{81}{20};$$

$$M_{xz} = 2\int_{-1}^2 \int_{y^2}^{y+2} \int_0^{\sqrt{z-y^2}} y\ dx\ dz\ dy = \int_{-1}^2 \int_{y^2}^{y+2} 2y(z-y^2)^{1/2}\ dz\ dy = \int_{-1}^2 \left[\frac{4}{3}y(z-y^2)^{3/2}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^2 \frac{4}{3}y(2+y-y^2)^{3/2}\ dy$$

$$= \left[\frac{4}{3}\left(-\frac{727}{640} - \frac{61}{320}y + \frac{63}{80}y^2 + \frac{11}{40}y^3 - \frac{1}{5}y^5\right)\cdot(2+y-y^2)^{1/2} + \frac{81}{128}\arcsin\left(\frac{2y-1}{3}\right)\right]_{-1}^2 = \frac{81}{64}\pi;$$

$$M_{xy} = 2\int_{-1}^2 \int_{y+2}^{y^2} \int_0^{\sqrt{z-y^2}} z\ dx\ dz\ dy = \int_{-1}^2 \int_{y+2}^{y^2} 2z(z-y^2)^{1/2}\ dz\ dy$$

$$= \int_{-1}^2 \left[\left(\frac{4}{5}z^2 - \frac{4}{15}y^2z - \frac{8}{15}y^4\right)\cdot(z-y^2)^{1/2}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^2 \frac{4}{15}(12 + 12y + y^2 - y^3 - 2y^4)(2+y-y^2)^{1/2}\ dy$$

$$= \left[-\frac{4}{15}\left(\frac{7075}{768} - \frac{2303}{384}y - \frac{427}{96}y^2 - \frac{23}{48}y^3 + \frac{1}{6}y^4 + \frac{1}{3}y^5\right)\cdot(2+y-y^2)^{1/2} + \frac{567}{128}\arcsin\left(\frac{2y-1}{3}\right)\right]_{-1}^2$$

$$= \frac{567}{128}\pi.$$

Therefore the centroid of the parabolic segment is located at the point $\left(\frac{8}{5\pi},\ \frac{1}{2},\ \frac{7}{4}\right)$.

C14S06.047: First we compute

$$\int_{x=0}^{2}\int_{y=0}^{(6-3x)/2}\int_{z=0}^{6-3x-2y} z\ dz\ dy\ dx = \int_0^2\int_0^{(6-3x)/2} \frac{1}{2}(6-3x-2y)^2\ dy\ dx$$

$$= \int_0^2\int_0^{(6-3x)/2}\left(18-18x+\frac{9}{2}x^2-12y+6xy+2y^2\right)\ dy\ dx$$

$$= \int_0^2\left[18y-18xy+\frac{9}{2}x^2y-6y^2+3xy^2+\frac{2}{3}y^3\right]_0^{(6-3x)/2}\ dx$$

$$= \int_0^2\left(18-27x+\frac{27}{2}x^2-\frac{9}{4}x^3\right)\ dx = \left[18x-\frac{27}{2}x^2+\frac{9}{2}x^3-\frac{9}{16}x^4\right]_0^2 = 9.$$

The volume of the pyramid is 6, and hence the average value of the density function $\delta(x,\,y,\,z) = z$ on the pyramid is $\overline{\delta} = \frac{9}{6} = \frac{3}{2}$.

C14S06.048: The volume of the cube is 1, and hence the average value of $f(x,\,y,\,z) = x^2+y^2+z^2$ on the cube is

$$\overline{f} = \int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\ dz\ dy\ dx$$

$$= \int_0^1\int_0^1\left(\frac{1}{3}+x^2+y^2\right)\ dy\ dx = \int_0^1\left(\frac{2}{3}+x^2\right)\ dx = \left[\frac{2}{3}x+\frac{1}{3}x^3\right]_0^1 = 1.$$

C14S06.049: The centroid of the cube of Problem 48 is, by symmetry, its midpoint $\left(\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\right)$. Because the cube has volume 1, the average value of

$$g(x,\,y,\,z) \ = \ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2$$

on the cube is

$$\overline{g} = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} g(x,\,y,\,z)\ dz\ dy\ dx = \int_0^1\int_0^1\left[\frac{3}{4}z-xz+x^2z-yz+y^2z-\frac{1}{2}z^2+\frac{1}{3}z^3\right]_0^1\ dy\ dx$$

$$= \int_0^1\int_0^1\left(\frac{7}{12}-x+x^2-y+y^2\right)\ dy\ dx = \int_0^1\left[\frac{7}{12}y-xy+x^2y-\frac{1}{2}y^2+\frac{1}{3}y^3\right]_0^1\ dx$$

$$= \int_0^1\left(\frac{5}{12}-x+x^2\right)\ dx = \left[\frac{5}{12}x-\frac{1}{2}x^2+\frac{1}{3}x^3\right]_0^1 = \frac{1}{4}.$$

C14S06.050: If the cube of Problem 48 has density $\delta(x,\,y,\,z) = x+y+z$ at the point $(x,\,y,\,z)$, then—because the volume of the cube is 1—the average value of the density function on the cube is

$$\overline{\delta} = \int_0^1 \int_0^1 \int_0^1 (x+y+z)\;dz\;dy\;dx = \int_0^1 \int_0^1 \left(\frac{1}{2}+x+y\right)\;dy\;dx$$
$$= \int_0^1 \left[\frac{1}{2}y + xy + \frac{1}{2}y^2\right]_0^1\;dx = \int_0^1 (1+x)\;dx = \frac{3}{2}.$$

C14S06.051: First we compute

$$J = \int_{x=0}^2 \int_{y=0}^{(6-3x)/2} \int_{z=0}^{6-3x-2y} (x^2+y^2+z^2)\;dz\;dy\;dx = \int_0^2 \int_0^{(6-3x)/2} \left[x^2z + y^2z + \frac{1}{3}z^3\right]_0^{6-3x-2y}\;dy\;dx$$
$$= \int_0^2 \int_0^{(6-3x)/2} \left(72 - 108x + 60x^2 - 12x^3 - 72y + 72xy - 20x^2y + 30y^2 - 15xy^2 - \frac{14}{3}y^3\right)\;dy\;dx$$
$$= \int_0^2 \left[72y - 108xy + 60x^2y - 12x^3y - 36y^2 + 36xy^2 - 10x^2y^2 + 10y^3 - 5xy^3 - \frac{7}{6}y^4\right]_0^{(6-3x)/2}\;dx$$
$$= \int_0^2 \left(\frac{135}{2} - 135x + \frac{441}{4}x^2 - \frac{171}{4}x^3 + \frac{207}{32}x^4\right)\;dx = \left[\frac{135}{2}x - \frac{135}{2}x^2 + \frac{147}{4}x^3 - \frac{171}{16}x^4 + \frac{207}{160}x^5\right]_0^2$$
$$= \frac{147}{5}.$$

Because the pyramid has volume $V = 6$, the average squared distance of its points from its centroid is

$$\overline{d} = \frac{J}{V} = \frac{147}{30} = \frac{49}{10} = 4.9.$$

C14S06.052: Suppose that the tetrahedron T has vertices at $(0,\,0,\,0)$, $(a,\,0,\,0)$, $(0,\,b,\,0)$, and $(0,\,0,\,c)$ where a, b, and c are positive constants. Suppose also that its density is $\delta = 1$. We plan first to find the centroid of T. Its mass is simply $m = \frac{1}{6}abc$. The equation of its bounding diagonal plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \quad \text{so that} \quad z = f(x,\,y) = c\left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

and therefore

$$M_{yz} = \int_{x=0}^a \int_{y=0}^{b-bx/a} \int_{z=0}^{f(x,y)} x\;dz\;dy\;dx = \int_0^a \int_0^{b-bx/a} \left(cx - \frac{c}{a}x^2 - \frac{c}{b}xy\right)\;dy\;dx$$
$$= \int_0^a \left[\frac{2abcxy - 2bcx^2y - acxy^2}{2ab}\right]_0^{b-bx/a}\;dx = \int_0^a \frac{a^2bcx - 2abcx^2 + bcx^3}{2a^2}\;dx$$
$$= \left[\frac{6a^2bcx^2 - 8abcx^3 + 3bcx^4}{24a^2}\right]_0^a = \frac{1}{24}a^2bc.$$

Therefore $\overline{x} = \frac{1}{4}a$. By symmetry, $\overline{y} = \frac{1}{4}b$ and $\overline{z} = \frac{1}{4}c$. It follows that the centroid of the pyramid of Example 2, with unit density, is located at the point $\left(\frac{1}{2},\,\frac{3}{4},\,\frac{3}{2}\right)$. Let $h(x,\,y,\,z)$ be the squared distance of the point $(x,\,y,\,z)$ of the pyramid from its centroid:

$$h(x,\,y,\,z) = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{4}\right)^2 + \left(z - \frac{3}{2}\right)^2.$$

Then we compute

$$
\begin{aligned}
J &= \int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} h(x,\,y,\,z)\;dz\;dy\;dx \\
&= \int_0^2 \int_0^{(6-3x)/2} \left[\frac{49}{16}z - xz + x^2z - \frac{3}{2}yz + y^2z - \frac{3}{2}z^2 + \frac{1}{3}z^3\right]_0^{6-3x-2} dy\;dx \\
&= \int_0^2 \int_0^{(6-3x)/2} \left(\frac{291}{8} - \frac{1107}{16}x + \frac{99}{2}x^2 - 12x^3 - \frac{409}{8}y\right. \\
&\qquad\qquad \left. + \frac{121}{2}xy - 20x^2y + 27y^2 - 15xy^2 - \frac{14}{3}y^3\right)\;dy\;dx \\
&= \int_0^2 \left[\frac{291}{8}y - \frac{1107}{16}xy + \frac{99}{2}x^2y - 12x^3y - \frac{409}{16}y^2\right. \\
&\qquad\qquad \left. + \frac{121}{4}xy^2 - 10x^2y^2 + 9y^3 - 5xy^3 - \frac{7}{6}y^4\right]_0^{(6-3x)/2} dx \\
&= \int_0^2 \left(\frac{441}{16} - \frac{1125}{16}x + \frac{4833}{64}x^2 - \frac{585}{16}x^3 + \frac{207}{32}x^4\right)\;dx \\
&= \left[\frac{441}{16}x - \frac{1125}{32}x^2 + \frac{1611}{64}x^3 - \frac{585}{64}x^4 + \frac{207}{160}x^5\right]_0^2 = \frac{441}{40}.
\end{aligned}
$$

Because the pyramid has volume $V = 6$, the average value of $h(x,\,y,\,z)$ on the pyramid is

$$\overline{h} = \frac{J}{V} = \frac{147}{80} = 1.8375.$$

C14S06.053: Using *Mathematica* 3.0, we find that the average distance of points of the cube of Problem 48 from the origin is

$$
\begin{aligned}
\overline{d} &= \int_{x=0}^1 \int_{y=1}^1 \int_{z=0}^1 \sqrt{x^2+y^2+z^2}\;\;dz\;dy\;dx \\
&= \int_0^1 \int_0^1 \left[\frac{1}{2}z\sqrt{x^2+y^2+z^2}\; + \frac{1}{2}(x^2+y^2)\ln\left(z + \sqrt{x^2+y^2+z^2}\,\right)\right]_0^1 dy\;dx \\
&= \int_0^1 \int_0^1 \left[\frac{1}{2}\sqrt{x^2+y^2+1}\; - \frac{1}{2}(x^2+y^2)\ln\left(\sqrt{x^2+y^2}\,\right) + \frac{1}{2}(x^2+y^2)\ln\left(1 + \sqrt{x^2+y^2+1}\,\right)\right] dy\;dx \\
&= \int_0^1 \left[\frac{1}{3}y\sqrt{x^2+y^2+1}\; - \frac{1}{3}x^2\arctan\left(\frac{y}{x\sqrt{x^2+y^2+1}}\right) - \frac{1}{6}y(3x^2+y^2)\ln\left(\sqrt{x^2+y^2}\,\right)\right. \\
&\qquad \left. + \frac{1}{6}y(3x^2+y^2)\ln\left(1 + \sqrt{x^2+y^2+1}\,\right) + \frac{1}{6}(3x^2+1)\ln\left(y + \sqrt{x^2+y^2+1}\,\right)\right]_0^1 dx \\
&= \int_0^1 \left[\frac{1}{3}\sqrt{x^2+2}\; - \frac{1}{3}x^2\arctan\left(\frac{1}{x\sqrt{x^2+2}}\right) - \frac{1}{3}(3x^2+1)\ln\left(\sqrt{x^2+1}\,\right)\right. \\
&\qquad\qquad \left. + \frac{1}{3}(3x^2+1)\ln\left(1 + \sqrt{x^2+2}\,\right)\right] dx
\end{aligned}
$$

$$= \left[\frac{1}{4} x\sqrt{x^2+2} + \frac{1}{3} \operatorname{arcsinh}\left(\frac{x}{\sqrt{2}} \right) - \frac{1}{12} x^4 \arctan\left(\frac{1}{x\sqrt{x^2+2}} \right) \right.$$

$$\left. - \frac{1}{6} \arctan\left(\frac{x}{\sqrt{x^2+2}} \right) - \frac{1}{3} x(x^2+1) \ln\left(\sqrt{x^2+1} \right) + \frac{1}{3} x(x^2+1) \ln\left(1 + \sqrt{x^2+2} \right) \right]_0^1$$

$$= \frac{1}{24} \left[6\sqrt{3} - \pi + 8 \operatorname{arcsinh}\left(\frac{\sqrt{2}}{2} \right) - 8\ln 2 + 16 \ln\left(1 + \sqrt{3} \right) \right] \approx 0.96059195645505295942510 8.$$

Section 14.7

C14S07.001: The volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=r^2}^{4} r\ dz\ dr\ d\theta = 2\pi \int_{r=0}^{2} (4r - r^3)\ dr = 2\pi \left[2r^2 - \frac{1}{4}r^4 \right]_0^2 = 8\pi.$$

C14S07.002: We assume unit density. Then the mass of the solid is $m = 8\pi$ by the result in the solution of Problem 1. It is clear by symmetry that $\overline{x} = \overline{y} = 0$, but we will also demonstrate this rigorously:

$$\begin{aligned} M_{yz} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cos\theta\ dz\ dr\ d\theta = \int_0^{2\pi} \int_0^2 (4r^2 \cos\theta - r^4 \cos\theta)\ dr\ d\theta \\ &= \int_0^{2\pi} \left[\frac{4}{3} r^3 \cos\theta - \frac{1}{5} r^5 \cos\theta \right]_0^2 d\theta = \int_0^{2\pi} \frac{64}{15} \cos\theta\ d\theta = \left[\frac{64}{15} \sin\theta \right]_0^{2\pi} = 0. \end{aligned}$$

Replacement of $\cos\theta$ with $\sin\theta$ in the first integral will clearly lead to the result $M_{xz} = 0$. There remains only this:

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 rz\ dz\ dr\ d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{1}{2} rz^2 \right]_{r^2}^4 dr\ d\theta = \int_0^{2\pi} \int_0^2 \left(8r - \frac{1}{2} r^5 \right) dr\ d\theta \\ &= \int_0^{2\pi} \left[4r^2 - \frac{1}{12} r^6 \right]_0^2 d\theta = \int_0^{2\pi} \frac{32}{3}\ d\theta = \frac{64}{3}\pi. \end{aligned}$$

Therefore the centroid of the solid is located at $\left(0, 0, \frac{8}{3}\right)$. Compare this answer with that obtained using the result in Problem 31 of Section 14.6.

C14S07.003: Place the center of the sphere at the origin. Then its volume is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\ dz\ dr\ d\theta = 2\pi \int_{r=0}^{a} 2r\sqrt{a^2 - r^2}\ dr = 2\pi \cdot \left[-\frac{2}{3}(a^2 - r^2)^{3/2} \right]_0^a = \frac{4}{3}\pi a^3.$$

C14S07.004: The moment of inertia of a solid sphere of density δ, radius a, and center $(0, 0, 0)$ with respect to the z-axis is

$$\begin{aligned} I_z &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \delta r^3\ dz\ dr\ d\theta = 2\pi\delta \int_0^a 2r^3(a^2 - r^2)^{1/2}\ dr \\ &= 2\pi\delta \cdot \left[-\frac{2}{15}(2a^4 + a^2r^2 - 3r^4)(a^2 - r^2)^{1/2} \right]_0^a = \frac{8}{15}\pi\delta a^5 = \frac{2}{5} ma^2 \end{aligned}$$

where $m = \frac{4}{3}\pi\delta a^3$ is the mass of the sphere.

C14S07.005: The volume is

$$\begin{aligned} V &= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta = 2\pi \int_{r=0}^{1} 2r(4 - r^2)^{1/2}\ dr = 2\pi \cdot \left[-\frac{2}{3}(4 - r^2)^{3/2} \right]_0^1 \\ &= 2\pi \left(\frac{16}{3} - 2\sqrt{3} \right) = \frac{4}{3}\pi \left(8 - 3\sqrt{3} \right) \approx 11.7447292674805137. \end{aligned}$$

C14S07.006: Assuming unit density, the result in the solution of Problem 5 shows that the mass of the solid is

$$m = \frac{2}{3}\pi\left(8 - 3\sqrt{3}\,\right).$$

By symmetry, the centroid lies on the z-axis, so we need only compute the moment with respect to the xy-plane:

$$M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} rz\;dz\;dr\;d\theta = 2\pi\int_0^1 \frac{1}{2}(4r - r^3)\;dr = 2\pi\cdot\left[r^2 - \frac{1}{8}r^4\right]_0^1 = 2\pi\cdot\frac{7}{8} = \frac{7}{4}\pi.$$

Therefore the centroid is located at the point $(\overline{x},\,\overline{y},\,\overline{z})$ where $\overline{x} = \overline{y} = 0$ and

$$\overline{z} = \frac{7\pi}{4}\cdot\frac{3}{2\pi\left(8 - 3\sqrt{3}\,\right)} = \frac{21}{296}\left(8 + 3\sqrt{3}\,\right) \approx 0.9362135164758083.$$

C14S07.007: The mass of the cylinder is

$$m = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} rz\;dz\;dr\;d\theta = 2\pi\int_{r=0}^{a}\frac{1}{2}rh^2\;dr = 2\pi\cdot\left[\frac{1}{4}r^2h^2\right]_0^a = \frac{1}{2}\pi a^2h^2.$$

C14S07.008: We saw in the solution of Problem 7 that the mass of the cylinder is $m = \frac{1}{2}\pi a^2h^2$. By symmetry, its centroid lies on the z-axis. So we need only compute its moment with respect to the xy-plane:

$$M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{h} rz^2\;dz\;dr\;d\theta = 2\pi\int_0^{a}\frac{1}{3}rh^3\;dr = 2\pi\cdot\left[\frac{1}{6}r^2h^3\right]_0^a = \frac{1}{3}\pi a^2h^3.$$

Therefore its centroid is located at the point $\left(0,\,0,\,\dfrac{2}{3}h\right)$.

C14S07.009: The moment of inertia of the cylinder of Problem 7 with respect to the z-axis is

$$\begin{aligned}
I_z &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} r^3z\;dz\;dr\;d\theta = 2\pi\int_{r=0}^{a}\left[\frac{1}{2}r^3z^2\right]_{z=0}^{h}\;dr\\
&= 2\pi\int_{r=0}^{a}\frac{1}{2}r^3h^2\;dr = 2\pi\cdot\left[\frac{1}{8}r^4h^2\right]_{r=0}^{a} = 2\pi\cdot\frac{1}{8}a^4h^2 = \frac{1}{4}\pi a^4h^2.
\end{aligned}$$

C14S07.010: The cylinder $x^2+y^2-2x=0$ meets the xy-plane in the circle with polar equation $r = 2\cos\theta$ and the entire circle is swept out as θ runs through the values from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. (We will integrate from 0 to $\pi/2$ and double the result.) Hence the volume of the region within both the cylinder and the sphere $r^2 + z^2 = 4$ is

$$\begin{aligned}
V &= 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\;dz\;dr\;d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} 2r(4-r^2)^{1/2}\;dr\;d\theta\\
&= 2\int_0^{\pi/2}\left[-\frac{2}{3}(4-r^2)^{3/2}\right]_0^{2\cos\theta}\;d\theta = 2\int_0^{\pi/2}\frac{16}{3}(1-\sin^3\theta)\;d\theta\\
&= \frac{32}{3}\int_0^{\pi/2}\left[1 - (1-\cos^2\theta)\sin\theta\right]\;d\theta = \frac{32}{3}\left[\theta + \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/2}
\end{aligned}$$

$$= \frac{32}{3}\left(\frac{\pi}{2} - 1 + \frac{1}{3}\right) = \frac{16}{9}(3\pi - 4) \approx 9.6440497080344528.$$

This problem has a pitfall. Here are the details of the simplification in the second line of this solution:

$$\left[-\frac{2}{3}(4 - r^2)^{3/2}\right]_0^{2\cos\theta} = \frac{16}{3} - \frac{2}{3}(4 - 4\cos^2\theta)\sqrt{4 - 4\cos^2\theta} = \frac{16}{3} - \frac{2}{3}(4\sin^2\theta)\sqrt{4\sin^2\theta}$$

$$= \frac{16}{3}\left[1 - (\sin^2\theta)\sqrt{\sin^2\theta}\,\right].$$

If $0 \leqq \theta \leqq \frac{1}{2}\pi$, then $\sin\theta \geqq 0$, and hence the last expression can be replaced with

$$\frac{16}{3}\left[1 - \sin^3\theta\,\right].$$

But if $-\frac{1}{2}\pi \leqq \theta \leqq 0$, then $\sin\theta \leqq 0$, so that

$$\frac{16}{3}\left[1 - (\sin^2\theta)\sqrt{\sin^2\theta}\,\right] = \frac{16}{3}\left[1 + \sin^3\theta\,\right].$$

If this important detail is overlooked by a student who integrates from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, he or she will obtain the incorrect answer $\frac{16}{3}\pi$ for the volume of the solid.

C14S07.011: The volume is

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{9-r^2} r\,dz\,dr\,d\theta = 2\pi\int_{r=0}^{3}(9r - r^3)\,dr = 2\pi\cdot\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3 = \frac{81}{2}\pi.$$

By symmetry, $\overline{x} = \overline{y} = 0$. The moment of the solid with respect to the xy-plane is

$$M_{xy} = \int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=0}^{9-r^2} rz\,dz\,dr\,d\theta = 2\pi\int_{r=0}^{3}\frac{1}{2}r(9 - r^2)^2\,dr$$

$$= 2\pi\int_{r=0}^{3}\left(\frac{81}{2}r - 9r^3 + \frac{1}{2}r^5\right)dr = 2\pi\cdot\left[\frac{81}{4}r^2 - \frac{9}{4}r^4 + \frac{1}{12}r^6\right]_0^3 = \frac{243}{2}\pi.$$

Therefore the z-coordinate of the centroid is $\overline{z} = 3$. *Suggestion:* Compare this answer with the answer obtained by using the result in Problem 31 of Section 14.6.

C14S07.012: The paraboloids meet in the circle $x^2 + y^2 = 4,\ z = 4$. Therefore the volume between them is

$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2(12r - 3r^3)\,dr = 2\pi\cdot\left[6r^2 - \frac{3}{4}r^4\right]_0^2 = 2\pi\cdot 12 = 24\pi.$$

We are to assume that the solid has unit density, so its mass is $m = 24\pi$ as well. By symmetry, the centroid lies on the z-axis. The moment of the solid with respect to the xy-plane is

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^{12-2r^2} rz\,dz\,dr\,d\theta = 2\pi\int_0^2\left(72r - 24r^3 + \frac{3}{2}r^5\right)dr = 2\pi\cdot\left[36r^2 - 6r^4 + \frac{1}{4}r^6\right]_0^2$$

$$= 2\pi\cdot 64 = 128\pi.$$

Therefore the centroid of the solid is located at the point $\left(0,\ 0,\ \dfrac{16}{3}\right)$.

C14S07.013: The curve formed by the intersection of the paraboloids lies on the cylinder $x^2+y^2=4$, and hence the solid projects vertically onto the disk D with boundary $x^2+y^2=4$ in the xy-plane. Therefore the volume of the solid is

$$V=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2+r^2\cos^2\theta}^{12-r^2-r^2\sin^2\theta} r\,dz\,dr\,d\theta=\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(12r-3r^3)\,dr\,d\theta=\int_{\theta=0}^{2\pi}\left[6r^2-\frac{3}{4}r^4\right]_{r=0}^{2}d\theta$$

$$=\int_0^{2\pi}12\,d\theta=2\pi\cdot 12=24\pi\approx 75.3982236861550377.$$

C14S07.014: The paraboloid $z=r^2$ and the plane $z=2r\cos\theta$ intersect in the cylinder $r=2\cos\theta$, which projects vertically onto the circle with the same polar equation in the xy-plane. Note that this circle is swept out as θ varies from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Hence the volume of the solid between the paraboloid and the plane is

$$V=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_{r^2}^{2r\cos\theta} r\,dz\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}(2r^2\cos\theta-r^3)\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left[\frac{2}{3}r^3\cos\theta-\frac{1}{4}r^4\right]_0^{2\cos\theta}d\theta=\int_{-\pi/2}^{\pi/2}\frac{4}{3}\cos^4\theta\,d\theta=\frac{1}{24}\Big[12\theta+8\sin 2\theta+\sin 4\theta\Big]_{-\pi/2}^{\pi/2}$$

$$=\frac{1}{2}\pi\approx 1.57079632679489661923.$$

C14S07.015: The spherical surface $r^2+z^2=2$ and the paraboloid $z=r^2$ meet in a horizontal circle that projects vertically onto the circle $x^2+y^2=1$ in the xy-plane. Hence the volume between the two surfaces is

$$V=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^1\left[r(2-r^2)^{1/2}-r^3\right]dr\,d\theta=2\pi\cdot\left[-\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{4}r^4\right]_0^1$$

$$=2\pi\cdot\left(\frac{2}{3}\sqrt{2}-\frac{7}{12}\right)=\frac{1}{6}\pi\left(8\sqrt{2}-7\right)\approx 2.2586524883563962.$$

C14S07.016: Choose a coordinate system in which the points of the cylinder are described by

$$0\leqq r\leqq a,\quad 0\leqq\theta\leqq 2\pi,\quad 0\leqq z\leqq h$$

and denote the (constant) density of the homogeneous cylinder by δ. Then the mass of the cylinder will be $m=\pi\delta a^2h$. So its moment of inertia with respect to its axis of symmetry—the z-axis—is

$$I_z=\int_0^{2\pi}\int_0^a\int_0^h\delta r^3\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^r\delta r^3h\,dr\,d\theta$$

$$=2\pi\delta\cdot\left[\frac{1}{4}r^4h\right]_0^a d\theta=\frac{1}{2}\pi\delta a^4h=\frac{1}{2}a^2\cdot\pi\delta a^2h=\frac{1}{2}ma^2.$$

C14S07.017: Set up a coordinate system in which the points of the cylinder are described by

$$0\leqq r\leqq a,\quad 0\leqq\theta\leqq 2\pi,\quad 0\leqq z\leqq h.$$

Because the cylinder has constant density δ, its mass is $m = \pi\delta a^2 h$. One diameter of its base coincides with the x-axis, so we will find the moment of inertia I of the cylinder with respect to that axis. The square of the distance of the point (x, y, z) from the x-axis is $y^2 + z^2 = z^2 + r^2 \sin^2\theta$. Therefore

$$\begin{aligned}
I = I_x &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} \delta(z^2 + r^2\sin^2\theta)\cdot r\,dz\,dr\,d\theta = \delta\int_0^{2\pi}\int_0^a \left(\frac{1}{3}h^3 r + hr^3\sin^2\theta\right)dr\,d\theta \\
&= \delta\int_0^{2\pi}\left[\frac{1}{6}h^3r^2 + \frac{1}{4}hr^4\sin^2\theta\right]_0^a d\theta = \delta\int_0^{2\pi}\left(\frac{1}{6}a^2h^3 + \frac{1}{4}a^4h\sin^2\theta\right)d\theta \\
&= \delta\left[\frac{6a^4h\theta + 8a^2h^3\theta - 3a^4h\sin 2\theta}{48}\right]_0^{2\pi} = \frac{1}{12}\delta\pi a^2h(3a^2 + 4h^2) = \frac{1}{12}m(3a^2 + 4h^2)
\end{aligned}$$

where m is the mass of the cylinder.

C14S07.018: By symmetry, the centroid lies on the axis of the cylinder midway between its two bases. This is so obvious that the intent of this problem must be to verify this by actually computing the integrals. Assume that the cylinder has unit density and is described in cylindrical coordinates by

$$0 \leqq r \leqq a, \quad 0 \leqq \theta \leqq 2\pi, \quad 0 \leqq z \leqq h.$$

Then its mass and moments are

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^a\int_0^h r\,dz\,dr\,d\theta = 2\pi\int_0^a rh\,dr = 2\pi\cdot\frac{1}{2}a^2h = \pi a^2 h; \\
M_{yz} &= \int_0^{2\pi}\int_0^a\int_0^h r^2\cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a r^2h\cos\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}a^3h\cos\theta\,d\theta \\
&= \left[\frac{1}{3}a^3h\sin\theta\right]_0^{2\pi} = 0; \\
M_{xz} &= \int_0^{2\pi}\int_0^a\int_0^h r^2\sin\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^a r^2h\sin\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}a^3h\sin\theta\,d\theta \\
&= \left[-\frac{1}{3}a^3h\cos\theta\right]_0^{2\pi} = 0; \\
M_{xy} &= \int_0^{2\pi}\int_0^a\int_0^h rz\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{1}{2}rh^2\,dr = 2\pi\cdot\left[\frac{1}{4}r^2h^2\right]_0^a = \frac{1}{2}\pi a^2h^2.
\end{aligned}$$

Therefore the centroid of the cylinder is located at the point

$$\left(0,\ 0,\ \frac{1}{2}h\right),$$

on its axis of symmetry and midway between its two bases.

C14S07.019: The volume is

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} r\,dz\,dr\,d\theta = 2\pi\int_0^1 (r - r^2)\,dr = 2\pi\cdot\left[\frac{1}{2}r^2 - \frac{1}{3}r^3\right]_0^1 = \frac{1}{3}\pi.$$

C14S07.020: Suppose that the cone has base radius R and height H. Set up a coordinate system in which its vertex is at the origin, its axis lies on the nonnegative z-axis, its base (at the top) is part of the horizontal plane $z = H$, and its curved side has cylindrical description

$$z = \frac{H}{R}r, \quad 0 \leqq r \leqq R, \quad 0 \leqq \theta \leqq 2\pi.$$

Assume that the cone has constant unit density. Then its mass and moments are

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^R \int_{Hr/R}^H r\,dz\,dr\,d\theta = 2\pi \int_0^R \left(Hr - \frac{H}{R}r^2 \right) dr \\
&= 2\pi \left[\frac{H}{2}r^2 - \frac{H}{3R}r^3 \right]_0^R = 2\pi \cdot \frac{1}{6}R^2H = \frac{1}{3}\pi R^2 H; \\
M_{yz} &= \int_0^{2\pi} \int_0^R \int_{Hr/R}^H r^2 \cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^R \left(Hr^2\cos\theta - \frac{H}{R}r^3\cos\theta \right) dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{H}{3}r^3\cos\theta - \frac{H}{4R}r^4\cos\theta \right]_0^R d\theta = \int_0^{2\pi} \frac{1}{12}R^2H\cos\theta\,d\theta = \left[\frac{1}{12}R^2H\sin\theta \right]_0^{2\pi} = 0; \\
M_{xz} &= \int_0^{2\pi} \int_0^R \int_{Hr/R}^H r^2 \sin\theta\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^R \left(Hr^2\sin\theta - \frac{H}{R}r^3\sin\theta \right) dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{H}{3}r^3\sin\theta - \frac{H}{4R}r^4\sin\theta \right]_0^R d\theta = \int_0^{2\pi} \frac{1}{12}R^2H\sin\theta\,d\theta = \left[-\frac{1}{12}R^2H\cos\theta \right]_0^{2\pi} = 0; \\
M_{xy} &= \int_0^{2\pi} \int_0^R \int_{Hr/R}^H rz\,dz\,dr\,d\theta = 2\pi \int_0^R \left(\frac{H^2}{2}r - \frac{H^2}{2R^2}r^3 \right) dr \\
&= 2\pi \cdot \left[\frac{H^2}{4}r^2 - \frac{H^2}{8R^2}r^4 \right]_0^R = 2\pi\frac{1}{8}R^2H^2 = \frac{1}{4}\pi R^2 H^2.
\end{aligned}$$

Therefore the centroid is located at the point

$$\left(0,\ 0,\ \frac{3}{4}H \right),$$

on the axis of the cone three-quarters of the way from its vertex to its base. Compare this with the solution of Problem 32 in Section 14.6.

C14S07.021: Without loss of generality we may assume that the hemispherical solid has density $\delta = 1$. Choose a coordinate system in which the solid is bounded above by the spherical surface $\rho = a$ and below by the xy-plane. Then its mass and moments are

$$\begin{aligned}
m &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^a \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi \int_{\phi=0}^{\pi/2} \frac{1}{3}a^3\sin\phi\,d\phi = \frac{2}{3}\pi a^3 \Big[-\cos\phi \Big]_0^{\pi/2} = \frac{2}{3}\pi a^3; \\
M_{yz} &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^a \rho^3 \sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \frac{1}{4}a^4\sin^2\phi\cos\theta\,d\phi\,d\theta \\
&= \int_{\theta=0}^{2\pi} \left(\int_{\phi=0}^{\pi/2} \frac{1}{4}a^2\sin^2\phi\,d\phi \right) \cos\theta\,d\theta = \left(\int_{\phi=0}^{\pi/2} \frac{1}{4}a^2\sin^2\phi\,d\phi \right) \cdot \Big[\sin\theta \Big]_{\theta=0}^{2\pi} = 0;
\end{aligned}$$

$$M_{xz} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \frac{1}{4} a^4 \sin^2\phi \sin\theta \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{\phi=0}^{\pi/2} \frac{1}{4} a^2 \sin^2\phi \, d\phi \right) \sin\theta \, d\theta = \left(\int_{\phi=0}^{\pi/2} \frac{1}{4} a^2 \sin^2\phi \, d\phi \right) \cdot \Big[-\cos\theta \Big]_{\theta=0}^{2\pi} = 0;$$

$$M_{yx} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_{\phi=0}^{\pi/2} \frac{1}{4} a^4 \sin\phi \cos\phi \, d\phi$$

$$= \frac{1}{2}\pi a^4 \left[\frac{1}{2} \sin^2\phi \right]_0^{\pi/2} = \frac{1}{4}\pi a^4.$$

Therefore the centroid of the hemispherical solid is located at the point $\left(0,\ 0,\ \frac{3}{8}a\right)$.

C14S07.022: We are given the information that the hemispherical solid has density $\delta = kz$ at the point $(x,\ y,\ z)$. Then its mass and moments are

$$m = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} k\rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_{\phi=0}^{\pi/2} \frac{1}{4} ka^4 \sin\phi \cos\phi \, d\phi$$

$$= \frac{1}{4} k\pi a^4 \Big[\sin^2\phi \Big]_0^{\pi/2} = \frac{1}{4} k\pi a^4;$$

$$M_{yz} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} k\rho^4 \sin^2\phi \cos\phi \cos\theta \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{15} ka^5 \sin^3\phi \cos\theta \right]_{\phi=0}^{\pi/2} d\theta$$

$$= \left[\frac{1}{15} ka^5 \sin\theta \right]_{\theta=0}^{2\pi} = 0;$$

$$M_{yz} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} k\rho^4 \sin^2\phi \cos\phi \sin\theta \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{15} ka^5 \sin^3\phi \sin\theta \right]_{\phi=0}^{\pi/2} d\theta$$

$$= \left[-\frac{1}{15} ka^5 \cos\theta \right]_{\theta=0}^{2\pi} = 0;$$

$$M_{xy} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{a} k\rho^4 \sin\phi \cos^2\phi \, d\rho \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \frac{1}{5} ka^5 \sin\phi \cos^2\phi \, d\phi \, d\theta$$

$$= 2\pi \cdot \left[-\frac{1}{15} ka^5 \cos^3\phi \right]_{\phi=0}^{\pi/2} = \frac{2}{15} k\pi a^5.$$

Therefore the centroid of this hemispherical solid is located at the point $\left(0,\ 0,\ \frac{8}{15}a\right)$.

C14S07.023: The plane $z = 1$ has the spherical-coordinates equation $\rho = \sec\phi$; the cone with cylindrical-coordinates equation $r = z$ has spherical-coordinates equation $\phi = \frac{1}{4}\pi$. Hence the volume bounded by the plane and the cone is

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_{\phi=0}^{\pi/4} \frac{1}{3} \sec^2\phi \tan\phi \, d\phi = 2\pi \cdot \left[\frac{1}{6} \sec^2\phi \right]_{\phi=0}^{\pi/4} = \frac{1}{3}\pi.$$

C14S07.024: Without loss of generality the cone has density $\delta = 1$. Assume that its vertex is at the origin and that its axis lies on the nonnegative z-axis. Assume that its curved side has spherical-coordinates equation $\phi = \alpha$ (where $0 < \alpha < \frac{1}{2}\pi$) and that its base lies on the plane $z = a > 0$; thus its base has spherical-coordinates equation $\rho = a \sec\phi$. Then its mass and moments are

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^{\alpha} \int_0^{a\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\alpha} \frac{1}{3} a^3 \sec^2\phi \tan\phi \, d\phi \\ &= 2\pi \left(\frac{1}{6} a^3 \sec^2\alpha - \frac{1}{6} a^3 \right) = \frac{1}{3}\pi a^3 \tan^2\alpha; \end{aligned}$$

$$\begin{aligned} M_{yz} &= \int_0^{2\pi} \int_0^{\alpha} \int_0^{a\sec\phi} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\alpha} \frac{1}{4} a^4 \sec^2\phi \tan^2\phi \cos\theta \, d\phi \, d\theta \\ &= \frac{1}{4} a^4 \int_0^{2\pi} \left(\int_0^{\alpha} \sec^2\phi \tan^2\phi \, d\phi \right) \cos\theta \, d\theta = \frac{1}{4} a^4 \left(\int_0^{\alpha} \sec^2\phi \tan^2\phi \, d\phi \right) \cdot \Big[\sin\theta \Big]_0^{2\pi} = 0; \end{aligned}$$

$$M_{xz} = \int_0^{2\pi} \left(\int_0^{\alpha} \int_0^{a\sec\phi} \rho^3 \sin^2\phi \, d\rho \, d\phi \right) \sin\theta \, d\theta = \left(\int_0^{\alpha} \int_0^{a\sec\phi} \rho^3 \sin^2\phi \, d\rho \, d\phi \right) \cdot \Big[-\cos\theta \Big]_0^{2\pi} = 0;$$

$$\begin{aligned} M_{yz} &= \int_0^{2\pi} \int_0^{\alpha} \int_0^{a\sec\phi} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\alpha} \frac{1}{4} a^4 \sec^3\phi \sin\phi \, d\phi \\ &= 2\pi \left(\frac{1}{8} a^4 \sec^2\alpha - \frac{1}{8} a^4 \right) = \frac{1}{4}\pi a^4 \tan^2\alpha. \end{aligned}$$

Therefore the centroid of the cone is on its axis of symmetry and three-quarters of the way from the vertex to the base, because it is located at the point

$$(\overline{x},\ \overline{y},\ \overline{z}) \ = \ \left(0,\ 0,\ \frac{3}{4}a \right).$$

C14S07.025: Assume unit density. Then the mass and the volume are numerically the same; they and the moments are

$$\begin{aligned} m = V &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot \frac{1}{3} a^3 \Big[-\cos\phi \Big]_0^{\pi/4} \\ &= \frac{2}{3}\pi a^2 \left(1 - \frac{1}{2}\sqrt{2} \right) = \frac{1}{3}\pi \left(2 - \sqrt{2} \right) a^3; \end{aligned}$$

$$M_{yz} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{a} \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta = \left(\int_0^{\pi/4} \int_0^{a} \rho^3 \sin^2\phi \, d\rho \, d\phi \right) \cdot \Big[\sin\theta \Big]_0^{2\pi} = 0;$$

$$M_{xz} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{a} \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta = \left(\int_0^{\pi/4} \int_0^{a} \rho^3 \sin^2\phi \, d\rho \, d\phi \right) \cdot \Big[-\cos\theta \Big]_0^{2\pi} = 0;$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot \frac{1}{4}a^4 \cdot \left[\frac{1}{2}\sin^2\phi \right]_0^{\pi/4} = \frac{1}{8}\pi a^4.$$

So the z-coordinate of the centroid is

$$\overline{z} = \frac{3\pi a^4}{8\pi\left(2 - \sqrt{2}\,\right)a^3} = \frac{3a}{8\left(2 - \sqrt{2}\,\right)} = \frac{3\left(2 + \sqrt{2}\,\right)a}{16} = \frac{3}{16}\left(2 + \sqrt{2}\,\right)a;$$

clearly $\overline{x} = \overline{y} = 0$. For a plausibility check, note that $\overline{z} \approx (0.6401650429449553)a$.

C14S07.026: Assume that the solid of Problem 25 has constant density δ. Then its moment of inertia with respect to the z-axis is

$$\begin{aligned} I_z &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \delta\rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = 2\pi\delta \int_0^{\pi/4} \frac{1}{5}a^5 \sin^3\phi \, d\phi \\ &= 2\pi\delta a^5 \cdot \frac{1}{60}\Big[\cos 3\phi - 9\cos\phi \Big]_0^{\pi/4} = 2\pi\delta \left(\frac{2}{15}a^5 - \frac{\sqrt{2}}{12}a^5 \right) = \frac{1}{30}\pi\delta\left(8 - 5\sqrt{2}\,\right)a^5. \end{aligned}$$

C14S07.027: Set up a coordinate system so that the center of the sphere is at the point with Cartesian coordinates $(a, 0, 0)$. Then its Cartesian equation is

$$(x-a)^2 + y^2 + z^2 = a^2; \quad x^2 - 2ax + y^2 + z^2 = 0;$$

and thus it has spherical-coordinates equation $\rho = 2a\sin\phi\cos\theta$ (but note that θ ranges from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$). We plan to find its moment of inertia with respect to the z-axis, and the square of the distance of a point of the sphere from the z-axis is

$$x^2 + y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi.$$

Moreover, if the sphere has mass m and constant density δ, then we also have $m = \frac{4}{3}\pi a^3\delta$. Finally,

$$\begin{aligned} I_z &= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^{2a\sin\phi\cos\theta} \delta\rho^2 \sin^2\phi \, d\rho \, d\phi \, d\theta = 2\delta \int_0^{\pi/2} \int_0^{\pi} \frac{1}{5}(2a\sin\phi\cos\theta)^5 \sin^3\phi \, d\phi \, d\theta \\ &= \frac{64}{5}\delta a^5 \int_0^{\pi/2} \int_0^{\pi} \sin^8\phi \cos^5\theta \, d\phi \, d\theta = \frac{128}{5}\delta a^5 \int_0^{\pi/2} \int_0^{\pi/2} \sin^8\phi \cos^5\theta \, d\phi \, d\theta \\ &= \frac{128}{5}\delta a^5 \cdot \left(\int_0^{\pi/2} \sin^8\phi \, d\phi \right) \cdot \left(\int_0^{\pi/2} \cos^5\theta \, d\theta \right). \end{aligned}$$

Then Formula (113) from the long table of integrals (see the endpapers) yields

$$I_z = \frac{128}{5}\delta a^5 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{\pi}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = \frac{28}{15}\pi\delta a^5 = \frac{4}{3}\pi\delta a^3 \cdot \frac{7}{5}a^2 = \frac{7}{5}ma^2.$$

C14S07.028: We will find its moment of inertia with respect to the z-axis (which contains a diameter) using density $\delta = \rho^2$ at the point (ρ, ϕ, θ):

$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_a^{2a} \delta\rho^6 \sin^3\phi \, d\rho \, d\phi \, d\theta = 2\pi\delta \int_0^{\pi} \frac{127}{7}\delta a^7 \sin^3\phi \, d\phi$$

$$= \frac{127}{42}\pi a^7 \Big[\cos 3\phi - 9\cos\phi\Big]_0^{\pi} = \frac{1016}{21}\pi a^7.$$

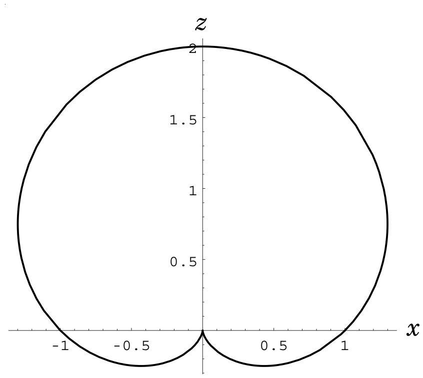
C14S07.029: The surface with spherical-coordinates equation $\rho = 2a\sin\phi$ is generated as follows. Draw the circle in the xz-plane with center $(a, 0)$ and radius a. Rotate this circle around the z-axis. This generates the surface with the given equation. It is called a *pinched torus*—a doughnut with an infinitesimal hole. Its volume is

$$V = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{2a\sin\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_{\phi=0}^{\pi} \frac{1}{3}(2a\sin\phi)^3 \sin\phi \, d\phi$$

$$= \frac{16}{3}\pi a^3 \cdot 2\int_{\phi=0}^{\pi/2} \sin^4\phi \, d\phi = \frac{32}{3}\pi a^3 \cdot \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2} = 2\pi^2 a^3.$$

We evaluated the last integral with the aid of Formula (113) from the long table of integrals (see the endpapers). The volume of the pinched torus is also easy to evaluate using the first theorem of Pappus (Section 14.5).

C14S07.030: Draw the cardioid with polar equation $r = 1 + \sin\theta$, then replace the y-axis with the z-axis. Such a cardioid is shown in the following figure.



To generate the surface with spherical-coordinates equation $\rho = 1 + \cos\phi$, rotate this cardioid around the x-axis. The resulting surface resembles an inverted apple. The volume that it bounds is

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \frac{1}{3}(1+\cos\phi)^3 \sin\phi \, d\phi$$

$$= 2\pi\left[-\frac{1}{12}(1+\cos\phi)^4\right]_0^{\pi} = 2\pi\cdot\frac{4}{3} = \frac{8}{3}\pi.$$

C14S07.031: Assuming constant density δ, we have

$$I_x = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{4a^2-r^2}} \delta(r^2\sin^2\theta + z^2)\cdot r \, dz \, dr \, d\theta$$

$$= 2\delta \int_0^{2\pi} \int_0^a \left(\frac{1}{3} r(4a^2 - r^2)^{3/2} + r^3(4a^2 - r^2)^{1/2} \sin^2\theta \right) \, dr \, d\theta$$

$$= 2\delta \int_0^{2\pi} \frac{1}{30} \Big[(4a^2 - r^2)^{3/2} \big[(8a^2 + 3r^2)\cos 2\theta - 16a^2 - r^2 \big] \Big]_{r=0}^{a} \, d\theta$$

$$= 2\delta \int_0^{2\pi} \frac{1}{30} a^5 \big[128 - 51\sqrt{3} + \left(33\sqrt{3} - 64 \right) \cos 2\theta \big] \, d\theta$$

$$= \frac{1}{30} \delta a^5 \left[\left(128 - 5\sqrt{3} \right) \cdot 2\theta + \left(33\sqrt{3} - 64 \right) \sin 2\theta \right]_0^{2\pi} = \frac{2}{15} \left(128 - 51\sqrt{3} \right) \delta \pi a^5.$$

C14S07.032: Assuming constant density δ, we have

$$I_z = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2a\cos\phi} \delta \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = 2\pi\delta \int_0^{\pi/6} \frac{1}{5} (2a\cos\phi)^5 \sin^3\phi \, d\phi$$

$$= \frac{64}{5} \pi\delta a^5 \int_0^{\pi/6} \cos^5\phi \sin^3\phi \, d\phi = \frac{64}{5} \pi\delta a^5 \int_0^{\pi/6} (\cos^5\phi - \cos^7\phi) \sin\phi \, d\phi$$

$$= \frac{64}{5} \pi\delta a^5 \left[\frac{1}{8} \cos^8\phi - \frac{1}{6} \cos^6\phi \right]_0^{\pi/6} = \frac{64}{5} \pi\delta a^5 \left[\frac{1}{8} \cdot \left(\frac{3}{4} \right)^4 - \frac{1}{6} \cdot \left(\frac{3}{4} \right)^3 - \frac{1}{8} + \frac{1}{6} \right]$$

$$= \frac{64}{5} \pi\delta a^5 \cdot \frac{67}{6144} = \frac{67}{480} \pi\delta a^5.$$

C14S07.033: If the density at (x, y, z) of the ice-cream cone is z, then its mass and moments are

$$m = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2a\cos\phi} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/6} 4a^4 \sin\phi \cos^5\phi \, d\phi$$

$$= \left[-\frac{4}{3} \pi a^4 \cos^6\phi \right]_0^{\pi/6} = \frac{37}{48} \pi a^4;$$

$$M_{yz} = \int_0^{2\pi} \left(\int_0^{\pi/6} \int_0^{2a\cos\phi} \rho^4 \sin^2\phi \cos\phi \, d\rho \, d\phi \right) \cos\theta \, d\theta$$

$$= \left(\int_0^{\pi/6} \int_0^{2a\cos\phi} \rho^4 \sin^2\phi \cos\phi \, d\rho \, d\phi \right) \cdot \Big[\sin\theta \Big]_0^{2\pi} = 0;$$

$M_{xz} = 0$ (by a similar computation);

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2a\cos\phi} \rho^4 \sin\phi \cos^2\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/6} \frac{32}{5} a^5 \cos^7\phi \sin\phi \, d\phi$$

$$= 2\pi \cdot \left[-\frac{4}{5} a^5 \cos^8\phi \right]_0^{\pi/6} = \frac{35}{32} \pi a^5.$$

Hence the centroid is located at the point $\left(0, \ 0, \ \dfrac{105}{74} a \right)$.

C14S07.034: The moment of inertia of the ice-cream cone of Problem 33 with respect to the z-axis is

$$I_z = \int_0^{2\pi} \int_0^{\pi/6} \int_0^{2a\cos\phi} (\rho\cos\phi)\cdot(\rho\sin\phi)^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/6} \frac{32}{3} a^6 \cos^7\phi \sin^3\phi \, d\phi$$

$$= \frac{1}{240}\pi a^6 \Big[\cos 10\phi + 5\cos 8\phi + 5\cos 6\phi - 20\cos 4\phi - 70\cos 2\phi \Big]_0^{\pi/6} = \frac{47}{240}\pi a^6.$$

C14S07.035: The similar star with uniform density k has mass $m_2 = \frac{4}{3}k\pi a^3$. The other star has mass

$$m_1 = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} k\left[1 - \left(\frac{\rho}{a}\right)^2\right] \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \left[\frac{5ka^2\rho^3\sin\phi - 3k\rho^5\sin\phi}{15a^2}\right]_0^a d\phi$$

$$= 2\pi \int_0^{\pi} \frac{2}{15} ka^3 \sin\phi \, d\phi = \left[-\frac{4}{15}k\pi a^3\cos\phi\right]_0^{\pi} = \frac{8}{15}k\pi a^3.$$

Finally, $\dfrac{m_1}{m_2} = \dfrac{2}{5}$.

C14S07.036: The moment of inertia of the first star of Problem 35 with respect to its diameter that lies on the z-axis is

$$I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} k\left[1 - \left(\frac{\rho}{a}\right)^2\right] \cdot \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \left[\frac{7ka^2\rho^5\sin^3\phi - 5k\rho^7\sin^3\phi}{35a^2}\right]_0^a d\phi$$

$$= 2\pi \int_0^{\pi} \frac{2}{35} ka^5 \sin^3\phi \, d\phi = \frac{1}{105}\pi k a^5 \Big[\cos 3\phi - 9\cos\phi \Big]_0^{\pi} = \frac{16}{105}\pi k a^5.$$

C14S07.037: The given triple integral takes the following form:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \exp\left(-\rho^3\right) \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \left[-\frac{1}{3}\exp\left(-\rho^3\right)\sin\phi\right]_0^a d\phi$$

$$= 2\pi \int_0^{\pi} \frac{1}{3}\big[1 - \exp\left(-a^3\right)\big]\sin\phi \, d\phi = \frac{4}{3}\pi\big[1 - \exp\left(-a^3\right)\big].$$

Clearly the value of the integral approaches $\dfrac{4}{3}\pi$ as $a \to +\infty$.

C14S07.038: If the given integral is evaluated over the ball B of Problem 37, the result is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \exp\left(-\rho^2\right) \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi} \left[-\frac{1}{2}(\rho^2+1)\exp\left(-\rho^2\right)\sin\phi\right]_0^a d\phi$$

$$= 2\pi \int_0^{\pi} \frac{1}{2}\big[1 - (a^2+1)\exp\left(-a^2\right)\big]\sin\phi \, d\phi = \pi\Big[\big\{(a^2+1)\exp\left(-a^2\right) - 1\big\}\cos\phi\Big]_0^{\pi}$$

$$= 2\pi\big[1 - (a^2+1)\exp\left(-a^2\right)\big].$$

Now let $a \to +\infty$ to see that the value of the integral given in the statement of this problem is 2π.

C14S07.039: Let $V = \frac{4}{3}\pi a^3$, the volume of the ball. The average distance of points of such a ball from its center is then

$$\overline{d} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2\pi}{V}\int_0^{\pi}\frac{1}{4}a^4\sin\phi\,d\phi = \frac{\pi a^4}{2V}\Big[-\cos\phi\Big]_0^{\pi} = \frac{\pi a^4}{V} = \frac{3}{4}a.$$

Note that the answer is both plausible and dimensionally correct.

C14S07.040: The key to solving such problems is to keep the integrand as simple as possible. To do so we set up a coordinate system in which the ball is centered at the point with Cartesian coordinates $(0,\,0,\,a)$. Thus the bounding spherical surface has spherical-coordinates equation $\rho = 2a\cos\phi$ where $0 \leqq \phi \leqq \frac{1}{2}\pi$ and $0 \leqq \theta \leqq 2\pi$. Let $V = \frac{4}{3}\pi a^3$, the volume of the ball. The average distance of points of the ball from its south pole (at the origin) is then

$$\begin{aligned}\overline{d} &= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \frac{2\pi}{V}\int_0^{\pi/2}4a^4\cos^4\phi\sin\phi\,d\phi = \frac{2\pi}{V}\left[-\frac{4}{5}a^4\cos^5\phi\right]_0^{\pi/2} = \frac{2\pi}{V}\cdot\frac{4}{5}a^4 = \frac{6}{5}a.\end{aligned}$$

As in the solution of Problem 39, note that the answer is both plausible and dimensionally correct.

C14S07.041: A *Mathematica* solution:

```
m = Integrate[ Integrate[ delta*a∧2*Sin[ phi ], { phi, 0, Pi } ],
              { theta, 0, 2*Pi } ]
```

$4\pi a^2\delta$

```
r = a*Sin[ phi ];
I0 = Integrate[ Integrate[ r∧2*delta*a∧2*Sin[ phi ], { phi, 0, Pi } ],
              { theta, 0, 2*Pi } ]
```

$\frac{8}{3}\pi a^4\delta$

```
I0/m
```

$\frac{2}{3}a^2$ —C.H.E.

Second solution, by hand: The spherical surface S of radius a is described by $\rho = a,\ 0 \leqq \phi \leqq \pi$, $0 \leqq \theta \leqq 2\pi$. Hence its moment of inertia with respect to the z-axis is

$$\begin{aligned}I_z &= \iint_S \delta(x^2+y^2)\,dA = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\delta a^4\sin^3\phi\,d\phi\,d\theta\\ &= 2\pi\delta a^4\int_0^{\pi}(1-\cos^2\phi)\sin\phi\,d\phi = 2\pi\delta a^4\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} = \left(\frac{2}{3}a^2\right)\cdot 4\pi\delta a^2 = \frac{2}{3}ma^2.\end{aligned}$$

C14S07.042: A *Mathematica* solution:

```
m = Integrate[ Integrate[ Integrate[ delta*rho∧2*Sin[ phi ],
  { rho, a, b } ], { phi, 0, Pi } ], { theta, 0, 2*Pi } ]
```

$$\frac{4}{3}\pi\delta(b^3 - a^3)$$

```
r = rho*Sin[ phi ];
I0 = Integrate[ Integrate[ Integrate[ delta*r∧2*delta*rho∧2*Sin[ phi ],
  { rho, a, b } ], { phi, 0, Pi } ], { theta, 0, 2*Pi } ]
```

$$\frac{8}{15}\pi\delta(b^5 - a^5)$$

```
Simplify[ I0/m ]
```

$$\frac{2}{5}\cdot\frac{b^5 - a^5}{b^3 - a^3}$$

—C.H.E.

Second solution, by hand: Let $r = \sqrt{x^2 + y^2}$ denote the usual radial polar coordinate. Then the moment of inertia of the solid with respect to the z-axis is

$$\begin{aligned} I_z &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=a}^{b} \delta r^2\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi\delta\int_0^{\pi}\int_a^b \rho^4 \sin^3\phi \, d\rho \, d\phi \\ &= 2\pi\delta\left(\frac{b^5 - a^5}{5}\right)\int_0^{\pi} \sin^3\phi \, d\phi = 2\pi\delta\left(\frac{b^5 - a^5}{5}\right)\cdot\frac{4}{3} = \frac{8}{15}\pi\delta(b^5 - a^5). \end{aligned}$$

The mass of the shell S is $m = \frac{4}{3}\pi\delta(b^3 - a^3)$, and therefore

$$I_z = \frac{4}{3}\pi\delta(b^3 - a^3)\cdot\frac{2(b^5 - a^5)}{5(b^3 - a^3)} = \frac{2}{5}m\cdot\frac{b^5 - a^5}{b^3 - a^3} = \frac{2}{5}mc^2$$

where $c^2 = \dfrac{b^5 - a^5}{b^3 - a^3}$.

C14S07.043: A *Mathematica* solution:

```
z = Sqrt[ b∧2 − a∧2 ];
m = 2*Integrate[ Integrate[ delta*r*z, { r, a, b } ], { theta, 0, 2*Pi } ]
```

$$\frac{4}{3}\pi\delta(b^2 - a^2)^{3/2}$$

```
I0 = 2*Integrate[ Integrate[ delta*r∧3*z, { r, a, b } ], { theta, 0, 2*Pi } ]
```

$$\frac{4}{15}\pi\delta(2b^4 + b^2a^2 - 3a^4)\sqrt{b^2 - a^2}$$

```
Simplify[ I0/m ]
```

$$\frac{1}{5}(3a^2 + 2b^2)$$

—C.H.E.

Second solution, by hand: Choose a coordinate system so that the z-axis is the axis of symmetry of the sphere-with-hole. The central cross section of the solid in the xz-plane is bounded by the circle with polar (or cylindrical coordinates) equation $r^2 + z^2 = b^2$. Hence the mass of the solid is

$$m = 2\int_{\theta=0}^{2\pi}\int_{r=a}^{b}\int_{z=0}^{\sqrt{b^2-r^2}} \delta r \, dz \, dr \, d\theta = 4\pi\delta\int_a^b r(b^2 - r^2)^{1/2} \, dr$$

$$= 4\pi\delta\left[-\frac{1}{3}(b^2-r^2)^{3/2}\right]_a^b = \frac{4}{3}\pi\delta(b^2-a^2)^{3/2}.$$

The moment of inertia of this solid with respect to the z-axis is

$$I_z = 2\int_{\theta=0}^{2\pi}\int_{r=a}^{b}\int_{z=0}^{\sqrt{b^2-r^2}} \delta r^3\,dz\,dr\,d\theta = 4\pi\delta\int_a^b r^3(b^2-r^2)^{1/2}\,dr.$$

Integration by parts with $u = r^2,\ dv = (b^2-r^2)^{1/2}\,dr$, so that

$$du = 2r\,dr \quad \text{and} \quad v = -\frac{1}{3}(b^2-r^2)^{3/2},$$

then yields

$$\begin{aligned}
I_z &= 4\pi\delta\left(\left[-\frac{1}{3}r^2(b^2-r^2)^{3/2}\right]_a^b + \frac{2}{3}\int_a^b r(b^2-r^2)^{3/2}\,dr\right)\\
&= 4\pi\delta\left(\frac{1}{3}a^2(b^2-a^2)^{3/2} + \frac{2}{3}\left[-\frac{1}{5}(b^2-r^2)^{5/2}\right]_a^b\right)\\
&= \frac{4}{15}\pi\delta\left[5a^2(b^2-a^2)^{3/2} + 2(b^2-a^2)(b^2-a^2)^{3/2}\right] = \frac{4}{15}\pi\delta(b^2-a^2)^{3/2}(5a^2+2b^2-2a^2)\\
&= \frac{4}{3}\pi\delta(b^2-a^2)^{3/2}\cdot\frac{1}{5}(3a^2+2b^2) = \frac{1}{5}m(3a^2+2b^2).
\end{aligned}$$

C14S07.044: As we examine Fig. 14.7.15(b), with the positive x-axis to the east and the positive y-axis to the north, we see that one-sixteenth of the solid is bounded below by the region R in the xy-plane described in polar (cylindrical) coordinates by

$$0 \leqq r \leqq 1, \qquad 0 \leqq \theta \leqq \frac{\pi}{4}$$

and above by part of the cylindrical surface with equation $x^2 + z^2 = 1$ (*not* $y^2 + z^2 = 1$; over R, that surface is *higher*). We get the total volume V of the solid of Problem 44 by multiplying by 16:

$$\begin{aligned}
V &= 16\int_{\theta=0}^{\pi/4}\int_{r=0}^{1}\int_{z=0}^{\sqrt{1-r^2\cos^2\theta}} r\,dz\,dr\,d\theta = 16\int_0^{\pi/4}\int_0^1 r\sqrt{1-r^2\cos^2\theta}\,dr\,d\theta\\
&= 16\int_0^{\pi/4}\left[-\frac{(1-r^2\cos^2\theta)^{3/2}}{3\cos^2\theta}\right]_{r=0}^{1} d\theta = \frac{16}{3}\int_0^{\pi/4}\left(\frac{1}{\cos^2\theta} - \frac{(1-\cos^2\theta)^{3/2}}{\cos^2\theta}\right)d\theta\\
&= \frac{16}{3}\int_0^{\pi/4}\frac{1-\sin^3\theta}{\cos^2\theta}\,d\theta = \frac{16}{3}\int_0^{\pi/4}\frac{1-(1-\cos^2\theta)\sin\theta}{\cos^2\theta}\,d\theta\\
&= \frac{16}{3}\int_0^{\pi 4}\frac{1+\cos^2\theta\,\sin\theta-\sin\theta}{\cos^2\theta}\,d\theta = \frac{16}{3}\int_0^{\pi/4}(\sec^2\theta+\sin\theta-\sec\theta\,\tan\theta)\,d\theta\\
&= \frac{16}{3}\Big[\tan\theta - \cos\theta - \sec\theta\Big]_0^{\pi/4} = \frac{16}{3}\left(3-\frac{3}{2}\sqrt{2}\right) = 8\left(2-\sqrt{2}\right) \approx 4.68629150101523961.
\end{aligned}$$

Replacement of $(1-\cos^2\theta)^{3/2}$ with $\sin^3\theta$ in the third line is permitted because $\sin\theta$ is nonnegative on R. (Thus we have avoided the "pitfall" indicated in the solution of Problem 10.) The plausibility of the answer is enhanced by the observation that a sphere of radius 1 will fit snugly within the boundary of the region bounded by all three cylinders, and the volume of such a sphere is approximately 4.18879020478639098.

Second solution: *Mathematica* 3.0 presents the last antiderivative in such a way as to produce a seemingly improper integral, so care must be taken in the computer algebra solution. We write t for θ here, as usual; recall also that the symbol "%" refers to the last output.

```
16*Integrate[ r, { z, 0, Sqrt[ 1 - (r*Cos[t])^2 ] } ]
```

$$16r\sqrt{1-r^2\cos^2 t}$$

```
Integrate[ %, r ]
```

$$8\left(\sqrt{2-r^2-r^2\cos 2t}\right)\cdot\frac{1}{3}\sqrt{2}\,\left(r^2-\sec^2 t\right)$$

```
(% /. r → 1) - (% /. r → 0)
```

$$\frac{16}{3}\sec^2 t+\left(8\sqrt{1-\cos 2t}\,\right)\cdot\frac{1}{3}\sqrt{2}\,\left(1-\sec^2 t\right)$$

```
Integrate[ %, t ]
```

$$\frac{16}{3}\tan t-\left(\sqrt{1-\cos 2t}\,\right)\cdot\frac{1}{3}\sqrt{2}\,\left(16\cot t+8\tan t\right)$$

```
Limit[ %, t → Pi/4 ] - Limit[ %, t → 0 ]
```

$$\frac{16}{3}-8\sqrt{2}+\frac{32}{3}=16-8\sqrt{2}$$

```
N[ %, 40 ]
```

4.686291501015239609586490206322241537144

C14S07.045: First we let $g(\phi,\theta)=6+3\cos 3\theta\,\sin 5\phi$, so that the boundary of the bumpy sphere B has spherical equation $\rho=g(\phi,\theta)$. Then the volume V of B is simply

$$V=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{g(\phi,\theta)}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

To evaluate V with the aid of *Mathematica* 3.0, we write `r` in place of ρ, `f` instead of ϕ, and `t` for θ, as usual. Then, to find the volume V of B step-by-step, we proceed as follows:

```
Integrate[ r*r*Sin[f], r ]
```

$$\frac{1}{3}\rho^3\sin\phi$$

```
% /. r → g[f,t]
```

$$\frac{1}{3}(\sin\phi)(6+3\cos 3\theta\,\sin 5\phi)^3$$

```
Integrate[ %, f ]
```

$$\frac{3}{78848}\left[-2247168\cos\phi-19712\cos 9\phi+16128\cos 11\phi-177408\cos(\phi-6\theta)\right.$$

$$- 9856\cos(9\phi - 6\theta) + 8064\cos(11\phi - 6\theta) - 177408\cos(\phi + 6\theta)$$
$$- 9856\cos(9\phi + 6\theta) + 8064\cos(11\phi + 6\theta) + 2772\sin(4\phi - 9\theta)$$
$$- 1848\sin(6\phi - 9\theta) - 264\sin(14\phi - 9\theta) + 231\sin(16\phi - 9\theta)$$
$$+ 185724\sin(4\phi - 3\theta) - 123816\sin(6\phi - 3\theta) - 792\sin(14\phi - 3\theta)$$
$$+ 693\sin(16\phi - 3\theta) + 185724\sin(4\phi + 3\theta) - 123816\sin(6\phi + 3\theta)$$
$$- 792\sin(14\phi + 3\theta) + 693\sin(16\phi + 3\theta) + 2772\sin(4\phi + 9\theta) - 1848\sin(6\phi + 9\theta)$$
$$- 264\sin(14\phi + 9\theta) + 231\sin(16\phi + 9\theta)\,]$$

```
(% /.  f → Pi) - (% /.  f → 0) // FullSimplify
```

$$\frac{12\,(157 + 25\cos 6\theta)}{11}$$

```
Integrate[ %, t ]
```

$$\frac{1884\theta + 50\sin 6\theta}{11}$$

```
(% /.  t → 2*Pi) - (% /.  t → 0)
```

$$\frac{3768\pi}{11}$$

```
N[ %, 60 ]
```

1076.138283520576447502476388017924266069590311789503624849396

Thus $V = \frac{3768}{11}\pi \approx 1076.13828352.$

C14S07.046: First we let $g(\phi, \theta) = 6 + 3\cos 3\theta \sin 5\phi$. To find the moment of the bumpy sphere with respect to the yz-plane, we computed

$$M_{yz} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{g(\phi,\theta)} \rho^3 \sin^2\phi \cos\theta \; d\rho \; d\phi \; d\theta$$

by executing the following *Mathematica* commands:

```
Integrate[ r∧3*(Sin[f])∧2*Cos[t], { r, 0, g[f,t] } ];
Integrate[ %, { f, 0, Pi } ];
Integrate[ %, { t, 0, 2*Pi } ]
```

0

Therefore $\overline{x} = 0$. Similarly, we found that

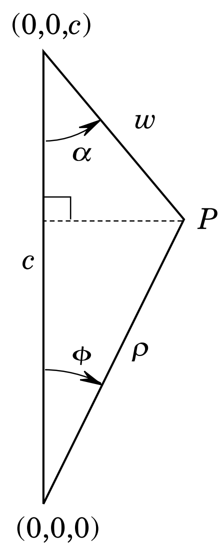
$$M_{xz} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{g(\phi,\theta)} \rho^3 \sin^2\phi \sin\theta \; d\rho \; d\phi \; d\theta = 0$$

and

$$M_{xy} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{g(\phi,\theta)} \rho^3 \sin\phi \cos\theta \; d\rho \; d\phi \; d\theta = 0.$$

Thus the centroid of the bumpy sphere is, indeed, located at its center $(0, 0, 0)$. (We suppressed the "intermediate output" in this solution because it is quite long, as in the solution of Problem 45.)

C14S07.047: The following figure makes some of the equations we use easy to derive.



A mass element $\delta\,dV$ located at the point $P(\rho,\ \phi,\ \theta)$ of the ball exerts a force on the mass m at $(0,\ 0,\ c)$ that has vertical component

$$dF = -\frac{Gm\delta\cos\alpha}{w^2}\,dV.$$

Note the following:

$$\begin{aligned}
&M = \frac{4}{3}\pi\delta a^3;\\
&w^2 = \rho^2 + c^2 - 2\rho c\cos\phi;\\
&2w\ dw = 2\rho c\sin\phi\ d\phi;\\
&w\cos\alpha + \rho\cos\phi = c;\\
&\rho\cos\phi = \frac{\rho^2 + c^2 - w^2}{2c};\\
&\rho\sin\phi\ d\phi = \frac{w}{c}\ dw.
\end{aligned}$$

Hence the total force exerted by the ball on m is

$$\begin{aligned}
F &= -\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a}\int_{\phi=0}^{\pi}\frac{Gm\delta\cos\alpha}{w^2}\cdot\rho^2\sin\phi\ d\phi\ d\rho\ d\theta = -\int_0^{2\pi}\int_0^a\int_0^\pi\frac{Gm\delta w\cos\alpha}{w^3}\cdot\rho^2\sin\phi\ d\phi\ d\rho\ d\theta\\
&= -2\pi\int_0^a\int_0^\pi\frac{Gm\delta}{w^3}(c - \rho\cos\phi)\cdot\rho^2\sin\phi\ d\phi\ d\rho = -2\pi\int_0^a\int_{c-\rho}^{c+\rho}\frac{Gm\delta}{w^3}\left(c - \frac{\rho^2 + c^2 - w^2}{2c}\right)\cdot\frac{\rho w}{c}\ dw\ d\rho\\
&= -2\pi\int_{\rho=0}^{a}\int_{w=c-\rho}^{c+\rho}\frac{Gm\delta}{w^3}\cdot\frac{2c^2 - \rho^2 - c^2 + w^2}{2c}\cdot\rho\cdot\frac{w}{c}\ dw\ d\rho
\end{aligned}$$

$$= -\pi Gm\delta \int_0^a \int_{c-\rho}^{c+\rho} \frac{1}{w^2} \cdot \frac{c^2 - \rho^2 + w^2}{c^2} \cdot \rho \, dw \, d\rho = -\frac{\pi Gm\delta}{c^2} \int_0^a \int_{c-\rho}^{c+\rho} \left(\frac{c^2 - \rho^2}{w^2} + 1 \right) \cdot \rho \, dw \, d\rho$$

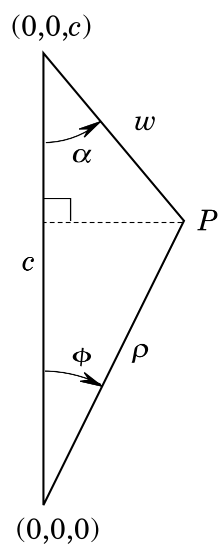
$$= -\frac{\pi Gm\delta}{c^2} \int_0^a \rho \cdot \left[\frac{\rho^2 - c^2}{w} + w \right]_{w=c-\rho}^{c+\rho} d\rho = -\frac{\pi Gm\delta}{c^2} \int_0^a \rho \cdot \left(\frac{\rho^2 - c^2}{\rho + c} + c + \rho + \frac{c^2 - \rho^2}{c - \rho} - c + \rho \right) d\rho$$

$$= -\frac{\pi Gm\delta}{c^2} \int_0^a \rho(\rho - c + \rho + c + c + \rho - c + \rho) \, d\rho = -\frac{\pi Gm\delta}{c^2} \int_0^a 4\rho^2 \, d\rho = -\frac{\pi Gm\delta}{c^2} \cdot \left[\frac{4}{3}\rho^3 \right]_0^a$$

$$= -\frac{\pi Gm\delta}{c^2} \cdot \frac{4}{3}a^3 = -\frac{4}{3}\pi\delta a^3 \cdot \frac{Gm}{c^2} = -\frac{GMm}{c^2}.$$

Magnificent! You can even extend this result to show that if the density of the ball varies only as a function of ρ, then the same conclusion follows: The ball acts, for purposes of gravitational attraction of an external mass m, as if all its mass M were concentrated at its center. And note one additional item of interest: This is one of the extremely rare spherical triple integrals *not* evaluated in the order $d\rho \, d\phi \, d\theta$.

C14S07.048: The following figure makes some of the equations we use easy to derive.



A mass element $\delta \, dV$ located at the point $P(\rho, \phi, \theta)$ of the ball exerts a force on the mass m at $(0, 0, c)$ that has vertical component

$$dF = \frac{Gm\delta \cos\alpha}{w^2} \, dV.$$

Note the following:

$$M = \frac{4}{3}\pi\delta a^3;$$

$$w^2 = \rho^2 + c^2 - 2\rho c \cos\phi;$$

$$2w \, dw = 2\rho c \sin\phi \, d\phi;$$

$$w \cos\alpha + \rho \cos\phi = c;$$

$$\rho \cos \phi = \frac{\rho^2 + c^2 - w^2}{2c};$$

$$\rho \sin \phi \, d\phi = \frac{w}{c} \, dw.$$

Hence the total force exerted by the ball on m is

$$F = \int_{\theta=0}^{2\pi} \int_{\rho=a}^{b} \int_{\phi=0}^{\pi} \frac{Gm\delta \cos \alpha}{w^2} \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta = \int_0^{2\pi} \int_a^b \int_0^{\pi} \frac{Gm\delta w \cos \alpha}{w^3} \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

$$= 2\pi \int_a^b \int_0^{\pi} \frac{Gm\delta}{w^3} (c - \rho \cos \phi) \cdot \rho^2 \sin \phi \, d\phi \, d\rho = 2\pi \int_a^b \int_{\rho-c}^{c+\rho} \frac{Gm\delta}{w^3} \left(c - \frac{\rho^2 + c^2 - w^2}{2c} \right) \cdot \frac{\rho w}{c} \, dw \, d\rho$$

$$= 2\pi \int_{\rho=a}^{b} \int_{w=\rho-c}^{c+\rho} \frac{Gm\delta}{w^3} \cdot \frac{2c^2 - \rho^2 - c^2 + w^2}{2c} \cdot \rho \cdot \frac{w}{c} \, dw \, d\rho$$

$$= \pi G m \delta \int_a^b \int_{\rho-c}^{c+\rho} \frac{1}{w^2} \cdot \frac{c^2 - \rho^2 + w^2}{c^2} \cdot \rho \, dw \, d\rho = \frac{\pi G m \delta}{c^2} \int_a^b \int_{\rho-c}^{c+\rho} \left(\frac{c^2 - \rho^2}{w^2} + 1 \right) \cdot \rho \, dw \, d\rho$$

$$= \frac{\pi G m \delta}{c^2} \int_a^b \rho \cdot \left[\frac{\rho^2 - c^2}{w} + w \right]_{w=\rho-c}^{c+\rho} d\rho = \frac{\pi G m \delta}{c^2} \int_a^b \rho \cdot \left(\frac{\rho^2 - c^2}{\rho + c} + c + \rho - \frac{\rho^2 - c^2}{\rho - c} + c - \rho \right) d\rho$$

$$= \frac{\pi G m \delta}{c^2} \int_a^b \rho(\rho - c + \rho + c - \rho - c + c - \rho) \, d\rho = \frac{\pi G m \delta}{c^2} \int_a^b 0 \, d\rho = 0.$$

The key difference between this derivation and that in the solution of Problem 47 is that the lower limit of integration on w is here $\rho - c$ rather than $c - \rho$. The reason for the change is that here we have $c \leqq \rho$, so that when $\phi = 0$ we have $w^2 = (\rho - c)^2$ and thus $w = \rho - c$.

C14S07.049: A *Mathematica* solution:

```
r = 6370*1000;    k = 0.371;

d1 = 11000;    d2 = 5000;

m1 = (4/3)*Pi*d1*x∧3;

m2 = (4/3)*Pi*d2*(r∧3 - x∧3);

m = m1 + m2;

i1 = (2/5)*m1*x∧2;
```

By Problem 42, the moment of inertia of the mantle with respect to the polar axis (through the poles of the planet) is

```
i2 = (2/5)*m2*(r∧5 - x∧5)/(r∧3 - x∧3);
```

Therefore we proceed to solve for x as follows.

```
i0 = i1 + i2;

eq1 = i0 == k*m*r∧2;
```

```
soln = NSolve[ eq1, x ];
```

We suppress the output, but there are five solutions. Only two are real and positive,

$$x_1 \approx 2.76447 \times 10^6 \quad \text{and} \quad x_2 \approx 5.87447 \times 10^6.$$

We are given the information that the mantle is "a few thousand kilometers thick," and x_2 does not satisfy this condition, as it implies that the mantle is less than 496 km thick. We conclude that the radius of the core is $x_1/1000$ km, and hence that the thickness of the mantle is $(r - x_1)/1000 \approx 3605.53$ km. —C.H.E.

Section 14.8

C14S08.001: The surface area element is

$$dS = \sqrt{1 + 1^2 + 3^2}\ dA = \sqrt{11}\ dA.$$

So the area in question is

$$A = 4\int_{x=0}^{2}\int_{y=0}^{3\sqrt{1-(x/2)^2}} \sqrt{11}\ dy\ dx = \int_0^2 6\sqrt{11}\sqrt{4-x^2}\ dx$$

$$= \left[3x\sqrt{11}\sqrt{4-x^2} + 12\sqrt{11}\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 6\pi\sqrt{11} \approx 62.5169044565658738.$$

C14S08.002: The surface area element is

$$dS = \sqrt{1 + 2^2 + 2^2}\ dA = 3\ dA.$$

So the area in question is

$$A = \int_0^1\int_{x^2}^{\sqrt{x}} 3\ dy\ dx = \int_0^1 (3x^{1/2} - 3x^2)\ dx = \left[2x^{3/2} - x^3\right]_0^1 = 1.$$

C14S08.003: The paraboloid meets the plane in the circle with equation $x^2 + y^2 = 4$, $z = 5$. Let D denote the circular disk $x^2 + y^2 \leqq 4$ in the xy-plane. The surface area element is

$$dS = \sqrt{1 + 4x^2 + 4y^2}\ dA,$$

so the surface area in question is

$$A = \iint_D \sqrt{1+4x^2+4y^2}\ dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r\sqrt{1+4r^2}\ dr\ d\theta = 2\pi\cdot\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2$$

$$= 2\pi\left(\frac{17\sqrt{17}-1}{12}\right) = \frac{1}{6}\pi\left(17\sqrt{17}-1\right) \approx 36.1769031974114084.$$

C14S08.004: The surface area element is

$$dS = \sqrt{1+x^2}\ dA,$$

so the area is

$$A = \int_0^1\int_0^x \sqrt{1+x^2}\ dy\ dx = \int_0^1 x(1+x^2)^{1/2}\ dx = \left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^1 = \frac{1}{3}\left(2\sqrt{2}-1\right) \approx 0.609475708249.$$

C14S08.005: The surface area element is

$$dS = \sqrt{2+4y^2}\ dA,$$

so the area is

$$A = \int_{y=0}^{2} \int_{x=0}^{1} (2+4y^2)^{1/2}\,dx\,dy = \int_{y=0}^{2} (2+4y^2)^{1/2}\,dy = \left[\frac{1}{2}y(2+4y^2)^{1/2} + \frac{1}{2}\operatorname{arcsinh}\left(y\sqrt{2}\,\right)\right]_0^2$$

$$= 3\sqrt{2} + \frac{1}{2}\operatorname{arcsinh}\left(2\sqrt{2}\,\right) = 3\sqrt{2} + \frac{1}{2}\ln\left(3+2\sqrt{2}\,\right) \approx 5.1240142741388282.$$

C14S08.006: The surface area element is $dS = \sqrt{2+4y^2}\,dA$, so the surface area is

$$A = \int_0^1 \int_0^y (2+4y^2)^{1/2}\,dx\,dy = \int_0^1 y(2+4y^2)^{1/2}\,dy = \left[\frac{1}{12}(2+4y^2)^{3/2}\right]_0^1$$

$$= \frac{1}{6}\left(3\sqrt{6} - \sqrt{2}\,\right) \approx 0.9890426199607321.$$

C14S08.007: The surface area element is $dS = \sqrt{14}\,dA$, so the surface area is

$$A = \int_{x=0}^{3} \int_{y=0}^{(6-2x)/3} \sqrt{14}\,dy\,dx = \int_0^3 \frac{1}{3}(6-2x)\sqrt{14}\,dx = \left[2x\sqrt{14} - \frac{1}{3}x^2\sqrt{14}\,\right]_0^3 = 3\sqrt{14}\,.$$

Alternatively, the vectors $\mathbf{u} = \langle -3, 2, 0 \rangle$ and $\mathbf{v} = \langle -3, 0, 6 \rangle$ span two adjacent sides of the triangular surface. So its area is half the magnitude of their cross product:

$$A = \frac{1}{2}|\mathbf{u}\times\mathbf{v}| = \frac{1}{2}|\langle 12, 18, 6 \rangle| = \frac{1}{2}\sqrt{504} = 3\sqrt{14} \approx 11.22497216032182415675.$$

C14S08.008: The surface area element is $dS = \sqrt{14}\,dA$. Hence the surface area is

$$A = \int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{14}\,dr\,d\theta = 2\pi \cdot \left[\frac{1}{2}r^2\sqrt{14}\,\right]_0^{\sqrt{2}} = 2\pi\sqrt{14} \approx 23.5095267170779957.$$

C14S08.009: The surface area element is $dS = \sqrt{1+x^2+y^2}\,dA$, so the surface area is

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r(1+r^2)^{1/2}\,dr\,d\theta = 2\pi \cdot \left[\frac{1}{3}(1+r^2)^{3/2}\right]_0^1 = \frac{2}{3}\pi\left(2\sqrt{2} - 1\right) \approx 3.8294488151512928.$$

C14S08.010: The surface area element is $dS = \sqrt{1+4x^2+4y^2}\,dA$, so the surface area is

$$\int_0^{2\pi} \int_0^2 r(1+4r^2)^{1/2}\,dr\,d\theta = 2\pi \cdot \left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2 = \frac{1}{6}\pi\left(17\sqrt{17} - 1\right) \approx 36.1769031974114084.$$

C14S08.011: The paraboloid meets the xy-plane in the circle with equation $x^2+y^2 = 16$. The surface area element is

$$dS = \sqrt{1+4x^2+4y^2}\,dA,$$

so the area is

$$A = \int_0^{2\pi} \int_0^4 r(1+4r^2)^{1/2}\ dr\ d\theta = 2\pi \cdot \left[\frac{1}{12}(1+4r^2)^{3/2} \right]_0^4 = \frac{1}{6}\pi \left(65\sqrt{65}\ - 1 \right) \approx 273.866639786258.$$

C14S08.012: Let $z(r,\ \theta) = br$. Then the surface area element is

$$dS = \sqrt{r^2 + (rz_r)^2 + (z_\theta)^2}\ dA = r\sqrt{1+b^2}\ dA,$$

so the surface area is

$$A = \int_0^{2\pi} \int_0^a r(1+b^2)^{1/2}\ dr\ d\theta = 2\pi \cdot \left[\frac{1}{2}r^2(1+b^2)^{1/2} \right]_0^a = \pi a\sqrt{a^2 + a^2b^2}\ = \pi a\sqrt{a^2+h^2}\ = \pi aL.$$

Note that when you use Eq. (10), you should see $dr\ d\theta$ where you are accustomed to see $r\ dr\ d\theta$.

C14S08.013: Let $\mathbf{r}(\theta,\ z) = \langle\, a\cos\theta,\ a\sin\theta,\ z\,\rangle$. Then

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta & a\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle\, a\cos\theta,\ a\sin\theta,\ 0\,\rangle$$

and hence

$$|\,\mathbf{r}_\theta \times \mathbf{r}_z\,| = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\ = a.$$

Therefore, by Eq. (8), the area of the zone is

$$A = \int_{\theta=0}^{2\pi} \int_{z=0}^{h} a\ dz\ d\theta = 2\pi \cdot \Big[\, az \,\Big]_{z=0}^{h} = 2\pi ah.$$

C14S08.014: Let $z(r,\ \theta) = \sqrt{a^2 - r^2}$. Then the surface area element in cylindrical coordinates is

$$dS = \sqrt{r^2 + (rz_r)^2 + (z_\theta)^2}\ \ dA = \frac{ar}{\sqrt{a^2 - r^2}}\ dA.$$

Thus by Eq. (10), the area of the zone is

$$A = \int_0^{2\pi} \int_{\sqrt{a^2-c^2}}^{\sqrt{a^2-b^2}} \frac{ar}{\sqrt{a^2-r^2}}\ dr\ d\theta = 2\pi \cdot \Big[-a(a^2-r^2)^{1/2} \Big]_{\sqrt{a^2-c^2}}^{\sqrt{a^2-b^2}} = 2\pi(ac - ab) = 2\pi a(c-b) = 2\pi ah.$$

C14S08.015: Let $z(x,\ y) = \sqrt{a^2 - x^2}$. Then the surface area element in Cartesian coordinates is

$$dS = \frac{a}{\sqrt{a^2 - x^2}}\ dA.$$

Let D be the disk in which the vertical cylinder meets the xy-plane. Then the area of the part of the horizontal cylinder—top *and* bottom—that lies within the vertical cylinder is

$$
\begin{aligned}
A &= 2\iint_D \frac{a}{\sqrt{a^2-x^2}}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{2a}{\sqrt{a^2-r^2\cos^2\theta}}\cdot r\,dr\,d\theta \\
&= \int_{\theta=0}^{2\pi}\left[\frac{-2a(a^2-r^2\cos^2\theta)^{1/2}}{\cos^2\theta}\right]_{r=0}^{a} d\theta = 4\int_{\theta=0}^{\pi/2} 2a^2(\sec^2\theta-\sec\theta\tan\theta)\,d\theta = 8a^2\Big[\tan\theta-\sec\theta\Big]_0^{\pi/2} \\
&= 8a^2+8a^2\cdot\left(\lim_{\theta\to(\pi/2)^-}\big[\tan\theta-\sec\theta\big]\right) = 8a^2+8a^2\cdot 0 = 8a^2.
\end{aligned}
$$

The change in the limits of integration in the second line was necessary because we needed the simplification $(1-\cos^2\theta)^{1/2} = \sin\theta$, which is not valid if $\pi < \theta < 2\pi$; moreover, we needed to avoid the discontinuity of the following improper integral at $\theta = \pi/2$.

C14S08.016: Let $z(r,\theta) = \sqrt{a^2-r^2}$. Then the surface area element in cylindrical coordinates is

$$dS = \sqrt{r^2+(rz_r)^2+(z_\theta)^2}\,dA = \frac{ar}{\sqrt{a^2-r^2}}\,dA.$$

Hence, by Eq. (10), the area of the part of the sphere (top *and* bottom) that lies within the cylinder is

$$
\begin{aligned}
A &= 2\int_0^{\pi}\int_0^{a\sin\theta}\frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = \int_0^{\pi}\Big[-2a(a^2-r^2)^{1/2}\Big]_0^{a\sin\theta} d\theta \\
&= 2\int_0^{\pi/2} 2a^2(1-\cos\theta)\,d\theta = 4a^2\Big[\theta-\sin\theta\Big]_0^{\pi/2} = 4a^2\left(\frac{\pi}{2}-1\right) = 2a^2(\pi-2) \approx (2.283185307180)a^2.
\end{aligned}
$$

The change of limits of integration in the second line is necessary because the important simplification $(a^2-a^2\sin^2\theta)^{1/2} = a\cos\theta$ is not valid on the interval $\frac{1}{2}\pi < \theta < \pi$. The student who fails to notice this will probably obtain the incorrect answer $2\pi a^2$.

C14S08.017: The surface $y = f(x,z)$ is parametrized by

$$\mathbf{r}(x,z) = \langle\, x,\ f(x,z),\ z\,\rangle$$

for (x,z) in the region R in the xz-plane. Then

$$\mathbf{r}_x\times\mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f_x & 0 \\ 0 & f_z & 1 \end{vmatrix} = \langle\, f_x,\ -1,\ f_z\,\rangle.$$

Therefore the area of the surface $y = f(x,z)$ lying "over" the region R is

$$A = \iint_R \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}\,dx\,dz.$$

The surface $x = f(y,z)$ is parametrized by

$$\mathbf{r}(y,z) = \langle\, f(y,z),\ y,\ z\,\rangle$$

for (y,z) in the region R in the xz-plane. Then

$$
\mathbf{r}_y \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_y & 1 & 0 \\ f_z & 0 & 1 \end{vmatrix} = \langle 1,\ -f_y,\ -f_z \rangle.
$$

Therefore the area of the surface $x = f(y, z)$ lying "over" the region R is

$$
A = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\ dy\, dz.
$$

C14S08.018: The equations in (6) take the form

$$
x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad z = a\cos\phi,
$$

and hence the spherical surface corresponding to the region R in the $\phi\theta$-plane is parametrized by

$$
\mathbf{r}(\phi, \theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle.
$$

Thus

$$
\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi \rangle.
$$

Therefore

$$
|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}\ = a^2\sin\phi.
$$

Thus, by Eq. (8), the surface area of the part of the sphere corresponding to R is

$$
A = \iint_R a^2 \sin\phi\ d\phi\ d\theta.
$$

C14S08.019: By Problem 50 of Section 14.2,

$$
A = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} a^2\sin\phi\ d\phi\ d\theta = a^2 \sin\hat{\phi}\ \Delta\phi\ \Delta\theta
$$

for some $\hat{\phi}$ in (ϕ_1, ϕ_2). Therefore

$$
\begin{aligned}
\Delta V &= \int_{\rho_1}^{\rho_2} (\rho^2 \sin\hat{\phi}\ \Delta\phi\ \Delta\theta)\ d\rho = \frac{1}{3}(\rho_2^3 - \rho_1^3)\sin\hat{\phi}\ \Delta\phi\ \Delta\theta \\
&= \frac{1}{3}\cdot\frac{\rho_2^3 - \rho_1^3}{\rho_2 - \rho_1}\sin\hat{\phi}\ \Delta\phi\ \Delta\theta\ \Delta\rho = \frac{1}{3}\cdot 3\hat{\rho}^2\sin\hat{\phi}\ \Delta\phi\ \Delta\theta\ \Delta\rho = \hat{\rho}^2\sin\hat{\phi}\ \Delta\rho\ \Delta\phi\ \Delta\theta
\end{aligned}
$$

for some $\hat{\rho}$ in (ρ_1, ρ_2), and this is Eq. (8) of Section 14.7.

C14S08.020: See the solution of Problem 29 of Section 14.7. The surface area of this pinched torus can be computed (without the second theorem of Pappus) as follows. First parametrize it via

$$\mathbf{r}(\phi,\,\theta) = \langle\, 2a\sin^2\phi\cos\theta,\ 2a\sin^2\phi\sin\theta,\ 2a\sin\phi\cos\phi\,\rangle$$

where $0 \leqq \phi \leqq \pi$ and $0 \leqq \theta \leqq 2\pi$. (To verify that this parametrization is correct, execute the *Mathematica* 3.0 command

```
ParametricPlot3D[ { 2*Sin[u]*Sin[u]*Cos[v], 2*Sin[u]*Sin[u]*Sin[v], 2*Sin[u]*Cos[u] },
                  {u, 0, Pi}, {v, 0, 2*Pi}, AspectRatio → Automatic ];
```

to see a copy of the pinched torus for the case $a = 1$.) Next,

$$\mathbf{r}_\phi = \langle\, 4a\sin\phi\cos\phi\cos\theta,\ 4a\sin\phi\cos\phi\sin\theta,\ 2a\cos^2\phi - 2a\sin^2\phi\,\rangle \quad \text{and}$$

$$\mathbf{r}_\theta = \langle\, -2a\sin^2\phi\sin\theta,\ 2a\sin^2\phi\cos\theta,\ 0\,\rangle.$$

After some lengthy computations, you will find that $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4a^2\sin^2\phi$. Hence the surface area of the pinched torus is

$$A = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 4a^2\sin^2\phi\ d\phi\ d\theta = 2\pi a^2\cdot\Big[2\phi - \sin 2\phi\Big]_0^\pi = 4\pi^2a^2.$$

C14S08.021: Given:

$$x = f(z)\cos\theta, \quad y = f(z)\sin\theta, \quad z = z$$

where $0 \leqq \theta \leqq 2\pi$ and $a \leqq z \leqq b$. The surface thereby generated is thereby parametrized by

$$\mathbf{r}(\theta,\,z) = \langle\, f(z)\cos\theta,\ f(z)\sin\theta,\ z\,\rangle,$$

and thus

$$\mathbf{r}_\theta = \langle\, -f(z)\sin\theta,\ f(z)\cos\theta,\ 0\,\rangle \quad \text{and}$$

$$\mathbf{r}_z = \langle\, f'(z)\cos\theta,\ f'(z)\sin\theta,\ 1\,\rangle.$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -f(z)\sin\theta & f(z)\cos\theta & 0 \\ f'(z)\cos\theta & f'(z)\sin\theta & 1 \end{vmatrix} = \langle\, f(z)\cos\theta,\ f(z)\sin\theta,\ -f(z)\cdot f'(z)\,\rangle$$

and hence

$$|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{[f(z)]^2 + [f(z)\cdot f'(z)]^2} = f(z)\sqrt{1 + [f'(z)]^2}\,.$$

Thus, by Eq. (8), the area of the surface of revolution is

$$A = \int_{\theta=0}^{2\pi} \int_{z=a}^{b} f(z)\sqrt{1 + [f'(z)]^2}\ dz\ d\theta = \int_{z=a}^{b} 2\pi f(z)\sqrt{1 + [f'(z)]^2}\ dz.$$

Compare this with Eq. (8) in Section 6.4.

C14S08.022: Part (a):

$$A = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\phi\ d\phi\ d\theta = 2\pi a^2 \cdot \Big[-\cos\phi\Big]_0^{\pi} = 4\pi a^2.$$

Part (b): The area is

$$A = \int_0^{2\pi} \int_0^{\pi/6} a^2 \sin\phi\ d\phi\ d\theta = 2\pi a^2 \cdot \Big[-\cos\phi\Big]_0^{\pi/6} = \left(2 - \sqrt{3}\,\right)\pi a^2,$$

a little less than 6.6987% of the total surface area of the sphere.

C14S08.023: In the result in Problem 21, take $f(z) = r$ (the constant radius of the cylinder). Then

$$f(z)\sqrt{1 + \big[f'(z)\big]^2} = r,$$

so the curved surface area of the cylinder is

$$A = \int_{\theta=0}^{2\pi} \int_{z=0}^{h} r\ dz\ d\theta = 2\pi \cdot \Big[rz\Big]_{z=0}^{h} = 2\pi rh.$$

C14S08.024: Let $z(x,\,y) = \sqrt{r^2 - x^2}$. Then

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\ dA = \frac{r}{\sqrt{r^2 - x^2}}\ dA.$$

Then Eq. (9) yields the curved surface area of the cylinder:

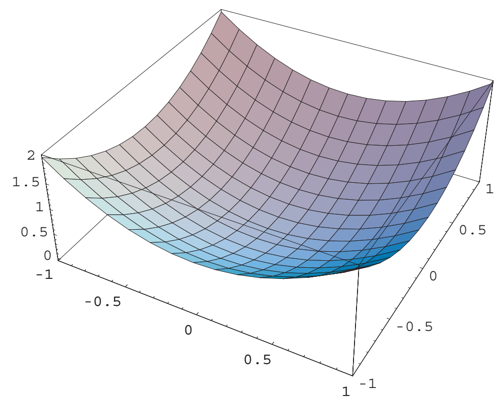
$$\begin{aligned} A &= 4\int_{x=0}^{r} \int_{y=0}^{h} \frac{r}{\sqrt{r^2 - x^2}}\ dy\ dx = 4\int_{x=0}^{r} \left[\frac{ry}{\sqrt{r^2 - x^2}}\right]_{y=0}^{h} dx \\ &= 4\int_0^{r} \frac{rh}{\sqrt{r^2 - x^2}}\ dx = 4 \cdot \left[rh \arctan\left(\frac{x}{\sqrt{r^2 - x^2}}\right)\right]_{x=0}^{r} = 4 \cdot \frac{\pi rh}{2} = 2\pi rh. \end{aligned}$$

Why the factor 4? We had to double twice; once because the integral gives only the area of the top of the cylinder, once again because we integrated only over the interval $0 \leqq x \leqq r$ rather than the interval $-r \leqq x \leqq r$.

C14S08.025: Part (a): The *Mathematica* 3.0 command

```
Plot3D[ x∗x + y∗y, {x, -1, 1}, {y, -1, 1}, AspectRatio → Automatic ];
```

produced the view of the surface that is shown next.



The surface area element is $dS = \sqrt{1+4x^2+4y^2}\ dA$, so the area of the surface is

$$
\begin{aligned}
A &= \int_{x=-1}^{1}\int_{y=-1}^{1} \sqrt{1+4x^2+4y^2}\ dy\ dx \\
&= \int_{-1}^{1}\left[\frac{1}{2}y\sqrt{1+4x^2+4y^2}+\frac{1}{4}(4x^2+1)\ln\left(2y+\sqrt{1+4x^2+4y^2}\,\right)\right]_{-1}^{1} dx \\
&= \int_{-1}^{1}\left[\sqrt{4x^2+5}-\frac{1}{4}(4x^2+1)\ln\left(-2+\sqrt{4x^2+5}\,\right)+\frac{1}{4}(4x^2+1)\ln\left(2+\sqrt{4x^2+5}\,\right)\right] dx \\
&= \left[\frac{2}{3}x\sqrt{4x^2+5}\ +\frac{7}{6}\operatorname{arcsinh}\left(\frac{2x}{\sqrt{5}}\right)-\frac{1}{6}\arctan\left(\frac{4x}{\sqrt{4x^2+5}}\right)\right. \\
&\qquad\left. +\frac{1}{12}x(4x^2+3)\ln\left(-2+\sqrt{4x^2+5}\,\right)+\frac{1}{12}x(4x^2+3)\ln\left(2+\sqrt{4x^2+5}\,\right)\right]_{-1}^{1} \\
&= 4+\frac{7}{3}\operatorname{arcsinh}\left(\frac{2\sqrt{5}}{5}\right)-\frac{1}{3}\arctan\left(\frac{4}{3}\right)+\frac{7}{6}\ln 5 \ \approx\ 7.44625672301236346326.
\end{aligned}
$$

Part (b): By symmetry, we integrate over the quarter of the square that lies in the first quadrant and multiply by 4. Thus the area is

$$
\begin{aligned}
A &= 4\int_{x=0}^{1}\int_{y=0}^{1-x} \sqrt{1+4x^2+4y^2}\ dy\ dx \\
&= \int_{0}^{1}\left[2y\sqrt{1+4x^2+4y^2}+(4x^2+1)\ln\left(2y+\sqrt{1+4x^2+4y^2}\,\right)\right]_{0}^{1-x} dx \\
&= \int_{0}^{1}\left[2(1-x)\sqrt{1+4(1-x)^2+4x^2}-(1+4x^2)\ln\left(\sqrt{1+4x^2}\,\right)\right. \\
&\qquad\left. +(1+4x^2)\ln\left(2(1-x)+\sqrt{1+4(1-x)^2+x^2}\,\right)\right] dx \\
&= \left[\frac{1}{3}(4x-2x^2-1)\sqrt{8x^2-8x+5}\ +\frac{5\sqrt{2}}{6}\operatorname{arcsinh}\left(\frac{\sqrt{6}}{3}[2x-1]\right)\right.
\end{aligned}
$$

$$- \frac{1}{6} \arctan \left(\frac{72 + 288x^2 - 25(4x+1)\sqrt{8x^2 - 8x + 5}}{184x^2 - 200x - 29} \right)$$

$$+ \frac{1}{6} \arctan \left(\frac{-72 - 288x^2 + 25(4x+1)\sqrt{8x^2 - 8x + 5}}{184x^2 - 200x - 29} \right)$$

$$- \frac{1}{3} x(4x^2+3) \ln \left(\sqrt{1+4x^2} \right) + \frac{1}{3} x(4x^2+3) \ln \left(2 - 2x + \sqrt{8x^2 - 8x + 5} \right) \Bigg]_0^1$$

$$= \frac{2\sqrt{5}}{3} + \frac{5\sqrt{2}}{3} \operatorname{arcsinh} \left(\frac{\sqrt{6}}{3} \right) - \frac{1}{6} \arctan \left(\frac{72 - 25\sqrt{5}}{71} \right) - \frac{1}{6} \arctan \left(\frac{72 - 25\sqrt{5}}{29} \right)$$

$$+ \frac{1}{6} \arctan \left(\frac{25\sqrt{5} - 72}{29} \right) + \frac{1}{6} \arctan \left(\frac{25\sqrt{5} - 72}{71} \right) \approx 3.0046254342814410.$$

Of course it was *Mathematica* 3.0 that computed and evaluated the antiderivatives in this solution.

C14S08.026: Part (a): The surface area element is

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\ dA = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\ dA = \sqrt{2}\ dA.$$

Hence the area of the surface is

$$\int_{x=-1}^{1} \int_{y=-1}^{1} \sqrt{2}\ dy\ dx = 4\sqrt{2} \approx 5.6568542494923802.$$

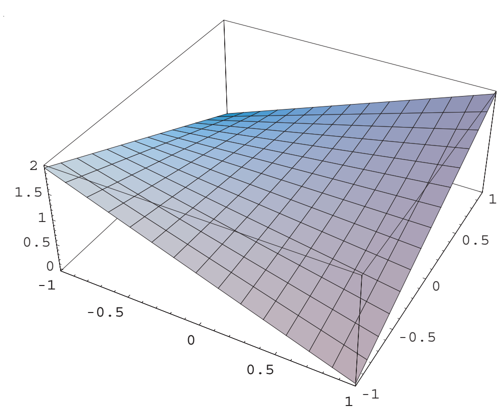
Part (b): By symmetry, we integrate over the quarter of the square that lies in the first quadrant, then multiply by 4. The area is thus

$$4 \int_{x=0}^{1} \int_{y=0}^{1-x} \sqrt{2}\ dy\ dx = \frac{1}{2} \cdot 4\sqrt{2} = 2\sqrt{2} \approx 2.8284271247461901.$$

In both cases we integrated the constant function by multiplying its value by the area of the domain of the integral.

C14S08.027: Part (a): The following graph of the surface was generated by the *Mathematica* 3.0 command

```
Plot3D[ 1 + x*y, {x, -1, 1}, {y, -1, 1} ];
```



The surface area element is $dS = \sqrt{1 + x^2 + y^2}\ dA$. Hence the surface area is

$$\int_{x=-1}^{1} \int_{y=-1}^{1} \sqrt{1+x^2+y^2}\ dy\,dx = \int_{-1}^{1} \left[\frac{1}{2} y \sqrt{1+x^2+y^2}\ + \frac{1}{2}(1+x^2) \ln \left(y + \sqrt{1+x^2+y^2}\ \right) \right]_{-1}^{1} dx$$

$$= \int_{-1}^{1} \left[\sqrt{2+x^2} - \frac{1}{2}(1+x^2) \ln \left(-1 + \sqrt{2+x^2}\ \right) + \frac{1}{2}(1+x^2) \ln \left(1 + \sqrt{2+x^2}\ \right) \right] dx$$

$$= \left[\frac{2}{3} x \sqrt{2+x^2}\ + \frac{4}{3} \operatorname{arcsinh} \left(\frac{x\sqrt{2}}{2} \right) - \frac{2}{3} \arctan \left(\frac{x}{\sqrt{2+x^2}} \right) \right.$$

$$\left. - \frac{1}{6} x(3+x^2) \ln \left(-1 + \sqrt{2+x^2}\ \right) + \frac{1}{6} x(3+x^2) \ln \left(1 + \sqrt{2+x^2}\ \right) \right]_{-1}^{1}$$

$$= \frac{4\sqrt{3}}{3} - \frac{2}{9}\pi + \frac{8}{3} \operatorname{arcsinh} \left(\frac{\sqrt{2}}{2} \right) - \frac{4}{3} \ln \left(-1 + \sqrt{3}\ \right) + \frac{4}{3} \ln \left(1 + \sqrt{3}\ \right) \approx 5.123157101094.$$

Part (b): Using symmetry, we integrate over the quarter of the square in the first quadrant and multiply the answer by 4. Thus the surface area is

$$4 \int_{x=0}^{1} \int_{y=0}^{1-x} \sqrt{1+x^2+y^2}\ dy\,dx = \int_{0}^{1} \left[2y\sqrt{1+x^2+y^2}\ + 2(1+x^2) \ln \left(y + \sqrt{1+x^2+y^2}\ \right) \right]_{0}^{1} dx$$

$$= \int_{0}^{1} \left[2(1-x)\sqrt{2x^2-2x+2}\ - 2(1+x^2) \ln \left(\sqrt{1+x^2}\ \right) \right.$$

$$\left. + 2(1+x^2) \ln \left(1 - x + \sqrt{2x^2-2x+2}\ \right) \right] dx$$

$$= \left[\frac{1}{3}(4x - 2x^2 - 1)\sqrt{2x^2-2x+2}\ + \frac{7\sqrt{2}}{6} \operatorname{arcsinh} \left(\frac{2x-1}{\sqrt{3}} \right) + \frac{4}{3} \arctan \left(\frac{\sqrt{2x^2-2x+2}}{x+1} \right) \right.$$

$$\left. - \frac{2}{3} x(3+x^2) \ln \left(\sqrt{1+x^2}\ \right) + \frac{2}{3} x(3+x^2) \ln \left(1 - x + \sqrt{2x^2-2x+2}\ \right) \right]_{0}^{1}$$

$$= \frac{2\sqrt{2}}{3} + \frac{7\sqrt{2}}{3} \operatorname{arcsinh} \left(\frac{\sqrt{3}}{3} \right) + \frac{4}{3} \arctan \left(\frac{\sqrt{2}}{2} \right) - \frac{4}{3} \arctan \left(\sqrt{2}\ \right) \approx 2.302310960471.$$

C14S08.028: The surface area element is

$$dS = \sqrt{1 + \frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2}}\ dA = \frac{2}{\sqrt{4-x^2-y^2}}\ dA.$$

Part (a): The surface area is *double* the integral of dS because half of the surface is above the xy-plane and half is below. Thus the area is

$$\int_{x=-1}^{1}\int_{y=-1}^{1}\frac{4}{\sqrt{4-x^2-y^2}}\,dy\,dx=\int_{-1}^{1}\left[4\arctan\left(\frac{y}{\sqrt{4-x^2-y^2}}\right)\right]_{-1}^{1}\,dx$$

$$=\int_{-1}^{1}8\arctan\left(\frac{1}{\sqrt{3-x^2}}\right)\,dx$$

$$=\left[8\arcsin\left(\frac{x}{\sqrt{3}}\right)+8x\arctan\left(\frac{1}{\sqrt{3-x^2}}\right)+8\arctan\left(\frac{3-2x}{\sqrt{3-x^2}}\right)\right.$$

$$\left.-8\arctan\left(\frac{3+2x}{\sqrt{3-x^2}}\right)\right]_{-1}^{1}$$

$$=16\arcsin\left(\frac{\sqrt{3}}{3}\right)+32\arctan\left(\frac{\sqrt{2}}{2}\right)-16\arctan\left(\frac{5\sqrt{2}}{2}\right)\approx 8.8205695749204929.$$

Part (b): Using symmetry, we integrate dS over the quarter of the square in the first quadrant, multiply by 4, then multiply by 2 as well because half of the surface is above the xy-plane and half is below. Thus the area is

$$8\int_{x=0}^{1}\int_{y=0}^{1-x}\frac{2}{\sqrt{4-x^2-y^2}}\,dy\,dx=\int_{0}^{1}\left[16\arctan\left(\frac{y}{\sqrt{4-x^2-y^2}}\right)\right]_{0}^{1-x}\,dx$$

$$=\int_{0}^{1}16\arctan\left(\frac{1-x}{\sqrt{3+2x-2x^2}}\right)\,dx$$

$$=\left[8\sqrt{2}\,\arcsin\left(\frac{2x-1}{\sqrt{7}}\right)+16x\arctan\left(\frac{1-x}{\sqrt{3+2x-2x^2}}\right)\right.$$

$$\left.-16\arctan\left(\frac{5-3x}{\sqrt{3+2x-2x^2}}\right)-16\arctan\left(\frac{5x+1}{3\sqrt{3+2x-2x^2}}\right)\right]_{0}^{1}$$

$$=16\sqrt{2}\,\arcsin\left(\frac{1}{\sqrt{7}}\right)+16\arctan\left(\frac{1}{3\sqrt{3}}\right)-32\arctan\left(\frac{2}{\sqrt{3}}\right)+16\arctan\left(\frac{5}{\sqrt{3}}\right)$$

$$\approx 4.1831896510064093986700437193627322 66.$$

C14S08.029: We readily verify that x, y, and z satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

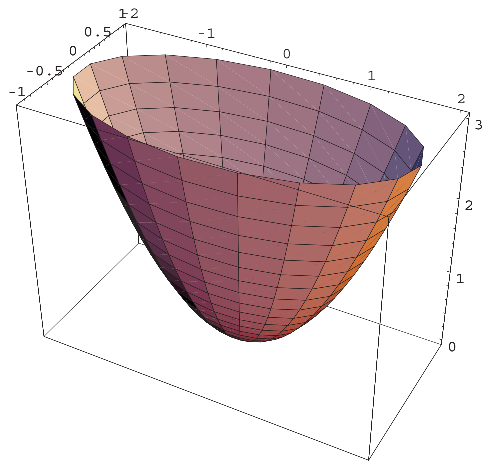
of an elliptic paraboloid. For a typical graph, we executed the *Mathematica* commmands

```
a = 2; b = 1; c = 3;

x = a*u*Cos[v]; y = b*u*Sin[v]; z = c*u∧2;

ParametricPlot3D[ { x, y, z }, { u, 0, 1 }, { v, 0, 2*Pi } ];
```

and the resulting graph is next. —C.H.E.



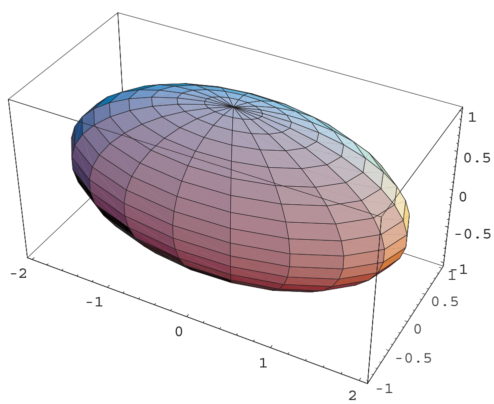
C14S08.030: We readily verify that x, y, and z satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

of an ellipsoid. For a typical example, we executed the *Mathematica* commands

```
a = 2; b = 1; c = 1;
x = a*Sin[u]*Cos[v]; y = b*Sin[u]*Sin[v]; z = c*Cos[u];
ParametricPlot3D[ { x, y, z }, { u, 0, Pi }, { v, 0, 2*Pi } ];
```

and the result is shown next. —C.H.E.



C14S08.031: We readily verify that x, y, and z satisfy the equation

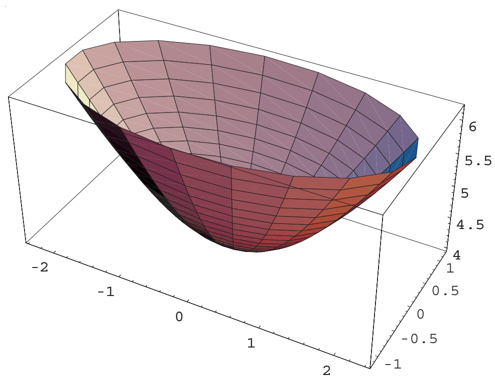
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

of a hyperboloid of two sheets. To see the upper half of a typical example, we executed the following *Mathematica* commands:

```
a = 2; b = 1; c = 4;
```

```
x = a*Sinh[u]*Cos[v]; y = b*Sinh[u]*Sin[v]; z = c*Cosh[u];
ParametricPlot3D[ { x, y, z }, { u, 0, 1 }, { v, 0, 2*Pi } ];
```

and the result is next. —C.H.E.



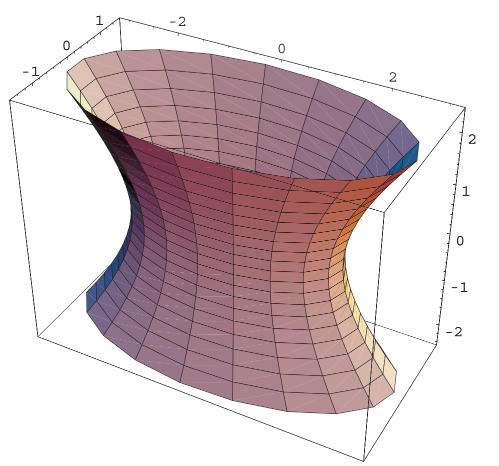
C14S08.032: It's easy to verify that x, y, and z satislfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

of a hyperboloid of one sheet. To graph a typical example, we executed the *Mathematica* commands

```
a = 2; b = 1; c = 2;
x = a*Cosh[u]*Cos[v]; y = b*Cosh[u]*Sin[v]; z = c*Sinh[u];
ParametricPlot3D[ { x, y, z }, { u, -1, 1 }, { v, 0, 2*Pi } ];
```

and the result is shown next. —C.H.E.



C14S08.033: The ellipsoid is parametrized via

$$\mathbf{r}(\phi,\,\theta) = \langle 4\sin\phi\cos\theta,\ 3\sin\phi\sin\theta,\ 2cos\phi\rangle.$$

Then

$$\mathbf{r}_\phi = \langle\, 4\cos\phi\cos\theta,\ 3\cos\phi\sin\theta,\ -2\sin\phi\,\rangle \qquad \text{and}$$

$$\mathbf{r}_\theta = \langle\, -4\sin\phi\sin\theta,\ 3\sin\phi\cos\theta,\ 0\,\rangle.$$

It now follows that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle\, 6\sin^2\phi\cos\theta, 8\sin^2\phi\sin\theta,\ 12\sin\phi\cos\phi\,\rangle,$$

so that

$$|\,\mathbf{r}_\phi \times \mathbf{r}_\theta\,| = \sqrt{36\sin^4\phi\cos^2\theta + 144\sin^2\phi\cos^2\phi + 64\sin^4\phi\sin^2\theta}\,.$$

We used the `NIntegrate` command in *Mathematica* 3.0 to approximate the surface area of the ellipsoid; it is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} (36\sin^4\phi\cos^2\theta + 144\sin^2\phi\cos^2\phi + 64\sin^4\phi\sin^2\theta)^{1/2}\,d\phi\,d\theta \ \approx\ 111.545774984838.$$

C14S08.034: Part (a): Begin with the ellipse

$$\left(\frac{x-b}{a}\right)^2 + \left(\frac{z}{c}\right)^2 \ =\ 1,$$

and note that this equation is satisfied if

$$\frac{x-b}{a} \ =\ \cos\psi \qquad \text{and} \qquad \frac{z}{c} \ =\ \sin\psi.$$

With the aid of a figure much like Fig. 14.8.13, we see that the ellipsoidal torus generated by rotation of this ellipse around the z-axis is parametrized by

$$x = (b + a\cos\psi)\cos\theta, \quad y = (b + a\cos\psi)\sin\theta, \quad z = c\sin\psi$$

where $0 \leqq \theta \leqq 2\pi$ and $0 \leqq \psi \leqq 2\pi$.

Part (b): Parametrize the surface of revolution via $\mathbf{r}(\psi,\,\theta)$ where the components of $\mathbf{r}$ are those given in Part (a), but using $a = 2$, $b = 3$, and $c = 1$. Then

$$\mathbf{r}_\psi = \langle\, -2\sin\psi\cos\theta,\ -2\sin\psi\sin\theta,\ \cos\psi\,\rangle;$$

$$\mathbf{r}_\theta = \langle\, -(3 + 2\cos\psi)\sin\theta,\ (3 + 2\cos\psi)\cos\theta,\ 0\,\rangle;$$

$$\mathbf{r}_\psi \times \mathbf{r}_\theta = \langle\, -(3 + 2\cos\psi)\cos\psi\cos\theta,\ -(3 + 2\cos\psi)\cos\psi\sin\theta,\ -2(3\sin\psi + \sin 2\psi)\,\rangle;$$

$$|\,\mathbf{r}_\psi \times \mathbf{r}_\theta\,| = \left[\frac{(5 - 3\cos 2\psi)(3 + 2\cos\psi)^2}{2}\right]^{1/2}.$$

It follows that the surface area of the ellipsoid is

$$A = \int_{\theta=0}^{2\pi}\int_{\psi=0}^{2\pi}\left[\frac{(5 - 3\cos 2\psi)(3 + 2\cos\psi)^2}{2}\right]^{1/2} d\psi\,d\theta = 2\pi\int_0^{2\pi}\left[\frac{(5 - 3\cos 2\psi)(3 + 2\cos\psi)^2}{2}\right]^{1/2} d\psi.$$

The command

```
2*Pi*NIntegrate[ Sqrt[ (1/2)*(5 - 3*Cos[2*t])*(3 + 2*Cos[t])∧2 ],
                  {t, 0, 2*Pi}, WorkingPrecision → 32 ]
```

in *Mathematica* 3.0 returns the approximation $A \approx 182.622946526146$.

Part (c): Let

$$f(x) = \left[1 - \frac{(x-3)^2}{4}\right]^{1/2}.$$

The graph of f is the top half of the ellipse. Hence the length of the ellipse is

$$L = 2\int_1^5 \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_1^5 \left[\frac{3x^2 - 18x + 11}{x^2 - 6x + 5}\right]^{1/2} dx.$$

Note that this integral is improper at each endpoint of the interval $[1, 5]$. A *Mathematica* 3.0 command similar to the one in Part (b) yields the approximation $L \approx 9.688448220548$.

Section 14.9

C14S09.001: It is easy to solve the given equations for

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Hence

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

C14S09.002: It is easy to solve the given equations for

$$x = \frac{u+2v}{7}, \quad y = \frac{v-3u}{7}.$$

Thus

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{7} & \dfrac{2}{7} \\ -\dfrac{3}{7} & \dfrac{1}{7} \end{vmatrix} = \frac{1}{7}.$$

C14S09.003: When we solve the equations $u = xy$ and $v = y/x$ for x and y, we find that there are two solutions:

$$x = \sqrt{\frac{u}{v}}, \quad y = \sqrt{uv} \qquad \text{and} \qquad x = -\sqrt{\frac{u}{v}}, \quad y = -\sqrt{uv}\,.$$

It doesn't matter which we choose; the value of the Jacobian will be the same. (Why?) So we choose the first solution. Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{2u^{1/2}v^{1/2}} & -\dfrac{u^{1/2}}{2v^{3/2}} \\ \dfrac{v^{1/2}}{2u^{1/2}} & \dfrac{u^{1/2}}{2v^{1/2}} \end{vmatrix} = \frac{1}{2v}.$$

C14S09.004: When we solve the equations $u = 2(x^2 + y^2)$, $v = 2(x^2 - y^2)$ for x and y, we get four solutions—all possible combinations of

$$x = \pm\frac{\sqrt{u+v}}{2}, \qquad y = \pm\frac{\sqrt{u-v}}{2}.$$

We choose the solution for which x and y are both nonnegative. Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{4\sqrt{u+v}} & \dfrac{1}{4\sqrt{u+v}} \\ \dfrac{1}{4\sqrt{u-v}} & -\dfrac{1}{4\sqrt{u-v}} \end{vmatrix} = -\frac{1}{8\sqrt{u^2-v^2}}.$$

C14S09.005: When we solve the equations $u = x + 2y^2$, $v = x - 2y^2$ for x and y, we get two solutions:

$$x = \frac{u+v}{2}, \qquad y = \pm\frac{\sqrt{u-v}}{2}.$$

We choose the solution for which y is nonnegative. Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \\ \dfrac{1}{4\sqrt{u-v}} & -\dfrac{1}{4\sqrt{u-v}} \end{vmatrix} = -\frac{1}{4\sqrt{u-v}}.$$

C14S09.006: Given

$$u = \frac{2x}{x^2+y^2}, \quad v = -\frac{2y}{x^2+y^2}, \tag{1}$$

note first that

$$u^2+v^2 = \frac{4x^2+4y^2}{(x^2+y^2)^2} = \frac{4}{x^2+y^2},$$

so that

$$x^2+y^2 = \frac{4}{u^2+v^2}. \tag{2}$$

Therefore, using the equations in (1), then Eq. (2), we have

$$x = \frac{1}{2}u(x^2+y^2) = \frac{2u}{u^2+v^2} \quad \text{and} \quad y = -\frac{1}{2}v(x^2+y^2) = -\frac{2v}{u^2+v^2}.$$

Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{2(v^2-u^2)}{(u^2+v^2)^2} & -\dfrac{4uv}{(u^2+v^2)^2} \\ \\ \dfrac{4uv}{(u^2+v^2)^2} & \dfrac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix} = \frac{4}{(u^2+v^2)^2}.$$

C14S09.007: First we solve the equations $u = x+y$ and $v = 2x-3y$ for

$$x = \frac{3u+v}{5}, \qquad y = \frac{2u-v}{5}.$$

Substitution in the equation $x+y=1$ then yields

$$1 = \frac{3u+v}{5} + \frac{2u-v}{5} = \frac{5u}{5} = u.$$

Similarly, $x+y=2$ yields $u=2$, $2x-3y=2$ yields $v=2$, and $2x-3y=5$ yields $v=5$. Moreover,

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{3}{5} & \dfrac{1}{5} \\ \\ \dfrac{2}{5} & -\dfrac{1}{5} \end{vmatrix} = -\frac{1}{5}.$$

Therefore

$$\iint_R 1\,dy\,dx = \int_{v=2}^{5}\int_{u=1}^{2} \frac{1}{5}\,du\,dv = 3\cdot 1\cdot \frac{1}{5} = \frac{3}{5}.$$

Note: Because R is a parallelogram with adjacent sides represented by the two vectors $\mathbf{a} = \langle \frac{3}{5}, \frac{2}{5}, 0 \rangle$ and $\mathbf{b} = \langle \frac{3}{5}, -\frac{3}{5}, 0 \rangle$, we have the following alternative method of finding the area A of R:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5} & \frac{2}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5} & 0 \end{vmatrix} = \left\langle 0,\ 0,\ -\frac{3}{5} \right\rangle,$$

and therefore $A = |\mathbf{a}\times\mathbf{b}| = \dfrac{3}{5}$.

C14S09.008: Given $u = xy$ and $v = \dfrac{y}{x}$, we have

$$uv = xy\cdot\frac{y}{x} = y^2 \quad \text{and} \quad \frac{u}{v} = xy\cdot\frac{x}{y} = x^2,$$

and thus we choose

$$x = \sqrt{\frac{u}{v}} \quad \text{and} \quad y = \sqrt{uv}\,. \tag{1}$$

Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{2u^{1/2}v^{1/2}} & -\dfrac{u^{1/2}}{2v^{3/2}} \\ \dfrac{v^{1/2}}{2u^{1/2}} & \dfrac{u^{1/2}}{2v^{1/2}} \end{vmatrix} = \frac{1}{2v}.$$

Also, if $y = x$, then substitution of the equations in (1) yields

$$(uv)^{1/2} = \frac{u^{1/2}}{v^{1/2}}; \quad uv = \frac{u}{v}; \quad v^2 = 1.$$

So we choose $v = 1$. (This choice implies that if we have a similar choice with u, we must choose $u > 0$ because of the equations in (1).) Similarly, $y = 2x$ yields $v = 2$, $xy = 1$ yields $u = 1$, and $xy = 2$ yields $u = 2$. Hence the area of the region of Fig. 14.9.7 is

$$A = \iint_R 1\,dx\,dy = \int_{v=1}^{2}\int_{u=1}^{2} \frac{1}{2v}\,du\,dv = \int_1^2 \frac{1}{2v}\,dv = \frac{1}{2}\ln 2 \approx 0.3465735902799727.$$

C14S09.009: If $u = xy$ and $v = xy^3$, then

$$uy^2 = xy^3 = v, \quad \text{so that} \quad y^2 = \frac{v}{u}; \quad y = \frac{v^{1/2}}{u^{1/2}}.$$

Then

$$x = \frac{u}{y} = u \cdot \frac{u^{1/2}}{v^{1/2}} = \frac{u^{3/2}}{v^{1/2}}.$$

(We do not need the solution in which x and y are negative.) Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{3u^{1/2}}{2v^{1/2}} & -\dfrac{u^{3/2}}{2v^{3/2}} \\ & \\ -\dfrac{v^{1/2}}{2u^{3/2}} & \dfrac{1}{2u^{1/2}v^{1/2}} \end{vmatrix} = \frac{3}{4v} - \frac{1}{4v} = \frac{1}{2v}.$$

We also find by substitution that $xy = 2$ corresponds to $u = 2$, $xy = 4$ corresponds to $u = 4$, $xy^3 = 3$ corresponds to $v = 3$, and $xy^3 = 6$ corresponds to $v = 6$. Hence the area of the region shown in Fig. 14.9.8 is

$$A = \iint_D 1\;dx\;dy = \int_{v=3}^{6}\int_{u=2}^{4} \frac{1}{2v}\;du\;dv = \int_3^6 \frac{1}{v}\;dv = \ln 2 \approx 0.6931471805599453.$$

C14S09.010: If $y = ux^2$ and $x = vy^2$, then

$$y = uv^2y^4; \quad y^3 = \frac{1}{uv^2}; \quad y = \frac{1}{u^{1/3}v^{2/3}}.$$

Then it follows that

$$x = vy^2 = \frac{v}{u^{2/3}v^{4/3}} = \frac{1}{u^{2/3}v^{1/3}}.$$

Next,

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} -\dfrac{2}{3u^{5/3}v^{1/3}} & -\dfrac{1}{3u^{2/3}v^{4/3}} \\ & \\ -\dfrac{1}{3u^{4/3}v^{2/3}} & -\dfrac{2}{3u^{1/3}v^{5/3}} \end{vmatrix} = \frac{1}{3u^2v^2}.$$

Next, $y = x^2$ corresponds to $u = 1$, $y = 2x^2$ corresponds to $u = 2$, $x = y^2$ corresponds to $v = 1$, and $x = 4y^2$ corresponds to $v = 4$. Therefore the area of the region shown in Fig. 14.9.9 is

$$A = \iint_R 1\;dx\;dy = \int_{v=1}^{4}\int_{u=1}^{2} \frac{1}{3u^2v^2}\;du\;dv = \int_{v=1}^{4}\left[-\frac{1}{3uv^2}\right]_{u=1}^{2} dv = \int_1^4 \frac{1}{6v^2}\;dv = \left[-\frac{1}{6v}\right]_1^4 = \frac{1}{8}.$$

C14S09.011: Given: the region R bounded by the curves $y = x^3$, $y = 2x^3$, $x = y^3$, and $x = 4y^3$. Choose u and v so that $y = ux^3$ and $x = vy^3$. Then

$$y = uv^3y^9; \qquad y^8 = \frac{1}{uv^3};$$
$$y = \frac{1}{u^{1/8}v^{3/8}}; \qquad x = vy^3 = \frac{v}{u^{3/8}v^{9/8}} = \frac{1}{u^{3/8}v^{1/8}}.$$

Then the curve $y = x^3$ can be written as

$$\frac{1}{u^{1/8}v^{3/8}} = \frac{1}{u^{9/8}v^{3/8}};$$

$$u^{1/8}v^{3/8} = u^{9/8}v^{3/8};$$

$$u = 1.$$

Similarly, the curve $y = 2x^3$ corresponds to $u = 2$, the curve $x = y^3$ corresponds to $v = 1$, and the curve $x = 4y^3$ corresponds to $v = 4$. Next,

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} -\dfrac{3}{8u^{11/8}v^{1/8}} & -\dfrac{1}{8u^{3/8}v^{9/8}} \\[2ex] -\dfrac{1}{8u^{9/8}v^{3/8}} & -\dfrac{3}{8u^{1/8}v^{11/8}} \end{vmatrix} = \frac{1}{8u^{3/2}v^{3/2}}.$$

Hence the area of R is

$$\iint_R 1\,dx\,dy = \int_{v=1}^{4}\int_{u=1}^{2} \frac{1}{8u^{3/2}v^{3/2}}\,du\,dv = \int_{v=1}^{4}\left[-\frac{1}{4u^{1/2}v^{3/2}}\right]_{u=1}^{2} dv = \int_1^4 \left(\frac{1}{4v^{3/2}} - \frac{1}{4\sqrt{2}\,v^{3/2}}\right) dv$$

$$= \left[\frac{\sqrt{2}\ -2}{4v^{1/2}}\right]_1^4 = \frac{2-\sqrt{2}}{8} \approx 0.07322330470336311890.$$

C14S09.012: The transformation

$$u = \frac{2x}{x^2+y^2}, \qquad v = \frac{2y}{x^2+y^2}$$

yields

$$u^2+v^2 = \frac{4(x^2+y^2)}{(x^2+y^2)^2} = \frac{4}{x^2+y^2}; \qquad x^2+y^2 = \frac{4}{u^2+v^2};$$

$$x = \frac{1}{2}u\cdot(x^2+y^2) = \frac{2u}{u^2+v^2}; \qquad y = \frac{1}{2}v\cdot(x^2+y^2) = \frac{2v}{u^2+v^2}.$$

The circle $x^2+y^2 = 2x$ is thereby transformed into

$$\frac{4}{u^2+v^2} = \frac{4u}{u^2+v^2}: \quad u = 1.$$

Similarly, the other three circles are transformed into $u = \frac{1}{3}$, $v = 1$, and $v = \frac{1}{4}$. The Jacobian of this transformation is

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} -\dfrac{2(u^2-v^2)}{(u^2+v^2)^2} & -\dfrac{4uv}{(u^2+v^2)^2} \\[2ex] -\dfrac{4uv}{(u^2+v^2)^2} & -\dfrac{2(v^2-u^2)}{(u^2+v^2)^2} \end{vmatrix} = -\frac{4}{(u^2+v^2)^2}.$$

Note also that

$$(x^2+y^2)^2 = \frac{16}{(u^2+v^2)^2}, \qquad \text{so that} \qquad \frac{1}{(x^2+y^2)^2} = \frac{(u^2+v^2)^2}{16}.$$

Therefore

$$\begin{aligned}\iint_R \frac{1}{(x^2+y^2)^2}\,dx\,dy &= \int_{v=1}^{1/4}\int_{u=1}^{1/3} \frac{(u^2+v^2)^2}{16}\cdot\frac{4}{(u^2+v^2)^2}\,du\,dv \\ &= \int_{v=1}^{1/4}\int_{u=1}^{1/3} \frac{1}{4}\,du\,dv = \left(-\frac{3}{4}\right)\cdot\left(-\frac{2}{3}\right)\cdot\frac{1}{4} = \frac{1}{8}.\end{aligned}$$

C14S09.013: The Jacobian of the transformation $x = 3r\cos\theta,\ y = 2r\sin\theta$ is

$$\frac{\partial(x,\,y)}{\partial(r,\,\theta)} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 6r\cos^2\theta + 6r\sin^2\theta = 6r.$$

The ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is transformed into

$$\frac{9r^2\cos^2\theta}{9} + \frac{4r^2\sin^2\theta}{4} = 1: \qquad \text{the circle} \quad r = 1.$$

The paraboloid has equation

$$z = x^2 + y^2 = 9r^2\cos^2\theta + 4r^2\sin^2\theta.$$

Therefore the volume of the solid is

$$\begin{aligned}V &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (9r^2\cos^2\theta + 4r^2\sin^2\theta)\cdot 6r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{27}{2}\cos^2\theta + 6\sin^2\theta\right)d\theta \\ &= \int_0^{2\pi}\left(6 + \frac{15}{2}\cdot\frac{1+\cos 2\theta}{2}\right)d\theta = \left[\frac{39}{4}\theta + \frac{15}{8}\sin 2\theta\right]_0^{2\pi} = \frac{39}{2}\pi \approx 61.26105674500096815.\end{aligned}$$

C14S09.014: The Jacobian of the transformation $x = au,\ y = bv, z = cw$ ($a,\ b,$ and c are positive constants) is

$$\frac{\partial(x,\,y,\,z)}{\partial(u,\,v,\,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The ellipsoid with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

becomes the sphere S with equation $u^2 + v^2 + w^2 = 1$. Let B denote the ball bounded by that sphere. Then the volume of the ellipsoid is

$$V = \iiint_R 1\,dx\,dy\,dz = \iiint_B abc\,dV = \frac{4}{3}\pi abc.$$

C14S09.015: We are given the transformation $u = xy,\ v = xz,\ w = yz$. Then $uvw = x^2y^2z^2$. Hence

$$\begin{aligned} u^{1/2}v^{1/2}w^{1/2} &= xyz = uz: & z &= \frac{v^{1/2}w^{1/2}}{u^{1/2}}; \\ u^{1/2}v^{1/2}w^{1/2} &= xyz = vy: & y &= \frac{u^{1/2}w^{1/2}}{v^{1/2}}; \\ u^{1/2}v^{1/2}w^{1/2} &= xyz = wx: & x &= \frac{u^{1/2}v^{1/2}}{w^{1/2}}. \end{aligned}$$

The surface $xy = 1$ corresponds to the plane $u = 1$. Similarly, the other surfaces correspond to the planes $u = 4,\ v = 1,\ v = 9,\ w = 4,$ and $w = 9$. The Jacobian of the given transformation is

$$\frac{\partial(x,\,y,\,z)}{\partial(u,\,v,\,w)} = \begin{vmatrix} \dfrac{v^{1/2}}{2u^{1/2}w^{1/2}} & \dfrac{u^{1/2}}{2v^{1/2}w^{1/2}} & -\dfrac{u^{1/2}v^{1/2}}{2w^{3/2}} \\ \dfrac{w^{1/2}}{2u^{1/2}v^{1/2}} & -\dfrac{u^{1/2}w^{1/2}}{2v^{3/2}} & \dfrac{u^{1/2}}{2v^{1/2}w^{1/2}} \\ -\dfrac{v^{1/2}w^{1/2}}{2u^{3/2}} & \dfrac{w^{1/2}}{2u^{1/2}v^{1/2}} & \dfrac{v^{1/2}}{2u^{1/2}w^{1/2}} \end{vmatrix} = -\frac{1}{2u^{1/2}v^{1/2}w^{1/2}}.$$

Therefore the volume bounded by the surfaces is

$$\begin{aligned} V &= \int_{w=4}^{9}\int_{v=1}^{9}\int_{u=1}^{4} \frac{1}{2u^{1/2}v^{1/2}w^{1/2}}\,du\,dv\,dw = \int_4^9\int_1^9 \left[\frac{u^{1/2}}{v^{1/2}w^{1/2}}\right]_{u=1}^{4} dv\,dw \\ &= \int_4^9\int_1^9 \frac{1}{v^{1/2}w^{1/2}}\,dv\,dw = \int_4^9 \left[\frac{2v^{1/2}}{w^{1/2}}\right]_{v=1}^{9} dw = \int_4^9 \frac{4}{w^{1/2}}\,dw = \Big[8w^{1/2}\Big]_4^9 = 8. \end{aligned}$$

C14S09.016: We are given the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 4(x^2 + y^2)$ and the planes $z = 1$ and $z = 4$. We are also given the transformation

$$x = \frac{r}{t}\cos\theta, \qquad y = \frac{r}{t}\sin\theta, \qquad z = r^2.$$

Under this transformation, the plane $z = 1$ corresponds to $r = 1$ and the plane $z = 4$ corresponds to $r = 2$. The paraboloid $z = x^2 + y^2$ corresponds to

$$r^2 = \frac{r^2}{t^2}\cos^2\theta + \frac{r^2}{t^2}\sin^2\theta = \frac{r^2}{t^2},$$

thus to $t = 1$. The other paraboloid yields $t = 2$. Finally, to obtain the entire solid, θ varies from 0 to 2π. The Jacobian of the given transformation is

$$\frac{\partial(x,\,y,\,z)}{\partial(r,\,\theta,\,t)} = \begin{vmatrix} \dfrac{1}{t}\cos\theta & -\dfrac{r}{t}\sin\theta & -\dfrac{r}{t^2}\cos\theta \\ \dfrac{1}{t}\sin\theta & \dfrac{r}{t}\cos\theta & -\dfrac{r}{t^2}\sin\theta \\ 2r & 0 & 0 \end{vmatrix} = 2r\left(\frac{r^2}{t^3}\sin^2\theta + \frac{r^2}{t^3}\cos^2\theta\right) = \frac{2r^3}{t^3}.$$

Hence the volume of the solid is

$$V = \int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{t=1}^{2} \frac{2r^3}{t^3}\,dt\,dr\,d\theta = 2\pi\int_1^2 \left[-\frac{r^3}{t^2}\right]_{t=1}^{2} dr$$
$$= 2\pi\int_1^2 \frac{3}{4}r^3\,dr = \frac{3}{2}\pi\left[\frac{1}{4}r^4\right]_1^2 = \frac{45}{8}\pi \approx 17.6714586764425869635.$$

C14S09.017: The substitution $x = u + v,\ y = u - v$ transforms the rotated ellipse $x^2 + xy + y^2 = 3$ into the ellipse S in "standard position," in which its axes lie on the coordinate axes. The resulting equation of S (in the uv-plane) is $3u^2 + v^2 = 3$. The Jacobian of this transformation is

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Therefore

$$I = \iint_R \exp(-x^2 - xy - y^2)\,dx\,dy = 2\iint_S \exp(-3u^2 - v^2)\,du\,dv. \tag{1}$$

The substitution $u = r\cos\theta,\ v = r\sqrt{3}\,\sin\theta$ has Jacobian

$$\frac{\partial(u,\,v)}{\partial(r,\,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sqrt{3}\,\sin\theta & r\sqrt{3}\,\cos\theta \end{vmatrix} = r\sqrt{3}\,(\cos^2\theta + \sin^2\theta) = r\sqrt{3}\,.$$

This transformation, applied to the bounding ellipse of the region S, yields

$$3 = 3u^2 + v^2 = 3r^2\cos^2\theta + 3r^2\sin^2\theta = 3r^2,$$

and thereby transforms it into the circle with polar equation $r = 1$. Then substitution in the second integral in Eq. (1) yields

$$I = 2\iint_S \exp(-3u^2 - u^2)\,du\,dv = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\sqrt{3}\,\exp(-3r^2)\,dr\,d\theta$$
$$= 4\pi\sqrt{3}\left[-\frac{1}{6}\exp(-3r^2)\right]_0^1 = \frac{2}{3}\pi\sqrt{3}\,\left(1 - e^{-3}\right) \approx 3.4469912225630013852\,8.$$

C14S09.018: Remember that $x = x(u,\,v),\ y = y(u,\,v),\ u = u(x,\,y),$ and $v = v(x,\,y)$. Then

$$\frac{\partial(x,\,y)}{\partial(u,\,v)}\cdot\frac{\partial(u,\,v)}{\partial(x,\,y)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\cdot\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} x_u u_x + x_v v_x & x_u u_y + x_v v_y \\ y_u u_x + y_v v_x & y_u u_y + y_v v_y \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

C14S09.019: Suppose that k is a positive constant. First we need an integration by parts with

$$u = \rho^2 \qquad \text{and} \qquad dv = \rho\exp(-k\rho^2)\,d\rho:$$

$$du = 2\rho\,d\rho \qquad \text{and} \qquad v = -\frac{1}{2k}\exp(-k\rho^2).$$

Thus

$$\int \rho^3\exp(-k\rho^2)\,d\rho = -\frac{1}{2k}\rho^2\exp(-k\rho^2) + \int\frac{1}{k}\rho\exp(-k\rho^2)\,d\rho$$
$$= -\frac{1}{2k}\rho^2\exp(-k\rho^2) - \frac{1}{2k^2}\exp(-k\rho^2) + C.$$

Then the improper triple integral given in Problem 19 will be the limit of I_a as $a \to +\infty$, where

$$I_a = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{a}\rho^3\exp(-k\rho^2)\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\Big[-\cos\phi\Big]_{\phi=0}^{\pi}\left[-\frac{1}{2k}\rho^2\exp(-k\rho^2) - \frac{1}{2k^2}\exp(-k\rho^2)\right]_{\rho=0}^{a}$$

$$= 4\pi\left[-\frac{1}{2k}a^2\exp(-ka^2) - \frac{1}{2k^2}\exp(-ka^2) + \frac{1}{2k^2}\right].$$

Because $k > 0$, it is clear that $I_a \to \dfrac{2\pi}{k^2}$ as $a \to +\infty$.

C14S09.020: Given: The solid ellipsoid R with constant density δ and boundary surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

(where a, b, and c are positive constants). The transformation

$$x = a\rho\sin\phi\cos\theta, \quad y = b\rho\sin\phi\sin\theta, \quad z = c\rho\cos\phi$$

has Jacobian

$$J = \frac{\partial(x,\, y,\, z)}{\partial(\rho,\, \phi,\, \theta)} = \begin{vmatrix} a\sin\phi\cos\theta & a\rho\cos\phi\cos\theta & -a\rho\sin\phi\sin\theta \\ b\sin\phi\sin\theta & b\rho\cos\phi\sin\theta & b\rho\sin\phi\cos\theta \\ c\cos\phi & -c\rho\sin\phi & 0 \end{vmatrix}$$

$$= abc\rho^2\cos^2\phi\sin\phi\cos^2\theta + abc\rho^2\sin^3\phi\cos^2\theta + abc\rho^2\cos^2\phi\sin\phi\sin^2\theta + abc\rho^2\sin^3\phi\sin^2\theta$$

$$= abc\rho^2\sin\phi.$$

This transformation also transforms the ellipsoidal surface of Eq. (1) into

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2 = 1,$$

and thereby transforms R into the solid ball B of radius 1 and center at the origin. Therefore the mass of the ellipsoid is

$$M = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} \delta abc\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\delta abc\int_{\phi=0}^{\pi}\frac{1}{3}\sin\phi\, d\phi = 2\pi\delta abc\left[-\frac{1}{3}\cos\phi\right]_0^{\pi} = \frac{4}{3}\pi\delta abc.$$

C14S09.021: Given: The solid ellipsoid R with constant density δ and boundary surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

(where a, b, and c are positive constants). The transformation

$$x = a\rho\sin\phi\cos\theta, \quad y = b\rho\sin\phi\sin\theta, \quad z = c\rho\cos\phi$$

has Jacobian

$$J = \frac{\partial(x,\, y,\, z)}{\partial(\rho,\, \phi,\, \theta)} = \begin{vmatrix} a\sin\phi\cos\theta & a\rho\cos\phi\cos\theta & -a\rho\sin\phi\sin\theta \\ b\sin\phi\sin\theta & b\rho\cos\phi\sin\theta & b\rho\sin\phi\cos\theta \\ c\cos\phi & -c\rho\sin\phi & 0 \end{vmatrix}$$

$$= abc\rho^2\cos^2\phi\sin\phi\cos^2\theta + abc\rho^2\sin^3\phi\cos^2\theta + abc\rho^2\cos^2\phi\sin\phi\sin^2\theta + abc\rho^2\sin^3\phi\sin^2\theta$$

$$= abc\rho^2\sin\phi.$$

This transformation also transforms the ellipsoidal surface of Eq. (1) into

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2 = 1,$$

and thereby transforms R into the solid ball B of radius 1 and center at the origin. Therefore the moment of inertia of the ellipsoid with respect to the z-axis is

$$
\begin{aligned}
I_z &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} (\rho^2 \sin^2 \phi)(a^2 \cos^2 \theta + b^2 \sin^2 \theta)\delta abc\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \left[\frac{1}{5}(\delta abc\rho^5 \sin^3 \phi)(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \right]_{\rho=0}^{1} d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{5}(\delta abc \sin^3 \phi)(a^2 \cos^2 \theta + b^2 \cos^2 \theta) \, d\phi \, d\theta \\
&= \frac{1}{60}\delta abc \int_0^{2\pi} \Big[(\cos 3\phi - 9\cos\phi)(a^2 \cos^2 \theta + b^2 \sin^2 \theta) \Big]_0^{\pi} d\theta \\
&= \frac{4}{15}\delta abc \int_0^{2\pi} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \, d\theta = \frac{1}{15}\delta abc \Big[2a^2\theta + 2b^2\theta + a^2 \sin 2\theta - b^2 \sin 2\theta \Big]_0^{2\pi} \\
&= \frac{1}{15}\delta abc(4\pi a^2 + 4\pi b^2).
\end{aligned}
$$

Because the mass of the sphere (found in the solution of Problem 20) is $M = \frac{4}{3}\pi\delta abc$, we see that

$$
\frac{I_z}{M} = \frac{1}{5}(a^2 + b^2), \qquad \text{and hence that} \quad I_z = \frac{1}{5}M(a^2 + b^2).
$$

C14S09.022: Given $u = xy$ and $v = \dfrac{y}{x}$, we have

$$
uv = xy \cdot \frac{y}{x} = y^2 \qquad \text{and} \qquad \frac{u}{v} = xy \cdot \frac{x}{y} = x^2,
$$

and thus we choose

$$
x = \sqrt{\frac{u}{v}} \qquad \text{and} \qquad y = \sqrt{uv}\,. \tag{1}
$$

The Jacobian of this transformation is

$$
\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} \dfrac{1}{2u^{1/2}v^{1/2}} & -\dfrac{u^{1/2}}{2v^{3/2}} \\[2ex] \dfrac{v^{1/2}}{2u^{1/2}} & \dfrac{u^{1/2}}{2v^{1/2}} \end{vmatrix} = \frac{1}{2v}.
$$

Also, if $y = x$, then substitution of the equations in (1) yields

$$
(uv)^{1/2} = \frac{u^{1/2}}{v^{1/2}}; \quad uv = \frac{u}{v}; \quad v^2 = 1.
$$

So we choose $v = 1$. (This choice implies that if we have a similar choice with u, we must choose $u > 0$ because of the equations in (1).) Similarly, $y = 2x$ yields $v = 2$, $xy = 1$ yields $u = 1$, and $xy = 2$ yields $u = 2$. Hence the area of the region of Fig. 14.9.7 is

$$
A = \iint_R 1 \, dx \, dy = \int_{v=1}^{2} \int_{u=1}^{2} \frac{1}{2v} \, du \, dv = \int_1^2 \frac{1}{2v} \, dv = \frac{1}{2}\ln 2 \approx 0.3465735902799727.
$$

Its moments with respect to the coordinate axes are

$$M_y = \int_{v=1}^2 \int_{u=1}^2 \frac{1}{2v} \cdot \frac{u^{1/2}}{v^{1/2}}\, du\, dv = \int_{v=1}^2 \left[\frac{u^{3/2}}{3v^{3/2}} \right]_{u=1}^2 dv = \int_1^2 \frac{2\sqrt{2}\, - 1}{3v^{3/2}}\, dv = \left[\frac{2 - 4\sqrt{2}}{3v^{1/2}} \right]_1^2 = \frac{5\sqrt{2}\, - 6}{3};$$

$$M_x = \int_{v=1}^2 \int_{u=1}^2 \frac{1}{2v} \cdot \sqrt{uv}\, du\, dv = \int_{v=1}^2 \left[\frac{u^{3/2}}{3v^{1/2}} \right]_{u=1}^2 dv$$

$$= \int_1^2 \frac{2\sqrt{2}\, - 1}{3v^{1/2}}\, dv = \left[\frac{4\sqrt{2}\, - 2}{3} v^{1/2} \right]_1^2 = \frac{10 - 6\sqrt{2}\,,}{3}.$$

Hence the coordinates of its centroid are

$$(\overline{x},\, \overline{y}) = \left(\frac{10\sqrt{2}\, - 12}{3 \ln 2},\, \frac{20 - 12\sqrt{2}}{3 \ln 2} \right) \approx (1.030149480423,\, 1.456851366485)\,.$$

C14S09.023: If $u = xy$ and $v = xy^3$, then

$$uy^2 = xy^3 = v, \qquad \text{so that} \qquad y^2 = \frac{v}{u}; \quad y = \frac{v^{1/2}}{u^{1/2}}.$$

Then

$$x = \frac{u}{y} = u \cdot \frac{u^{1/2}}{v^{1/2}} = \frac{u^{3/2}}{v^{1/2}}.$$

(We do not need the solution in which x and y are negative.) Then

$$\frac{\partial(x,\, y)}{\partial(u,\, v)} = \begin{vmatrix} \dfrac{3u^{1/2}}{2v^{1/2}} & -\dfrac{u^{3/2}}{2v^{3/2}} \\ -\dfrac{v^{1/2}}{2u^{3/2}} & \dfrac{1}{2u^{1/2}v^{1/2}} \end{vmatrix} = \frac{3}{4v} - \frac{1}{4v} = \frac{1}{2v}.$$

We also find by substitution that $xy = 2$ corresponds to $u = 2$, $xy = 4$ corresponds to $u = 4$, $xy^3 = 3$ corresponds to $v = 3$, and $xy^3 = 6$ corresponds to $v = 6$. Hence the area of the region shown in Fig. 14.9.8 is

$$A = \iint_D 1\, dx\, dy = \int_{v=3}^6 \int_{u=2}^4 \frac{1}{2v}\, du\, dv = \int_3^6 \frac{1}{v}\, dv = \ln 2 \approx 0.6931471805599453.$$

Its moments with respect to the coordinate axes are

$$M_y = \int_{v=3}^6 \int_{u=2}^4 \frac{1}{2v} \cdot \frac{u^{3/2}}{v^{1/2}}\, du\, dv = \int_{v=3}^6 \left[\frac{u^{5/2}}{5v^{3/2}} \right]_{u=2}^4 dv = \int_3^6 \frac{32 - 4\sqrt{2}}{5v^{3/2}}\, dv$$

$$= \left[\frac{8\sqrt{2}\, - 64}{5v^{1/2}} \right]_3^6 = \frac{72\sqrt{3}\, - 40\sqrt{6}}{15};$$

$$M_x = \int_{v=3}^6 \int_{u=2}^4 \frac{1}{2v} \cdot \frac{v^{1/2}}{u^{1/2}}\, du\, dv = \int_{v=3}^6 \left[\frac{u^{1/2}}{v^{1/2}} \right]_{u=2}^4 dv = \int_3^6 \frac{2 - \sqrt{2}}{v^{1/2}}\, dv$$

$$= \left[\left(4 - 2\sqrt{2}\right) v^{1/2} \right]_3^6 = 6\sqrt{6}\, - 8\sqrt{3}\,.$$

Therefore its centroid is located at the point

$$(\overline{x},\,\overline{y}) = \left(\frac{72\sqrt{3}\,-40\sqrt{6}}{15\ln 2},\;\frac{6\sqrt{6}\,-8\sqrt{3}}{\ln 2}\right) \approx (2.570696785449,\;1.212631342551)\,.$$

C14S09.024: If $y = ux^2$ and $x = vy^2$, then

$$y = uv^2y^4;\quad y^3 = \frac{1}{uv^2};\quad y = \frac{1}{u^{1/3}v^{2/3}}.$$

Then it follows that

$$x = vy^2 = \frac{v}{u^{2/3}v^{4/3}} = \frac{1}{u^{2/3}v^{1/3}}.$$

Next,

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \begin{vmatrix} -\dfrac{2}{3u^{5/3}v^{1/3}} & -\dfrac{1}{3u^{2/3}v^{4/3}} \\[2ex] -\dfrac{1}{3u^{4/3}v^{2/3}} & -\dfrac{2}{3u^{1/3}v^{5/3}} \end{vmatrix} = \frac{1}{3u^2v^2}.$$

Next, $y = x^2$ corresponds to $u = 1$, $y = 2x^2$ corresponds to $u = 2$, $x = y^2$ corresponds to $v = 1$, and $x = 4y^2$ corresponds to $v = 4$. Therefore the area of the region shown in Fig. 14.9.9 is

$$A = \iint_R 1\,dx\,dy = \int_{v=1}^{4}\int_{u=1}^{2} \frac{1}{3u^2v^2}\,du\,dv = \int_{v=1}^{4}\left[-\frac{1}{3uv^2}\right]_{u=1}^{2} dv = \int_1^4 \frac{1}{6v^2}\,dv = \left[-\frac{1}{6v}\right]_1^4 = \frac{1}{8}.$$

Its moments with respect to the coordinate axes are

$$\begin{aligned}
M_y &= \int_{v=1}^{4}\int_{u=1}^{2} \frac{1}{3u^2v^2}\cdot\frac{1}{u^{2/3}v^{1/3}}\,du\,dv = \int_{v=1}^{4}\left[-\frac{1}{5u^{5/3}v^{7/3}}\right]_{u=1}^{2} dv \\
&= \int_{v=1}^{4} \frac{4-2^{1/3}}{20v^{7/3}}\,dv = \left[\frac{3\cdot 2^{1/2}-12}{80v^{4/3}}\right]_1^4 = \frac{96-36\cdot 2^{1/3}+3\cdot 2^{2/3}}{640};\\
M_x &= \int_{v=1}^{4}\int_{u=1}^{2} \frac{1}{3u^2v^2}\cdot\frac{1}{u^{1/3}v^{2/3}}\,du\,dv = \int_{v=1}^{4}\left[-\frac{1}{4u^{4/3}v^{8/3}}\right]_{u=1}^{2} dv \\
&= \int_1^4 \frac{4-2^{2/3}}{16v^{8/3}}\,dv = \left[\frac{3\cdot 2^{2/3}-12}{80v^{5/3}}\right]_1^4 = \frac{96+3\cdot 2^{1/3}-30\cdot 2^{2/3}}{640}.
\end{aligned}$$

Therefore its centroid is located at the point with coordinates

$$(\overline{x},\,\overline{y}) = \left(\frac{96-36\cdot 2^{1/3}+3\cdot 2^{2/3}}{80},\;\frac{96+3\cdot 2^{1/3}-30\cdot 2^{2/3}}{80}\right) \approx (0.692563066996,\,0.651971644883)\,.$$

C14S09.025: Given: The solid ellipsoid R with constant density δ and boundary surface with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \;=\; 1 \qquad (1)$$

(where a, b, and c are positive constants). The transformation

$$x = a\rho \sin\phi \cos\theta, \quad y = b\rho \sin\phi \sin\theta, \quad z = c\rho \cos\phi$$

has Jacobian

$$J = \frac{\partial(x,\,y,\,z)}{\partial(\rho,\,\phi,\,\theta)} = \begin{vmatrix} a\sin\phi\cos\theta & a\rho\cos\phi\cos\theta & -a\rho\sin\phi\sin\theta \\ b\sin\phi\sin\theta & b\rho\cos\phi\sin\theta & b\rho\sin\phi\cos\theta \\ c\cos\phi & -c\rho\sin\phi & 0 \end{vmatrix}$$

$$= abc\rho^2\cos^2\phi\sin\phi\cos^2\theta + abc\rho^2\sin^3\phi\cos^2\theta + abc\rho^2\cos^2\phi\sin\phi\sin^2\theta + abc\rho^2\sin^3\phi\sin^2\theta$$

$$= abc\rho^2\sin\phi.$$

This transformation also transforms the ellipsoidal surface of Eq. (1) into

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = \rho^2\sin^2\phi + \rho^2\cos^2\phi = \rho^2 = 1,$$

and thereby transforms R into the solid ball B of radius 1 and center at the origin. Note also that

$$x^2 + y^2 = a^2\rho^2\sin^2\phi\cos^2\theta + b^2r^2\sin^2\phi\sin^2\theta = (\rho^2\sin^2\phi)(a^2\cos^2\theta + b^2\sin^2\theta).$$

Assume that the solid R has constant density δ. Then its moment of inertia with respect to the z-axis is

$$I_z = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1}(\rho^2\sin^2\phi)(a^2\cos^2\theta + b^2\sin^2\theta)\delta abc\rho^2\sin\phi\;d\rho\;d\phi\;d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{5}(\delta abc\sin^3\phi)(a^2\cos^2\theta + b^2\sin^2\theta)\;d\phi\;d\theta$$

$$= \int_0^{2\pi}\frac{4}{15}(\delta abc)(a^2\cos^2\theta + b^2\sin^2\theta)\;d\theta = \frac{4}{15}\pi\delta abc(a^2+b^2) = \frac{1}{5}M(a^2+b^2)$$

where M is the mass of the ellipsoid. By symmetry,

$$I_y = \frac{4}{15}\pi\delta abc(a^2+c^2) = \frac{1}{5}M(a^2+c^2) \quad \text{and} \quad I_x = \frac{4}{15}\pi\delta abc(b^2+c^2) = \frac{1}{5}M(b^2+c^2).$$

C14S09.026: Assume that the solid of Problem 16 has constant density δ. By symmetry its centroid lies on the z-axis. A consequence of the solution of Problem 16 is that the solid has mass $M = \frac{45}{8}\pi\delta$. Its moment with respect to the xy-plane is

$$M_{xy} = \int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{t=1}^{2} 2\left(\frac{r}{t}\right)^3\cdot r^2\;dt\;dr\;d\theta = \int_0^{2\pi}\int_1^2\left[-\frac{r^5}{t^2}\right]_{t=1}^{2}\;dr\;d\theta$$

$$= 2\pi\int_1^2\frac{3}{4}r^5\;dr = 2\pi\left[\frac{1}{8}r^6\right]_1^2 = 2\pi\cdot\frac{63}{8} = \frac{63}{4}\pi.$$

Therefore the centroid of the solid is located at the point $\left(0,\,0,\,\frac{14}{5}\right)$. Next, by symmetry,

$$\begin{aligned}
I_y = I_x &= \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \int_{t=1}^{2} \left[\left(\frac{r}{t} \sin\theta \right)^2 + r^4 \right] \cdot 2 \left(\frac{r}{t} \right)^3 \, dt \, dr \, d\theta \\
&= \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \left[-\frac{r^5(1 + 4r^2t^2 - \cos 2\theta)}{4t^4} \right]_{t=1}^{2} dr \, d\theta \\
&= \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \frac{3}{64} r^5 (5 + 16r^2 - 5\cos 2\theta) \, dr \, d\theta = \delta \int_{\theta=0}^{2\pi} \left[\frac{3}{32} r^8 + \frac{5}{64} r^2 \sin^2\theta \right]_{r=1}^{2} d\theta \\
&= \delta \int_0^{2\pi} \left(\frac{765}{32} + \frac{315}{64} \sin^2\theta \right) d\theta = \delta \left[\frac{45}{64} (7 \sin 2\theta - 150\theta) \right]_0^{2\pi} = \frac{3375}{64} \pi\delta.
\end{aligned}$$

Thus $I_x = I_y \approx (165.6699250916492528)\delta$. Finally, to compute I_z, note that

$$x^2 + y^2 = \left(\frac{r}{t} \cos\theta \right)^2 + \left(\frac{r}{t} \sin\theta \right)^2 = \frac{r^2}{t^2}.$$

Therefore the moment of inertia of the solid with respect to the z-axis is

$$\begin{aligned}
I_z &= \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \int_{t=1}^{2} \frac{2r^5}{t^5} \, dt \, dr \, d\theta = \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \left[-\frac{r^5}{2t^4} \right]_{t=1}^{2} dr \, d\theta \\
&= \delta \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \frac{15}{32} r^5 \, dr = 2\pi\delta \left[\frac{5}{64} r^6 \right]_1^2 = \frac{315}{32} \pi\delta \approx (30.9250526837745272)\delta.
\end{aligned}$$

C14S09.027: The average distance of points of the ellipsoid from its center at (0, 0, 0) is

$$\overline{d} = \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^1 (abc\rho^2 \sin\phi) \sqrt{(a\rho \sin\phi \cos\theta)^2 + (b\rho \sin\phi \sin\theta)^2 + (c\rho \cos\phi)^2} \, d\rho \, d\phi \, d\theta$$

where $V = \frac{4}{3}\pi abc$ is the volume of the ellipsoid. In particular, if $a = 4$, $b = 3$, and $c = 2$, we find (using the `NIntegrate` command in *Mathematica* 3.0) that $\overline{d} \approx 2.300268522983$.

C14S09.028: Following the instructions in Problem 28, we have

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{1 - xy} \, dx \, dy \; &= \; \int_0^1 \int_0^1 \left(1 + \sum_{n=1}^{\infty} x^n y^n \right) dx \, dy \; = \; 1 + \sum_{n=1}^{\infty} \left(\int_0^1 x^n dx \right) \cdot \left(\int_0^1 y^n \, dy \right) \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \right) \cdot \left(\frac{1}{n+1} \right) \; = \; 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \; = \; \zeta(2). \qquad \text{—C.H.E.}
\end{aligned}$$

C14S09.029: Part (a): First note that

$$\int_0^1 \int_0^1 \left(\frac{1}{1 - xy} - \frac{1}{1 + xy} \right) dx \, dy \; = \; \int_0^1 \int_0^1 \frac{2xy}{1 - x^2y^2} \, dx \, dy.$$

The Jacobian of the substitution $u = x^2$, $v = y^2$ is

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy,$$

so

$$\int_0^1\int_0^1 \frac{2xy}{1-x^2y^2}\,dx\,dy = \frac{1}{2}\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\cdot 4xy\,dx\,dy = \frac{1}{2}\int_0^1\int_0^1 \frac{1}{1-uv}\,du\,dv = \frac{1}{2}\zeta(2).$$

Part (b): Addition as indicated in Problem 29, and cancellation of the integrals involving $1/(1+xy)$, yields the equation

$$2\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \frac{1}{2}\zeta(2) + 2\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy,$$

which we readily solve for

$$\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy = \int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2).$$

Part (c): The Jacobian of the transformation $T : \boldsymbol{R}^2_{uv} \to \boldsymbol{R}^2_{xy}$ that we define by $x = (\sin v)/(\cos u)$, $y = (\sin u)/(\cos v)$ is

$$J_T = \begin{vmatrix} \dfrac{\cos u}{\cos v} & -\dfrac{\sin u \sin v}{\cos^2 v} \\[2ex] -\dfrac{\sin u \sin v}{\cos^2 u} & \dfrac{\cos v}{\cos u} \end{vmatrix} = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} = 1 - \tan^2 u \tan^2 v.$$

Reading the limits for the transformed integral from Fig. 14.9.10(a) in the text, we therefore find that

$$\begin{aligned} \zeta(2) &= \frac{4}{3}\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy \\ &= \frac{4}{3}\int_0^{\pi/2}\int_0^{(\pi/2)-v} \left(1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v}\right)^{-1}\cdot(1-\tan^2 u\tan^2 v)\,du\,dv = \frac{4}{3}\int_0^{\pi/2}\int_0^{(\pi/2)-v} 1\,du\,dv \\ &= \frac{4}{3}\int_0^{\pi/2}\left(\frac{\pi}{2} - v\right)dv = \frac{4}{3}\left[\frac{\pi}{2}v - \frac{1}{2}v^2\right]_0^{\pi/2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}. \end{aligned}$$

—C.H.E.

Chapter 14 Miscellaneous Problems

C14S0M.001: The domain of the given integral is bounded above by the graph of $y = x^3$, below by the x-axis, and on the right by the vertical line $x = 1$. When its order of integration is reversed, the given integral becomes

$$\int_{x=0}^{1} \int_{y=0}^{x^3} \frac{1}{\sqrt{1+x^2}}\, dy\, dx = \int_0^1 \frac{x^3}{\sqrt{1+x^2}}\, dx$$

$$= \left[\frac{1}{3}(x^2-2)\sqrt{1+x^2}\, \right]_0^1 = \frac{2-\sqrt{2}}{3} \approx 0.1952621458756350.$$

Mathematica 3.0 can evaluate the given integral without first reversing the order of integration. It obtains

$$\int_{y=0}^{1} \int_{x=y^{1/3}}^{1} \frac{1}{\sqrt{1+x^2}}\, dx\, dy = \int_0^1 \left[\operatorname{arcsinh}(1) - \operatorname{arcsinh}(y^{1/3}) \right] dy$$

$$= \left[\frac{y^{2/3}-2}{3} \sqrt{1+y^{2/3}}\, + y \operatorname{arcsinh}(1) - y \operatorname{arcsinh}(y^{1/3}) \right]_0^1 = \frac{2-\sqrt{2}}{3}.$$

C14S0M.002: The given integral is improper, but the only discontinuity occurs at (0, 0), a corner point of the domain of the integral. If we define the value of the integrand to be 1 at that point, the integrand will be continuous on its entire domain and the problem vanishes. The domain of the given integral is bounded above by the line $y = x$, below by the x-axis, and on the right by the vertical line $x = 1$. When its order of integration is reversed, the given integral becomes

$$\int_{x=0}^{1} \int_{y=0}^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^1 \left[\frac{y \sin x}{x} \right]_{y=0}^{x} dx$$

$$= \int_0^1 \sin x\, dx = \Big[-\cos x \Big]_0^1 = 1 - \cos(1) \approx 0.4596976941318602.$$

Mathematica 3.0 can evaluate the given integral without first reversing the order of integration. It obtains

$$\int_{y=0}^{1} \int_{x=y}^{1} \frac{\sin x}{x}\, dx\, dy = \int_0^1 \Big[\text{SinIntegral}(x) \Big]_{x=y}^{1} dy$$

$$= \Big[y\, \text{SinIntegral}(1) - y\, \text{SinIntegral}(y) - \cos y \Big]_0^1 = 1 - \cos(1).$$

C14S0M.003: The domain of the given integral is bounded above by the line $y = 1$, below and on the right by the line $y = x$, and on the left by the y-axis. When its order of integration is reversed, it becomes

$$\int_{y=0}^{1} \int_{x=0}^{y} \exp(-y^2)\, dx\, dy = \int_0^1 y \exp(-y^2)\, dy = \left[-\frac{1}{2} \exp(-y^2) \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2e} = \frac{e-1}{2e} \approx 0.3160602794142788.$$

Mathematica 3.0 can evaluate the given integral without first reversing the order of integration:

$$\int_{x=0}^{1}\int_{y=x}^{1} \exp(-y^2)\,dy\,dx = \int_{x=0}^{1}\left[\frac{1}{2}\sqrt{\pi}\,\operatorname{erf}(y)\right]_{y=x}^{1} dx = \int_0^1 \frac{1}{2}\sqrt{\pi}\,\big[\operatorname{erf}(1)-\operatorname{erf}(x)\big]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\pi}\,x\big[\operatorname{erf}(1)-\operatorname{erf}(x)\big] - \exp(-x^2)\right]_0^1 = \frac{e-1}{2e}.$$

Note: By definition, the *error function* is $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x \exp(-t^2)\,dt.$

C14S0M.004: The domain of the given integral is bounded above by the line $y = 4$, below and on the right by the graph of $y = x^{2/3}$, and on the left by the y-axis. When its order of integration is reversed, the integral takes the form

$$\int_{y=0}^{4}\int_{x=0}^{y^{3/2}} x\cos y^4\,dx\,dy = \int_{y=0}^{4}\left[\frac{1}{2}x^2\cos y^4\right]_{x=0}^{y^{3/2}} dy = \int_0^4 \frac{1}{2}y^3\cos y^4\,dy$$

$$= \left[\frac{1}{8}\sin y^4\right]_0^4 = \frac{1}{8}\sin 256 \approx -0.1249010042633828.$$

Mathematica 3.0 can evaluate the given integral without first reversing the order of integration, but all the antiderivatives that it uses involve the gamma function with complex arguments—we omit the details—and it expresses the final answer in the form

$$\frac{1}{16}\,i\,\big[\,1-\exp\left(512i\right)\big]\cdot\exp\left(-256i\right)$$

where $i^2 = -1$. You will need *Euler's formula* $e^{i\theta} = \cos\theta + i\,\sin\theta$ to show that this answer is equal to the previous answer.

C14S0M.005: The domain of the given integral is bounded above by the graph of $y = x^2$, below by the x-axis, and on the right by the line $x = 2$. When its order of integration is reversed, it becomes

$$\int_{x=0}^{2}\int_{y=0}^{x^2} \frac{y\,\exp(x^2)}{x^3}\,dy\,dx = \int_{x=0}^{2}\left[\frac{y^2\exp(x^2)}{2x^3}\right]_{y=0}^{x^2} dx$$

$$= \int_0^2 \frac{1}{2}x\exp(x^2)\,dx = \left[\frac{1}{4}\exp(x^2)\right]_0^2 = \frac{e^4-1}{4} \approx 13.3995375082860598.$$

Mathematica 3.0 can evaluate the given integral without first reversing the order of integration, but the intermediate antiderivatives involve the *exponential integral function*

$$\operatorname{Ei}(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt$$

and hence we omit the details.

C14S0M.006: Here we obtain

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{1}{y}e^{-y}\,dy\,dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{1}{y}e^{-y}\,dx\,dy$$

$$= \int_0^\infty \left[\frac{x}{y}e^{-y}\right]_{x=0}^{y} dy = \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = 1 - \left(\lim_{y\to\infty} e^{-y}\right) = 1.$$

C14S0M.007: The volume is

$$V = \int_{y=0}^{1} \int_{x=y}^{2-y} (x^2 + y^2)\ dx\ dy = \int_{y=0}^{1} \left[\frac{1}{3}x^3 + xy^2 \right]_{x=y}^{2-y} dy$$

$$= \int_0^1 \frac{4}{3}(2 - 3y + 3y^2 - 2y^3)\ dy = \left[\frac{8}{3}y - 2y^2 + \frac{4}{3}y^3 - \frac{2}{3}y^4 \right]_0^1 = \frac{4}{3}.$$

C14S0M.008: The paraboloids intersect in the circle $x^2 + y^2 = 16,\ z = 32$. Hence the volume between them is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \int_{z=2r^2}^{48-r^2} r\ dz\ dr\ d\theta = 2\pi \int_0^4 (48r - 3r^3)\ dr$$

$$= 2\pi \left[24r^2 - \frac{3}{4}r^4 \right]_0^4 = 2\pi(384 - 192) = 384\pi \approx 1206.3715789784806036.$$

C14S0M.009: By symmetry, the centroid lies on the z-axis. Assume that the solid has unit density. Then its mass and volume are

$$m = V = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/3}^{\pi/2} \int_{\rho=0}^{3} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= 2\pi \cdot \Big[-\cos\phi \Big]_{\pi/3}^{\pi/2} \cdot \left[\frac{1}{3}\rho^3 \right]_0^3 = 18\pi \cdot \left[0 - \left(-\frac{1}{2} \right) \right] = 9\pi \approx 28.2743338823081391.$$

The moment of the solid with respect to the xy-plane is

$$M_{xy} = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/3}^{\pi/2} \int_{\rho=0}^{3} \rho^3 \sin\phi\cos\phi\ d\rho\ d\phi\ d\theta = 2\pi \int_{\phi=\pi/3}^{\pi/2} \left[\frac{1}{4}\rho^4 \sin\phi\cos\phi \right]_{\rho=0}^{3} d\phi$$

$$= 2\pi \int_{\phi=\pi/3}^{\pi/2} \frac{81}{4} \sin\phi\cos\phi\ d\phi = 2\pi \left[-\frac{81}{8}\cos^2\phi \right]_{\phi=\pi/3}^{\pi/2} = \frac{81}{16}\pi.$$

Therefore the centroid of the solid is located at the point with coordinates

$$(\overline{x}, \overline{y}, \overline{z}) = \left(0,\ 0,\ \frac{9}{16} \right).$$

C14S0M.010: The elliptic paraboloids intersect in a curve that lies on the elliptical cylinder with Cartesian equation $x^2 + 4y^2 = 4$. The intersection of that cylinder with the xy-plane forms the elliptical boundary of a region R suitable for the domain of the volume integral, which is

$$V = \iint_R (8 - 2x^2 - 8y^2)\ dx\ dy. \tag{1}$$

Let us use the substitution $x = 2r\cos\theta,\ y = r\sin\theta$. The boundary of R takes the form

$$4r^2\cos^2\theta + 4r^2\sin^2\theta = 4,$$

and hence is transformed into the circle $r = 1$. The Jacobian of this transformation is

$$\frac{\partial(x,\, y)}{\partial(r,\, \theta)} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 2r.$$

Therefore the volume intgral in Eq. (1) is transformed into

$$\begin{aligned} V &= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (8 - 8r^2\cos^2\theta - 8r^2\sin^2\theta) \cdot 2r\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^1 16(r - r^3)\ dr\ d\theta = 32\pi\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = 8\pi \approx 25.1327412287183459. \end{aligned}$$

C14S0M.011: First interchange y and z: We are to find the volume bounded by the paraboloid $z = x^2 + 3y^2$ and the cylinder $z = 4 - y^2$. These surfaces intersect in a curve that lies on the elliptic cylinder $x^2 + 4y^2 = 4$, bounding the region R in the xy-plane. Hence the volume is

$$V = \iint_R (4 - x^2 - 4y^2)\ dx\ dy.$$

Apply the transformation $x = 2r\cos\theta,\ y = r\sin\theta$. This transforms R into the region $0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq 2\pi$. Moreover, the Jacobian of this transformation is

$$\frac{\partial(x,\, y)}{\partial(r,\, \theta)} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 2r.$$

Hence

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4 - 4r^2) \cdot 2r\ dr\ d\theta = 2\pi \cdot \Big[4r^2 - 2r^4\Big]_0^1 = 4\pi \approx 12.5663706143591729538 5.$$

C14S0M.012: The volume is

$$V = \int_{x=-1}^{1} \int_{z=x^2}^{2-x^2} (4 - z)\ dz\ dx = \int_{-1}^{1} \left[4z - \frac{1}{2}z^2\right]_{x^2}^{2-x^2} dx = \int_{-1}^{1} (6 - 6x^2)\ dx = \Big[6x - 2x^3\Big]_{-1}^{1} = 8.$$

C14S0M.013: First interchange x and z: We are to find the volume enclosed by the elliptical cylinder $4x^2 + y^2 = 4$ and between the planes $z = 0$ and $z = y + 2$. Let R denote the plane region in which the elliptical cylinder meets the xy-plane. Then the volume is

$$V = \iint_R (y + 2)\ dx\ dy.$$

The transformation $x = r\cos\theta,\ y = 2r\sin\theta$ transforms R into the rectangle $0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq 2\pi$. The Jacobian of this transformation is

$$\frac{\partial(x,\, y)}{\partial(r,\, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 2r.$$

Therefore

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}(2+2r\sin\theta)\cdot 2r\ dr\ d\theta = \int_0^{2\pi}\left[2r^2+\frac{4}{3}r^3\sin\theta\right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi}\left(2+\frac{4}{3}\sin\theta\right) d\theta = \left[2\theta-\frac{4}{3}\cos\theta\right]_0^{2\pi} = 4\pi \approx 12.5663706143591729 5385.$$

C14S0M.014: Let R denote the bounded plane region with boundary the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \;=\; 1; \tag{1}$$

we assume that $a > 0,\ b > 0$, and that $a < h$ (this ensures that the plane $z = h + x$ is *above* R). Then the volume within the elliptical cylinder and between the planes $z = 0$ and $z = h + x$ is

$$V = \iint_R (h+x)\ dx\ dy.$$

Apply the transformation $x = ar\cos\theta,\ y = br\sin\theta$. This transforms R into the region $0 \leqq r \leqq 1$, $0 \leqq \theta \leqq 2\pi$ (you can see this by substitution into Eq. (1)). Moreover, the Jacobian of this transformation is

$$\frac{\partial(x,\,y)}{\partial(r,\,\theta)} \;=\; \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} \;=\; abr(\cos^2\theta+\sin^2\theta) \;=\; abr.$$

Consequently,

$$V = \int_0^{2\pi}\int_0^1 (h+ar\cos\theta)\cdot abr\ dr\ d\theta = \int_0^{2\pi}\left[\frac{1}{2}abhr^2+\frac{1}{3}a^2br^3\cos\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}abh+\frac{1}{3}a^2b\cos\theta\right) d\theta = \left[\frac{1}{2}abh\theta-\frac{1}{3}a^2b\sin\theta\right]_0^{2\pi} = \pi abh.$$

C14S0M.015: The graph of $x^4 + x^2y^2 = y^2$ in the first quadrant is the graph of

$$y = \frac{x^2}{\sqrt{1-x^2}}, \quad 0 \leqq x < 1.$$

This curve meets the line $y = x$ at the point $\left(\frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{2}\,\right)$ and, of course, at the point $(x,\,y) = (0,\,0)$. Conversion of the first equation into polar form yields

$$r^4\cos^4\theta + r^4\cos^2\theta\,\sin^2\theta = r^2\sin^2\theta;$$

$$(r^2\cos^2\theta)(\cos^2\theta+\sin^2\theta) = \sin^2\theta;$$

$$r^2\cos^2\theta = \sin^2\theta;$$

thus $r = \tan\theta$. Noting that the line $y = x$ has polar equation $\theta = \frac{1}{4}\pi$, we find that

$$\iint_R \frac{1}{(1+x^2+y^2)^2}\,dA = \int_{\theta=0}^{\pi/4}\int_{r=0}^{\tan\theta} \frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\pi/4}\left[-\frac{1}{2(1+r^2)}\right]_{r=0}^{\tan\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(1-\cos^2\theta)\,d\theta = \frac{1}{8}\Big[2\theta - \sin 2\theta\Big]_0^{\pi/4} = \frac{\pi-2}{16} \approx 0.0713495408493621.$$

C14S0M.016: The mass and moments are

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{1}{3}y^3\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left(\frac{1}{3}x^{3/2} + x^{5/2} - x^4 - \frac{1}{3}x^6\right) dx$$

$$= \left[\frac{2}{15}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{1}{5}x^5 - \frac{1}{21}x^7\right]_0^1 = \frac{6}{35} \approx 0.1714285714285714;$$

$$M_y = \int_0^1 \int_{x^2}^{\sqrt{x}} x\cdot(x^2+y^2)\,dy\,dx = \int_0^1 \left[x^3y + \frac{1}{3}xy^3\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{1}{3}x^{5/2} + x^{7/2} - x^5 - \frac{1}{3}x^7\right) dx = \left[\frac{2}{21}x^{7/2} + \frac{2}{9}x^{9/2} - \frac{1}{6}x^6 - \frac{1}{24}x^8\right]_0^1 = \frac{55}{504};$$

$$M_x = \int_0^1 \int_{x^2}^{\sqrt{x}} y\cdot(x^2+y^2)\,dy\,dx = \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{1}{4}y^4\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{1}{4}x^2 + \frac{1}{2}x^3 - \frac{1}{2}x^6 - \frac{1}{4}x^8\right) dx = \left[\frac{1}{12}x^3 + \frac{1}{8}x^4 - \frac{1}{14}x^7 - \frac{1}{36}x^9\right]_0^1 = \frac{55}{504}.$$

Therefore the centroid of the lamina is located at the point with coordinates

$$(\overline{x},\,\overline{y}) \;=\; \left(\frac{275}{432},\,\frac{275}{432}\right) \;\approx\; (0.636574074074,\;0.636574074074).$$

C14S0M.017: The mass and and moments are

$$m = \int_{-2}^{2}\int_{2y^2}^{4+y^2} y^2\,dx\,dy = \int_{-2}^{2}\Big[xy^2\Big]_{2y^2}^{4+y^2} dy$$

$$= \int_{-2}^{2} (4y^2 - y^4)\,dy = \left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]_{-2}^{2} = \frac{128}{15} \approx 8.533333333333333;$$

$$M_x = \int_{-2}^{2}\int_{2y^2}^{4+y^2} y^3\,dx\,dy = \int_{-2}^{2}\Big[xy^3\Big]_{2y^2}^{4+y^2} dx = \int_{-2}^{2}(4y^3 - y^5)\,dy = \left[y^4 - \frac{1}{6}y^6\right]_{-2}^{2} = 0;$$

$$M_y = \int_{-2}^{2}\int_{2y^2}^{4+y^2} xy^2\,dx\,dy = \int_{-2}^{2}\left(8y^2 + 4y^4 - \frac{3}{2}y^6\right) dy = \left[\frac{8}{3}y^3 + \frac{4}{5}y^5 - \frac{3}{14}y^7\right]_{-2}^{2} = \frac{4096}{105}.$$

Therefore the centroid of the lamina is located at the point with coordinates

$$(\overline{x},\,\overline{y}) \;=\; \left(\frac{32}{7},\,0\right) \;\approx\; (4.5714285714285714,\,0).$$

C14S0M.018: The mass and moments are

$$\begin{aligned} m &= \int_1^2 \int_0^{\ln x} \frac{1}{x}\,dy\,dx = \int_1^2 \left[\frac{y}{x}\right]_0^{\ln x} dx = \int_1^2 \frac{\ln x}{x}\,dx \\ &= \left[\frac{1}{2}(\ln x)^2\right]_1^2 = \frac{1}{2}(\ln 2)^2 \;\approx\; 0.2402265069591007; \end{aligned}$$

$$\begin{aligned} M_y &= \int_1^2 \int_0^{\ln x} \frac{x}{x}\,dy\,dx = \int_1^2 \ln x\,dx = \Big[-x + x\ln x\Big]_1^2 = -1 + 2\ln 2; \\ M_x &= \int_1^2 \int_0^{\ln x} \frac{y}{x}\,dy\,dx = \int_1^2 \left[\frac{y^2}{2x}\right]_0^{\ln x} dx = \int_1^2 \frac{(\ln x)^2}{2x}\,dx = \left[\frac{1}{6}(\ln x)^3\right]_1^2 = \frac{1}{6}(\ln 2)^3. \end{aligned}$$

Therefore the centroid of the lamina is located at the point

$$(\overline{x},\,\overline{y}) \;=\; \left(\frac{-2+4\ln 2}{(\ln 2)^2},\,\frac{\ln 2}{3}\right) \;\approx\; (1.608042201545,\,0.231049060187).$$

C14S0M.019: The mass and moments are

$$\begin{aligned} m &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} kr\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{2}kr^2\right]_{r=0}^{2\cos\theta} d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} 2k\cos^2\theta\,d\theta = \left[\frac{1}{2}k(2\theta + \sin 2\theta)\right]_{\theta=-\pi/2}^{\pi/2} = k\pi; \\ M_y &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} kr^2\cos\theta\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{3}kr^3\cos\theta\right]_{r=0}^{2\cos\theta} d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \frac{8}{3}k\cos^4\theta\,d\theta = \left[\frac{1}{12}k(12\theta + 8\sin 2\theta + \sin 4\theta)\right]_{\theta=-\pi/2}^{\pi/2} = k\pi; \end{aligned}$$

$$\begin{aligned} M_x &= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} kr^2\sin\theta\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{1}{3}kr^3\sin\theta\right]_{r=0}^{2\cos\theta} d\theta \\ &= \int_{\theta=-\pi/2}^{\pi/2} \frac{8}{3}k\cos^3\theta\,\sin\theta\,d\theta = \left[-\frac{2}{3}k\cos^4\theta\right]_{\theta=-\pi/2}^{\pi/2} = 0. \end{aligned}$$

Therefore the centroid of the lamina is located at the point $(1,\,0)$.

C14S0M.020: The mass and moments are

$$
\begin{aligned}
m &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}r^3\right]_0^{2\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2}\frac{8}{3}\cos^3\theta\,d\theta = \left[\frac{2}{9}(9\sin\theta+\sin 3\theta)\right]_{-\pi/2}^{\pi/2} = \frac{32}{9} \approx 3.5555555555555555;
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{4}r^4\cos\theta\right]_0^{2\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4\cos^5\theta\,d\theta = \frac{1}{60}\Big[150\sin\theta+25\sin 3\theta+3\sin 5\theta\Big]_{-\pi/2}^{\pi/2} = \frac{64}{15};
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\sin\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{4}r^4\sin\theta\right]_0^{2\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\,\sin\theta\,d\theta = \left[-\frac{4}{5}\cos^5\theta\right]_{-\pi/2}^{\pi/2} = 0.
\end{aligned}
$$

Therefore the centroid of the lamina is located at the point $\left(\frac{6}{5},\,0\right)$.

C14S0M.021: By the first theorem of Pappus, the y-coordinate $\overline{y}$ of the centroid must satisfy the equation

$$
2\pi\overline{y}\cdot\frac{1}{2}\pi ab \;=\; \frac{4}{3}\pi ab^2,
$$

and it follows immediately that $\overline{y} \;=\; \dfrac{4b}{3\pi}$.

C14S0M.022: Part (a): By symmetry, $\overline{x} = \overline{y}$, and by the first theorem of Pappus, $\overline{y}$ must satisfy the equation

$$
2\pi\overline{y}\left(\frac{1}{4}\pi b^2 - \frac{1}{4}\pi a^2\right) = \frac{2}{3}\pi(b^3-a^3),
$$

and therefore

$$
\overline{y} = \frac{4(b^3-a^3)}{3\pi(b^2-a^2)} = \frac{4(a^2+ab+b^2)}{3\pi(a+b)}.
$$

Part (b): $\displaystyle\lim_{b\to a}\frac{4(a^2+ab+b^2)}{3\pi(a+b)} = \frac{4\cdot 3a^2}{3\pi\cdot 2a} = \frac{2a}{\pi}.$

Therefore the centroid of the region is located at $\left(\dfrac{2a}{\pi},\,\dfrac{2a}{\pi}\right)$.

C14S0M.023: Assume that the lamina has constant density δ. By symmetry, $\overline{x} = 0$. The mass of the lamina and its moment with respect to the x-axis are

$$m = \int_{x=-2}^{2} \int_{y=0}^{4-x^2} \delta \, dy \, dx = \int_{x=-2}^{2} \delta(4 - x^2) \, dx = \delta \left[4x - \frac{1}{3}x^3 \right]_{-2}^{2} = \frac{32}{3}\delta \quad \text{and}$$

$$M_x = \int_{x=-2}^{2} \int_{y=0}^{4-x^2} \delta y \, dy \, dx = \int_{x=-2}^{2} \left[\frac{1}{2}\delta y^2 \right]_{y=0}^{4-x^2} dx = \frac{1}{2}\delta \int_{x=-2}^{2} (16 - 8x^2 + x^4) \, dx$$

$$= \frac{1}{2}\delta \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{-2}^{2} = \frac{256}{15}\delta.$$

Therefore the centroid of the lamina is at the point $\left(0, \frac{8}{5}\right)$.

C14S0M.024: The volume of the solid is

$$V = \int_0^1 \int_0^{1-y} x^2 \, dx \, dy = \int_0^1 \left[\frac{1}{3}x^3 \right]_0^{1-y} dy = \int_0^1 \frac{1}{3}(1-y)^3 \, dy = \left[-\frac{1}{12}(1-y)^4 \right]_0^1 = \frac{1}{12}.$$

C14S0M.025: The volume of the ice-cream cone is

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=2r}^{\sqrt{5-r^2}} r \, dz \, dr \, d\theta = 2\pi \int_{r=0}^{1} \left(r\sqrt{5-r^2} - 2r^2 \right) dr$$

$$= -\frac{2}{3}\pi \cdot \left[(5-r^2)^{3/2} + 2r^3 \right]_0^1 = \frac{10}{3}\pi\left(\sqrt{5} - 2\right) \approx 2.4720980795371330541036 26.$$

C14S0M.026: Assume constant density $\delta = 1$. Clearly $\overline{x} = \overline{y} = 0$. The mass and moment with respect to the xy-plane are

$$m = \int_0^{2\pi} \int_0^{\pi/3} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \int_0^{\pi/3} \frac{1}{3}a^3 \sin\phi \, d\phi = 2\pi \left[-\frac{1}{3}a^3 \cos\phi \right]_0^{\pi/3} = \frac{1}{3}\pi a^3 \qquad \text{and}$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\pi/3} \int_0^a \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\pi/3} \frac{1}{4}a^4 \sin\phi \cos\phi \, d\phi = 2\pi \left[\frac{1}{8}a^4 \sin^2\phi \right]_0^{\pi/3} = \frac{3}{16}\pi a^4.$$

Therefore the centroid is located at the point

$$(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{9}{16}a\right).$$

Because the density is 1, the volume is $V = \dfrac{1}{3}\pi a^3$ (numerically the same as the mass).

C14S0M.027: Let δ be the [constant] density of the cone and let h denote its height. Place the cone with its vertex at the origin and with its axis lying on the nonnegative z-axis. Then its mass is $M = \frac{1}{3}\pi\delta a^2 h$. The side of the cone has cylindrical equation $z = hr/a$, so the moment of inertia of the cone with respect to the z-axis is

$$I_z = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=hr/a}^{h} \delta r^3 \, dz \, dr \, d\theta = 2\pi\delta \int_{r=0}^{a} \left(r^3 h - \frac{r^4 h}{a} \right) dr$$

$$= 2\pi\delta \left[\frac{1}{4} r^4 h - \frac{1}{5a} r^5 h \right]_{r=0}^{a} = \frac{1}{10} \pi \delta a^4 h = \frac{3}{10} M a^2.$$

Note that the answer is plausible and dimensionally correct. One of our physics teachers, Prof. J. J. Kyame of Tulane University (retired), always insisted that we express moment of inertia in terms of mass, as here, so that the answer can be inspected for plausibility and dimensional accuracy.

C14S0M.028: The mass is

$$m = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{a} (\rho^3 \sin^2 \phi \, \cos \phi \sin \theta \cos \theta) \cdot (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{6} a^6 \sin^3 \phi \, \cos \phi \sin \theta \cos \theta \, d\phi \, d\theta = \int_0^{\pi/2} \left[\frac{1}{24} a^6 \sin^4 \phi \, \sin \theta \cos \theta \right]_0^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{24} a^6 \sin \theta \cos \theta \, d\theta = \left[\frac{1}{48} a^6 \sin^2 \theta \right]_0^{\pi/2} = \frac{1}{48} a^6.$$

C14S0M.029: We are given the solid ellipsoid E with constant density $\delta = 1$ and boundary the surface with Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Its moment of inertia with respect to the x-axis is then

$$I_x = \iiint_E (y^2 + z^2) \, dV.$$

We use the transformation

$$x = a\rho \sin \phi \cos \theta, \quad y = b\rho \sin \phi \sin \theta, \quad z = c\rho \cos \phi. \tag{1}$$

Under this transformation, E is replaced with the solid B determined by

$$0 \leqq \theta \leqq 2\pi, \quad 0 \leqq \phi \leqq \pi, \quad 0 \leqq \rho \leqq 1.$$

The Jacobian of the transformation in (1) is

$$\frac{\partial(x,\, y,\, z)}{\partial(\rho,\, \phi,\, \theta)} = \begin{vmatrix} a \sin \phi \cos \theta & a\rho \cos \phi \cos \theta & -a\rho \sin \phi \sin \theta \\ b \sin \phi \sin \theta & b\rho \cos \phi \sin \theta & b\rho \sin \phi \cos \theta \\ c \cos \phi & -c\rho \sin \phi & 0 \end{vmatrix} = abc\rho^2 \sin \phi.$$

Therefore

$$I_x = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \left[(b\rho \sin \phi \sin \theta)^2 + (c\rho \cos \phi)^2 \right] \cdot abc\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} (b^2 \sin^2\phi \, \sin^2\theta + c^2 \cos^2\phi) \cdot abc\rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{5} abc \int_0^{2\pi} \int_0^{\pi} (b^2 \sin^3\phi \, \sin^2\theta + c^2 \sin\phi \cos^2\phi) \, d\phi \, d\rho$$

$$= \frac{1}{5} abc \int_0^{2\pi} \int_0^{\pi} \left[b^2(1-\cos^2\phi) \sin\phi \sin^2\theta + c^2 \sin\phi \cos^2\phi \right] d\phi \, d\theta$$

$$= \frac{1}{5} abc \int_0^{2\pi} \left[\frac{1}{3} b^2 \cos^3\phi \, \sin^2\theta - b^2 \cos\phi \sin^2\theta - \frac{1}{3} c^2 \cos^3\phi \right]_0^{\pi} d\theta$$

$$= \frac{1}{5} abc \int_0^{2\pi} \left[-\frac{1}{3} b^2(1-\cos 2\theta) + b^2(1-\cos 2\theta) + \frac{2}{3} c^2 \right] d\theta$$

$$== \frac{1}{5} abc \left[-\frac{1}{3} b^2\theta + \frac{1}{6} b^2 \sin 2\theta + b^2\theta - \frac{1}{2} b^2 \sin 2\theta + \frac{2}{3} c^2\theta \right]_0^{2\pi}$$

$$= \frac{1}{5} abc \left(-\frac{2}{3}\pi b^2 + 2\pi b^2 + \frac{4}{3}\pi c^2 \right) = \frac{4}{15}\pi abc(b^2+c^2) = \frac{1}{5} M(b^2+c^2)$$

where $M = \frac{4}{3}\pi abc$ is the mass of E.

C14S0M.030: The region of Problem 30 is in the first octant, within the cylinder with cylindrical equation $r = a$, outside the sphere with spherical equation $\rho = a$, and below the plane with Cartesian equation $z = a$ (where $a > 0$). Thus its volume is

$$V = \int_0^{\pi/2} \int_0^{a} \int_{\sqrt{a^2-r^2}}^{a} r \, dz \, dr \, d\theta = \int_0^{\pi/2} \int_0^{a} \left(ar - r\sqrt{a^2-r^2} \right) dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} ar^2 + \frac{1}{3}(a^2-r^2)^{3/2} \right]_0^{a} d\theta = \frac{\pi}{2} \cdot \frac{1}{6} a^3 = \frac{1}{12}\pi a^3.$$

To check this answer, begin with a solid sphere of radius a. Circumscribe a right circular cylinder of the same radius. Subtract the volume of the sphere from that of the cylinder, then divide by 8 to get the volume of the part of the region in the first octant:

$$V = \frac{1}{8} \left(2\pi a^3 - \frac{4}{3}\pi a^3 \right) = \frac{1}{8} \cdot \frac{2}{3}\pi a^3 = \frac{1}{12}\pi a^3.$$

C14S0M.031: The cylinder $r = 2\cos\theta$ meets the xy-plane in the circle with equation $r = 2\cos\theta$, $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$. With density $\delta = 1$, the moment of inertia of the solid region with respect to the z-axis is

$$I_z = 2 \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{\sqrt{4-r^2}} r^3 \, dz \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^{2\cos\theta} r^3 \sqrt{4-r^2} \, dr \, d\theta$$

$$= 4 \int_0^{\pi/2} \left[\frac{1}{15}(3r^4 - 4r^2 - 32)\sqrt{4-r^2} \right]_0^{2\cos\theta} d\theta$$

$$= \frac{4}{15} \int_0^{\pi/2} (64 - 64\sin\theta - 32\sin\theta \, \cos^2\theta + 96\sin\theta \, \cos^4\theta) \, d\theta$$

$$= \frac{8}{225}\left[480\theta + 450\cos\theta - 25\cos 3\theta - 9\cos 5\theta\right]_0^{\pi/2} = \frac{128}{225}(15\pi - 26) \approx 12.0171461995217912.$$

The student who obtains the incorrect answer $\frac{128}{15}\pi$ may well have overlooked the fact that

$$\sqrt{4 - 4\cos^2\theta} = |2\sin\theta|.$$

C14S0M.032: The area element $dA = r\ dr\ d\theta$ moves around a circle of radius $x = r\cos\theta$, and hence of circumference $2\pi r\cos\theta$. So the volume generated is

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} 2\pi r^2\cos\theta\ dr\ d\theta = 2\int_0^{\pi/2}\left[\frac{2}{3}\pi r^3\cos\theta\right]_0^{2a\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}\left(\frac{16}{3}\pi a^3\cos^4\theta\right) d\theta = \left[\frac{1}{3}\pi a^3(12\theta + 8\sin 2\theta + \sin 4\theta)\right]_0^{\pi/2} = 2\pi^2 a^3.$$

C14S0M.033: The area element $r\ dr\ d\theta$ moves around a circle of radius $y = r\sin\theta$, and therefore of circumference $2\pi r\sin\theta$. Hence the volume swept out is

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{1+\cos\theta} 2\pi r^2\sin\theta\ dr\ d\theta = \int_{\theta=0}^{\pi}\left[\frac{2}{3}\pi r^3\sin\theta\right]_{r=0}^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi}\frac{2}{3}\pi(1+\cos\theta)^3\sin\theta\ d\theta = \left[-\frac{1}{6}\pi(1+\cos\theta)^4\right]_0^{\pi} = \frac{8}{3}\pi \approx 8.37758040957278196923.$$

C14S0M.034: Assume that $b \geqq 0$. Then the area element $r\ dr\ d\theta$ moves around a circle having radius $b + x = b + r\cos\theta$, and hence the volume swept out is

$$V = \int_0^{2\pi}\int_0^a 2\pi(b + r\cos\theta)r\ dr\ d\theta = \int_0^{2\pi}\left[\pi b r^2 + \frac{2}{3}\pi r^3\cos\theta\right]_0^a d\theta$$

$$= \int_0^{2\pi}\left(\pi a^2 b + \frac{2}{3}\pi a^3\cos\theta\right) d\theta = \left[\pi a^2 b\theta + \frac{2}{3}\pi a^3\sin\theta\right]_0^{2\pi} = 2\pi^2 a^2 b.$$

C14S0M.035: The moment of inertia of the torus of Problem 34 with respect to the line $x = -b$ (where $b \geqq 0$), its natural axis of symmetry, is

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} 2\pi\delta(b + r\cos\theta)^3\cdot r\ dr\ d\theta = \int_{\theta=0}^{2\pi}\pi\delta\left[b^3r^2 + 2b^2r^3\cos\theta + \frac{3}{2}br^4\cos^2\theta + \frac{2}{5}r^5\cos^3\theta\right]_{r=0}^{a} d\theta$$

$$= \pi\delta\int_0^{2\pi}\left(a^2b^3 + 2a^3b^2\cos\theta + \frac{3}{2}a^4b\cos^2\theta + \frac{2}{5}a^5\cos^3\theta\right) d\theta$$

$$= \frac{1}{120}\pi\delta a^2\left[90a^2b\theta + 120b^3\theta + 36a^3\sin\theta + 240ab^2\sin\theta + 45a^2b\sin 2\theta + 4a^3\sin 3\theta\right]_0^{2\pi}$$

$$= \frac{1}{120}\pi\delta a^2(180\pi a^2 b + 240\pi b^3) = \frac{1}{2}\pi^2\delta a^2 b(3a^2 + 4b^2) = \frac{1}{4}M(3a^2 + 4b^2)$$

where $M = 2\pi^2\delta a^2 b$ is the mass of the torus.

C14S0M.036: The average distance of points of a circular disk of radius a from its center is

$$\overline{d} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r^2\ dr\ d\theta = \frac{2}{a^2}\cdot\left[\frac{1}{3}r^3\right]_0^a = \frac{2a^3}{3a^2} = \frac{2}{3}a.$$

C14S0M.037: Use the disk bounded by the circle with polar equation $r = 2a\sin\theta,\ 0 \leqq \theta \leqq \pi$. Then the origin is a point on the boundary of the disk, and the average distance of points of this disk from the origin is

$$\begin{aligned}\overline{d} = \frac{1}{\pi a^2}\int_{\theta=0}^{\pi}\int_{r=0}^{2a\sin\theta} r^2\ dr\ d\theta &= \frac{1}{\pi a^2}\int_0^{\pi}\frac{8}{3}a^3\sin^3\theta\ d\theta \\ &= \frac{2a^3}{9\pi a^2}\left[\cos 3\theta - 9\cos\theta\right]_0^{\pi} = \frac{32}{9\pi}a \ \approx\ (1.131768484209)a.\end{aligned}$$

C14S0M.038: Use the circular disks bounded by $r = 2\cos\theta$ and $r = 4\cos\theta,\ -\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$. The two circles are tangent at the origin, and the average distance of points outside the small circle and inside the large circle from the origin is

$$\begin{aligned}\overline{d} = \frac{1}{3\pi}\int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^2\ dr\ d\theta = \frac{1}{3\pi}\int_{-\pi/2}^{\pi/2}\left[\frac{1}{3}r^3\right]_{2\cos\theta}^{4\cos\theta} d\theta &= \frac{1}{9\pi}\int_{-\pi/2}^{\pi/2} 56\cos^3\theta\ d\theta \\ &= \frac{14}{27\pi}\left[9\sin\theta + \sin 3\theta\right]_{-\pi/2}^{\pi/2} = \frac{224}{27\pi} \ \approx\ 2.6407931298210782.\end{aligned}$$

C14S0M.039: We use the ball bounded by the surface with spherical-coordinates equation $\rho = a$. Then the average distance of points of this ball from its center is

$$\begin{aligned}\overline{d} &= \frac{3}{4\pi a^3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{a}\rho^3\sin\phi\ d\rho\ d\phi\ d\theta \\ &= \frac{3}{4\pi a^3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\frac{1}{4}a^4\sin\phi\ d\phi\ d\theta = \frac{3}{4\pi a^3}\cdot 2\pi\cdot\left[-\frac{1}{4}a^4\cos\phi\right]_0^{\pi} = \frac{3}{4\pi a^3}\cdot 2\pi\cdot\frac{1}{2}a^4 = \frac{3}{4}a.\end{aligned}$$

C14S0M.040: We use the ball of radius a centered at the point $(0,\,0,\,a)$ on the z-axis. The average distance of points of this ball from the origin (its "south pole") is then

$$\begin{aligned}\overline{d} = \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi}\rho^3\sin\phi\ d\rho\ d\phi\ d\theta &= \frac{3}{4\pi a^3}\cdot 2\pi\cdot\int_0^{\pi/2} 4a^4\sin\phi\cos^4\phi\ d\phi \\ &= \frac{3}{2a^3}\left[-\frac{4}{5}a^4\cos^5\phi\right]_0^{\pi/2} = \frac{3}{2a^3}\cdot\frac{4}{5}a^4 = \frac{6}{5}a.\end{aligned}$$

C14S0M.041: We will use the spheres with spherical-coordinates equations $\rho = 2\cos\phi$ and $\rho = 4\cos\phi$, which have a mutual point of tangency at the origin. Then the average distance of points outside the smaller and inside the larger sphere from the origin is

$$\overline{d} = \frac{3}{28\pi}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=2\cos\phi}^{4\cos\phi}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{28\pi}\cdot 2\pi\cdot\int_{\phi=0}^{\pi/2}\left[\frac{1}{4}\rho^4\sin\phi\right]_{\rho=2\cos\phi}^{4\cos\phi}d\phi$$

$$= \frac{3}{14}\int_0^{\pi/2} 60\sin\phi\cos^4\phi\,d\phi = \frac{3}{14}\Big[-12\cos^5\phi\Big]_0^{\pi/2} = \frac{18}{7} \approx 2.5714285714285714.$$

C14S0M.042: Place the cone with its vertex at the origin and its axis on the nonnegative z-axis. Then a spherical-coordinates equation of its side is

$$\phi = \arctan\left(\frac{R}{H}\right)$$

and an equation of its base (at the top) is $\rho = H\sec\phi$. Hence the average distance of points of the cone from its vertex is

$$\overline{d} = \frac{3}{\pi R^2 H}\int_0^{2\pi}\int_0^{\arctan(R/H)}\int_0^{H\sec\phi}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3}{\pi R^2 H}\cdot 2\pi\cdot\int_0^{\arctan(R/H)}\frac{1}{4}H^4\sec^3\phi\,\tan\phi\,d\phi$$

$$= \frac{6}{R^2 H}\left[\frac{1}{12}H^4\sec^3\phi\right]_0^{\arctan(R/H)} = \frac{6}{R^2 H}\cdot\frac{1}{12}\cdot\left[H(H^2+R^2)^{3/2} - H^4\right] = \frac{(H^2+R^2)^{3/2} - H^3}{2R^2}.$$

Thus the average distance of points of the cone from its vertex is

$$\overline{d} = \frac{L^3 - H^3}{2R^2}$$

where $L = \sqrt{R^2 + H^2}$ is the slant height of the cone.

C14S0M.043: The part of the paraboloid that lies between the two given planes also is the part between the cylinders $r = 2$ and $r = 3$. Let R denote the part of the xy-plane between those two cylinders. Then the surface area in question is

$$A = \iint_R \sqrt{r^2 + (rz_r)^2 + (z_\theta)^2}\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=2}^{3}\sqrt{r^2+4r^4}\,dr\,d\theta = 2\pi\int_2^3 r(1+4r^2)^{1/2}\,dr$$

$$= 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_2^3 = \frac{1}{6}\pi\left(37\sqrt{37} - 17\sqrt{17}\right) \approx 81.1417975124065455.$$

C14S0M.044: Let D denote the circular disk $x^2 + y^2 \leqq 4$. Then the surface area is

$$A = \iint_D \sqrt{1 + (z_x)^2 + (z_y)^2}\,dx\,dy = \iint_D \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^2 r(1+4r^2)^{1/2}\,dr\,d\theta = 2\pi\left[\frac{1}{12}(1+4r^2)^{1/2}\right]_0^2 = \frac{1}{6}\pi\left(17\sqrt{17} - 1\right) \approx 36.1769031974114084.$$

C14S0M.045: Let R be the region in the $\phi\theta$-plane determined by the inequalities $\phi_2 \leqq \phi \leqq \phi_1$ and $0 \leqq \theta \leqq 2\pi$, where

$$\cos\phi_2 = \frac{z_2}{a} \quad\text{and}\quad \cos\phi_1 = \frac{z_1}{a}.$$

Thus the part of the sphere $\rho = a$ for which the spherical coordinates ϕ and θ satisfy these inequalities is the part of the sphere between the planes $z = z_1$ and $z = z_2$. Thus the formula in Problem 18 of Section 14.8 yields the area of this surface to be

$$A = \iint_R a^2 \sin\phi \, d\phi \, d\theta = \int_{\theta=0}^{2\pi} \int_{\phi=\phi_2}^{\phi_1} a^2 \sin\phi \, d\phi \, d\theta$$

$$= 2\pi a^2 \Big[+\cos\phi \Big]_{\phi_1}^{\phi_2} = 2\pi a^2 \left(\frac{z_2 - z_1}{a} \right) = 2\pi a(z_2 - z_1) = 2\pi a h$$

because $h = z_2 - z_1$.

C14S0M.046: The graph of $z(r, \theta) = \sqrt{4 - r^2}$ is the top half of the sphere, so we will need to double the area integral. Let D be the plane region bounded by the circle with Cartesian equation $x^2 + y^2 = 2x$; the polar-coordinates equation of this circle is $r = 2\cos\theta$, $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$. The area of the part of the sphere above this circle plus the area of the part below it is then

$$A = 2 \iint_D \sqrt{r^2 + (rz_r)^2 + (z_\theta)^2} \, dr \, d\theta = 2 \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{2r}{(4 - r^2)^{1/2}} \, dr \, d\theta$$

$$= 4 \int_0^{\pi/2} \Big[-2(4 - r^2)^{1/2} \Big]_0^{2\cos\theta} d\theta = 4 \int_0^{\pi/2} \big[4 - 2(4 - 4\cos^2\theta)^{1/2} \big] \, d\theta$$

$$= 4 \int_0^{\pi/2} (4 - 4\sin\theta) \, d\theta = 4 \Big[4\theta + 4\cos\theta \Big]_0^{\pi/2} = 4 \cdot 4 \cdot \frac{\pi}{2} - 16 = 8(\pi - 2) \approx 9.13274123.$$

The student who obtains the incorrect answer $2\pi^2$ likely forgot that $\sqrt{4 - 4\cos^2\theta} \neq 2\sin\theta$ if $-\pi < \theta < 0$.

C14S0M.047: Position the cone with its vertex at the origin and its axis on the nonnegative z-axis. The side of the cone has Cartesian equation $z = \sqrt{x^2 + y^2}$, and hence

$$dS = \sqrt{1 + (z_x)^2 + (z_y)^2} \, dx \, dy = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} \, dx \, dy = \sqrt{2} \, dx \, dy.$$

Let S be the square with vertices at $(\pm 1, \pm 1)$. Because the area of S is 4, we see with no additional computations that the area of the part of the cone that lies directly above S is

$$\iint_S \sqrt{2} \, dx \, dy = 4\sqrt{2} \approx 5.6568542494923802.$$

C14S0M.048: Let $z(x, y) = \frac{1}{2}x^2$. Let D be the disk in the xy-plane bounded by the circle $x^2 + y^2 = 1$. Then the area of the part of the parabolic cylinder that lies over D is

$$A = \iint_D \sqrt{1 + (z_x)^2 + (z_y)^2} \, dA = 4 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1 + x^2} \, dy \, dx = 4 \int_0^1 \sqrt{1 - x^4} \, dx.$$

The antiderivative of $\sqrt{1 - x^4}$ is known to be nonelementary. (See Sherman K. Stein and Anthony Barcellos: *Calculus and Analytic Geometry* (McGraw-Hill: New York, 1992), page 460, Problems 164–174.) Therefore we next used *Mathematica* 3.0 for a numerical integration. The command

```
NIntegrate[ 4*Sqrt[1 - x∧4], {x,0,1}, AccuracyGoal → 24, WorkingPrecision → 30 ]
```

yielded the approximation $A \approx 3.49607673905615974729$.

C14S0M.049: Given: $x = x(t),\ y = y(t),\ z = z,\ a \leqq t \leqq b,\ 0 \leqq z \leqq h(t)$: Let

$$\mathbf{r}(t,\, z) = \langle\, x(t),\, y(t),\, z\,\rangle, \quad \text{so that} \quad \mathbf{r}_t = \langle\, x'(t),\, y'(t),\, 0\,\rangle \quad \text{and} \quad \mathbf{r}_z = \langle\, 0,\, 0,\, 1\,\rangle.$$

Then

$$\mathbf{r}_t \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle\, y'(t),\, -x'(t),\, 0\,\rangle,$$

and hence

$$|\mathbf{r}_t \times \mathbf{r}_z| = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\,.$$

Therefore the area of the "fence" is

$$A = \int_{t=a}^{b} \int_{z=0}^{h(t)} \Big(\big[x'(t)\big]^2 + \big[y'(t)\big]^2 \Big)^{1/2}\ dz\ dt.$$

C14S0M.050: The "fence" stands above the plane curve $r = a\sin\theta,\ 0 \leqq \theta \leqq \pi$. Thus we take the height function $h(\theta) = \sqrt{a^2 - r^2}$ as the upper limit of integration in the formula in Problem 49, where

$$z = \sqrt{a^2 - r^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$$

provided that $0 \leqq \theta \leqq \frac{1}{2}\pi$. Moreover, we have

$$x(\theta) = a\sin\theta\cos\theta, \quad y(\theta) = a\sin^2\theta, \quad \text{and} \quad z = z$$

where $0 \leqq \theta \leqq \pi$ and $0 \leqq z \leqq |a\cos\theta|$. Thus

$$\begin{aligned} \big[x'(\theta)\big]^2 + \big[y'(\theta)\big]^2 &= a^2(\cos^2\theta - \sin^2\theta)^2 + 4a^2\sin^2\theta\,\cos^2\theta \\ &= a^2(\cos^4\theta + 2\sin^2\theta\,\cos^2\theta + \sin^4\theta) = a^2(\cos^2\theta + \sin^2\theta)^2 = a^2. \end{aligned}$$

We double the integral to allow for the fact that an equal height of the "fence" stands *below* the xy-plane, and double it again in order to restrict the range of θ to the interval $0 \leqq \theta \leqq \frac{1}{2}\pi$. Thus the area in question is

$$A = 4\int_{\theta=0}^{\pi/2} \int_{z=0}^{h(\theta)} a\ dz\ d\theta = 4a\int_0^{\pi/2} \int_0^{a\cos\theta} 1\ dz\ d\theta = 4a\int_0^{\pi/2} a\cos\theta\ d\theta = 4a^2\Big[\ \sin\theta\ \Big]_0^{\pi/2} = 4a^2.$$

C14S0M.051: We are given the region R bounded by the curves $x^2 - y^2 = 1,\ x^2 - y^2 = 4,\ xy = 1$, and $xy = 3$, of constant density δ. Its polar moment of inertia is

$$I_0 = \iint_R (x^2 + y^2)\,\delta\ dx\ dy.$$

The hyperbolas bounding R are u-curves and v-curves if we let $u = xy$ and $v = x^2 - y^2$. If we make this substitution, then

$$4u^2 + v^2 = 4x^2y^2 - (x^2 - y^2)^2 = (x^2 + y^2)^2,$$

and therefore we will substitute $\sqrt{4u^2 + v^2}$ for $x^2 + y^2$ in the integral for I_0. Moreover, it is not necessary to solve for x and y in terms of u and v because of a result in Section 14.9 (see the proof in Problem 18 there). Thus

$$\frac{\partial(u,\, v)}{\partial(x,\, y)} = \begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix} = -2(x^2 + y^2),$$

and therefore

$$\frac{\partial(x,\, y)}{\partial(u,\, v)} = -\frac{1}{2(x^2 + y^2)} = -\frac{1}{2\sqrt{4u^2 + v^2}}.$$

Therefore

$$I_0 = \int_{v=1}^{4} \int_{u=1}^{3} \frac{\sqrt{4u^2 + v^2}}{2\sqrt{4u^2 + v^2}}\, \delta\, du\, dv = \int_1^4 \int_1^3 \frac{1}{2}\delta\, du\, dv = 3\delta.$$

C14S0M.052: The equations $u = x - y,\ v = x + y$ are easy to solve for

$$x = \frac{u + v}{2}, \quad y = \frac{-u + v}{2}. \tag{1}$$

Then the curves that bound the region R are transformed as follows:

$$\begin{aligned} x + y = 1 \quad &\text{becomes} \quad v = 1; \\ y = 0 \quad &\text{becomes} \quad u = v; \\ x = 0 \quad &\text{becomes} \quad u = -v. \end{aligned}$$

The Jacobian of the transformation in (1) is

$$\frac{\partial(x,\, y)}{\partial(u,\, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Therefore

$$\begin{aligned} \iint_R \exp\left(\frac{x - y}{x + y}\right) dx\, dy &= \int_{v=0}^{1} \int_{u=-v}^{v} \frac{1}{2} \exp\left(\frac{u}{v}\right) du\, dv = \frac{1}{2} \int_{v=0}^{1} \left[v \exp\left(\frac{u}{v}\right) \right]_{u=-v}^{v} dv \\ &= \frac{1}{2} \int_0^1 v \cdot \left(e - \frac{1}{e}\right) dv = \frac{1}{2} \left[\frac{1}{2} v^2 \left(e - \frac{1}{e}\right) \right]_0^1 = \frac{e^2 - 1}{4e} \approx 0.5876005968219007. \end{aligned}$$

C14S0M.053: We use the transformation

$$x = a\rho \sin\phi \cos\theta, \quad y = b\rho \sin\phi \sin\theta, \quad z = c\rho \cos\phi.$$

We saw in the solution of Problem 29 that the Jacobian of this transformation is

$$\frac{\partial(x,\, y,\, z)}{\partial(\rho,\, \phi,\, \theta)} = abc\rho^2 \sin\phi.$$

Moreover, it follows from work shown in the solution of Problem 29 that the density function takes the form $\delta(\rho,\, \phi,\, \theta) = 1 - \rho^2$. Finally, the ellipsoidal surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is transformed into the surface $\rho = 1$, and therefore the mass of the solid ellipsoid is

$$\begin{aligned} m &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} (1-\rho^2)abc\rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = 2\pi abc \int_{\phi=0}^{\pi} \int_{\rho=0}^{1} (\rho^2 - \rho^4)\sin\phi \; d\rho \; d\phi \\ &= 2\pi abc \int_0^{\pi} \frac{2}{15} \sin\phi \; d\phi = \frac{4}{15}\pi abc \Big[-\cos\phi\Big]_0^{\pi} = \frac{8}{15}\pi abc. \end{aligned}$$

C14S0M.054: If $r^2 = u^{1/2}\cos 2\theta = v^{1/2}\sin 2\theta$, then $r^4 = u\cos^2 2\theta = v\sin^2 2\theta$. Hence

$$u = \frac{r^4}{\cos^2 2\theta} \qquad \text{and} \qquad v = \frac{r^4}{\sin^2 2\theta}.$$

Thus

$$\frac{uv}{u+v} = \frac{r^8}{\sin^2 2\theta \, \cos^2 2\theta} \cdot \frac{\sin^2 2\theta \, \cos^2 2\theta}{r^4 \cos^2 2\theta + r^4 \sin^2 2\theta} = \frac{r^4}{\cos^2 2\theta + \sin^2 2\theta} = r^4$$

and

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{r^2}{v^{1/2}} \cdot \frac{u^{1/2}}{r^2} = \frac{u^{1/2}}{v^{1/2}}.$$

Therefore

$$\theta = \frac{1}{2}\arctan\left(\frac{u^{1/2}}{v^{1/2}}\right) \qquad \text{and} \qquad r = \left(\frac{uv}{u+v}\right)^{1/4}.$$

To find the Jacobian of this transformation from the uv-plane to the $r\theta$-plane, note first that

$$\cos 2\theta = \frac{r^2}{u^{1/2}} \qquad \text{and} \qquad \sin 2\theta = \frac{r^2}{v^{1/2}}.$$

After the next computation, we will use a result in Section 14.9 (see Problem 18 there).

$$\frac{\partial(u,\, v)}{\partial(r,\, \theta)} = \begin{vmatrix} 4r^3 \sec^2 2\theta & 4r^4 \sec^2 2\theta \, \tan 2\theta \\ 4r^3 \csc^2 2\theta & -4r^4 \csc^2 2\theta \, \cot 2\theta \end{vmatrix} = -16r^7 \sec^3 2\theta \, \csc^3 2\theta,$$

and thus

$$\begin{aligned} \frac{\partial(r,\, \theta)}{\partial(u,\, v)} &= -\frac{\sin^3 2\theta \, \cos^3 2\theta}{16r^7} = -\frac{r^{12}}{16r^7 u^{3/2} v^{3/2}} \\ &= -\frac{r^6}{16ru^{3/2}v^{3/2}} = -\frac{u^{3/2}v^{3/2}}{16r(u+v)^{3/2}u^{3/2}v^{3/2}} = -\frac{1}{16r(u+v)^{3/2}}. \end{aligned}$$

The transformation we are using has the following effect on the lemniscates that bound the region R:

$$r^2 = 4\sin 2\theta \quad \text{becomes} \quad v^{1/2} = 4;$$

$$r^2 = 3\sin 2\theta \quad \text{becomes} \quad v^{1/2} = 3;$$

$$r^2 = 4\cos 2\theta \quad \text{becomes} \quad u^{1/2} = 4;$$

$$r^2 = 3\cos 2\theta \quad \text{becomes} \quad u^{1/2} = 3.$$

Therefore the area of R is

$$\begin{aligned} A &= \iint_R r\,dr\,d\theta = \int_{v=9}^{16}\int_{u=9}^{16} \frac{1}{16(u+v)^{3/2}}\,du\,dv = \int_9^{16}\left[-\frac{1}{8(u+v)^{1/2}}\right]_9^{16} dv \\ &= \frac{1}{8}\int_9^{16}\left[\frac{1}{(v+9)^{1/2}} - \frac{1}{(v+16)^{1/2}}\right] dv = \frac{1}{4}\Big[(v+9)^{1/2} - (v+16)^{1/2}\Big]_9^{16} \\ &= \frac{10-7\sqrt{2}}{4} \approx 0.0251262658470836645\,97. \end{aligned}$$

C14S0M.055: The spherical surface with radius $\sqrt{3}$ centered at the origin has equation $x^2+y^2+z^2=3$, and hence the upper hemisphere has equation $z=\sqrt{3-x^2-y^2}$. Next,

$$1 + (z_x)^2 + (z_y)^2 = 1 + \frac{x^2}{3-x^2-y^2} + \frac{y^2}{3-x^2-y^2} = \frac{3}{3-x^2-y^2}.$$

We integrate over the unit square in the xy-plane, quadruple the result to find the area of the part of the surface above the 2-by-2 square, then double it to account for the spherical surface *below* the xy-plane. Thus the surface area is

$$\begin{aligned} A &= 8\int_{x=0}^1\int_{y=0}^1 \frac{\sqrt{3}}{\sqrt{3-x^2-y^2}}\,dy\,dx = 8\int_{x=0}^1\left[\sqrt{3}\ \arctan\left(\frac{y}{\sqrt{3-x^2-y^2}}\right)\right]_{y=0}^1 \\ &= 8\int_0^1 \sqrt{3}\ \arctan\left(\frac{1}{\sqrt{2-x^2}}\right) dx = \int_{x=0}^1 8\sqrt{3}\ \arcsin\left(\frac{1}{\sqrt{3-x^2}}\right) dx. \end{aligned}$$

Now use integration by parts with

$$u = 8\sqrt{3}\ \arcsin\frac{1}{\sqrt{3-x^2}}, \qquad dv = dx;$$

$$du = \frac{8x\sqrt{3}}{(3-x^2)\sqrt{2-x^2}}\,dx, \qquad v = x.$$

Thus we find that

$$\begin{aligned} A &= \left[8x\sqrt{3}\ \arcsin\frac{1}{\sqrt{3-x^2}}\right]_0^1 - 8\sqrt{3}\int_0^1 \frac{x^2}{(3-x^2)\sqrt{2-x^2}}\,dx \\ &= 8\sqrt{3}\ \arcsin\frac{1}{\sqrt{2}} - 8\sqrt{3}\int_0^1 \frac{x^2}{(3-x^2)\sqrt{2-x^2}}\,dx = 2\pi\sqrt{3} - 8\sqrt{3}\int_0^1 \frac{x^2}{(3-x^2)\sqrt{2-x^2}}\,dx. \end{aligned}$$

Now make the substitution $x = \sqrt{2}\ \sin\theta,\ dx = \sqrt{2}\ \cos\theta\ d\theta$. This yields

$$\begin{aligned}
A &= 2\pi\sqrt{3}\ - 8\sqrt{3}\int_0^{\pi/4} \frac{2\sin^2\theta}{(3-2\sin^2\theta)}\ d\theta \\
&= 2\pi\sqrt{3}\ - 8\sqrt{3}\int_0^{\pi/4} \frac{1-\cos 2\theta}{3-2\sin^2\theta}\ d\theta = 2\pi\sqrt{3}\ - 4\sqrt{3}\int_{\phi=0}^{\pi/2} \frac{1-\cos\phi}{2+\cos\phi}\ d\phi
\end{aligned}$$

where $\phi = 2\theta$. Now substitute

$$u = \tan\frac{\phi}{2}, \quad \sin\phi = \frac{2u}{1+u^2}, \quad \cos\phi = \frac{1-u^2}{1+u^2}, \quad d\phi = \frac{2\ du}{1+u^2}$$

(see the discussion immediately following Miscellaneous Problem 134 of Chapter 8 (Chapter 7 of the "early transcendentals version")). This yields

$$\begin{aligned}
A &= 2\pi\sqrt{3}\ - 4\sqrt{3}\int_0^1 \frac{4u^2}{(u^2+1)(u^2+3)}\ du = 2\pi\sqrt{3}\ - 8\sqrt{3}\int_0^1 \left(\frac{3}{u^2+3} - \frac{1}{u^2+1}\right) du \\
&= 2\pi\sqrt{3}\ - 8\sqrt{3}\left[\sqrt{3}\ \arctan\left(\frac{u}{\sqrt{3}}\right) - \arctan u\right]_0^1 = 2\pi\sqrt{3}\ - \left(8\sqrt{3}\right)\cdot\frac{\pi}{12}\left(2\sqrt{3}\ - 3\right) \\
&= 4\pi\left(\sqrt{3}\ - 1\right)\ \approx\ 9.1992217564514412532 8.
\end{aligned}$$

C14S0M.056: We will find the volume of the part of the solid that lies in the first octant, then multiply by 8. Thus the volume is

$$V = 8\int_{x=0}^{a}\int_{y=0}^{(a^{2/3}-x^{2/3})^{3/2}} (a^{2/3} - x^{2/3} - y^{2/3})^{3/2}\ dy\ dx.$$

The substitution $y = b\sin^3\theta$ transforms the integrand into

$$(a^{2/3} - x^{2/3} - b^{2/3}\sin^2\theta)^{3/2},$$

and hence will be useful provided that $b^{2/3} = a^{2/3} - x^{2/3}$; for this reason, we let $b = (a^{2/3} - x^{2/3})^{3/2}$. Then the substitution

$$y = (a^{2/3} - x^{2/3})^{3/2}\sin^3\theta, \quad dy = 3(a^{2/3} - x^{2/3})^{3/2}\sin^2\theta\ \cos\theta\ d\theta$$

yields

$$\begin{aligned}
V &= 8\int_0^a\int_0^{\pi/2} \left[(a^{2/3} - x^{2/3})^{3/2}\cos^3\theta\right]\cdot 3(a^{2/3} - x^{2/3})^{3/2}\sin^2\theta\ \cos\theta\ d\theta\ dx \\
&= 24\int_0^a\int_0^{\pi/2} (a^{2/3} - x^{2/3})^3\sin^2\theta\ \cos^4\theta\ d\theta\ dx = 24\int_0^a (a^{2/3} - x^{2/3})^3\int_0^{\pi/2} (\cos^4\theta - \cos^6\theta)\ d\theta\ dx.
\end{aligned}$$

Then Formula 117 of the long table of integrals yields

$$\begin{aligned}
V &= 24\int_0^a (a^{2/3} - x^{2/3})^3\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2}\ -\ \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{\pi}{2}\right) dx = 24\int_0^a \frac{\pi}{32}(a^{2/3} - x^{2/3})^3\ dx \\
&= \frac{3}{4}\pi\left[a^2x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3\right]_0^a = \frac{3}{4}\pi\cdot\frac{16}{105}a^3 = \frac{4}{35}\pi a^3.
\end{aligned}$$

C14S0M.057: We will find the volume of the part of the solid that lies in the first octant, then multiply by 8. Thus the volume of the entire solid is

$$V = 8\int_{x=0}^{a}\int_{y=0}^{(a^{1/3}-x^{1/3})^3} (a^{1/3} - x^{1/3} - y^{1/3})^3\ dy\ dx.$$

The substitution $y = b\sin^6\theta$ transforms the integrand into

$$(a^{1/3} - x^{1/3} - b^{1/3}\sin^2\theta)^3,$$

and hence will be useful provided that $b^{1/3} = a^{1/3} - x^{1/3}$. Thus we choose $b = (a^{1/3} - x^{1/3})^3$, and the substitution

$$y = (a^{1/3} - x^{1/3})^3\sin^6\theta, \quad dy = 6(a^{1/3} - x^{1/3})^3\sin^5\theta\ \cos\theta\ d\theta$$

then yields

$$\begin{aligned}
V &= 8\int_{x=0}^{a}\int_{\theta=0}^{\pi/2} \left[(a^{1/3} - x^{1/3})\cos^2\theta\right]^3 \cdot 6(a^{1/3} - x^{1/3})^3\sin^5\theta\ \cos\theta\ d\theta \\
&= 48\int_0^a\int_0^{\pi/2} (a^{1/3} - x^{1/3})^6\sin^5\theta\ \cos^7\theta\ d\theta \\
&= 48\int_0^a (a^{1/3} - x^{1/3})^6\left[\frac{1}{122880}(-600\cos 2\theta - 75\cos 4\theta\right. \\
&\qquad\qquad\qquad \left. + 100\cos 6\theta + 30\cos 8\theta - 12\cos 10\theta - 5\cos 12\theta)\right]_0^{\pi/2} dx \\
&= 48\int_0^a \frac{1}{120}(a^{1/3} - x^{1/3})^6\ dx = \frac{2}{5}\int_0^a (a^{1/3} - x^{1/3})^6\ dx \\
&= \frac{2}{5}\left[a^2x - \frac{9}{2}a^{5/3}x^{4/3} + 9a^{4/3}x^{5/3} - 10ax^2 + \frac{45}{7}a^{2/3}x^{7/3} - \frac{9}{4}a^{1/3}x^{8/3} + \frac{1}{3}x^3\right]_0^a = \frac{2}{5}\cdot\frac{1}{84}a^3 = \frac{1}{210}a^3.
\end{aligned}$$

C14S0M.058: The average squared distance of points of the ellipsoid from its center (at (0, 0, 0)) is

$$\frac{1}{V}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \left[(a\sin\phi\cos\theta)^2 + (b\sin\phi\sin\theta)^2 + (c\cos\phi)^2\right]\cdot\rho^2\cdot abc\rho^2\sin\phi\ d\rho\ d\phi\ d\theta \tag{1}$$

where $V = \frac{4}{3}\pi abc$ is the volume of the ellipsoid. To obtain this integral, we began with the solid ellipsoid E given in Problem 58. We set up the triple integral

$$\frac{1}{V}\iiint_E (x^2 + y^2 + z^2)\ dV,$$

then converted it—using ellipsoidal coordinates, as in Problems 20, 21, 25, and 27 of Section 14.9—to the integral shown in (1). (The expression $abc\rho^2\sin\phi$ that appears there is the Jacobian of the transformation.) Then we evaluated the integral in (1) using *Mathematica* 3.0, as follows.

```
Integrate[ ((a*Sin[phi]*Cos[theta])^2 + (b*Sin[phi]*Sin[theta])^2
     + (c*Cos[phi])^2)*(a*b*c*rho^4*Sin[phi]), rho ]
```

$$\frac{1}{5}(abc\rho^5 \sin\phi)(c^2\cos^2\phi + a^2\cos^2\theta\,\sin^2\phi + b^2\sin^2\phi\,\sin^2\theta)$$

```
(% /.  rho → 1) - (% /.  rho → 0)
```

$$\frac{1}{5}(abc\sin\phi)(c^2\cos^2\phi + a^2\cos^2\theta\,\sin^2\phi + b^2\sin^2\phi\,\sin^2\theta)$$

```
Simplify[ Integrate[ %, phi ] ]
```

$$\frac{1}{120}(abc\cos\phi)(-10a^2 - 10b^2 - 4c^2 + 2(a^2+b^2-2c^2)\cos 2\phi + (a^2-b^2)\cos 2(\phi-\theta) - 10a^2\cos 2\theta + 10b^2\cos 2\theta + a^2\cos 2(\phi+\theta) - b^2\cos 2(\phi+\theta))$$

```
Simplify[ (% /.  phi → Pi) - (% /.  phi → 0) ]
```

$$\frac{2}{15}abc(a^2+b^2+c^2+(a^2-b^2)\cos 2\theta$$

```
Integrate[ %, theta ]
```

$$\frac{1}{15}(2a^3bc\theta + 2ab^3c\theta + 2abc^3\theta + a^3bc\sin 2\theta - ab^3c\sin 2\theta))$$

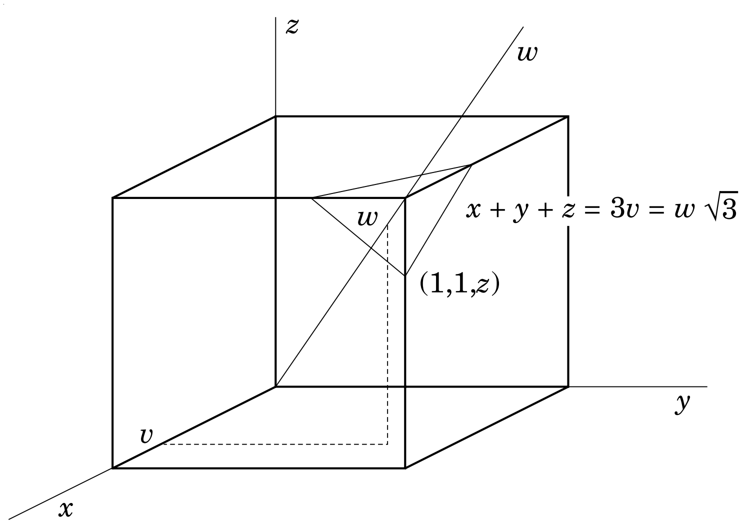
```
Factor[ (% /.  theta → 2*Pi) - (% /.  theta → 0) ]
```

$$\frac{4}{15}\pi abc(a^2+b^2+c^2)$$

```
%/((4/3)*Pi*a*b*c)
```

$$\frac{1}{5}(a^2+b^2+c^2)$$

C14S0M.059: Locate the cube C as shown in the next figure, with one vertex at the origin and the opposite vertex at the point (1, 1, 1) in space. Let L be the line through these two points; we will rotate C around the line L to generate the solid S. We also install a coordinate system on L; it becomes the w-axis, with $w = 0$ at the origin and $w = \sqrt{3}$ at the point with Cartesian coordinates (1, 1, 1). Thus distance is measured on the w-axis in exactly the same way it is measured on the three Cartesian coordinate axes.



Choose a point v on the x-axis with $\frac{1}{2} \leqq v \leqq 1$. We first deal with the case $\frac{2}{3} \leqq v \leqq 1$. Then the point with Cartesian coordinates $(v,\, v,\, v)$ determines a point $w = v\sqrt{3}$ on the w-axis, and the plane normal to the w-axis at this point intersects C in a triangle, also shown in the preceding figure. It is clear that the plane has equation $x+y+z = 3v = w\sqrt{3}$ and that it meets one edge of the cube at the point $(1,\, 1,\, z)$ for some z between 0 and 1. In fact, because $(1,\, 1,\, z)$ satisfies the equation of the plane, it follows that $z = -2+w\sqrt{3}$.

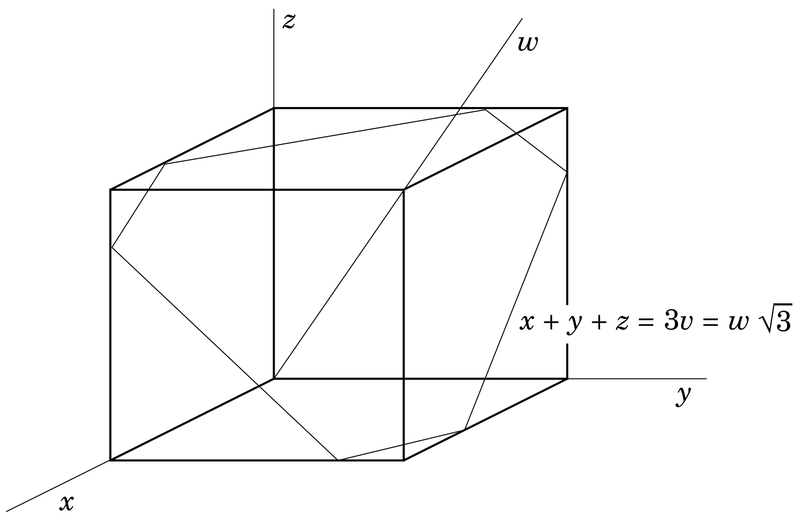
When the cube is rotated around L, the resulting solid S meets the plane $x+y+z = 3v$ in a circular disk centered at $(v,\, v,\, v)$, and the radius of this disk is the distance from $(v,\, v,\, v)$ to $(1,\, 1,\, z)$, which is

$$\begin{aligned}\sqrt{(v-1)^2+(v-1)^2+(w\sqrt{3}\,-2-v)^2} &= \sqrt{2(v-1)^2+(3v-2-v)^2} \\ &= \sqrt{6(v-1)^2} = (1-v)\sqrt{6} = \left(1-\tfrac{1}{3}w\sqrt{3}\right)\cdot\sqrt{6}\,.\end{aligned}$$

Now we turn to the case $\frac{1}{2} \leqq v \leqq \frac{2}{3}$. In this case the plane through $(v,\, v,\, v)$ meets the surface of the cube in a semi-regular hexagon, one in which each interior angle is $2\pi/3$ and whose sides are of only two different lengths a and b, alternating as one moves around the hexagon. Such a hexagon is shown in the next figure.

One of the vertices of the hexagon is located at the point $(1,\, y,\, 0)$ on one edge of the cube. The distance from $(v,\, v,\, v)$ to this point is the radius of the circular disk in which the plane normal to the w-axis at $(v,\, v,\, v)$ meets the solid S. It is easy to show that $y = 3v-1$, and it follows that the distance in question is

$$\sqrt{(v-1)^2+(2v-1)^2+v^2} = \sqrt{6v^2-6v+2} = \sqrt{2w^2-2w\sqrt{3}\,+2}\,.$$



By considering only values of v for which $\frac{1}{2} \leqq v \leqq 1$, we obtain only half of the solid S, so we now can find the volume V of S as follows. We shift to coordinates on the w-axis and remember that $dw = \sqrt{3}\; dv$. The result is that

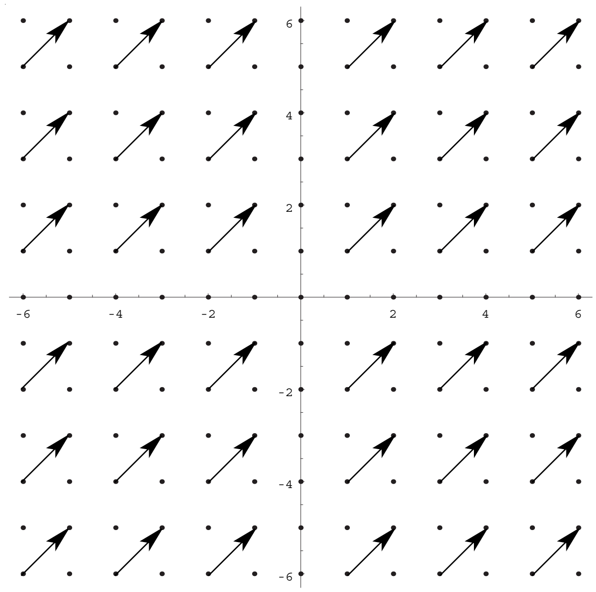
$$V = 2\int_{v=1/2}^{2/3} \pi(2w^2-2w\sqrt{3}\,+2)\sqrt{3}\; dw + 2\int_{v=2/3}^{1} 6\pi\left(1-\tfrac{1}{3}w\sqrt{3}\right)^2\sqrt{3}\; dw.$$

The adjusted limits of integration are $w = \frac{1}{2}\sqrt{3}$ to $\frac{2}{3}\sqrt{3}$ in the first integral and $w = \frac{2}{3}\sqrt{3}$ to $\sqrt{3}$ in the second. Then *Mathematica* 3.0 promptly reports that

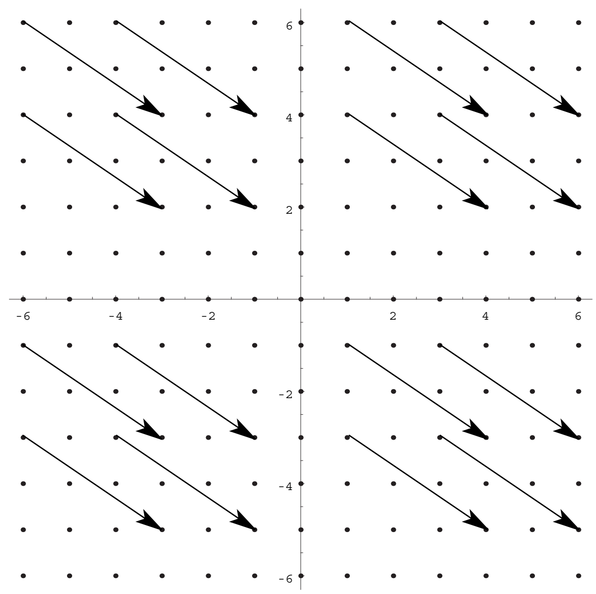
$$V = \frac{\pi}{\sqrt{3}} \approx 1.8137993642342178505940782576421557 32.$$

Section 15.1

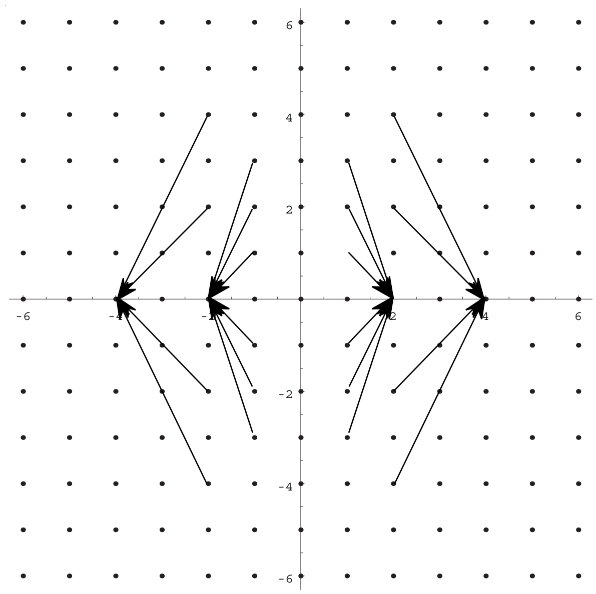
C15S01.001: $\mathbf{F}(x, y) = \langle 1, 1 \rangle$ is a constant vector field; some vectors in this field are shown next.



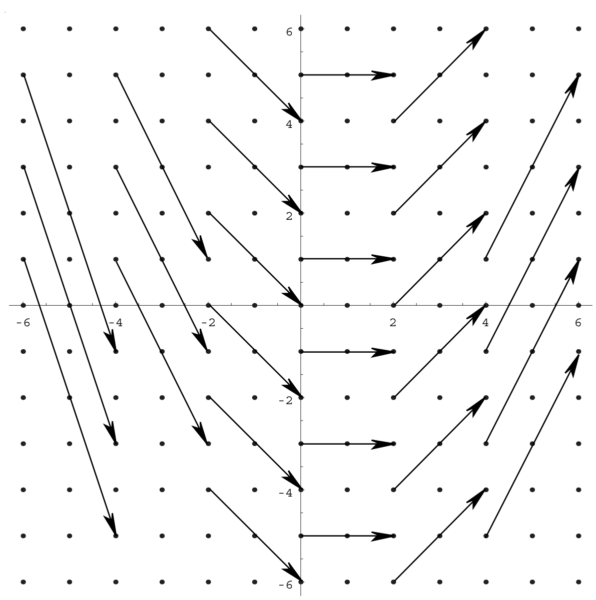
C15S01.002: The vector field $\mathbf{F}(x, y) = \langle 3, -2 \rangle$ is a constant vector field. Some typical vectors in this field are shown next.



C15S01.003: Some typical vectors in the field $\mathbf{F}(x, y) = \langle x, -y \rangle$ are shown next.

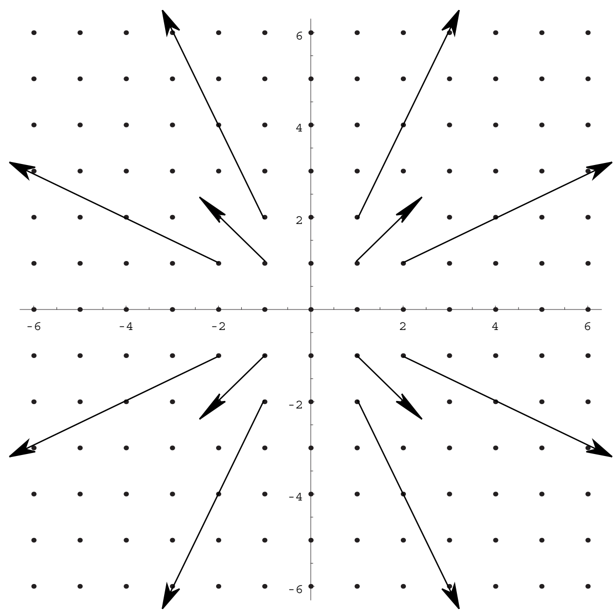


C15S01.004: Some typical vectors in the field $\mathbf{F}(x, y) = \langle 2, x \rangle$ are shown next.

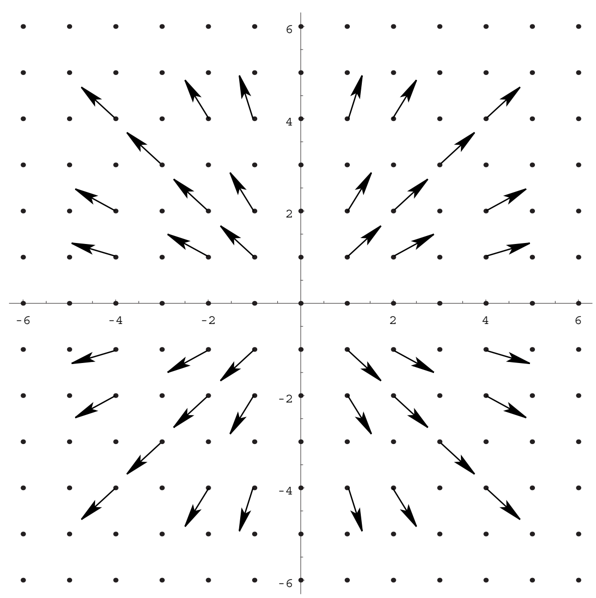


C15S01.005: Some typical vectors in the field $\mathbf{F}(x, y) = \langle (x^2 + y^2)^{1/2} \langle x, y \rangle$ are shown next. Note that the length of each vector is proportional to the square of the distance from the origin to its initial point and

that each vector points directly away from the origin.

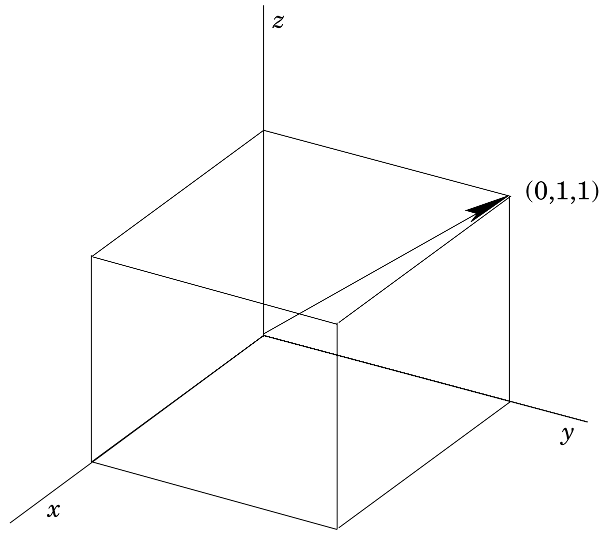


C15S01.006: Some typical vectors in the field $\mathbf{F}(x, y) = (x^2 + y^2)^{-1/2}\langle\, x,\, y\,\rangle$ are shown next. Note that each is a unit vector that points directly away from the origin.

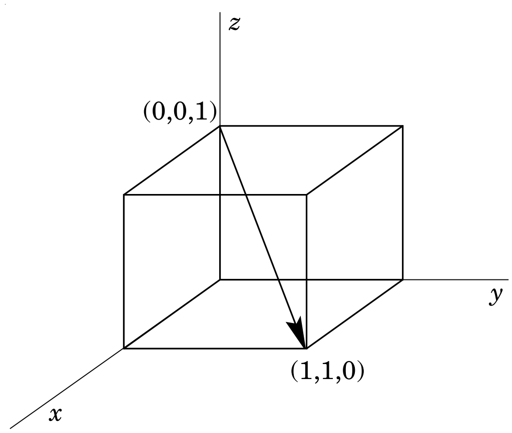


C15S01.007: The vector field $\mathbf{F}(x, y, z) = \langle\, 0,\, 1,\, 1\,\rangle$ is a constant vector field. All vectors in this field are

parallel translates of the one shown in the next figure.



C15S01.008: The vector field $\mathbf{F}(x, y, z) = \langle 1, 1, -1 \rangle$ is a constant vector field. All vectors in this field are parallel translates of the one shown in the next figure.



C15S01.009: Each vector in the field $\mathbf{F}(x, y, z) = \langle -x, -y \rangle$ is parallel to the xy-plane and reaches from its initial point at (x, y, z) to its terminal point $(0, 0, z)$ on the z-axis.

C15S01.010: Each vector in the field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ points directly away from the origin and its length is the same as the distance from the origin to its initial point.

C15S01.011: The vector field $\boldsymbol{\nabla}(xy) = \langle y, x \rangle$ is shown in Fig. 15.1.8. To verify this, evaluate the gradient at $(2, 0)$.

C15S01.012: The gradient vector field $\boldsymbol{\nabla}(2x^2 + y^2) = \langle 4x, 2y \rangle$ is shown in Fig. 15.1.9. To verify this, evaluate the gradient at $(2, 2)$.

C15S01.013: The gradient vector field

$$\boldsymbol{\nabla}\left(\sin \tfrac{1}{2}(x^2 + y^2)\right) = \left\langle x \cos \tfrac{1}{2}(x^2 + y^2),\ y \cos \tfrac{1}{2}(x^2 + y^2) \right\rangle$$

is shown in Fig. 15.1.10. To verify this, evaluate the gradient at $(1, 1)$ and at $(0, 1)$.

C15S01.014: The gradient vector field

$$\nabla\left(\sin \tfrac{1}{2}(y^2 - x^2)\right) = \left\langle -x\cos \tfrac{1}{2}(y^2 - x^2),\ y\cos \tfrac{1}{2}(y^2 - x^2) \right\rangle$$

is shown in Fig. 15.1.7. To verify this, evaluate the gradient at the point (1, 1).

C15S01.015: If $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, then

$$\nabla\cdot\mathbf{F} = 1 + 1 + 1 = 3 \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \langle 0, 0, 0 \rangle = \mathbf{0}.$$

C15S01.016: If $\mathbf{F}(x, y, z) = \langle 3x, -2y, -4z \rangle$, then

$$\nabla\cdot\mathbf{F} = 3 - 2 - 4 = -3 \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x & -2y & -4z \end{vmatrix} = \langle 0, 0, 0 \rangle = \mathbf{0}.$$

C15S01.017: If $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$, then

$$\nabla\cdot\mathbf{F} = 0 + 0 + 0 = 0 \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \langle x - x, y - y, z - z \rangle = \mathbf{0}.$$

C15S01.018: If $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$, then

$$\nabla\cdot\mathbf{F} = 2x + 2y + 2z \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \langle 0, 0, 0 \rangle = \mathbf{0}.$$

C15S01.019: If $\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle$, then

$$\nabla\cdot\mathbf{F} = y^2 + z^2 + x^2 \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & yz^2 & zx^2 \end{vmatrix} = \langle -2yz, -2xz, -2xy \rangle.$$

C15S01.020: If $\mathbf{F}(x, y, z) = \langle 2x - y, 3y - 2z, 7z - 3x \rangle$, then

$$\nabla\cdot\mathbf{F} = 2 + 3 + 7 = 12 \quad \text{and} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & 3y - 2z & 7z - 3x \end{vmatrix} = \langle 2, 3, 1 \rangle.$$

C15S01.021: If $\mathbf{F}(x,\,y,\,z) = \langle\, y^2+z^2,\ x^2+z^2,\ x^2+y^2 \,\rangle$, then

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = 0+0+0 = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = \langle\, 2y-2z,\ 2z-2x,\ 2x-2y \,\rangle.$$

C15S01.022: If $\mathbf{F}(x,\,y,\,z) = \langle\, 0,\ e^{xz}\sin y,\ e^{xy}\cos z \,\rangle$, then

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = e^{xz}\cos y - e^{xy}\sin z \quad \text{and}$$

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & e^{xz}\sin y & e^{xy}\cos z \end{vmatrix} = \langle\, xe^{xy}\cos z - xe^{xz}\sin y,\ -ye^{xy}\cos z,\ ze^{xz}\sin y \,\rangle.$$

C15S01.023: If $\mathbf{F}(x,\,y,\,z) = \langle\, x+\sin yz,\ y+\sin xz,\ z+\sin xy \,\rangle$, then

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = 1+1+1 = 3 \quad \text{and}$$

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+\sin yz & y+\sin xz & z+\sin xy \end{vmatrix}$$

$$= \langle\, x\cos xy - x\cos xz,\ y\cos yz - y\cos xy,\ z\cos xz - z\cos yz \,\rangle.$$

C15S01.024: If $\mathbf{F}(x,\,y,\,z) = \langle\, x^2e^{-z},\ y^3\ln x,\ z\cosh y \,\rangle$, then

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F} = 2xe^{-z} + 3y^2\ln x + \cosh y \quad \text{and}$$

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2e^{-z} & y^3\ln x & z\cosh y \end{vmatrix} = \langle\, z\sinh y,\ -x^2e^{-z},\ x^{-1}y^3 \,\rangle.$$

C15S01.025: If a and b are constants and f and g are differentiable functions of two variables, then

$$\boldsymbol{\nabla}(af+bg) = \left\langle \frac{\partial}{\partial x}(af+bg),\ \frac{\partial}{\partial y}(af+bg) \right\rangle = \langle\, af_x+bg_x,\ af_y+bg_y \,\rangle$$

$$= \langle\, af_x,\ af_y \,\rangle \ + \ \langle\, bg_x,\ bg_y \,\rangle = a\,\langle\, f_x,\ f_y \,\rangle \ + \ b\,\langle\, g_x,\ g_y \,\rangle = a\boldsymbol{\nabla}f + b\boldsymbol{\nabla}g.$$

If f and g are functions of three variables, then the proof is similar, merely longer.

C15S01.026: If a and b are constants and $\mathbf{F} = \langle P, Q, R \rangle$ and $\mathbf{G} = \langle S, T, U \rangle$ where P, Q, R, S, T, and U are each differentiable functions of three variables, then

$$\begin{aligned}\nabla\cdot(a\mathbf{F} + b\mathbf{G}) &= \nabla\cdot\langle aP + bS,\ aQ + bT,\ aR + bU \rangle = aP_x + bS_x + aQ_y + bT_y + aR_z + bU_z \\ &= a(P_x + Q_y + R_z) + b(S_x + T_y + U_z) = a\nabla\cdot\mathbf{F} \ + \ b\nabla\cdot\mathbf{G}.\end{aligned}$$

C15S01.027: If a and b are constants and $\mathbf{F} = \langle P, Q, R \rangle$ and $\mathbf{G} = \langle S, T, U \rangle$ where P, Q, R, S, T, and U are each differentiable functions of three variables, then

$$\begin{aligned}\nabla\times(a\mathbf{F} + b\mathbf{G}) &= \nabla\times\langle aP + bS,\ aQ + bT,\ aR + bU \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ aP + bS & aQ + bT & aR + bU \end{vmatrix} \\ &= \langle aR_y + bU_y - aQ_z - bT_z,\ aP_z + bS_z - aR_x - bU_x,\ aQ_x + bT_x - aP_y - bS_y \rangle \\ &= a\langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle \ + \ b\langle U_y - T_z,\ S_z - U_x,\ T_x - S_y \rangle \\ &= a\,(\nabla\times\mathbf{F}) \ + \ b\,(\nabla\times\mathbf{G}).\end{aligned}$$

C15S01.028: Suppose that $\mathbf{G} = \langle S, T, U \rangle$ where S, T, U, and f are differentiable functions of x, y, and z. Then

$$\begin{aligned}\nabla\cdot(f\mathbf{G}) &= \nabla\cdot\langle fS, fT, fU \rangle = f_xS + fS_x + f_yT + fT_y + f_zU + fU_z \\ &= (f)(S_x + T_y + U_z) + \langle f_x, f_y, f_z \rangle\cdot\langle S, T, U \rangle = (f)(\nabla\cdot\mathbf{G}) \ + \ (\nabla f)\cdot\mathbf{G}.\end{aligned}$$

C15S01.029: Suppose that $\mathbf{G} = \langle S, T, U \rangle$ where S, T, U, and f are differentiable functions of x, y, and z. Then

$$\nabla\times(f\mathbf{G}) = \nabla\times\langle fS, fT, fU \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fS & fT & fU \end{vmatrix}$$

$$\begin{aligned}&= \langle f_yU + fU_y - f_zT - fT_z,\ f_zS + fS_z - f_xU - fU_x,\ f_xT + fT_x - f_yS - fS_y \rangle \\ &= \langle f_yU - f_zT,\ f_zS - f_xU,\ f_xT - f_yS \rangle \ + \ \langle fU_y - fT_z,\ fS_z - fU_x,\ fT_x - fS_y \rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ S & T & U \end{vmatrix} \ + \ (f)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ S & T & U \end{vmatrix} = (\nabla f)\times\mathbf{G} \ + \ (f)(\nabla\times\mathbf{G}).\end{aligned}$$

C15S01.030: Suppose that f and g are differentiable functions of x, y, and z. Then

$$\begin{aligned}
\nabla\left(\frac{f}{g}\right) &= \left\langle \frac{\partial}{\partial x}\left(\frac{f}{g}\right),\ \frac{\partial}{\partial y}\left(\frac{f}{g}\right),\ \frac{\partial}{\partial z}\left(\frac{f}{g}\right)\right\rangle = \left\langle \frac{gf_x - fg_x}{g^2},\ \frac{gf_y - fg_y}{g^2},\ \frac{gf_z - fg_z}{g^2}\right\rangle \\
&= \left\langle \frac{gf_x}{g^2},\ \frac{gf_y}{g^2},\ \frac{gf_z}{g^2}\right\rangle - \left\langle \frac{fg_x}{g^2},\ \frac{fg_y}{g^2},\ \frac{fg_z}{g^2}\right\rangle = \frac{g}{g^2}\langle f_x,\ f_y,\ f_z\rangle - \frac{f}{g^2}\langle g_x,\ g_y,\ g_z\rangle \\
&= \frac{g}{g^2}\,\nabla f - \frac{f}{g^2}\,\nabla g = \frac{g\nabla f - f\nabla g}{g^2}.
\end{aligned}$$

C15S01.031: If $\mathbf{F} = \langle P,\ Q,\ R\rangle$ and $\mathbf{G} = \langle S,\ T,\ U\rangle$ where P, Q, R, S, T, and U are each differentiable functions of three variables, then

$$\begin{aligned}
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \nabla\cdot\langle QU - RT,\ RS - PU,\ PT - QS\rangle \\
&= Q_xU + QU_x - R_xT - RT_x + R_yS + RS_y - P_yU - PU_y + P_zT + PT_z - Q_zS - QS_z
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}) &= \mathbf{G}\cdot\langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle - \mathbf{F}\cdot\langle U_y - T_z,\ S_z - U_x,\ T_x - S_y\rangle \\
&= SR_y - SQ_z + TP_z - TR_x + UQ_x - UP_y - PU_y + PT_z - QS_z + QU_x - RT_x + RS_y.
\end{aligned}$$

Then comparison of the last expressions in each of the two computations reveals that

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}).$$

C15S01.032: Suppose that $\mathbf{F} = \langle P,\ Q,\ R\rangle$ where the component functions P, Q, and R of the three variables x, y, and z have continuous second-order partial derivatives. Then

$$\nabla\cdot(\nabla\times\mathbf{F}) = \nabla\cdot\langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

because $P_{zy} = P_{yz}$, $Q_{zx} = Q_{xz}$, and $R_{yx} = R_{xy}$.

C15S01.033: Suppose that f and g are twice-differentiable functions of the three variables x, y, and z. Then

$$\begin{aligned}
\nabla\cdot\big[\nabla(fg)\big] &= \nabla\cdot\langle f_xg + fg_x,\ f_yg + fg_y,\ f_zg + fg_z\rangle \\
&= f_{xx}g + 2f_xg_x + fg_{xx} + f_{yy}g + 2f_yg_y + fg_{yy} + f_{zz}g + 2f_zg_z + fg_{zz} \\
&= (f)(g_{xx} + g_{yy} + g_{zz}) + (g)(f_{xx} + f_{yy} + f_{zz}) + 2(f_xg_x + f_yg_y + f_zg_z) \\
&= f\nabla\cdot\langle g_x,\ g_y,\ g_z\rangle + g\nabla\cdot\langle f_x,\ f_y,\ f_z\rangle + 2\langle f_x,\ f_y,\ f_z\rangle\cdot\langle g_x,\ g_y,\ g_z\rangle \\
&= f\nabla\cdot(\nabla g) + g\nabla\cdot(\nabla f) + 2(\nabla f)\cdot(\nabla g).
\end{aligned}$$

C15S01.034: Suppose that f and g are twice-differentiable functions of x, y, and z. Then

$$\nabla \cdot (\nabla f \times \nabla g) = \nabla \cdot (\langle f_x,\, f_y,\, f_z \rangle \times \langle g_x,\, g_y,\, g_z \rangle) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix}$$

$$\begin{aligned}
&= \nabla \cdot \langle f_y g_z - f_z g_y,\; f_z g_x - f_x g_z,\; f_x g_y - f_y g_x \rangle \\
&= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} + f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy} + f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz} \\
&= (f_{zy} - f_{yz}) g_x + (f_{xz} - f_{zx}) g_y + (f_{yx} - f_{xy}) g_z + (g_{yz} - g_{zy}) f_x + (g_{zx} - g_{xz}) f_y + (g_{xy} - g_{yx}) f_z = 0.
\end{aligned}$$

C15S01.035: If $\mathbf{r} = \langle x,\, y,\, z \rangle$, then $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = \mathbf{0}$ by the solution of Problem 15.

C15S01.036: Write the constant vector $\mathbf{a}$ in the form $\langle a_1,\, a_2,\, a_3 \rangle$. Then

$$\nabla \cdot (\mathbf{a} \times \mathbf{r}) = \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \nabla \cdot \langle a_2 z - a_3 y,\; a_3 x - a_1 z,\; a_1 y - a_2 x \rangle = 0 + 0 + 0 = 0$$

and

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix} = \langle a_1 + a_1,\; a_2 + a_2,\; a_3 + a_3 \rangle = 2\mathbf{a}.$$

C15S01.037: By the results in Problems 28 and 35,

$$\nabla \cdot \frac{\mathbf{r}}{r^3} = \frac{1}{r^3}\,(\nabla \cdot \mathbf{r}) - \frac{3}{2}(x^2 + y^2 + z^2)^{-5/2} \langle 2x,\, 2y,\, 2z \rangle \cdot \mathbf{r} = \frac{3}{r^3} - 3r^{-5}(x^2 + y^2 + z^2) = \frac{3}{r^3} - \frac{3r^2}{r^5} = 0.$$

C15S01.038: By the results in Problem 29 and 35,

$$\nabla \times \frac{\mathbf{r}}{r^3} = \frac{1}{r^3}\,(\nabla \times \mathbf{r}) - \frac{3}{2}(x^2 + y^2 + z^2)^{-5/2} \langle 2x,\, 2y,\, 2z \rangle \times \mathbf{r} = \mathbf{0} - 3r^{-5}(\mathbf{r} \times \mathbf{r}) = \mathbf{0} - \mathbf{0} = \mathbf{0}.$$

C15S01.039: If $r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, then

$$\nabla r = \nabla (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \langle 2x,\, 2y,\, 2z \rangle = \frac{\mathbf{r}}{r}.$$

C15S01.040: If $r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, then

$$\nabla \left(\frac{1}{r} \right) = \nabla (x^2 + y^2 + z^2)^{-1/2} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} \langle 2x,\, 2y,\, 2z \rangle = -\frac{\mathbf{r}}{r^3}.$$

C15S01.041: By the results in Problems 28, 35, and 39,

$$\nabla \cdot (r\mathbf{r}) = r\nabla \cdot \mathbf{r} + (\nabla r) \cdot \mathbf{r} = 3r + \frac{\mathbf{r} \cdot \mathbf{r}}{r} = 3r + \frac{r^2}{r} = 4r.$$

C15S01.042: By the results in Problems 28, 39, and 40,

$$\nabla \cdot (\nabla r) = \nabla \cdot \left(\frac{\mathbf{r}}{r}\right) = \frac{1}{r}\nabla \cdot \mathbf{r} + \left(\nabla \frac{1}{r}\right) \cdot \mathbf{r} = \frac{3}{r} - \frac{\mathbf{r} \cdot \mathbf{r}}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

C15S01.043: Here we have

$$\begin{aligned}\nabla(\ln r) = \nabla\left(\ln\sqrt{x^2+y^2+z^2}\,\right) &= \frac{1}{2}\left\langle \frac{2x}{x^2+y^2+z^2},\ \frac{2y}{x^2+y^2+z^2},\ \frac{2z}{x^2+y^2+z^2}\right\rangle \\ &= \frac{1}{x^2+y^2+z^2}\langle\, x,\ y,\ z\,\rangle = \frac{\mathbf{r}}{r^2}.\end{aligned}$$

C15S01.044: If $\mathbf{r} = \langle\, x,\ y,\ z\,\rangle$ and $r = |\mathbf{r}|$, then

$$\begin{aligned}\nabla(r^{10}) &= \nabla\big([x^2+y^2+z^2]^5\big) \\ &= \langle\, 10x(x^2+y^2+z^2)^4,\ 10y(x^2+y^2+z^2)^4,\ 10z(x^2+y^2+z^2)^4\,\rangle = 10r^8\langle\, x,\ y,\ z\,\rangle = 10r^8\,\mathbf{r}.\end{aligned}$$

Section 15.2

C15S02.001: If $x(t) = 4t - 1$ and $y(t) = 3t + 1$, $-1 \leqq t \leqq 1$, then

$$\int_C (x^2 + y^2)\ ds = \int_{-1}^{1} (125t^2 - 10t + 10)\ dt = \left[\frac{125}{3}t^3 - 5t^2 + 10t\right]_{-1}^{1} = \frac{310}{3},$$

$$\int_C (x^2 + y^2)\ dx = \int_{-1}^{1} (100t^2 - 8t + 8)\ dt = \left[\frac{100}{3}t^3 - 4t^2 + 8t\right]_{-1}^{1} = \frac{248}{3}, \quad \text{and}$$

$$\int_C (x^2 + y^2)\ dy = \int_{-1}^{1} (75t^2 - 6t + 6)\ dt = \left[25t^3 - 3t^2 + 6t\right]_{-1}^{1} = 62.$$

C15S02.002: If $x(t) = t$ and $y(t) = t^2$, $0 \leqq t \leqq 1$, then

$$\int_C x\ ds = \int_0^1 t(1 + 4t^2)^{1/2}\ dt = \left[\frac{1}{12}(1 + 4t^2)^{3/2}\right]_0^1 = \frac{5\sqrt{5}\ - 1}{12} \approx 0.8483616572915790,$$

$$\int_C x\ dx = \int_0^1 t\ dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}, \quad \text{and} \quad \int_C x\ dy = \int_0^1 2t^2\ dt = \left[\frac{2}{3}t^3\right]_0^1 = \frac{2}{3}.$$

C15S02.003: If $x(t) = e^t + 1$ and $y(t) = e^t - 1$, $0 \leqq t \leqq \ln 2$, then

$$\int_C (x + y)\ ds = \int_0^{\ln 2} 2^{3/2}e^{2t}\ dt = \left[2^{1/2}e^{2t}\right]_0^{\ln 2} = 3\sqrt{2} \approx 4.2426406871192851,$$

$$\int_C (x + y)\ dx = \int_0^{\ln 2} 2e^{2t}\ dt = \left[e^{2t}\right]_0^{\ln 2} = 3, \quad \text{and} \quad \int_C (x + y)\ dy = \int_0^{\ln 2} 2e^{2t}\ dt = \left[e^{2t}\right]_0^{\ln 2} = 3.$$

C15S02.004: If $x(t) = \sin t$ and $y(t) = \cos t$, $0 \leqq t \leqq \frac{1}{2}\pi$, then

$$\int_C (2x - y)\ ds = \int_0^{\pi/2} (2\sin t - \cos t)\ dt = \left[-2\cos t - \sin t\right]_0^{\pi/2} = 1,$$

$$\int_C (2x - y) dx = \int_0^{\pi/2} (2\sin t\ \cos t - \cos^2 t)\ dt = -\frac{1}{4}\left[2t + 2\cos 2t + \sin 2t\right]_0^{\pi/2} = \frac{4 - \pi}{4}, \quad \text{and}$$

$$\int_C (2x - y)\ dy = \int_0^{\pi/2} (\sin t\ \cos t\ - 2\sin^2 t)\ dt = \left[\frac{1}{4}(2\sin 2t - \cos 2t - 4t)\right]_0^{\pi/2} = \frac{1 - \pi}{2}.$$

C15S02.005: If $x(t) = 3t$ and $y(t) = t^4$, $0 \leqq t \leqq 1$, then

$$\int_C xy\ ds = \int_0^1 3t^5\sqrt{9 + 16t^6}\ dt = \left[\frac{1}{48}(9 + 16t^6)^{3/2}\right]_0^1 = \frac{49}{24} \approx 2.0416666666666667,$$

$$\int_C xy\ dx = \int_0^1 9t^5\ dt = \left[\frac{3}{2}t^6\right]_0^1 = \frac{3}{2}, \quad \text{and} \quad \int_C xy\ dy = \int_0^1 12t^8\ dt = \left[\frac{4}{3}t^9\right]_0^1 = \frac{4}{3}.$$

C15S02.006: The path C is self-parametrizing: $x = x,\ y(x) = x^2$. Hence

$$\int_C P(x,\,y)\,dx + Q(x,\,y)\,dy = \int_C xy\,dx + (x+y)\,dy = \int_{-1}^{2} (2x^2 + 3x^3)\,dx = \left[\frac{2}{3}x^3 + \frac{3}{4}x^4\right]_{-1}^{2} = \frac{69}{4}.$$

C15S02.007: One parametrization of the path C is this: Let $y(t) = t$ and $x(t) = t^3,\ -1 \leqq t \leqq 1$. Then

$$\int_C P(x,\,y)\,dx + Q(x,\,y)\,dy = \int_C y^2\,dx + x\,dy = \int_{-1}^{1} (t^3 + 3t^4)\,dt = \left[\frac{1}{4}t^4 + \frac{3}{5}t^5\right]_{-1}^{1} = \frac{6}{5}.$$

C15S02.008: One parametrization of the path C is this: Let $x(t) = t^2$ and $y(t) = t^3,\ 1 \leqq t \leqq 2$. Then

$$\int_C P(x,\,y)\,dx + Q(x,\,y)\,dy = \int_C yx^{1/2}\,dx + x^{3/2}\,dy = \int_1^2 5t^5\,dt = \left[\frac{5}{6}t^6\right]_1^2 = \frac{105}{2}.$$

C15S02.009: Parametrize the path C in two parts. Let

$$x_1(t) = t,\ y_1(t) = 1,\ -1 \leqq t \leqq 2 \quad \text{and let} \quad x_2(t) = 2,\ y_2(t) = t,\ 1 \leqq t \leqq 5.$$

Then

$$\int_C P(x,\,y)\,dx + Q(x,\,y)\,dy = \int_C x^2y\,dx + xy^3\,dy = \int_{-1}^{2} t^2\,dt + \int_1^5 2t^3\,dt$$
$$= \left[\frac{1}{3}t^3\right]_{-1}^{2} + \left[\frac{1}{2}t^4\right]_1^5 = 3 + 312 = 315.$$

C15S02.010: Parametrize the path C in two parts: Let

$$x_1(t) = 3,\ y_1(t) = 2 - t,\ -3 \leqq t \leqq 0 \quad \text{and let} \quad x_2(t) = 3 - t,\ y_2(t) = -1,\ 0 \leqq t \leqq 5.$$

Then

$$\int_C P(x,\,y)\,dx + Q(x,\,y)\,dy = \int_C (x+2y)\,dx + (2x-y)\,dy = \int_{-3}^{0} (-t+4)\,dt + \int_0^5 (t-1)\,dt$$
$$= \left[-\frac{1}{2}t^2 + 4t\right]_{-3}^{0} + \left[\frac{1}{2}t^2 - t\right]_0^5 = \frac{33}{2} + \frac{15}{2} = 24.$$

C15S02.011: Because $\mathbf{r}(t) = \langle\, t,\, t^2,\, t^3\,\rangle$ for $0 \leqq t \leqq 1$, we have

$$d\mathbf{r} = \langle\, 1,\, 2t,\, 3t^2\,\rangle \quad \text{and} \quad \mathbf{F}(t) = \langle\, t^3,\, t,\, -t^2\,\rangle.$$

Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2t^2 + t^3 - 3t^4)\,dt = \left[\frac{2}{3}t^3 + \frac{1}{4}t^4 - \frac{3}{5}t^5\right]_0^1 = \frac{19}{60} \approx 0.316666666667.$$

C15S02.012: Parametrize the path C as follows: $\mathbf{r}(t) = \langle\, 2+2t,\, -1+3t,\, 3-4t\,\rangle,\ 0 \leqq t \leqq 1$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (20 + 4t - 72t^2)\,dt = \Big[20t + 2t^2 - 24t^3\Big]_0^1 = -2.$$

To use *Mathematica* 3.0 to solve this problem, one method is to proceed as follows.

```
x[t_] := 2 + 2*t;  y[t_] := -1 + 3*t; z[t_] := 3 - 4*t
r[t_] := { x[t], y[t], z[t] }
f[t_] := { y[t]*z[t], x[t]*z[t], x[t]*y[t] }
f[t].r'[t]
```

(`a.b` gives the dot product of the vectors **a** and **b**.)

$$3(3 - 4t)(2 + 2t) + 2(3 - 4t)(-1 + 3t) - 4(2 + 2t)(-1 + 3t)$$

```
Expand[%]
```

$$20 + 4t - 72t^2$$

```
Integrate[%, t]
```

$$20t + 2t^2 - 24t^3$$

```
(% /.  t → 1) - (% /.  t → 0)
```

$$-2$$

C15S02.013: The path C is parametrized by $\mathbf{r}(t) = \langle \sin t,\, \cos t,\, 2t \rangle$, $0 \leqq t \leqq \pi$. Hence

$$d\mathbf{r} = \langle \cos t,\, -\sin t,\, 2 \rangle \quad \text{and} \quad \mathbf{F}(t) = \langle \cos t,\, -\sin t,\, 2t \rangle,$$

and thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi (4t + 1)\,dt = \Big[2t^2 + t\Big]_0^\pi = 2\pi^2 + \pi \approx 22.8808014557685105.$$

C15S02.014: Parametrize the path C in three sections, as follows:

$$\begin{aligned}
&x_1(t) = 4t, \quad y_1(t) = 0, \quad z_1(t) = 0, \quad 0 \leqq t \leqq 1;\\
&x_2(t) = 4, \quad y_2(t) = 2t, \quad z_2(t) = 0, \quad 0 \leqq t \leqq 1;\\
&x_3(t) = 4, \quad y_3(t) = 2, \quad z_3(t) = 3t, \quad 0 \leqq t \leqq 1.
\end{aligned}$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} \;=\; I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^1 32t\ dt = \Big[16t^2\Big]_0^1 = 16,$$

$$I_2 = \int_0^1 (8t + 24)\ dt = \Big[4t^2 + 24t\Big]_0^1 = 28,$$

$$I_3 = \int_0^1 81t^2\ dt = \Big[27t^3\Big]_0^1 = 27,$$

and therefore $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 71.$

C15S02.015: Parametrize the path C in three sections, as follows:

$$\begin{aligned}
&x_1(t) = -1, && y_1(t) = 2, && z_1(t) = -2 + 4t, && 0 \leqq t \leqq 1;\\
&x_2(t) = -1 + 2t, && y_2(t) = 2, && z_2(t) = 2, && 0 \leqq t \leqq 1;\\
&x_3(t) = 1, && y_3(t) = 2 + 3t, && z_3(t) = 2, && 0 \leqq t \leqq 1.
\end{aligned}$$

Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^1 (32 - 64t)\ dt = \Big[32t - 32t^2\Big]_0^1 = 0,$$

$$I_2 = \int_0^1 16\ dt = \Big[16t\Big]_0^1 = 16, \quad \text{and}$$

$$I_3 = \int_0^1 12\ dt = \Big[12t\Big]_0^1 = 12.$$

Therefore $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = 0 + 16 + 12 = 28.$

C15S02.016: Parametrize the path C by

$$\mathbf{r}(t) = \langle\, 1 + 2t,\ -1 + 3t,\ 2 + 3t\,\rangle, \quad 0 \leqq t \leqq 1.$$

Then $f(x, y, z) = xyz$ becomes $f(t) = 18t^3 + 15t^2 - t - 2$ and $|\mathbf{r}'(t)| = \sqrt{22}$. Hence

$$\begin{aligned}
\int_C f(x, y, z)\ ds &= \int_0^1 (18t^3 + 15t^2 - t - 2)\sqrt{22}\ dt\\
&= \sqrt{22}\left[\frac{9}{2}t^4 + 5t^3 - \frac{1}{2}t^2 - 2t\right]_0^1 = 7\sqrt{22} \approx 32.8329103187640069.
\end{aligned}$$

C15S02.017: Here we have

$$\int_C (2x + 9xy)\ ds = \int_0^1 (2t + 9t^3)(1 + 4t^2 + 9t^4)^{1/2}\ dt$$

$$= \left[\frac{1}{6}(1 + 4t^2 + 9t^4)^{3/2}\right]_0^1 = \frac{14\sqrt{14} - 1}{6} \approx 8.5638672358058632.$$

C15S02.018: Here we have

$$\int_C xy\ ds = \int_0^{5\pi/2} (36\sin t\ \cos t)(16\sin^2 t + 81\cos^2 t + 49)^{1/2}\ dt = \left[-\frac{12}{65}(16\sin^2 t + 81\cos^2 t + 49)^{3/2}\right]_0^{5\pi/2}$$

$$= -\frac{12}{65}\left[(16 + 49)^{3/2} - (81 + 49)^{3/2}\right] = 12\left(2\sqrt{130} - \sqrt{65}\right) \approx 176.8950090442105192.$$

C15S02.019: Because the wire W is uniform, we may assume that its density is $\delta = 1$. Moreover, $\overline{x} = 0$ by symmetry. Parametrize the wire by

$$\mathbf{r}(t) = \langle\, a\cos t,\ a\sin t\,\rangle, \quad 0 \leqq t \leqq \pi.$$

Then $ds = a\ dt$. Also the mass of the wire is πa, so it remains only to compute the moment

$$M_x = \int_W ay\ dt = \int_0^\pi a^2 \sin t\ dt = \Big[-a^2\cos t\Big]_0^\pi = 2a^2.$$

Therefore the centroid of the wire is located at the point $\left(0,\ \dfrac{2a}{\pi}\right)$.

C15S02.020: Parametrize the wire W by

$$\mathbf{r}(t) = \langle a\cos t,\ a\sin t\,\rangle, \quad 0 \leqq t \leqq \pi.$$

Because the wire is uniform, assume that its density is δ, a positive constant. The length of the wire is πa, so its mass is $m = \pi\delta a$. Finally, $ds = a\ dt$. Therefore the moments of inertia of W with respect to the coordinate axes are

$$I_x = \int_C a\delta y^2\ dt = \int_0^\pi a^3\delta\sin^2 t\ dt = \frac{1}{2}a^3\delta\left[t - \frac{1}{2}\sin 2t\right]_0^\pi = \frac{1}{2}\pi\delta a^3 = \frac{1}{2}ma^2 \quad \text{and}$$

$$I_y = \int_C a\delta x^2\ dt = \int_0^\pi a^3\delta\cos^2 t\ dt = \frac{1}{2}a^3\delta\left[t + \frac{1}{2}\sin 2t\right]_0^\pi = \frac{1}{2}\pi\delta a^3 = \frac{1}{2}ma^2.$$

C15S02.021: First, the arc length element is

$$ds = \sqrt{9\sin^2 t + 9\cos^2 t + 16}\ dt = 5\ dt.$$

The mass and moments of the helical wire W are

$$m = \int_W 5k\ dt = \int_0^{2\pi} 5k\ dt = 10k\pi;$$

$$M_{yz} = \int_0^{2\pi} 15k\cos t\ dt = \Big[15k\sin t\Big]_0^{2\pi} = 0;$$

$$M_{xz} = \int_0^{2\pi} 15k\sin t\ dt = \Big[-15k\cos t\Big]_0^{2\pi} = 0;$$

$$M_{xy} = \int_0^{2\pi} 20kt\ dt = \Big[10kt^2\Big]_0^{2\pi} = 40k\pi^2.$$

Therefore the coordinates of the centroid are

$$\overline{x} = \overline{y} = 0, \quad \overline{z} = \frac{40k\pi^2}{10k\pi} = 4\pi.$$

C15S02.022: The moment of inertia of the helical wire W of Problem 21 with respect to the z-axis is

$$I_z = \int_W (x^2+y^2)\ dm = \int_0^{2\pi} 9\cdot 5k\ dt = \Big[45kt\Big]_0^{2\pi} = 90k\pi = 9m$$

where m is the mass of the wire.

C15S02.023: Parametrize the wire W via

$$\mathbf{r}(t) = \langle\, a\cos t,\ a\sin t,\ 0\,\rangle, \quad 0 \leqq t \leqq \frac{\pi}{2}.$$

Then the arc-length element is $ds = |\mathbf{r}'(t)|\ dt = a\ dt$. The mass element is $dm = a\cdot k\cdot a^2\sin t\,\cos t\ dt$, and hence the mass of W is

$$m = \int_0^{\pi/2} ka^3\sin t\,\cos t\ dt = \left[\frac{1}{2}ka^3\sin^2 t\right]_0^{\pi/2} = \frac{1}{2}ka^3.$$

Clearly the z-coordinate of the centroid is $\overline{z} = 0$, and $\overline{y} = \overline{x}$ by symmetry. The moment of the wire with respect to the y-axis is

$$M_y = \int_0^{\pi/2} ka^4\sin t\,\cos^2 t\ dt = -\frac{1}{3}ka^4\Big[\cos^3 t\Big]_0^{\pi/2} = \frac{1}{3}ka^4.$$

Therefore the x-coordinate of the centroid is

$$\overline{x} = \frac{2ka^4}{3ka^3} = \frac{2}{3}a.$$

The moments of W with respect to the coordinate axes are

$$I_x = \int_0^{\pi/2} ka^5\sin^2 t\,\cos t\ dt = \frac{1}{4}ka^5\Big[\sin^4 t\Big]_0^{\pi/2} = \frac{1}{4}ka^5 = \frac{1}{2}ma^2;$$

$I_y = I_x$ by symmetry, and $I_0 = I_x + I_y = \dfrac{1}{2}ka^5 = ma^2$.

C15S02.024: The wire W has constant density k and is parametrized via $x = t - \sin t$, $y = 1 - \cos t$, $0 \leqq t \leqq 2\pi$. Hence the arc-length element is

$$ds = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt = \sqrt{2 - 2\cos t}\ dt$$
$$= 2\left(\frac{1-\cos t}{2}\right)^{1/2} dt = 2\left(\sin^2 \frac{t}{2}\right)^{1/2} dt = 2\sin\frac{t}{2}\ dt;$$

the last equality is justified because $0 \leqq \frac{1}{2}t \leqq \pi$. Hence the mass of W and its moment with respect to the y-axis are

$$m = \int_0^{2\pi} 2k\sin\frac{t}{2}\ dt = \left[-4k\cos\frac{t}{2}\right]_0^{2\pi} = 8k \quad \text{and}$$
$$M_y = \int_0^{2\pi} 2k(t - \sin t)\sin\frac{t}{2}\ dt = \frac{2}{3}k\left[\sin\frac{3t}{2} + 9\sin\frac{t}{2} - 6t\cos\frac{t}{2}\right]_0^{2\pi} = 8\pi k.$$

Hence the x-coordinate of the centroid of W is $\overline{x} = \pi$. Its moment with respect to the x-axis is

$$M_x = \int_0^{2\pi} 2k(1-\cos t)\sin\frac{t}{2}\ dt = \frac{2}{3}k\left[\cos\frac{3t}{2} - 9\cos\frac{t}{2}\right]_0^{2\pi} = \frac{32}{3}k,$$

and hence the y-coordinate of its centroid is $\overline{y} = \frac{4}{3}$. Its moment of inertia with respect to the x-axis is

$$I_x = \int_0^{2\pi} 2k(1-\cos t)^2\sin\frac{t}{2}\ dt = \int_0^{2\pi} 8k\sin^5\frac{t}{2}\ dt$$
$$= \frac{1}{15}k\left[-150\cos\frac{t}{2} + 25\cos\frac{3t}{2} - 3\cos\frac{5t}{2}\right]_0^{2\pi} = \frac{256}{15}k = \frac{32}{15}m.$$

C15S02.025: Using the given parametrization we find that the arc-length element is $ds = \frac{3}{2}|\sin 2t|\ dt$, and hence the polar moment of inertia of the wire is

$$I_0 = 4\int_0^{\pi/2} k(\cos^6 t + \sin^6 t)\ ds = -\frac{3}{16}k\left[\cos 6t + 7\cos 2t\right]_0^{\pi/2} = 3k.$$

Because the mass of the wire is

$$m = 4\int_0^{\pi/2} k\ ds = -3k\left[\cos 2t\right]_0^{\pi/2} = 6k,$$

we can also write $I_0 = \dfrac{1}{2}m$.

C15S02.026: The standard parametrization of the circle C with center $(0, 0)$ and radius a is

$$\mathbf{r}(t) = \langle\, a\cos t,\ a\sin t\,\rangle, \quad 0 \leqq t \leqq 2\pi.$$

With this parametrization, we have arc-length element $ds = a\ dt$, and hence the average distance of points of the circle from its center is

$$\overline{d} = \frac{1}{2\pi a}\int_0^{2\pi} a^2\ dt = \frac{1}{2\pi a}\left[a^2 t\right]_0^{2\pi} = \frac{2\pi a^2}{2\pi a} = a.$$

C15S02.027: We are given the circle C of radius a centered at the origin and the point $(a, 0)$ on C. Suppose that (x, y) is a point of C. Let t be the angular polar coordinate of (x, y). Then, by the law of cosines, the distance w between $(a, 0)$ and (x, y) satisfies the equation

$$w^2 = a^2 + a^2 - 2a^2 \cos t = 2a^2(1 - \cos t) = 4a^2 \cdot \frac{1 - \cos t}{2} = 4a^2 \sin^2 \frac{t}{2}.$$

Because $0 \leqq t \leqq 2\pi$, it now follows that $w = 2a \sin \dfrac{t}{2}$, and hence the average value of w on C is

$$\overline{d} = \frac{1}{2\pi a} \int_0^{2\pi} 2a^2 \sin \frac{t}{2}\, dt = -\frac{4a^2}{2\pi a} \left[\cos \frac{t}{2} \right]_0^{2\pi} = \frac{8a^2}{2\pi a} = \frac{4}{\pi} a \approx (1.2732395447351627)a.$$

C15S02.028: We use the parametrization given in the statement of Problem 24. Then the arc-length element is

$$\begin{aligned} ds &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\; dt = \sqrt{2 - 2\cos t}\; dt \\ &= 2\left(\frac{1 - \cos t}{2} \right)^{1/2} dt = 2\left(\sin^2 \frac{t}{2} \right)^{1/2} dt = 2 \sin \frac{t}{2}\, dt; \end{aligned}$$

the last equality is justified because $0 \leqq \frac{1}{2}t \leqq \pi$. An immediate consequence of Example 2 in Section 10.5 is that this cycloid has length 8. Hence the average distance of points of the cycloid from the origin is

$$\overline{d} = \frac{1}{8} \int_0^{2\pi} 2(t^2 + 2 - 2t \sin t - 2\cos t)^{1/2} \sin \frac{t}{2}\, dt.$$

This integral appears to involve a nonelementary antiderivative, but the *Mathematica* 3.0 command

```
NIntegrate[ 2*Sin[t/2]*Sqrt[t*t + 2 - 2*t*Sin[t] - 2*Cos[t]], {t, 0, 2*Pi},
            WorkingPrecision → 28, AccuracyGoal → 24 ]
```

(the options ask *Mathematica* to carry 28 decimal digits in its computations and to select enough sampling points to assure that 24 digits are correct in the final answer) yielded the result $\overline{d} \approx 3.552523608078470787$.

C15S02.029: The parametrization $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $0 \leqq t \leqq 2\pi$ of the astroid yields the arc-length element

$$ds = \sqrt{9\cos^4 t \sin^2 t + 9\cos^2 t \sin^4 t}\; dt = \frac{3}{2}\sqrt{\sin^2 2t}\; dt = \frac{3}{2} \sin 2t\, dt,$$

although the last equality is valid only if $\sin 2t$ is nonnegative. Hence we will find the average distance of points of the astroid in the first quadrant from the origin; by symmetry, this will be the same as the average distance of all of its points from the origin. Noting that

$$[x(t)]^2 + [y(t)]^2 = \cos^6 t + \sin^6 t$$

and noting also that the length of the first-quadrant arc of the astroid is $\frac{3}{2}$ (a consequence of the solution of Problem 30 in Section 10.5), we find that the average distance of points of the astroid from the origin is

$$\begin{aligned}\overline{d} &= \frac{2}{3}\int_0^{\pi/2} \frac{3}{2}\sqrt{\cos^6 t + \sin^6 t}\ \sin 2t\,dt\\ &= \left[\frac{\sqrt{3}}{24}\operatorname{arctanh}\left(\frac{\sqrt{6}\ \cos 2t}{\sqrt{5+3\cos 4t}}\right) - \frac{\sqrt{2}}{16}(\cos 2t)\sqrt{5+3\cos 4t}\right]_0^{\pi/2}\\ &= \frac{1}{2}+\frac{\sqrt{3}}{12}\operatorname{arctanh}\frac{\sqrt{3}}{2} \approx 0.69008649907523658688277356372.\end{aligned}$$

Of course we used *Mathematica* 3.0 to find and simplify both the antiderivative and the value of the definite integral. By contrast, *Derive* 2.56 yields the result

$$\begin{aligned}\overline{d} &= \frac{2}{3}\int_0^{\pi/2} \frac{3}{2}\sqrt{\cos^6 t + \sin^6 t}\ \sin 2t\,dt\\ &= \left[-\frac{\sqrt{3}}{24}\ln\left(2\sqrt{3\cos^4 t - 3\cos^2 t + 1}\ + 2\sqrt{3}\ \cos^2 t - \sqrt{3}\right) - \frac{1-2\cos^2 t}{4}\sqrt{3\cos^4 t - 3\cos^2 t + 1}\,\right]_0^{\pi/2}\\ &= \frac{6+\sqrt{3}\ \ln\left(2+\sqrt{3}\,\right)}{12} \approx 0.69008649907523658688277356372.\end{aligned}$$

C15S02.030: Using the parametrization of the helix given in the statement of Problem 21, we find that the arc-length element is

$$ds = \sqrt{9\sin^2 t + 9\cos^2 t + 16}\ dt = 5\ dt.$$

Hence the length of the helix is

$$s = \int_0^{2\pi} 5\ dt = 10\pi.$$

Next, the distance from the origin to the point $(x(t),\ y(t), z(t))$ of the helix is

$$\sqrt{16t^2 + 9\cos^2 t + 9\sin^2 t}\ = \sqrt{16t^2+9}\,.$$

Therefore the average distance of points of the helix from the origin is

$$\begin{aligned}\overline{d} &= \frac{1}{10\pi}\int_0^{2\pi} 5\sqrt{16t^2+9}\ \ dt = \frac{1}{10\pi}\left[\frac{5}{2}t\sqrt{16t^2+9}\ + \frac{45}{8}\operatorname{arcsinh}\frac{4t}{3}\right]_0^{2\pi}\\ &= \frac{1}{10\pi}\left(5\pi\sqrt{64\pi^2+9}\ + \frac{45}{8}\operatorname{arcsinh}\frac{8\pi}{3}\right) = \frac{1}{2}\sqrt{64\pi^2+9}\ + \frac{9}{16\pi}\operatorname{arcsinh}\frac{8\pi}{3} \approx 13.1609004583093278.\end{aligned}$$

C15S02.031: With the given parametrization, we find that $ds = \sqrt{2}\,e^{-t}\,dt$ and that

$$\sqrt{[x(t)]^2+[y(t)]^2}\ = e^{-t}.$$

The length of the spiral is

$$\int_0^{\infty} \sqrt{2}\,e^{-t}\,dt = \left[-\sqrt{2}\,e^{-t}\right]_0^{\infty} = \sqrt{2}\,,$$

and thus the average distance of points of the spiral from the origin is

$$\overline{d} = \frac{1}{\sqrt{2}} \int_0^\infty \sqrt{2}\, e^{-2t}\, dt = \left[-\frac{1}{2} e^{-2t} \right]_0^\infty = \frac{1}{2}.$$

C15S02.032: The work done in moving along a path on the sphere is zero because $\mathbf{F}$ is normal to the sphere. Therefore $\mathbf{F} \cdot \mathbf{T}$ is identically zero on any path on the sphere. Let C denote the straight line segment from $(1, 0, 0)$ to $(5, 0, 0)$. Parametrize C by $x(t) = 1 + 4t,\ y = 0,\ z = 0,\ 0 \leqq t \leqq 1$. Then the work done in moving along C is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{4k}{(1+4t)^2}\, dt = -k \left[\frac{1}{1+4t} \right]_0^1 = \frac{4}{5} k.$$

C15S02.033: Part (a): Parametrize the path by $x(t) = 1,\ y(t) = t,\ 0 \leqq t \leqq 1$. Then the force is

$$\mathbf{F}(t) = \left\langle \frac{k}{1+t^2},\ \frac{kt}{1+t^2} \right\rangle,$$

and so the work is

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{kt}{1+t^2}\, dt = \left[\frac{1}{2} k \ln(1+t^2) \right]_0^1 = \frac{1}{2} k \ln 2.$$

Part (b): Parametrize the path by $x(t) = 1 - t,\ y(t) = 1,\ 0 \leqq t \leqq 1$. Then the force is

$$\mathbf{F}(t) = \left\langle \frac{k(1-t)}{1+(1-t)^2},\ \frac{k}{1+(1-t)^2} \right\rangle,$$

and thus the work is

$$W = \int_0^1 -\frac{k(1-t)}{1+(1-t)^2}\, dt = \left[\frac{1}{2} k \ln\left(1 + (1-t)^2\right) \right]_0^1 = -\frac{1}{2} k \ln 2.$$

C15S02.034: Parametrize the unit circle C in the usual way: $x(t) = \cos t,\ y(t) = \sin t,\ 0 \leqq t \leqq 2\pi$. Suppose that the force function has the form $\mathbf{F}(x, y) = \mathbf{F}(t) = \langle k_1, k_2 \rangle$. Then the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (k_2 \cos t - k_1 \sin t)\, dt = \Big[k_1 \cos t + k_2 \sin t \Big]_0^{2\pi} = 0.$$

C15S02.035: Parametrize the unit circle C in the usual way: $x(t) = \cos t,\ y(t) = \sin t,\ 0 \leqq t \leqq 2\pi$. Then the force function is $\mathbf{F}(t) = \langle k \cos t,\ k \sin t \rangle$. Hence $\mathbf{F} \cdot d\mathbf{r} = 0$, and therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

C15S02.036: Parametrize the unit circle C in the usual way: $x(t) = \cos t,\ y(t) = \sin t,\ 0 \leqq t \leqq 2\pi$. Then $\mathbf{F}(t) = \langle -\sin t,\ \cos t \rangle$, so that $\mathbf{F} \cdot d\mathbf{r} = \sin^2 t + \cos^2 t = 1$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1\, dt = 2\pi \approx 6.2831853071795864769252867665590 1.$$

C15S02.037: The work done in moving along a path on the sphere is zero because $\mathbf{F}$ is normal to the sphere. Therefore $\mathbf{F} \cdot \mathbf{T}$ is identically zero on any path on the sphere.

C15S02.038: The force function is $\mathbf{F}(t) = \langle\, 0,\, -150\,\rangle$ and the path may be parametrized as follows: $x(t) = 100(1-t),\ y(t) = 100(1-t),\ 0 \leqq t \leqq 1$. Hence the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 15000\,dt = 15000 \quad (\text{ft·lb}).$$

C15S02.039: The force function is $\mathbf{F}(t) = \langle\, 0,\, -150\,\rangle$ and the path may be parametrized as follows: $x(t) = 100\sin t,\ y(t) = 100\cos t,\ 0 \leqq t \leqq \frac{1}{2}\pi$. Hence the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} 15000 \sin t\,dt = -15000\Big[\cos t\Big]_0^{\pi/2} = 15000 \quad (\text{ft·lb}).$$

C15S02.040: The force function is $\mathbf{F}(t) = \langle\, 0,\, -150\,\rangle$ and the path may be parametrized as follows: $x(t) = 100(1-t),\ y(t) = 100(1-t)^2,\ 0 \leqq t \leqq 1$. Hence the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 30000(1-t)\,dt = \Big[30000t - 15000t^2\Big]_0^1 = 15000 \quad (\text{ft·lb}).$$

C15S02.041: The force function is $\mathbf{F}(t) = \langle\, 0,\, 0,\, -200\,\rangle$ and the path may be parametrized as follows: $x(t) = 25\cos t,\ y(t) = 25\sin t,\ z(t) = 100 - 100t/(10\pi),\ 0 \leqq t \leqq 10\pi$. Hence the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{10\pi} \frac{2000}{\pi}\,dt = \left[\frac{2000}{\pi}t\right]_0^{10\pi} = 20000 \quad (\text{ft·lb}).$$

C15S02.042: We parametrize the typical circle C as follows: $x(t) = a\cos t,\ y(t) = a\sin t,\ z(t) \equiv 0$. Moreover, we are given

$$\mathbf{B}(t) = \langle\, -aB\sin t,\ aB\cos t,\ 0\,\rangle$$

where a is a constant. Hence

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \int_0^{2\pi} a^2|\mathbf{B}|\,dt = 2\pi a^2|\mathbf{B}| = \mu I$$

by Ampere's law. Therefore

$$B = |\mathbf{B}| = \frac{\mu I}{2\pi a^2},$$

which shows that B is proportional to the current I and inversely proportional to the square of the radius a of the distance from the wire.

Section 15.3

Note: In the solutions of Problems 1 through 16, $P(x, y)$ always denotes the first component of the given vector field $\mathbf{F}(x, y)$, $Q(x, y)$ always denotes its second component, and $\phi(x, y)$ denotes a scalar potential function with gradient $\mathbf{F}(x, y)$.

C15S03.001: If $P(x, y) = 2x + 3y$ and $Q(x, y) = 3x + 2y$, then

$$\frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = x^2 + 3xy + y^2$.

C15S03.002: If $P(x, y) = 4x - y$ and $Q(x, y) = 6y - x$, then

$$\frac{\partial P}{\partial y} = -1 = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = 2x^2 - xy + 3y^2$.

C15S03.003: If $P(x, y) = 3x^2 + 2y^2$ and $Q(x, y) = 4xy + 6y^2$, then

$$\frac{\partial P}{\partial y} = 4y = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = x^3 + 2xy^2 + 2y^3$.

C15S03.004: If $P(x, y) = 2xy^2 + 3x^2$ and $Q(x, y) = 2x^2y + 4y^3$, then

$$\frac{\partial P}{\partial y} = 4xy = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = x^3 + x^2y^2 + y^4$.

C15S03.005: If $P(x, y) = 2y + \sin 2x$ and $Q(x, y) = 3x + \cos 3y$, then

$$\frac{\partial P}{\partial y} = 2 \neq 3 = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is not conservative.

C15S03.006: If $P(x, y) = 4x^2y - 5y^4$ and $Q(x, y) = x^3 - 20xy^3$, then

$$\frac{\partial P}{\partial y} = 4x^2 - 20y^3 \neq 3x^2 - 20y^3 = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is not conservative.

C15S03.007: If $P(x, y) = x^3 + \dfrac{y}{x}$ and $Q(x, y) = y^2 + \ln x$, then

$$\frac{\partial P}{\partial y} = \frac{1}{x} = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = \dfrac{1}{4}x^4 + y \ln x + \dfrac{1}{3}y^3$.

C15S03.008: If $P(x, y) = 1 + ye^{xy}$ and $Q(x, y) = 2y + xe^{xy}$, then

$$\frac{\partial P}{\partial y} = (1 + xy)e^{xy} = \frac{\partial Q}{\partial x}.$$

Hence **F** is conservative. By inspection, $\phi(x, y) = x + y^2 + e^{xy}$.

C15S03.009: If $P(x, y) = \cos x + \ln y$ and $Q(x, y) = \dfrac{x}{y} + e^y$, then

$$\frac{\partial P}{\partial y} = \frac{1}{y} = \frac{\partial Q}{\partial x}.$$

Hence **F** is conservative. By inspection, $\phi(x, y) = \sin x + x \ln y + e^y$.

C15S03.010: If $P(x, y) = x + \arctan y$ and $Q(x, y) = \dfrac{x + y}{1 + y^2}$, then

$$\frac{\partial P}{\partial y} = \frac{1}{1 + y^2} = \frac{\partial Q}{\partial x}.$$

Hence **F** is conservative. By inspection, $\phi(x, y) = \dfrac{1}{2}x^2 + x \arctan y + \dfrac{1}{2}\ln(1 + y^2)$.

C15S03.011: If $P(x, y) = x \cos y + \sin y$ and $Q(x, y) = y \cos x + \sin x$, then

$$\frac{\partial P}{\partial y} = \cos y - x \sin y \;\neq\; \cos x - y \sin x = \frac{\partial Q}{\partial x}.$$

Hence **F** is not conservative.

C15S03.012: If $P(x, y) = (xy + y)e^{x-y}$ and $Q(x, y) = (xy + x)e^{x-y}$, then

$$\frac{\partial P}{\partial y} = (1 + x)(1 - y)e^{x-y} \;\neq\; (1 + x)(1 + y)e^{x-y} = \frac{\partial Q}{\partial x}.$$

Hence **F** is not conservative

C15S03.013: If $P(x, y) = 3x^2y^3 + y^4$ and $Q(x, y) = 3x^3y^2 + y^4 + 4xy^3$, then

$$\frac{\partial P}{\partial y} = 9x^2y^2 + 4y^3 = \frac{\partial Q}{\partial x}.$$

Hence **F** is conservative. By inspection, $\phi(x, y) = x^3y^3 + xy^4 + \dfrac{1}{5}y^5$.

C15S03.014: If $P(x, y) = e^x \sin y + \tan y$ and $Q(x, y) = e^x \cos y + x \sec^2 y$, then

$$\frac{\partial P}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial Q}{\partial x}.$$

Hence **F** is conservative. By inspection, $\phi(x, y) = e^x \sin y + x \tan y$.

C15S03.015: If $P(x, y) = \dfrac{2x}{y} - \dfrac{3y^2}{x^4}$ and $Q(x, y) = \dfrac{2y}{x^3} - \dfrac{x^2}{y^2} + \dfrac{1}{\sqrt{y}}$, then

$$\frac{\partial P}{\partial y} = -\frac{2x}{y^2} - \frac{6y}{x^4} = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. By inspection, $\phi(x, y) = \dfrac{x^2}{y} + 2\sqrt{y} + \dfrac{y^2}{x^3}$.

C15S03.016: If $P(x, y) = \dfrac{2x^{5/2} - 3y^{5/3}}{2x^{5/2}y^{2/3}}$ and $Q(x, y) = \dfrac{3y^{5/3} - 2x^{5/2}}{3x^{3/2}y^{5/3}}$, then

$$\frac{\partial P}{\partial y} = -\frac{4x^{5/2} + 9y^{5/3}}{6x^{5/2}y^{5/3}} = \frac{\partial Q}{\partial x}.$$

Hence $\mathbf{F}$ is conservative. To find a scalar potential for $\mathbf{F}$, we compute

$$\int P(x, y)\,dx = \int \left(y^{-2/3} - \frac{3}{2}x^{-5/2}y\right)\,dx = xy^{-2/3} + x^{-3/2}y + g(y)$$

and

$$\int Q(x, y)\,dy = \int \left(x^{-3/2} - \frac{2}{3}xy^{-5/3}\right)\,dy = x^{-3/2}y + xy^{-2/3} + h(x).$$

Thus a scalar potential for $\mathbf{F}(x, y)$ is $\phi(x, y) = \dfrac{x}{y^{2/3}} + \dfrac{y}{x^{3/2}}$.

C15S03.017: We let $x(t) = x_1 t$ and $y(t) = y_1 t$ for $0 \leqq t \leqq 1$. Also let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{F}(x, y) = \langle 3x^2 + 2y^2,\ 4xy + 6y^2 \rangle$. Then

$$\int_{t=0}^1 \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^1 3(x_1^3 + 2x_1y_1^2 + 2y_1^3)t^2\,dt = \Big[(x_1^3 + 2x_1y_1^2 + 2y_1^3)t^3\Big]_0^1 = x_1^3 + 2x_1y_1^2 + 2y_1^3.$$

Then, as in Example 3, a scalar potential for $\mathbf{F}(x, y)$ is $\phi(x, y) = x^3 + 2xy^2 + 2y^3$.

C15S03.018: We let $x(t) = x_1 t$ and $y(t) = y_1 t$ for $0 \leqq t \leqq 1$. Also let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{F}(x, y) = \langle 2xy^2 + 3x^2,\ 2x^2y + 4y^3 \rangle$. Then

$$\int_{t=0}^1 \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^1 (3x_1^3t^2 + 4x_1^2y_1^2t^3 + 4y_1^4t^3)\,dt = \Big[x_1^3t^3 + x_1^2y_1^2t^4 + y_1^4t^4\Big]_0^1 = x_1^3 + x_1^2y_1^2 + y_1^4.$$

Therefore a scalar potential for $\mathbf{F}$ is $\phi(x, y) = x^3 + x^2y^2 + y^4$.

C15S03.019: We let $x(t) = x_1 t$ and $y(t) = y_1 t$ for $0 \leqq t \leqq 1$. Also let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{F}(x, y) = \langle 3x^2y^3 + y^4,\ 3x^3y^2 + y^4 + 4xy^3 \rangle$. Then

$$\begin{aligned}\int_{t=0}^1 \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)\,dt &= \int_0^1 (6x_1^3y_1^3t^5 + 5x_1y_1^4t^4 + y_1^5t^4)\,dt \\ &= \left[x_1^3y_1^3t^6 + x_1y_1^4t^5 + \frac{1}{5}y_1^5t^5\right]_0^1 = x_1^3y_1^3 + x_1y_1^4 + \frac{1}{5}y_1^5.\end{aligned}$$

Therefore a scalar potential for $\mathbf{F}$ is $\phi(x, y) = x^3y^3 + xy^4 + \dfrac{1}{5}y^5$.

C15S03.020: We let $x(t) = x_1 t$ and $y(t) = y_1 t$ for $0 \leqq t \leqq 1$. Also let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{F}(x, y) = \langle 1 + ye^{xy},\ 2y + xe^{xy} \rangle$. Then

$$\int_{t=0}^{1} \mathbf{F}(x(t),\, y(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 \left[x_1 + 2x_1y_1t\exp\left(x_1y_1t^2\right) + 2y_1^2t \right] dt$$

$$= \Big[\exp\left(x_1y_1t^2\right) + x_1t + y_1^2t^2 \Big]_0^1 = \exp(x_1y_1) - 1 + x_1 + y_1^2.$$

Therefore a scalar potential for $\mathbf{F}$ is $\phi(x,\, y) = e^{xy} + x + y^2$.

C15S03.021: Let $P(x,\, y) = y^2 + 2xy$ and $Q(x,\, y) = x^2 + 2xy$. Then

$$\frac{\partial P}{\partial y} = 2x + 2y = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x,\, y)$ is conservative with potential function $\phi(x,\, y) = x^2y + xy^2$. Therefore

$$\int_{(0,0)}^{(1,2)} P\, dx \ + \ Q\, dy = \Big[x^2y + xy^2 \Big]_{(0,0)}^{(1,2)} = 6 - 0 = 6.$$

C15S03.022: Let $P(x,\, y) = 2x - 3y$ and $Q(x,\, y) = 2y - 3x$. Then

$$\frac{\partial P}{\partial y} = -3 = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x,\, y)$ is conservative with potential function $\phi(x,\, y) = x^2 - 3xy + y^2$. Thus

$$\int_{(0,0)}^{(1,1)} P\, dx \ + \ Q\, dy = \Big[x^2 - 3xy + y^2 \Big]_{(0,0)}^{(1,1)} = -1 - 0 = -1.$$

C15S03.023: Let $P(x,\, y) = 2xe^y$ and $Q(x,\, y) = x^2e^y$. Then

$$\frac{\partial P}{\partial y} = 2xe^y = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x,\, y)$ is conservative with potential function $\phi(x,\, y) = x^2e^y$. Consequently

$$\int_{(0,0)}^{(1,-1)} P\, dx \ + \ Q\, dy = \Big[x^2e^y \Big]_{(0,0)}^{(1,-1)} = \frac{1}{e} - 0 = \frac{1}{e}.$$

C15S03.024: Let $P(x,\, y) = \cos y$ and $Q(x,\, y) = -x\sin y$. Then

$$\frac{\partial P}{\partial y} = -\sin y = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x,\, y)$ is conservative with potential function $\phi(x,\, y) = x\cos y$. So

$$\int_{(0,0)}^{(2,\pi)} P\, dx \ + \ Q\, dy = \Big[x\cos y \Big]_{(0,0)}^{(2,\pi)} = -2 - 0 = -2.$$

C15S03.025: Let $P(x,\, y) = \sin y + y\cos x$ and $Q(x,\, y) = \sin x + x\cos y$. Then

$$\frac{\partial P}{\partial y} = \cos x + \cos y = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x, y)$ is conservative with potential function $\phi(x, y) = y \sin x + x \sin y$. Therefore

$$\int_{(\pi/2,\pi/2)}^{(\pi,\pi)} P\,dx \;+\; Q\,dy = \Big[y \sin x + x \sin y\Big]_{(\pi/2,\pi/2)}^{(\pi,\pi)} = 0 - \pi = -\pi.$$

C15S03.026: Let $P(x, y) = e^y + ye^x$ and $Q(x, y) = e^x + xe^y$. Then

$$\frac{\partial P}{\partial y} = e^x + e^y = \frac{\partial Q}{\partial x},$$

and therefore $\mathbf{F}(x, y)$ is conservative with potential function $\phi(x, y) = xe^y + ye^x$. It then follows that

$$\int_{(0,0)}^{(1,-1)} P\,dx \;+\; Q\,dy = \Big[xe^y + ye^x\Big]_{(0,0)}^{(1,-1)} = \frac{1}{e} - e - 0 = \frac{1}{e} - e \approx -2.3504023872876029.$$

C15S03.027: By inspection, $\phi(x, y, z) = xyz$. For an analytic solution, write

$$\mathbf{F}(x, y, z) = \langle\, P(x, y, z),\; Q(x, y, z),\; R(x, y, z)\,\rangle$$

where $P(x, y, z) = yz$, $Q(x, y, z) = xz$, and $R(x, y, z) = xy$. Then let

$$g(x, y, z) = \int P(x, y, z)\,dx = \int yz\,dx = xyz + h(y, z).$$

Then

$$\frac{\partial g}{\partial y} = Q(x, y, z) = xz = xz + \frac{\partial h}{\partial y} \quad \text{and} \quad \frac{\partial g}{\partial z} = R(x, y, z) = xy = xy + \frac{\partial h}{\partial z},$$

and therefore the choice $h(y, z) \equiv 0$ yields a scalar potential for $\mathbf{F}$.

C15S03.028: Write $\mathbf{F}(x, y, z) = \langle\, P(x, y, z),\; Q(x, y, z),\; R(x, y, z)\,\rangle$ where $P(x, y, z) = 2x - y - z$, $Q(x, y, z) = 2y - x$, and $R(x, y, z) = 2z - x$. Let

$$\phi(x, y, z) = \int P(x, y, z)\,dx = x^2 - xy - xz + g(y, z).$$

Then

$$Q(x, y, z) = 2y - x = \frac{\partial \phi}{\partial y} = -x + \frac{\partial g}{\partial y}.$$

It follows that $g_y(y, z) = 2y$, and thus that $g(y, z) = y^2 + h(z)$. Then

$$R(x, y, z) = 2z - x = \frac{\partial \phi}{\partial z} = -x + h'(z).$$

Therefore $h(z) = z^2 + C$ where C is a constant. Therefore

$$\phi(x, y, z) = x^2 + y^2 + z^2 - xy - xz + C;$$

for a particular scalar potential for $\mathbf{F}$, choose $C = 0$.

C15S03.029: Write

$$\mathbf{F}(x,\,y,\,z) = \langle\, P(x,\,y,\,z),\ Q(x,\,y,\,z),\ R(x,\,y,\,z)\,\rangle$$

where $P(x,\,y,\,z) = y\cos z - yze^x$, $Q(x,\,y,\,z) = x\cos z - ze^x$, and $R(x,\,y,\,z) = -xy\sin z - ye^x$. Let

$$\phi(x,\,y,\,z) = \int P(x,\,y,\,z)\,dx = xy\cos z - yze^x + g(y,\,z).$$

Then

$$Q(x,\,y,\,z) = x\cos z - ze^x = \frac{\partial\phi}{\partial y} = x\cos z - ze^x + \frac{\partial g}{\partial y}.$$

Hence $g_y(x,z) = 0$, and so g is a function of z alone. Thus

$$\phi(x,\,y,\,z) = xy\cos z - yze^x + g(z),$$

and therefore

$$R(x,\,y,\,z) = -xy\sin z - ye^x = \frac{\partial\phi}{\partial z} = -xy\sin z - ye^x + g'(z).$$

Thus $g(z) = C$, a constant. So every scalar potential for $\mathbf{F}$ has the form $\phi(x,\,y,\,z) = xy\cos z - yze^x + C$. For a particular scalar potential, simply choose $C = 0$.

C15S03.030: Parametrize the circle C via $x(t) = \cos t$, $y(t) = \sin t$, $-\pi \leqq t \leqq \pi$. Let $\mathbf{r}(t) = \langle\, x(t),\, y(t)\,\rangle$. Then

$$\mathbf{F}(x(t),\,y(t)) = \langle -\sin t,\, \cos t\,\rangle,$$

and hence $\mathbf{F}(x(t),\,y(t))\cdot\mathbf{r}'(t) = 1$. So integration from $P(1,\,0)$ to $Q(-1,\,0)$ along the top half of the circle yields

$$\int_P^Q \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{\pi} 1\;dt = \pi,$$

whereas integration from P to Q along the bottom half of the circle yields

$$\int_P^Q \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{-\pi} 1\;dt = -\pi.$$

Next, if $\nabla f = \mathbf{F}$ for some function $f = f(x,\,y)$, then

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2},$$

and it follows that

$$f(x,\,y) = \arctan\left(\frac{y}{x}\right) + g(x).$$

But then

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2} + g'(x),$$

and the choice of $g(x) \equiv 0$ yields the function

$$f(x,\,y) = \arctan\left(\frac{y}{x}\right)$$

for which $\nabla f = \mathbf{F}$ provided that $x \neq 0$. There can be no function $\phi(x,\,y)$ such that $\nabla\phi = \mathbf{F}$ for all $(x,\,y) \neq (0,\,0)$ because our previous work shows that the line integral of $\mathbf{F}$ from P to Q is not independent of the path.

C15S03.031: Suppose that the force field $\mathbf{F} = \langle\, P,\, Q\,\rangle$ is conservative in the plane region D. Then there exists a potential function $\phi(x,\,y)$ for $\mathbf{F}$; that is, $\nabla\phi = \mathbf{F}$, so that

$$\frac{\partial\phi}{\partial x} = P(x,\,y) \quad\text{ and }\quad \frac{\partial\phi}{\partial y} = Q(x,\,y)$$

on the interior of D. But then

$$\frac{\partial P}{\partial y} = \frac{\partial^2\phi}{\partial y\,\partial x} = \frac{\partial^2\phi}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

on the interior of D, under the assumption that the second-order mixed partial derivatives of ϕ are continuous there. Of course, continuity of P_y and Q_x on D is enough to guarantee this.

C15S03.032: Suppose that the force field $\mathbf{F} = \langle\, P,\, Q,\, R\,\rangle$ is conservative in the space region D. Then there exists a potential function $\phi(x,\,y,\,z)$ for $\mathbf{F}$; that is, $\nabla\phi = \mathbf{F}$, so that

$$\frac{\partial\phi}{\partial x} = P(x,\,y,\,z), \quad \frac{\partial\phi}{\partial y} = Q(x,\,y,\,z), \quad\text{and}\quad \frac{\partial\phi}{\partial z} = R(x,\,y,\,z)$$

on the interior of D. But then

$$\frac{\partial P}{\partial y} = \frac{\partial^2\phi}{\partial y\,\partial x} = \frac{\partial^2\phi}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial^2\phi}{\partial z\,\partial x} = \frac{\partial^2\phi}{\partial x\,\partial z} = \frac{\partial R}{\partial x}, \quad\text{and}\quad \frac{\partial Q}{\partial z} = \frac{\partial^2\phi}{\partial z\,\partial y} = \frac{\partial^2\phi}{\partial y\,\partial z} = \frac{\partial R}{\partial y}$$

provided that $\phi(x,\,y,\,z)$ has continuous second-order mixed partial derivatives. But continuity of P_y, P_z, Q_x, Q_z, R_x, and R_y on D is enough to guarantee this.

C15S03.033: The given integral is not independent of the path because

$$\frac{\partial x^2}{\partial z} \;=\; 0 \;\neq\; 2y \;=\; \frac{\partial y^2}{\partial y}.$$

C15S03.034: Parametrize the path C from $(0,\,0,\,0)$ to $(x_1,\,y_1,\,z_1)$ by $x(t) = x_1t$, $y(t) = y_1t$, $z(t) = z_1t$, $0 \leqq t \leqq 1$. Let $\mathbf{r}(t) = \langle\, x(t),\, y(t),\, z(t)\,\rangle$. Given

$$\mathbf{F}(x,\,y,\,z) = \langle\, yz,\; xz + y,\; xy + 1\,\rangle,$$

note that $\mathbf{F}(x(t),\,y(t),\,z(t)) = \langle\, y_1z_1t^2,\; y_1t + x_1z_1t^2,\; 1 + x_1y_1t^2\,\rangle$, so that

$$\mathbf{F}(t)\cdot\mathbf{r}'(t) = z_1 + y_1^2t + 3x_1y_1z_1t^2.$$

Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (z_1 + y_1^2t + 3x_1y_1z_1t^2)\,dt = \left[z_1t + \frac{1}{2}y_1^2t^2 + x_1y_1z_1t^3\right]_0^1 = z_1 + \frac{1}{2}y_1^2 + x_1y_1z_1.$$

The subscripts are now superfluous; thus we obtain

$$f(x,\, y,\, z) = z + \frac{1}{2}y^2 + xyz,$$

and therefore $\nabla f = \langle\, yz,\, y + xz,\, 1 + xy\,\rangle = \mathbf{F}(x,\, y,\, z)$.

C15S03.035: Part (a): If

$$f(x,\, y) = \arctan\left(\frac{y}{x}\right),$$

then

$$\frac{\partial f}{\partial x} = \frac{-\dfrac{y}{x^2}}{1 + \dfrac{y^2}{x^2}} = -\frac{y}{x^2 + y^2} \quad \text{and}$$

$$\frac{\partial f}{\partial y} = \frac{\dfrac{1}{x}}{1 + \dfrac{y^2}{x^2}} = \frac{x}{x^2 + y^2}.$$

Part (b): Because $\mathbf{F} = \nabla f$ has a potential on the right half-plane $x > 0$, line integrals of $\mathbf{F}$ will be independent of the path C provided that the path always remains in the right half-plane. Hence if C is such a path from $A(x_1,\, y_1) = (r_1,\, \theta_1)$ to $B(x_2,\, y_2) = (r_2,\, \theta_2)$, then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \Big[f(x,\, y) \Big]_A^B = \theta_2 - \theta_1.$$

Part (c): Parametrize the unit circle using $x(t) = \cos t$, $y(t) = \sin t$, $-\pi \leqq t \leqq \pi$. It is easy to verify that

$$\mathbf{F}(x(t),\, y(t)) = \langle\, -\sin t,\, \cos t\,\rangle$$

and that $\mathbf{F}(t) \cdot \mathbf{r}'(t) = 1$. Therefore

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} 1\; dt = \pi \quad \text{and} \quad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{-\pi} 1\; dt = -\pi.$$

This does not contradict the fundamental theorem of calculus for line integrals because $\mathbf{F}$ does not have a scalar potential defined in a region containing both C_1 and C_2 (see the solution of Problem 30).

C15S03.036: If

$$\mathbf{F} = \frac{k\mathbf{r}}{r^3}$$

is the inverse-square force field of Example 7 in Section 15.2, then in Cartesian coordinates we have

$$\mathbf{F}(x,\, y,\, z) = \left\langle \frac{kx}{(x^2 + y^2 + z^2)^{3/2}},\, \frac{ky}{(x^2 + y^2 + z^2)^{3/2}},\, \frac{kz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle,$$

and it is easy to show that a scalar potential for $\mathbf{F}$ is

$$\phi(x,\,y,\,z) \;=\; -\frac{k}{(x^2+y^2+z^2)^{1/2}}.$$

Therefore $\mathbf{F}$ is conservative on any space region not containing the origin. Let $r = \sqrt{x^2+y^2+z^2}$. Then $\phi(x,\,y,\,z) = -k/r$, and therefore the work done by $\mathbf{F}$ in moving from a point P at distance r_1 from the origin to a point Q at distance r_2 from the origin is

$$W = \int_P^Q \mathbf{F}\cdot d\mathbf{r} = \Big[\phi(x,\,y,\,z)\Big]_P^Q = \left[-\frac{k}{r}\right]_{r_1}^{r_2} = k\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

C15S03.037: The units are mks units throughout. We use $M = 5.97\times10^{24}$, $G = 6.67\times10^{-11}$, $m = 10000$, and let $k = GMm = 3.98199\times10^{18}$ in the formula in Problem 36. We also must convert r_1 and r_2 into meters: $r_1 = 9000\cdot1000$ and $r_2 = 11000\cdot1000$. Then substitution in the formula in Problem 36 yields $W = 8.04442\times10^{10}$ N·m.

C15S03.038: The units are mks units throughout. We use $M = 1.99\times10^{30}$, $G = 6.67\times10^{-11}$, $m = 10000$, and let $k = GMm = 1.32733\times10^{24}$ in the formula in Problem 36. We also must convert r_1 and r_2 into meters: $r_1 = 1.5\times10^{11}$ and $r_2 = 2.29\times10^{11}$. Then substitution in the formula in Problem 36 yields $W = 3.05267\times10^{12}$ N·m.

Section 15.4

Note: As in the text, the notation

$$\oint_C P(x, y)\, dx \,+\, Q(x, y)\, dy$$

(and variations thereof) always denotes an integral around the *closed* path C with *counterclockwise* (positive) orientation.

C15S04.001: By Green's theorem,

$$\oint_C (x + y^2)\, dx + (y + x^2)\, dy = \int_{y=-1}^{1} \int_{x=-1}^{1} (2x - 2y)\, dx\, dy$$

$$= \int_{y=-1}^{1} \Big[x^2 - 2xy \Big]_{x=-1}^{1} dy = \int_{-1}^{1} -4y\, dy = \Big[-2y^2 \Big]_0^1 = 0.$$

C15S04.002: By Green's theorem,

$$\oint_C (x^2 + y^2)\, dx - 2xy\, dy = \int_0^1 \int_0^{1-x} (-2y - 2y)\, dy\, dx$$

$$= \int_0^1 \Big[-2y^2 \Big]_0^{1-x} dx = \int_0^1 -2(1-x)^2\, dx = \left[\frac{2}{3}(1-x)^3 \right]_0^1 = -\frac{2}{3}.$$

C15S04.003: By Green's theorem,

$$\oint_C (y + e^x)\, dx + (2x^2 + \cos y)\, dy = \int_0^1 \int_y^{2-y} (4x - 1)\, dx\, dy$$

$$= \int_0^1 \Big[2x^2 - x \Big]_y^{2-y} dy = \int_0^1 (6 - 6y)\, dy = \Big[6y - 3y^2 \Big]_0^1 = 3.$$

C15S04.004: By Green's theorem,

$$\oint_C (x^2 - y^2)\, dx + xy\, dy = \int_0^1 \int_{x^2}^{x} 3y\, dy\, dx$$

$$= \int_0^1 \left[\frac{3}{2}y^2 \right]_{x^2}^{x} dx = \frac{3}{2} \int_0^1 (x^2 - x^4)\, dx = \frac{3}{2} \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \frac{1}{5}.$$

C15S04.005: By Green's theorem,

$$\oint_C \big[-y^2 + \exp(e^x) \big]\, dx + (\arctan y)\, dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 2y\, dy\, dx = \int_0^1 \Big[y^2 \Big]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 (x - x^4)\, dx = \left[\frac{1}{2}x^2 - \frac{1}{5}x^5 \right]_0^1 = \frac{3}{10}.$$

C15S04.006: Let D be the disk bounded by the circle C. Then by Green's theorem,

$$\begin{aligned}
\oint_C y^2\,dx + (2x-3y)\,dy &= \iint_D (2-2y)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{3} (2-2r\sin\theta)\cdot r\,dr\,d\theta \\
&= \int_0^{2\pi}\left[r^2 - \frac{2}{3}r^3\sin\theta\right]_0^3 d\theta = \int_0^{2\pi}(9-18\sin\theta)\,d\theta \\
&= \Big[9\theta + 18\cos\theta\Big]_0^{2\pi} = 18\pi \approx 56.5486677646162783.
\end{aligned}$$

C15S04.007: By Green's theorem,

$$\oint_C (x-y)\,dx + y\,dy = \int_0^{\pi}\int_0^{\sin x} 1\,dy\,dx = \int_0^{\pi}\sin x\,dx = \Big[-\cos x\Big]_0^{\pi} = 2.$$

C15S04.008: Let R be the region bounded by the curve C. By Green's theorem,

$$\oint_C e^x\sin y\,dx + e^x\cos y\,dy = \iint_R (e^x\cos y - e^x\cos y)\,dA = \iint_R 0\,dA = 0.$$

C15S04.009: Let D be the bounded region bounded by the curve C. Then by Green's theorem,

$$\oint_C y^2dx + xy\,dy = \iint_D (y-2y)\,dA = -\iint_D y\,dA = 0$$

by symmetry.

C15S04.010: Let R denote the bounded plane region bounded by the curve C. Then by Green's theorem,

$$\oint_C \frac{y}{1+x^2}\,dx + (\arctan x)\,dy = \iint_R \left(\frac{1}{1+x^2} - \frac{1}{1+x^2}\right)dA = \iint_R 0\,dA = 0.$$

C15S04.011: Let R denote the bounded plane region bounded by the curve C. Then by Green's theorem,

$$\begin{aligned}
\oint_C xy\,dx + x^2\,dy &= \iint_R (2x-x)\,dA = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{3}r^3\cos\theta\right]_0^{\sin 2\theta} d\theta \\
&= \int_0^{\pi/2}\frac{8}{3}\sin^3\theta\,\cos^4\theta\,d\theta = \frac{1}{840}\Big[5\cos 7\theta + 7\cos 5\theta - 35\cos 3\theta - 105\cos\theta\Big]_0^{\pi/2} \\
&= \frac{16}{105} \approx 0.15238095238095238095.
\end{aligned}$$

C15S04.012: Let D be the bounded plane region bounded by the curve C. Then by Green's theorem,

$$\oint_C x^2\,dx - y^2\,dy = \iint_D 0\,dA = 0.$$

C15S04.013: The given parametrization and the corollary to Green's theorem yield area

$$A = \oint_C x\,dy = \int_0^{2\pi} a^2 \cos^2 t\,dt = a^2\left[\frac{1}{2}t + \frac{1}{2}\sin t\,\cos t\right]_0^{2\pi} = \pi a^2.$$

C15S04.014: The given parametrization and the corollary to Green's theorem yield area

$$A = \oint_C y\,dx = a^2\int_0^{2\pi}(1-\cos t)^2\,dt = \frac{1}{4}a^2\Big[6t - 8\sin t + \sin 2t\Big]_0^{2\pi} = 3\pi a^2 \approx (9.4247779607693797)a^2.$$

There's no minus sign in front of the first integral because the counterclockwise direction around the region bounded by the cycloid and the x-axis is opposite to the direction of the parametrization along the cycloid. There's no need to evaluate the line integral along the x-axis because $x\,dy = 0$ there.

C15S04.015: We'll use the given parametrization, find the area of the part of the astroid in the first quadrant, then multiply by 4. The corollary to Green's theorem yields area

$$A = \oint_C x\,dy = 4\int_0^{\pi/2} 3\sin^2 t\,\cos^4 t\,dt$$

$$= \frac{1}{16}\Big[12t + 3\sin 2t - 3\sin 4t - \sin 6t\Big]_0^{\pi/2} = \frac{3}{8}\pi \approx 1.17809724509617246442.$$

There's no need to evaluate the line integral along the x- or y-axes because $x\,dy = 0$ there.

C15S04.016: Parametrize the lower part of C with $x(t) = t$, $y(t) = t^3$, $0 \leqq t \leqq 1$. Parametrize the upper part of C with $x(t) = t$, $y(t) = t^2$, $0 \leqq t \leqq 1$. Note that the latter parametrization is opposite the counterclockwise direction around C, so an extra minus sign will be required. The corollary to Green's theorem then yields area

$$A = \oint_C x\,dy = \int_0^1 t\cdot 3t^2\,dt - \int_0^1 t\cdot 2t\,dt = \left[\frac{3}{4}t^4\right]_0^1 - \left[\frac{2}{3}t^3\right]_0^1 = \frac{3}{4} - \frac{2}{3} = \frac{1}{12} \approx 0.0833333333333333.$$

C15S04.017: Denote by E the bounded plane region bounded by the given curve C. Then the work is

$$W = \oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C -2y\,dx + 3x\,dy = \iint_E 5\,dA = 5\cdot 6\pi = 30\pi \approx 94.2477796076937972.$$

(The area of the ellipse is 6π because it has semiaxes of lengths 2 and 3.)

C15S04.018: Denote by D the disk bounded by the given circle C. Then the work is

$$W = o\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \oint_C (y^2 - x^2)\,dx + 2xy\,dy = \iint_D (2y - 2y)\,dA = 0.$$

C15S04.019: Denote by T the triangular region bounded by the given triangle C. Then the work done is

$$W = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C 5x^2y^3\, dx + 7x^3y^2\, dy = \iint_T (21x^2y^2 - 15x^2y^2)\, dA = \int_{x=0}^{3} \int_{y=0}^{6-2x} 6x^2y^2\, dy\, dx$$

$$= \int_0^3 \left[2x^2y^3 \right]_{y=0}^{6-2x} dx = \int_0^3 (432x^2 - 432x^3 + 144x^4 - 16x^5)\, dx$$

$$= \left[144x^3 - 108x^4 + \frac{144}{5}x^5 - \frac{8}{3}x^6 \right]_0^3 = \frac{972}{5} = 194.4.$$

C15S04.020: Denote by S the semicircular plane region bounded by the given curve C. Then the work done is

$$W = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C xy^2\, dx + 3x^2y\, dy = \iint_S (6xy - 2xy)\, dA = \iint_S 4xy\, dA = 0$$

by symmetry. Or, without using symmetry, the last integral becomes

$$\int_{\theta=0}^{\pi} \int_{r=0}^{2} 4r^3 \sin\theta\, \cos\theta\, dr\, d\theta = \int_0^{\pi} 16 \sin\theta\, \cos\theta\, d\theta = \left[8\sin^2\theta \right]_0^{\pi} = 0.$$

C15S04.021: Denote by R the bounded plane region bounded by the given curve C. Then the outward flux of $\mathbf{F}$ across C is

$$\phi = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA = \iint_R \nabla \cdot \langle\, 2x,\, 3y \,\rangle\, dA = \iint_R 5\, dA = 30\pi \approx 94.2477796076937972.$$

C15S04.022: Denote by D the disk bounded by the given circular path C. Then the outward flux of $\mathbf{F}$ across C is

$$\phi = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA = \iint_D \nabla \cdot \langle\, x^3,\, y^3 \,\rangle\, dA = \iint_D (3x^2 + 3y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^3 3r^3\, dr\, d\theta = 6\pi \left[\frac{1}{4}r^4 \right]_0^3 = \frac{243}{2}\pi \approx 381.7035074111598785.$$

C15S04.023: Denote by T the triangular region bounded by the given path C. Then the outward flux of $\mathbf{F}$ across C is

$$\phi = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_T \nabla \cdot \mathbf{F}\, dA$$

$$= \iint_T \nabla \cdot \langle\, 3x + \sqrt{1+y^2}\,,\ 2y - (1+x^4)^{1/3} \,\rangle\, dA = \iint_T 5\, dA = 5 \cdot \frac{1}{2} \cdot 3 \cdot 6 = 45$$

because T is a triangle with base 3 and height 6.

C15S04.024: Denote by S the semicircular region bounded by the given path C. Then the outward flux of $\mathbf{F}$ across C is

$$\phi = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_S \nabla \cdot \mathbf{F}\, dA = \iint_S (3x^2 + 3y^2)\, dA$$

$$= 3 \int_0^{\pi} \int_0^2 r^3\, dr\, d\theta = 3\pi \cdot \frac{16}{4} = 12\pi \approx 37.6991118430775189.$$

C15S04.025: Given f, a twice-differentiable function of x and y, we have

$$\nabla^2 f = \nabla \cdot (\nabla f) = \nabla \cdot \langle f_x,\, f_y \rangle = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

C15S04.026: Given $f(x,\, y) = \ln(x^2 + y^2)$, we have

$$\nabla f = \left\langle \frac{2x}{x^2 + y^2},\, \frac{2y}{x^2 + y^2} \right\rangle, \quad \text{and thus}$$

$$\nabla^2 f = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} + \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$$

provided that $(x,\, y) \neq (0,\, 0)$.

C15S04.027: If f and g are twice-differentiable functions of x and y, then

$$\begin{aligned}
\nabla^2(fg) &= \nabla \cdot \big[\nabla(fg)\big] = \nabla \cdot \langle f_x g + f g_x,\ f_y g + f g_y \rangle \\
&= f_{xx} g + f_x g_x + f_x g_x + f g_{xx} + f_{yy} g + f_y g_y + f_y g_y + f g_{yy} \\
&= (f)(g_{xx} + g_{yy}) + 2(f_x g_x + f_y g_y) + (g)(f_{xx} + f_{yy}) \\
&= f \nabla^2 g + g \nabla^2 f + 2\langle f_x,\, f_y \rangle \cdot \langle g_x,\, g_y \rangle = f \nabla^2 g + g \nabla^2 f + 2 \nabla f \cdot \nabla g.
\end{aligned}$$

Compare this with Problem 33 of Section 15.1.

C15S04.028: By Green's theorem,

$$\oint_C f_x\, dy - f_y\, dx = \oint_C -f_y\, dx + f_x\, dy = \iint_R (f_{xx} + f_{yy})\, dA = \iint_R \nabla^2 f\, dx\, dy.$$

C15S04.029: We may assume constant density $\delta = 1$. Then

$$A = \oint_C x\, dy = \oint_C -y\, dx.$$

Now

$$M_y = \iint_R x\, dA \quad \text{and} \quad M_x = \iint_R y\, dA.$$

Hence, by Green's theorem,

$$M_y = \oint_C \frac{1}{2} x^2\, dy \quad \text{and} \quad M_x = -\oint_C \frac{1}{2} y^2\, dx.$$

Therefore

$$\overline{x} = \frac{M_y}{A} = \frac{1}{2A}\oint_C x^2\, dy \quad \text{and} \quad \overline{y} = \frac{M_x}{A} = -\frac{1}{2A}\oint_C y^2\, dx.$$

C15S04.030: Part (a): We use the semicircular region R bounded by the semicircle C described by $x^2+y^2 = a^2$, $y \geqq 0$ and the line segment L with endpoints $(-a, 0)$ and $(a, 0)$. Both the integrals in Problem 29 will be zero on L, so we will parametrize only C and integrate only along C. Then the parametrization $x = a\cos t$, $y = a\sin t$, $0 \leqq t \leqq \pi$ of C and the results in Problem 29 yield

$$\overline{x} = \frac{1}{\pi a^2}\int_C x^2\, dy = \frac{1}{\pi a^2}\int_0^{\pi} a^3\cos^3 t\, dt = \frac{a}{\pi}\left[\sin t - \frac{1}{3}\sin^3 t\right]_0^{\pi} = 0 \quad \text{and}$$

$$\overline{y} = -\frac{1}{\pi a^2}\int_C y^2\, dx = \frac{1}{\pi a^2}\int_0^{\pi} a^3\sin^3 t\, dt = \frac{a}{\pi}\left[-\cos t + \frac{1}{3}\cos^3 t\right]_0^{\pi} = \frac{4a}{3\pi} \approx (0.4244131815783876)a.$$

Part (b): We use the quarter circle R bounded by the circular arc C described by $x^2 + y^2 = a^2$, $x \geqq 0$, $y \geqq 0$ and the coordinate axes. Both the integrals in Problem 29 will be zero on the coordinate axes, so we parametrize only C and integrate only along C. The parametrization $x = a\cos t$, $y = a\sin t$, $0 \leqq t \leqq \pi/2$ and the results in Problem 29 yield

$$\overline{x} = \frac{2}{\pi a^2}\int_C x^2\, dy = \frac{2}{\pi a^2}\int_0^{\pi/2} a^3\cos^3 t\, dt = \frac{2a}{\pi}\left[\sin t - \frac{1}{3}\sin^3 t\right]_0^{\pi/2} = \frac{4a}{3\pi} \approx (0.4244131815783876)a \quad \text{and}$$

$$\overline{y} = -\frac{2}{\pi a^2}\int_C y^2\, dx = \frac{2}{\pi a^2}\int_0^{\pi/2} a^3\sin^3 t\, dt = \frac{2a}{\pi}\left[-\cos t + \frac{1}{3}\cos^3 t\right]_0^{\pi/2} = \frac{4a}{3\pi} \approx (0.4244131815783876)a.$$

C15S04.031: Suppose that the plane region R is bounded by the piecewise smooth simple closed curve C, oriented counterclockwise, and that R has constant density δ. Then, by Green's theorem,

$$I_x = \iint_R \delta y^2\, dA = \delta\oint_C -\frac{1}{3}y^3\, dx = -\frac{1}{3}\delta\oint_C y^3\, dx \quad \text{and}$$

$$I_y = \iint_R \delta x^2\, dA = \delta\oint_C \frac{1}{3}x^3\, dy = \frac{1}{3}\delta\oint_C x^3\, dy.$$

C15S04.032: By the results in Problem 31, we have

$$\begin{aligned} I_0 = I_x + I_y &= \frac{1}{3}\delta\oint_C -y^3\, dx + x^3\, dy = \frac{1}{3}\delta\iint_R (3x^2 + 3y^2)\, dA = \delta\iint_R r^2\, dA \\ &= \delta\int_0^{2\pi}\int_0^a r^3\, dr\, d\theta = \delta\cdot 2\pi\cdot\frac{1}{4}a^4 = \pi\delta a^2\cdot\frac{1}{2}a^2 = \frac{1}{2}Ma^2. \end{aligned}$$

C15S04.033: As in Problem 30 of Section 10.4, the substitution $y = tx$ in the equation $x^3 + y^3 = 3xy$ of the folium yields $x^3 + t^3x^3 = 3tx^2$, and thereby the parametrization

$$x(t) = \frac{3t}{1+t^3}, \quad y(t) = \frac{3t^2}{1+t^3}, \quad 0 \leqq t < +\infty$$

of the first-quadrant loop of the folium. If C is the half of its loop that stretches from (0, 0) to $\left(\frac{3}{2}, \frac{3}{2}\right)$ along the lower half of the folium, then C is swept out by this parametrization as t varies from 0 to 1. Let J be the straight line segment joining $\left(\frac{3}{2}, \frac{3}{2}\right)$ with (0, 0); parametrize J with $x = \frac{3}{2}(1-t),\ y = \frac{3}{2}(1-t),\ 0 \leqq t \leqq 1$. Then the area of the folium is

$$A = 2 \cdot \frac{1}{2} \oint_{C \cup J} x\,dy - y\,dx = \int_C x\,dy - y\,dx + \int_J x\,dy - y\,dx.$$

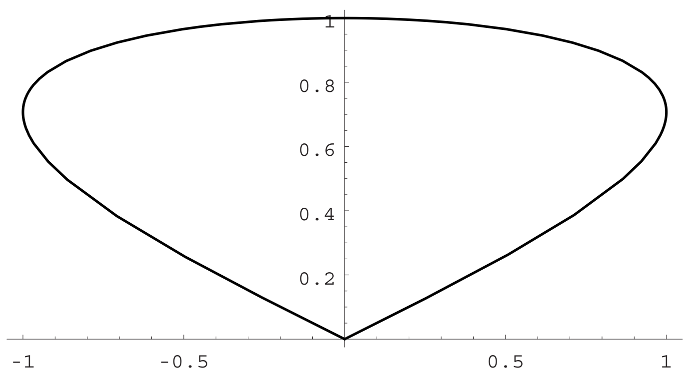
The last integral is

$$\int_{t=0}^{1} \left[-\frac{3}{2}(1-t) + \frac{3}{2}(1-t) \right] dt = 0,$$

and hence the area of the folium is

$$\begin{aligned} A = \int_0^1 \big[x(t)y'(t) - y(t)x'(t) \big]\,dt &= \int_0^1 \left[\frac{9t(2t-t^4)}{(1+t^3)^3} - \frac{9t^2(2t^3-1)}{(1+t^3)^3} \right] dt \\ &= \int_0^1 \frac{9t^2}{(1+t^3)^2}\,dt = \left[-\frac{3}{1+t^3} \right]_0^1 = \frac{3}{2}. \end{aligned}$$

C15S04.034: One complete loop of the curve is swept out as t varies from 0 to π, as indicated in the graph of the loop, shown next.



This graph was generated by *Mathematica* 3.0 in response to the command

```
ParametricPlot[ {Sin[2*t], Sin[t]}, {t, 0, Pi}, AspectRatio → Automatic ];
```

The area enclosed by the loop is

$$A = \oint_C x\,dy = \int_0^\pi 2\cos^2 t \sin t\,dt = \left[-\frac{2}{3}\cos^3 t \right]_0^\pi = \frac{4}{3}.$$

C15S04.035: We substitute $f\nabla g$ for $\mathbf{F}$ in Eq. (9),

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_R \nabla \cdot \mathbf{F}\,dA.$$

With the aid of the result in Problem 28 of Section 15.1, this yields

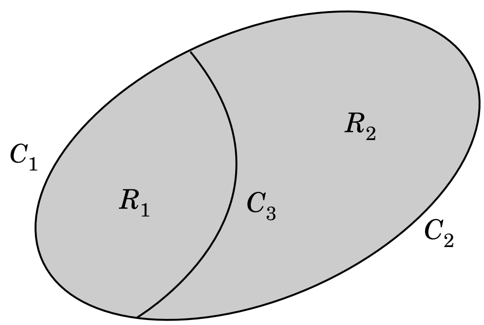
$$\oint_C f \nabla g \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot (f \nabla g)\, dA = \iint_R (f \nabla \cdot \nabla g + \nabla f \cdot \nabla g)\, dA.$$

C15S04.036: Suppose that the plane region R is bounded below by the horizontal line $y = a,$ on the right by the graph of $x = g_2(y),$ above by the horizontal line $y = b,$ and on the left by the graph of $x = g_1(y)$; let C denote the boundary of $R,$ oriented counterclockwise. On the top and bottom we have $dy = 0$, so we will neither parametrize these two parts of C nor will we bother to integrate $Q(x,\, y)\, dy$ over them. The right-hand part of C may be parametrized by $x = g_2(t),\ y = t,\ a \leqq t \leqq b.$ The left-hand part may be parametrized, though in the "wrong" direction, by $x = g_1(t),\ y = t,\ a \leqq t \leqq b.$ Thus we have

$$\oint_C Q(x,\, y)\, dy = \int_a^b Q(g_2(t), t)\, dt - \int_a^b Q(g_1(t),\, t)\, dt = \int_a^b \big[Q(g_2(t),\, t) - Q(g_1(t),\, t)\big]\, dt$$

$$= \int_a^b \Big[Q(x,\, t)\Big]_{x=g_1(t)}^{g_2(t)}\, dt = \int_{t=a}^b \int_{x=g_1(t)}^{g_2(t)} \frac{\partial Q}{\partial x}\, dx\, dt = \int_{y=a}^b \int_{x=g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x}\, dA = \iint_R Q_x\, dA.$$

C15S04.037: It suffices to show the result in the case that $R = R_1 \cup R_1$ is the union of two regions, with C the boundary of $R,$ $C_1 \cup C_3$ the boundary of $R_1,$ and $-C_3 \cup C_2$ the boundary of R_2. Then $C = C_1 \cup C_2$. Perhaps the next figure will clarify all this.



Under the assumption that Green's theorem holds for R_1 and for $R_2,$ we have

$$\oint_{C_1 \cup C_3} P\, dx + Q\, dy = \iint_{R_1} (Q_x - P_y)\, dA \quad \text{and}$$

$$\oint_{-C_3 \cup C_2} P\, dx + Q\, dy = \iint_{R_2} (Q_x - P_y)\, dA.$$

Addition of these equations yields

$$\int_{C_1} (P\, dx + Q\, dy) + \int_{C_2} (P\, dx + Q\, dy) + \int_{C_3} (P\, dx + Q\, dy) - \int_{C_3} (P\, dx + Q\, dy) = \iint_R (Q_x - P_y)\, dA.$$

Therefore

$$\oint_C P\, dx + Q\, dy \ = \ \iint_R (Q_x - P_y)\, dA.$$

C15S04.038: Part (a): First parametrize C as follows:

$$x(t) = x_1 + (x_2 - x_1)t, \quad y(t) = y_1 + (y_2 - y_1)t, \quad 0 \leqq t \leqq 1.$$

Then

$$\begin{aligned}
\int_C x\,dy - y\,dx &= \int_{t=0}^1 \Big\{ \big[x_1 + (x_2 - x_1)t\big](y_2 - y_1) - \big[y_1 + (y_2 - y_1)t\big](x_2 - x_1)\Big\}\,dt \\
&= \left[x_1(y_2 - y_1)t + \frac{1}{2}(x_2 - x_1)(y_2 - y_1)t^2 - y_1(x_2 - x_1)t - \frac{1}{2}(y_2 - y_1)(x_2 - x_1)t^2 \right]_0^1 \\
&= x_1y_2 - x_1y_1 + \frac{1}{2}(x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1) - x_2y_1 + x_1y_1 - \frac{1}{2}(x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1) \\
&= x_1y_2 - x_2y_1.
\end{aligned}$$

Part (b): Let C denote the boundary of the given triangle T, oriented counterclockwise. Then the area of T is given by

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}(0 \cdot y_1 - x_2 \cdot 0 + x_1y_2 - x_2y_1 + x_2 \cdot 0 - 0 \cdot y_2) = \frac{1}{2}(x_1y_2 - x_2y_1).$$

C15S04.039: Part (a): With $x_1 = 1,\ y_1 = 0,\ x_2 = \cos(2\pi/3),\ y_2 = \sin(2\pi/3),\ x_3 = \cos(4\pi/3),$ and $y_3 = \sin(4\pi/3),$ we obtain area

$$A = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = \frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{3} = \frac{3}{4}\sqrt{3} \approx 1.29903810567666.$$

Part (b): With the points $(x_i,\ y_i)$ $(1 \leqq i \leqq 5)$ chosen in a way analogous to that in part (a), we obtain area

$$\begin{aligned}
A &= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_5 - x_5y_4 + x_5y_1 - x_1y_5) \\
&= \frac{5}{8}\sqrt{10 + 2\sqrt{5}} \approx 2.37764129073788393029.
\end{aligned}$$

(We used a computer algebra program to evaluate the first line, then simplified the result by hand.)

C15S04.040: By the corollary to Green's theorem,

$$A = \iint_R 1\,dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_J \left|\frac{\partial(x,\,y)}{\partial(u,\,v)}\right|(u\,dv - v\,du) = \iint_S \left|\frac{\partial(x,\,y)}{\partial(u,\,v)}\right|\,du\,dv.$$

C15S04.041: Using *Mathematica* 3.0, we entered the parametric functions

$$x(t) = \frac{(2n+1)t^n}{t^{2n+1}+1} \quad \text{and} \quad y(t) = \frac{(2n+1)t^{n+1}}{t^{2n+1}+1}.$$

Then we computed one of the integrands for area in Eq. (4):

```
(−y′[t]∗x[t] + x[t]∗y′[t])/2 // Together
```

$$\frac{(2n+1)^2 t^{2n}}{2(t^{2n+1}+1)^2}$$

Then we set $n = 1$, then 2, then 3, and found the areas of the corresponding loops as follows:

```
Integrate[ (9/2)*(t∧2)/(1 + t∧3)∧2, t ]
```

$$-\frac{3}{2(t^3+1)}$$

```
( % /. t → Infinity ) − ( % /. t → 0 )
```

$$\frac{3}{2}$$

```
Integrate[ (25/2)*(t∧4)/(1 + t∧5)∧2, t ]
```

$$-\frac{5}{2(t^5+1)}$$

```
( % /. t → Infinity ) − ( % /. t → 0 )
```

$$\frac{5}{2}$$

```
Integrate[ (49/2)*(t∧6)/(1 + t∧7)∧2, t ]
```

$$-\frac{7}{2(t^7+1)}$$

```
( % /. t → Infinity ) − ( % /. t → 0 )
```

$$\frac{7}{2}$$

In fact, the integral is easy to find in the general case:

```
Integrate[ (1/2)*((2*n + 1)∧2)*(t∧(2*n))/(1 + t∧(2*n + 1))∧2, t ]
```

$$-\frac{2n+1}{2(1+t^{2n+1})}$$

Therefore

$$A_n = \left[-\frac{2n+1}{2(1+t^{2n+1})} \right]_0^\infty = 0 + \frac{2n+1}{2} = n + \frac{1}{2}.$$

To avoid the improper integral, let C denote the simple closed curve consisting of the lower half of the loop (swept out as t ranges from 0 to 1) together with the "return path" along the line $y = x$ back to the origin. On the return path P we have the parametrization $x = y = n + \frac{1}{2} - t$ as t ranges from 0 to $n + \frac{1}{2}$; but on this path, $-y\,dx + x\,dy = 0$, and hence

$$\frac{1}{2} \int_P -y\,dx + x\,dy = 0.$$

The area enclosed by the loop is double that enclosed by C, and thus

$$A_n = 2 \cdot \left[-\frac{2n+1}{2(1+t^{2n+1})} \right]_0^1 = -\frac{2n+1}{2} + (2n+1) = n + \frac{1}{2}.$$

The only danger in this "short cut" is that you may use one of the other two formulas in Eq. (4) of the text and forget the line integral along P.

Section 15.5

C15S05.001: Here S is the surface $z = h(x, y) = 1 - x - y$ over the plane triangle bounded by the nonnegative coordinate axes and the graph of $y = 1 - x$. So

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dx\ dy = \sqrt{3}\ dx\ dy.$$

Therefore

$$\iint_S (x+y)\ dS = \int_{x=0}^1 \int_{y=0}^{1-x} (x+y)\sqrt{3}\ dy\ dx = \sqrt{3}\ \int_0^1 \left[xy + \frac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \sqrt{3}\ \int_0^1 \left(\frac{1}{2} - \frac{1}{2}x^2 \right) dx = \sqrt{3}\left[\frac{1}{2}x - \frac{1}{6}x^3 \right]_0^1 = \frac{1}{3}\sqrt{3} \approx 0.5773502691896258.$$

C15S05.002: Here S is the surface $z = h(x, y) = 6 - 2x - 3y$ over the plane triangle bounded by the nonnegative coordinate axes and the graph of $y = \frac{1}{3}(6 - 2x)$. Also

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dx\ dy = \sqrt{1+4+9}\ dx\ dy = \sqrt{14}\ dx\ dy,$$

and therefore

$$\iint_S xyz\ dS = \int_0^3 \int_0^{(6-2x)/3} xy(6 - 2x - 3y)\sqrt{14}\ dy\ dx = \sqrt{14}\ \int_0^3 \left[3xy^2 - x^2y^2 - xy^3 \right]_0^{(6-2x)/3} dx$$

$$= \sqrt{14}\ \int_0^3 \left(4x - 4x^2 + \frac{4}{3}x^3 - \frac{4}{27}x^4 \right) dx = \sqrt{14}\left[2x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4 - \frac{4}{135}x^5 \right]_0^3$$

$$= \frac{9}{5}\sqrt{14} \approx 6.7349832961930945.$$

C15S05.003: First, S is the surface $z = h(x, y) = 2x + 3y$ lying over the circular disk D with center $(0, 0)$ and radius 3 in the xy-plane. Also

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dx\ dy = \sqrt{14}\ dx\ dy,$$

and thus

$$\iint_S (y + z + 3)\ dS = \iint_D (y + 2x + 3y + 3)\sqrt{14}\ dA = \int_{\theta=0}^{2\pi} \int_{r=0}^3 (4r\sin\theta + 2r\cos\theta + 3)\left(r\sqrt{14} \right) dr\ d\theta$$

$$= \sqrt{14}\ \int_0^{2\pi} \left[\frac{3}{2}r^2 + \frac{2}{3}r^3(\cos\theta + 2\sin\theta) \right]_0^3 d\theta = \sqrt{14}\ \int_0^{2\pi} \left(\frac{27}{2} + 18\cos\theta + 36\sin\theta \right) d\theta$$

$$= \sqrt{14}\left[\frac{27}{2}\theta + 18\sin\theta - 36\cos\theta \right]_0^{2\pi} = 27\pi\sqrt{14} \approx 317.3786106805529421.$$

C15S05.004: The surface S is the part of the cone $z = h(x, y) = \sqrt{x^2 + y^2}$ that lies over the circular disk D with center $(0, 0)$ and radius 2 in the xy-plane. Next,

$$dS = \left[1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2\right]^{1/2} dx\,dy = \left[1 + \frac{x^2+y^2}{x^2+y^2}\right]^{1/2} dx\,dy = \sqrt{2}\,dx\,dy.$$

Therefore

$$\iint_S z^2\,dS = \iint_S (x^2+y^2)\,dS = \iint_D r^3\sqrt{2}\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

$$= 2\pi\sqrt{2}\left[\frac{1}{4}r^4\right]_0^2 = 8\pi\sqrt{2} \approx 35.54306350526692997613.$$

C15S05.005: The surface S is the part of the paraboloid $z = h(x,\,y) = x^2 + y^2$ that lies over the circular disk D with center $(0,\,0)$ and radius 2 in the xy-plane. Also

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\,dx\,dy = \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy,$$

and thus

$$\iint_S (xy+1)\,dS = \iint_D (xy+1)\sqrt{1+4x^2+4y^2}\,dx\,dy = \int_0^{2\pi}\int_0^2 (1 + r^2\sin\theta\,\cos\theta)\cdot r(1+4r^2)^{1/2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{240}(1+4r^2)^{3/2}\Big[20 + (6r^2-1)\sin 2\theta\Big]_0^2\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{240}\left[340\sqrt{17} - 20 + \left(1 + 391\sqrt{17}\right)\sin 2\theta\right]\,d\theta$$

$$= \frac{1}{480}\left[40\left(17\sqrt{17}-1\right)\theta - \left(1+391\sqrt{17}\right)\cos 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{480}\left[1 + 391\sqrt{17} - 1 - 391\sqrt{17} + 80\pi\left(17\sqrt{17}-1\right)\right]$$

$$= \frac{1}{6}\pi\left(-1 + 17\sqrt{17}\right) \approx 36.17690319741140836475 6.$$

C15S05.006: Here S is the hemisphere $z = h(x,\,y) = (1 - x^2 - y^2)^{1/2}$, described in spherical coordinates by

$$\rho = 1, \quad 0 \leqq \phi \leqq \frac{1}{2}\pi, \quad 0 \leqq \theta \leqq 2\pi$$

and parametrized for such ϕ and θ by

$$x = \sin\phi\,\cos\theta, \quad y = \sin\phi\,\sin\theta, \quad z = \cos\phi.$$

Thus with $\mathbf{r}(\phi,\,\theta) = \langle\,\sin\phi\,\cos\theta,\ \sin\phi\,\sin\theta,\ \cos\phi\,\rangle$ we have

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\,\cos\theta & \cos\phi\,\sin\theta & -\sin\phi \\ -\sin\phi\,\sin\theta & \sin\phi\,\cos\theta & 0 \end{vmatrix} = \langle \sin^2\phi\,\cos\theta,\ \sin^2\phi\,\sin\theta,\ \sin\phi\,\cos\phi\,\rangle.$$

Therefore

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = (\sin^4\phi\,\cos^2\theta + \sin^4\phi\,\sin^2\theta + \sin^2\phi\,\cos^2\phi)^{1/2} = (\sin^4\phi + \sin^2\phi\,\cos^2\phi)^{1/2} = |\sin\phi|.$$

Hence

$$\iint_S (x^2+y^2)z\,dS = \int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi\,\cos\phi\,d\phi\,d\theta = 2\pi\left[\frac{1}{4}\sin^4\phi\right]_0^{\pi/2} = \frac{1}{2}\pi \approx 1.5707963267948966.$$

C15S05.007: The surface S is the part of the graph of $z = h(x,\,y) = x + y$ that lies over the circular disk D with center $(0,\,0)$ and radius 3 in the xy-plane. Also $dS = \sqrt{1+1+1}\;dx\;dy = \sqrt{3}\;dx\;dy,$ and hence

$$I_z = \iint_S \delta(x^2+y^2)\,dS = \iint_D \delta r^2\sqrt{3}\;dA = \delta\sqrt{3}\int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta = 2\pi\delta\sqrt{3}\left[\frac{1}{4}r^4\right]_0^3 = \frac{81}{2}\pi\delta\sqrt{3}\,.$$

The mass of S is

$$m = \delta\sqrt{3}\int_0^{2\pi}\int_0^3 r\,dr\,d\theta = 2\pi\delta\sqrt{3}\left[\frac{1}{2}r^2\right]_0^3 = 9\pi\delta\sqrt{3}\,,$$

and therefore $I_z = \dfrac{9}{2}m.$

C15S05.008: The surface S has equation $z = h(x,\,y) = xy$ and lies over (and under) the circular disk D with center $(0,\,0)$ and radius 5 in the xy-plane. The surface area element is

$$dS = \sqrt{1+(h_x)^2+(h_y)^2}\;dA = \sqrt{1+x^2+y^2}\;dA,$$

and therefore the moment of inertia of S (with constant density δ) with respect to the z-axis is

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\delta\,dS = \iint_D \delta r^2\sqrt{1+r^2}\;dA = \int_0^{2\pi}\int_0^5 \delta r^3\sqrt{1+r^2}\;dr\,d\theta \\ &= \frac{2}{15}\pi\delta\left[(3r^4+r^2-2)\sqrt{1+r^2}\,\right]_0^5 = \frac{4}{15}\pi\delta\left(1+949\sqrt{26}\,\right). \end{aligned}$$

Because the mass of S is

$$m = \iint_S \delta\,dS = \int_0^{2\pi}\int_0^5 \delta r\sqrt{1+r^2}\;dr\,d\theta = \frac{2}{3}\pi\delta\left(-1+26\sqrt{26}\,\right),$$

the moment of inertia may also be expressed in the form

$$I_z = \frac{2+1898\sqrt{26}}{-5+130\sqrt{26}}\cdot m \approx m\cdot(3.8358837113445991)^2.$$

C15S05.009: Suppose that (x, z) is a point in the xz-plane. Let $\mathbf{w}$ be the radius vector from the origin in the xz-plane to (x, z) and let θ be the angle that $\mathbf{w}$ makes with the nonnegative x-axis. Then points in the cylindrical surface S are described by

$$x = \cos\theta, \quad y = y, \quad z = \sin\theta, \quad -1 \leqq y \leqq 1, \quad 0 \leqq \theta \leqq 2\pi.$$

Thus the cylindrical surface S is parametrized by $\mathbf{r}(y, \theta) = \langle \cos\theta,\ y,\ \sin\theta \rangle$, for which $\mathbf{r}_y = \langle 0, 1, 0 \rangle$ and $\mathbf{r}_\theta = \langle -\sin\theta, 0, \cos\theta \rangle$. Hence

$$\mathbf{r}_y \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} = \langle \cos\theta, 0, \sin\theta \rangle.$$

Therefore $|\mathbf{r}_y \times \mathbf{r}_\theta| = 1$, so that $dS = dy\,d\theta$. Let D denote the rectangle $-1 \leqq y \leqq 1$, $0 \leqq \theta \leqq 2\pi$. Because S has constant density δ, its moment of inertia with respect to the z-axis is therefore

$$I_z = \iint_S (x^2 + y^2)\delta\,dS = \iint_D (y^2 + \cos^2\theta)\delta\,dA = \int_{\theta=0}^{2\pi} \int_{y=-1}^{1} (y^2 + \cos^2\theta)\delta\,dy\,d\theta$$

$$= \delta \int_0^{2\pi} \left[\frac{1}{3}y^3 + y\cos^2\theta \right]_{-1}^{1} d\theta = \delta \int_0^{2\pi} \left(\frac{2}{3} + 2\cos^2\theta \right) d\theta = \frac{1}{6}\delta \left[10\theta + 3\sin 2\theta \right]_0^{2\pi} = \frac{10}{3}\pi\delta.$$

Because the mass m of S is the product of its surface area and its density, we have $m = 4\pi\delta$, and hence we may also express I_z in the form

$$I_z = \frac{5}{6}m = m \cdot (0.9128709291752769)^2.$$

C15S05.010: The surface S has equation $z = h(x, y) = \sqrt{x^2 + y^2}$ and lies over the annular region R in the xy-plane described in polar coordinates by $2 \leqq r \leqq 5$, $0 \leqq \theta \leqq 2\pi$. We have surface area element

$$dS = \left(1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} \right)^{1/2} dA = \sqrt{2}\ dA,$$

and therefore the moment of inertia of the constant-density surface S with respect to the z-axis is

$$I_z = \iint_S (x^2 + y^2)\delta\,dS = \int_0^{2\pi} \int_2^5 \delta r^3 \sqrt{2}\,dr\,d\theta = \delta\sqrt{2} \int_0^{2\pi} \left[\frac{1}{4}r^4 \right]_2^5 = \frac{609}{2}\pi\delta\sqrt{2}\,.$$

The mass of S is

$$m = \iint_S \delta\,dS = \int_0^{2\pi} \int_2^5 \delta r\sqrt{2}\,dr\,d\theta = 21\delta\pi\sqrt{2}\,,$$

and therefore its moment of inertia with respect to the z-axis may also be expressed in the form

$$I_z = \frac{29}{2}m \approx m \cdot (3.8078865529319541)^2.$$

C15S05.011: The surface S has the spherical-coordinates parametrization

$$\mathbf{r}(\phi,\,\theta) = \langle\, 5\sin\phi\,\cos\theta,\ 5\sin\phi\,\sin\theta,\ 5\cos\phi\,\rangle, \quad 0 \leqq \phi \leqq \arccos\left(\frac{3}{5}\right),\ \ 0 \leqq \theta \leqq 2\pi.$$

Therefore

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5\cos\phi\,\cos\theta & 5\cos\phi\,\sin\theta & -5\sin\phi \\ -5\sin\phi\,\sin\theta & 5\sin\phi\,\cos\theta & 0 \end{vmatrix} = \langle\, 25\sin^2\phi\,\cos\theta,\ 25\sin^2\phi\,\sin\theta,\ 25\sin\phi\,\cos\phi\,\rangle,$$

and thus

$$|\,\mathbf{r}_\phi \times \mathbf{r}_\theta\,| = \sqrt{625\sin^2\phi\,\cos^2\phi + 625\sin^4\phi\,\cos^2\theta + 625\sin^4\phi\,\sin^2\theta}\; = 25\sin\phi$$

(because $0 \leqq \phi \leqq \pi/2$). Hence the mass of S is

$$m = \iint_S \delta\; dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\arccos(3/5)} 25\delta\sin\phi\; d\phi\; d\theta = 2\pi\delta\Big[-25\cos\phi\Big]_0^{\arccos(3/5)}$$
$$= 20\pi\delta \;\approx\; (62.8318530717958648)\delta.$$

Next,

$$x^2 + y^2 = (5\sin\phi\,\cos\theta)^2 + (5\sin\phi\,\sin\theta)^2 = 25\sin^2\phi,$$

and hence the moment of inertia of S with respect to the z-axis is

$$I_z = \iint_S (x^2+y^2)\delta\; dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\arccos(3/5)} 625\delta\sin^3\phi\; d\phi\; d\theta = 2\pi\delta\left[\frac{625}{3}\cos^3\phi - 625\cos\phi\right]_0^{\arccos(3/5)}$$
$$= \frac{520}{3}\pi\delta \;\approx\; (544.5427266222308280)\delta.$$

The moment of inertia may also be expressed in the form $I_z = \dfrac{26}{3}m \;\approx\; m\cdot(2.9439202887759490)^2$.

C15S05.012: The upper half of the surface S has the spherical-coordinates parametrization

$$\mathbf{r}(\phi,\,\theta) = \langle\, 5\sin\phi\,\cos\theta,\ 5\sin\phi\,\sin\theta,\ 5\cos\phi\,\rangle, \quad \arccos\left(\frac{4}{5}\right) \leqq \phi \leqq \frac{1}{2}\pi,\ \ 0 \leqq \theta \leqq 2\pi.$$

To find the mass and moment of inertia with respect to the z-axis, we will integrate over the top half of S and then double the result. But first,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5\cos\phi\,\cos\theta & 5\cos\phi\,\sin\theta & -5\sin\phi \\ -5\sin\phi\,\sin\theta & 5\sin\phi\,\cos\theta & 0 \end{vmatrix} = \langle\, 25\sin^2\phi\,\cos\theta,\ 25\sin^2\phi\,\sin\theta,\ 25\sin\phi\,\cos\phi\,\rangle,$$

and thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{625 \sin^2\phi \, \cos^2\phi + 625 \sin^4\phi \, \cos^2\theta + 625 \sin^4\phi \, \sin^2\theta} \, = 25 \sin\phi$$

(because $0 \leqq \phi \leqq \pi/2$). Hence the mass of S is

$$\begin{aligned} m &= \iint_S \delta \, dS = 2\int_{\theta=0}^{2\pi} \int_{\phi=\arccos(4/5)}^{\pi/2} 25\delta \sin\phi \, d\phi \, d\theta = 4\pi\delta \Big[-25\cos\phi \Big]_{\arccos(4/5)}^{\pi/2} \\ &= 80\pi\delta \approx (251.3274122871834591)\delta. \end{aligned}$$

Next,

$$x^2 + y^2 = (5\sin\phi\,\cos\theta)^2 + (5\sin\phi\,\sin\theta)^2 = 25\sin^2\phi,$$

and hence the moment of inertia of S with respect to the z-axis is

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\delta \, dS = 2\int_{\theta=0}^{2\pi} \int_{\phi=\arccos(4/5)}^{\pi/2} 625\delta \sin^3\phi \, d\phi \, d\theta = 4\pi\delta \left[\frac{625}{3}\cos^3\phi - 625\cos\phi \right]_{\arccos(4/5)}^{\pi/2} \\ &= \frac{4720}{3}\pi\delta \approx (4942.7724416479413618)\delta. \end{aligned}$$

The moment of inertia may also be expressed in the form $I_z = \dfrac{59}{3}m \approx m \cdot (4.4347115652166902)^2$.

C15S05.013: An upward unit vector normal to S is

$$\mathbf{n} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle.$$

The surface has equation $z = h(x, y) = \sqrt{9 - x^2 - y^2}$, and therefore

$$dS = \sqrt{1 + \frac{x^2}{9 - x^2 - y^2} + \frac{y^2}{9 - x^2 - y^2}} \, dA = \frac{3}{\sqrt{9 - x^2 - y^2}} \, dA.$$

Next, $\mathbf{F} \cdot \mathbf{n} = \dfrac{1}{3}(x^2 + y^2)$, and S lies over the circular disk D in the plane with center $(0, 0)$ and radius 3. Therefore

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_D \frac{x^2+y^2}{\sqrt{9 - x^2 - y^2}} \, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \frac{r^3}{\sqrt{9 - r^2}} \, dr \, d\theta \\ &= 2\pi \left[-\frac{1}{3}(r^2 + 18)\sqrt{9 - r^2} \right]_0^3 = 36\pi \approx 113.0973355292325566. \end{aligned}$$

C15S05.014: An upward unit vector normal to S is $\mathbf{n} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$. The surface S has equation $z = h(x, y) = 3 - 2x - 2y$, and therefore $dS = \sqrt{1 + 4 + 4} \, dA = 3 \, dA$. Also, the surface S lies over the triangle T in the first quadrant bounded by the coordinate axes and the line $y = \frac{1}{2}(3 - 2x)$. Thus

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \iint_T 3 \, dy \, dx = \int_0^{3/2} \int_0^{(3-2x)/2} 3 \, dy \, dx \\ &= \int_0^{3/2} \frac{3}{2}(3 - 2x) \, dx = \frac{1}{2}\Big[9x - 3x^2 \Big]_0^{3/2} = \frac{27}{8} = 3.375. \end{aligned}$$

C15S05.015: An upward unit vector normal to S is

$$\mathbf{n} = \left\langle -\frac{3}{10}\sqrt{10},\ 0,\ \frac{1}{10}\sqrt{10} \right\rangle.$$

The surface S has equation $z = h(x, y) = 3x+2$, and therefore $dS = \sqrt{10}\ dA$. Also, S lies over the circular disk S in the xy-plane with center $(0, 0)$ and radius 2. Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iint_D 3z\ dA = \iint_D (9x+6)\ dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (9r^2\cos\theta + 6r)\ dr\ d\theta = \int_0^{2\pi} \Big[3r^3\cos\theta + 3r^2\Big]_0^2$$

$$= \int_0^{2\pi} (24\cos\theta + 12)\ d\theta = \Big[24\sin\theta + 12\theta\Big]_0^{2\pi} = 24\pi \approx 75.3982236861550377.$$

C15S05.016: An upward unit vector normal to the surface S is $\mathbf{n} = \left\langle \frac{1}{2}x,\ \frac{1}{2}y,\ \frac{1}{2}z \right\rangle$. The surface may be parametrized in spherical coordinates by

$$\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi \rangle, \quad 0 \leqq \phi \leqq \frac{1}{2}\pi,\ \ 0 \leqq \theta \leqq 2\pi.$$

Then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix} = \langle 4\sin^2\phi\,\cos\theta,\ 4\sin^2\phi\,\sin\theta,\ 4\sin\phi\cos\phi \rangle.$$

Therefore

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\,\cos^2\theta + 16\sin^4\phi\,\sin^2\theta + 16\sin^2\phi\,\cos^2\phi} = 4\sin\phi$$

(the last equality because $0 \leqq \phi \leqq \pi$). Next,

$$\mathbf{F}\cdot\mathbf{n} = \frac{1}{2}z^2 = 2\cos^2\phi,$$

and therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} 8\sin\phi\cos^2\phi\ d\phi\ d\theta = 2\pi\left[-\frac{8}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{16}{3}\pi \approx 16.7551608191455639.$$

C15S05.017: The surface S has Cartesian equation $z = h(x, y) = \sqrt{x^2+y^2}$, and thus has normal vector

$$\mathbf{n}_1 = \langle h_x,\ h_y,\ -1 \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}},\ -1 \right\rangle,$$

and thus (in polar coordinates) a upward-pointing vector normal to S is $\mathbf{n}_2 = \langle -\cos\theta,\ -\sin\theta,\ 1 \rangle$. Therefore an upward-pointing unit vector normal to S is

$$\mathbf{n} = \frac{\mathbf{n}_2}{|\mathbf{n}_2|} = \frac{\sqrt{2}}{2}\langle -\cos\theta,\ -\sin\theta,\ 1 \rangle.$$

Next,

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \left(1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}\right)^{1/2} dA = \sqrt{2}\ dA,$$

and in polar coordinates we have $\mathbf{F} = \langle\, r\sin\theta,\ -r\cos\theta,\ 0\,\rangle$. But then $\mathbf{F}\cdot\mathbf{n}\,dS = 0$, so the surface integral is zero as well.

C15S05.018: The surface S has equation $z = h(x,\,y) = 4 - x^2 - y^2$ and lies above the circular disk D in the xy-plane with center $(0,\,0)$ and radius 2. Also

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{1 + 4x^2 + 4y^2}\ dA,$$

and because $\langle h_x,\ h_y,\ -1\,\rangle$ is normal to S, we find that an upward-pointing unit vector normal to S is

$$\mathbf{n} = \left\langle \frac{2x}{\sqrt{1 + 4x^2 + 4y^2}},\ \frac{2y}{\sqrt{1 + 4x^2 + 4y^2}},\ \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} \right\rangle.$$

Therefore

$$\mathbf{F}\cdot\mathbf{n}\,dS = \frac{4x^2 + 4y^2 + 3}{\sqrt{1 + 4x^2 + 4y^2}} \cdot \sqrt{1 + 4x^2 + 4y^2}\ dA,$$

and consequently

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (4x^2 + 4y^2 + 3)\ dA = \int_0^{2\pi}\int_0^2 (4r^3 + 3r)\ dr\ d\theta$$

$$= 2\pi\left[r^4 + \frac{3}{2}r^2\right]_0^2 = 44\pi \approx 138.2300767579509025.$$

C15S05.019: On the face of the cube in the xy-plane, $z = 0$, and so

$$\mathbf{F}(x,\,y,\,z)\cdot\mathbf{n} = \langle\, x,\ 2y,\ 0\,\rangle\cdot\langle\, 0,\ 0,\ -1\,\rangle = 0,$$

and hence the flux of $\mathbf{F}$ across that face is zero. Similarly, the flux across the faces in the other two coordinate planes is zero. On the top face we have $z = 1$, and hence

$$\mathbf{F}(x,\,y,\,z)\cdot\mathbf{n} = \langle x,\ 2y,\ 3\,\rangle\cdot\langle\, 0,\ 0,\ 1\,\rangle = 3.$$

Similarly, the flux across the face in the plane $y = 1$ is 2 and the flux across the face in the plane $x = 1$ is 1. Hence the total flux of $\mathbf{F}$ across S is $3 + 2 + 1 = 6$.

C15S05.020: The hemispherical surface $z = \sqrt{4 - x^2 - y^2}$ has unit normal vector

$$\mathbf{n} = \frac{1}{2}\langle\, x,\ y,\ z\,\rangle$$

and parametrization

$$x = 2\sin\phi\,\cos\theta,\ \ y = 2\sin\phi\,\sin\theta,\ \ z = 2\cos\phi, \quad 0 \leqq \phi \leqq \frac{1}{2}\pi,\ \ 0 \leqq \theta \leqq 2\pi.$$

The usual computation of $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$ (see the solution of Problem 6, 11, 12, 13, or 16) yields $dS = 4\sin\phi\, dA$ (provided that $0 \leqq \phi \leqq \pi$), and thereby

$$\begin{aligned}\mathbf{F}(x,\,y,\,z)\cdot\mathbf{n} &= \langle\, 4\sin\phi\,\cos\theta,\ -6\sin\phi\,\sin\theta,\ 2\cos\phi\,\rangle\cdot\langle\,\sin\phi\,\cos\theta,\,\sin\phi\,\sin\theta,\,\cos\phi\,\rangle\\ &= 2\cos^2\phi + 4\sin^2\phi\,\cos^2\theta - 6\sin^2\phi\,\sin^2\theta,\end{aligned}$$

and hence the flux of $\mathbf{F}$ across the upper hemispherical surface H is

$$\begin{aligned}\iint_H \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi/2}(8\cos^2\phi\,\sin\phi + 16\sin^3\phi\,\cos^2\theta - 24\sin^3\phi\,\sin^2\theta)\,d\phi\,d\theta\\ &= \int_0^{2\pi}\left[-\frac{8}{3}\cos^3\phi - 16\cos\phi\,\cos^2\theta + \frac{16}{3}\cos^3\phi\,\cos^2\theta + 24\cos\phi\,\sin^2\theta - 8\cos^3\phi\,\sin^2\theta\right]_0^{\pi/2}d\theta\\ &= \int_0^{2\pi}\left(\frac{8}{3} + \frac{32}{3}\cos^2\theta - 16\sin^2\theta\right)d\theta = \left[\frac{40}{3}\sin\theta\,\cos\theta\right]_0^{2\pi} = 0.\end{aligned}$$

On the circular disk D that forms the base of the hemispherical solid,

$$\mathbf{F}\cdot\mathbf{n} = \langle\,2x,\ -3y,\ 0\,\rangle\cdot\langle\,0,\ 0,\ -1\,\rangle = 0,$$

and therefore

$$\iint_D \mathbf{F}\cdot\mathbf{n}\,dS = 0.$$

Hence the total flux of $\mathbf{F}$ across the surface S is zero.

C15S05.021: For the same reasons given in the solution of Problem 19, $\mathbf{F}\cdot\mathbf{n} = 0$ on the three faces of the pyramid in the coordinate planes. On the fourth face T a unit normal vector is

$$\mathbf{n} \ = \ \frac{1}{\sqrt{26}}\langle\,3,\,4,\,1\,\rangle,$$

and because this face is the graph of $z = h(x,\,y) = 12 - 3x - 4y$, we have

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{26}\ dx\,dy.$$

Therefore $\mathbf{F}\cdot\mathbf{n}\,dS = (3x - 4y)\,dy\,dx$, and consequently

$$\begin{aligned}\iint_T \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^4\int_0^{(12-3x)/4}(3x - 4y)\,dy\,dx = \int_0^4\left[3xy - 2y^2\right]_0^{(12-3x)/4}dx\\ &= \int_0^4\left(-18 + 18x - \frac{27}{8}x^2\right)dx = \left[-18x + 9x^2 - \frac{9}{8}x^3\right]_0^4 = 0.\end{aligned}$$

Thus the total flux of $\mathbf{F}$ across S is zero. If you now turn two pages ahead in your textbook, you will see how the divergence theorem enables you to obtain the same result in less than two seconds and without need of pencil, paper, or computer.

C15S05.022: On the circular disk D that forms the base of the given parabolic solid, we easily see that $\mathbf{F}\cdot\mathbf{n} = \langle\,2x,\,2y,\,3\,\rangle\cdot\langle\,0,\,0,\,-1\,\rangle = -3$, a constant, and therefore the flux of $\mathbf{F}$ across D is simply the product

of -3 and the area of D: -12π. An upward-pointing unit vector normal to the upper curved surface C described by $z = h(x, y) = 4 - x^2 - y^2$ is

$$\mathbf{n} = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} \langle\, 2x,\ 2y,\ 1\,\rangle,$$

and

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{1 + 4x^2 + 4y^2}\ dA.$$

Therefore on C we have $\mathbf{F} \cdot \mathbf{n}\, dS = 4x^2 + 4y^2 + 3$. Thus the flux of $\mathbf{F}$ across C is

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (4x^2 + 4y^2 + 3)\, dA = \int_0^{2\pi} \int_0^2 (4r^2 + 3) \cdot r\, dr\, d\theta = 2\pi \left[r^4 + \frac{3}{2} r^2 \right]_0^2 = 44\pi,$$

and therefore the total flux of $\mathbf{F}$ across S is

$$44\pi - 12\pi = 32\pi \approx 100.5309649148733836308045 88.$$

C15S05.023: The paraboloids meet in the circle $x^2 + y^2 = 9$, $z = 9$, so both the upper surface and the lower surface lie over the disk D in the xy-plane with center $(0, 0)$ and radius 3. The lower surface L is the graph of $h(x, y) = x^2 + y^2$ and the upper surface U is the graph of $j(x, y) = 18 - x^2 - y^2$ for (x, y) in D. A vector normal to L is

$$\langle\, h_x,\ h_y,\ -1\,\rangle = \langle\, 2x,\ 2y,\ -1\,\rangle$$

and hence the outer unit vector normal to L is

$$\mathbf{n}_1 = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} \langle 2x,\ 2y,\ -1\,\rangle;$$

similarly, the outer unit vector normal to U is

$$\mathbf{n}_2 = \frac{1}{\sqrt{1 + 4x^2 + 4y^2}} \langle\, 2x,\ 2y,\ 1\,\rangle.$$

The surface area element for L is

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{1 + 4x^2 + 4y^2}\ dA$$

and the surface area element for U is the same. Next,

$$\mathbf{F} \cdot \mathbf{n}_1\, dS = -z^2 = -(x^2 + y^2)^2\, dA \quad \text{and} \quad \mathbf{F} \cdot \mathbf{n}_2\, dS = z^2 = (18 - x^2 - y^2)^2\, dA.$$

Thus

$$\iint_L \mathbf{F} \cdot \mathbf{n}_1\, dS = -\iint_D (x^2 + y^2)^2\, dA = -\int_0^{2\pi} \int_0^3 r^5\, dr\, d\theta = -2\pi \left[\frac{1}{6} r^6 \right]_0^3 = -243\pi$$

and

$$\iint_U \mathbf{F} \cdot \mathbf{n}_2\, dS = \iint_D (18 - x^2 - y^2)^2\, dA = \int_0^{2\pi} \int_0^3 (18 - r^2)^2 \cdot r\, dr\, d\theta = 2\pi \left[-\frac{1}{6} (18 - r^2)^3 \right]_0^3 = 1701\pi.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 1701\pi - 243\pi = 1458\pi \approx 4580.4420889339185417.$$

C15S05.024: As we saw in the solutions of Problems 4, 10, and 17, the surface area element for the conical surface $z = \sqrt{x^2 + y^2}$ is $dS = \sqrt{2}\ dA$ and the outer unit normal vector for that surface C is

$$\mathbf{n}_1 = \frac{\sqrt{2}}{2} \left\langle \frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}},\ -1 \right\rangle.$$

Let D denote the circular disk in the xy-plane centered at the origin and having radius 3. Then

$$\begin{aligned}
\iint_C \mathbf{F} \cdot \mathbf{n}_1\, dS &= \iint_D \frac{x^3 + 2y^3 - 3z^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\, dA = \iint_D \frac{x^3 + 2y^3 - 3(x^2+y^2)^{3/2}}{\sqrt{x^2+y^2}}\, dA \\
&= \int_0^{2\pi} \int_0^3 (r^3 \cos^3\theta + 2r^3 \sin^3\theta - 3r^3)\ dr\ d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^2 (\cos^3\theta + 2\sin^3\theta - 3) \right]_0^3 d\theta \\
&= \int_0^{2\pi} \frac{81}{4} (\cos^3\theta + 2\sin^3\theta - 3)\ d\theta \\
&= \left[\frac{27}{16} (2\cos 3\theta - 18\cos\theta + \sin 3\theta + 9\sin\theta - 36\theta) \right]_0^{2\pi} = -\frac{243\pi}{2}.
\end{aligned}$$

The outer unit vector normal to the circular disk $z = 3,\ x^2 + y^2 \leqq 9$ that forms the top T of the solid is $\mathbf{n}_2 = \langle 0, 0, 1 \rangle$ and it should be clear that the surface area element is $dS = dA = dx\ dy$. Hence

$$\iint_T \mathbf{F} \cdot \mathbf{n}_2\ dS = \iint_D 27\ dA = \pi \cdot 3^2 \cdot 27 = 243\pi.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = 243\pi - \frac{243}{2}\pi = \frac{243}{2}\pi \approx 381.7035074111598784732111171.$$

C15S05.025: The surface S may be parametrized by

$$x(\phi, \theta) = a\sin\phi\, \cos\theta,\ y(\phi, \theta) = a\sin\phi\, \sin\theta,\ z(\phi, \theta) = a\cos\phi, \quad 0 \leqq \phi \leqq \frac{1}{2}\pi,\ 0 \leqq \theta \leqq \frac{1}{2}\pi.$$

Then, if $\mathbf{r}(\phi, \theta) = \langle x(\phi, \theta),\ y(\phi, \theta),\ z(\phi, \theta) \rangle$, we find that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\, \cos\theta & a\cos\phi\, \sin\theta & -a\sin\phi \\ -a\sin\phi\, \sin\theta & a\sin\phi\, \cos\theta & 0 \end{vmatrix} = \langle a^2 \sin^2\phi\, \cos\theta,\ a^2\sin^2\phi\, \sin\theta,\ a^2 \sin\phi\, \cos\phi \rangle,$$

and therefore

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4 \cos^2\phi\, \sin^2\phi + a^4 \sin^4\phi\, \cos^2\theta + a^4 \sin^4\phi\, \sin^2\theta} = a^2 \sin\phi$$

because $0 \leqq \phi \leqq \pi$. Because the surface has unit density, its mass is

$$m = \iint_S 1\ dS = \int_0^{\pi/2} \int_0^{\pi/2} a^2 \sin\phi\ d\phi\ d\theta = \frac{1}{2}\pi\Big[-a^2\cos\phi\Big]_0^{\pi/2} = \frac{1}{2}\pi a^2.$$

The moment of S with respect to the yz-plane is

$$\begin{aligned} M_{yz} &= \iint_S x(\phi,\,\theta)\ dS = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \sin^2\phi\,\cos\theta\ d\phi\ d\theta \\ &= \int_0^{\pi/2} \left[\frac{1}{4}a^3(\cos\theta)(2\phi - \sin 2\phi)\right]_0^{\pi/2} d\theta = \int_0^{\pi/2} \frac{1}{4}\pi a^3\cos\theta\ d\theta = \left[\frac{1}{4}\pi a^3\sin\theta\right]_0^{\pi/2} = \frac{1}{4}\pi a^3. \end{aligned}$$

Therefore (by symmetry) $(\overline{x},\,\overline{y},\,\overline{z}) = \left(\frac{1}{2}a,\ \frac{1}{2}a,\ \frac{1}{2}a\right)$.

C15S05.026: Given the conical surface $z = r$, we saw in the solutions of Problems 4, 10, 17, and 24 that $dS = \sqrt{2}\ dA = \sqrt{2}\ dx\ dy$. The surface S lies over the circular disk D with center $(0, 0)$ and radius a in the xy-plane and because the surface has constant density $\delta = k$, it has mass

$$m = \iint_S k\ dS = \int_0^{2\pi} \int_0^a kr\sqrt{2}\ dr\ d\theta = 2\pi\left[\frac{1}{2}kr^2\sqrt{2}\right]_0^a = \pi ka^2\sqrt{2}\,.$$

The moment of S with respect to the xy-plane is

$$M_{xy} = \iint_S kz\ dS = \int_0^{2\pi} \int_0^a kr^2\sqrt{2}\ dr\ d\theta = 2\pi\left[\frac{1}{3}kr^3\sqrt{2}\right]_0^a = \frac{2}{3}\pi ka^3\sqrt{2}\,.$$

The centroid of S lies on the z-axis by symmetry, and therefore

$$\overline{x} = 0, \quad \overline{y} = 0, \quad \text{and} \quad \overline{z} = \frac{M_{xy}}{m} = \frac{2}{3}a.$$

The moment of inertia of S with respect to the z-axis is

$$I_z = \iint_S k(x^2 + y^2)\ dS = \int_0^{2\pi} \int_0^a kr^3\sqrt{2}\ dr\ d\theta = 2\pi\left[\frac{1}{4}kr^4\sqrt{2}\right]_0^a = \frac{1}{2}\pi ka^4\sqrt{2} = \frac{1}{2}ma^2.$$

C15S05.027: The surface $z = r^2$, $0 \leqq r \leqq a$ is described in Cartesian coordinates by $z = h(x,\,y) = x^2 + y^2$, and thus

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{1 + 4x^2 + 4y^2}\ dA.$$

Because the surface S lies over the circular disk D in the xy-plane with center $(0, 0)$ and radius a and because S has constant density δ, its mass is

$$\begin{aligned} m &= \iint_S \delta\ dS = \iint_D \delta\sqrt{1 + 4x^2 + 4y^2}\ dA = \int_0^{2\pi} \int_0^a \delta r\sqrt{1 + 4r^2}\ dr\ d\theta \\ &= 2\pi\delta\left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^a = \frac{1}{6}\pi\delta\big[(1 + 4a^2)^{3/2} - 1\big]. \end{aligned}$$

The moment of S with respect to the xy-plane is

$$M_{xy} = \iint_S \delta z \, dS = \iint_D \delta(x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\ dA = \int_0^{2\pi} \int_0^a \delta r^3 \sqrt{1 + 4r^2}\ dr\ d\theta$$

$$= 2\pi\delta\left[\left(\frac{1}{5}r^4 + \frac{1}{60}r^2 - \frac{1}{120}\right)\sqrt{1+4r^2}\ \right]_0^a = \frac{1}{60}\pi\delta\big[(24a^4 + 2a^2 - 1)\sqrt{1+4a^2}\ + 1\big].$$

Hence the z-coordinate of the centroid of S is

$$\overline{z} = \frac{M_{yx}}{m} = \frac{(24a^4 + 2a^2 - 1)\sqrt{1+4a^2}\ + 1}{10\big[(1+4a^2)^{3/2} - 1\big]}.$$

For example, if $a = 1$, then

$$\overline{z} = \frac{1 + 25\sqrt{5}}{10\left(-1 + 5\sqrt{5}\,\right)} \approx 0.5589371284878981.$$

By symmetry, $\overline{x} = \overline{y} = 0$. Finally, the moment of inertia of S with respect to the z-axis is

$$I_z = \iint_S \delta(x^2 + y^2)\ dS = \iint_D \delta(x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\ dA = \frac{1}{60}\pi\delta\big[(24a^4 + 2a^2 - 1)\sqrt{1+4a^2}\ + 1\big]$$

(the computations are exactly the same as those in the evaluation of M_{xy}).

C15S05.028: The surface S may be parametrized by

$$x(\phi,\,\theta) = a\sin\phi\,\cos\theta,\ \ y(\phi,\,\theta) = a\sin\phi\,\sin\theta,\ \ z(\phi,\,\theta) = a\cos\phi, \quad 0 \leqq \phi \leqq \frac{1}{4}\pi,\ \ 0 \leqq \theta \leqq 2\pi.$$

Then, as in the solution of Problem 25, we find that $dS = a^2 \sin\phi\ dA$. We may assume that S has constant density $\delta = 1$. Hence its mass is

$$m = \iint_S 1\ dS = \int_0^{2\pi} \int_0^{\pi/4} a^2 \sin\phi\ d\phi\ d\theta = 2\pi\Big[-\cos\phi\Big]_0^{\pi/4} = \left(2 - \sqrt{2}\,\right)\pi a^2$$

and its moment with respect to the xy-plane is

$$M_{xy} = \iint_S z\ dS = \int_0^{2\pi} \int_0^{\pi/4} a^3 \sin\phi\,\cos\phi\ d\phi\ d\theta = 2\pi\left[\frac{1}{2}a^3\sin^2\phi\right]_0^{\pi/4} = \frac{1}{2}\pi a^3.$$

Thus—by symmetry—$\overline{x} = \overline{y} = 0$ and

$$\overline{z} = \frac{M_{xy}}{m} = \frac{a}{2\left(2 - \sqrt{2}\,\right)} \approx (0.8535533905932738)a.$$

C15S05.029: The surface S is described by $z = h(x,\,y) = \sqrt{4 - x^2 - y^2}\,$, and

$$1 + (h_x)^2 + (h_y)^2 = 1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} = \frac{4}{4 - x^2 - y^2}.$$

Thus

$$dS = \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA.$$

In cylindrical coordinates, S is described by

$$z = \sqrt{4 - r^2}, \quad 0 \leqq r \leqq 2\cos\theta, \quad -\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi.$$

We may without loss of generality assume that S has constant density $\delta = 1$. Hence its mass is

$$\begin{aligned} m &= \iint_S 1\, dS = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta = 2\int_0^{\pi/2} \Big[-2(4 - r^2)^{1/2} \Big]_0^{2\cos\theta} d\theta \\ &= 4\int_0^{\pi/2} \Big(2 - \sqrt{4 - 4\cos^2\theta}\, \Big)\, d\theta = 8\int_0^{\pi/2} (1 - \sin\theta)\, d\theta = 8\Big[\theta + \cos\theta \Big]_0^{\pi/2} = 4\pi - 8. \end{aligned}$$

Clearly $\overline{y} = 0$ by symmetry. The moment of S with respect to the xy-plane is

$$\begin{aligned} M_{xy} &= \iint_S z\, dS = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2r\sqrt{4 - r^2}}{\sqrt{4 - r^2}}\, dr\, d\theta = 2\int_0^{\pi/2} \Big[r^2 \Big]_0^{2\cos\theta} d\theta \\ &= 2\int_0^{\pi/2} 4\cos^2\theta\, d\theta = 4\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta = 4\Big[\theta + \frac{1}{2}\sin 2\theta \Big]_0^{\pi/2} = 2\pi. \end{aligned}$$

The moment of S with respect to the yz-plane is

$$M_{yz} = \iint_S x\, dS = 2\int_0^{\pi/2} \int_0^{2\cos\theta} \frac{2r^2\cos\theta}{\sqrt{4 - r^2}}\, dr\, d\theta.$$

Let $r = 2\sin\psi$. Then $dr = 2\cos\psi\, d\psi$ and $\sqrt{4 - r^2} = 2\cos\psi$. Thus

$$\begin{aligned} \int \frac{r^2}{\sqrt{4 - r^2}}\, dr &= \int \frac{4\sin^2\psi}{2\cos\psi} \cdot 2\cos\psi\, d\psi = 4\int \frac{1 - \cos 2\psi}{2}\, d\psi \\ &= 2\psi - \sin 2\psi + C = 2\psi - 2\sin\psi\, \cos\psi + C = 2\arcsin\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{4 - r^2} + C. \end{aligned}$$

Therefore

$$\begin{aligned} M_{yz} &= 2\int_0^{\pi/2} \left[\left(4\arcsin\frac{r}{2} - r\sqrt{4 - r^2}\, \right)\cos\theta \right]_0^{2\cos\theta} d\theta \\ &= 2\int_0^{\pi/2} \Big[4(\cos\theta)\arcsin(\cos\theta) - 4\cos^2\theta\, \sin\theta \Big]\, d\theta. \end{aligned}$$

To evaluate

$$J = \int (\cos\theta)\arcsin(\cos\theta)\, d\theta,$$

we use integration by parts. Let

$$u = \arcsin(\cos\theta), \qquad dv = \cos\theta\ d\theta; \quad \text{then}$$

$$du = -\frac{\sin\theta}{\sqrt{1-\cos^2\theta}}\ d\theta, \qquad v = \sin\theta.$$

Thus

$$J = (\sin\theta)\arcsin(\cos\theta) + \int \frac{\sin^2\theta}{\sin\theta}\ d\theta = (\sin\theta)\arcsin(\cos\theta)\ -\ \cos\theta\ +\ C.$$

Consequently,

$$M_{yz} = 2\left[4(\sin\theta)\arcsin(\cos\theta) - 4\cos\theta + \frac{4}{3}\cos^3\theta\right]_0^{\pi/2} = 2\left(4 - \frac{4}{3}\right) = \frac{16}{3}.$$

Therefore

$$\overline{x} = \frac{M_{yz}}{m} = \frac{4}{3\pi - 6} \approx 1.1679589292560724408026 06 \quad \text{and}$$

$$\overline{z} = \frac{M_{xy}}{m} = \frac{\pi}{2\pi - 4} \approx 1.3759691969420543306019 55.$$

C15S05.030: As a consequence of Example 5 in Section 14.8, if the toroidal surface has uniform density δ, then its mass is

$$M = \int_0^{2\pi}\int_0^{2\pi} a\delta(b + a\cos\psi)\ d\theta\ d\psi = 2\pi\delta a\Big[b\psi + a\sin\psi\Big]_0^{2\pi} = 4\pi^2\delta ab.$$

Figure 14.8.13 shows that the distance of the mass element dM of the toroidal surface from the z-axis is $r = (b + a\cos\psi)$. We will use *Mathematica* 3.0 to find the moment of inertia of the surface with respect to the z-axis. The computations can be carried out with a single command, but we split the process into several steps so that you may check your work if you solved this problem by another method. We need to evaluate

$$I_z = \int_0^{2\pi}\int_0^{2\pi} a\delta(b + a\cos\psi)^3\ d\theta\ d\psi.$$

```
Integrate[ a*delta*(b + a*Cos[psi])∧3, theta ]
```

$$a\delta\theta(b + a\cos\psi)^3$$

```
(% /. theta → 2*Pi) - (% /. theta → 0)
```

$$2a\delta\pi(b + a\cos\psi)^3$$

```
Integrate[ %, psi ]
```

$$\frac{1}{6}a\delta\pi(18a^2b\psi + 12b^3\psi + 9a^3\sin\psi + 36ab^2\sin\psi + 9a^2b\sin 2\psi + a^3\sin 3\psi)$$

```
isubz = (% /. psi → 2*Pi) - (% /. psi → 0)
```

$$\frac{1}{6}a\delta\pi(36a^2b\pi + 24b^3\pi)$$

```
mass = 4*Pi*Pi*delta*a*b;
```

```
isubz/mass
```

$$\frac{36a^2b\pi + 24b^3\pi}{24b\pi}$$

```
Together[ Simplify[ % ] ]
```

$$\frac{1}{2}(3a^2 + 2b^2)$$

Therefore $I_z = \dfrac{1}{2}M(3a^2 + 2b^2)$.

C15S05.031: The surface S is described by $h(x, y) = 4 - y^2$, and hence $dS = \sqrt{1 + 4y^2}\ dA$. Thus the moment of inertia of S with respect to the z-axis is

$$\begin{aligned} I_z &= \int_{-2}^{2}\int_{-1}^{1}(x^2 + y^2)\sqrt{1 + 4y^2}\ dx\ dy \int_{-2}^{2}\left[\left(\frac{1}{3}x^3 + xy^2\right)\sqrt{1 + 4y^2}\right]_{-1}^{1} dy \\ &= \int_{-2}^{2}\left(\frac{2}{3} + 2y^2\right)\sqrt{1 + 4y^2}\ dy = \left[\frac{24y^3 + 19y}{48}\sqrt{1 + 4y^2} + \frac{13}{96}\operatorname{arcsinh}(2y)\right]_{-2}^{2} \\ &= \frac{460\sqrt{17} + 13\operatorname{arcsinh}(4)}{48} \approx 40.0804135603857952029798l7. \end{aligned}$$

C15S05.032: The surface S is described by $h(x, y) = 4 - x^2 - y^2$, and thus

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\ dA = \sqrt{1 + 4x^2 + 4y^2}\ dA.$$

Therefore the moment of inertia of S with respect to the z-axis is

$$\begin{aligned} I_z &= \int_{-1}^{1}\int_{-1}^{1}(x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\ dx\ dy \\ &= \int_{-1}^{1}\left[\left\{\frac{1}{4}x^3 + \frac{1}{32}x(20y^2 + 1)\right\}\sqrt{1 + 4x^2 + 4y^2}\right. \\ &\qquad\qquad \left. + \frac{1}{64}(48y^4 + 8y^2 - 1)\ln\left(2x + \sqrt{1 + 4x^2 + 4y^2}\right)\right]_{-1}^{1} dy \\ &= \int_{-1}^{1}\left[\frac{20y^2 + 9}{16}\sqrt{4y^2 + 5} + \frac{48y^4 + 8y^2 - 1}{64}\left\{\ln\left(2 + \sqrt{4y^2 + 5}\right) - \ln\left(-2 + \sqrt{4y^2 + 5}\right)\right\}\right] dy \\ &= \left[\frac{42y^3 + 49y}{120}\sqrt{4y^2 + 5} + \frac{169}{480}\operatorname{arcsinh}\left(\frac{2y}{\sqrt{5}}\right) + \frac{1}{60}\arctan\left(\frac{4y}{\sqrt{4y^2 + 5}}\right)\right. \\ &\qquad\qquad \left. + \frac{144y^5 + 40y^3 - 15y}{960}\left\{\ln\left(2 + \sqrt{4y^2 + 5}\right) - \ln\left(-2 + \sqrt{4y^2 + 5}\right)\right\}\right]_{-1}^{1} \end{aligned}$$

$$= \frac{91}{20} + \frac{169}{240}\,\text{arcsinh}\left(\frac{2}{\sqrt{5}}\right) + \frac{1}{30}\arctan\left(\frac{4}{3}\right) + \frac{169}{480}\ln 5 \approx 5.7142223706057327548623318.$$

All of the antiderivatives and evaluations were computed using *Mathematica* 3.0.

C15S05.033: By Eq. (12) in Section 15.5,

$$\cos\gamma = \frac{1}{|\mathbf{N}|}\cdot\frac{\partial(x,\,y)}{\partial(x,\,y)} = \frac{1}{|\mathbf{N}|}.$$

Therefore $|\mathbf{N}|\,dS = \sec\gamma\,dx\,dy$.

C15S05.034: We compute the three Jacobians in Eq. (17) using the parameters y and z. The result is

$$\frac{\partial(y,\,z)}{\partial(y,\,z)} = \begin{vmatrix} y_y & y_z \\ z_y & z_z \end{vmatrix} = 1,$$

$$\frac{\partial(z,\,x)}{\partial(y,\,z)} = \begin{vmatrix} z_y & z_z \\ x_y & x_z \end{vmatrix} = -\frac{\partial x}{\partial y}, \quad \text{and}$$

$$\frac{\partial(x,\,y)}{\partial(y,\,z)} = \begin{vmatrix} x_y & x_z \\ y_y & y_z \end{vmatrix} = -\frac{\partial x}{\partial z}.$$

Therefore

$$\iint_S P\,dy\;dz + Q\,dz\,dx + R\,dx\,dy = \iint_D \left(P - Q\frac{\partial x}{\partial y} - R\frac{\partial x}{\partial z}\right)dy\,dz.$$

C15S05.035: The temperature within the ball is $u(x,\,y,\,z) = 4(x^2+y^2+z^2)$. With position vector $\mathbf{r} = \langle\, x,\,y,\,z\,\rangle$ for points of B, we find that

$$\mathbf{q} = -K\,\boldsymbol{\nabla} u = -2\cdot 4\langle\, 2x,\,2y,\,2z\,\rangle = -16\langle\, x,\,y,\,z\,\rangle = -16\mathbf{r}.$$

A unit vector normal to the concentric spherical surface S of radius 3 is $\mathbf{n} = \frac{1}{3}\mathbf{r}$, so

$$\mathbf{q}\cdot\mathbf{n} = -\frac{16}{3}(x^2+y^2+z^2) = -\frac{16}{3}\cdot 9 = -48.$$

Because S is a spherical surface of radius 3, its surface area is $4\pi\cdot 9 = 36\pi$. Therefore the rate of heat flow across S is

$$\iint_S \mathbf{q}\cdot\mathbf{n}\,dS = -\iint_S 48\,dS = -48\cdot 36\pi = -1728\pi.$$

C15S05.036: The temperature within the cylinder is $u(x,\,y) = 4(x^2+y^2)$, so

$$\mathbf{q} = -K\,\boldsymbol{\nabla} u = -2\cdot 4\langle\, 2x,\,2y,\,0\,\rangle = -16\langle\, x,\,y,\,0\,\rangle.$$

A unit vector normal to the inner cylindrical surface is $\mathbf{n} = \frac{1}{3}\langle\, x,\,y,\,0\,\rangle$, and hence $\mathbf{q}\cdot\mathbf{n} = -48$. Therefore the rate of flow of heat across the inner surface is

$$\iint_S \mathbf{q} \cdot \mathbf{n}\, dS = -48 \cdot 2\pi \cdot 3 \cdot 10 = -2880\pi.$$

C15S05.037: The given parametrization yields $\mathbf{N} = \langle -2bu^2 \cos v,\ -2au^2 \sin v,\ abu \rangle$, so the area of the paraboloid is

$$A = \int_0^{2\pi} \int_0^c u\sqrt{(2au \sin v)^2 + (2bu \cos v)^2 + (ab)^2}\ du\ dv$$

$$= \int_0^{2\pi} \frac{\left[(ab)^2 + 2(ac)^2 + 2(bc)^2 - 2(a^2 - b^2)c^2 \cos 2v\right]^{3/2} - (ab)^3}{6\left[a^2 + b^2 - (a^2 - b^2) \cos 2v\right]}\ dv.$$

We believe the last integral to be nonelementary (because *Mathematica* 3.0 uses elliptic functions to compute the antiderivative). With $a = 4,\ b = 3,$ and $c = 2$ it reduces to

$$\int_0^{2\pi} \frac{-1728 + (344 - 56 \cos 2v)^{3/2}}{150 - 42 \cos 2v}\ dv.$$

The *Mathematica* 3.0 `NIntegrate` command yields the result $A \approx 194.702812872043$. To compute the moment of inertia of the paraboloid with respect to the z-axis, we insert the factor

$$x^2 + y^2 = (au \cos v)^2 + (bu \sin v)^2 = (4u \cos v)^2 + (3u \sin v)^2$$

into the first integral, and *Mathematica* yields the result $I_z \approx 5157.168115181396$.

C15S05.038: Using the given parametrization, we find that

$$\mathbf{N} = \langle\, bc \sin^2 u\ \cos v,\ ac \sin^2 u\ \sin v,\ ab \sin u \cos u \,\rangle,$$

and thus that

$$|\mathbf{N}| = (\sin u)\sqrt{(bc \sin u \cos v)^2 + (ac \sin u \sin v)^2 + (ab \cos u)^2}\,.$$

Hence (using $a = 4,\ b = 3,\ c = 2,$ and density $\delta = 1$) the area of the ellipsoid is

$$A = \int_0^{2\pi} \int_0^{\pi} |\mathbf{N}|\ du\ dv \approx 111.545774984838$$

and its moment of inertia with respect to the z-axis is

$$I_z = \int_0^{2\pi} \int_0^{\pi} \left[(a \sin u \cos v)^2 + (b \sin u \sin v)^2\right] \cdot |\mathbf{N}|\ du\ dv \approx 847.811218594696.$$

C15S05.039: The given parametrization yields

$$|\mathbf{N}| = (\cosh u)\sqrt{(b \cosh u \cos v)^2 + (a \cosh u \sin v)^2 + (ab \sinh u)^2}\ ,$$

and hence (using $a = 4,\ b = 3,\ c = 2,$ and density $\delta = 1$) we find that the hyperboloid has surface area

$$A = \int_0^{2\pi} \int_{-c}^{c} |\mathbf{N}|\ du\ dv \approx 1057.350512779488$$

and moment of inertia with respect to the z-axis

$$I_z = \int_0^{2\pi} \int_{-c}^{c} (\cosh^2 u)\big[(a\cos v)^2 + (b\sin v)^2\big] \cdot |\mathbf{N}|\, du\, dv \approx 98546.9348740325.$$

C15S05.040: The given parametrization of the Möbius strip yields

$$|\mathbf{N}| = \sqrt{16 + \frac{3}{4}t^2 + 8t\cos\left(\frac{1}{2}\theta\right) + \frac{1}{2}t^2\cos\theta}\,,$$

and thus the Möbius strip has area

$$A = \int_0^{2\pi} \int_{-1}^{1} |\mathbf{N}|\, dt\, d\theta \approx 50.398571814841$$

and its moment of inertia with respect to the z-axis is

$$I_z = \int_0^{2\pi} \int_{-1}^{1} (x^2 + y^2) \cdot |\mathbf{N}|\, dt\, d\theta \approx 831.469864671567.$$

C15S05.041: We use Fig. 14.7.15 of the text and the notation there; the only change is replacement of the variable ρ with the constant radius a of the spherical surface. The spherical shell has constant density δ and total mass $M = 4\pi a^2\delta$. The "sum" of the vertical components of the gravitational forces exerted by mass elements $\delta\, dS$ of the spherical surface S on the mass m is

$$F = \iint_S \frac{Gm\delta\cos\alpha}{w^2}\, dS.$$

We saw in the solution of Problem 25 (among others) that $dS = a^2\sin\phi\, dA$. Figure 14.7.15 also shows us that

$$w\cos\alpha = c - a\cos\phi \quad \text{and} \quad w^2 = a^2 + c^2 - 2ac\cos\phi \tag{1}$$

(by the law of cosines (Appendix L, page A-49)). Note that

$$F = 2\pi Gm\delta \int_{\phi=0}^{\pi} \frac{a^2\cos\alpha\,\sin\phi}{w^2}\, d\phi.$$

Substitute $\cos\alpha = \dfrac{c - a\cos\phi}{w}$ to obtain

$$F = 2\pi Gm\delta \int_0^{\pi} \frac{a^2(c - a\cos\phi)\sin\phi}{w^3}\, d\phi.$$

Next note that $\phi = 0$ corresponds to $w = c - a$ and that $\phi = \pi$ corresponds to $w = c + a$. Moreover, by the second equation in (1),

$$\cos\phi = \frac{a^2 + c^2 - w^2}{2ac} \qquad \text{and thus}$$

$$-\sin\phi\, d\phi = -\frac{w}{ac}\, dw.$$

These substitutions yield

$$
\begin{aligned}
F &= 2\pi G m \delta \int_{w=c-a}^{c+a} \frac{a^2}{w^3} \left(c - \frac{a^2+c^2-w^2}{2c} \right) \cdot \frac{w}{ac}\, dw \\
&= 2\pi G m \delta \int_{c-a}^{c+a} \frac{a}{w^2 c} \cdot \frac{1}{2c} \cdot (c^2 + w^2 - a^2)\, dw = \frac{2\pi G m \delta a}{2c^2} \int_{c-a}^{c+a} \left(\frac{c^2 - a^2}{w^2} + 1 \right) dw \\
&= \frac{\pi G m \delta a}{c^2} \left[\frac{a^2 - c^2}{w} + w \right]_{c-a}^{c+a} = \frac{\pi G m \delta a}{c^2} (a - c + a + c + c + a - c + a) = \frac{4\pi G m \delta a^2}{c^2} = \frac{GMm}{c^2}.
\end{aligned}
$$

Section 15.6

C15S06.001: The right-hand side in the divergence theorem (Eq. (1)) is

$$\iiint_B \nabla \cdot \mathbf{F}\, dV = \iiint_B 3\, dV = 3 \cdot \frac{4}{3}\pi \cdot 1^3 = 4\pi$$

and the left-hand side in the divergence theorem is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \langle\, x,\, y,\, z\,\rangle \cdot \langle\, x,\, y,\, z\,\rangle\, dS = \iint_S (x^2 + y^2 + z^2)\, dS = \iint_S 1\, dS = 1 \cdot 4\pi \cdot 1^2 = 4\pi.$$

Note that we integrate a constant function by multiplying its value by the size (length, area, or volume) of the domain of the integral. We will continue to do so without further comment.

C15S06.002: Here we have

$$\mathbf{F}(x,\, y,\, z) = (x^2 + y^2 + z^2)^{1/2} \langle\, x,\, y,\, z\,\rangle, \qquad \text{and} \qquad \mathbf{n} = \frac{1}{3}\langle\, x,\, y,\, z\,\rangle$$

is a unit vector normal to the surface S. Because $\mathbf{F} \cdot \mathbf{n} = \frac{1}{3}(x^2 + y^2 + z^2)^{3/2}$, $\mathbf{F} \cdot \mathbf{n}$ takes on the constant value 9 on S. Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 9 \cdot \text{area}(S) = 9 \cdot 4\pi \cdot 3^2 = 324\pi \approx 1017.8760197630930093.$$

Next, let B denote the solid ball bounded by S. Then

$$\nabla \cdot \mathbf{F} = \frac{x^2}{\sqrt{x^2 + y^2 + z^2}} + \frac{y^2}{\sqrt{x^2 + y^2 + z^2}} + \frac{z^2}{\sqrt{x^2 + y^2 + z^2}} + 3\sqrt{x^2 + y^2 + z^2} = 4\sqrt{x^2 + y^2 + z^2},$$

and thus

$$\begin{aligned}\iiint_B \nabla \cdot \mathbf{F}\, dV &= \iiint_B 4\sqrt{x^2 + y^2 + z^2}\ dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^3 4\rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi} 81 \sin\phi\, d\phi = 2\pi \Big[-81\cos\phi\Big]_0^{\pi} = 324\pi.\end{aligned}$$

C15S06.003: On the face F of the cube in the plane $x = 2$, a unit vector normal to F is $\mathbf{n} = \mathbf{i}$, and $\mathbf{F} \cdot \mathbf{i} = x = 2$. Hence

$$\iint_F \mathbf{F} \cdot \mathbf{n}\, dS = 2 \cdot \text{area}(F) = 8.$$

By symmetry, the same result obtains on the faces in the planes $y = 2$ and $z = 2$. On the face G of the cube in the plane $x = 0$, a unit vector normal to G is $\mathbf{n} = -\mathbf{i}$, and $\mathbf{F} \cdot (-\mathbf{i}) = -x = 0$. Hence

$$\iint_G \mathbf{F} \cdot \mathbf{n}\, dS = 0.$$

By symmetry, the same result holds on the faces in the other two coordinate planes. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 3\cdot 8 + 3\cdot 0 = 24.$$

Let B denote the solid cube bounded by S and let V denote the volume of B. Because $\nabla\cdot\mathbf{F} = 3$, we also have

$$\iiint_B \nabla\cdot\mathbf{F}\,dV = 3\cdot V = 3\cdot 8 = 24.$$

C15S06.004: On the face F of the cube in the plane $x = 2$, a unit vector normal to F is $\mathbf{n} = \mathbf{i}$, and $\mathbf{F}\cdot\mathbf{i} = xy = 2y$. Hence

$$\iint_F \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^2\int_0^2 2y\,dy\,dz = \int_0^2 4\,dz = 8.$$

By symmetry, the same result holds on the faces in the planes $y = 2$ and $z = 2$. On the face G of the cube in the plane $x = 0$, a unit vector normal to G is $\mathbf{n} = -\mathbf{i}$, and $\mathbf{F}\cdot(-\mathbf{i}) = -xy = 0$. Hence

$$\iint_G \mathbf{F}\cdot\mathbf{n}\,dS = 0.$$

By symmetry, the same result holds on the faces in the other two coordinate planes. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 3\cdot 8 + 3\cdot 0 = 24.$$

Let B denote the solid cube bounded by S. Because $\nabla\cdot\mathbf{F} = y + z + x$, we see that

$$\begin{aligned}\iiint_B \nabla\cdot\mathbf{F}\,dV &= \int_0^2\int_0^2\int_0^2 (x+y+z)\,dx\,dy\,dz\\ &= \int_0^2\int_0^2 (2+2y+2z)\,dy\,dz = \int_0^2 (8+4z)\,dz = \Big[8z+2z^2\Big]_0^2 = 24.\end{aligned}$$

C15S06.005: On the face F of the tetrahedron that lies in the plane $x = 0$, a unit vector normal to F is $\mathbf{n} = -\mathbf{i}$, and $\mathbf{F}\cdot\mathbf{n} = -x - y = -y$. Hence

$$\iint_F \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^1\int_0^{1-z} (-y)\,dy\,dz = \int_0^1 -\frac{1}{2}(1-z)^2\,dz = \left[\frac{1}{6}(1-z)^3\right]_0^1 = -\frac{1}{6}.$$

By symmetry the same result holds on the faces in the other two coordinate planes. On the fourth face G of the tetrahedron, a unit vector normal to G is

$$\mathbf{n} = \frac{\sqrt{3}}{3}\langle 1, 1, 1\rangle,$$

and G is part of the graph of $z = h(x, y) = 1 - x - y$, so that

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\;dA = \sqrt{3}\;dA.$$

Therefore

$$\iint_G \mathbf{F}\cdot\mathbf{n}\,dS = \iint_G (2x+2y+2z)\,dA = \iint_G 2\,dA = \int_0^1 \int_0^{1-x} 2\,dy\,dx = \int_0^1 \Big[2y\Big]_0^{1-x} dx = \Big[2x - x^2\Big]_0^1 = 1,$$

and therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 1 - 3\cdot\frac{1}{6} = \frac{1}{2}.$$

Let B denote the solid tetrahedron itself, with volume V. Then

$$\iiint_B \nabla\cdot\mathbf{F}\,dV = \iiint_B 3\,dV = 3\cdot V = 3\cdot\frac{1}{6}\cdot 1\cdot 1 = \frac{1}{2}.$$

C15S06.006: Let B denote the cube bounded by the surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \nabla\cdot\mathbf{F}\,dV = \iiint_B (2x+2y+2z)\,dV = \int_0^2\int_0^2\int_0^2 (2x+2y+2z)\,dz\,dy\,dx$$
$$= \int_0^2\int_0^2 (4+4x+4y)\,dy\,dx = \int_0^2 (16+8x)\,dx = \Big[16x+4x^2\Big]_0^2 = 48.$$

C15S06.007: Let B denote the solid cylinder bounded by the surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \nabla\cdot\mathbf{F}\,dV = \iiint_B 3(x^2+y^2+z^2)\,dV = \int_0^{2\pi}\int_0^3\int_{-1}^4 3(r^2+z^2)\cdot r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^3 \Big[3r^3z + rz^3\Big]_{-1}^4 dr = 2\pi\int_0^3 (65r+15r^3)\,dr = 2\pi\left[\frac{65}{2}r^2 + \frac{15}{4}r^4\right]_0^3$$
$$= 2\pi\cdot\frac{2385}{4} = \frac{2385}{2}\pi \approx 3746.3492394058284367.$$

C15S06.008: Denote by B the solid paraboloid bounded by the given surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \nabla\cdot\mathbf{F}\,dV = \iiint_B 4(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^5\int_0^{25-r^2} 4r^3\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^5 (100r^3 - 4r^5)\,dr = 2\pi\left[25r^4 - \frac{2}{3}r^6\right]_0^5$$
$$= 2\pi\cdot\frac{15625}{3} = \frac{31250}{3}\pi \approx 32724.9234748936795673.$$

C15S06.009: Let B denote the tetrahedron bounded by the given surface S. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \nabla\cdot\mathbf{F}\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} (2x+1)\,dz\,dy\,dx$$
$$= \int_0^1\int_0^{1-x} (1+x-2x^2-y-2xy)\,dy\,dx = \int_0^1 \left[(1+x-2x^2)y - \frac{1}{2}(2x+1)y^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{3}{2}x^2 + x^3 \right) dx = \left[\frac{1}{2}x - \frac{1}{2}x^3 + \frac{1}{4}x^4 \right]_0^1 = \frac{1}{4}.$$

C15S06.010: Let B denote the solid region bounded by the given surface S. Then

$$\iint_B \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iiint_B (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^3 \int_{r^2}^9 r^3\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^3 (9r^3 - r^5)\, dr = 2\pi \left[\frac{9}{4}r^4 - \frac{1}{6}r^6 \right]_0^3 = 2\pi \cdot \frac{243}{4} = \frac{243}{2}\pi \approx 381.7035074111598785.$$

C15S06.011: Let B denote the solid paraboloid bounded by the surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iiint_B 5(x^2 + y^2 + z^2)\, dV = \int_0^{2\pi} \int_0^5 \int_0^{25-r^2} 5(r^2 + z^2) \cdot r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^5 \left[5r^3 z + \frac{5}{3}rz^3 \right]_0^{25-r^2} dr = 2\pi \int_0^5 \left(\frac{78125}{3}r - 3000r^3 + 120r^5 - \frac{5}{3}r^7 \right) dr$$

$$= 2\pi \left[\frac{78125}{6}r^2 - 750r^4 + 20r^6 - \frac{5}{24}r^8 \right]_0^5 = 2\pi \cdot \frac{703125}{8} = \frac{703125}{4}\pi \approx 552233.08363883.$$

C15S06.012: First note that

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \boldsymbol{\nabla} \cdot \left\langle \frac{x}{\sqrt{x^2 + y^2 + z^2}},\ \frac{y}{\sqrt{x^2 + y^2 + z^2}},\ \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

$$= -\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}} - \frac{y^2}{(x^2 + y^2 + z^2)^{3/2}} - \frac{z^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}.$$

Let B denote the solid ball bounded by the surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iiint_B \frac{2}{\sqrt{x^2 + y^2 + z^2}}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^2 \frac{2}{\rho} \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi} \left[\rho^2 \sin\phi \right]_0^2 d\phi = 2\pi \int_0^{\pi} 4 \sin\phi\, d\phi = 2\pi \left[-4\cos\phi \right]_0^{\pi} = 16\pi \approx 50.2654824574366918.$$

C15S06.013: Here is a step-by-step illustration of the solution using *Mathematica* 3.0.

```
f = {x, y, 3}      (* First we define the vector function F. *)
```

$\{x,\ y,\ 3\}$

```
D[%[[1]],x] + D[%[[2]],y] + D[%[[3]],z]      (* Then we compute div F. *)
```

2

```
Integrate[ 2*r, z ]     (* Begin the triple integral in cylindrical coordinates.  *)
```

$2rz$

```
(% /.  z → 4) - (% /.  z → r∧2)     (* Substitute the limits on z.  *)
```

$8r - 2r^3$

```
Integrate[ %, r]
```

$4r^2 - \frac{1}{2}r^4$

```
(% /.  r → 2) - (% /.  r → 0)     (* Substitute the limits on r.  *)
```

8

```
2*Pi*%     (* Integrate the constant by multiplying by 2π.  *)
```

16π

```
N[%, 18]     (* Approximate the answer.  *)
```

50.2654824574366918

C15S06.014: Let B denote the solid bounded by the surface S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \int_{-2}^{2} \int_0^{4-x^2} \int_0^{5-z} 4x^2\, dy\, dz\, dx = \int_{-2}^{2} \int_0^{4-x^2} \Big[4x^2 y \Big]_0^{5-z} dz\, dx$$

$$= \int_{-2}^{2} \int_0^{4-x^2} 4x^2(5-z)\, dz\, dx = \int_{-2}^{2} \Big[20x^2 z - 2x^2 z^2 \Big]_0^{4-x^2} dx$$

$$= \int_{-2}^{2} (48x^2 - 4x^4 - 2x^6)\, dx = \left[16x^3 - \frac{4}{5}x^5 - \frac{2}{7}x^7 \right]_{-2}^{2} = \frac{4608}{35} \approx 131.657142857143.$$

C15S06.015: Compare this problem and its solution with Problem 25 of Section 15.4. If f is a twice-differentiable scalar function, then

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \boldsymbol{\nabla} \cdot \langle f_x,\, f_y,\, f_z \rangle = f_{xx} + f_{yy} + f_{zz}.$$

C15S06.016: By the divergence theorem and the result in Problem 15,

$$\iint_S \frac{\partial f}{\partial n}\, dS = \iint_S (\boldsymbol{\nabla} f) \cdot \mathbf{n}\, dS = \iiint_T \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)\, dV = \iiint_T \boldsymbol{\nabla}^2 f\, dV.$$

C15S06.017: If $\boldsymbol{\nabla}^2 f \equiv 0$ in the region T with boundary surface S, then by the divergence theorem and Problem 28 in Section 15.1,

$$\iint_S f\frac{\partial f}{\partial n}\,dS = \iint_S (f)(\boldsymbol{\nabla} f)\boldsymbol{\cdot}\mathbf{n}\,dS = \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}\big[(f)(\boldsymbol{\nabla} f)\big]\,dV$$

$$= \iiint_T \big[(f)\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\nabla} f) + (\boldsymbol{\nabla} f)\boldsymbol{\cdot}(\boldsymbol{\nabla} f)\big]\,dV = \iiint_T \big[(f)\boldsymbol{\nabla}^2 f + |\boldsymbol{\nabla} f|^2\big]\,dV = \iiint_T |\boldsymbol{\nabla} f|^2\,dV.$$

C15S06.018: By the divergence theorem and Problem 28 of Section 15.1,

$$\iint_S f\frac{\partial g}{\partial n}\,dS = \iint_S (f)(\boldsymbol{\nabla} g)\boldsymbol{\cdot}\mathbf{n}\,dS = \iint_S \mathbf{F}\boldsymbol{\cdot}\mathbf{n}\,dV = \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}\,dV = \iiint_T \boldsymbol{\nabla}\boldsymbol{\cdot}(f\boldsymbol{\nabla} g)\,dV$$

$$= \iiint_T \big[(f)(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla} g) + (\boldsymbol{\nabla} f)\boldsymbol{\cdot}(\boldsymbol{\nabla} g)\big]\,dV = \iiint_T \left(f\boldsymbol{\nabla}^2 g + \boldsymbol{\nabla} f\boldsymbol{\cdot}\boldsymbol{\nabla} g\right)\,dV.$$

C15S06.019: Green's first identity states that if the space region T has surface S with a piecewise smooth parametrization, if f and g are twice-differentiable scalar functions, and if $\partial f/\partial n = (\boldsymbol{\nabla} f)\boldsymbol{\cdot}\mathbf{n}$ where $\mathbf{n}$ is the unit vector normal to S with outer direction, then

$$\iint_S f\frac{\partial g}{\partial n}\,dS = \iiint_T (f\boldsymbol{\nabla}^2 g + \boldsymbol{\nabla} f\boldsymbol{\cdot}\boldsymbol{\nabla} g)\,dV.$$

Interchanging the roles of f and g yields the immediate consequence

$$\iint_S g\frac{\partial f}{\partial n}\,dS = \iiint_T (g\boldsymbol{\nabla}^2 f + \boldsymbol{\nabla} g\boldsymbol{\cdot}\boldsymbol{\nabla} f)\,dV.$$

Then substraction of the second of these equations from the first yields

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS = \iiint_B \left(f\boldsymbol{\nabla}^2 g + \boldsymbol{\nabla} f\boldsymbol{\cdot}\boldsymbol{\nabla} g - g\boldsymbol{\nabla}^2 f - \boldsymbol{\nabla} g\boldsymbol{\cdot}\boldsymbol{\nabla} f\right)\,dV = \iiint_B \left(f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla}^2 f\right)\,dV,$$

and this is Green's second identity.

C15S06.020: Let $\mathbf{a}$ be an arbitrary constant vector and let $\mathbf{F} = f\mathbf{a}$. Then the divergence theorem yields

$$\iint_S \mathbf{F}\boldsymbol{\cdot}\mathbf{n}\,dS = \iiint_B \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{F}\,dV; \qquad \text{that is,}$$

$$\iint_S f\mathbf{a}\boldsymbol{\cdot}\mathbf{n}\,dS = \iiint_B \boldsymbol{\nabla}\boldsymbol{\cdot}(f\mathbf{a})\,dV = \iiint_B \big[(f)(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{a}) + (\boldsymbol{\nabla} f)\boldsymbol{\cdot}\mathbf{a}\big]\,dV$$

$$= \iiint_B \big[(f)(0) + \mathbf{a}\boldsymbol{\cdot}\boldsymbol{\nabla} f\big]\,dV = \mathbf{a}\iiint_B \boldsymbol{\nabla} f\,dV, \qquad \text{and so}$$

$$\mathbf{a}\iint_S f\mathbf{n}\,dS = \mathbf{a}\iiint_B \boldsymbol{\nabla} f\,dV.$$

The last equation holds for every constant vector $\mathbf{a}$, including $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Therefore the x-, y-, and z-components of the last two integrals are the same, and consequently

$$\iint_S f\,\mathbf{n}\,dS = \iiint_B \nabla f\;dV.$$

C15S06.021: By the result in Problem 20, we have

$$\mathbf{B} = -\iint_S p\,\mathbf{n}\,dS = -\iiint_T \nabla(\delta g z)\;dV = -\iiint_T \langle\, 0,\ 0,\ \delta g\,\rangle\;dV = -\mathbf{k}\iiint_T \delta g\;dV = -W\,\mathbf{k}$$

because

$$\iiint_T \delta g\;dV = mg = W$$

is the weight of the fluid displaced by the body.

C15S06.022: We use *Mathematica* 3.0 to solve this problem. We begin by defining $\mathbf{F}$ and $\mathbf{r}$ as in the statement of the problem:

```
r = {x, y, z}; r0 = {a, b, c}; r − r0
```

$$\{x-a,\ y-b,\ z-c\}$$

```
f = %/((r − r0).(r − r0))∧(3/2)
```

$$\left\{\frac{x-a}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}},\ \frac{y-b}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}},\ \frac{z-c}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}}\right\}$$

Now we compute the divergence of $\mathbf{F}$ (called `f` here to avoid capitals).

```
D[%[[1]], x] + D[%[[2]], y] + D[%[[3]], z]
```

$$-\frac{3(x-a)^2}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{5/2}} - \frac{3(y-b)^2}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{5/2}} - \frac{3(z-c)^2}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{5/2}} + \frac{3}{\left((x-a)^2+(y-b)^2+(z-c)^2\right)^{3/2}}$$

```
Simplify[ % ]
```

0

Therefore $\nabla\cdot\mathbf{F} = 0$ except at the point $(a,\ b,\ c)$.

C15S06.023: Let B denote the region bounded by the paraboloid and the plane. Because

$$\mathbf{F}(x,\ y,\ z) = \left\langle\, x\sqrt{x^2+y^2+z^2}\ ,\ y\sqrt{x^2+y^2+z^2}\ ,\ z\sqrt{x^2+y^2+z^2}\,\right\rangle,$$

we have

$$\nabla \cdot \mathbf{F} = \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2}{\sqrt{x^2+y^2+z^2}} + 3\sqrt{x^2+y^2+z^2} = 4\sqrt{x^2+y^2+z^2}\,.$$

Thus in cylindrical coordinates, $\nabla \cdot \mathbf{F} = 4\sqrt{r^2+z^2}$. Then, by the divergence theorem,

$$\begin{aligned} I &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_B \nabla \cdot \mathbf{F}\, dV \\ &= \int_0^{2\pi} \int_0^{25} \int_0^{\sqrt{25-z}} 4r\sqrt{r^2+z^2}\ dr\ dz\ d\theta = \frac{8}{3}\pi \int_0^{25} \left[(25 - z + z^2)^{3/2} - z^3 \right] dz. \end{aligned}$$

Let

$$J = \int_0^c \left((z-a)^2 + b^2 \right)^{3/2}\ dz.$$

Later we will use the following values: $b = \frac{3}{2}\sqrt{11}$, $a = \frac{1}{2}$, and $c = 25 = a^2 + b^2$. The substitution $z = a + b\tan u$ yields

$$J = b^4 \int_{z=0}^{c} \sec^5 u\ du,$$

and the integral formulas in 37 and 28 of the endpapers of the text then yield

$$\begin{aligned} J = b^4 \int_{z=0}^{c} \sec^5 u\, du &= b^4 \left(\left[\frac{1}{4} \sec^3 u\ \tan u \right]_{z=0}^{c} + \frac{3}{4} \int_{z=0}^{c} \sec^3 u\, du \right) \\ &= b^4 \left(\left[\frac{1}{4} \sec^3 u\ \tan u \right]_{z=0}^{c} + \frac{3}{4} \left[\frac{1}{2} \sec u\ \tan u + \frac{1}{2} \ln |\sec u + \tan u| \right]_{z=0}^{c} \right). \end{aligned}$$

The substitution $z = a + b\tan u$ also yields

$$\sec u \ = \ \frac{1}{b} \left[(z-a)^2 + b^2 \right]^{1/2} \qquad \text{and} \qquad \tan u \ = \ \frac{1}{b}(z-a).$$

It now follows that

$$\begin{aligned} J &= b^4 \left[\frac{3(z-a)\left[(z-a)^2+b^2\right]^{3/2}}{4b^4} + \frac{3(z-a)\left[(z-a)^2+b^2\right]^{1/2}}{8b^2} + \frac{3}{8} \ln \left| \frac{z - a + \left[(z-a)^2+b^2\right]^{1/2}}{b} \right| \right]_{z=0}^{c} \\ &= b^4 \left[\frac{3a(a^2+b^2)^{1/2}}{8b^2} + \frac{a(a^2+b^2)^{3/2}}{4b^4} + \frac{3(c-a)\left[(c-a)^2+b^2\right]^{1/2}}{8b^2} \right. \\ &\qquad \left. + \frac{(c-a)\left[(c-a)^2+b^2\right]^{3/2}}{4b^4} - \frac{3}{8} \ln \left(\frac{-a + (a^2+b^2)^{1/2}}{b} \right) + \frac{3}{8} \ln \left(\frac{c - a + \left[(c-a)^2+b^2\right]^{1/2}}{b} \right) \right]. \end{aligned}$$

Then substitution of the numerical values of a, b, and c yields

$$J = \frac{9801}{16} \left[\frac{1081885}{6534} - \frac{3}{8} \ln \left(\frac{3}{\sqrt{11}} \right) + \frac{3}{8} \ln \left(3\sqrt{11} \right) \right] = \frac{3}{128} \left(4327540 + 9801 \ln 11 \right).$$

And, finally,

$$I = \frac{8}{3}\pi\left(J - \int_0^{25} z^3\,dz\right) = \frac{8}{3}\pi\left[\frac{1}{128}\,(482620 + 29403\ln 11)\right]$$

$$= \frac{482620 + 29403\ln 11}{48}\pi \approx 36201.96719156658969933477411 5.$$

We integrated $\sec^5 u$ using the integral formulas mentioned earlier; all the subsequent work was done by *Mathematica* 3.0.

C15S06.024: We begin with Gauss's law in the form

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = -4\pi GM.$$

Because $\mathbf{F}$ and $\mathbf{n}$ have opposite directions and $\mathbf{n}$ is a unit vector, this law in this special case takes the form

$$\iint_S |\mathbf{F}|\,dS = 4\pi GM;$$

$$|\mathbf{F}| \cdot 4\pi r^2 = 4\pi GM;$$

$$|\mathbf{F}| = \frac{GM}{r^2}.$$

C15S06.025: We begin with Gauss's law in the form

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = -4\pi GM.$$

Because $\mathbf{F}$ and $\mathbf{n}$ have opposite directions and $\mathbf{n}$ is a unit vector, we may in this case deduce that

$$\iint_S |\mathbf{F}|\,dS = 4\pi GM = 4\pi G \cdot 0 = 0.$$

Therefore $4\pi r^2|\mathbf{F}| = 0$, so that $|\mathbf{F}| = 0$. Therefore $\mathbf{F} = \mathbf{0}$.

C15S06.026: We begin with Gauss's law in the form

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = -4\pi GM_r = -4\pi G \cdot \frac{4}{3}\pi\delta r^3.$$

Because $\mathbf{F}$ and $\mathbf{n}$ have opposite directions and $\mathbf{n}$ is a unit vector, we may now conclude that

$$\iint_S |\mathbf{F}|\,dS = 4\pi GM_r;$$

$$4\pi r^2|\mathbf{F}| = 4\pi GM_r;$$

$$|\mathbf{F}| = \frac{GM_r}{r^2}.$$

C15S06.027: Imagine a cylindrical surface of radius r and length L concentric around the wire. Because the electric field $\mathbf{E}$ is normal to the wire, there is no flux of $\mathbf{E}$ across the top and bottom of the cylinder, so the surface S of Gauss's law may be regarded as the curved side of the cylinder. It follows that

$$\iint_S \mathbf{E}\cdot\mathbf{n}\,dS = \frac{Q}{\epsilon_0} = \frac{Lq}{\epsilon_0}.$$

Because $\mathbf{E}$ and $\mathbf{n}$ are parallel and $\mathbf{n}$ is a unit vector, it now follows that

$$\begin{aligned}\iint_S |\mathbf{E}|\,dS &= \frac{Lq}{\epsilon_0};\\ 2\pi r L|\mathbf{E}| &= \frac{Lq}{\epsilon_0};\\ |\mathbf{E}| &= \frac{q}{2\pi\epsilon_0 r}.\end{aligned}$$

Section 15.7

C15S07.001: Because $\mathbf{n}$ is to be the upper unit normal vector, we have

$$\mathbf{n} = \mathbf{n}(x,\, y,\, z) = \frac{1}{2}\langle\, x,\, y,\, z\,\rangle.$$

The boundary curve C of the hemispherical surface S has the parametrization

$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = 0, \quad 0 \leqq \theta \leqq 2\pi.$$

Therefore

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \oint_C 3y\; dx - 2x\; dy + xyz\; dz$$

$$= \int_0^{2\pi} (-12\sin^2\theta - 8\cos^2\theta)\; d\theta = \Big[-10\theta + \sin 2\theta\Big]_0^{2\pi} = -20\pi.$$

C15S07.002: Parametrize the boundary curve C as follows:

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 4, \quad 0 \leqq t \leqq 2\pi.$$

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C 2y\; dx + 3x\; dy + e^z\; dz$$

$$= \int_0^{2\pi} (12\cos^2 t - 8\sin^2 t)\; dt = \Big[2t + 5\sin 2t\Big]_0^{2\pi} = 4\pi.$$

C15S07.003: Parametrize the boundary curve C of the surface S as follows:

$$x = 3\cos t, \quad y = 3\sin t, \quad z = 0, \quad 0 \leqq t \leqq 2\pi.$$

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \oint_C \mathbf{F} \cdot \mathbf{T}\; ds = \int_0^{2\pi} (-6\cos t - 27\sin^2 t\, \cos t)\; dt = \left[\frac{3}{4}(3\sin 3t - 17\sin t)\right]_0^{2\pi} = 0.$$

C15S07.004: Parametrize the boundary curves as follows:

$$C_1: \qquad x = \cos t, \quad y = \sin t, \qquad z = 1, \quad 0 \leqq t \leqq 2\pi;$$

$$C_2: \qquad x = \cos t, \quad y = -\sin t, \quad z = 3, \quad 0 \leqq t \leqq 2\pi.$$

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \oint_{C_1} \mathbf{F} \cdot \mathbf{T}\; ds + \oint_{C_2} \mathbf{F} \cdot \mathbf{T}\; ds = \int_0^{2\pi} (\cos^2 t - \sin^2 t)\; dt + \int_0^{2\pi} (3\sin^2 t - 3\cos^2 t)\; dt$$

$$= \left[\frac{1}{2}\sin 2t\right]_0^{2\pi} - \left[\frac{3}{2}\sin 2t\right]_0^{2\pi} = 0 + 0 = 0.$$

C15S07.005: Parametrize the boundary curves of the surface S as follows:

$$C_1: \qquad x = \cos t, \qquad y = -\sin t, \quad z = 1, \quad 0 \leqq t \leqq 2\pi;$$

$$C_2: \qquad x = 3\cos t, \quad y = 3\sin t, \quad z = 3, \quad 0 \leqq t \leqq 2\pi.$$

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds + \oint_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt + \int_0^{2\pi} -27(\cos^2 t + \sin^2 t)\, dt = \Big[-26t \Big]_0^{2\pi} = -52\pi.$$

C15S07.006: Use for the surface S the disk $x^2 + y^2 \leqq 9,\ z = 4;$ use $\mathbf{n} = \mathbf{k}$ for the normal. Then we have $\nabla \times \mathbf{F} = \langle 3, 0, -5 \rangle$, and hence $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -5$. Therefore

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (-5)\, dS = -5 \cdot \text{area}(S) = -45\pi.$$

C15S07.007: Parametrize S (the elliptical region bounded by C) as follows:

$$x = z = r\cos t, \quad y = r\sin t, \quad 0 \leqq t \leqq 2\pi.$$

Then $\mathbf{r}_r \times \mathbf{r}_t = \langle -r, 0, r \rangle$, $dS = r\sqrt{2}\, dr\, dt$, the upper unit normal for S is

$$\mathbf{n} = \frac{1}{2}\sqrt{2}\, \langle -1, 0, 1 \rangle,$$

and $\nabla \times \mathbf{F} = \langle 3, 2, 1 \rangle$. Therefore $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -\sqrt{2}$. Consequently,

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^2 (-2r)\, dr\, dt$$

$$= 2\pi \Big[-r^2 \Big]_0^2 = 2\pi \cdot (-4) = -8\pi \approx -25.1327412287183459.$$

C15S07.008: The upper unit normal to the triangle bounded by C is

$$\mathbf{n} = \frac{1}{2}\sqrt{2}\, \langle 0, -1, 1 \rangle,$$

and consequently $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 0$. Therefore the value of the line integral is zero.

C15S07.009: If $\mathbf{F}(x, y, z) = \langle y - x, x - z, x - y \rangle$, then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - x & x - z & x - y \end{vmatrix} = \langle 0, -1, 0 \rangle.$$

The surface S bounded by C is part of the plane with equation $x + 2y + z = 2$, so an upward normal to S is $\langle\, 1,\ 2,\ 1\,\rangle$. Hence the unit normal vector we need is

$$\mathbf{n} = \frac{1}{6}\sqrt{6}\,\langle\, 1,\ 2,\ 1\,\rangle.$$

Then $(\boldsymbol{\nabla}\times\mathbf{F})\boldsymbol{\cdot}\mathbf{n} = -\frac{1}{3}\sqrt{6}$, a constant, so it remains only to find the area of S. Its vertices are at $A(2,\ 0,\ 0)$, $B(0,\ 1,\ 0)$, and $C(0,\ 0,\ 2)$, so we compute the "edge vectors"

$$\mathbf{u} = \overrightarrow{AB} = \langle\, -2,\ 1,\ 0\,\rangle \qquad \text{and} \qquad \mathbf{v} = \overrightarrow{AC} = \langle\, -2,\ 0,\ 2\,\rangle$$

and their cross product

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 0 \\ -2 & 0 & 2 \end{vmatrix} = \langle\, 2,\ 4,\ 2\,\rangle;$$

the area of S is then $\frac{1}{2}|\,\mathbf{u}\times\mathbf{v}\,| = \sqrt{6}$. Therefore

$$\oint_C \mathbf{F}\boldsymbol{\cdot}\mathbf{T}\ ds = \iint_S (\boldsymbol{\nabla}\times\mathbf{F})\boldsymbol{\cdot}\mathbf{n}\ dS = \left(-\frac{1}{3}\sqrt{6}\right)\cdot\sqrt{6} \ = -2.$$

C15S07.010: Let E denote the elliptical region bounded by C. Now E lies in the plane with equation $-y + z = 0$, so has upper unit normal

$$\mathbf{n} = \frac{1}{2}\sqrt{2}\,\langle\, 0,\ -1,\ 1\,\rangle.$$

Next, $\boldsymbol{\nabla}\times\mathbf{F} = \langle\, -2z,\ -2x,\ -2y\,\rangle$, so $(\boldsymbol{\nabla}\times\mathbf{F})\boldsymbol{\cdot}\mathbf{n} = \sqrt{2}\,(x-y)$. The projection of D into the xy-plane may be described in this way:

$$x^2 + (y-1)^2 \ = \ 1, \quad z = 0;$$

alternatively, by $r = 2\sin t \quad (0 \leqq y \leqq \pi)$, $z = 0$. Thus

$$J = \int_C \mathbf{F}\boldsymbol{\cdot}\mathbf{T}\ ds = \iint_E (\boldsymbol{\nabla}\times\mathbf{F})\boldsymbol{\cdot}\mathbf{n}\ dS = \iint_E \sqrt{2}\,(x-y)\ dS.$$

A parametrization of E is

$$\mathbf{w}(r,\ t) = \langle\, r\cos t,\ r\sin t,\ r\sin t\,\rangle, \quad 0 \leqq t \leqq \pi, \quad 0 \leqq r \leqq 2\sin t.$$

Next we find that $\mathbf{w}_r\times\mathbf{w}_t = \langle\, 0,\ -r,\ r\,\rangle$. Therefore $dS = r\sqrt{2}\ dr\ dt$. Consequently,

$$\begin{aligned} J &= \iint_E \sqrt{2}\,(x-y)\ dS = \int_0^{\pi}\int_0^{2\sin t} r\sqrt{2}\,(\cos t - \sin t)\cdot r\sqrt{2}\ dr\ dt \\ &= \int_0^{\pi}\int_0^{2\sin t} 2r^2(\cos t - \sin t)\ dr\ dt = \int_0^{\pi}\left[\frac{2}{3}r^3(\cos t - \sin t)\right]_0^{2\sin t} dt \\ &= \int_0^{\pi}\frac{16}{3}(\sin^3 t\,\cos t - \sin^4 t)\ dt = \frac{1}{6}\Big[-12t - 4\cos 2t + \cos 4t + 8\sin 2t - \sin 4t\Big]_0^{\pi} = -2\pi. \end{aligned}$$

C15S07.011: If $\mathbf{F}(x,\,y,\,z) = \langle\, 3y - 2z,\, 3x + z,\, y - 2x \,\rangle$, then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y - 2z & 3x + z & y - 2x \end{vmatrix} = \langle\, 1 - 1,\; -2 + 2,\; 3 - 3 \,\rangle = \mathbf{0}.$$

Therefore $\mathbf{F}$ is irrotational. Next, let C be the straight line segment from $(0,\,0,\,0)$ to the point $(u,\,v,\,w)$ of space. Parametrize C as follows: $x = tu,\; y = tv,\; z = tw,\; 0 \leqq t \leqq 1$. Then

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{t=0}^{1} (6tuv - 4tuw + 2tvw)\, dt = \Big[(3uv - 2uw + vw)t^2\Big]_0^1 = 3uv - 2uw + vw.$$

Now replace u with x, v with y, and w with z (because $(u,\,v,\,w)$ represents an *arbitrary* point of space). This yields the potential function

$$\phi(x,\,y,\,z) = 3xy - 2xz + yz.$$

To be absolutely certain of this, verify for yourself that $\nabla\phi = \mathbf{F}$.

C15S07.012: If $\mathbf{F}(x,\,y,\,z) = \langle\, 3y^3 - 10xz^2,\, 9xy^2,\, -10x^2z \,\rangle$, then we can use *Mathematica* 3.0 to verify that $\mathbf{F}$ is irrotational and to find a scalar potential $\phi(x,\,y,\,z)$ for $\mathbf{F}$, as follows. Recall that % refers to the "last output," that `expr1 → expr2` requests replacement of `expr1` with `expr2`, that `expr[[k]]` refers to the kth entry in the vector `expr`, and that `u.v` computes the dot product of the vectors `u` and `v`. We begin by constructing $\mathbf{F}$ and computing its curl.

```
ff = { 3*y^3 - 10*x*z^2, 9*x*y^2, -10*x^2*z };
```

The semicolon at the end of the line suppresses the output, which in this case would be the echo

$$\{3y^3 - 10xz^2,\; 9xy^2,\; -10x^2z\}$$

```
{D[ff[[3]],y] - D[ff[[2]],z], D[ff[[1]],z] - D[ff[[3]],x],
 D[ff[[2]],x] - D[ff[[1]],y]}
```

$$\{0,\; 0,\; 0\}$$

Thus $\mathbf{F}$ (alias `ff` to avoid encroaching on *Mathematica's* capital letters) is irrotational. Now we follow the technique illustrated in Example 3.

```
ff /.  {x → t*u, y → t*v, z →t*w}
```

$$\{3t^3v^3 - 10t^3uw^2,\; 9t^3uv^2,\; -10t^3u^2w\}$$

```
%.{u, v, w}
```

$$9t^3uv^3 - 10t^3u^2w^2 + u(3t^3v^3 - 10t^3uw^2)$$

```
Simplify[%]
```

$$4t^3u(5uw^2 - 3v^3)$$

```
Integrate[%, t]
```

$$t^4u(3v^3 - 5uw^2)$$

```
(% /.  t → 1) - (% /.  t → 0)
```

$$3uv^3 - 5u^2w^2$$

```
% /.  {u → x, v → y, w → z}
```

$$3xy^3 - 5x^2z^2$$

Finally we check the result $\phi(x, y, z) = 3xy^3 - 5x^2z^2$:

```
phi = 3*x*y∧3 - 5*x∧2*z∧2;   {D[phi,x], D[phi,y], D[phi,z]}
```

$$\{3y^3 - 10xz^2,\ 9xy^2,\ -10x^2z\}$$

```
% - ff
```

$$\{0,\ 0,\ 0\}$$

C15S07.013: If $\mathbf{F}(x, y, z) = \langle\, 3e^z - 5y\sin x,\ 5\cos x,\ 17 + 3xe^z\,\rangle$, then

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3e^z z - 5y\sin x & 5\cos x & 17 + 3xe^z \end{vmatrix} = \langle\, 0,\ 3e^z - 3e^z,\ -5\sin x + 5\sin x\,\rangle = \mathbf{0}.$$

Therefore $\mathbf{F}$ is irrotational. Then the method of Example 3 yields

$$x(t) = tu, \quad y(t) = tv, \quad z(t) = tw, \quad 0 \leqq t \leqq 1,$$
$$\mathbf{r}(t) = \langle\, x(t),\, y(t),\, z(t)\,\rangle = \langle\, tu,\, tv,\, tw\,\rangle, \qquad \text{and}$$
$$\mathbf{F}(t) = \langle\, 3\exp(tw) - 5tv\sin tu,\ 5\cos tu,\ 17 + 3tu\exp(tw)\,\rangle,$$

so that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} (17w + 3tuwe^{tw} + 5v\cos tu + 3ue^{tw} - 5tuv\sin tu)\, dt$$
$$= \Big[3tue^{tw} + 17tw + 5tv\cos tu\Big]_0^1 = 3ue^w + 17w + 5v\cos u.$$

Therefore $\phi(x, y, z) = 3xe^z + 17z + 5y\cos x$.

C15S07.014: First write $\mathbf{F}$ in the form

$$\mathbf{F} = (x^2 + y^2 + z^2)^{3/2}\langle\, x,\, y,\, z\,\rangle.$$

Then $\mathbf{F}(x, y, z) = \langle\, P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\,\rangle$ where

$P(x,\,y,\,z) = (x^2+y^2+z^2)^{3/2}\cdot x, \quad Q(x,\,y,\,z) = (x^2+y^2+z^2)^{3/2}\cdot y, \quad \text{and} \quad R(x,\,y,\,z) = (x^2+y^2+z^2)^{3/2}\cdot z.$

Then

$$\begin{aligned}
\frac{\partial P}{\partial y} &= 3(x^2+y^2+z^2)^{1/2}\cdot xy, & \frac{\partial P}{\partial z} &= 3(x^2+y^2+z^2)^{1/2}\cdot xz,\\
\frac{\partial Q}{\partial x} &= 3(x^2+y^2+z^2)^{1/2}\cdot xy, & \frac{\partial Q}{\partial z} &= 3(x^2+y^2+z^2)^{1/2}\cdot yz,\\
\frac{\partial R}{\partial x} &= 3(x^2+y^2+z^2)^{1/2}\cdot xz, & \frac{\partial R}{\partial y} &= 3(x^2+y^2+z^2)^{1/2}\cdot yz.
\end{aligned}$$

Therefore

$$\nabla\times\mathbf{F} = \langle\, R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\,\rangle = \langle\, 0,\,0,\,0\,\rangle = \mathbf{0},$$

and hence $\mathbf{F}$ is irrotational. Then the method of Example 3 yields

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (u^2+v^2+w^2)^{5/2}t^4\,dt = \left[\frac{1}{5}(u^2+v^2+w^2)^{5/2}t^5\right]_0^1 = \frac{1}{5}(u^2+v^2+w^2)^{5/2}.$$

Therefore a scalar potential for $\mathbf{F}$ is $\phi(x,\,y,\,z) = \dfrac{1}{5}(x^2+y^2+z^2)^{5/2}$; that is, $\phi(r) = \dfrac{1}{5}r^5$.

C15S07.015: We are given $\mathbf{r} = \langle\, x,\,y,\,z\,\rangle$; suppose that $\mathbf{a} = \langle\, b,\,c,\,d\,\rangle$ is a constant vector. Part (a):

$$\mathbf{a}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ b & c & d\\ x & y & z \end{vmatrix} = \langle\, cz - dy,\, dx - bz,\, by - cx\,\rangle,$$

and hence $\nabla\cdot(\mathbf{a}\times\mathbf{r}) = 0 + 0 + 0 = 0$. Part (b): Using some of the results in part (a), we have

$$\nabla\times(\mathbf{a}\times\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ cz - dy & dx - bz & by - cx \end{vmatrix} = \langle\, 2b,\,2c,\,2d\,\rangle = 2\mathbf{a}.$$

Part (c): First,

$$(\mathbf{r}\cdot\mathbf{r})\mathbf{a} = \langle\, b(x^2+y^2+z^2),\ c(x^2+y^2+z^2),\ d(x^2+y^2+z^2)\,\rangle.$$

Thus $\nabla\cdot\big[(\mathbf{r}\cdot\mathbf{r})\mathbf{a}\big] = 2bx + 2cy + 2dz = 2\mathbf{r}\cdot\mathbf{a}$. Part (d): Using some of the results in parts (a) and (c), we have

$$\nabla \times \left[(\mathbf{r} \cdot \mathbf{r})\mathbf{a}\right] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ b(x^2+y^2+z^2) & c(x^2+y^2+z^2) & d(x^2+y^2+z^2) \end{vmatrix}$$

$$= \langle\, 2dy - 2cz,\ 2bz - 2dx,\ 2cx - 2by \,\rangle$$

$$= -2\langle\, cz - dy,\ dx - bz,\ by - cx \,\rangle = -2(\mathbf{a} \times \mathbf{r}) = 2(\mathbf{r} \times \mathbf{a}).$$

C15S07.016: Suppose that S and T are oriented surfaces having the same oriented boundary curve C; also assume that the following integrals exist (because S and T have piecewise smooth parametrizations, $\mathbf{n}$ represents both a unit vector normal to S and a unit vector normal to T, C has a piecewise differentiable parametrization, and $\mathbf{F}$ is continuous and has continuous first-order partial derivatives). Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_T (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

C15S07.017: Assume that S is a closed surface having a piecewise smooth parametrization, that $\mathbf{n}$ is the outer unit normal vector for S, and that $\mathbf{F}$ is continuous and has continuous first-order partial derivatives on an open region containing S and the solid T that it bounds. Part (a): By the divergence theorem,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iiint_T \nabla \cdot (\nabla \times \mathbf{F})\, dV = \iiint_T 0\, dV = 0$$

by Problem 32 of Section 15.1. Part (b): Let C be a simple closed curve on S having a suitably differentiable parametrization and a given orientation. Then C is the common boundary of the two surfaces S_1 and S_2 into which it divides S; that is, S is the union of S_1 and S_2, S_1 and S_2 meet in the curve C, and C has positive orientation on (say) S_1 and the opposite orientation on S_2. Then

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = -\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

Therefore

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$

C15S07.018: Following the *Suggestion* given in the statement of the problem, let $\mathbf{F} = \phi\mathbf{a}$ where $\mathbf{a}$ is an arbitrary constant vector. Write $\mathbf{T} = \langle\, T_1, T_2, T_3 \,\rangle$ and $\mathbf{n} = \langle\, n_1, n_2, n_3 \,\rangle$. Then

$$\oint_C (\phi\mathbf{a}) \cdot \mathbf{T}\, ds = \iint_S \left[\nabla \times (\phi\mathbf{a})\right] \cdot \mathbf{n}\, dS = \iint_S \left[(\phi)(\nabla \times \mathbf{a}) + (\nabla\phi) \times \mathbf{a}\right] \cdot \mathbf{n}\, dS = \iint_S (\nabla\phi \times \mathbf{a}) \cdot \mathbf{n}\, dS.$$

If $\mathbf{a} = \mathbf{i}$, then

$$\oint_C (\phi\mathbf{a}) \cdot \mathbf{T}\, ds = \oint_C \phi T_1\, ds = \iint_S \langle\, 0, \phi_z, -\phi_y \,\rangle\, dS = \iint_S (n_2\phi_z - n_3\phi_y)\, dS.$$

Similarly, if $\mathbf{a} = \mathbf{j}$, then

$$\oint_C (\phi\mathbf{a})\cdot\mathbf{T}\,ds = \oint_C \phi T_2\,ds = \iint_S (n_3\phi_x - n_1\phi_z)\,dS,$$

and if $\mathbf{a} = \mathbf{k}$, then

$$\oint_C (\phi\mathbf{a})\cdot\mathbf{T}\,ds = \oint_C \phi T_3\,ds = \iint_S (n_1\phi_y - n_2\phi_x)\,dS.$$

Hence

$$\begin{aligned}\oint_C \phi\mathbf{T}\,ds \;&=\; \oint_C \phi\langle\, T_1,\, T_2,\, T_3\,\rangle\,ds \;=\; \left\langle \oint_C \phi T_1\,ds,\; \oint_C \phi T_2\,ds,\; \oint_C \phi T_3\,ds \right\rangle \\ &= \left\langle \iint_S (n_2\phi_z - n_3\phi_y)\,dS,\; \iint_S (n_3\phi_x - n_1\phi_z)\,dS,\; \iint_S (n_1\phi_y - n_2\phi_x)\,dS \right\rangle \\ &= \iint_S \langle\, n_2\phi_z - n_3\phi_y,\; n_3\phi_x - n_1\phi_z,\; n_1\phi_y - n_2\phi_x\,\rangle\,dS \;=\; \iint_S \mathbf{n}\times(\boldsymbol{\nabla}\phi)\;dS.\end{aligned}$$

C15S07.019: We are given the constant vector $\mathbf{a}$ and the vector $\mathbf{r} = \langle\, x,\, y,\, z\,\rangle$. Let $\mathbf{F} = \mathbf{a}\times\mathbf{r}$. Then by Stokes' theorem,

$$\int_C (\mathbf{a}\times\mathbf{r})\cdot\mathbf{T}\,ds = \iint_S \big[\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{r})\big]\cdot\mathbf{n}\,dS = \iint_S 2\mathbf{a}\cdot\mathbf{n}\,dS$$

by part (b) of Problem 15. But because integration is carried out componentwise, it now follows that

$$\int_C (\mathbf{a}\times\mathbf{r})\cdot\mathbf{T}\,ds = 2\mathbf{a}\iint_S \mathbf{n}\,dS.$$

C15S07.020: Write $\mathbf{F}(x,\, y,\, z)$ in the form $\langle\, P,\, Q,\, R\,\rangle$ where $P,\ Q,$ and R are functions of $x,\ y,$ and z. Write the unit normal vector $\mathbf{n}$ in the form $\langle\, n_1,\, n_2,\, n_3\,\rangle$. Then

$$\mathbf{n}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ n_1 & n_2 & n_3 \\ P & Q & R \end{vmatrix} = \langle\, n_2R - n_3Q,\; n_3P - n_1R,\; n_1Q - n_2P\,\rangle$$

and

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle\, R_y - Q_z,\; P_z - R_x,\; Q_x - P_y\,\rangle.$$

Equation (4) from Section 15.6 tells us that

$$\iint_S P\,dy\,dz \;+\; Q\,dz\,dx \;+\; R\,dx\,dy \;=\; \iiint_T (P_x + Q_y + R_z)\,dV.$$

Then Eq. (3) of Section 15.6 implies that

$$\iint_S (n_2 R - n_3 Q)\, dS = \iint_S R\, dz\, dx \;-\; Q\, dx\, dy = \iiint_T (R_y - Q_z)\, dV. \tag{1}$$

Similarly,

$$\iint_S (n_3 P - n_1 R)\, dS = \iint_S P\, dx\, dy \;-\; R\, dy\, dz = \iiint_T (P_z - R_x)\, dV \tag{2}$$

and

$$\iint_S (n_1 Q - n_2 P)\, dS = \iint_S Q\, dy\, dz \;-\; P\, dz\, dx = \iiint_T (Q_x - P_y)\, dV. \tag{3}$$

Addition of the results in Eqs. (1), (2), and (3) then yields

$$\begin{aligned}\iint_S (\mathbf{n} \times \mathbf{F})\, dS &= \iint_S \langle\, n_2 R - n_3 Q,\ n_3 P - n_1 R,\ n_1 Q - n_2 P\,\rangle\, dS \\ &= \iiint_T \langle\, R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\,\rangle\, dV = \iiint_T (\boldsymbol{\nabla} \times \mathbf{F})\, dV.\end{aligned}$$

C15S07.021: Beginning with the *Suggestion* given in the statement of Problem 21, we find that

$$\phi_x = \lim_{h\to 0} \frac{\phi(x+h,\, y,\, z) - \phi(x,\, y,\, z)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{(x,y,z)}^{(x+h,y,z)} P(t,\, y,\, z)\, dt = \lim_{h\to 0} \frac{h \cdot P(x^\star,\, y,\, z)}{h} = P(x,\, y,\, z)$$

where $x^\star$ is between x and $x+h$. A similar argument shows that $\phi_y = Q$ and $\phi_z = R$. Adding these results establishes that $\boldsymbol{\nabla}\phi = \mathbf{F} = \langle\, P,\, Q,\, R\,\rangle$.

C15S07.022: First note that

$$\mathbf{L} = \iint_S (\mathbf{r} - \mathbf{r}_0) \times (-\delta g z\, \mathbf{n})\, dS = \delta g \iiint_V \boldsymbol{\nabla} \times \big[z(\mathbf{r} - \mathbf{r}_0) \big]\, dV$$

by Problem 20. But $\boldsymbol{\nabla} \times \big[z(\mathbf{r} - \mathbf{r}_0) \big] = (\boldsymbol{\nabla} z) \times (\mathbf{r} - \mathbf{r}_0) + z\big[\boldsymbol{\nabla} \times (\mathbf{r} - \mathbf{r}_0) \big]$. It follows immediately that $\boldsymbol{\nabla} z = \mathbf{k}$ and that $\boldsymbol{\nabla} \times (\mathbf{r} - \mathbf{r}_0) = \mathbf{0}$. Thus

$$\mathbf{L} = \delta g \iiint_V \mathbf{k} \times (\mathbf{r} - \mathbf{r}_0)\, dV = \delta g\, \mathbf{k} \times \left(\iiint_V \mathbf{r}\, dV \;-\; \mathbf{r}_0 V \right).$$

Consequently $\mathbf{L} = \mathbf{0}$, because $\mathbf{r}_0 = \dfrac{1}{V} \displaystyle\iiint_V \mathbf{r}\, dV$.

Chapter 15 Miscellaneous Problems

C15S0M.001: Parametrize C: $x = t,\ y = \frac{4}{3}t,\ 0 \leqq t \leqq 3$. Then

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt = \frac{5}{3}\ dt \quad \text{and} \quad [x(t)]^2 + [y(t)]^2 = \frac{25}{9}t^2.$$

Therefore

$$\int_C (x^2 + y^2)\ ds = \int_0^3 \frac{125}{27}t^2\ dt = \left[\frac{125}{81}t^3\right]_0^3 = \frac{125}{3} \approx 41.666666666667.$$

C15S0M.002: Parametrize C: $x = t,\ y = t^2,\ -1 \leqq t \leqq 1$. Then

$$\int_C y^2\ dx + x^2\ dy = \int_{-1}^1 (2t^3 + t^4)\ dt = \left[\frac{1}{2}t^4 + \frac{1}{5}t^5\right]_{-1}^1 = \frac{2}{5}.$$

C15S0M.003: We are given the curve C parametrized by $\mathbf{r}(t) = \langle\, e^{2t},\ e^t,\ e^{-t}\,\rangle,\ 0 \leqq t \leqq \ln 2$. Because $\mathbf{F}(t) = \langle\, x,\ y,\ z\,\rangle$, we can also write

$$\mathbf{F}(t) = \langle\, e^{2t},\ e^t,\ e^{-t}\,\rangle,$$

and therefore

$$\int_C \mathbf{F} \cdot \mathbf{T}\ ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\ln 2} (2e^{4t} + e^{2t} - e^{-2t})\ dt = \frac{1}{2}\left[e^{4t} + e^{2t} + e^{-2t}\right]_0^{\ln 2} = \frac{69}{8} = 8.625.$$

C15S0M.004: Parametrize the three line segments separately, as follows:

$$\begin{aligned}
C_1: &\quad x_1(t) = 1 + t, \quad y_1(t) = 1, \quad && z_1(t) = 2;\\
C_2: &\quad x_2(t) = 2, \quad y_2(t) = 1 + 2t, \quad && z_2(t) = 2;\\
C_3: &\quad x_3(t) = 2, \quad y_2(t) = 3, \quad && z_2(t) = 2 + 4t;
\end{aligned}$$

in each case the range for the parameter is $0 \leqq t \leqq 1$. Hence

$$\begin{aligned}
\int_C xyz\ ds &= \int_{C_1} xyz\ ds + \int_{C_2} xyz\ ds + \int_{C_3} xyz\ ds\\
&= \int_0^1 (2t + 2)\ dt + \int_0^1 (16t + 8)\ dt + \int_0^1 (96t + 48)\ dt\\
&= \Big[t^2 + 2t\Big]_0^1 + \Big[8t^2 + 8t\Big]_0^1 + \Big[48t^2 + 48t\Big]_0^1\\
&= 3 + 16 + 96 = 115.
\end{aligned}$$

C15S0M.005: Given the curve C with parametrization

$$x(t) = t, \quad y(t) = t^{3/2}, \quad z(t) = t^2, \quad 0 \leqq t \leqq 4,$$

we substitute and find that

$$\int_C z^{1/2}\,dx + x^{1/2}\,dy + y^2\,dz = \int_0^4 \left(t + \frac{3}{2}t + 2t^4\right)dt = \left[\frac{5}{4}t^2 + \frac{2}{5}t^5\right]_0^4 = \frac{2148}{5} = 429.6.$$

C15S0M.006: If $\phi(x,\,y,\,z) = xy^2 + \frac{1}{2}z^2,$ then

$$\boldsymbol{\nabla}\phi \;=\; \langle\, y^2,\; 2xy,\; z\,\rangle,$$

and therefore the given integral is independent of the path C from the fixed point A to the fixed point B.

C15S0M.007: Suppose that there exists a function $\phi(x,\,y)$ such that $\boldsymbol{\nabla}\phi = \langle\, x^2y,\; xy^2\,\rangle$. Then

$$\phi(x,\,y) = \int x^2y\;dx = \frac{1}{3}x^3y + g(y),$$

and hence

$$\frac{\partial\phi}{\partial y} = \frac{1}{3}x^3 + g'(y). \tag{1}$$

But there is no choice of $g(y)$ such that the last expression in Eq. (1) can equal xy^2 unless x is constant. This is not possible on any path C from $(0,\,0)$ to $(1,\,1)$. Thus there is no such function ϕ, and thus by Theorem 2 the given integral is not independent of the path from $(0,\,0)$ to $(1,\,1)$.

C15S0M.008: Let δ denote the (constant) density of the wire. The length of the wire is $2\pi a$, and hence $2\pi a\delta = M$, the mass of the wire. Therefore (for future reference)

$$\delta \;=\; \frac{M}{2\pi a}. \tag{1}$$

Parametrize the wire using $x(t) = a\cos t,\; y(t) = a\sin t,\; 0 \leqq t \leqq 2\pi$. Then

$$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\;\,dt = a\;dt.$$

Part (a): The moment of inertia of the wire with respect to the z-axis is then

$$I_z = \int_0^{2\pi} \delta a^3\;dt = 2\pi\delta a^3 = Ma^2$$

(by Eq. (1)). Part (b): The moment of inertia of the wire with respect to the x-axis is

$$I_x = \int_0^{2\pi} \delta a y^2\;dt = \frac{1}{4}\delta a^3\Big[2t - \sin 2t\Big]_0^{2\pi} = \pi\delta a^3 = \frac{1}{2}Ma^2$$

(by Eq. (1)). It is also possible to solve part (b) mentally if you note that $I_y = I_x$ and recall that $I_0 = I_x + I_y$.

C15S0M.009: Parametrize the wire W using $x(t) = t,\; y(t) = \frac{1}{2}t^2,\; 0 \leqq t \leqq 2$. Then

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\;\,dt = \sqrt{1+t^2}\;\,dt.$$

The density of the wire is $\delta(t) = x(t) = t$. Therefore the mass of the wire is

$$m = \int_W \delta\, ds = \int_0^2 t\sqrt{1+t^2}\ dt = \left[\frac{1}{3}(1+t^2)^{3/2}\right]_0^2 = \frac{5\sqrt{5}\ -1}{3} \approx 3.3934466291663162.$$

Its moment of inertia with respect to the y-axis is

$$\begin{aligned} I_y &= \int_W \delta x^2\ ds = \int_0^2 t^3(1+t^2)^{1/2}\ dt \\ &= \left[\frac{1}{15}(1+t^2)^{1/2}(3t^4+t^2-2)\right]_0^2 = \frac{50\sqrt{5}\ +2}{15} \approx 7.5868932583326323. \end{aligned}$$

If you prefer, $I_y \approx m \cdot (1.4952419583303542)^2$.

C15S0M.010: Parametrize the given path C as follows: $x = t,\ y = t^2,\ z = t^3,\ 1 \leqq t \leqq 2$. Then

$$\mathbf{F}(x,\, y,\, z) = \langle\, t^3,\, -t,\, t^2\, \rangle \quad \text{and} \quad \mathbf{r}(t) = \langle\, t,\, t^2,\, t^3\, \rangle.$$

Hence the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (3t^4+t^3-2t^2)\ dt = \left[\frac{3}{5}t^5 + \frac{1}{4}t^4 - \frac{2}{3}t^3\right]_1^2 = \frac{1061}{60} \approx 17.683333333333.$$

C15S0M.011: Let R denote the region bounded by C. Then Green's theorem yields

$$\begin{aligned} \oint_C x^2y\, dx\, +\, xy^2\, dy &= \iint_R (y^2-x^2)\ dA = \int_{x=-2}^2 \int_{y=x^2}^{8-x^2} (y^2-x^2)\ dy\ dx \\ &= \int_{-2}^2 \left[\frac{1}{3}y^3 - x^2y\right]_{x^2}^{8-x^2} dx = \int_{-2}^2 \left(\frac{512}{3} - 72x^2 + 10x^4 - \frac{2}{3}x^6\right) dx \\ &= \left[\frac{512}{3}x - 24x^3 + 2x^5 - \frac{2}{21}x^7\right]_{-2}^2 = \frac{2816}{7} \approx 402.2857142857142857. \end{aligned}$$

C15S0M.012: Let R denote the plane region bounded by the given cardioid. Then

$$\begin{aligned} \oint_C x^2\ dy &= \iint_R 2x\ dA = \iint_R 2r^2\cos\theta\ dr\ d\theta = \int_0^{2\pi} \left[\frac{2}{3}r^3\cos\theta\right]_0^{1+\cos\theta} d\theta \\ &= \int_0^{2\pi} \left(\frac{2}{3}\cos\theta + 2\cos^2\theta + 2\cos^3\theta + \frac{2}{3}\cos^4\theta\right) d\theta \\ &= \frac{1}{48}\Big[60\theta + 104\sin\theta + 32\sin 2\theta + 8\sin 3\theta + \sin 4\theta\Big]_0^{2\pi} = \frac{5}{2}\pi \approx 7.8539816339744831. \end{aligned}$$

C15S0M.013: Suppose that C is any positively oriented piecewise smooth simple closed curve in the xy-plane. Let $\mathbf{n}$ be the outwardly directed unit vector normal to C. We will apply the vector form of Green's theorem in Eq. (9) of Section 15.4,

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

where R denotes the bounded plane region with boundary C and $\mathbf{F}$ is a two-dimensional vector function of x and y with continuously differentiable component functions. If $\mathbf{F}(x, y) = \langle x^2y, -xy^2 \rangle$, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R (2xy - 2xy)\, dA = 0.$$

Therefore the integrals given in the statement of Problem 13 are equal because each is equal to zero. It is also possible to verify this by a direct computation. For example, using the parametrization $x = \cos t$, $y = \sin t$, $0 \leqq t \leqq 2\pi$ and the outer unit normal vector $\mathbf{n} = \langle \cos t, \sin t \rangle$, the first integral in Problem 13 becomes

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} (\cos^3 t \sin t - \cos t \sin^3 t)dt = -\frac{1}{4}\Big[\cos^4 t + \sin^4 t\Big]_0^{2\pi} = 0 - 0 = 0.$$

C15S0M.014: Part (a): Parametrize C as follows: $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leqq t \leqq 1$. Then

$$\begin{aligned}\int_C -y\, dx + x\, dy &= \int_0^1 \big\{\big[x_1 + (x_2 - x_1)t\big] \cdot (y_2 - y_1) - (x_2 - x_1) \cdot \big[y_1 + (y_2 - y_1)t\big]\big\}\, dt \\ &= \int_0^1 (x_1y_2 - x_2y_1)\, dt = x_1y_2 - x_2y_1.\end{aligned}$$

Part (b): By the corollary to Green's theorem (Eq. (4) in Section 15.4) and part (a), the area of the polygon with boundary C is

$$\begin{aligned}A &= \frac{1}{2}\oint_C -y\, dx + x\, dy \\ &= \frac{1}{2}\big[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\big] \\ &= \frac{1}{2}\sum_{i=1}^{n}(x_iy_{i+1} - x_{i+1}y_i).\end{aligned}$$

C15S0M.015: Suppose that

$$\int_C P\, dx + Q\, dy$$

is independent of the path in the plane region D. By Theorem 2 of Section 15.3, $\mathbf{F} = \langle P, Q \rangle = \nabla\phi$ in D where ϕ is some differentiable scalar potential function. Suppose that C is a simple closed curve in D. Choose a point (a, b) on C. Then by Theorem 1 of Section 15.3,

$$\oint_C P\, dx + Q\, dy = \Big[\phi(x, y)\Big]_{(a,b)}^{(a,b)} = \phi(a, b) - \phi(a, b) = 0.$$

C15S0M.016: If $Q_x = P_y$ on $D,$ C is a piecewise smooth simple closed curve in $D,$ and R is the region bounded by $C,$ then

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA = \iint_R 0\,dA = 0.$$

If

$$\oint_C P\,dx + Q\,dy = 0$$

for every piecewise smooth simple closed curve C in $D,$ then

$$\int_J P\,dx + Q\,dy$$

is independent of the path in D. Here's why: Suppose that A and B are two points of D. Let J and K be two paths in D from A to B. Let $C = J \cup (-K)$. Then C is a closed path in D, and it follows that

$$0 = \oint_C P\,dx + Q\,dy = \int_J P\,dx + Q\,dy \; - \int_K P\,dx + Q\,dy,$$

and therefore

$$\int_J P\,dx + Q\,dy = \int_K P\,dx + Q\,dy.$$

Consequently $\mathbf{F} = \langle P, Q \rangle = \boldsymbol{\nabla}\phi$ for some scalar potential ϕ defined on D (by Theorem 2 of Section 15.3). Therefore

$$P_y - Q_x = \phi_{xy} - \phi_{yx} = 0$$

on $D,$ and thus $P_y = Q_x$ on $D.$

C15S0M.017: The surface S is described by $h(x, y) = 2 - x^2 - y^2,$ and thus

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\;dA = \sqrt{1 + 4x^2 + 4y^2}\;dA.$$

Therefore

$$\iint_S (x^2 + y^2 + 2z)\,dS = \iint_S (4 - x^2 - y^2)\,dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} r(4 - r^2)\sqrt{1 + 4r^2}\;dr\,d\theta$$

$$= 2\pi \left[\frac{1}{120}(41 + 158r^2 - 24r^4)\sqrt{1 + 4r^2} \right]_0^{\sqrt{2}} = 2\pi \cdot \frac{371}{60} = \frac{371}{30}\pi \approx 38.8510291493937764.$$

C15S0M.018: A unit vector normal to S is

$$\mathbf{n}(x, y, z) = \frac{1}{a}\langle x, y, z \rangle,$$

and therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \frac{1}{a}(x^2 + y^2 + z^2)^2\, dS = \iint_S a^3\, dS = 4\pi a^2 \cdot a^3 = 4\pi a^5.$$

C15S0M.019: The upper surface S_1 is described by $h(x, y) = 12 - 2x^2 - y^2$, and thus

$$dS = \sqrt{1 + (h_x)^2 + (h_y)^2}\, dA = \sqrt{1 + 16x^2 + 4y^2}\, dA.$$

An outwardly pointing vector normal to S_1 is

$$\left\langle -\frac{\partial h}{\partial x},\, -\frac{\partial h}{\partial y},\, 1 \right\rangle = \langle\, 4x,\, 2y,\, 1\,\rangle,$$

and therefore a outwardly pointing unit vector normal to S_1 is

$$\mathbf{n} = \frac{1}{\sqrt{1 + 16x^2 + 4y^2}}\, \langle 4x,\, 2y,\, 1\,\rangle.$$

Hence the outward flux of $\mathbf{F} = \langle\, x,\, y,\, z\,\rangle$ across S_1 is

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_{S_1} \frac{4x^2 + 2y^2 + z}{\sqrt{1 + 16x^2 + 4y^2}} \cdot \sqrt{1 + 16x^2 + 4y^2}\, dS = \iint_{S_1} (4x^2 + 2y^2 + z)\, dS \\
&= \iint_S (12 + 2x^2 + y^2)\, dS = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (12 + 2r^2\cos^2\theta + r^2\sin^2\theta) \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[6r^2 + \frac{1}{8}r^4(3 + \cos 2\theta) \right]_0^2 d\theta = \int_0^{2\pi} (30 + 2\cos 2\theta)\, d\theta = \Big[30\theta + \sin 2\theta \Big]_0^{2\pi} = 60\pi.
\end{aligned}$$

The lower surface S_2 is described by $h(x, y) = x^2 + 2y^2$. By computations similar to those shown earlier in this solution, we find that

$$dS = \sqrt{1 + 4x^2 + 16y^2}\, dA$$

and that an outer unit vector normal to S_2 is

$$\mathbf{n} = \frac{1}{\sqrt{1 + 4x^2 + 16y^2}}\, \langle\, 2x,\, 4y,\, -1\,\rangle.$$

Thus the flux of $\mathbf{F}$ across S_2 is

$$\begin{aligned}
\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_{S_2} (2x^2 + 4y^2 - z)\, dS = \iint_{S_2} (x^2 + 2y^2)\, dS = \int_0^{2\pi} \int_0^2 (r^2\cos^2\theta + 2r^2\sin^2\theta) \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{4}r^4\cos^2\theta + \frac{1}{2}r^4\sin^2\theta \right]_0^2 d\theta = \int_0^{2\pi} (4\cos^2\theta + 8\sin^2\theta)\, d\theta = \Big[6\theta - \sin 2\theta \Big]_0^{2\pi} = 12\pi.
\end{aligned}$$

Therefore the total outward flux of $\mathbf{F}$ across the boundary of T is

$$\phi = 60\pi + 12\pi = 72\pi \approx 226.19467105846511316931.$$

C15S0M.020: Suppose that the surface S has area $a(S)$ and that (x, y, z) is a point of S. Let $f(x, y, z)$ denote the distance between (x, y, z) and the fixed point P of space. Then (in analogy with the definition used in Problems 47 through 53 of Section 14.6, Problems 39 and 40 of Section 14.7, and Miscellaneous Problems 36 through 42 of Chapter 14) we define the *average distance* of points of S from the point P to be

$$\overline{d} = \frac{1}{a(S)} \iint_S f(x, y, z)\, dS.$$

To find the average distance of points of the spherical surface S of radius $a > 0$ from a fixed point P on S, we choose for S the spherical surface of radius a tangent to the xy-plane at the origin and otherwise lying above the xy-plane. A spherical coordinates equation of S is $\rho = 2a\cos\phi$. We parametrize S as follows:

$$\begin{aligned}
x(\phi, \theta) &= \rho \sin\phi\, \cos\theta = 2a\cos\phi\, \sin\phi\, \cos\theta,\\
y(\phi, \theta) &= \rho \sin\phi\, \sin\theta = 2a\cos\phi\, \sin\phi\, \sin\theta,\\
z(\phi, \theta) &= \rho\cos\phi = 2a\cos^2\phi, \quad 0 \leqq \phi \leqq \tfrac{1}{2}\pi, \quad 0 \leqq \theta \leqq 2\pi.
\end{aligned}$$

With $\mathbf{r}(\phi, \theta) = \langle\, x(\phi, \theta),\ y(\phi, \theta),\ z(\phi, \theta)\,\rangle$, we have

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2a(\cos^2\phi - \sin^2\phi)\cos\theta & 2a(\cos^2\phi - \sin^2\phi)\sin\theta & -4a\sin\phi\,\cos\phi \\ -2a\sin\phi\,\cos\phi\,\sin\theta & 2a\sin\phi\,\cos\phi\,\cos\theta & 0 \end{vmatrix}\\
&= \langle\, 8a^2\sin^2\phi\,\cos^2\phi\,\cos\theta,\ 8a^2\sin^2\phi\,\cos^2\phi\,\sin\theta,\ 4a^2(\sin\phi\,\cos^3\phi - \sin^3\phi\,\cos\phi)\,\rangle\\
&= 4a^2(\sin\phi\,\cos\phi)\langle\, 2\sin\phi\,\cos\phi\,\cos\theta,\ 2\sin\phi\,\cos\phi\,\sin\theta,\ \cos^2\phi - \sin^2\phi\,\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
&|\,\mathbf{r}_\phi \times \mathbf{r}_\theta\,|\\
&\quad = 4a^2|\sin\phi\cos\phi|(4\sin^2\phi\,\cos^2\phi\,\cos^2\theta + 4\sin^2\phi\,\cos^2\phi\,\sin^2\theta + \cos^4\phi - 2\sin^2\phi\,\cos^2\phi + \sin^4\phi)^{1/2}\\
&\quad = 4a^2|\sin\phi\cos\phi|(4\sin^2\phi\,\cos^2\phi + \cos^4\phi - 2\sin^2\phi\,\cos^2\phi + \sin^4\phi)^{1/2}\\
&\quad = 4a^2|\sin\phi\cos\phi|\big([\cos^2\phi + \sin^2\phi]\big)^{1/2} = 4a^2|\sin\phi\cos\phi| = 4a^2\sin\phi\cos\phi.
\end{aligned}$$

(We may drop the absolute value symbols in the last step because $0 \leqq \phi \leqq \pi/2$.) Choose P to be the origin. The distance of the typical point (ρ, ϕ, θ) of S from P is then ρ; moreover, $\rho = 2a\cos\phi$. Therefore the average distance of points of S from the fixed point P is

$$\overline{d} = \frac{1}{4\pi a^2}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}(2a\cos\phi)\cdot(4a^2\sin\phi\cos\phi)\,d\phi\,d\theta = \frac{2\pi\cdot 8a^3}{4\pi a^2}\left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{16\pi a^3}{12\pi a^2} = \frac{4}{3}a.$$

Alternatively, we may choose for S the spherical surface of radius a centered at the origin; we will compute the average distance of points of S from its "north pole" $P(0, 0, a)$. If a point of S is at distance w from P and its spherical coordinates are (ρ, ϕ, θ), then the law of cosines implies that

$$w^2 = a^2 + \rho^2 - 2a\rho\cos\phi = a^2 + a^2 - 2a^2\cos\phi = 2a^2(1 - \cos\phi).$$

Therefore the average distance of points of S from the fixed point P is

$$\overline{d} = \frac{1}{4\pi a^2} \int_0^{2\pi} \int_0^{\pi} \sqrt{2a^2(1-\cos\phi)}\ (a^2 \sin\phi)\ d\phi\ d\theta = \frac{2\pi a^3}{4\pi a^2} \int_0^{\pi} (\sin\phi)\sqrt{2(1-\cos\phi)}\ d\phi$$

$$= \frac{a}{2}\left[\frac{2}{3} \cdot 2^{1/2} \cdot (1-\cos\phi)^{3/2}\right]_0^{\pi} = \frac{a}{2} \cdot \frac{2}{3} \cdot 2^{1/2} \cdot 2^{3/2} = \frac{4}{3}a.$$

C15S0M.021: We compute the three Jacobians in Eq. (17) of Section 15.5 using the parameters y and z. The result is

$$\frac{\partial(y,\,z)}{\partial(y,\,z)} = \begin{vmatrix} y_y & y_z \\ z_y & z_z \end{vmatrix} = 1,$$

$$\frac{\partial(z,\,x)}{\partial(y,\,z)} = \begin{vmatrix} z_y & z_z \\ x_y & x_z \end{vmatrix} = -\frac{\partial x}{\partial y}, \quad \text{and}$$

$$\frac{\partial(x,\,y)}{\partial(y,\,z)} = \begin{vmatrix} x_y & x_z \\ y_y & y_z \end{vmatrix} = -\frac{\partial x}{\partial z}.$$

Therefore

$$\iint_S P\ dy\ \ dz\ +\ Q\ dz\ dx\ +\ R\ dx\ dy\ =\ \iint_D \left(P\ -\ Q\frac{\partial x}{\partial y}\ -\ R\frac{\partial x}{\partial z}\right) dy\ dz.$$

C15S0M.022: Suppose that the surface S is described by $y = g(x,\,z)$ for $(x,\,z)$ in the region D of the xz-plane. Then S is parametrized by $\mathbf{r}(x,\,z) = \langle\, x,\ g(x,\,z),\ z\,\rangle$. Thus

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & g_x & 0 \\ 0 & g_z & 1 \end{vmatrix} = \langle\, g_x,\ -1,\ g_z\,\rangle.$$

Therefore

$$|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{1 + (g_x)^2 + (g_z)^2}\,,$$

and consequently

$$\iint_S f(x,\,y,\,z)\ dS = \iint_D f(x,\,g(x,\,z),\,z)\sqrt{1+(g_x)^2+(g_z)^2}\ \ dA = \iint_D f(x,\,g(x,\,z),\,z)\sec\beta\ dx\ dz.$$

C15S0M.023: Here we have

$$\overline{z} = \frac{1}{V}\iiint_T z\ dV = \frac{1}{V}\iiint_T \boldsymbol{\nabla}\cdot \langle\, 0,\ 0,\ \tfrac{1}{2}z^2\,\rangle\ dV = \frac{1}{V}\iint_S \frac{1}{2}z^2\ dx\ dy = \frac{1}{2V}\iint_S z^2\ dx\ dy$$

by Eq. (4) of Section 15.6.

C15S0M.024: By symmetry, $\overline{x} = \overline{y} = 0$. Parametrize the curved surface S of the hemisphere in the usual way:

$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad z = a\cos\phi, \quad 0 \leqq \phi \leqq \pi/2, \quad 0 \leqq \theta \leqq 2\pi.$$

Then $z^2 = a^2 \cos^2 \phi$ and

$$\frac{\partial(x,\,y)}{\partial(\phi,\,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\phi\cos\theta & -a\sin\phi\sin\theta \\ a\cos\phi\sin\theta & a\sin\phi\cos\theta \end{vmatrix} = a^2 \sin\phi\cos\phi.$$

There is no need to integrate on the bottom face of the solid; it is in the xy-plane, where $z^2 = 0$. Consequently,

$$\begin{aligned} \overline{z} &= \frac{3}{4\pi a^3} \iint_S z^2 \, dx \, dy = \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} (a\cos\phi)^2 \frac{\partial(x,\,y)}{\partial(\phi,\,\theta)} \, d\phi \, d\theta \\ &= \frac{3}{4\pi a^3} \int_0^{2\pi} \int_0^{\pi/2} a^4 \cos^3\phi \, \sin\phi \, d\phi \, d\theta = \frac{3a}{4\pi} \cdot 2\pi \cdot \left[-\frac{1}{4}\cos^4\phi \right]_0^{\pi/2} = \frac{3}{8} a. \end{aligned}$$

C15S0M.025: By Eq. (23) of Section 15.5, the heat flow across the boundary sphere S *into* B is given by

$$R = \iint_S K(\nabla u) \cdot \mathbf{n} \, dS.$$

The divergence theorem then gives

$$R = \iiint_B \nabla \cdot (K\nabla u) \, dV = \iiint_B K\nabla^2 u \, dV.$$

C15S0M.026: Let the ball B be subdivided into small volume elements $\Delta V_1, \ \Delta V_2, \ \ldots, \ \Delta V_n$. The heat capacity c is measured in units such as calories per degree per cubic centimeter, so if Δu is small, then approximately $(c\,\Delta u)\,\Delta V_i$ calories of heat are required to raise the temperature of the volume element ΔV_i by Δu degrees. It follows that the rate at which heat is flowing into this volume element is given by

$$\Delta R_i \approx \lim_{\Delta t \to 0} \frac{(c\,\Delta u)\,\Delta V_i}{\Delta t} = c \cdot \frac{\partial u}{\partial t} \, \Delta V_i.$$

The total rate of heat flow into B is therefore

$$R = \sum_{i=1}^n \Delta R_i \approx \sum_{i=1}^n c \cdot \frac{\partial u}{\partial t} \, \Delta V_i \to \iiint_B c \cdot \frac{\partial u}{\partial t} \, dV.$$

C15S0M.027: Problems 25 and 26 imply that

$$\iiint_B c \cdot \frac{\partial u}{\partial t} \, dV = \iiint_B K\nabla^2 u \, dV$$

for *any* small ball B within the body. This can be so only if $cu_t \equiv K\nabla^2 u$; that is, $u_t = k\nabla^2 u$ (because $k = K/c$).

C15S0M.028: By Problem 17 of Section 15.6,

$$\iint_S f\frac{\partial f}{\partial n}\,dS = \iiint_T |\nabla f|^2\,dV$$

where $\partial f/\partial n = (\nabla f)\cdot \mathbf{n}$. Part (a): Let $f = u_1 - u_2$. Because $f \equiv 0$ on S,

$$\iiint_T |\nabla f|^2\,dV = \iint_S f\frac{\partial f}{\partial n}\,dS = 0.$$

Therefore $\nabla f = \mathbf{0}$ at each point of T. Part (b): $f_x = f_y = f_z = 0$ at each point of T. Therefore f is constant on T. Because $f \equiv 0$ on the boundary S of T, it now follows that $f \equiv 0$ on T. Therefore $u_1 \equiv u_2$ on T.

C15S0M.029: We begin with $\mathbf{r} = \langle\, x,\ y,\ z\,\rangle$ and $\phi = \phi(r)$ where $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. Part (a):

$$\begin{aligned}\nabla\phi(r) &= \nabla\phi\left(\sqrt{x^2+y^2+z^2}\,\right)\\ &= \left\langle \phi'(r)\cdot\frac{x}{\sqrt{x^2+y^2+z^2}},\ \phi'(r)\cdot\frac{y}{\sqrt{x^2+y^2+z^2}},\ \phi'(r)\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle\\ &= \phi'(r)\left\langle\frac{x}{r},\ \frac{y}{r},\ \frac{z}{r}\right\rangle = \frac{\mathbf{r}}{r}\phi'(r).\end{aligned}$$

Part (b): We use the result in part (a) and the result in Problem 28 of Section 15.1:

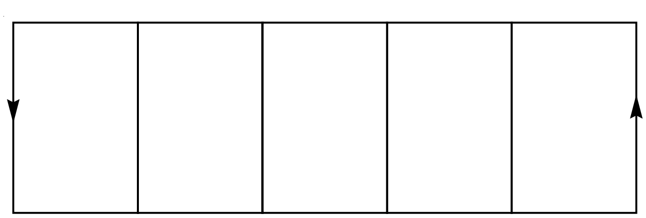
$$\nabla\cdot\big[\phi(r)\mathbf{r}\big] = \phi(r)(\nabla\cdot\mathbf{r}) + (\nabla\phi)\cdot\mathbf{r} = \phi(r)(1+1+1) + \frac{\mathbf{r}\cdot\mathbf{r}}{r}\cdot\frac{d\phi}{dr} = 3\phi(r) + r\frac{d\phi}{dr}.$$

Part (c): We use the result in part (a) and the results in Problems 29 and 35 of Section 15.1:

$$\nabla\times(\phi(r)\mathbf{r}) = \big(\phi(r)\big)(\nabla\times\mathbf{r}) + (\nabla\phi)\times\mathbf{r} = \mathbf{0} + \frac{\mathbf{r}\times\mathbf{r}}{r}\cdot\frac{d\phi}{dr} = \mathbf{0}.$$

C15S0M.030: Cut the upper half of the torus using the two semicircles in which any plane containing the z-axis intersects the torus. The outer boundary circle is oriented counterclockwise; the inner boundary circle, clockwise. Zeugma!

C15S0M.031: Let us envision a Möbius strip M in space constructed from a long narrow rectangular strip of paper by matching its ends with a half-twist. Let the strip of paper be subdivided into smaller rectangles $R_1,\ R_2,\ \dots,\ R_n$ as indicated in the following figure, with the boundary curve of each of these rectangles oriented in the positive fashion described in Section 15.7. Then the arrows cancel along any interior segment indicated in the figure, and the arrows on the ends of the strip are as indicated there—upward on the right and downward on the left.



The Möbius strip is formed by matching these two ends of the rectangular paper strip, with the two end arrows matching in direction (thus providing the necessary half-twist). If we denote by S_i the ith curvilinear rectangle on the Möbius strip (corresponding to the original R_i), then Stokes' theorem gives

$$\oint_{C_i} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_{S_i} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

for each i. Taking account of cancellation of line integrals in opposite directions along segments corresponding to interior segments in the figure, summation then yields

$$\begin{aligned}\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \sum_{i=1}^{n} \iint_{S_i} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \\ &= \sum_{i=1}^{n} \oint_{C_i} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds + 2\int_J \mathbf{F} \cdot \mathbf{T}\, ds,\end{aligned}$$

where C denotes the boundary curve of the Möbius strip and J denotes the single interior segment along which the arrows match, so that the line integrals along J do *not* cancel. Because of the final term in the last equation, this calculation does *not* yield Stokes' theorem in the form

$$\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS \;=\; \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$$

for the Möbius strip. Greater generality would take us too far afield, but surely you can well imagine that a similar "failure to cancel" would occur for any subdivision of the original narrow rectangular strip.

C15S0M.032: Let θ denote the angle between $\mathbf{u}$ and $\mathbf{r}$. The point P with position vector $\mathbf{r}$ is at distance $|\mathbf{r}|\sin\theta$ from the line of rotation determined by $\mathbf{u}$. Hence, because $|\mathbf{u}| = 1$, the velocity of P is

$$\mathbf{v} = \omega|\mathbf{r}|\sin\theta = \omega|\mathbf{u}|\,|\mathbf{r}|\sin\theta = (\omega\mathbf{u}) \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}.$$

Finally,

$$\nabla \times \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega}$$

by the result in Problem 36 of Section 15.1.

C15S0M.033: Part (a): If ΔS_i is a small piece of the boundary sphere S of the small ball B, and δ_i, $\mathbf{v}_i$, and $\mathbf{n}_i$ denote (respectively) the density, fluid flow velocity vector, and outward unit normal at time t at a typical point of ΔS_i, then the rate of *outward* fluid flow across this area element ΔS_i is approximately $\delta_i \mathbf{v}_i \cdot \mathbf{n}_i\, \Delta S_i$. Hence the rate of flow of fluid *into* B at time t is given by

$$Q'(t) \;\approx\; -\sum_{i=1}^{n} \delta_i \mathbf{v}_i \cdot \mathbf{n}_i\, \Delta S_i \;\to\; -\iint_S \delta\mathbf{v} \cdot \mathbf{n}\, dS.$$

Part (b): Equating our two expressions for $Q'(t)$, we get

$$\iiint_B \frac{\partial \delta}{\partial t}\, dV = -\iint_S \delta\mathbf{v} \cdot \mathbf{n}\, dS = -\iiint_B \nabla \cdot (\delta\mathbf{v})\, dV$$

(applying the divergence theorem on the right). The fact that

$$\iiint_B \left[\frac{\partial \delta}{\partial t} + \boldsymbol{\nabla} \cdot (\delta \mathbf{v}) \right] dV = 0$$

therefore holds for *any* small ball B within the fluid flow region implies that the integrand must vanish identically; that is,

$$\frac{\partial \delta}{\partial t} + \boldsymbol{\nabla} \cdot (\delta \mathbf{v}) \; = \; 0.$$

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